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Relativistic Magnetohydrodynamics*

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The relativistic equations of motion for a conducting fluid in a magnetic field are formulated. A relativistically correct equation for the production of entropy due to Joule heating is derived from the conservation laws for mass, momentum, and energy. It is shown that both the energy density of the magnetic field and the magnetic pressure contribute to the inertia of a perfectly conducting fluid. The general dispersion relation for small-amplitude oscillations is found.

INTRODUCTION

It has been shown by Northrop[1] that in many cases a perfectly conducting fluid in a magnetic field behaves as if it had a mass density of

$$\mu+B^2/4\pi c^2,$$

where μ is the usual mass density, and B is the magnetic field in Gaussian units. This suggests the interesting question of why it is that twice the energy density of the magnetic field should contribute to the inertia of the fluid. One also is inclined to ask if the internal energy of the fluid should not also contribute to its inertia. The work reported here grew out of an attempt to answer these questions.

Northrop treated the fluid motion nonrelativistically as have most authors. Clearly any investigation of the inertia of energy should be conducted within the framework of the theory of relativity. In the next section we formulate the relativistic magnetohydrodynamic equations for a perfect fluid.

FUNDAMENTAL EQUATIONS

We shall use the summation convention. We denote the spatial coordinates by x_1, x_2, and x_3 and let $x_4=ict$. We adopt the convention that Latin subscripts take on the values 1, 2, 3, 4, and Greek subscripts take on the values 1, 2, 3.

Since the number of particles in the fluid must be conserved, we have

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x_\alpha}(nu_\alpha)=0, \tag{1}$$

where n is the particle density and u_α is the fluid velocity. We define

$$n^0=n(1-u^2/c^2)^{\frac{1}{2}}, \tag{2}$$

and the usual relativistic four-velocity

$$U_i=\left(\frac{u_\alpha}{(1-u^2/c^2)^{\frac{1}{2}}},\frac{ic}{(1-u^2/c^2)^{\frac{1}{2}}}\right). \tag{3}$$

Then Eq. (1) can be written in the form

$$(\partial/\partial x_i)(n^0U_i)=0. \tag{4}$$

If we multiply this equation by m_0, the rest mass of each particle, and let $\mu^0=n^0m_0$, then we may write

$$(\partial/\partial x_i)(\mu^0U_i)=0. \tag{5}$$

n^0 may be interpreted as the proper number density. It is the number of particles per unit volume in a reference system which moves with the local velocity of the fluid. It follows that μ^0 is the proper density of proper mass. We also define μ_0, the relative density of proper mass, and μ, the relative density of relative mass. These mass densities have been discussed by Møller[2] and by Synge.[3] The relation between μ^0, μ_0, and

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[1] T. G. Northrop, Phys. Rev. **103**, 1150 (1956).

[2] C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), Sec. 50.

[3] J. L. Synge, *Relativity: The Special Theory* (Interscience Publishers, Inc., New York, 1956), Chap. 8.

μ is

$$\mu^0=\mu_0(1-u^2/c^2)^{\frac{1}{2}}=\mu(1-u^2/c^2). \tag{6}$$

Clearly μ^0 is an invariant and by Eq. (5) it is conserved.

The energy-momentum tensor for a perfect fluid has been given by Taub.[4] It is

$$T_{ik}^{(1)}=(\mu^0+\mu^0\epsilon^0/c^2+p/c^2)U_iU_k+p\delta_{ik}, \tag{7}$$

where ϵ^0 is the internal energy per unit proper mass in a reference system which moves with the local velocity of the fluid and p is the pressure. The quantities ϵ^0 and p as well as μ^0 are invariants.

The energy-momentum tensor for the electromagnetic field is (in Gaussian units)

$$T_{\alpha\beta}^{(2)}=(1/8\pi)(E^2+B^2)\delta_{\alpha\beta}-(1/4\pi)(E_\alpha E_\beta+B_\alpha B_\beta), \tag{8}$$

$$T_{\alpha 4}^{(2)}=T_{4\alpha}^{(2)}=(i/4\pi)(\mathbf{E}\times\mathbf{B})_\alpha, \tag{9}$$

$$T_{44}^{(2)}=-(1/8\pi)(E^2+B^2). \tag{10}$$

The total energy-momentum tensor for the system consisting of both fluid and field is just

$$T_{ik}=T_{ik}^{(1)}+T_{ik}^{(2)}. \tag{11}$$

The equations for the conservation of energy and momentum are then

$$\partial T_{ik}/\partial x_k=0. \tag{12}$$

The equations for the conservation of proper mass, energy, and momentum must be supplemented by Maxwell's equations:

$$\nabla\times\mathbf{B}=\frac{4\pi}{c}\mathbf{J}+\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}, \tag{13}$$

$$\nabla\times\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{14}$$

$$\nabla\cdot\mathbf{B}=0, \tag{15}$$

$$\nabla\cdot\mathbf{E}=4\pi\rho_e. \tag{16}$$

In order to complete this set of equations an equation of state relating μ^0, ϵ^0, and p must be given. Also some equation relating $\mathbf{J}$ and the electric and magnetic fields (Ohm's law, perhaps) must be given. For the time being we shall leave these equations unspecified.

It might be thought that because of the equivalence of mass and energy there should not be separate conservation laws for them in a relativistic theory. However, Eckart[5] has shown that both laws are necessary. This becomes clear if it is remembered that Eq. (5) is equivalent to Eq. (4) which expresses the conservation of the number of particles.

It is well known that in the ordinary hydrodynamics of an ideal fluid the conservation of mass, momentum, and energy also implies the conservation of entropy. This has been shown from the relativistic hydrodynamic equations by Taub.[4] We shall show that it is also true in magnetohydrodynamics if the conductivity of the gas is infinite. To show this, we calculate

$$U_i(\partial T_{ik}/\partial x_k)=0. \tag{17}$$

Consider first

$$\frac{\partial T_{ik}^{(1)}}{\partial x_k}=\frac{\partial}{\partial x_k}\mu^0\left(1+\frac{\epsilon^0}{c^2}+\frac{p}{\mu^0c^2}\right)U_iU_k+\frac{\partial p}{\partial x_i}. \tag{18}$$

By using Eq. (5) this may be written

$$\frac{\partial T_{ik}^{(1)}}{\partial x_k}=\mu^0U_k\frac{\partial}{\partial x_k}\left(1+\frac{\epsilon^0}{c^2}+\frac{p}{\mu^0c^2}\right)U_i+\frac{\partial p}{\partial x_i}. \tag{19}$$

We multiply by U_i and carry out the summation to obtain

$$U_i\frac{\partial T_{ik}^{(1)}}{\partial x_k}=-\mu^0\left[U_k\frac{\partial\epsilon^0}{\partial x_k}+pU_k\frac{\partial}{\partial x_k}\left(\frac{1}{\mu^0}\right)\right]. \tag{20}$$

In deriving Eq. (20) we have used

$$U_iU_i=-c^2. \tag{21}$$

Now, by the equation

$$d\epsilon^0+pd(1/\mu^0)=\theta ds, \tag{22}$$

we define the absolute temperature θ and the specific entropy s as measured by an observer moving with the fluid. Equation (20) may now be written

$$U_i\partial T_{ik}^{(1)}/\partial x_k=-\mu^0\theta U_k\partial s/\partial x_k. \tag{23}$$

Now

$$\begin{aligned}U_k\frac{\partial s}{\partial x_k}&=\frac{1}{(1-u^2/c^2)^{\frac{1}{2}}}\left(\frac{\partial s}{\partial t}+u_\alpha\frac{\partial s}{\partial x_\alpha}\right)\\&=\frac{1}{(1-u^2/c^2)^{\frac{1}{2}}}\frac{ds}{dt}\\&=ds/d\tau,\end{aligned} \tag{24}$$

where τ is the proper time. That is to say, it is the time recorded by a clock which moves with the element of fluid.

By using Maxwell's equations it is easily shown that

$$U_i\frac{\partial T_{ik}^{(2)}}{\partial x_k}=(\mathbf{J}-\rho_e\mathbf{u})\cdot\left[\frac{\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{B}}{(1-u^2/c^2)^{\frac{1}{2}}}\right], \tag{25}$$

so that finally Eq. (17) becomes

$$\frac{ds}{d\tau}=\frac{1}{\mu^0\theta}\mathbf{J}_c\cdot\mathbf{E}', \tag{26}$$

where

$$\mathbf{J}_c=\mathbf{J}-\rho_e\mathbf{u} \tag{27}$$

[4] A. H. Taub, Phys. Rev. **74**, 328 (1948).
[5] C. Eckart, Phys. Rev. **58**, 919 (1940).

is the conduction current, and

$$\mathbf{E}'=\frac{\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{B}}{(1-u^2/c^2)^{\frac{1}{2}}} \tag{28}$$

is the electric field measured by an observer moving with the fluid.

Equation (26) has just the form one would expect. It gives the increase in entropy per unit proper time due to the irreversible Joule heating. The convection current has to be subtracted from the total current since it produces no irreversible effects.

If the fluid is a perfect conductor, $\mathbf{E}'$ will vanish and the entropy will be constant along a streamline. For a perfect conductor Ohm's law is replaced by

$$\mathbf{E}+(1/c)\mathbf{u}\times\mathbf{B}=0. \tag{29}$$

For a perfect conductor the equation for the conservation of energy,

$$\partial T_{4k}/\partial x_k=0, \tag{30}$$

may be replaced by the simpler equation

$$ds/d\tau=ds/dt=0. \tag{31}$$

For instance, if the equation of state of the fluid is

$$\epsilon^0=p/[(\gamma-1)\mu^0], \tag{32}$$

then Eq. (31) is equivalent to

$$d/dt(p/\mu^{0\gamma}). \tag{33}$$

In what follows, we shall for simplicity use the perfect gas equation of state, Eq. (32). The velocity distribution and equation of state of a monatomic relativistic gas have been investigated by Jüttner.[6] He finds

$$\epsilon^0=c^2\left[\frac{k\theta}{m_0c^2}-\frac{iH_2^{(1)\prime}(imc^2/k\theta)}{H_2^{(1)}(imc^2/k\theta)}\right], \tag{34}$$

and

$$p=\mu^0k\theta/m_0. \tag{35}$$

In Eq. (34), $H_2^{(1)}$ is the Hankel function of the first kind,[7] and the prime denotes differentiation with respect to the argument. It can be shown from Eqs. (34) and (35) that the relation between ϵ^0, μ^0, and p is given by Eq. (32) in both the high-temperature and low-temperature limits. In the low-temperature limit the gas constant γ has the value 5/3. In the high-temperature limit it has the value $\frac{4}{3}$.

EFFECTIVE MASS

We assume the fluid to be perfectly conducting. Using Eq. (29), we find

$$\begin{aligned}(\mathbf{E}\times\mathbf{B})&=(1/c)B^2\mathbf{u}-(1/c)\mathbf{B}(\mathbf{u}\cdot\mathbf{B})\\&=(1/c)B^2(\mathbf{u}-\mathbf{u}_B),\end{aligned} \tag{36}$$

[6] F. Jüttner, Ann. Physik **34**, 856 (1911).

[7] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945), fourth edition, p. 133.

where we have defined $\mathbf{u}_\mathbf{B}$ as the velocity along the magnetic field. We can without loss of generality take the magnetic field to be in the x_3 direction. Using Eqs. (7) and (9), we calculate $T_{\alpha4}$ and find

$$T_{\alpha4}=\left[\mu^0+\frac{\mu^0\epsilon^0}{c^2}+\frac{p}{c^2}+\frac{B^2}{4\pi c^2}\left(1-\frac{u^2}{c^2}\right)^{\frac{1}{2}}\right]U_\alpha U_4 \tag{37}$$

if $\alpha=1$, 2, and

$$T_{34}=[\mu^0+\mu^0\epsilon^0/c^2+p/c^2]U_3U_4. \tag{38}$$

Now

$$T_{\alpha4}=icg_\alpha, \tag{39}$$

where g_α is the momentum density. We see that for fluid motions along the magnetic field, the magnetic field does not contribute to the inertia of the fluid. For fluid motions perpendicular to the field, the field contributes the momentum density

$$(B^2/4\pi c^2)U_\alpha.$$

If we assign to the magnetic field its usual energy density $B^2/8\pi$ and a pressure of the same amount, we see that this energy density and pressure contribute to the inertia in the same way that the internal energy and pressure of the fluid do. The factor $(1-u^2/c^2)^{\frac{1}{2}}$ indicates that the mass density $B^2/4\pi c^2$ is analogous to the relative density of proper mass μ_0.

The factor of two in the mass density of the magnetic field, which seemed to us anomalous and prompted this investigation, is now seen to be due to the fact that both magnetic energy density and magnetic pressure contribute to the inertia of the fluid.

WAVES OF SMALL AMPLITUDE

We shall now linearize the equations and show that there are three wave velocities which must be considered. Since u_i is now considered to be infinitesimal, we shall not need the superscripts on μ^0 and ϵ^0 any longer. We write

$$\begin{aligned}B_i&=B_{0i}+\tilde{B}_i,\\p&=p_0+\tilde{p},\\\mu&=\mu_0+\tilde{\mu},\end{aligned} \tag{40}$$

where now the subscript denotes an unperturbed quantity. $\tilde{B}_i$, $\tilde{p}$, $\tilde{\mu}$, and E_i are considered small quantities and their products are neglected. We assume the equation of state given by Eq. (32). The unperturbed magnetic field is taken to be in the x_3 direction.

The elements of the linearized energy-momentum tensor are

$$T_{\alpha\beta}=\delta_{\alpha\beta}\left[p_0+\tilde{p}+\frac{1}{8\pi}B_0^2+\frac{1}{4\pi}B_0\tilde{B}_3\right]-\frac{1}{4\pi}B_0\tilde{B}_\beta\delta_{\alpha3}-\frac{1}{4\pi}B_0\tilde{B}_\alpha\delta_{\beta3}, \tag{41}$$

$$T_{\alpha 4}=\left(\mu_0+\frac{\gamma p_0}{(\gamma-1)c^2}+\frac{B_0^2}{4\pi c^2}\right)icu_\alpha-\frac{B_0^2}{4\pi c^2}icu_3\delta_{\alpha 3}, \tag{42}$$

$$T_{44}=-(\mu_0+\tilde{\mu})c^2-\frac{1}{(\gamma-1)}(p_0+\tilde{p})-\frac{1}{8\pi}(B_0^2+2B_0\tilde{B}_3). \tag{43}$$

The set of equations

$$\partial T_{\alpha k}/\partial x_k=0 \tag{44}$$

become

$$\left(\mu_0+\frac{\gamma p_0}{(\gamma-1)c^2}+\frac{B_0^2}{4\pi c^2}\right)\frac{\partial u_\alpha}{\partial t}=-\frac{\partial}{\partial x_\alpha}\left(\tilde{p}+\frac{B_0\tilde{B}_3}{4\pi}\right)+\frac{B_0}{4\pi}\frac{\partial B_\alpha}{\partial x_3} \tag{45}$$

for $\alpha=1, 2$, and

$$\left(\mu_0+\frac{\gamma p_0}{(\gamma-1)c^2}\right)\frac{\partial u_3}{\partial t}=-\frac{\partial\tilde{p}}{\partial x_3}. \tag{46}$$

From Eqs. (5) and (33), we find

$$\partial\tilde{p}/\partial t=-\gamma p_0(\partial u_\alpha/\partial x_\alpha). \tag{47}$$

From Eqs. (14) and (29) we find

$$\frac{\partial B_\alpha}{\partial t}=B_0\frac{\partial u_\alpha}{\partial x_3}-B_0\frac{\partial u_\beta}{\partial x_\beta}\delta_{\alpha 3}. \tag{48}$$

If Eqs. (45) and (46) are now differentiated with respect to time and Eqs. (47) and (48) used to eliminate $\tilde{p}$ and $\tilde{B}_\alpha$, it is found that

$$\frac{\partial^2 u_\alpha}{\partial t^2}=V_1^2\left[\frac{\partial^2 u_\alpha}{\partial x_3^2}+\frac{\partial}{\partial x_i}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\right)\right]+V_2^2\frac{\partial^2 u_\beta}{\partial x_\alpha\partial x_\beta} \tag{49}$$

for $\alpha=1, 2$, and

$$\frac{\partial^2 u_3}{\partial t^2}=V_3^2\frac{\partial^2 u_\beta}{\partial x_3\partial x_\beta}, \tag{50}$$

where we have defined the three velocities V_1, V_2, and V_3 by

$$V_1^2=\frac{B_0^2}{4\pi[\mu_0+\gamma p_0/(\gamma-1)c^2+B_0^2/4\pi c^2]}, \tag{51}$$

$$V_2^2=\frac{\gamma p_0}{[\mu_0+\gamma p_0/(\gamma-1)c^2+B_0^2/4\pi c^2]}, \tag{52}$$

$$V_3^2=\frac{\gamma p_0}{[\mu_0+\gamma p_0/(\gamma-1)c^2]}. \tag{53}$$

Equations (49) and (50) describe the propagation of any small-amplitude disturbance in the fluid. If it is assumed that u_α is of the form $\exp(ik_\beta x_\beta)$, then Eqs. (49) and (50) become homogeneous algebraic equations. The requirement that the determinant of this set of equations vanish gives the dispersion relation:

$$\begin{aligned}\omega^6-\omega^4[(V_1^2+V_2^2)(k_1^2+k_2^2)+(2V_1^2+V_3^2)k_3^2]\\+\omega^2V_1^2k_3^2[(2V_3^2+V_1^2)k_3^2+(V_1^2+V_2^2+V_3^2)\\\times(k_1^2+k_2^2)]-V_1^4V_3^2k_3^4(k_1^2+k_2^2+k_3^2)=0.\end{aligned} \tag{54}$$

From Eqs. (49), (50), and (54) it is easily shown that transverse waves exist which move in the direction of the magnetic field with the velocity V_1. Longitudinal waves that move in the direction of the field have the velocity V_3. Longitudinal waves that move perpendicularly to the field have the velocity $(V_1^2+V_2^2)^{\frac{1}{2}}$.

We shall now examine V_1, V_2, and V_3 in several limiting cases. In the limit of infinite light velocity, V_1 becomes the usual magnetohydrodynamic wave velocity[8] and V_2 and V_3 become equal to the usual velocity of sound.

In the limit of low pressures, V_1, V_2, and V_3 become the velocities found by Northrop.[1]

If the magnetic field vanishes, then V_1 vanishes and $V_2=V_3$ is the sound velocity found by Taub[4] for a relativistic gas. In the limit of very high temperatures this velocity becomes equal to

$$(\gamma-1)^{\frac{1}{2}}c.$$

In this limit $\gamma=\frac{4}{3}$, so the velocity becomes $c/\sqrt{3}$. This agrees with the conclusion of de Hoffmann and Teller.[9] It is also interesting to note that $c/\sqrt{3}$ is the velocity found by Synge[3] for the case of waves in an incompressible fluid.‡

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[8] H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, New York, 1950).

[9] F. de Hoffman and E. Teller, Phys. Rev. **80**, 692 (1950).

‡ *Note added in proof.*—It has been called to the author's attention by the referee of this paper that Eq. (54) may be written in the much simpler form

$$[V^2-V_1^2n_3^2]\left[V^4-V^2\left(V_1^2+V_2^2+\frac{V_1^2V_3^2}{c^2}n_3^2\right)+V_1^2V_3^2n_3^2\right]$$

where $V=\omega/k$ is the phase velocity and $n_3=k_3/k$. This form of the dispersion relation allows one to solve explicitly for the phase velocity V.