Field and Particle Equations for the Classical Yang-Mills Field and Particles with Isotopic Spin (*).

S. K. WONG

Institute for Theoretical Physics
State University of New York - Stony Brook, N. Y.

(ricevuto il 20 Settembre 1969)

Summary. — A complete system of equations describing the interaction between the Yang-Mills field and isotopic-spin-carrying particles in the classical limit is extracted from the equations of motion for the quantum fields. Some simple consequences are derived. The consistency of the equations is investigated.

1. – Introduction.

Difficulties still persist in the quantum theory of the Yang-Mills [1,2] field. Investigation of the theory at the classical level that might furnish insight into the quantum theory is desirable. By analogy with electromagnetism, in the classical limit, one expects that there should emerge a picture of isotopic-spin-carrying particles interacting with each other through a C-number Yang-Mills field. It is the purpose of this note to write down the equations describing this interaction.

2. – Extraction of the classical equations of motion.

Our starting point will be the field equations for the quantum fields, from which we shall extract the desired equations by taking appropriate limits.

(*) Partially supported by the U.S. AEC under Contract No. AT(30-1)-3668B.
(1) C. N. YANG and R. L. MILLS: *Phys. Rev.*, **96**, 191 (1954).
(2) B. ZUMINO: *Acta Phys. Austriaca, Suppl.* II, 212, (1965).

Consider the interaction between the Yang-Mills field $\boldsymbol{b}_\mu(x)$[3] and a spin-$\frac{1}{2}$ field $\psi(x)$ which transforms under a particular irreducible representation of SU_2 in which X_α with $\alpha = 1, 2, 3$ form a set of generators such that

$$[X_\alpha, X_\beta] = i\varepsilon_{\alpha\beta\gamma} X_\gamma \,. \tag{1}$$

The Lagrangian density is [4]

$$\mathscr{L} = -\frac{1}{4}\boldsymbol{f}_{\mu\nu}\cdot\boldsymbol{f}_{\mu\nu} - \bar{\psi}\gamma_\mu(\partial_\mu - ig\boldsymbol{b}_\mu\cdot\boldsymbol{X})\psi - \frac{mc}{\hbar}\bar{\psi}\psi \,, \tag{2}$$

where

$$\boldsymbol{f}_{\mu\nu} = \partial_\mu \boldsymbol{b}_\nu - \partial_\nu \boldsymbol{b}_\mu + g\boldsymbol{b}_\mu \times \boldsymbol{b}_\nu \,. \tag{3}$$

The field equations are

$$\partial_\mu \boldsymbol{f}_{\mu\nu} + g\boldsymbol{b}_\mu \times \boldsymbol{f}_{\mu\nu} = -ig\bar{\psi}\gamma_\nu \boldsymbol{X}_\psi \psi \,, \tag{4}$$

$$\gamma_\mu(\partial_\mu - ig\boldsymbol{X}\cdot\boldsymbol{b}_\mu)\psi + \frac{mc}{\hbar}\psi = 0 \,. \tag{5}$$

Let us concentrate first on (5), and regard it as a one-particle Dirac equation for an isotopic-spin-carrying particle in a given external C-number field $\boldsymbol{b}_\mu(x)$. It can be written in the form

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{6}$$

with

$$H = c\alpha_i(p_i - g\boldsymbol{b}_i\cdot\boldsymbol{I}) + mc^2\beta - igc\boldsymbol{b}_4\cdot\boldsymbol{I} \,, \tag{7}$$

where α_i, β are the Dirac matrices, $p_i = (\hbar/i)\partial_i$, and $\boldsymbol{I} = \hbar\boldsymbol{X}$ satisfy the commutation relations

$$[I_\alpha, I_\beta] = i\hbar\varepsilon_{\alpha\beta\gamma} I_\gamma \,. \tag{8}$$

In the Heisenberg picture, the following equations of motions are obtained:

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{i}{\hbar}[H, x_i] = \alpha_i c \,, \tag{9}$$

---

(3) Bold-face letters will represent a triple in the isotopic-spin space. Thus, $\boldsymbol{b}_\mu = (b_\mu^1, b_\mu^2, b_\mu^3)$.

(4) Notations: A four-vector a_μ has components $(a_1, a_2, a_3, a_4 = ia_0)$. Greek indices run from 1 to 4, Latin indices from 1 to 3. The γ-matrices are Hermitian and satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$.

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = \frac{i}{\hbar}[H, p_i] = gc(\alpha_j \partial_i \boldsymbol{b}_j + i\partial_i \boldsymbol{b}_4)\cdot \boldsymbol{I}\,, \tag{10}$$

$$\frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}t} = \frac{i}{\hbar}[H, \boldsymbol{I}] = g\left(\frac{\mathrm{d}x_i}{\mathrm{d}t}\boldsymbol{b}_i + ic\boldsymbol{b}_4\right)\times \boldsymbol{I}\,. \tag{11}$$

Defining the mechanical momenta by

$$\pi_i = p_i - g\boldsymbol{b}_i\cdot \boldsymbol{I} \tag{12}$$

we further find

$$\frac{\mathrm{d}\pi_i}{\mathrm{d}t} = g\left(\frac{\mathrm{d}x_j}{\mathrm{d}t}\boldsymbol{f}_{ij} + ic\boldsymbol{f}_{i4}\right)\cdot \boldsymbol{I}\,. \tag{13}$$

When (13) is compared with a similar equation for a charged particle moving in an external electromagnetic field, the following equation governing the world line $\xi_\mu(\tau)$ of the particle in space-time suggests itself:

$$m\ddot{\xi}_\mu = g\boldsymbol{f}_{\mu\nu}\cdot \boldsymbol{I}(\tau)\,\dot{\xi}_\nu\,, \tag{14}$$

where the dot denotes differentiation with respect to the proper time τ. The right-hand side obviously represents a generalization of the Lorentz force. Equation (14) is to be supplemented by

$$\dot{\boldsymbol{I}} + g\boldsymbol{b}_\mu \times \boldsymbol{I}\dot{\xi}_\mu = 0 \tag{15}$$

obtainable from (11). In the limit we are considering, a particle is thus described by an internal vector $\boldsymbol{I}$ as well as its space-time co-ordinates.

The isotopic spin current carried by a point particle, analogous to the electric current, is [5]

$$g\int \boldsymbol{I}(\tau)\,\dot{\xi}_\mu\,\delta^{(4)}(x-\xi(\tau))\,\mathrm{d}\tau. \tag{16}$$

Hence the field equation is

$$\partial_\mu \boldsymbol{f}_{\mu\nu} + g\boldsymbol{b}_\mu \times \boldsymbol{f}_{\mu\nu} = -\boldsymbol{j}_\nu\,, \tag{17}$$

with $\boldsymbol{j}_\nu$ being a sum of terms of the form (16), coming from each particle.

---

[5] In the corresponding expression $e\int \dot{\xi}_\mu\,\delta^{(4)}(x-\xi)\,\mathrm{d}\tau$ for the electric current, we may perform the τ-integration and obtain $j_\mu = \big(e(\mathrm{d}\xi_i/\mathrm{d}t)\,\delta^{(3)}(x-\xi(t)),\; ice\delta^{(3)}(x-\xi(t))\big)$, a perhaps more familiar expression.

Equation (17), *together with* (14), (15) *for each particle, completely describes the interaction among a system of particles with isotopic spin, when all relevant qauntitites are treated as C-numbers.*

3. – Some simple consequences.

We notice an immediate consequence of (15):

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\boldsymbol{I}^2 = 0 \,. \tag{18}$$

The isotopic spin of each particle thus performs a precessional motion.

To investigate further the consequences of (14), (15) and (17), it is convenient to introduce the « covariant derivatives »

$$\nabla_\mu \boldsymbol{A} = \partial_\mu \boldsymbol{A} + g\boldsymbol{b}_\mu \times \boldsymbol{A} \tag{19}$$

for any vector $\boldsymbol{A}$ in the isotopic-spin space. The following identities are found to be valid [6]

$$\nabla_\mu \nabla_\nu \boldsymbol{f}_{\mu\nu} = 0 \,, \tag{20}$$

$$\nabla_\mu \boldsymbol{f}_{\varrho\sigma} + \nabla_\varrho \boldsymbol{f}_{\sigma\mu} + \nabla_\sigma \boldsymbol{f}_{\mu\varrho} = 0 \,. \tag{21}$$

From the free Yang-Mills field part of the Lagrangian (2), the symmetric energy-momentum tensor for the field can be obtained by a familiar technique [7] to be

$$T_{\mu\nu} = \boldsymbol{f}_{\mu\sigma}\cdot\boldsymbol{f}_{\nu\sigma} - \tfrac{1}{4}\delta_{\mu\nu}\boldsymbol{f}_{\varrho\sigma}\cdot\boldsymbol{f}_{\varrho\sigma} \,. \tag{22}$$

The divergence of this tensor is

$$\begin{aligned}\partial_\nu T_{\mu\nu} &= -\tfrac{1}{2}\partial_\mu \boldsymbol{f}_{\varrho\sigma}\cdot\boldsymbol{f}_{\varrho\sigma} + \partial_\nu \boldsymbol{f}_{\mu\varrho}\cdot\boldsymbol{f}_{\nu\varrho} + \boldsymbol{f}_{\mu\varrho}\cdot\partial_\nu \boldsymbol{f}_{\nu\varrho} = \\ &= -\tfrac{1}{2}\nabla_\mu \boldsymbol{f}_{\varrho\sigma}\cdot\boldsymbol{f}_{\varrho\sigma} + \nabla_\nu \boldsymbol{f}_{\mu\varrho}\cdot\boldsymbol{f}_{\nu\varrho} + \boldsymbol{f}_{\mu\varrho}\cdot\nabla_\nu \boldsymbol{f}_{\nu\varrho} \,,\end{aligned}$$

---

[6] Notice that for any internal vector $\boldsymbol{A}$, $[\nabla_\mu, \nabla_\nu]\boldsymbol{A} = g\boldsymbol{f}_{\mu\nu}\times\boldsymbol{A}$. Replacing $\boldsymbol{A}$ by $\boldsymbol{f}_{\mu\nu}$ and using the antisymmetry of $\boldsymbol{f}_{\mu\nu}$ (20) is proved. To prove (21), we compute

$$\begin{aligned}\nabla_\mu \boldsymbol{f}_{\varrho\sigma} &= \partial_\mu(\partial_\varrho \boldsymbol{b}_\sigma - \partial_\sigma \boldsymbol{b}_\varrho) + g\partial_\mu(\boldsymbol{b}_\varrho\times\boldsymbol{b}_\sigma) + g\boldsymbol{b}_\mu\times(\partial_\varrho\boldsymbol{b}_\sigma - \partial_\sigma\boldsymbol{b}_\varrho) + g^2\boldsymbol{b}_\mu\times(\boldsymbol{b}_\varrho\times\boldsymbol{b}_\sigma) = \\ &= \partial_\mu(\partial_\varrho \boldsymbol{b}_\sigma - \partial_\sigma \boldsymbol{b}_\varrho) + g^2\boldsymbol{b}_\mu\times(\boldsymbol{b}_\varrho\times\boldsymbol{b}_\sigma) + g(\boldsymbol{b}_\varrho\times\partial_\mu\boldsymbol{b}_\sigma + \boldsymbol{b}_\mu\times\partial_\varrho\boldsymbol{b}_\sigma) - g(\boldsymbol{b}_\sigma\times\partial_\mu\boldsymbol{b}_\varrho + \boldsymbol{b}_\mu\times\partial_\sigma\boldsymbol{b}_\varrho).\end{aligned}$$

Upon cyclic permutation of μ, ϱ, σ and adding, (21) is obtained.

[7] See, for instance, J. M. JAUCH and F. ROHRLICH: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955).

where in the second line the replacement of all ordinary derivatives by covariant derivatives is justified because the extra terms add up to zero. Making use of (21), (17) and then (14),

$$\partial_\nu T_{\mu\nu} = -\boldsymbol{j}_\varrho \cdot \boldsymbol{f}_{\mu\varrho} = -\sum_{\text{all particles}} g \int \boldsymbol{f}_{\mu\varrho} \cdot \boldsymbol{I}\dot{\xi}_\varrho \, \delta^{(4)}(x-\xi)\, \mathrm{d}\tau = -\sum \int \dot{p}_\mu \, \delta^{(4)}(x-\xi)\, \mathrm{d}\tau \,, \tag{23}$$

which justifies, as a corresponding equation in electrodynamics does, the interpretation of $T_{\mu\nu}$ as the energy-momentum tensor of the field.

4. – Consistency.

In virtue of (20), a consequence of the field equation (17) is that

$$\nabla_\mu \boldsymbol{j}_\mu = 0 \,. \tag{24}$$

Through (19) and (17), this is also equivalent to the conservation of isotopic spin

$$\partial_\mu(\boldsymbol{j}_\mu + g\boldsymbol{b}_\nu \times \boldsymbol{f}_{\mu\nu}) = 0 \,. \tag{25}$$

On the other hand, one finds, using the expression (16) for $\boldsymbol{j}_\mu$,

$$\begin{aligned} \nabla_\mu \boldsymbol{j}_\mu &= \sum_{\text{all particles}} g \int \{\boldsymbol{I}\dot{\xi}_\mu \partial_\mu \delta^{(4)}(x-\xi) + g\boldsymbol{b}_\mu \times \boldsymbol{I}\dot{\xi}_\mu \delta^{(4)}(x-\xi)\}\, \mathrm{d}\tau = \\ &= g \sum \int \left\{ -\boldsymbol{I} \frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(4)}(x-\xi) + g\boldsymbol{b}_\mu \times \boldsymbol{I}\dot{\xi}_\mu \delta^{(4)}(x-\xi) \right\} \mathrm{d}\tau = \\ &= g \sum \int (\dot{\boldsymbol{I}} + g\boldsymbol{b}_\mu \times \boldsymbol{I}\dot{\xi}_\mu)\, \delta^{(4)}(x-\xi)\, \mathrm{d}\tau \,. \end{aligned} \tag{26}$$

The requirement (24) is therefore consistent with the eq. (15) for the precession of the isotopic spin.

5. – Validity.

In arriving at the eqs. (14), (15) and (17), the usual conditions for the classical limit are, of course, to be assumed. In addition, we also require the isotopic spins of the particles to be large (or, more precisely, the matrix elements of $\boldsymbol{I}$ should be large on the average compared with $\hbar$) in order that they can be treated as C-numbers.

The above considerations can be generalized to any semi-simple Lie group.

* * *

I should like to thank Prof. C. N. YANG for stimulating discussions and comments on the manuscript.

RIASSUNTO (*)

Si ricava dalle equazioni di moto per i campi quantici un sistema completo di equazioni che descrivono l'interazione fra il campo di Yang-Mills e particelle provviste di spin isotopico nel limite classico. Si ricavano alcune semplici conseguenze. Si studia la coerenza delle equazioni.

(*) *Traduzione a cura della Redazione.*

Уравнения полей и частиц для классического поля Янга-Миллса и частицы с изотопическим спином.

Резюме (*). — Из уравнений движения для квантовых полей извлекается полная система уравнений, описывающих взаимодействие между полем Янга-Миллса и частицами, обладающими изотопическим спином в классическом пределе. Выводятся некоторые следствия. Исследуется непротиворечивость этих уравнений.

(*) *Переведено редакцией.*