Creation and evolution of magnetic helicity

R. Jackiw*
Center for Theoretical Physics
Massachusetts Institute of Technology
Cambridge, MA 02139–4307, USA

So-Young Pi
Physics Department
Boston University
Boston, MA 02215, USA

Abstract

Projecting a non-Abelian $SU(2)$ vacuum gauge field – a pure gauge constructed from the group element U – onto a fixed (electromagnetic) direction in isospace gives rise to a nontrivial magnetic field, with nonvanishing magnetic helicity, which coincides with the winding number of U. Although the helicity is not conserved under Maxwell (vacuum) evolution, it retains one-half its initial value at infinite time.

I. INTRODUCTION

It has been suggested that primordial magnetic fields can develop large correlation lengths provided they carry nonvanishing "magnetic helicity" $\int \mathrm{d}^3r\, \mathbf{a}\cdot\mathbf{b}$, a quantity known to particle physicists as the Abelian, Euclidean Chern-Simons term. Here $\mathbf{a}$ is an Abelian gauge potential for the magnetic field $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$. If there exists a period of decaying turbulence in the early universe, which can occur after a first-order phase transition, a magnetic field with nonvanishing helicity could have relaxed to a large-scale configuration, which enjoys force-free dynamics (source currents for the magnetic fields proportional to the fields themselves) thereby avoiding dissipation [1].

In this paper we accomplish two things. First we show that configurations of $(\mathbf{a}, \mathbf{b})$, with quantized helicity, arise from vacuum configurations of a non-Abelian $SU(2)$ vector potential. The quantization occurs because $(\mathbf{a}, \mathbf{b})$ wind and intertwine in an intricate manner. We relate this "winding number" of our Abelian fields to the topological properties of the structures in a non-Abelian $SU(2)$ gauge group. Our construction is based on $\Pi_3(S_3)$, which is relevant to a non-Abelian gauge theory in 3-space. [The construction

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can also be described in terms of Hopf maps and $\Pi_3(S_2)$, which have already appeared in the literature [2]; we show how our expressions appear in that formalism.] Only integer winding numbers occur in the mathematical setting of $\Pi_3(S_3)$ or $\Pi_3(S_2)$; yet we find that half-integer windings also lead to interesting Abelian gauge fields.

Second, we study the time evolution of the magnetic helicity. Since $\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}^3r\,\mathbf{a}\cdot\mathbf{b} = -2\int \mathrm{d}^3r\,\mathbf{E}\cdot\mathbf{b}$ as a consequence of the definition for the electric field, $\mathbf{E}$, time variation is determined by the specific physical situation that fixes $\mathbf{E}\cdot\mathbf{b}$. For plasma physics or magnetohydrodynamics the projection of $\mathbf{E}$ on $\mathbf{b}$ is proportional to the resistivity of the medium, and vanishes for infinite conductivity (zero resistivity), see Sec. IV. In that approximation, helicity is conserved. However, cosmological electromagnetic fields can be expected also to experience evolution in vacuum, where $\mathbf{E}\cdot\mathbf{b}\neq 0$. Therefore, it is interesting to determine what happens to the magnetic helicity under Maxwell evolution. Specifically, we posit that at $t = 0$ there are magnetic fields with quantized helicity, and no electric fields. With this initial data, we solve the Maxwell equations and find that at $t = \infty$ the helicity is precisely half its initial value. The calculation is carried out explicitly for two interesting cases (integer and half-integer quanta) and then a general argument is given, which requires a regularity hypothesis.

In Section II, we describe how to construct from non-Abelian vacuum fields Abelian gauge fields with quantized helicity. In Section III, we study time evolution. Concluding remarks comprise Section IV, where we also speculate on the applicability of these mathematical considerations to the $SU(2)\times U(1)$ "standard model", before and after its electroweak phase transition.

II. HELICITY OF GAUGE FIELDS

A. Chern-Simons structures

The Chern-Simons number of a non-Abelian gauge potential A_i^a is given by

$$\begin{aligned} CS(A) &= \frac{1}{16\pi^2}\int \mathrm{d}^3r\,\epsilon^{ijk}(A_i^a\partial_j A_k^a + \tfrac{1}{3}f^{abc}A_i^aA_j^bA_k^c) \\ &= -\frac{1}{8\pi^2}\int \mathrm{d}^3r\,\epsilon^{ijk}\,\mathrm{tr}(A_i\partial_j A_k + \tfrac{1}{3}A_i[A_j, A_k]) \\ &= -\frac{1}{8\pi^2}\int \mathrm{tr}(AdA + \tfrac{2}{3}A^3) \end{aligned} \tag{2.1}$$

In the second equality, the A_i are elements of the Lie algebra with generators T^a

$$A_i = A_i^aT^a\ , \quad [T^a, T^b] = f_{abc}T^c\ , \quad \mathrm{tr}\,T^aT^b = -\delta_{ab}/2$$

while form notation is used in the third equality: $A \equiv A_i\,\mathrm{d}x^i$. The normalization factor $\frac{1}{16\pi^2}$ is chosen for later convenience, and is also maintained in the Abelian limit, where only the bilinear part of (2.1) survives. Thus our Chern-Simons quantity is $\frac{1}{16\pi^2}$ times the magnetic helicity, which entered physics in the work of Woltier [3] and was subsequently further elaborated by many people, including Moffatt, Berger and Field, as well as Arnold and Khesin [4]. Particle physicists have considered both Abelian and non-Abelian Chern-Simons terms on 3-dimensional Minkowski space in studies of $(2+1)$-dimensional gauge theories [5].

To show how Abelian fields with nonvanishing and quantized helicity arise from non-Abelian vacuum configurations, we recall first that under a gauge transformation

$$A \to A^U \equiv U^{-1}AU + U^{-1}\mathrm{d}U \tag{2.2}$$

where U is an element of the gauge group, the Chern-Simons term transforms as

$$CS(A^U) = CS(A) + \frac{1}{8\pi^2}\int \mathrm{d}(\mathrm{tr}\,\mathrm{d}U\,U^{-1}A) + \frac{1}{24\pi^2}\int \mathrm{tr}(U^{-1}\mathrm{d}U)^3 \tag{2.3}$$

With sufficiently regular A and U, the second term on the right side integrates to zero [6]. However, the last term, also a total derivative – although that is not apparent from its formula (but see below) – depends only on the group element, and is not damped by any fall-off of A. Indeed its value is the winding number of U, which effects a mapping from 3-space into the group.

$$W(U) = \frac{1}{24\pi^2}\int \mathrm{d}r(U^{-1}\mathrm{d}U)^3 \tag{2.4}$$

$W(U)$ is an integer when U is nonsingular for finite $\mathbf{r}$ and tends to $\pm I$ at infinity, for then 3-space can be compactified to the 3-sphere, and we assume that the gauge group contains $SU(2)$. Thus we are relying on $\Pi_3(SU(2)) = \Pi_3(S_3) =$ integers. It follows therefore that the Chern-Simons number of a non-Abelian vacuum gauge potential, that is, one which is a pure gauge

$$\mathcal{A} = U^{-1}\mathrm{d}U \tag{2.5}$$

is the winding number of U.

Let us now specialize to $SU(2)$. Then U involves the Pauli matrices $T^a = \sigma^a/2i$.

$$\begin{aligned} U &= e^{\omega^a T^a} \\ &= \cos f/2 - i\sigma^a\widehat{\omega}^a \sin f/2 \end{aligned} \tag{2.6}$$

and the winding number is explicitly seen to involve a total derivative [7]

$$W(U) = -\frac{1}{16\pi^2}\int \mathrm{d}\Big(\epsilon_{abc}\widehat{\omega}^a\,\mathrm{d}\widehat{\omega}^b\,\mathrm{d}\widehat{\omega}^c\,(f - \sin f)\Big) \tag{2.7}$$

where $\widehat{\omega}^a$ is a unit vector in isospace: $\omega^a = \widehat{\omega}^a f$.

B. Constructing an Abelian gauge field

Consider next the Abelian gauge potential 1-form a, constructed from a non-Abelian vacuum configuration $U^{-1}\mathrm{d}U$ by projecting the latter on a fixed direction in isospace, specified by a constant unit vector $\widehat{n}^a$

$$a = i \text{ tr } \widehat{n}^a \sigma^a U^{-1} \mathrm{d}U \tag{2.8a}$$

In components, this reads

$$a_i = \widehat{n}^a \mathcal{A}_i^{\ a} \tag{2.8b}$$

where $\mathcal{A}_i^{\ a}$ is the vacuum (pure gauge) non-Abelian potential

$$U^{-1} \partial_i U = \mathcal{A}_i^{\ a} \frac{\sigma^a}{2i} \tag{2.9}$$

which satisfies

$$\epsilon^{ijk} \partial_j \mathcal{A}_k^{\ a} = -\tfrac{1}{2} \epsilon^{ijk} \epsilon_{abc} \mathcal{A}_j^{\ b} \mathcal{A}_k^{\ c} \tag{2.10}$$

and carries the winding number

$$W(U) = -\frac{1}{96\pi^2} \int \mathrm{d}^3 r \, \epsilon^{ijk} \epsilon_{abc} \mathcal{A}_i^{\ a} \mathcal{A}_j^{\ b} \mathcal{A}_k^{\ c} \tag{2.11}$$

Note that **a** is **not** an Abelian pure gauge.

We now show that the magnetic helicity of **a** coincides with $W(U)$. The magnetic helicity, with our normalization, is given by

$$\begin{aligned} H(a) &= \frac{1}{16\pi^2} \int \mathrm{d}^3 r \, \epsilon^{ijk} a_i \partial_j a_k \\ &= \frac{1}{16\pi^2} \widehat{n}^a \widehat{n}^b \int \mathrm{d}^3 r \, \epsilon^{ijk} \mathcal{A}_i^{\ a} \partial_j \mathcal{A}_k^{\ b} \\ &= -\frac{1}{32\pi^2} \widehat{n}^a \widehat{n}^b \int \mathrm{d}^3 r \, \mathcal{A}_i^{\ a} \epsilon^{ijk} \epsilon_{bcd} \mathcal{A}_j^{\ c} \mathcal{A}_k^{\ d} \end{aligned} \tag{2.12a}$$

where (2.8) and (2.10) have been used. But

$$\epsilon^{ijk} \mathcal{A}_i^{\ a} \mathcal{A}_j^{\ c} \mathcal{A}_k^{\ d} = \tfrac{1}{6} \epsilon_{acd} \epsilon^{ijk} \epsilon_{a'b'c'} \mathcal{A}_i^{\ a'} \mathcal{A}_j^{\ b'} \mathcal{A}_k^{\ c'}$$

Thus

$$H(a) = -\frac{1}{96\pi^2} \int \mathrm{d}^3 r \, \epsilon^{ijk} \epsilon_{abc} \mathcal{A}_i^{\ a} \mathcal{A}_j^{\ b} \mathcal{A}_k^{\ c} = W(U) \tag{2.12b}$$

and we conclude that the magnetic helicity is quantized by the winding number of the non-Abelian vacuum configuration.

The form of the Abelian potential **a** may be given explicitly. Parameterizing the unit vector $\widehat{\omega}^a$ as

$$\widehat{\omega}^a = (\sin\Theta \cos\Phi, \sin\Theta \sin\Phi, \cos\Theta) \tag{2.13}$$

and taking $\widehat{n}^a$ to point in the third (electromagnetic) direction, we find

$$a = \cos\Theta \, \mathrm{d}f - (\sin f) \sin\Theta \, \mathrm{d}\Theta - (1 - \cos f) \sin^2\Theta \, \mathrm{d}\Phi \tag{2.14a}$$

An alternate formula presents (2.14a) in "Clebsch" form, which will be useful later. (The Clebsch form for an arbitrary 3-vector $\mathbf{V}$ is $\mathbf{V} = \boldsymbol{\nabla}\gamma + \alpha\boldsymbol{\nabla}\beta$ where γ, α, β are three scalar functions.)

$$a = \mathrm{d}(-2\Phi) + 2\Big(1 - (\sin^2 f/2)\sin^2\Theta\Big)\,\mathrm{d}(\Phi + \tan^{-1}[(\tan f/2)\cos\Theta]\Big) \tag{2.14b}$$

The magnetic field is determined by the 2-form obtained from (2.14a)

$$\mathrm{d}a = \sin\Theta\Big[(1-\cos f)\,\mathrm{d}f\,\mathrm{d}\Theta - 2(1-\cos f)\cos\Theta\,\mathrm{d}\Theta\,\mathrm{d}\Phi + (\sin f)\sin\Theta\,\mathrm{d}\Phi\,\mathrm{d}f\Big] \tag{2.15a}$$

or from the Clebsch expression (2.14b)

$$\mathrm{d}a = -2\,\mathrm{d}\Big((\sin^2 f/2)\sin^2\Theta\Big)\,\mathrm{d}\Big(\Phi + \tan^{-1}[(\tan f/2)\cos\Theta]\Big) \tag{2.15b}$$

Finally the magnetic helicity becomes, according to (2.14a) and (2.15a)

$$H(a) = -\frac{1}{8\pi^2}\int(1-\cos f)\sin\Theta\,\mathrm{d}f\,\mathrm{d}\Theta\,\mathrm{d}\Phi \tag{2.16a}$$

or from (2.14b) and (2.15b)

$$H(a) = \frac{1}{4\pi^2}\int \mathrm{d}\Phi\,\mathrm{d}\Big((\sin^2 f/2)\sin^2\Theta\Big)\,\mathrm{d}\Big(\tan^{-1}[(\tan f/2)\cos\Theta]\Big) \tag{2.16b}$$

In the Clebsch parameterization, the magnetic helicity is seen to involve integration of a total derivative and is therefore given by a surface integral.

The explicit Clebsch expressions for a and $\mathrm{d}a$ demonstrate that one may find two magnetic surfaces, S_n $(n = 1, 2)$, which satisfy $\mathrm{d}a\,\mathrm{d}S_n = \mathbf{b}\cdot\boldsymbol{\nabla}S_n = 0$

$$\begin{aligned} S_1 &= (\sin f/2)\sin\Theta = c \\ S_2 &= \Phi + \tan^{-1}\Big[(\tan f/2)\cos\Theta\Big] = \phi_0 \\ &\quad c, \phi_0 \text{ constants} \end{aligned} \tag{2.17}$$

The intersection of these surfaces forms magnetic lines, that is, integral curves of $\mathbf{b}$ that solve the dynamical system

$$\frac{\mathrm{d}\mathbf{r}(\tau)}{\mathrm{d}\tau} = \mathbf{b}\Big(\mathbf{r}(\tau)\Big) \tag{2.18}$$

where τ is an evolution parameter for the dynamical system. Evidently, for our configuration this problem in integrable, leading to curves given by

$$\begin{aligned} \cos f/2 &= \sqrt{1-c^2}\,\cos(\Phi - \phi_0) \\ \sin\Theta &= \frac{c}{\sqrt{\sin^2(\Phi-\phi_0) + c^2\cos^2(\Phi-\phi_0)}} \end{aligned} \tag{2.19}$$

C. Hopf Mapping

It is known that the Abelian Chern-Simons term can be related to the degree of the Hopf map, which is quantized according to $\Pi_3(S_2)$ [2]. The construction of the relevant vector potential proceeds in the following manner. Consider a complex spinor $u = \binom{u_1}{u_2}$ with $u^* u = |u_1|^2 + |u_2|^2 = 1$. Then $N^a \equiv u^* \sigma^a u$ is a unit vector and the Hopf curvature is defined by

$$b^i = \frac{1}{4}\epsilon^{ijk}\epsilon_{abc}N^a \partial_j N^b \partial_k N^c \tag{2.20}$$

Because b^i is divergence-free, it can be written as the curl of a potential a_i given by $-iu^*\partial_i u$. By comparison with (2.14b) we find that our potential arises when u is chosen as

$$u = \begin{pmatrix} \sqrt{1 - (\sin^2 f/2)\sin^2\Theta}\, e^{2i\tan^{-1}[(\tan f/2)(\cos\Theta)]} \\ (\sin f/2)\sin\Theta e^{-2i\Phi} \end{pmatrix} \tag{2.21}$$

and the Hopf index coincides with the helicity of the here-constructed $(\mathbf{a}, \mathbf{b})$.

Evidently, the present expressions are awkward when compared to those based on $\Pi_3(S_3)$, presumably because the latter construction is directly related to gauge fields.

D. Explicit Expressions

Henceforth we work with explicit expressions obtained by identifying $\widehat{\omega}^a$ in (2.6) with the radial unit vector $\widehat{r}^a$ and taking f to depend only on r. This further means that Θ in Φ in (2.13) are identified with polar and azimuthal angles of spherical coordinates θ and ϕ, with ranges $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$ respectively. (A simple generalization would be to identify Θ and Φ with integer multiples of θ and ϕ.) The magnetic helicity (2.16) then becomes

$$H = -\frac{1}{2\pi}\int_0^\infty \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r}(f - \sin f) = -\frac{1}{2\pi}(f - \sin f)\Big|_{r=\infty} \tag{2.22}$$

where we have taken $f(0)$ to vanish. Observe that when $\sin f(\infty)$ is nonvanishing H is an irrational/transcendental number. When $f(\infty)$ is an even integer multiple of π, $\sin f(\infty)$ vanishes and H is an integer. But note that also an odd integer multiple of π for $f(\infty)$ leads to vanishing $\sin f(\infty)$, and a half-integer value for H. Therefore it appears to us that configurations with half-integer magnetic helicity are also singled out. The half-integer fields share further good properties with the integer-valued configurations: we shall show below that only for these two the winding number does not change under the gauge transformation to the Coulomb gauge.

When reference is made back to the non-Abelian vacuum configuration, which determines $\mathbf{a}$, we see from (2.6) that for the integer windings $U \xrightarrow[r\to\infty]{} \pm I$, as expected. For the half-integer ones, a "hedgehog" asymptote is attained: $U \xrightarrow[r\to\infty]{} \pm i\boldsymbol{\sigma}\cdot\widehat{r}$.

The vector potential $\mathbf{a}$ and the magnetic field $\mathbf{b} = \boldsymbol{\nabla} \times \mathbf{a}$ are neatly described by spherical components, which are ϕ-independent

$$a_r = (\cos\theta) f' \tag{2.23a}$$

$$a_\theta = -(\sin\theta)\frac{1}{r}\sin f \tag{2.23b}$$

$$a_\phi = -(\sin\theta)\frac{1}{r}(1-\cos f) \tag{2.23c}$$

$$b_r = -2(\cos\theta)\frac{1}{r^2}(1-\cos f) \tag{2.24a}$$

$$b_\theta = (\sin\theta)\frac{f'}{r}\sin f \tag{2.24b}$$

$$b_\phi = (\sin\theta)\frac{f'}{r}(1-\cos f) \tag{2.24c}$$

(The dash denotes differentiation with respect to r.) The Clebsch representation (2.14b), (2.15b) and (2.16b) gives a clear picture of the helicity. From (2.16b) we have

$$H = -\frac{1}{8\pi^2}\int \mathrm{d}^3r\, \partial_i\phi b^i = -\frac{1}{8\pi^2}\int \mathrm{d}^3r\, \frac{1}{r\sin\theta}\Big(\frac{\partial}{\partial\phi}\phi\Big) b_\phi \tag{2.25a}$$

Since b_ϕ is ϕ-independent, the ϕ integration is trivial, leaving

$$H = -\frac{1}{4\pi}\int_0^\pi \mathrm{d}\theta \int_0^\infty r\,\mathrm{d}r\, b_\phi\Big|_{\phi=2\pi} \tag{2.25b}$$

The integral is over the positive-x (x, z) half-plane and b_ϕ is the toroidal magnetic field, perpendicular to that plane. So H measures the flux of the toroidal magnetic field through a half-plane.

Note that the functions occurring in our Clebsch parameterization are multivalued, owing to the presence of the naked azimuthal angle ϕ and the $\tan^{-1}$. This is as it must be, because the helicity is nonvanishing [8]. Indeed the multivaluedness of ϕ is responsible for the nonzero value of the ϕ "surface" integral (2.25), which reproduces the helicity.

When using $\mathbf{a}$ in a description of electromagnetic fields, we are effectively in the $a^0 = 0$, Weyl gauge. But also the vector potential must be transverse since there are no sources. However, the potential in (2.23) is not transverse for general f. Transversality may be achieved in one of two ways. We may perform a gauge transformation, transforming $\mathbf{a}$ to a transverse expression $\mathbf{a}^T$. Alternatively, we may choose the function f so that $\mathbf{a}$ is transverse. We discuss these two possibilities in turn.

The transversality condition affects only the poloidal components of $\mathbf{a}$ (a_r and a_θ), because the toroidal component (a_ϕ) is ϕ-independent. By using a gauge function proportional to $\cos\theta$ times a function of r, one can choose that function so that the resulting, gauge equivalent potential is transverse. The components of the transverse potential then read

$$a_r^T = (\cos\theta) F$$

$$a_\theta^T = -(\sin\theta)\frac{1}{2r}\Big(r^2 F\Big)'$$

$$a_\phi^T = -(\sin\theta)\frac{1}{r}\Big(1-\cos f\Big)$$

$$F(r) \equiv \tfrac{2}{3}\int_r^\infty \frac{\mathrm{d}r'}{r'^2}(f-\sin f) - \frac{4}{3r^3}\int_0^r \mathrm{d}r'\, r'(f-\sin f) \tag{2.26}$$

Of course the magnetic field remains unchanged, but we must still check the helicity integral: while the integrand is gauge dependent, the integral changes only be a surface term. It may be that the above gauge transformation contributes from the surface. Indeed this happens for general f: one finds

$$H(a^T) = H(a) + \frac{1}{6\pi}(1-\cos f)\sin f\Big|_{r=\infty} \tag{2.27}$$

and the helicity of the gauge-equivalent, transverse configuration differs from the original expression. However if, and only if, f goes at infinite r to an even or odd multiple of π, the gauge variance vanishes. Thus both integer and half-integer windings are stable against this gauge transformation, which renders the potential transverse, but for other, irrational windings, the "winding number" loses its meaning because it is gauge dependent [9].

The other way to achieve transversality is to impose that condition on the original configuration (2.23), thereby determining f. Transversality requires that f satisfy

$$r^2 f'' + 2rf' - 2\sin f = 0 \tag{2.28}$$

Although this equation cannot be solved by elementary functions, it is easily analyzed by analogy to a mechanical problem where "time" is $\ln r/r_0$. One finds two solutions that are regular at the origin, vanishing linearly with r, and tending to $\pm\pi$ in an oscillatory manner for large r. Moreover, the scale of r is arbitrary: the solutions are a universal function of r/r_0 and its negative; the positive solution is plotted in Fig. 1. Evidently this transverse potential necessarily corresponds to half-integer winding [10].

III. TIME EVOLUTION

According to Maxwell's vacuum equations the transverse vector potential satisfies the wave equation, which we integrate subject to the initial condition that $\mathbf{a}$ is given at time $t=0$ by (2.26), with a definite choice for f, and that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}$ is zero. Thus at initial time, there is no electric field and only the magnetic field (2.24) is present, maintained for $t<0$ by a steady current $\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{b}$.

$$j_r = 2(\cos\theta)\frac{1}{r^2}(f-\sin f)'$$

$$j_\theta = -(\sin\theta)\frac{1}{r}(f-\sin f)''$$

$$j_\phi = (\sin\theta)\Big(\frac{1}{r^2}[r(1-\cos f)]'\Big)' \tag{3.1}$$

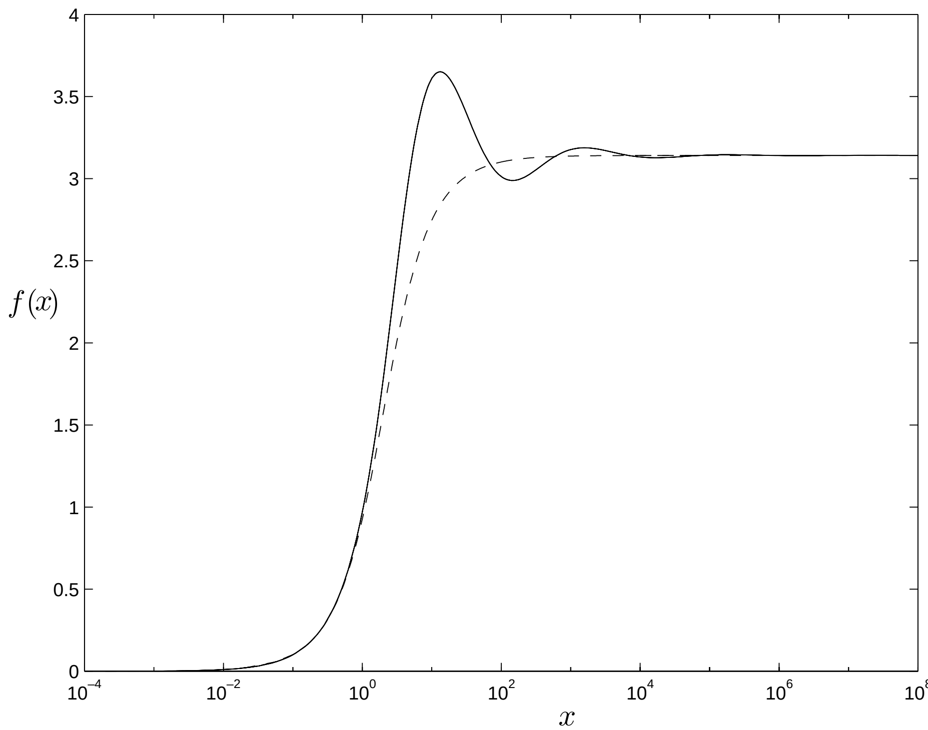

FIG. 1. Profile of regular solution to (2.28) (*solid line*); profile of $2\tan^{-1} x/2$, which possesses the same $x = 0$ and $x = \infty$ asymptotes (*dashed line*).

and carrying energy $\mathcal{E} = \frac{1}{2}\int \mathrm{d}^3r\, \mathbf{b}^2$

$$\begin{aligned}\mathcal{E} &= \int_0^\infty \frac{\mathrm{d}r}{r^2}\sin^2 f/2 \int_0^\pi \sin\theta\,\mathrm{d}\theta\Big[\Big(\sin^2 f/2 + r^2 f'^2/4\Big) + \cos 2\theta\Big(\sin^2 f/2 - r^2 f'^2/4\Big)\Big] \\ &= \frac{8\pi}{3}\int_0^\pi \frac{\mathrm{d}r}{r^2}(1-\cos f)(1-\cos f + r^2 f'^2) \qquad (3.2)\end{aligned}$$

We determine the time asymptote for the helicity in two cases

$$f = 2\tan^{-1} r/r_0 \qquad (3.3a)$$

$$f = 4\tan^{-1} r/r_0 \qquad (3.3b)$$

The former, with $f \xrightarrow[r\to\infty]{} \pi$, corresponds to half-integer winding; for the latter $f \xrightarrow[r\to\infty]{} 2\pi$, one has integer winding, also the energy density is spherically symmetric; see (3.2) [10]. The corresponding magnetic fields are

half-integer winding

$$\begin{aligned} b_r &= -4(\cos\theta)\frac{1}{r^2+r_0^2} \\ b_\theta &= 4(\sin\theta)\frac{r_0^2}{(r^2+r_0^2)^2} \\ b_\phi &= 4(\sin\theta)\frac{r r_0}{(r^2+r_0^2)^2} \qquad (3.4a)\end{aligned}$$

$$\text{integer winding} \qquad b_r = -16(\cos\theta)\frac{r_0^2}{(r^2+r_0^2)^2}$$

$$b_\theta = 16(\sin\theta)\frac{r_0^2(r_0^2-r^2)}{(r^2+r_0^2)^3}$$

$$b_\phi = 32(\sin\theta)\frac{rr_0^3}{(r^2+r_0^2)^3} \tag{3.4b}$$

The time-evolved fields are obtained by standard Fourier transform techniques, and the helicity integral is evaluated as function of time. We find for the two cases

$$H = -\tfrac{1}{2} + \tfrac{1}{4}\frac{t^2}{r_0^2+t^2}\Big(1 + \tfrac{2}{3}\frac{r_0^2}{r_0^2+t^2}\Big) \tag{3.5a}$$

$$H = -1 + \tfrac{1}{2}\frac{t^2}{r_0^2+t^2}\Big(1 + \frac{r_0^2}{r_0^2+t^2} + \frac{8r_0^6}{(r_0^2+t^2)^3}\Big) \tag{3.5b}$$

It is seen that for $t \to \infty$, H attains one-half its value at $t = 0$.

While one would like to have an understanding how the localization of the helicity changes with time, it does not seem possible to pose such a question in a meaningful way, because the helicity density, namely, the integrand that defines H, is gauge dependent, and without invariant meaning. Indeed, as we have noted earlier, even the integrated helicity can be gauge dependent when the fields and the gauge function survive on the surfaces bounding the integration region. (This points to an analogy with the energy in general relativity, whose density is diffeomorphism-dependent. Only the integrated quantity is invariant and a unique value is determined only after asymptotic conditions are prescribed. Furthermore, the energy may be presented as a surface integral and so also can the helicity, when the Clebsch parameterization is used for the gauge potential.)

The result that under Maxwell evolution H decreases to half its value at infinite time can also be understood from a general argument. In the Weyl-Coulomb gauge, which we are using, the vector potential satisfies the wave equation, which is uniquely solved with our posited initial conditions by

$$\mathbf{a}^T(t,\mathbf{r}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos kt e^{-i\mathbf{k}\cdot\mathbf{r}}\tilde{\mathbf{a}}^T(\mathbf{k}) \tag{3.6}$$

Here $\tilde{\mathbf{a}}^T(\mathbf{k})$ is Fourier transform of the initial data (2.26) and $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}^T(t,\mathbf{r}) = -\mathbf{E}(t,\mathbf{r})$ vanishes at $t = 0$. It follows that

$$H = -i\int \frac{\mathrm{d}^3k}{(2\pi)^3}\cos^2 kt\epsilon^{ijk}k^i\tilde{a}_j^T(\mathbf{k})\tilde{a}_k^{T^*}(\mathbf{k}) \tag{3.7}$$

$$= \frac{1}{2i}\int \frac{\mathrm{d}^3k}{(2\pi)^3}(1+\cos 2kt)\epsilon^{ijk}k^i\tilde{a}_j^T(\mathbf{k})\tilde{a}_k^{T^*}(\mathbf{k}) \tag{3.8}$$

By appealing to the Riemann-Lebesgue lemma, we can argue that the term involving the cosine disappears at large t, owing to destructive interference, leaving half the value at

$t = 0$. However, this step is justified provided the rest of integrand is well behaved at $k = 0$, which in turn depends on the behavior of $\mathbf{a}^T(\mathbf{r})$ at large r. A dimensional estimate shows that a large-distance decrease of $\mathbf{a}^T$ faster than $1/r$ is sufficient, which would mean that $\mathbf{b}$ should decrease faster than $1/r^2$. But (3.4a) exhibits a large r behavior for $\mathbf{b}$ of order $1/r^2$, modulated by an angular factor. Since our explicit calculation supports the general argument, the angular factor evidently provides sufficient large-r damping.

IV. CONCLUSION

Our investigation is based on the connection between the winding number of non-Abelian gauge group elements and the Chern-Simons number of Abelian gauge fields that are obtained from the former by projection. This mathematical fact suggests a physical scenario for the $SU(2)\times U(1)$ "standard model". Before its (first-order) phase transition, its vacuum could be populated by pure gauge, vacuum configurations $U^{-1}dU$ of the $SU(2)$ gauge group, which carry nonvanishing winding numbers. After the phase transition, one direction in isospace is identified with electromagnetism, and the projection of the vacuum configuration becomes a magnetic field with nonvanishing helicity. It remains to be shown whether the above scenario is energetically stable and can be justified on physical grounds.

Another problem deserving further study concerns nontrivial evolution of an initial configuration with magnetic helicity. Rather than using the free Maxwell equations, one would rely on the magnetohydrodynamical ones, which make use of Ohm's law to express $\mathbf{E}$ in terms of $\mathbf{b}$ and $\mathbf{j}$. Its nonrelativistic form is

$$\mathbf{j} = \eta(\mathbf{E} + \mathbf{v} \times \mathbf{b}) \tag{4.1}$$

where η is conductivity and $\mathbf{v}$ is the fluid velocity, taken to be divergenceless in a fluid of constant density. Inserting this in the Maxwell equation

$$\frac{\partial \mathbf{b}}{\partial t} + \boldsymbol{\nabla} \times \mathbf{E} = 0 \tag{4.2}$$

produces an evolution equation for $\mathbf{B}$

$$\frac{\partial \mathbf{b}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{b}) = -\frac{1}{\eta}\boldsymbol{\nabla} \times \mathbf{j}\ . \tag{4.3}$$

Further, approximating $\mathbf{j}$ by $\boldsymbol{\nabla} \times \mathbf{b}$, that is, ignoring $\partial\mathbf{E}/\partial t$ because it is negligible on the relevant time scales, converts the above into

$$\frac{\partial \mathbf{b}}{\partial t} - \boldsymbol{\nabla} \times (\mathbf{v} \times \mathbf{b}) = \frac{1}{\eta}\nabla^2 \mathbf{b} \tag{4.4}$$

which can be analyzed either with $\mathbf{v}$ prescribed externally or determined self-consistently by its Euler equation, with initial $\mathbf{b}$ of the form (2.24). An interesting choice for an external $\mathbf{v}$ could be a transverse form that carries nonvanishing "kinetic helicity" $\int \mathrm{d}^3 r\, \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{v})$; for example, what one gets by taking $\mathbf{v}$ in the form $\mathbf{a}$ of (2.23) with f solving

(2.28). For zero resistivity (infinite conductivity) the right side of (4.4) is absent – there is no dissipation. Then $\mathbf{E} = -\mathbf{v} \times \mathbf{b}$ and magnetic helicity is conserved, since $\mathbf{E} \cdot \mathbf{b}$ vanishes.

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[9] The response to more general gauge transformations is as follows. Choosing the gauge function $(\cos\theta)\omega(r)$ [so that the form (2.23) is preserved] a_r, a_θ change into $\tilde{a}_r = (\cos\theta)(f' + \omega')$, $\tilde{a}_\theta = -(\sin\theta)\frac{1}{r}(\sin f + \omega)$. Consequently the winding number becomes $H(\tilde{a}) = H(a) - \frac{1}{6\pi}(1 - \cos f)\omega|_{r=\infty}$. Configurations with integer winding remain gauge invariant [provided $\omega(r)$ does not increase too rapidly at infinity]. For all other configurations, $\omega(r)$ must vanish at infinity to maintain gauge invariance of H. Moreover, the gauge function $(\cos\theta)\omega(r)$ is "well-defined" at infinity only for vanishing ω. (We thank A. Polychronakos for discussion.)

[10] Eq. (2.28) was integrated numerically by Y. Bergner.

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