Yang-Mills magnetohydrodynamics: Nonrelativistic theory

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The equations of non-Abelian (Yang-Mills) magnetohydrodynamics in the nonrelativistic case are derived in two ways. Noncanonical Poisson brackets are given in the spaces of both magnetic potentials and magnetic fields. The Yang-Mills magnetic field for this theory turns out to be "frozen" into the fluid.

INTRODUCTION

In the cores of neutron stars that are sufficiently cold (temperature $\lesssim 1$ MeV) and dense (baryon density >1 fm^{-3}), hadronic matter exists in the deconfined quark-gluon plasma phase.[1–3] Neutron-star characteristics such as thermal evolution are sensitive to the adopted nuclear (quark) equation of state, neutron-star mass, the assumed magnetic (Yang-Mills) field strength, the possible existence of superfluidity, pion (gluon) condensation, and other issues as yet unresolved. Under reasonable assumptions about the dependence on temperature and density of the confinement-deconfinement transition, models of neutron stars have been developed which predict the existence of quark-gluon cores as large as ~ 8 km in radius for neutron stars with radius ~ 10 km.[1] There is some hope of detecting these hypothetical quark-gluon cores of neutron stars by measuring cooling rates,[2] but the data and computations on this issue are as yet inconclusive.[3]

Since free quarks are present in the core of such a dense, cold neutron star, the Yang-Mills field lines will thread throughout the entire quark-gluon configuration. As a result, Yang-Mills interactions will influence the material dynamics in the interior region, just as ordinary magnetic fields influence the material motion in the exterior region.

One way to model this effect is by extending classical magnetohydrodynamics[4] (MHD) to incorporate non-Abelian Yang-Mills interactions. Our starting point for deriving such a model is the theory of classical quark-gluon plasma physics, called chromohydrodynamics (CHD).[5,6] We work within the framework of the Hamiltonian formalism. The resulting model of ideal Yang-Mills MHD can be regarded as a prototype showing how in principle the quark-gluon interactions can be incorporated phenomenologically into a fluid-dynamical theoretical picture of neutron stars that contain quark-matter cores. Only the nonrelativistic theory is discussed in the present work.

Here, we derive the equations of motion and give the noncanonical Poisson brackets for the limit of CHD corresponding to the non-Abelian extension of MHD. The derived equations have the interesting property that Yang-Mills magnetic fields are "frozen" into the fluid motion, just as occurs in classical Abelian MHD. Consequently, in Yang-Mills MHD, dynamo effects are possible, just as in classical MHD. These dynamo effects could be expected to influence, for example, the convective stability properties of the hypothetical quark-matter cores of neutron stars.

We use two different methods to attack the problem. First, the equations of Yang-Mills MHD are derived using a non-Abelian version of the phenomenological arguments[4] used in classical plasma physics to derive Abelian MHD from the multifluid plasma equations. Next, an alternative derivation of the equations is given, by using the Hamiltonian formalism in the space of magnetic potentials. Then, we discuss the frozen-in property for both magnetic potentials and magnetic fields. This leads to new motion equations and another noncanonical Poisson bracket for the theory, in terms of non-Abelian magnetic fields.

In conclusion, we discuss the two related Hamiltonian representations of Yang-Mills MHD in terms of either magnetic potentials, or magnetic fields. These two representations are shown to be compatible under the nonlinear map that defines the Yang-Mills magnetic field in terms of its vector potential. Since the Hamiltonian matrices are linear in the dependent variables of each representation, we associate their corresponding noncanonical Poisson brackets to appropriate Lie algebras.

DERIVATION VIA PLASMA-PHYSICS ARGUMENTS

The equations of classical multifluid chromodynamics form a dynamical system consisting of (1) conservation laws for mass, entropy, and gauge charge of each species, (2) dynamical Yang-Mills equations for the self-consistent fields, (3) the motion equation for each species, along with (4) the nondynamical Gauss's law, which is a nonholonomic constraint, preserved by the dynamics

$$\rho^s{}_{,t} = -\nabla\cdot\rho^s\vec{v}^{\,s} , \tag{1a}$$

$$\sigma^s{}_{,t} = -\nabla\cdot\sigma^s\vec{v}^{\,s} , \tag{1b}$$

$$D_t^* G^s = -D_i^* G^s v_i^s , \tag{1c}$$

$$\vec{A}_{,t} = -\vec{E} + \vec{D} A_0 , \tag{2a}$$

$$\epsilon^*(D_t\vec{E}) = {}^*(\vec{D}\times\vec{B}) - \sum_s G^s\vec{v}^{\,s} , \tag{2b}$$

$$\vec{v}^{\,s}{}_{,t} = -(\vec{v}^{\,s}\cdot\vec{\nabla})\vec{v}^{\,s} - \frac{1}{\rho^s}\vec{\nabla}p^s + \frac{1}{\rho^s}\langle G^s, (\vec{E}+\vec{v}^{\,s}\times\vec{B})\rangle\ , \tag{3}$$

$$\epsilon^*(\vec{D}\cdot\vec{E}) = \sum_s G^s\ , \tag{4}$$

where we use the following notation.

Eulerian coordinates in $\mathbb{R}^n$ are x_i, $1 \le i \le n$, while $x_0 = t$ is the time; derivatives are denoted by, e.g., $\partial_i = \partial/\partial x_i$, $\partial\alpha/\partial x_i = \alpha_{,i}$, $\partial\alpha/\partial t = \alpha_{,t}$; superscript s labels species type, $1 \le s \le S$; ρ^s is the mass density of the corresponding species; $\sigma^s = \rho^s\eta^s$ the entropy density; G_a^s, $1 \le a \le N$ the gauge-charge density; A_μ^a, $0 \le \mu \le n$, $1 \le a \le N$, the self-consistent Yang-Mills vector potentials; v_i^s, $1 \le i \le n$, the velocity $\vec{v}^{\,s}$ components; $p^s = p^s(\rho^s, \eta^s)$, partial pressure. The vector potentials A_μ^a have both a spacetime index μ and an internal gauge-symmetry index a, $1 \le a \le N$, where N is the dimension of the gauge algebra $\mathcal{G}$. Multiplication in the Lie algebra $\mathcal{G}$ is denoted for $\alpha,\beta \in \mathcal{G}$ by $[\alpha,\beta] = \mathrm{ad}_\alpha\beta = \mathrm{ad}(\alpha)\beta$. The algebra $\mathcal{G}$ has a dual $\mathcal{G}^*$, with pairing between $\alpha \in \mathcal{G}$, $\gamma^* \in \mathcal{G}^*$ denoted $\langle\gamma^*,\alpha\rangle$; the map $\mathcal{G} \to \mathcal{G}^*$, $\gamma \to {}^*\gamma$, is defined by the equation $\langle {}^*\gamma, \alpha\rangle = (\gamma,\alpha)$ for all $\alpha \in \mathcal{G}$, where $(\ ,\ )$ is an invariant, nondegenerate, symmetric bilinear form on $\mathcal{G}$ (e.g., Killing form for $\mathcal{G}$ semisimple). Thus, e.g., operators ad_α^*: $\mathcal{G}^* \to \mathcal{G}^*$ are defined by $\langle \mathrm{ad}_\alpha^*\gamma, \beta\rangle = \langle\gamma, \mathrm{ad}_\alpha\beta\rangle$. Covariant derivatives are denoted

$$\vec{D} = \vec{\nabla} - \mathrm{ad}_{\vec{A}},\quad D_t = \partial/\partial t - \mathrm{ad}_{A_0},\quad \vec{D}^* = \vec{\nabla} + \mathrm{ad}^*_{\vec{A}}\ ,$$

$$D_t^* = \partial/\partial t + \mathrm{ad}^*_{A_0},\quad [D_i, D_t] = \mathrm{ad}_{E_i},\quad [D_i, D_j] = \mathrm{ad}_{B_{ij}}\ ,$$

so that $E_i = A_{i,t} - A_{0,i} + [A_i, A_0]$,

$$B_{ij} = A_{i,j} - A_{j,i} + [A_i, A_j]\ , \tag{5}$$

$$(\vec{D}\times\vec{B})_j = D_k B_{kj},\quad (\vec{v}\times\vec{B})_j = v_k B_{kj}\ ,$$

and the scalar parameter ϵ is the "electrical permittivity" of the fluids. Components of A_i are given in a basis of the Lie algebra $\mathcal{G}$ with elements $\hat{e}_a$ satisfying the commutation relation $[\hat{e}_a, \hat{e}_b] = \epsilon_{ab}^c\hat{e}_c$ by $A_i = A_i^a\hat{e}_a$; e_c are elements of the dual basis, and

$$G = G_c e^c,\quad G_c \in \Lambda^n(\mathbb{R}^n)\ .$$

Just as in the Abelian MHD case,[4] one can derive the Yang-Mills MHD equations from (1)—(4) by first setting $\epsilon = 0$. For $\epsilon = 0$, one finds from (4) and (2b) that $\sum_s G^s = 0$ and

$$^*(\vec{D}\times\vec{B}) = \sum_s G^s\vec{v}^{\,s} \equiv \vec{J}\ ,$$

respectively. Then, by summing the charge equations (1c) over species index s it follows that $D_t^*\sum_s G^s = 0$, since $\vec{D}^*\cdot\vec{J} = 0$, by definition of $\vec{J}$. In the *two-species case,* with negligible second density ρ^2, summing the two motion equations (3), dropping terms in the sum multiplied by ρ^2, and defining

$$G = G^1 = -G^2,\quad \rho = \rho^1,\quad \vec{v} = \vec{v}^{\,1},\quad p = p^1 + p^2\ ,$$

leads to the motion equation in the MHD form

$$v_{i,t} = -v_j v_{i,j} - p_{,i}/\rho - \langle J_j, B_{ji}\rangle/\rho\ ,$$

where the terms in (3) multiplying $\vec{E}$ have summed to zero and $\langle J_j, B_{ji}\rangle$ is a non-Abelian "$\vec{J}\times\vec{B}$" force density. Next, neglecting drift effects and inertial effects in the motion equation (3) for the second species leads to $\vec{E} + \vec{v}\times\vec{B} = 0$, just as in the Abelian case. Finally, assuming that the second species is isentropic removes its corresponding entropy equation. Note that the non-Abelian nature of the fields does not interfere with this approximation scheme. The resulting Yang-Mills MHD equations are the following:

$$\rho_{,t} = -\nabla\cdot\rho\vec{v}\ , \tag{6}$$

$$\sigma_{,t} = -\nabla\cdot\sigma\vec{v}\ , \tag{7}$$

$$D_t^* G = -D_i^* G v_i\ , \tag{8}$$

$$A_{j,t} = v_k B_{kj} + D_j A_0\ , \tag{9}$$

$$v_{i,t} = -v_j v_{i,j} - p_{,i}/\rho + \langle J_j, B_{ij}\rangle/\rho\ , \tag{10}$$

where

$$J_j = \sum_s G^s v_j^s = {}^*(D_k B_{kj})\ .$$

An appropriate choice of gauge is the hydrodynamic gauge,[3] $A_0 + \vec{v}\cdot\vec{A} = 0$, for which the charge density and field equations simplify to

$$G_{,t} = -\nabla\cdot G\vec{v}\ , \tag{8$'$}$$

$$A_{i,t} = -(A_{i,k} + A_k\partial_i)v_k\ . \tag{9$'$}$$

Equations (8$'$) and (9$'$) are homogeneous in the noncommuting variables $G \in \mathcal{G}^*$, $\vec{A} \in \mathcal{G}$, just as they would be in the Abelian case (of course, in the Abelian MHD the charge density G is proportional to the mass density ρ, so there is no additional equation for G in that case). However, in the motion equation (10), the $\vec{J}\times\vec{B}$ force density is quite nonlinear, being fifth degree in $\vec{A}$.

HAMILTONIAN DERIVATION

The Yang-Mills MHD equations (6)—(10) also arise from reasoning which focuses upon the Hamiltonian structure of the theory. First, following the Abelian case, one postulates that the dynamical equation for $\vec{A}$ in Yang-Mills MHD must be (2a) from CHD, but with $\vec{E} = -\vec{v}\times\vec{B}$. Substituting this relation for $\vec{E}$ and using the hydrodynamic gauge $A_0 + v_j A_j = 0$ in (2a) gives

$$\begin{aligned} A_{i,t} &= v_j B_{ji} + D_i A_0 \\ &= -v_j(A_{i,j} - A_{j,i}) - \partial_i(v_j A_j) \\ &= -(A_{i,j} + A_j\partial_i)v_j\ . \end{aligned} \tag{11}$$

As occurs in the Eulerian descriptions of most fluid theories,[6,7] we assume that the Hamiltonian form of the theory eventually will reflect its Lie algebra (or group) of symmetries and, thus, the corresponding Hamiltonian matrix should be linear in the field variables. Since this matrix is known in the Abelian case,[7] we need only determine what difference noncommutativity makes. But Eq. (2a) is the *same* as in the Abelian case; therefore, the only non-Abelian change in the Hamiltonian matrix could occur due to the interaction of the charge G with other variables. Taking this interaction to be the same as in CHD, where the charges have nonzero Poisson brackets only with themselves and with the momentum density $\vec{M}$, fixes the Hamiltonian matrix. To obtain the motion equations, we need only to choose the Hamiltonian, which we take to be the CHD Hamiltonian with all terms involving $\vec{E}$ dropped, exactly as in the Abelian case.

The resulting Hamiltonian form of Eqs. (6)—(10) is

$$\partial_t \begin{bmatrix} \rho \\ \sigma \\ G \\ M_i \\ A_i \end{bmatrix} = - \begin{bmatrix} 0 & 0 & 0 & \partial_j \rho & 0 \\ 0 & 0 & 0 & \partial_j \sigma & 0 \\ 0 & 0 & -\mathrm{ad}^*(\ )G & \partial_j G & 0 \\ \rho\partial_i & \sigma\partial_i & G\partial_i & (\partial_j M_i + M_j \partial_i) & (-A_{j,i} + \partial_j A_i) \\ 0 & 0 & 0 & (A_{i,j} + A_j \partial_i) & 0 \end{bmatrix} \begin{bmatrix} \delta H/\delta\rho \\ \delta H/\delta\sigma \\ \delta H/\delta G \\ \delta H/\delta M_j \\ \delta H/\delta A_j \end{bmatrix} \tag{12}$$

with $\vec{M} \equiv \rho\vec{v}$ and Hamiltonian H consisting of the sum of kinetic, internal, and magnetic energies,

$$H = \int d^n x [\, |\vec{M}|^2/2\rho + U(\rho,\sigma) + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij} \rangle] \,, \tag{13}$$

where $U(\rho,\sigma)$ is the internal energy density of the fluid. The term in the middle of the Hamiltonian matrix in (12) is to be read as

$$-\mathrm{ad}^* \left[\frac{\delta H}{\delta G} \right] G = -G_a \epsilon^a_{bc} \frac{\delta H}{\delta G_b} e^c \,. \tag{14}$$

Expanding (12) we arrive again at Eqs. (6)—(10).

The Hamiltonian matrix in (12) produces the following Poisson bracket for Yang-Mills MHD:

$$\{H,F\} = -\int d^n x \left[\frac{\delta F}{\delta M_i} \left[(\partial_j M_i + M_j \partial_i) \frac{\delta H}{\delta M_j} + \rho \partial_i \frac{\delta H}{\delta \rho} + \sigma \partial_i \frac{\delta H}{\delta \sigma} + G \partial_i \frac{\delta H}{\delta G} + (-A_{j,i} + \partial_j A_i) \frac{\delta H}{\delta A_j} \right] \right.$$
$$\left. + \left[\frac{\delta F}{\delta \rho} \partial_j \rho + \frac{\delta F}{\delta \sigma} \partial_j \sigma + \frac{\delta F}{\delta G} \partial_j G + \frac{\delta F}{\delta A_i} (A_{i,j} + A_j \partial_i) \right] \frac{\delta H}{\delta M_j} - \frac{\delta F}{\delta G_a} \frac{\delta H}{\delta G_b} \epsilon^c_{ab} G_c \right] . \tag{15}$$

Let Λ^K denote the set of K-forms on $\mathbb{R}^n$, and $\Lambda^K \otimes \mathcal{G}$ the set of K-forms on $\mathbb{R}^n$ with values in $\mathcal{G}$. The Lie algebra $D = D(\mathbb{R}^n)$ of vector fields on $\mathbb{R}^n$ acts naturally on Λ^K and $\Lambda^K \otimes \mathcal{G}$. Let $\ominus$ denote the semidirect product and $\oplus$ denote the direct sum. The Poisson bracket (15) is the natural one associated to the dual space of the Lie algebra,

$$D \ominus [\Lambda^0 \oplus \Lambda^0 \oplus (\Lambda^0 \otimes \mathcal{G}) \oplus (\Lambda^{n-1} \otimes \mathcal{G}^*)] \,, \tag{16}$$

provided one identifies the dual coordinates as follows: M_i is dual to $\partial_i \in D(\mathbb{R}^n)$; ρ and σ are dual to the first and second summand, respectively; G_a is dual to $\Lambda^0 \otimes \hat{e}_a \in (\Lambda^0 \otimes \mathcal{G})$; A_i^a is dual to $[\partial_i \lrcorner (dx_1 \wedge \cdots \wedge dx_n)] \otimes e^a \in (\Lambda^{n-1} \otimes \mathcal{G}^*)$. For the reader who might have been expecting to see nonzero brackets between $\vec{A}$'s, we note that the $\vec{A}$-$\vec{A}$ brackets must vanish, since $(\Lambda^1 \otimes \mathcal{G}) \approx (\Lambda^{n-1} \otimes \mathcal{G}^*)^*$ where $\vec{A}$'s belong, is not a Lie algebra. For further explanations and examples of how a Poisson bracket that is linear in its variables corresponds to the dual of a Lie algebra, see Refs. 6 and 7. Since the Poisson bracket (15) has now been associated with a Lie algebra, clearly it satisfies the Jacobi identity.

FROZEN-IN VARIABLES

The hallmark of classical MHD is that the $\vec{A}$ and $\vec{B}$ fields are frozen into the fluid. For both classical and Yang-Mills MHD, the frozen-in property for $\vec{A}$ follows from (9′), in the Eulerian description. (For a discussion of the classical case, see, e.g., Parker.[8]) In the Lagrangian description, the corresponding statement for the freezing-in of $\vec{A}$ is that the one-form $A = A_i dx_i$ is preserved in time for line elements dx_i that are comoving with the fluid. Thus, $A_i(\vec{l},t)dx_i = A_i(\vec{l},0)dl_i$, where $\vec{x}(\vec{l},t)$ is the trajectory of the fluid particle element that is initially located at $\vec{x}(\vec{l},0) = \vec{l}$. Denoting $F_{ij} = \partial x_i/\partial l_j$, $\vec{A}^0 = \vec{A}(\vec{l},0)$, (9′) implies that

$$A_i = A_j^0 F_{ji}^{-1} \,, \tag{17}$$

which, by (9′), holds for *both* classical and Yang-Mills MHD.

Now, in classical MHD it also follows from (9′) that magnetic flux is frozen-in since the comoving two-form $B = dA = -B_{ij} dx_i \wedge dx_j = (\vec{B} \cdot d\vec{S})$, with components $B_{ij} = A_{i,j} - A_{j,i}$ satisfies

$$B_{ij} = B_{kl}^0 F_{ki}^{-1} F_{lj}^{-1} \tag{18}$$

which is the classical condition for the magnetic flux through every comoving surface element to be preserved in time by the fluid dynamics.

For the Yang-Mills case, though, the definition of B_{ij} (5) involves the commutator $[A_i, A_j]$. Nonetheless, as one can easily show, since (17) holds for the Yang-Mills case, the frozen-in condition (18) follows for Yang-Mills MHD, as well. Consequently, the Eulerian equation for classical MHD which describes the freezing-in of B_{ij}, namely,

$$B_{ij,t} = -(B_{ij,k} + B_{kj}\partial_i + B_{ik}\partial_j)v_k \ , \tag{19}$$

also applies to Yang-Mills MHD. We shall show next that the result (19) is gauge invariant.

HAMILTONIAN MATRICES

The Eulerian equation (19) for the freezing-in of B_{ij} can also be gotten by simply relating the Hamiltonian matrix from (12) in the A_i representation to a Hamiltonian matrix in the B_{ij} representation, using (5). Two Hamiltonian matrices $b_1 = b_1(\vec{u})$ and $b_2 = b_2(\vec{w})$, say, are compatible under a map φ: $\vec{u}(\vec{x}) \rightarrow \vec{w}(\vec{x})$ from one set of dependent variables to another, according to a standard formula

$$(b_2)^* = \frac{D\vec{w}}{D\vec{u}} b_1 \left[\frac{D\vec{w}}{D\vec{u}} \right]^\dagger . \tag{20}$$

In (20), $D\vec{w}/D\vec{u}$ is the Fréchet derivative operator and the dagger denotes the adjoint with respect to measure $d^n x$. For the map φ':

$$\{M_i, \rho, \sigma, G, A_i\} \rightarrow \{M_i, \rho, \sigma, G, B_{ij}\} \ ,$$

with B_{ij} given in (5) and b_1 given in (12), straightforward calculation of the Fréchet derivative operator $D\vec{w}/D\vec{u}$, followed by matrix multiplication as in (20), leads to the following Hamiltonian matrix, b_2, in the B_{ij} representation:

$$b_2 = \begin{bmatrix} 0 & 0 & 0 & \partial_k \rho & 0 \\ 0 & 0 & 0 & \partial_k \sigma & 0 \\ 0 & 0 & -\mathrm{ad}^*(\ )G & \partial_k G & 0 \\ \rho\partial_i & \sigma\partial_i & G\partial_i & (\partial_k M_i + M_k \partial_i) & (-B_{jl,i} + \partial_j B_{il} + \partial_l B_{ji}) \\ 0 & 0 & 0 & (B_{mn,k} + B_{kn}\partial_m + B_{mk}\partial_n) & 0 \end{bmatrix} \tag{21}$$

with the corresponding Poisson brackets given by

$$\{H,F\} = -\int d^n x \left[\frac{\delta F}{\delta M_i} \left[(\partial_k M_i + M_k \partial_i) \frac{\delta H}{\delta M_k} + \rho \partial_i \frac{\delta H}{\delta \rho} + \sigma \partial_i \frac{\delta H}{\delta \sigma} + G \partial_i \frac{\delta H}{\delta G} + (-B_{jl,i} + \partial_j B_{il} + \partial_l B_{ji}) \frac{\delta H}{\delta B_{jl}} \right] \right.$$
$$\left. + \left[\frac{\delta F}{\delta \rho} \partial_k \rho + \frac{\delta F}{\delta \sigma} \partial_k \sigma + \frac{\delta F}{\delta G} \partial_k G + \frac{\delta F}{\delta B_{ml}} (B_{ml,k} + B_{kl}\partial_m + B_{mk}\partial_l) \right] \frac{\delta H}{\delta M_k} - \frac{\delta F}{\delta G_a} \frac{\delta H}{\delta G_b} \epsilon^c_{ab} G_c \right] . \tag{22}$$

Remarkably, even though the map φ' is nonlinear in A_i, it results in a Hamiltonian matrix which is linear in B_{ij}. The resulting Poisson brackets (22) correspond to the dual of the semidirect product Lie algebra

$$D \ominus [\Lambda^0 \oplus \Lambda^0 \oplus (\Lambda^0 \otimes \mathscr{G}) \oplus (\Lambda^{n-2} \otimes \mathscr{G}^*)] \ , \tag{23}$$

where dual coordinates are: M_i dual to $\partial_i \in D$; ρ, σ dual to Λ^0; G_a dual to $1 \otimes \hat{e}_a \in (\Lambda^0 \otimes \mathscr{G})$; B^a_{ij} dual to $[\partial_i \lrcorner \partial_j \lrcorner (dx_1, \wedge \cdots \wedge dx_n)] \otimes e^a \in (\Lambda^{n-2} \otimes \mathscr{G}^*)$. Thus, the Yang-Mills MHD equations are expressible in Hamiltonian form $F_{,t} = \{H,F\}$ with Hamiltonian H given in (13) in terms of B_{ij}, Hamiltonian matrix (21), and corresponding Poisson brackets (22). In addition, the resulting dynamical equation (19) for B_{ij} is the condition that B_{ij} be frozen into the fluid.

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