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Topologically Massive Gauge Theories

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Gauge vector and gravity models are studied in three-dimensional space-time, where novel, gauge invariant, P and T odd terms of topological origin give rise to masses for the gauge fields. In the vector case, the massless Maxwell excitation, which is spinless, becomes massive with spin 1. When interacting with fermions, the quantum theory is infrared and ultraviolet finite in perturbation theory. For non-Abelian models, topological considerations lead to a quantization condition on the dimensionless coupling constant–mass ratio. Ordinary Einstein gravity is trivial, but when augmented by our mass term, it acquires a propagating, massive, spin 2 mode. This theory is ghost-free and causal, although of third-derivative order. Quantum calculations are presented in both the Abelian and non-Abelian vector models, to exhibit some of the delicate aspects of infrared behavior, and regularization dependence. © 1982 Academic Press

I. INTRODUCTION

The study of vector and tensor gauge theories in three-dimensional space-time is motivated by their connection to high temperature behavior of four-dimensional models [1], and is justified by the special properties which they enjoy. We shall exhibit and discuss some of the more striking aspects arising from the topology of odd-dimensional spaces, where topologically non-trivial, gauge invariant terms provide masses for the gauge fields [2].

The massive gauge models have a number of interesting features. Einstein's gravity, which is trivial and without propagation in three dimensions, becomes a dynamical theory, with a propagating particle. In general, both for vector and

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tensor fields, spin and number of degrees of freedom are different in the massless and massive cases. Moreover, the particle content also differs from that in conventionally massive [gauge non-invariant] theories. For vector fields, our models possess a single, parity-violating, massive, spin 1 excitation, in contrast to a single, massless, spin 0 excitation in the Maxwell theory, and to a pair of spin 1 degrees of freedom in gauge non-invariant models with mass [3]. Similarly, with second-rank tensor fields, describing gravity, the topological mass term leads to a single, parity-violating, spin 2 particle, whereas a conventional [gauge non-invariant] mass term gives rise to a spin 2 doublet. Furthermore, the topological term is of third-derivative order, yet the single propagating mode is governed by the Klein–Gordon equation.

The topological mass terms are closely related to the Chern–Simons secondary characteristic classes [4].[3] These are in turn obtained from the odd-parity, four-dimensional Pontryagin invariants, which govern topological structure of gauge theories in four dimensions [5]. The topological aspects of our mass terms are put into evidence by an interesting analogy between our Abelian theory and vortex excitations in the Abelian Higgs model [6]. More dramatic is an effect in non-Abelian examples: gauge invariance implies that the mass term [in a dimensionless combination with the dimensionful coupling constant] is quantized.

At the quantum level, the mass term provides an infrared cutoff for the super-renormalizable vector theories. Thus the topological mass suggests a new cure for the infrared problem of three-dimensional vector gauge theories,[4] without disturbing the ultraviolet or gauge aspects. This may be understood from the fact that the mass term in the action is bilinear in the mass μ and in the derivative. Thus, for small momenta p, the propagator behaves as $(\mu p)^{-1}$, while retaining its p^{-2} large momentum form. Nevertheless, some delicate points remain in perturbative calculations. Gauges must be used which do not artificially introduce infrared singularities. Low orders of perturbation theory produce answers which are regulator dependent. Gauge-variant amplitudes are calculable, but one cannot extract sensible on-shell parameters from them in the non-Abelian theory. For gravity there is partial improvement in ultraviolet behavior due to the third derivatives, but they do not dominate the entire propagator: one mode retains a p^{-2} propagation. It also appears likely that there exists a supergravity version of the model, with a corresponding topological mass term providing the Rarita–Schwinger action with dynamical content in the form of a massive spin $\frac{3}{2}$ excitation.

[3] The Chern–Simons structure, which is intrinsically odd-dimensional, occurred previously in a fixed-time [Schrödinger picture] formulation of four-dimensional Yang–Mills and gravitational quantum theories; see [5].

[4] Since three-dimensional gauge theories are super-renormalizable, perturbation theory for unrenormalized amplitudes is infrared divergent when no mass terms are present. The divergences are more severe than in four-dimensional electrodynamics, where the integrals arising in perturbation theory are infrared finite, and divergences appear only after mass shell renormalization is enforced. A cure for the three-dimensional divergences has been found by resumming the perturbative expansion; see [7]. Also, non-perturbative arguments have been put forward that the massless theory generates a mass dynamically; see [8]. Here we are considering another alternative: a mass is introduced gauge invariantly into the Lagrangian.

In the succeeding three sections, we shall deal with Abelian vector gauge theories, both free and coupled to fermions; then with Yang–Mills theory; and finally gravity will be discussed. Concluding remarks comprise Section V.

II. ABELIAN GAUGE THEORY

A. *General Properties*

Topologically massive spinor electrodynamics in three dimensions is governed by a Lagrangian composed of gauge field, fermion and interaction terms:

$$\mathscr{L} = \mathscr{L}_G + \mathscr{L}_F + \mathscr{L}_I. \tag{2.1}$$

These are respectively

$$\mathscr{L}_G = \frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{\mu}{4} \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.2a}$$

$$\mathscr{L}_F = i\bar{\psi}\partial\!\!\!/ \psi - m\bar{\psi}\psi, \tag{2.2b}$$

$$\mathscr{L}_I = -J^\mu A_\mu, \qquad J^\mu = -e\bar{\psi}\gamma^\mu\psi. \tag{2.2c}$$

The coupling constant e has dimension $[\text{mass}]^{1/2}$. The equations of motion

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2} \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} = J^\nu, \tag{2.3a}$$

$$(i\partial\!\!\!/ + eA\!\!\!/ - m)\, \psi = 0 \tag{2.3b}$$

are invariant against gauge transformations,

$$A_\mu \to A_\mu + \frac{1}{e} \partial_\mu \Omega, \tag{2.4a}$$

$$\psi \to e^{i\Omega} \psi, \tag{2.4b}$$

while the gauge Lagrangian changes by a total derivative[5]

$$\mathscr{L}_G \to \mathscr{L}_G + \partial_\alpha \left(\frac{\mu}{4e} \varepsilon^{\alpha\mu\nu} F_{\mu\nu} \Omega \right). \tag{2.5}$$

[Of course $\mathscr{L}_F \to \mathscr{L}_I$ is gauge invariant.] The theory is obviously Poincaré invariant.

[5] Apart from a total derivative, $\mathscr{L}_G$ may be written in a gauge invariant form, which, however, is spatially non-local, and Lorentz non-invariant. This follows from the identity $(\mu/4)\, \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha = (\mu/2)\, B\, \mathbf{\nabla}^{-1} \cdot \mathbf{E} - (\mu/2)\, \mathbf{E} \cdot \mathbf{\nabla}^{-1} B - (\mu/4)\, \varepsilon^{\mu\nu\alpha}\, \partial_\alpha (F_{\mu\nu}\, \mathbf{\nabla}^{-1} \cdot \mathbf{A})$. Here $B = -\frac{1}{2} \varepsilon^{ij} F_{ij} = \varepsilon^{ij}\, \partial_i A^j$, $E^i = F^{i0}$ and $\mathbf{\nabla}^{-1} = \mathbf{\nabla}/\nabla^2$. The above is a generic to Abelian theories: if a Lagrangian changes by a total derivative under gauge transformations, then there exists [up to a divergence] a gauge invariant form for it. Another example is provided by the linearized gravitational action in the gauge invariant form (4.24).

We use a two-dimensional realization of the Dirac algebra

$$\begin{gathered}\gamma^0 = \sigma^3, \qquad \gamma^1 = i\sigma^1, \qquad \gamma^2 = i\sigma^2, \\ \gamma^\mu \gamma^\nu = g^{\mu\nu} - i\varepsilon^{\mu\nu\alpha}\gamma_\alpha, \qquad g^{\mu\nu} = \mathrm{diag}(1, -1, -1).\end{gathered} \tag{2.6}$$

The σ^i's are Pauli matrices and ψ is a two-component spinor.

Discrete transformations are somewhat unusual. Charge conjugation

$$\mathscr{C} A_\mu \mathscr{C}^{-1} = -A_\mu, \tag{2.7a}$$

$$\mathscr{C} \psi \mathscr{C}^{-1} = \sigma^1 \psi^\dagger \tag{2.7b}$$

leaves the equations invariant. However, the mass terms for both the gauge and Fermi fields change sign under a parity transformation

$$\begin{aligned}\mathscr{P} A^0(t, \mathbf{r})\, \mathscr{P}^{-1} &= A^0(t, \mathbf{r}'), \\ \mathscr{P} A^1(t, \mathbf{r})\, \mathscr{P}^{-1} &= -A^1(t, \mathbf{r}'), \\ \mathscr{P} A^2(t, \mathbf{r})\, \mathscr{P}^{-1} &= A^2(t, \mathbf{r}'),\end{aligned} \tag{2.8a}$$

$$\begin{gathered}\mathscr{P} \psi(t, \mathbf{r})\, \mathscr{P}^{-1} = \sigma^1 \psi(t, \mathbf{r}'), \\ \mathbf{r} = (x, y), \qquad \mathbf{r}' = (-x, y).\end{gathered} \tag{2.8b}$$

Similarly, time-inversion reverses the sign of both masses

$$\begin{aligned}\mathscr{T} A^0(t, \mathbf{r})\, \mathscr{T}^{-1} &= A^0(-t, \mathbf{r}), \\ \mathscr{T} \mathbf{A}(t, \mathbf{r})\, \mathscr{T}^{-1} &= -\mathbf{A}(-t, \mathbf{r}),\end{aligned} \tag{2.9a}$$

$$\mathscr{T} \psi(t, \mathbf{r})\, \mathscr{T}^{-1} = \sigma^2 \psi(-t, \mathbf{r}), \tag{2.9b}$$

The combined time- and space-inversion leaves the mass terms invariant, so CPT symmetry is valid.

[By considering a doublet of models, one with masses (μ, m), the other with $(-\mu, -m)$, and defining parity and time-inversion to include an interchange of the two fields, one obtains a P and T conserving system.]

It is seen, therefore, that the gauge-field and fermion mass terms belong together, since both violate P and T, and in perturbation theory one can be generated from the other. The relationship between them is further highlighted by noting that for $\mu = m$, $\mathscr{L}_G + \mathscr{L}_F$ changes only by a total divergence, when an infinitesimal supersymmetry transformation is performed. That transformation, parametrized by a constant spinor ε, has the form

$$\delta A_\mu = i\bar{\varepsilon}\gamma_\mu \psi - i\bar{\psi}\gamma_\mu \varepsilon, \tag{2.10a}$$

$$\delta\psi = -i{}^*\!\not{F}\varepsilon. \tag{2.10b}$$

Here $^*F^\mu$ is the dual field strength, which is a vector in three dimensions:

$$^*F^\mu = \tfrac{1}{2}\, \varepsilon^{\mu\alpha\beta} F_{\alpha\beta}, \tag{2.11a}$$

$$F^{\mu\nu} = \varepsilon^{\mu\nu\alpha}\, {}^*F_\alpha. \tag{2.11b}$$

[The interaction is not supersymmetric, because there is a mismatch between the boson and fermion degrees of freedom, just as in four-dimensional quantum electrodynamics.] Since the sign of both masses may be reversed by a discrete transformation, we take m to be positive, without loss of generality.

The Bianchi identity on the dual field strength,

$$\partial_\mu\, {}^*F^\mu = 0, \tag{2.12}$$

also follows from (2.3a) [when the current is conserved]. Thus the Abelian gauge field equations (2.3a) and (2.12), are of the "Maxwell" type, making no reference to the vector potential.

Equation (2.3a) may be restated in equivalent, dual form,

$$\partial_\alpha\, {}^*F_\beta - \partial_\beta\, {}^*F_\alpha - \mu F_{\alpha\beta} = -\varepsilon_{\alpha\beta\mu} J^\mu, \tag{2.13a}$$

whose divergence yields

$$(\Box + \mu^2)\, {}^*F^\mu = \mu \left(g^{\mu\nu} - \varepsilon^{\mu\nu\alpha} \frac{\partial_\alpha}{\mu} \right) J_\nu. \tag{2.13b}$$

Equation (2.13b) clearly demonstrates that the gauge field excitations are massive, but it does not expose their number or spin. The differential operator occurring on the right-hand side of (2.13b) may be used to rewrite the field equation (2.3a) as

$$\left(g^{\mu\nu} + \varepsilon^{\mu\nu\alpha} \frac{\partial_\alpha}{\mu} \right) {}^*F_\nu = \frac{J^\mu}{\mu} \tag{2.13c}$$

and (2.13b) follows by operating on (2.13c) with $(g^{\nu\mu} - \varepsilon^{\nu\mu\alpha}\, \partial_\alpha/\mu)$. We see, therefore, that the two operators $(g^{\mu\nu} \pm \varepsilon^{\mu\nu\alpha}\, \partial_\alpha/\mu)$ factorize $g^{\mu\nu}(1 + \Box/\mu^2)$, in three dimensions. In our topological theory, a transverse vector field $({}^*F^\mu)$ satisfies field equations linear in these operator factors, while in the conventionally massive [gauge non-invariant] theory, the transverse vector potential (A^μ) satisfies field equations bilinear in the factors.

It is gratifying that μ^2 occurs in (2.13b) with the correct sign for a propagating particle. Although we have no a priori control over this sign [the Lagrangian is linear in μ], we may understand that it must emerge the way it does by considering the energy-momentum tensor $\theta^{\mu\nu}$. Since the mass term is topological, it does not contribute explicitly to $\theta^{\mu\nu}$. [When coupling our theory to an external metric $(\mu/4) \int dx\, \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha$ is already coordinate invariant, without additional metric factors. Hence, its variation with respect to the metric vanishes.][6] Since the energy has its conventional, positive Maxwell form, the system's excitations cannot be tachyonic.

The solution to (2.13b) is

$$
\begin{aligned}
{}^*F^{\mu} &= \frac{\mu}{\Box + \mu^2}\left(g^{\mu\nu} - \varepsilon^{\mu\nu\alpha}\frac{\partial_\alpha}{\mu}\right) J_\nu \\
&= \frac{1}{\mu}\left(g^{\mu\nu} + \varepsilon^{\mu\nu\alpha}\frac{\partial_\alpha}{\mu}\right)^{-1} J_\nu .
\end{aligned}
\tag{2.14a}
$$

The vector potential is therefore given by

$$
A^{\mu} = \frac{1}{\Box + \mu^2}\left(J^{\mu} - \frac{\mu}{\Box}\,\varepsilon^{\mu\alpha\beta}\,\partial_\alpha J_\beta\right) + \text{gauge term.} \tag{2.14b}
$$

One sees that these quantities are similar to those occurring for a conventionally massive [gauge non-invariant] vector field, coupled to a conserved current $\tilde{J}^{\mu}$, given by a "twisted" transformation of J^{μ}:

$$
\tilde{J}^{\mu} = J^{\mu} - \frac{\mu}{\Box}\,\varepsilon^{\mu\alpha\beta}\,\partial_\alpha J_\beta . \tag{2.15}
$$

Upon integrating the time component of (2.3a) over all space, we find

$$
-\int d\mathbf{r}\; B = Q/\mu, \tag{2.16}
$$

where B is the magnetic field,

$$
B = -{}^*F^0 = \varepsilon^{ij}\,\partial_i A^j \tag{2.17}
$$

and Q is the total charge

$$
\begin{aligned}
Q &= \int d\mathbf{r}\; \rho, \\
\rho &\equiv J^0.
\end{aligned}
\tag{2.18}
$$

A surface integral of the longitudinal electric field has been dropped, since it decreases exponentially due to the mass term. It is seen that the flux passing out of

[6] Of course, the conventional Maxwell energy-momentum tensor remains conserved in our theory, in the absence of sources: $\theta^{\mu\nu} = -F^{\mu\alpha}F_{\alpha}{}^{\nu} + \frac{1}{4}\,g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} \equiv {}^*F^{\mu}\,{}^*F^{\nu} - \frac{1}{2}\,g^{\mu\nu}\,{}^*F^{\alpha}\,{}^*F_{\alpha}$, $\partial_\mu\theta^{\mu\nu} = {}^*F^{\mu}(\partial_\mu\,{}^*F^{\nu} - \partial_\nu\,{}^*F^{\mu}) = \mu\,{}^*F^{\mu}F_{\mu}{}^{\nu} = 0$. In verifying conservation we used the Bianchi identity (2.12) and the sourceless version of the field equation (2.13a). This derivation also applies to the Yang–Mills case.

our two-dimensional space, $\int d\mathbf{r}\, B$, is time independent—this follows from the conservation of $^*F^\mu$—and proportional to the total charge.

For classical fields, Eq. (2.16) shows that the magnetic potential is long range, even though the magnetic field is short range:

$$\mathbf{A} \xrightarrow[r \to \infty]{} -\frac{Q}{2\pi\mu} \boldsymbol{\nabla} \tan^{-1} y/x. \tag{2.19}$$

This behavior brings out an analogy between the present theory and the three-dimensional Abelian Higgs model. Both describe massive vector fields in a gauge invariant way. Both lead to topologically nontrivial solutions for the gauge potentials. In the Higgs model, these are supported by static electric currents, constructed from the [vortex] Higgs field, while the charge density vanishes. In the present model, a nonvanishing static charge density ρ produces an effective twisted current, with nonvanishing spatial components, according to (2.15). In both cases, flux is related to charge. In the Higgs model, $(\text{flux})^{-1}$ is quantized in integer units of the Higgs field's charge, the vortex being charge neutral [6]. Here, flux is directly proportional to the charge, but we see no reason for quantizing any of the parameters.

B. *Non-interacting Quantum Theory*

We shall now quantize the free gauge theory to expose the particle content. Quantization will be performed in the Weyl gauge ($A^0 = 0$), which is appropriate to the canonical formalism for a gauge theory. A subsequent change of variables will produce a conventional Coulomb gauge quantization.

Since the topological term contains time derivatives, the canonical momentum changes

$$\Pi^i = \frac{\delta \mathscr{L}_G}{\delta \dot{A}^i} = -E^i + \frac{\mu}{2} \varepsilon^{ij} A^j, \qquad E^i = F^{i0}. \tag{2.20}$$

The Hamiltonian,

$$H = \tfrac{1}{2} \int d\mathbf{r} [\mathbf{E}^2 + B^2], \tag{2.21a}$$

when written in canonical variables, is

$$H = \frac{1}{2} \int d\mathbf{r} \left[\left(\Pi^i - \frac{\mu}{2} \varepsilon^{ij} A^j \right)^2 + B^2 \right]. \tag{2.21b}$$

The only nonvanishing canonical commutator[7]

$$i[\Pi^i(\mathbf{r}), A^j(\mathbf{r}')] = \delta^{ij} \delta(\mathbf{r} - \mathbf{r}') \tag{2.22}$$

[7] Here and throughout, we suppress a common time argument in local operators satisfying equal-time commutation relations.

implies the gauge invariant algebra

$$i[E^i(\mathbf{r}), E^j(\mathbf{r}')] = \mu\varepsilon^{ij}\delta(\mathbf{r}-\mathbf{r}'), \tag{2.23a}$$

$$i[E^i(\mathbf{r}), B(\mathbf{r}')] = -\varepsilon^{ij}\,\partial_j\delta(\mathbf{r}-\mathbf{r}'), \tag{2.23b}$$

$$i[B(\mathbf{r}), B(\mathbf{r}')] = 0. \tag{2.23c}$$

The Hamiltonian equations of motion

$$\dot{A}^i = i[H, A^i] = \Pi^i - \frac{\mu}{2}\,\varepsilon^{ij}A^j, \tag{2.24a}$$

$$\dot{\Pi}^i = i[H, \Pi^i] = -\frac{\mu}{2}\,\varepsilon^{ij}\Pi^j - \frac{\mu^2}{4}\,A^i - \varepsilon^{ij}\,\partial_j B \tag{2.24b}$$

produce the spatial components of the field equation. The time component, corresponding to the Gauss law,

$$G \equiv \partial_i \Pi^i + \frac{\mu}{2}\,B = 0 \tag{2.25}$$

must be imposed as a condition on the physical states

$$G\,|\Psi\rangle = 0. \tag{2.26}$$

This can be done, since G is a constant of motion arising from the remaining static gauge symmetry of the Weyl gauge

$$\delta\mathbf{A} = -\frac{1}{e}\,\nabla\Omega, \qquad \dot{\Omega} = 0. \tag{2.27}$$

It is easy to construct the states of the theory. For example, in a [functional] Schrödinger representation, the energy eigenstates satisfy

$$\frac{1}{2}\int d\mathbf{r}\left[\left(\frac{1}{i}\frac{\delta}{\delta A^i(\mathbf{r})} - \frac{\mu}{2}\,\varepsilon^{ij}A^j(\mathbf{r})\right)^2 + B^2(\mathbf{r})\right]\,\Psi(\mathbf{A}) = E\Psi(\mathbf{A}), \tag{2.28a}$$

where the wave functionals $\Psi(\mathbf{A})$ are functionals of the vector potentials $\mathbf{A}$, and obey the subsidiary condition (2.26)

$$\left[\frac{1}{i}\,\partial_i\,\frac{\delta}{\delta A^i(\mathbf{r})} + \frac{\mu}{2}\,B(\mathbf{r})\right]\,\Psi(\mathbf{A}) = 0. \tag{2.28b}$$

The lowest energy solution to (2.28) is

$$\Psi_0(\mathbf{A}) = e^{i\chi(\mathbf{A})} \exp -\frac{1}{2} \int d\mathbf{r}\, d\mathbf{r}'\, A^i(\mathbf{r})\, K^{ij}(\mathbf{r}-\mathbf{r}')\, A^j(\mathbf{r}'), \tag{2.29a}$$

$$\chi(\mathbf{A}) = \frac{\mu}{2} \int d\mathbf{r}\, B\, \boldsymbol{\nabla}^{-1} \cdot \mathbf{A}, \tag{2.29b}$$

$$K^{ij}(\mathbf{r}-\mathbf{r}') = \int \frac{d\mathbf{k}}{(2\pi)^2}\, e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} (\delta^{ij} - \hat{k}^i \hat{k}^j)\, \omega(\mathbf{k}), \tag{2.29c}$$

$$\omega(\mathbf{k}) \equiv \sqrt{\mathbf{k}^2 + \mu^2}, \qquad \boldsymbol{\nabla}^{-1} \equiv \boldsymbol{\nabla}/\nabla^2$$

with eigenvalue

$$E_0 = \frac{1}{2} \operatorname{Tr} K = \frac{1}{2} \int d\mathbf{r} \int \frac{d\mathbf{k}}{(2\pi)^2}\, \omega(\mathbf{k}). \tag{2.30}$$

E_0 is, of course, the infinite zero-point energy. The one-particle states are described by

$$\Psi_{\mathbf{p}}(\mathbf{A}) = \int d\mathbf{r}\, e^{i\mathbf{p}\cdot\mathbf{r}} B(\mathbf{r})\, \Psi_0(\mathbf{A}). \tag{2.31}$$

They are eigenstates of the momentum operator $\mathbf{P}$

$$\mathbf{P} = -\int d\mathbf{r} (\boldsymbol{\nabla} A^i)\, \Pi^i \tag{2.32}$$

with eigenvalue $\mathbf{p}$, and the energy eigenvalue is $E(\mathbf{p}) = \omega(\mathbf{p}) + E_0$. It is now clear that the theory describes a particle of mass $|\mu|$.

To pass to the more familiar Coulomb gauge description, we first solve the Gauss constraint, viz., we find the most general functional obeying (2.28b). It is easy to see that any functional of the form

$$\Psi(\mathbf{A}) = e^{i\chi(\mathbf{A})} \Phi(\mathbf{A}_T) \tag{2.33}$$

with χ given by (2.29b) and Φ an arbitrary functional of transverse potentials, $\mathbf{A}_T$, satisfies (2.28b).

The Coulomb gauge Hamiltonian, H_C, operates on the transverse functional Φ. Evidently, it is

$$H_C = e^{-i\chi} H e^{i\chi} = \tfrac{1}{2} \int d\mathbf{r} [\boldsymbol{\Pi}_T^2 + A_T^i(-\nabla^2 + \mu^2)\, A_T^i], \tag{2.34}$$

where $\mathbf{\Pi}_T$ is the momentum conjugate to $\mathbf{A}_T$. The Hamiltonian field equations now simply read

$$\begin{aligned} \dot{\mathbf{A}}_T &= \mathbf{\Pi}_T, \\ (\Box + \mu^2)\, \mathbf{A}_T &= 0, \end{aligned} \tag{2.35}$$

and the familiar Coulomb gauge commutation relations hold:

$$\begin{gathered} i[\Pi^i_T(\mathbf{r}), A^j_T(\mathbf{r}')] = (\delta^{ij} + \hat{\partial}_i \hat{\partial}_j)\, \delta(\mathbf{r} - \mathbf{r}'), \\ \hat{\partial}_i \equiv \partial_i / \sqrt{-\nabla^2}. \end{gathered} \tag{2.36}$$

To expose the number and spin of the gauge field excitations, we use the gauge invariant statement that the field equations (2.3a) [with the current set to zero] and commutation relations (2.23) are solved by a canonical, free, massive field φ,

$$(\Box + \mu^2)\, \varphi = 0, \tag{2.37a}$$

$$i[\dot{\varphi}(\mathbf{r}), \varphi(\mathbf{r}')] = \delta(\mathbf{r} - \mathbf{r}') \tag{2.37b}$$

in terms of which the electromagnetic fields are simply expressed

$$E^i = -\varepsilon^{ij}\, \hat{\partial}_j \dot{\varphi} - \mu\, \hat{\partial}_i \varphi, \tag{2.38a}$$

$$B = \sqrt{-\nabla^2}\; \varphi. \tag{2.38b}$$

The field φ is related to the transverse vector potential

$$A^i_T = \varepsilon^{ij}\, \hat{\partial}_j \varphi. \tag{2.39}$$

The translation generators of the gauge theory, when expressed in terms of φ, coincide, as they must, with those of a scalar field

$$H = \tfrac{1}{2} \int d\mathbf{r} [\mathbf{E}^2 + B^2] = \tfrac{1}{2} \int d\mathbf{r} [\dot{\varphi}^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2], \tag{2.40a}$$

$$P^i = \int d\mathbf{r}\, \varepsilon^{ij} E^j B = -\int d\mathbf{r}\, \dot{\varphi}\, \partial_i \varphi. \tag{2.40b}$$

These formulas confirm that our theory possesses a single excitation with mass $|\mu|$.

Next, we examine the spin of the particle. In two spatial dimensions "spin" is a rotational pseudoscalar, which can be viewed as the projection of a three-dimensional spin along the [missing] third axis; hence it can have positive or negative values. The electromagnetic angular momentum generator, when expressed in terms of φ, shows that φ is a rotational scalar, as is expected from (2.38) and (2.39):

$$M^{ij} = \varepsilon^{ij} M = -\varepsilon^{ij} \int d\mathbf{r}\, \mathbf{r} \cdot \mathbf{E} B,$$
$$M = -\int d\mathbf{r}\, \dot{\varphi} \varepsilon^{ij} r^i\, \partial_j \varphi. \tag{2.41}$$

But we cannot conclude that the excitations are spinless. One reason for this is that rotations in two spatial dimensions, being described by the $O(2)$ group, do not require definite values of spin. Only by considering Lorentz boosts does the full non-Abelian $O(2, 1)$ structure of the Lorentz group come into play, and spin becomes determined.

Since φ clearly is not a Lorentz scalar, as is evident from (2.38) and (2.39), we do not expect the usual scalar form for the boost generators. Indeed, we find

$$\begin{aligned} M^{i0} + tP^i &= \frac{1}{2} \int d\mathbf{r}\, r^i [\mathbf{E}^2 + B^2] \\ &= \frac{1}{2} \int d\mathbf{r}\, r^i [\dot{\varphi}^2 + (\nabla \varphi)^2 + \mu^2 \varphi^2] + \mu \varepsilon^{ij} \int d\mathbf{r}\, \dot{\varphi} \frac{\partial_j}{-\nabla^2} \varphi. \end{aligned} \tag{2.42}$$

The last term is required so that M^{i0} generate the correct boosts on components of the gauge invariant $F^{\mu\nu}$. Note that M^{i0} is infrared singular. As a consequence, the commutator of two boosts does not reproduce the rotation generator; there is a zero-momentum anomaly

$$i[M^{i0}, M^{j0}] = \varepsilon^{ij}(M - \Delta),$$
$$\Delta = \frac{\mu^3}{4\pi} \left(\int d\mathbf{r}\, \varphi \right)^2 + \frac{\mu}{4\pi} \left(\int d\mathbf{r}\, \dot{\varphi} \right)^2. \tag{2.43}$$

This does not mean that the theory is Lorentz non-invariant but rather that the field φ, owing to the Laplacians in its definition (2.38) and to the zero-frequency ambiguity in the transverse–longitudinal decomposition (2.39), transforms in a singular way under Lorentz boosts.

One may remove the singularity, thus regaining the conventional algebra, and determining the spin of the excitation unambiguously by the following momentum-space procedure. We make a mode expansion for φ:

$$\varphi(x) = \int \frac{d\mathbf{k}}{2\pi \sqrt{2\omega(\mathbf{k})}} [e^{-ikx} a(\mathbf{k}) + e^{ikx} a^\dagger(\mathbf{k})],$$
$$k = (\omega(\mathbf{k}), \mathbf{k}), \qquad [a(\mathbf{k}), a^\dagger(\mathbf{k}')] = \delta(\mathbf{k} - \mathbf{k}'). \tag{2.44}$$

The translation and rotation generators are conventional,

$$P^\mu = \int d\mathbf{k}\, k^\mu a^\dagger(\mathbf{k})\, a(\mathbf{k}), \tag{2.45}$$

$$M = \int d\mathbf{k}\, a^\dagger(\mathbf{k}) \frac{1}{i}\frac{\partial}{\partial\theta} a(\mathbf{k}), \qquad \theta = \tan^{-1} k_2/k_1, \tag{2.46}$$

but the Lorentz boosts have an extra term, which involves an infrared singular integrand

$$M^{i0} = \frac{i}{2} \int d\mathbf{k}\, \omega(\mathbf{k}) [a^\dagger(\mathbf{k})\, \overleftrightarrow{\partial}_i\, a(\mathbf{k})] - \mu \int d\mathbf{k}\, \varepsilon^{ij} (k^j/\mathbf{k}^2)\, a^\dagger(\mathbf{k})\, a(\mathbf{k}). \tag{2.47}$$

To remove the singularity, we redefine the phase of the creation and annihilation operators

$$a(\mathbf{k}) \to e^{i(\mu/|\mu|)\,\theta} a(\mathbf{k}). \tag{2.48}$$

Commutation relations of a and $a^\dagger$ are unaffected and the translation generators retain their form. The rotation generator acquires a spin term,

$$M = \int d\mathbf{k}\, a^\dagger(\mathbf{k}) \frac{1}{i}\frac{\partial}{\partial\theta} a(\mathbf{k}) + \frac{\mu}{|\mu|} \int d\mathbf{k}\, a^\dagger(\mathbf{k})\, a(\mathbf{k}), \tag{2.49}$$

while the boost generators become non-singular:

$$\begin{aligned} M^{i0} = {} & \frac{i}{2} \int d\mathbf{k}\, \omega(\mathbf{k}) [a^\dagger(\mathbf{k})\, \overleftrightarrow{\partial}_i a(\mathbf{k})] \\ & + \frac{\mu}{|\mu|} \int d\mathbf{k}\, \frac{1}{\omega(\mathbf{k}) + |\mu|}\, \varepsilon^{ij} k^j a^\dagger(\mathbf{k})\, a(\mathbf{k}). \end{aligned} \tag{2.50}$$

One may check that the Poincaré algebra is now satisfied.

We also see that the rotation generator (2.49) puts into evidence a non-vanishing spin: $1[\mu > 0]$ or $-1[\mu < 0]$. It must be emphasized that no further phase redefinition, which would change the spin term in (2.49) [e.g., $a \to e^{is\theta} a$], is allowed. Such changes would introduce new singularities into the Lorentz boosts. We thus see that, as stated before, the spin content of the theory is not determined solely by the rotation group; rather, the full Lorentz group must be considered and the spin is uniquely determined by the requirement that the boost generators be non-singular and satisfy the conventional algebra.[8]

The spin is not smooth in the zero-mass limit, which may be directly taken in (2.46) and (2.47), or in (2.48), (2.49) and (2.50) with the convention that $\mu/|\mu|$ vanishes in the limit. The massless theory is spinless; this reflects the circumstance that massless representations of the three-dimensional Poincaré group describe particles with no spin [3].

Our results are also consistent with the fact that the angular momentum M, as well as the Pauli–Lubanski invariant $\varepsilon^{\mu\nu\alpha} M_{\mu\nu} P_{\alpha}$, are odd under a parity transformation. Consequently, in a parity conserving theory, with one degree of freedom [this is the situation when $\mu = 0$], spin must be zero, since additional degrees of freedom are unavailable to form the parity doubled, non-zero spin states. Correspondingly, when parity is violated [$\mu \neq 0$], a single degree of freedom may carry nonvanishing spin. When we consider two degrees of freedom, with topological masses $\pm\mu$, both spins ± 1 are present and parity is conserved, as explained previously. This is the same kinematical structure as in a conventionally massive [gauge non-invariant] vector theory, which also possesses two degrees of freedom with spin ± 1 [3]. Similarly, the massive, parity violating Dirac theory has spin 1/2; in the doubled, parity conserving case, both spins $\pm 1/2$ occur; while the massless Dirac theory describes a spinless fermion [3]. These statements may be checked by explicit computations, which are entirely similar to those presented above.

C. *Interacting Theory*

C(1). *Quantization.* We now turn to the interacting theory. The full Hamiltonian may be written as

$$H = H_G + H_M - \int d\mathbf{r}\, \mathbf{J} \cdot \mathbf{A}. \tag{2.51}$$

Here, H_G is the gauge field Hamiltonian (2.21), H_M is the matter field Hamiltonian [the Dirac Hamiltonian in our problem] and the last term in (2.51) describes an interaction of the vector potential with the spatial current. The subsidiary condition (2.25) now includes the matter charge density

$$\boldsymbol{\nabla} \cdot \boldsymbol{\Pi} + (\mu/2)\, B + \rho = 0. \tag{2.52}$$

[8] Observe that the effective free scalar Hamiltonian (2.40a) and also the scalar action and commutation relations (2.37) do not specify the Lorentz transformation properties of the theory, since different generators [functionals of φ] satisfy the Lorentz algebra: the trivial scalar ones [Eq. (2.41) and the conventional part of Eq. (2.42)], the ones that we have here constructed with spin 1, and those in Section IV, appropriate to spin 2. This ambiguity is a feature of three-dimensional free fields, when written in terms of their independent degrees of freedom, since all spatial indices can be removed in a rotationally invariant way, leaving only the [invisible] temporal indices to specify Lorentz transformation properties.

We pass directly to the Coulomb gauge: the longitudinal component of $\mathbf{A}$ is set to zero; the longitudinal component of $\mathbf{\Pi}$ is evaluated from (2.52). In this way, one finds for $H_G - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A}$

$$H_G - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A} = \frac{1}{2}\int d\mathbf{r}\left[\mathbf{\Pi}_T^2 + A_T^i(-\nabla^2+\mu^2)\, A_T^i + \rho\,\frac{1}{-\nabla^2}\,\rho\right]$$
$$-\int d\mathbf{r}\left[J^i + \frac{\mu}{-\nabla^2}\,\varepsilon^{ij}\,\partial_j\,\rho\right] A_T^i. \tag{2.53a}$$

The last term in (2.53a) exhibits the effective spatial current, interacting with the transverse vector potential; apart from a gradient term, it is the twisted combination appearing in (2.15). There also appears an instantaneous Coulomb interaction; however, that is misleading, since by itself it is infrared divergent, owing to the occurrence of the Green's function for the two-dimensional Laplacian. In fact, from the alternative expression for the gauge field energy,

$$H_G - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A} = \frac{1}{2}\int d\mathbf{r}[\mathbf{E}^2 + B^2] - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A} \tag{2.53b}$$

and from the formulas for the solutions (2.14), we see that no infrared divergences are present. The canonical Hamiltonian (2.53a) may be presented in a manifestly infrared finite form. To this end, we use (2.39) and an analogous formula for $\Pi_T^i = -\varepsilon^{ij}\hat{\partial}_j \Pi$. Then it follows that

$$H_G - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A} = \frac{1}{2}\int d\mathbf{r}\left[\Pi^2 + \left(\sqrt{-\nabla^2+\mu^2}\,\varphi - \frac{\mu}{\sqrt{-\nabla^2}}\,\frac{1}{\sqrt{-\nabla^2+\mu^2}}\,\rho\right)^2\right.$$
$$\left. + \rho\,\frac{1}{-\nabla^2+\mu^2}\,\rho\right] - \int d\mathbf{r}\,\varepsilon^{ij}\,\hat{\partial}_i\, J^j \varphi. \tag{2.53c}$$

After a canonical transformation, which shifts φ by $[\mu/(-\nabla^2+\mu^2)\sqrt{-\nabla^2}]\,\rho$, this becomes free of infrared divergences:

$$H_G - \int d\mathbf{r}\, \mathbf{J}\cdot\mathbf{A}$$
$$= \frac{1}{2}\int d\mathbf{r}\left[\Pi^2 + \varphi(-\nabla^2+\mu^2)\,\varphi + \rho\,\frac{1}{-\nabla^2+\mu^2}\,\rho\right]$$
$$+ \int d\mathbf{r}\left[\mu\,\hat{\partial}_i J^i\,\frac{1}{-\nabla^2+\mu^2}\,\Pi - \varepsilon^{ij}\,\hat{\partial}_i J^j \varphi - \mu\varepsilon^{ij} J^i\,\frac{\partial_j}{-\nabla^2(-\nabla^2+\mu^2)}\,\rho\right]. \tag{2.53d}$$

Here φ stands for the shifted field.

The above consideration means that gauge non-invariant quantities will be infrared divergent, if the Coulomb gauge is used for computation. This is also seen from the photon propagator in the Coulomb gauge

$$D^{C}_{\mu\nu}(p) = \frac{-i}{p^2 - \mu^2 + i\varepsilon}\,[g_{\mu\nu} + \bar{p}_\mu \bar{p}_\nu/\mathbf{p}^2 - n_\mu n_\nu p_0^2/\mathbf{p}^2 + i\mu\varepsilon_{\mu\nu\alpha}\bar{p}^\alpha/\mathbf{p}^2],$$
$$\bar{p}^\mu = (0, \mathbf{p}), \qquad n^\mu(1, \mathbf{0}). \tag{2.54a}$$

The $1/\mathbf{p}^2$ singularity of the time–time component is not integrable on a two-dimensional space. However, terms proportional to p^μ may be changed at will in a photon propagator, so an infrared safe version of (2.54a) may be adopted:

$$D'^{C}_{\mu\nu}(p) = \frac{-i}{p^2 - \mu^2 + i\varepsilon}\,[g_{\mu\nu} + i\mu\varepsilon_{\mu\nu\alpha}\bar{p}^\alpha/\mathbf{p}^2]. \tag{2.54b}$$

But we shall use a Lorentz covariant gauge in our perturbative calculations, which we now discuss.

C(2). *Lowest-Order Perturbation Theory.* Rather than pursuing the above Lorentz non-covariant development, we pass to covariant perturbation theory based on the free vector propagator,

$$D_{\mu\nu}(p) = \frac{-i}{p^2 - \mu^2 + i\varepsilon}\,[P_{\mu\nu}(p) - i\mu\varepsilon_{\mu\nu\alpha}p^\alpha/p^2] - i\alpha p_\mu p_\nu/p^4$$
$$P_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu/p^2, \tag{2.55}$$

where α is a gauge parameter. The singularities at $p^2 = 0$ disappear when $D_{\mu\nu}$ is contracted into conserved currents. The free fermion propagator is conventional,

$$S(p) = \frac{i}{\not{p} - m}, \qquad m > 0, \tag{2.56}$$

as is the electromagnetic vertex. We have used the freedom of fixing the sign of one mass to take m positive.

We shall calculate one-loop corrections to the two propagators

$$(\mathscr{D}^{-1})_{\mu\nu} = (D^{-1})_{\mu\nu} - i\Pi_{\mu\nu}, \tag{2.57a}$$

$$\Pi_{\mu\nu}(p) = -ie^2 \int \frac{dk}{(2\pi)^3}\,\mathrm{tr}\,\gamma_\mu S(p+k)\,\gamma_\nu S(k) + O(e^4), \tag{2.57b}$$

$$\mathscr{S}^{-1} = S^{-1} + i\Sigma, \tag{2.58a}$$

$$\Sigma(p) = -ie^2 \int \frac{dk}{(2\pi)^3}\,\gamma^\mu S(p+k)\,\gamma^\nu D_{\mu\nu}(k) + O(e^4). \tag{2.58b}$$

The integral for the photon polarization tensor is linearly divergent and dimensional arguments show the divergent part to be proportional to $g_{\mu\nu}$. Consequently, the divergence is not gauge invariant and will be removed by any gauge invariant integration procedure. The integral for the fermion self-energy is logarithmically divergent. But we shall maintain Lorentz invariance, so logarithmic divergences cannot arise. Gauge invariance may be enforced by several Lorentz invariant methods: Pauli–Villars regularization; dimensional regularization; and gauge invariant projection, where the coefficient of $g_{\mu\nu}$ is not calculated separately, but determined from the $p_\mu p_\nu$ term. It happens that the Pauli–Villars method gives a result which differs from the other two, and the consequent ambiguity affects the calculation of the mass shifts.

While there is no ultimate way to decide on a unique regulator, there are various observations which point out virtues and defects of the different regularization procedures. The Pauli–Villars method is not appropriate to the massless case, since it introduces masses which violate symmetries of the massless theory: space and time inversion. Dimensional continuation is awkward in the massive case, since masses are introduced with the help of the epsilon tensor, which has no direct generalization to continuous dimensions. Gauge invariant projection does least violence to the naturally arising integral. In what follows, we shall adopt the Pauli–Villars method, but we shall always call attention to the extra terms arising in that context, and absent in the others. We shall also show that the Pauli–Villars expressions produce results that are the most regular, in a well-defined sense explained below.

Evaluation of (2.57b), with a cut-off Λ, yields

$$\begin{aligned}\Pi_{\mu\nu}(p) = {} & \frac{-e^2}{3\pi^2}\Lambda g_{\mu\nu} + P_{\mu\nu}(p)\left[\frac{e^2}{8\pi}p^2\int_{2m}^{\infty} da\, \frac{1+4m^2/a^2}{p^2-a^2+i\varepsilon}\right] \\ & + im\varepsilon_{\mu\nu\alpha}p^\alpha\left[\frac{e^2}{2\pi}\int_{2m}^{\infty} da\,\frac{1}{p^2-a^2+i\varepsilon}\right] + O(e^4). \end{aligned} \tag{2.59}$$

The Pauli–Villars regulation removes the gauge non-invariant, cut-off dependent term, which is also removed by dimensional regularization and never arises when gauge invariant projections are made. Furthermore, a constant is added to the axial structure function by the Pauli–Villars procedure; this modification does not occur in the other methods. Thus, the Pauli–Villars regulated polarization tensor is

$$\Pi_{\mu\nu}(p) = P_{\mu\nu}(p)\,\Pi^{(1)}(p^2) + im\varepsilon_{\mu\nu\alpha}\,p^\alpha\Pi^{(2)}(p^2), \tag{2.60a}$$

$$\Pi^{(1)}(p^2) = \frac{e^2}{8\pi}p^2\int_{2m}^{\infty} da\,\frac{1+4m^2/a^2}{p^2-a^2+i\varepsilon} + O(e^4), \tag{2.60b}$$

$$\Pi^{(2)}(p^2) = \frac{e^2}{2\pi}p^2\int_{2m}^{\infty} da\,\frac{1/a^2}{p^2-a^2+i\varepsilon} + O(e^4). \tag{2.60c}$$

Note that $\Pi^{(1)}(-p^2)$ satisfies the requirement of positivity. Also, it vanishes at $p^2=0$, so that the vacuum polarization tensor is non-singular there. That $\Pi^{(2)}(p^2)$ vanishes at that point as well is a consequence of the Pauli–Villars procedure.

The gauge-field propagator may be presented in spectral form

$$\begin{aligned}\mathscr{D}_{\mu\nu}(p) = -iP_{\mu\nu}(p)\left[\frac{Z_G}{p^2-\mu_{ph}^2+i\varepsilon}+\tilde{\Pi}^{(1)}(p^2)\right]\\ -\mu_{ph}\varepsilon_{\mu\nu\alpha}p^\alpha/p^2\left[\frac{Z_G}{p^2-\mu_{ph}^2+i\varepsilon}+\tilde{\Pi}^{(2)}(p^2)\right]-i\alpha p_\mu p_\nu/p^4.\end{aligned} \tag{2.61}$$

The physical mass, μ_{ph}, is given by

$$\mu_{ph}=\mu-\frac{e^2\mu}{8\pi}\int_{2m}^{\infty}da\,\frac{1+(4m/a^2)(m-\mu)}{a^2-\mu^2}+O(e^4), \tag{2.62}$$

while the residue at the pole, Z_G, is

$$Z_G=1-\frac{e^2}{8\pi}\int_{2m}^{\infty}da\,\frac{(1/a^2)(a^2-2m\mu)^2+(2m-\mu)^2}{(a^2-\mu^2)^2}+O(e^4). \tag{2.63}$$

The continuum contributions are

$$\tilde{\Pi}^{(1)}(p^2)=\frac{e^2}{8\pi}\int_{2m}^{\infty}da\,\frac{(1/a^2)(a^2-2m\mu)^2+(2m-\mu)^2}{(p^2-a^2+i\varepsilon)(a^2-\mu^2)^2}+O(e^4), \tag{2.64a}$$

$$\tilde{\Pi}^{(2)}(p^2)=\frac{e^2}{4\pi}\left(1-\frac{2m}{\mu}\right)\int_{2m}^{\infty}da\,\frac{a^2-2m\mu}{(p^2-a^2+i\varepsilon)(a^2-\mu^2)^2}+O(e^4). \tag{2.64b}$$

The regularization ambiguity affects only the physical mass, Eq. (2.62). That quantity, derived by the Pauli–Villars method, should be decreased by $(e^2/2\pi)\,\mu$, if dimensional regularization or gauge invariant projection is used.

The formulas satisfy general requirements of unitarity and analyticity. Of course, $2m$ must be taken greater than $|\mu|$; otherwise, the gauge particle is unstable. The expressions are derived under the hypothesis that the radiative corrections are small; i.e., $2m \geqslant |\mu| \gg e^2$.

When $\mu^2=0$, and the polarization tensor is of the form (2.60a), one gets for the exact propagator

$$\mathscr{D}_{\mu\nu}(p)=\frac{-i}{p^2-\Pi(p^2)}\left[P_{\mu\nu}(p)-\frac{i\varepsilon_{\mu\nu\alpha}p^\alpha}{p^2}\mathscr{M}(p^2)\right], \tag{2.65a}$$

$$\Pi(p^2)=\Pi^{(1)}(p^2)+\frac{[m\Pi^{(2)}(p^2)]^2}{1-\Pi^{(1)}(p^2)/p^2}, \tag{2.65b}$$

$$\mathscr{M}(p^2)=\frac{m\Pi^{(2)}(p^2)}{1-\Pi^{(1)}(p^2)/p^2}. \tag{2.65c}$$

An axial structure in $\mathscr{D}_{\mu\nu}$ is induced by the fermion mass term, even when none is present in the free theory. Whether a gauge-field mass is generated depends on the behavior of $\Pi(p^2)$ at $p^2=0$. If $\Pi^{(1)}(0)\neq 0$, then the vacuum polarization tensor has a pole [see (2.60a)] and the familiar Schwinger mechanism creates a mass for the gauge particle [9]. This does not happen for our model in perturbation theory, where $\Pi^{(1)}(0)=0$ and the vacuum polarization tensor is non-singular [see (2.60b)]. Rather, when $\Pi^{(1)}(p^2)$ vanishes at $p^2=0$, we have

$$\Pi(0)=\left.\frac{[m\Pi^{(2)}(p^2)]^2}{1-\Pi^{(1)}(p^2)/p^2}\right|_{p^2=0}. \tag{2.66}$$

But it is $\Pi^{(2)}(p^2)$ that contains the regularization ambiguity. With the Pauli–Villars method $\Pi^{(2)}(0)$ vanishes to $O(e^2)$ [see (2.60c)], and the gauge field remains massless. However, with the other methods, $\Pi^{(2)}(0)$ is non-vanishing, and the gauge field acquires a mass of order e^2. This is the sense in which the Pauli–Villars regulators are the smoothest: they keep the gauge field massless, at least to lowest order.

Next, the fermion propagator is computed. The $O(e^2)$ self-energy $\Sigma(p)$, Eq. (2.58b), naturally falls into three pieces: Σ_{I} arising from the exchange of a conventional, transverse vector meson; Σ_{II}, where the axial portion of $D_{\mu\nu}$ is used to propagate the vector meson; Σ_{III}, proportional to the gauge parameter α, coming from the exchange of the longitudinal vector meson. The respective amplitudes are

$$\begin{aligned}\Sigma_{\mathrm{I}}(p)=\frac{-e^2}{16\pi}\int_{-\infty}^{\infty}\frac{da}{\not p-a}\bigg[&\left(\frac{\mu^2}{a^2}+\frac{4m}{a}\right)\theta(a^2-M^2)\\&+\frac{1}{\mu^2a^2}(a^2-m^2)^2\,\theta(M^2-a^2)\,\theta(a^2-m^2)\bigg],\end{aligned} \tag{2.67a}$$

$$\begin{aligned}\Sigma_{\mathrm{II}}(p)=\frac{-e^2}{8\pi}\int_{-\infty}^{\infty}\frac{da}{\not p-a}\bigg[&(a+m)\,\frac{\mu}{a^2}\,\theta(a^2-M^2)\\&+\frac{(a-m)(a^2-m^2)}{\mu a^2}\,\theta(M^2-a^2)\,\theta(a^2-m^2)\bigg],\end{aligned} \tag{2.67b}$$

$$\Sigma_{\mathrm{III}}(p)=\frac{-\alpha e^2}{16\pi}\int_{-\infty}^{\infty}\frac{da}{\not p-a}\,\frac{(a+m)^2}{a^2}\,\theta(a^2-m^2), \tag{2.67c}$$

$$M=m+|\mu|.$$

Lorentz invariant integration procedures cannot lead to logarithmically divergent integrals at one loop, and so achieve finite results. Dimensional regularization reproduces the above.

Pauli–Villars regularization offers alternate possibilities. First, one must decide which fields will be regulator fields. The fermion regulator fields, which are necessary for gauge invariance of the vacuum polarization tensor, do not contribute here. Thus, if they are the only regulators, the Pauli–Villars method also gives (2.67).

However, one may have additional massive vector fields, as further regulators. They do not contribute to $\Pi_{\mu\nu}$, but they add a constant to Σ_{II}, and remove Σ_{III} altogether, since it does not depend on the vector meson mass. Thus, the fermion self-energy, evaluated with Pauli–Villars vector meson regulator fields, gives a Landau gauge ($\alpha = 0$) result.

There is another reason for preferring the Landau gauge. Observe that in an arbitrary [covariant] gauge there are massless longitudinal photons propagating with an infrared singular propagator. While the singularity is integrable, it gives rise to pathological mass shell behavior. $\Sigma_{\mathrm{III}}(m)$ is nonvanishing; so the renormalized mass is gauge dependent [α-dependent]. Also, $\Sigma'_{\mathrm{III}}(m)$ is infinite, so the wavefunction renormalization cannot be defined.

It is instructive to see explicitly how these pathologies arise. $\Sigma_{\mathrm{III}}(p)$ is given by

$$\begin{aligned}\Sigma_{\mathrm{III}}(p) &= -\alpha e^2 \int \frac{dk}{(2\pi)^3}\, \gamma^\mu S(p+k)\, \gamma^\nu \frac{k_\mu k_\nu}{k^4} \\ &= i\alpha e^2 \int \frac{dk}{(2\pi)^3} \left[\not{p} - m - (\not{p} - m) \frac{1}{\not{p} + \not{k} - m} (\not{p} - m) - \not{k} \right] \frac{1}{k^4}. \end{aligned} \tag{2.68a}$$

The last term in the bracket vanishes by symmetric integration, and formally it appears that the other terms vanish on mass shell. Yet each is infrared divergent and explicit evaluation gives

$$\Sigma_{\mathrm{III}}(m) = \frac{\alpha e^2}{8\pi}. \tag{2.68b}$$

The cure is simple: one works in the Landau gauge $\alpha = 0$ where no longitudinal, massless vector mesons are present. A similar result is achieved by a Pauli–Villars regularization of the massive vector mesons, which, however, also adds a constant to Σ_{II}.

We shall therefore always use the Landau gauge, but shall not introduce additional Pauli–Villars regulator fields. Therefore, the fermion self-energy is from (2.67)

$$\begin{aligned}\Sigma &= \Sigma_{\mathrm{I}} + \Sigma_{\mathrm{II}} + O(e^4), \\ \Sigma(p) &= \frac{-e^2}{16\pi} \int_{-\infty}^{\infty} \frac{da}{\not{p} - a} \left[\frac{(\mu + 2m)(\mu + 2a)}{a^2}\, \theta(a^2 - M^2) \right. \\ &\quad \left. + \frac{(a + m + 2\mu)(a^2 - m^2)(a - m)}{\mu^2 a^2}\, \theta(M^2 - a^2)\, \theta(a^2 - m^2) \right] + O(e^4). \end{aligned} \tag{2.69}$$

The fermion propagator, written in spectral form, becomes

$$\mathscr{S}(p) = \frac{iZ_F}{\not{p} - m_{ph}} + i\tilde{\Sigma}(p). \tag{2.70}$$

The physical mass, m_{ph}, is

$$m_{ph} = m + \frac{e^2}{16\pi} \int_{-\infty}^{\infty} da \left[\frac{(\mu + 2m)(\mu + 2a)}{a^2(a-m)} \theta(a^2 - M^2) \right. \\ \left. + \frac{(a+m+2\mu)(a^2 - m^2)}{\mu^2 a^2} \theta(M^2 - a^2)\, \theta(a^2 - m^2) \right] + O(e^4), \tag{2.71}$$

while the renormalization constant, Z_F, is

$$Z_F = 1 - \frac{e^2}{16\pi} \int_{-\infty}^{\infty} da \left[\frac{(\mu + 2m)(\mu + 2a)}{a^2(a-m)^2} \theta(a^2 - M^2) \right. \\ \left. + \frac{(a+m+2\mu)(a+m)}{\mu^2 a^2} \theta(M^2 - a^2)\, \theta(a^2 - m^2) \right] + O(e^4). \tag{2.72}$$

The continuum contribution is

$$\tilde{\Sigma}(p) = \frac{-e^2}{16\pi} \int_{-\infty}^{\infty} \frac{da}{\not p - a} \left[\frac{(\mu + 2m)(\mu + 2a)}{a^2(a-m)^2} \theta(a^2 - M^2) \right. \\ \left. + \frac{(a+m+2\mu)(a+m)}{\mu^2 a^2} \theta(M^2 - a^2)\, \theta(a^2 - m^2) \right] + O(e^4). \tag{2.73}$$

We have checked that the same result is obtained for m_{ph} with the infrared safe, Coulomb gauge propagator (2.54b).

Observe that the cut in $\tilde{\Sigma}(p)$ begins at $p^2 = m^2$, even though the physical two-particle threshold is at $p^2 = M^2$. Evidently, the gauge variant fermion propagator, when evaluated in the unphysical Landau gauge, contains contributions from virtual, unphysical processes. [The threshold at m^2 arises from the $k_\mu k_\nu / k^2$ and $\varepsilon_{\mu\nu\alpha} k^\alpha / k^2$ parts of the vector meson propagator.]

C(3). *Survey of Higher Orders.* We foresee no difficulty in carrying out perturbative calculations to arbitrary order. Infrared divergences will not arise in the Landau gauge. To be sure, the propagator has an $O(p^{-1})$ singularity, but this is integrable in low orders. When the propagator is iterated, $D^{\mu\nu}$ is always multiplied by $\Pi^{\mu\nu}$, which vanishes at $p = 0$. So the product $D\Pi$ is non-singular at $p = 0$, and will not lead to off-mass-shell infrared divergences. However, we do not know whether a mass shall can be defined for gauge variant quantities.

In our super-renormalizable theory, one-particle irreducible diagrams with three or more loops are convergent. The ultraviolet divergences of the one- and two-loop graphs will be absent, as in the examples computed here. The regulator ambiguity, which affects the computation of the mass shift is therefore present only in low orders of perturbation theory.

III. NON-ABELIAN GAUGE THEORIES

A. *General Features*

Our topological mass term can be generalized to a non-Abelian gauge theory. The gauge field Lagrangian is

$$\mathscr{L}_G = \frac{1}{2g^2} \operatorname{tr} F^{\mu\nu} F_{\mu\nu} - \frac{\mu}{2g^2} \varepsilon^{\mu\nu\alpha} \operatorname{tr} \left(F_{\mu\nu} A_\alpha - \frac{2}{3} A_\mu A_\nu A_\alpha \right). \tag{3.1}$$

We use a matrix notation

$$A_\mu = g T^a A^a_\mu, \tag{3.2a}$$

$$F_{\mu\nu} = g T^a F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.2b}$$

which employs the representation matrices of the group

$$[T^a, T^b] = f^{abc} T^c. \tag{3.3}$$

The coupling constant is g, while μ/g^2 is dimensionless. The field equations which follow from (3.1) are gauge covariant,

$$\mathscr{D}_\mu F^{\mu\nu} + \frac{\mu}{2} \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} = 0, \tag{3.4a}$$

$$\mathscr{D}_\mu = \partial_\mu + [A_\mu, \ ] \tag{3.4b}$$

and from our previous consideration of the non-interacting limit ($g = 0$), we know that μ indeed provides a mass for the field. The dual field

$$*F^\mu = \tfrac{1}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} \tag{3.5}$$

satisfies the Bianchi identity

$$\mathscr{D}_\mu \, {*F^\mu} = 0. \tag{3.6}$$

The dual of (3.4a) is

$$\mathscr{D}_\alpha \, {*F_\beta} - \mathscr{D}_\beta \, {*F_\alpha} - \mu F_{\alpha\beta} = 0 \tag{3.7}$$

and another covariant divergence converts this, with the help of (3.4a) and the Ricci identity $|\mathscr{D}_\alpha, \mathscr{D}_\beta| = F_{\alpha\beta}$, to

$$(\mathscr{D}_\alpha \mathscr{D}^\alpha + \mu^2) \, {*F_\mu} = \varepsilon_{\mu\alpha\beta} [{*F^\alpha}, {*F^\beta}], \tag{3.8}$$

which is the non-Abelian analogue of (2.13b).

The field Lagrangian (3.1) is not invariant against gauge transformations; rather it changes by a total derivative. Consider a finite transformation

$$A_\mu \to U^{-1} A_\mu U + U^{-1} \partial_\mu U. \tag{3.9}$$

The response of the action to the gauge transformation (3.9) is

$$\begin{aligned}\int dx\, \mathscr{L}_G \to \int dx\, \mathscr{L}_G &+ \frac{\mu}{g^2} \int dx\, \varepsilon^{\alpha\mu\nu} \operatorname{tr} \partial_\mu [A_\alpha \partial_\nu U U^{-1}] \\ &+ \frac{\mu}{3g^2} \int dx\, \varepsilon^{\alpha\beta\gamma} \operatorname{tr}[\partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U U^{-1}].\end{aligned} \tag{3.10}$$

The second term on the right-hand side is the analogue of the Abelian term (2.5). We shall only consider gauge transformations which tend to the identity at temporal and spatial infinity:

$$U(x) \xrightarrow[x \to \infty]{} I. \tag{3.11}$$

This restriction is made to avoid convergence problems in (3.10); also, it reflects our assumption of asymptotic space-time uniformity. With Eq. (3.11), we may conclude that the A-dependent surface integral in (3.10) vanishes. The last term in (3.10) can also be converted to a surface integral once the integrand is rewritten as a total derivative. This can be made manifest by introducing an explicit parametrization for U. We choose the gauge group to be $SU(2)$ [more generally, we consider a $SU(2)$ subgroup of the gauge group] and make use of the exponential parametrization

$$U(x) = \exp i\sigma^a \theta^a(x). \tag{3.12}$$

It follows that

$$\int dx\, \mathscr{L}_G \to \int dx\, \mathscr{L}_G + \mu \frac{8\pi^2}{g^2} w(U), \tag{3.13}$$

$$\begin{aligned} w(U) &= \frac{1}{24\pi^2} \int dx\, \varepsilon^{\alpha\beta\gamma} \operatorname{tr}[\partial_\alpha U U^{-1} \partial_\beta U U^{-1} \partial_\gamma U U^{-1}] \\ &= \frac{1}{16\pi^2} \int dx\, \varepsilon^{\alpha\beta\gamma} \varepsilon^{abc} \partial_\alpha \left[\theta^a \partial_\beta \theta^b \partial_\gamma \theta^c \frac{1}{\theta^2} \left(1 - \frac{\sin 2\theta}{2\theta}\right)\right], \\ \theta^2 &= \theta^a \theta^a. \end{aligned} \tag{3.14}$$

It is recognized that $w(U)$ is the winding number of the gauge transformation U [5]. Consequently, the surface integral is not zero, but takes an integer value which characterizes the homotopic equivalence class to which U belongs.[9] Only for homotopically trivial U's—those continuously deformable to I—does $w(U)$ vanish. These considerations are, of course, familiar from the analysis of the topological structure in four-dimensional Yang–Mills theory. That they should reappear in the three-dimensional theory is not surprising, in view of the further mathematical/topological connections which we shall draw in Section IIIB.

We conclude that the action is not gauge invariant, but changes by $\mu(8\pi^2/g^2)\, w(U)$. However, it is the exponential of the action, $\exp i \int dx\, \mathscr{L}$, that should be gauge invariant. Otherwise, the expectation of a gauge invariant operator $\mathcal{O}$ would be undefined, as can be seen from a functional integral representation: $\langle \mathcal{O} \rangle = Z^{-1} \int \mathscr{D}A \mathcal{O}(A) \exp iI(A)$, with gauge invariant measure $\mathscr{D}A$ and normalization factor Z. Changing variables $A \to A^U$, where A^U is a gauge transform of A, implies that $\langle \mathcal{O} \rangle = \exp[i\mu(8\pi^2/g^2)\, w(U)] \langle \mathcal{O} \rangle$, which can only be tolerated if the change in the action is an integral multiple of 2π.[10] This gives a quantization condition on the dimensionless ratio $4\pi\mu/g^2$:

$$4\pi \frac{\mu}{g^2} = n, \qquad n = 0, \pm 1, \ldots. \tag{3.15}$$

A Euclidean formulation leads to the same conclusion. The functional integral requires $\exp -\int dx\, \mathscr{L}$ to be gauge invariant, but the mass term's contribution to the action is purely imaginary; a factor of i appears when the continuation to imaginary time [Euclidean space] is performed. The quantization condition (3.15) is entirely due to the internal group.

When gauge fields are coupled to fermions by a Lagrangian

$$\mathscr{L}_F + \mathscr{L}_I = i\bar{\psi}\gamma^\mu D_\mu \psi - m\bar{\psi}\psi \tag{3.16}$$

then the fermion mass term mixes with the topological mass term. Here D_μ is the covariant derivative in the fermion representation. The symmetries of the theory are

[9] That $w(U)$ is an integer can also be established by taking the integration in (3.14) to be over Euclidean three-space and by recognizing that the gauge transformations satisfying condition (3.11) provide a mapping of S_3 [three-dimensional space with the points at infinity identified] to S_3 [the $SU(2)$ manifold]. Such mappings are characterized by integral winding numbers, whose analytic expression is provided by Eq. (3.14), in Euclidean space. However, once we know that $w(U)$ is an integer in Euclidean space, it will also be an integer in Minkowski space, since the integral in Eq. (3.14) is a metric independent coordinate invariant. The quantization of $w(U)$ comes entirely from the homotopic properties of the internal group.

[10] An alternate argument is based on the fact that the phase exponential changes by $\exp[i\mu(8\pi^2/g^2)\, N]$ when the functional integration [without change of variables] ranges over gauge copies with relative winding number N. If there is no quantization, then the sum of these contributions is the vanishing overall factor $\sum_N \exp[i\mu(8\pi^2/g^2)\, N]$, rather than $\sum_N 1$ which is harmlessly cancelled by the normalizing denominator. However, J. Schonfeld, in a private communication, suggests that perhaps one may also cancel the vanishing factor by the normalization of the functional integral. Whether this "cancellation of zero" is acceptable is unclear to us.

the same as in the Abelian case, except that for Majorana fermions in the adjoint representation and $m=\mu$, there is also a supersymmetry, since the degrees of freedom match.

B. *Topological Connection*

We have repeatedly mentioned that the gauge invariant mass makes use of topologically non-trivial structures. Some aspects have been exhibited already: I_{CS}, the topological mass term in the action, is a metric-independent scalar, which is invariant against "small" gauge transformations [those homotopic to the identity], while "large" ones [those not homotopic to the identity] change it by discrete amounts. In fact, I_{CS} is proportional to a well-known topological entity, the Chern–Simons secondary characteristic class [4], which in turn is related to four-dimensional topology. We now explain these topological ideas.

In even dimensions, one may construct from gauge fields a gauge invariant Pontryagin density, $\mathscr{P}_{2n}$, whose integral over the even dimensional space is an invariant that measures the topological content of the fields. Examples in two and four dimensions are

$$\mathscr{P}_2=\frac{1}{2\pi}\,{}^*F\equiv\frac{1}{4\pi}\,\varepsilon^{\mu\nu}F_{\mu\nu}, \tag{3.17a}$$

$$\mathscr{P}_4=-\frac{1}{16\pi^2}\,\mathrm{tr}\;{}^*F^{\mu\nu}F_{\mu\nu}. \tag{3.17b}$$

These gauge invariant objects can also be written as total derivatives of gauge variant quantities

$$\mathscr{P}_{2n}=\partial_\mu X^\mu_{2n}. \tag{3.18}$$

The two- and four-dimensional expressions are

$$X^\mu_2=\frac{1}{2\pi}\,\varepsilon^{\mu\nu}A_\nu, \tag{3.19a}$$

$$X^\mu_4=-\frac{1}{16\pi^2}\,\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{tr}\left(A_\alpha F_{\beta\gamma}-\frac{2}{3}A_\alpha A_\beta A_\gamma\right). \tag{3.19b}$$

The Chern–Simons secondary characteristic class is gotten by integrating one component of X_{2n}^{μ} over the $2n-1$ dimensional space which does not include that component. The integral is known to be gauge invariant against homotopically trivial gauge transformations; otherwise, it changes by the winding number of the transformation.

We see, therefore, that I_{CS} is precisely the Chern–Simons structure, since it is proportional to $\int dx^0\, dx^1\, dx^2\, X_4^3$. This observation has an immediate parallel in the construction of a topological mass for three-dimensional gravity, from the four-dimensional $^*RR$ Hirzebruch–Pontryagin density; see Section IV.

It is only in three dimensions that a topological term is bilinear and can produce a mass. The one-dimensional case is vacuous. In two dimensions, the Pontryagin density is the anomalous divergence of the axial vector current [10] and is responsible for mass generation in the Schwinger model.[11] In four dimensions, the Pontryagin density is linked to the θ vacua [5]. In higher dimensions, expressions with higher field powers appear, e.g., εFFF... [even dimensions] or εAFF... [odd dimensions]; bilinear topological expressions require higher-rank antisymmetric tensors.

Since a [Euclidean] three-dimensional theory describes a four-dimensional one at high temperature [1], perhaps the topological mass is a phenomenological description of magnetic screening due to topological excitations in four dimensions. Note also, that massive gauge particles in two and three dimensions emerge for topological reasons [11]. One may therefore speculate that a topological mass generating mechanism—as yet undiscovered—can also operate in four dimensions. This would provide a welcome alternative to the conventional Higgs procedure.

C. *Perturbation Theory*

Thus far, we have been concerned with kinematical properties of three-dimensional, topologically massive Yang–Mills fields. Let us now examine dynamics, when the gauge fields are coupled to massive fermions in the fundamental representation. We compute the gauge field and ghost self-energies for a $SU(N)$ gauge group to lowest non-trivial order in perturbation theory. [The fermion propagator is identical to the one in the Abelian theory, save for a multiplicative factor of $(N^2-1)/2N$.] The calculation is straightforward, but enormously lengthy. One difference from the familiar massless theory is the presence of new three-point vertices coming from the topological mass term. The free propagators are obvious generalizations of the Abelian ones. The Landau gauge is used exclusively and the regulator procedure is the following: for fermion loops we use Pauli–Villars regulators; for gauge field loops, Pauli–Villars regularization is unavailable, and we use dimensional regularization. [Gauge invariant projection methods would be equally effective.]

[11] An analysis of mass generation in the Schwinger model [9] in terms of the anomalous divergence of the two-dimensional axial vector current is given in [11].

For the gauge field propagator, the results are

$$\mathscr{D}_{\mu\nu}^{-1} = D_{\mu\nu}^{-1} - i\Pi_{\mu\nu}, \tag{3.20a}$$

$$\Pi_{\mu\nu} = P_{\mu\nu}\Pi^{(1)}(p^2) + i\mu\varepsilon_{\mu\nu\alpha}p^\alpha\Pi^{(2)}(p^2). \tag{3.20b}$$

The $O(g^2)$ invariant functions are presented in a spectral representation

$$\Pi^{(1)}(p^2) = -\frac{Ng^2}{64\pi}p^2\int_0^\infty da\,\frac{\rho^{(1)}(a)}{p^2-a^2+i\varepsilon} + \frac{g^2}{16\pi}p^2\int_{2m}^\infty da\,\frac{1+4m^2/a^2}{p^2-a^2+i\varepsilon}, \tag{3.21a}$$

$$\Pi^{(2)}(p^2) = -\frac{Ng^2}{32\pi}\int_0^\infty da\,\frac{\rho^{(2)}(a)}{p^2-a^2+i\varepsilon} + \frac{m}{\mu}\frac{g^2}{4\pi}p^2\int_{2m}^\infty da\,\frac{1/a^2}{p^2-a^2+i\varepsilon}. \tag{3.21b}$$

The spectral functions are positive, continuous and possess zero-mass thresholds

$$\rho^{(1)}(a) = \begin{cases} -a^4/\mu^4 + 31a^2/\mu^2, & 0 \leqslant a \leqslant \mu, \\ a^4/\mu^4 - 23a^2/\mu^2 + 102 - 50\mu^2/a^2, & \mu \leqslant a \leqslant 2\mu, \\ 22 - 34\mu^2/a^2, & 2\mu \leqslant a \leqslant \infty, \end{cases} \tag{3.22a}$$

$$\rho^{(2)}(a) = \begin{cases} -3a^4/\mu^4 + 4a^2/\mu^2, & 0 \leqslant a \leqslant \mu, \\ 3a^4/\mu^4 - 6a^2/\mu^2 + 2 + 2\mu^2/a^2, & \mu \leqslant a \leqslant 2\mu, \\ 26 + 2\mu^2/a^2, & 2\mu \leqslant a \leqslant \infty. \end{cases} \tag{3.22b}$$

The second integrals in (3.21a) and (3.21b) are the fermion loop contributions to $\Pi_{\mu\nu}$ [compare (2.60)].

For the ghost self-energy, we find to $O(g^2)$

$$G^{-1} = G_0^{-1} - iM, \tag{3.23}$$

$$M(p^2) = -\frac{Ng^2}{16\pi}p^2\int_0^\infty da\,\frac{\rho(a)}{p^2-a^2+i\varepsilon}. \tag{3.24}$$

The spectral function is again positive and continuous with a threshold at zero mass:

$$\rho(a) = \begin{cases} a^2/\mu^2, & 0 \leqslant a \leqslant \mu, \\ 2 - \mu^2/a^2, & \mu \leqslant a \leqslant \infty. \end{cases} \tag{3.25}$$

Observe that the gauge-field contribution to $\Pi^{(1)}$ is negative for space-like p^2. This is opposite in sign from the fermion contribution and is the three-dimensional relic of the familiar reversal of sign in the four-dimensional vacuum polarization tensor, which is responsible for asymptotic freedom. The same is true of $M(p^2)$, and this behavior has already been noted in the massless theory [12].

At $p^2 = 0$ $\Pi^{(1)}$ vanishes; the vacuum polarization is non-singular, and so is $D\Pi$. Therefore, no infrared divergences will arise from iterating the p^{-1} singularity of the bare propagator. The bare ghost propagator is massless, even in the presence of the topological mass. But the p^{-2} singularity is never enhanced in higher orders, since $G_0 M$ is regular, owing to the vanishing of M at $p^2 = 0$. Therefore, there should be no infrared divergences in higher orders.

Since the spectral function is non-zero for $a < \mu$, the propagator has an imaginary part at $p^2 = \mu^2$, and mass-shell characteristics cannot be sensibly extracted. To determine them, we must calculate vacuum expectation values of gauge-invariant operators. The unphysical thresholds come from the $p_\mu p_\nu / p^2$ and $\varepsilon_{\mu\nu\alpha} p^\alpha / p^2$ parts of the bare vector propagator, as well as from the massless ghost propagator. That they contribute to a gauge variant quantity is not surprising; a similar phenomenon was seen in the fermion propagator of the Abelian theory.

While our inability to extract physical information from the gauge variant propagators of the Yang–Mills theory is disappointing, it does not obviously negate perturbative calculations of vacuum expectation values of gauge invariant operators, which also contain all the physical information. Indeed, the propagators that we have examined contribute to these calculations, and it is gratifying that well-defined expressions are obtained for them. Nevertheless, the present results indicate that as usual the non-Abelian theory, even when massive, possesses a more intricate infrared structure than the Abelian one.

As we have seen explicitly, the one-loop corrections are ultraviolet and infrared finite, because the propagator is dominated by the massless contributions for large p, and by μp for small p. At higher loops, the divergence could be logarithmic [at most], but this is forbidden by gauge invariance. So the theory is ultraviolet finite, and should also be infrared finite, but we have not studied the latter question in higher orders.

IV. GRAVITATION

A. *Introduction*

In this section, we study the dynamics of three-dimensional gravity with a Chern–Simons topological mass term. We shall consider both the full theory and its linearized approximation. First we develop the non-linear non-Abelian model in order to understand the geometrical significance of the topological term. Then we analyze the kinematics of the free field to extract the particle content. While there is considerable resemblance to vector gauge models [as we shall be noting], the effect of the miss term is here more dramatic:

(a) The massless Einstein theory is trivial, but acquires a propagating, massive, spin 2 degree of freedom when the mass term is present.

(b) The topological term has third time derivative dependence, yet the theory is ghost-free and unitary. As in the vector case, the propagation is causal rather than tachyonic, although there is no a priori way to control this in the action.

(c) The topological mass term's contribution to the field equations also has a natural geometrical significance: it is the three-dimensional analog of the Weyl tensor. Its action also responds non-trivially to "large" gauge transformations [local rotations].

B. *Non-linear Theory*

We begin with some general remarks on three-dimensional geometry and its bearing to the absence of dynamics in ordinary Einstein theory. This absence corresponds to a simple degree-of-freedom count in the linearized approximation. In d space-time dimensions, the excitations of a symmetric tensor gauge theory are described by the transverse–traceless part of its spatial components; these number $\frac{1}{2}d(d-3)$. [A gauge vector is described by its transverse spatial components, of which there are $(d-2)$.]

The components of the Riemann tensor $R_{\alpha\beta\gamma\delta}$, defined from the connection $\Gamma^{\alpha}_{\gamma\beta}$ by

$$R^{\alpha}{}_{\beta\gamma\delta} = \partial_{\gamma}\Gamma^{\alpha}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}_{\beta\gamma} + \Gamma^{\alpha}_{\mu\gamma}\Gamma^{\mu}_{\beta\delta} - \Gamma^{\alpha}_{\mu\delta}\Gamma^{\mu}_{\beta\gamma}, \tag{4.1}$$

can be decomposed into a symmetric Ricci trace $R_{\mu\nu} \equiv R_{\alpha\mu}{}^{\alpha}{}_{\nu}$, and a traceless part which is the conformably invariant [Weyl] tensor $C^{\alpha}{}_{\beta\gamma\delta}$. However, for $d=3$ the latter vanishes identically and is replaced by a new conformal [Weyl] second-rank tensor $C^{\mu\nu}$, which we shall discuss shortly. The vanishing of $C^{\alpha}{}_{\beta\gamma\delta}$ is due to the fact that Riemann and Ricci tensors each have the same number of components (6) and are fully equivalent:

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= g_{\alpha\gamma}\tilde{R}_{\beta\delta} + g_{\beta\delta}\tilde{R}_{\alpha\gamma} - g_{\alpha\delta}\tilde{R}_{\beta\gamma} - g_{\beta\gamma}\tilde{R}_{\alpha\delta}, \\ \tilde{R}_{\alpha\beta} &\equiv R_{\alpha\beta} - \tfrac{1}{4}\, g_{\alpha\beta}R, \qquad R \equiv R^{\alpha}_{\alpha} \end{aligned} \tag{4.2a}$$

or

$$\begin{aligned} R^{\alpha\beta}{}_{\gamma\delta} &= -\varepsilon^{\alpha\beta\mu}\varepsilon_{\gamma\delta\nu}G^{\nu}_{\mu}, \\ G^{\mu\nu} &\equiv R^{\mu\nu} - \tfrac{1}{2}\, g^{\mu\nu}R. \end{aligned} \tag{4.2b}$$

Note that the sign in the first equation of (4.2b) is signature-dependent; we have taken the one appropriate to signature $(+--)$; with this signature the volume element is $\sqrt{+g}$, $g \equiv \det g_{\mu\nu}$.

Owing to the equivalence (4.2), the source-free Einstein equations $G^{\mu\nu}=0$ simply state that the full curvature tensor vanishes, so that source-free space-time is flat. In presence of sources, the curvature is locally determined by the energy-momentum tensor. Matter acquires a gravitational self-interaction, but there are no gravitons [just as there are no photons in a vector gauge theory at $d=2$]. We note incidentally that for $d=2$, $G^{\mu\nu}$ vanishes identically and the Einstein action is the Euler topological invariant. [This fact is useful in some of our later calculations which concern two-dimensional space in $d=3$ space-time.]

A nonvanishing Weyl tensor for $d=3$, which is invariant under local conformal changes of the metric and vanishes if and only if the metric is conformally flat, is the density $\sqrt{g}\, C^{\mu}_{\ \nu}$, where the odd parity tensor $C^{\mu\nu}$ is defined by [13]

$$C^{\mu\nu} = \frac{1}{\sqrt{g}}\, \varepsilon^{\mu\alpha\beta} \mathscr{D}_\alpha \tilde{R}^{\nu}_{\beta}\,. \tag{4.3}$$

This quantity is manifestly traceless, identically covariantly conserved and symmetric, by virtue of the Bianchi identities satisfied by $G_{\alpha\beta}$:

$$\varepsilon_{\alpha\mu\nu} \sqrt{g}\, C^{\mu\nu} = \mathscr{D}_\beta G^{\beta}_{\alpha} \equiv 0. \tag{4.4}$$

In an equivalent formula for $C^{\mu\nu}$, the scalar curvature may be omitted in (4.3) and replaced by explicit $(\mu\nu)$ symmetrization of the remainder.

That $C^{\mu\nu}$ satisfies a Bianchi identity, viz., that it is identically covariantly conserved, suggests that it is the functional derivative of a geometrical invariant. This is indeed true and the invariant is [just as in the vector case] the Chern–Simons characteristic, obtained from the $d=4$ Hirzebruch–Pontryagin density

$$^{*}RR \equiv \tfrac{1}{2}\, \varepsilon^{\mu\nu\alpha\beta} R_{\mu\nu\rho\sigma} R_{\alpha\beta}{}^{\rho\sigma} = \partial_\mu X^\mu \tag{4.5}$$

by taking the $d=3$ integral of X^3, and omitting all x^3 dependence. The integral may be written as

$$\begin{aligned} I_{\mathrm{CS}} &= -\tfrac{1}{4} \int dx\, X^3 \\ &= -\tfrac{1}{4} \int dx\, \varepsilon^{\mu\nu\alpha} [R_{\mu\nu ab}\, \omega_\alpha{}^{ab} + \tfrac{2}{3}\, \omega_{\mu b}{}^{c} \omega_{\nu c}{}^{a} \omega_{\alpha a}{}^{b}]. \end{aligned} \tag{4.6}$$

Here we have used the Ricci connection definition of the curvature, $R_{\mu\nu ab} \equiv \partial_\mu \omega_{\nu ab} + \omega_{\mu a}{}^{c} \omega_{\nu cb} - (\mu \leftrightarrow \nu)$, $\omega_{\mu ab} = -\omega_{\mu ba}$. [One could alternatively have introduced matrix notation as in the Yang–Mills theory, in terms of the representation matrices t^{ab} of the $O(2, 1)$ group, to avoid explicit local Lorentz indices.] The $\omega_{\mu ab}$ are themselves curls of the fundamental dynamical variables, the *dreibeins*, so that I_{CS} is of third derivative order, rather than first as in the vector case.

Thus, our action is [omitting a possible cosmological term]

$$I = \frac{1}{\kappa^2} \int dx\, \sqrt{g}\, R + \frac{1}{\kappa^2 \mu}\, I_{\mathrm{CS}}\,. \tag{4.7}$$

The resulting field equations, obtained by varying (4.7) with respect to the metric, are[12]

$$\mathscr{G}^{\mu\nu} \equiv G^{\mu\nu} + \frac{1}{\mu} C^{\mu\nu} = 0. \tag{4.8}$$

The sign of the Einstein part of (4.7) is opposite to the conventional one in four dimensions; this choice is required for the physical mode to not be ghost-like, as we shall see below. Matter coupling would be included by the usual addition of the covariantized matter action. Note that the Einstein part of the action has coefficient κ^{-2} with dimension of mass, while the topological part [being of third-derivative order] has a dimensionless coefficient which we have written as the product of κ^{-2} and an inverse mass μ^{-1}. The field equations (4.8) simply express a balance between Einstein and Weyl tensors; in the absence of sources they imply that the scalar curvature vanishes [since $C^{\mu\nu}$ is traceless].

A suggestive "first-order" form for the field equations is

$$\begin{aligned} &\mathscr{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)\, R_{\lambda\sigma} = 0, \\ &\mathscr{O}_{\mu\nu}{}^{\lambda\sigma}(\mu) \equiv \left(\delta^\lambda_\mu \delta^\sigma_\nu - \frac{1}{2} g_{\mu\nu} g^{\lambda\sigma}\right) + \frac{1}{\mu\sqrt{g}} \varepsilon_\mu{}^{\alpha\beta} \left(\delta^\lambda_\beta \delta^\sigma_\nu - \frac{1}{4} g^{\lambda\sigma} g_{\nu\beta}\right) \mathscr{D}_\alpha. \end{aligned} \tag{4.9}$$

[The operator $\mathscr{O}(\mu)$ is the tensor equivalent of the analogous vector operator occurring in (2.13b).] $\mathscr{O}(\mu)$ may be multiplied by $\mathscr{O}(-\mu)$ to yield a second-order equation for the Ricci tensor. In the source-free case, using $R=0$, we find [analogously to (3.8)]

$$0 = \mu^2 \mathscr{O}_{\alpha\beta}{}^{\mu\nu}(-\mu)\, \mathscr{O}_{\mu\nu}{}^{\lambda\sigma}(\mu)\, R_{\lambda\sigma}, \tag{4.10a}$$

which explicitly reads

$$(\mathscr{D}_\alpha \mathscr{D}^\alpha + \mu^2)\, R_{\mu\nu} = -g_{\mu\nu} R^{\alpha\beta} R_{\alpha\beta} + 3 R^\alpha_\mu R_{\nu\alpha}. \tag{4.10b}$$

[12] To express $*RR$ as a total divergence one must use the *vierbein* description, which is appropriate to a gauge theoretic viewpoint. However, the variation of $I_{\rm CS}$, like that of the Einstein action, depends only on the metric, rather than on the *dreibein*. To see this, observe first that [as in the Yang–Mills case] $-\delta I_{\rm CS} = \frac{1}{2}\int dx\, \varepsilon^{\alpha\beta\nu} R^{ab}_{\alpha\beta} \delta\omega_{\nu ab}$. Next obtain $\delta\omega_{\nu ab}$ by varying the equation $0 = \mathscr{D}_\nu e_{a\mu} = [\delta^b_a \delta^\sigma_\mu \partial_\nu + \delta^\sigma_\mu \omega_{\nu a}{}^b - \delta^b_a \Gamma^\sigma_{\mu\nu}]\, e_{b\sigma}$ which defines the connection in terms of the *dreibein* $e_{a\mu}$. This implies that $\delta\omega_{\nu ab} = e^\mu_b(e_{a\sigma} \delta\Gamma^\sigma_{\mu\nu} - \mathscr{D}_\nu \delta e_{a\mu})$. When inserted into $\delta I_{\rm CS}$, the second term in $\delta\omega$ does not contribute by the Bianchi identity on the curvature. The total variation therefore has the purely metric form $-\delta I_{\rm CS} = \frac{1}{2}\int dx\, \varepsilon^{\mu\alpha\beta} R_{\alpha\beta\sigma}{}^\nu \delta\Gamma^\sigma_{\mu\nu}$. Inserting $\delta\Gamma^\sigma_{\mu\nu}$ in terms of $\delta g_{\mu\nu}$ easily yields $-\delta I_{\rm CS} = \int dx \sqrt{g}\, C^{\mu\nu} \delta g_{\mu\nu}$. This shows that $I_{\rm CS}$ is coordinate invariant [because $C^{\mu\nu}$ is covariantly conserved] and conformably invariant [because $C^{\mu\nu}$ is traceless]. The above results hold for infinitesimal variations. $I_{\rm CS}$ still changes under finite local rotations, since for these only the connection, rather than the *dreibein* dependence, is relevant and everything is identical [up to notation] to the vector case in (3.10).

This exhibits the massive character of the excitations, but gives no information as to their number [if any] or spins.

In the linearized limit, the right side of (4.10b) vanishes, the differential operator is just the flat space d'Alembertian, and the curvature satisfies the Klein–Gordon equation with mass $|\mu|$. Note that propagation is causal, rather than tachyonic, although we have no control over the sign of the μ^2 term in (4.10), since it would be the same in the parity-conjugate model with $\mu \to -\mu$ in (4.7) and (4.8).[13] [Parity and time-inversion conservation are restored in a doublet of gravities, with opposite masses.]

The definition and measurement of energy has some important distinctions from [but also similarities to] that familiar in normal Einstein theory, which we first review [15]. Energy is defined, for asymptotically flat solutions at spatial infinity, by the following considerations. [More generally, similar results hold in presence of a cosmological term or for geometries asymptotic to non-flat solutions of the Einstein equations [15]]. We decompose the metric into its asymptotic, Minkowski part $\eta_{\mu\nu}$ plus a deviation $h_{\mu\nu}$ which is not necessarily small:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,. \tag{4.11}$$

The Bianchi identities (4.4) are written in the form

$$0 = \mathscr{D}_\mu G^{\mu\nu} = \partial_\mu G_L^{\mu\nu} + (\Gamma^{\nu}_{\alpha\beta} G^{\alpha\beta} + \Gamma^{\beta}_{\alpha\beta} G^{\alpha\nu}) + \partial_\mu G_N^{\mu\nu}\,, \tag{4.12}$$

where $G_L^{\mu\nu}$ is the part linear in $h_{\mu\nu}$ and $G_N^{\mu\nu}$ is the non-linear remainder. Then, by virtue of the field equations, $G^{\mu\nu}=0$, and the linearized Bianchi identity, $\partial_\mu G_L^{\mu\nu} \equiv 0$, it follows that $G_N^{\mu\nu}$ is an ordinarily conserved quantity, and hence may be identified with a multiple of the energy-momentum pseudo-tensor [it is not a tensor]. Furthermore, since $G_L^{\mu\nu}$ is identically conserved, it may be written in terms of a superpotential as $G_L^{\mu\nu} \equiv -\partial_\alpha \partial_\beta K^{\mu\alpha\nu\beta}$, where K has the algebraic symmetries of the Riemann tensor. Consequently, $G_L^{0\nu} \equiv -\partial_i \partial_\beta K^{0i\nu\beta}$, and we may integrate the Einstein constraint equations, $G^{0\nu} = G_L^{0\nu} + G_N^{0\nu} = 0$, as

$$\frac{\kappa^2}{2}\, P^\mu = \int d\mathbf{r}\; G_N^{0\mu} = -\int d\mathbf{r}\; G_L^{0\mu} = \oint dS^i\, \partial_\beta K^{0i\mu\beta}. \tag{4.13a}$$

[13] The Bel–Robinson tensor $B^{\mu\nu\alpha\beta}$ provides a clue [similar to that given by the Maxwell energy-momentum tensor in the vector case] for the causal behavior. We recall its definition [for any d]: $B^{\mu\nu\alpha\beta} \equiv R^{\mu}{}_{\rho}{}^{\alpha}{}_{\sigma} R^{\nu\rho\beta\sigma} + R^{\mu}{}_{\rho}{}^{\beta}{}_{\sigma} R^{\nu\rho\alpha\sigma} - \frac{1}{2} g^{\mu\nu} R^{\alpha}{}_{\lambda\kappa\tau} R^{\beta\lambda\kappa\tau}$. As a consequence of the Bianchi identities, the covariant divergence of $B^{\mu\nu\alpha\beta}$ on its (μ, ν) indices is proportional to the Ricci tensor's curl $(\mathscr{D}_\alpha R_{\mu\nu} - \mathscr{D}_\nu R_{\mu\alpha})$. In our model, this curl equals [a dual of] the Ricci tensor itself, and as in ordinary Einstein theory B is then conserved by virtue of the identity $\varepsilon^{\mu\nu\alpha} R_{\nu\lambda} R_{\alpha\sigma} + (\sigma \leftrightarrow \lambda) = 0$. [The above derivation is identical to that of footnote 4.] In particular, the linearized Bel–Robinson tensor is ordinarily conserved by virtue of the linearized equations. But the integral of its time component $\int d\mathbf{r}\, B^{0000}$ is a positive definite "superenergy," related to the normal energy in a positive way; see [14].

[The sign taken above is appropriate to our present theory, with a negative Einstein action.] That is, the energy and momentum are expressible by flux integrals at spatial infinity over first derivatives of $h_{\mu\nu}$. In particular, the energy has the form

$$\begin{aligned} \frac{\kappa^2}{2} P^0 &= \oint d\mathbf{S} \cdot \mathbf{G}, \\ \nabla \cdot \mathbf{G} &= -G_L^{00} = G_N^{00}, \end{aligned} \tag{4.13b}$$

where $\mathbf{G}$ is itself a gradient [of the gauge-invariant component of the spatial metric]. Asymptotically $\mathbf{G}$ behaves as

$$\frac{\kappa^2 P^0}{4\pi} \frac{\mathbf{r}}{r} = \frac{\kappa^2 P^0}{4\pi} \nabla \ln r,$$

which is an expression of the two-dimensional Coulomb [Newtonian] law. [These properties have obvious counterparts in the usual, massless vector theory: read longitudinal electric field for $\mathbf{G}$ and charge for energy.]

The same argument follows step by step in our massive model, with the replacement of $G^{\mu\nu}$ by our field operator $\mathscr{G}^{\mu\nu} \equiv G^{\mu\nu} + (1/\mu)\, C^{\mu\nu}$. This is so because $G^{\mu\nu}$ and $C^{\mu\nu}$ separately satisfy Bianchi identities. The energy-momentum pseudo-tensor is now $(2/\kappa^2)\, \mathscr{G}_N^{\mu\nu}$ and equals [by the field equations] $-(2/\kappa^2)\, \mathscr{G}_L^{\mu\nu}$; the latter is still identically conserved and expressible as a superpotential, $\mathscr{G}_L^{\mu\nu} \equiv -\partial_\alpha \partial_\beta \mathscr{K}^{\mu\alpha\nu\beta}$:

$$\theta^{\mu\nu} = \frac{2}{\kappa^2} \mathscr{G}_N^{\mu\nu} = -\frac{2}{\kappa^2} \mathscr{G}_L^{\mu\nu}. \tag{4.14}$$

The flux-integral form for energy and momentum continues to hold:

$$\frac{\kappa^2}{2} P^\mu = \int d\mathbf{r}\, \mathscr{G}_N^{0\mu} = -\int d\mathbf{r}\, \mathscr{G}_L^{0\mu} = \oint dS^i\, \partial_\beta \mathscr{K}^{0i\mu\beta}. \tag{4.15a}$$

However, the contributions from the two parts of $-(2/\kappa^2)\, \mathscr{G}_L^{00}$ to P^0,

$$\frac{\kappa^2}{2} P^0 = -\int d\mathbf{r} \left(G_L^{00} + \frac{1}{\mu} C_L^{00} \right), \tag{4.15b}$$

are quite different, because of the short-range nature of this massive theory. Since $-C_L^{00} = \varepsilon^{ij} \partial_i R_L^{0j}$, its volume integral is also the surface integral of $\varepsilon^{ij} R_L^{0j}$. The latter quantity is short range, due to the mass term, as is seen from (4.10b), and this surface integral vanishes. Consequently, the total energy comes from the volume integral of $-G_L^{00}$. Thus, the potential $\mathbf{G}$ in (4.13b) possesses the same long-range component as in the absence of mass. This will be demonstrated explicitly, when the linearized approximation is discussed. [The above may be contrasted with the vector case, Eqs. (2.16)–(2.19), where the integral of the topological term survives to give the charge, while the longitudinal electric field is short range. For both the vector and the tensor theories, it is the lowest derivative term in the constraint equation that carries the asymptotic information.]

Our discussion thus shows that there is no clash between general gauge invariance arguments, which imply flux integral form for conserved generators (P^μ, $M^{\alpha\beta}$) associated with asymptotic [Poincaré] symmetries, and the short-range nature of the theory. [This conclusion is unaltered in presence of matter sources.]

We conclude with a brief description of the Chern–Simons characteristics, although it is in fact quite similar to the vector discussion [4]. There are no $d=2$ invariants [only the Euler number $\varepsilon^{\mu\nu}R_{\mu\nu ab}\varepsilon^{ab}$] and the higher-dimensional ones, e.g., $P_6=\varepsilon RRR\equiv\partial_\mu X_6^\mu$ are at least cubic in curvature and so would not alter the kinematics; also, they involve at least six derivatives. Three dimensions is the only interesting case except if higher-rank tensor fields are involved. [Euler characteristics do not yield any odd-dimensional invariants.] Finally, we mention that a discussion of the effect of large gauge transformations on the topological term and possible quantization of the dimensionless quantity $(\kappa^2\mu)$ formally is the same as in the vector case.[10] However, when the "internal" group is the non-compact $SO(2, 1)$, rather than the compact $SO(3)$ or $SU(2)$, quantization will not occur, since the maximal compact subgroup, $SO(2)$, is homotopically trivial. This is fortunate since in the quantum gravity theory infinite renormalization of the coupling constants may be required; see below.

B. *Linearized Approximation*

In order to analyze the particle content of our model, we must consider its free-field kinematics which are defined by the linearized approximation to the full theory (4.7). This approximation consists of retaining terms to order κ^2 in the expansion (4.11). The conserved linearized energy-momentum tensor $\theta^{\mu\nu}$ is obtained by keeping in $\mathscr{G}_N^{\mu\nu}$—which is proportional to the conserved pseudo-tensor of the full theory—terms quadratic in κ^2; we call them $\mathscr{G}_Q^{\mu\nu}$.[14] It should be remembered that here, as in linearized gravity for any dimension, $\theta^{\mu\nu}$ is not invariant under Abelian gauge transformations,

$$\delta h_{\mu\nu}=\partial_\mu\xi_\nu+\partial_\nu\xi_\mu. \tag{4.16}$$

Only the Poincaré generators, formed as spatial moments of $\theta^{0\nu}$, are gauge invariant, but they cannot be given by a flux integral in linearized approximation. The energy is manifestly positive only after all constraints have been used. [This is even true of conventionally massive spin-2 theory.] Consequently, an a priori argument against tachyonic excitations cannot be given [in contrast to the vector case] from the energy alone.[14]

[14] One could also obtain an energy-momentum tensor by conventional Noether or Belinfante methods; these, however, are considerably more complicated than the above procedure. When carried out, they yield [as they must] the same $\theta^{\mu\nu}$ up to a superpotential. Note that [unlike the vector case] the topological mass term contributes explicitly here, since for the linearized theory $I_{\rm CS}^L$ is not external-metric independent.

We now construct the linearized action. Rather than directly expanding (4.7), which depends on the *dreibeins*, one may simply linearize the [purely metric] field equations (4.8) in terms of (4.11) and deduce the form of I_E^L, I_{CS}^L as functionals of $h_{\mu\nu}$. Alternatively, one would find that I_{CS}^L depends only on the symmetric part of the deviation $e_\mu^a - \delta_\mu^a$, in quadratic order. The Einstein part has the form

$$I_E^L = -\tfrac{1}{2}\int dx\, h_{\mu\nu} G_L^{\mu\nu}, \tag{4.17}$$

where

$$G_L^{\mu\nu} \equiv R_L^{\mu\nu} - \tfrac{1}{2}\eta^{\mu\nu} R_L \tag{4.18a}$$

is the variational derivative of $-I_E^L$, and $R_L^{\mu\nu}$ is the linearized Ricci tensor

$$\begin{aligned} R_L^{\mu\nu} &= \tfrac{1}{2}(-\Box h^{\mu\nu} + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} - \partial^\mu \partial^\nu h), \\ h &\equiv h_\alpha^\alpha, \end{aligned} \tag{4.18b}$$

Throughout, all index and differential operations are with respect to the Minkowski metric $\eta_{\mu\nu}$. The topological term (4.6) becomes

$$\frac{1}{\mu} I_{\text{CS}}^L = \frac{1}{2\mu}\int dx\, \varepsilon_{\mu\alpha\beta} G_L^{\alpha\nu}\, \partial^\mu h_\nu^\beta. \tag{4.19}$$

Its variational derivative may be presented as

$$-\frac{1}{\mu}\frac{\delta I_{\text{CS}}^L}{\delta h_{\mu\nu}} = \frac{1}{\mu} C_L^{\mu\nu} = \frac{1}{\mu}\varepsilon^\mu{}_{\alpha\beta}\,\partial^\alpha \tilde{R}_L^{\beta\nu}, \tag{4.20}$$

which of course agrees with (4.3) in the linearized approximation.

Einstein's action is invariant against the gauge transformation (4.16), as are the local curvature components $R_L^{\mu\nu}$. The topological Lagrangian changes by a total derivative

$$\begin{aligned} \delta \mathscr{L}_{\text{CS}}^L &= \frac{1}{\mu}\varepsilon_{\mu\alpha\beta} G_L^{\alpha\nu}\, \partial^\mu \partial_\nu \xi^\beta \\ &= \frac{1}{\mu}\varepsilon_{\mu\alpha\beta}\,\partial_\nu [G_L^{\alpha\nu}\,\partial^\mu \xi^\beta]. \end{aligned} \tag{4.21}$$

Consequently, the field equations

$$\begin{aligned} &-\Box h^{\mu\nu} + \partial^\mu \partial_\alpha h^{\alpha\nu} + \partial^\nu \partial_\alpha h^{\alpha\mu} - \partial^\mu \partial^\nu h + \eta^{\mu\nu}(\Box h - \partial_\alpha \partial_\eta h^{\alpha\beta}) \\ &-\frac{1}{2\mu}\varepsilon^{\mu\alpha\beta}\,\partial_\alpha(\Box h_\beta^\nu - \partial_\lambda \partial^\nu h_\beta^\lambda) - \frac{1}{2\mu}\varepsilon^{\nu\alpha\beta}\,\partial_\alpha(\Box h_\beta^\mu - \partial_\lambda \partial^\mu h_\beta^\lambda) = 0 \end{aligned} \tag{4.22}$$

are gauge invariant.

To obtain the physical content of the theory, we perform a space-time decomposition of the action $I^L = I_E^L + (1/\mu)\, I_{\text{CS}}^L$. The following notation is used,

$$h^{\mu\nu} = (h^{00} \equiv N,\, h^{0i} \equiv N^i,\, h^{ij}), \tag{4.23a}$$

where $i, j = 1, 2$ are spatial indices. The spatial tensor h^{ij} and the vector $\mathbf{N}^i$ are further decomposed into their orthogonal and irreducible parts:

$$\begin{aligned} h^{ij} &= (\delta^{ij} + \hat{\partial}_i \hat{\partial}_j)\, \varphi - \hat{\partial}_i \hat{\partial}_j \chi + (\partial_i \xi_T^j + \partial_j \xi_T^i), \\ \mathbf{N} &= \mathbf{N}_T + \boldsymbol{\nabla} N_L, \qquad \boldsymbol{\nabla} \cdot \mathbf{N}_T = 0 = \boldsymbol{\nabla} \cdot \boldsymbol{\xi}_T. \end{aligned} \tag{4.23b}$$

The component φ—the transverse-trace part of h^{ij}—is gauge invariant; gauge transformations affect only χ and $\boldsymbol{\xi}_T$. Absent from h^{ij} in (4.23b) is the transverse-traceless part. As we noted earlier, it vanishes in two-dimensional space. Since any transverse two-vector has only one component, we see that $h^{\mu\nu}$ is decomposable into six space-scalar functions, only three of which are gauge invariant.

When the decomposition (4.23) is inserted into the action I^L, it is relatively straightforward [after integrations by parts] to obtain its form as a functional of the variables (φ, χ, $\boldsymbol{\xi}_T$, $\mathbf{N}_T$, N_L, N). One finds

$$I_L = -\frac{1}{2} \int dx \left\{ [\varphi \,\Box\, \varphi + \lambda \varphi + \sigma^2] + \frac{1}{\mu} \sigma \lambda \right\}, \tag{4.24a}$$

where the variables λ and σ are the other two gauge-invariant metric combinations

$$\begin{aligned} \lambda &= \nabla^2 (N + 2\dot{N}_L) + \ddot{\chi} - \Box\, \varphi, \\ \sigma &= \varepsilon^{ij}\, \partial_j (N_T^i + \dot{\xi}_T^i). \end{aligned} \tag{4.24b}$$

Unlike φ, they are also locally Weyl invariant [i.e., under $\delta h_{\mu\nu} \rho \eta_{\mu\nu}$]. The triple time derivatives in the action are safely hidden in the Lagrange multiplier λ [which enforces the constraint associated with gauge invariance] and in the constrained function σ. It is clear from (4.24) that neither part of the action alone yields dynamics. Without the μ^{-1} terms, the field equations state that the three gauge invariants φ, λ and σ all vanish. In the purely topological theory, without I_E, λ and σ vanish; φ is undetermined, being a [Weyl] gauge variable.

Upon elimination of the λ-constraint, the action takes on its final form:

$$I_L = -\tfrac{1}{2} \int dx\, \varphi(\Box + \mu^2)\, \varphi. \tag{4.25}$$

This represents a single massive degree of freedom of mass μ, with positive definite Hamiltonian,

$$H = \tfrac{1}{2} \int d\mathbf{r} [\dot{\varphi}^2 + (\boldsymbol{\nabla} \varphi)^2 + \mu^2 \varphi^2] \tag{4.26}$$

and canonical commutation relations [or Poisson brackets]

$$i[\dot{\varphi}(\mathbf{r}), \varphi(\mathbf{r}')] = \delta(\mathbf{r}-\mathbf{r}'). \tag{4.27}$$

Note that changing the sign of the Einstein part of the action would have led to an opposite overall [ghost] sign in (4.25).

The analysis could also have included coupling to a conserved matter energy-momentum tensor, through a term $\frac{1}{2}\kappa h_{\mu\nu} T^{\mu\nu}$. We mention the simplest and most important case in which only a static density $T^{00} = \rho(\mathbf{r})$ is present. In the action, this changes the constraint [the coefficient of λ in (4.24a)] to $(\varphi - \kappa \nabla^{-2}\rho) + \mu^{-1}\sigma$, and adds $-\kappa\rho\varphi$. The final effect of this coupling is entirely summarized by an interaction $\kappa \int \rho\varphi$ added to the free action (4.25), where φ now stands for the shifted combination $\varphi - \kappa \nabla^{-2}\rho$. The static interaction Hamiltonian is attractive, of Yukawa type: $H_I = -\kappa^2 \int \rho[1/(-\nabla^2 + \mu^2)]\, \rho$. [The situation is here again as in the vector theory; compare (2.53d).] Although all three gauge-invariant quantities—λ, σ and the shifted φ—are short range, G_L^{00} is proportional to the Laplacian of the untranslated φ, which has the long-range component $\kappa \nabla^{-2}\rho$. This demonstrates explicitly our earlier statements that the total energy in the non-linear theory comes entirely from the volume integral of G_L^{00}, which is also the surface integral of the gradient of the untranslated φ. Note that in the Einstein limit, $\mu \to \infty$ [not $\mu \to 0$!], the interaction between static T^{00} sources vanishes.

We now revert to the sourceless case. The variable φ is a scalar under spatial rotations, as is seen from its definition in (4.23b), but has complicated Lorentz transformation properties. The former fact leads to the immediate prediction that the spatial translation and rotation generators have the simple scalar forms

$$P^i = \int d\mathbf{r}\; \theta^{0i} = -\int d\mathbf{r}\; \dot{\varphi}\, \partial_i \varphi, \tag{4.28}$$

$$\begin{aligned} M = \tfrac{1}{2}\, \varepsilon^{ij} M^{ij} &= \varepsilon^{ij} \int d\mathbf{r}\; r^i \theta^{0j} \\ &= -\int d\mathbf{r}\; \dot{\varphi} \varepsilon^{ij} r^i\, \partial_j \varphi. \end{aligned} \tag{4.29}$$

However, the boosts

$$M^{i0} + tP^i = \int d\mathbf{r}\; x^i \theta^{00} \tag{4.30}$$

[as in the vector theory] are more complicated because φ is not a three-scalar; rather, $\nabla^2\varphi$ is proportional to the time–time component of the gauge invariant tensor $R_{\mu\nu}$. Their form will determine the particle's spin.[6]

To verify our statements about P^μ and M and to compute M^{i0} we must evaluate $\theta^{0\mu}$ in terms of the single dynamical variable. The first step is to obtain the form of $h^{\mu\nu}$ which solves the linearized constraints. Since the values of the Poincaré generators are gauge invariant, there is a great deal of simplification by use of a convenient gauge, which we shall take to be the Weyl gauge, i.e., Gaussian coordinates

$$h^{0\nu}=0. \tag{4.31}$$

To determine h^{ij}, i.e., to relate χ and ξ_T, to φ, it suffices to insert the general form of $h^{\mu\nu}$, in the chosen gauge, into the constraint equations $\mathscr{G}_L^{0\nu}=0$. These, together with the single dynamical equation $R_L=0$, determine everything. We find

$$\begin{gathered} h^{ij}=\left[\delta^{ij}+\left(1+\frac{\mu^2}{-\nabla^2+\mu^2}\right)\hat{\partial}_i\hat{\partial}_j\right]\varphi-\frac{\mu}{-\nabla^2(-\nabla^2+\mu^2)}(\varepsilon^{i\kappa}\,\partial_j\,\partial_k+\varepsilon^{ik}\,\partial_i\,\partial_k)\,\dot{\varphi}, \\ (\square+\mu^2)\,\varphi=0. \end{gathered} \tag{4.32}$$

In any other gauge, $h_{\mu\nu}$ will differ from this by an arbitrary $(\partial_\mu\xi_\mu+\partial_\nu\xi_\mu)$ term. We may now insert (4.32) into the formula for the energy-momentum tensor, which is symmetric in (μ, ν), though not explicitly so:

$$\theta^{\mu\nu}=\frac{2}{\kappa^2}\mathscr{G}_Q^{\mu\nu}\equiv\frac{2}{\kappa^2}\left[G^{\mu\nu}+\frac{1}{\mu\sqrt{g}}\varepsilon^{\mu\alpha\beta}\mathscr{D}_\alpha\tilde{R}_\beta^\nu\right]_Q. \tag{4.33}$$

Here Q stands for terms quadratic in $h^{\mu\nu}$. Some simplification is achieved by recalling that the intrinsic spatial Einstein tensor, ${}^{(2)}G^{ij}$, vanishes identically, and by using the embedding formulas for ${}^{(3)}G^{ij}$ in terms of ${}^{(2)}G^{ij}$ and $\ddot{h}^{ij}$. After evaluation of (4.33) in terms of (4.32) and lengthy manipulations, we arrive at (4.26) as well as the predicted expressions (4.28) and (4.29). But the boosts acquire an additional contribution [as they did for the vector system (Eq. (2.42))], which is required so that M^{i0} generate the correct Lorentz transformations on gauge invariant quantities, such as $R^{\mu\nu}$. Their form is

$$M^{i0}+tP^i=\frac{1}{2}\int d\mathbf{r}\,r^i[\dot{\varphi}^2+(\nabla\varphi)^2+\mu^2\varphi^2]+2\mu\varepsilon^{ij}\int d\mathbf{r}\,\dot{\varphi}\,\frac{\partial_j}{-\nabla^2}\,\varphi. \tag{4.34}$$

Note that the infrared singular term, involving the inverse Laplacian, is just twice that of the vector case. The same reasoning as in the vector case then immediately assures us that the graviton has spin ± 2, the sign being correlated with that of μ. After this term is removed [by a procedure analogous to that in the vector case, Eqs. (2.44)–(2.50)], the Poincaré algebra may be verified.

This completes the main task of this subsection, viz., the determination of the kinematics of our theory and verification that it contains no ghosts. Before turning to the quantum theory, we discuss several related points.

The topologically massive theory may be contrasted with the normal [gauge non-invariant] massive model, where $-\frac{1}{4}\mu^2 \int dx(h_{\mu\nu}^2 - h^2)$ replaces the topological mass term. Here one expects the number of degrees of freedom corresponding to a symmetric tensor which is transverse-traceless, $\partial_\nu h^{\mu\nu} = 0 = h$, since these equations are a consequence of the Pauli–Fierz theory. This allows $\frac{1}{2}d(d+1) - d - 1 = \frac{1}{2}(d+1)(d-2)$ excitations. [Massive vector theory has $(d-1)$ degrees of freedom since $\partial_\mu A^\mu = 0$.] The counting is verified by explicit reduction of the corresponding action. The two degrees of freedom are φ and $\mu\varepsilon^{ij}\,\partial_i(1/\sqrt{-\nabla^2+\mu^2})\,\xi_T^j$, each having Klein–Gordon dynamics and spin ± 2. This corresponds to achieving parity conserving, topologically massive excitations by forming a doublet of topological actions with opposite masses.

C. *Ultraviolet Behavior*

Even though its action is of third derivative order, the ultraviolet properties of topological quantum gravity are not manifestly improved over those of normal Einstein gravity, which is off-shell non-renormalizable in three dimensions. The reason is that the mass term possesses a larger gauge symmetry than the conventional Einstein term: in addition to the gauge invariance (4.16), the tracelessness of $C_{\mu\nu}$ ensures that I_{CS} is invariant against local conformal transformations.[10]

$$\delta g_{\mu\nu} = \rho g_{\mu\nu}, \tag{4.35}$$

This means that there will be a component of $h_{\mu\nu}$, present only in I_E and not in I_{CS}, whose propagation is governed by I_E. Consequently, the free propagator will have the conventional p^{-2} ultraviolet asymptote, which is also consistent with the absence of ghosts [16].[15]

This is verified by computing the free propagator for $I^L = I_E^L + (1/\mu)\, I_{\mathrm{CS}}^L$. In the Landau gauge, $\partial_\mu h^{\mu\nu} = 0$, we find

$$\begin{aligned} D_{\mu\nu,\,\alpha\beta} = {} & \frac{-i/2}{p^2}\,(P_{\mu\alpha}P_{\nu\beta} + P_{\nu\alpha}P_{\mu\beta} - 2P_{\mu\nu}P_{\alpha\beta}) \\ & + \frac{i/2}{p^2-\mu^2}\,(P_{\mu\alpha}P_{\nu\beta} + P_{\nu\alpha}P_{\mu\beta} - P_{\mu\nu}P_{\alpha\beta}) \\ & + \frac{\mu/4}{p^2-\mu^2}\,\frac{p^\gamma}{p^2}\,(\varepsilon_{\mu\alpha\gamma}P_{\nu\beta} + \varepsilon_{\nu\alpha\gamma}P_{\mu\beta} + \varepsilon_{\mu\beta\gamma}P_{\nu\alpha} + \varepsilon_{\nu\beta\gamma}P_{\mu\alpha}). \end{aligned} \tag{4.36}$$

[15] There is a close parallel between the present model and the four-dimensional one in which the Einstein Lagrangian is supplemented by $(R_{\mu\nu}^2 - \frac{1}{3}R^2)$. Apart from total derivatives, this addition is the square of the [four-dimensional] Weyl tensor and is conformally invariant. The conformal parts of the metric decrease as p^{-4}, while those associated with the scale factor remain at p^{-2}. The difference between our model and the four-dimensional one is that in the latter there are ghosts due to the fourth derivatives in the addition. In either dimension, adding a further [non-unitary] R^2 term removes the conformal invariance and leads to a p^{-4} behavior in the scale factor's propagator [17].

Equation (4.36) behaves as $(i/2)\, P_{\mu\nu} P_{\alpha\beta}/p^2$ for large p. There is no massless propagation; the $p^2=0$ poles in (4.36) disappear when the propagator is contracted into conserved sources [just as in the vector case].

There are a number of further consequences to be drawn from (4.36). First, it follows, as noted earlier, that pure Einstein theory [$\mu \to \infty$] has no $T^{00}-T^{00}$ interaction, because the source–source coupling has the form $\int dp[T_{\mu\nu}(p)\, T^{\mu\nu}(-p) - T^{\mu}_{\mu}(p)\, T^{\nu}_{\nu}(-p)]\, p^{-2}$. Second, the absence of an on-shell pole in the Einstein contribution to D means that all interactions in pure Einstein theory are of contact type for general [non-static] sources. Third, the residues in D are such that the p^{-2} part looks like a ghost [and it is therefore important that it has no pole on shell], while the $(p^2-\mu^2)^{-1}$ part has the non-ghost sign. This is a consequence of the "opposite" sign we took for the Einstein action.

Conceivably, despite the p^{-2} behavior of the propagator and p^3 growth of vertices, cancellations may occur and the theory may be renormalizable. An infinite renormalization of $\kappa^2\mu$ renders a quantization condition on that combination problematical; fortunately, in gravity theory, with Minkowski signature, none seems required, because the compact subgroup $SO(2)$ of the Lorentz group is homotopically trivial.

A supergravity model can also be constructed, and possibly it may be less divergent. Its [P and T conserving] Rarita–Schwinger part, $I=(i/2)\int dx\, \bar{\psi}_{\mu}\varepsilon^{\mu\nu\alpha}\,\partial_{\nu}\psi_{\alpha}$, by itself, gives trivial dynamics like the Einstein action. [It implies that $\partial_{\mu}\psi_{\nu}-\partial_{\nu}\psi_{\mu}=0$, hence ψ_{μ} is a pure gauge.] When augmented by a topological mass term, which is the supersymmetric companion to X^3, the theory gives rise to a massive spin $\frac{3}{2}$ excitation. Since supergravity has positive energy, positivity of energy in our model will follow just as for Einstein gravity in four dimensions [18].

V. CONCLUSION

In addition to providing yet another theoretical laboratory, the study of gauge theories in three dimensions has particular interest because novel topological structures are available in odd-dimensional spaces. This is especially true for $d=3$, since it is only there that the new terms affect the kinematics of the theory. We have seen that the Chern–Simons characteristics provide a gauge invariant mass-generating mechanism which alters the behavior of these theories. In the vector case they give an effective infrared cutoff, while for gravity, they lead to a ghost-free, causally propagating excitation.

Our results may have physical significance if these models [in their Euclidean version] emerge as high-temperature limits of four-dimensional theories. Clearly, the question here is how to fit the topological mass terms into this framework. Presumably, they owe their origin to the unavoidable θ-angles of four-dimensional physics. Therefore, naturalness would imply that the high-temperature effective Lagrangian contains the corresponding three-dimensional topological structures.

Obviously, numerous interesting problems still remain, both in three and higher odd dimensions. Similar approaches might be of interest in supersymmetry and supergravity, in Kaluza–Klein theory, and in other attempts at dimensional reduction. Indeed, Chern–Simons-like terms occur automatically in $d=7$, 11 supergravity. For the present models, it would be instructive to express the full non-linear actions in terms of a single dynamical field variable. For gravity, we expect that an appropriate redefinition $\varphi \to \Phi(\varphi)$ will lead to a form $I = -\int dx\{\frac{1}{2}\Phi(\Box + \mu^2)\Phi + V(\Phi)\}$, where the [space non-local] potential V and the difference $(\Phi - \varphi)$ will vanish as $\kappa \to 0$. There should also exist a generalization of (4.32) which expresses the metric $g_{\mu\nu}$ as a functional of this variable. In a similar analysis of the Yang–Mills case, the corresponding independent variable would be a color multiplet Φ^a. Finally, a detailed discussion of the theories, divergence properties would check that Yang–Mills theory is infrared finite to all orders and would determine the ultraviolet behavior in gravity.

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ERRATUM‡

Volume **140**, Number 2 (1982), in the article "Topologically Massive Gauge Theories," by S. Deser, R. Jackiw, and S. Templeton, pages 372–411: Pisarski and Rao [1] have pointed out that our evaluation [2] of the parity-even portion of the vacuum polarization tensor in the topologically massive three-dimensional Yang–Mills theory is wrong, and they have published a corrected formula, An independent check is now available thanks to the recalculation by de Mello [3] who agrees with Pisarski and Rao. The principal discussion in our paper [2] remains unaltered, being based on the parity-odd contribution to the vacuum polarization, which stands unmodified, and has been confirmed by Pisarski and Rao.

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‡ Erratum reprinted from Volume 185, page 406.