The Central Idea

THE KINETIC mixing matrix can be specified in an explicitly diagonalized form

$$\mathcal{M} \quad = \quad \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a^2 & \\ & b^2 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} . \tag{1}$$

The quantity a^2 will be the effective coupling constant of the photon, and therefore we choose

$$a^2 \quad \simeq \quad 1 \, .$$

Meanwhile, the quantity b^2 will be the coupling constant of the hidden photon. We would like to have the hidden photon completely decoupled by the end of inflation, and also never to be at strong coupling. Therefore, we choose b^2 to be growing,

$$b^2 \quad \propto \quad a^{2n}(\tau) \, ,$$

where $a(\tau)$ is the scale factor[1]. One cannot consistently impose the alternative condition, $b^2 \quad \propto \quad a^{-2n}(\tau)$. In this case $b(\tau)$ would ultimately become very small, and the quantum corretions would take over and alter its value.

The kinetic matrix (1) can be evaluated to the following form,

$$\mathcal{M} \quad = \quad \begin{pmatrix} 1 & \epsilon \\ \epsilon & f^2 \end{pmatrix} , \tag{2}$$

where

$$1 \quad \ll \quad f^2 \qquad\qquad \text{and} \qquad\qquad \epsilon \quad \ll \quad f \, .$$

We define our Lagrangian such that this matrix mixes the observable photon with the hidden photon. The small value of the coupling of the hidden photon to matter is ensured by the growing $f^2(\tau) \;\simeq\; b^2(\tau)$. On the other hand, the coupling constant of the photon to matter must remain of order one at all times. Exactly how this is realized we will show in the next subsection.

[1]To distinguish the scale factor $a(\tau)$ from the inverse coupling constant a^2 we will always write it as $a(\tau)$ — a function of τ. Note that on the contrary, a^2 is practically constant in our scenario.

* * *

We have already defined that

$$a^2 \;=\; 1\,, \qquad\qquad \text{and} \qquad\qquad b^2 \;\simeq\; a^{2n}(\tau).$$

Let us accept that these are in fact exact equalities. For the coupling constant of the real photon to remain of order one at all times, θ must decrease in time,

$$\theta \;\simeq\; \sin\theta \;\propto\; a^{-n-\lambda}(\tau)\,.$$

Here λ is a small "excess" power, say $\lambda \sim 1/2$. These requirements ensure that the mixing matrix takes the form as that of Eq. (2).

The time dependences of $\epsilon(\tau)$ and $f^2(\tau)$ are,

$$f^2 \;=\; a^{2n}(\tau)$$

and

$$\epsilon \;=\; -\,a^{n-\lambda}(\tau)\,.$$

The condition $\epsilon \ll f(\tau)$ is also met.

At this point we can write down the Lagrangian which we have in mind,

$$\mathcal{L} \;=\; \begin{pmatrix} F_{\mu\nu} & G_{\mu\nu} \end{pmatrix} \mathcal{M} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix} \;+\; F^{\mu} J_{\mu}\,,$$

where J^{μ} is the matter current, and we have suppressed factors like $1/4\pi$. It is our free will to define how the gauge fields couple to matter. Explicitly, F^{μ} becomes the real photon and G^{μ} the hidden one. The behaviour of F^{μ} is slightly altered as compared to that of the usual U(1) photon by the presence of G^{μ}.

* * *

Here we sketch the underlying ideas of the scenario. The kinetic term written explicitly is

$$\begin{pmatrix} F_{\mu\nu} & G_{\mu\nu} \end{pmatrix} \begin{pmatrix} 1 & \epsilon \\ \epsilon & f^2 \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}, \tag{3}$$

where again F_{μ} and G_{μ} are the real and the hidden photons. Although we will not need to diagonalize the kinetic matrix in the actual calculations, we assume we do so just to provide a supporting argument.

The form of the kinetic matrix (1) already gives the diagonalized form,

$$a^2\, A_{\mu\nu}^2 \quad + \quad b^2\, B_{\mu\nu}^2 \quad + \quad \text{one-derivative} \;\; \text{and} \;\; \text{zero-derivative terms.} \tag{4}$$

Here $A_{\mu\nu}$ and $B_{\mu\nu}$ are the rotated fieldstrengths. Now the rotation $\sin\theta$ is applied to the gauge fields A_μ and B_μ rather than to the fieldstrengths. Since this rotation is time-dependent, it produces terms where less than two derivatives are applied to A_μ and B_μ. Ignore these terms now.

What we have come to, is two decoupled instances of electrodynamics, one with a finite gauge coupling, the other one with a gauge coupling becoming very small. It is known that when the coupling constant decreases with time (*i.e.* the inverse coupling $b^2(\tau)$ increases), the electrodynamics *can* produce the magnetic field for itself. This way, B_μ generates the magnetic field $\vec{B}_{(B)}$, and it does not suffer from the strong coupling problem — there is no constraint that $b^2(\tau)$ nowadays be of order one.

But then, as we said, A_μ and B_μ are not the observable fields — rather, F_μ and G_μ are. Since the observable fields are linear combinations of A_μ and B_μ, by generating $\vec{B}_{(B)}$ we also generate $\vec{B}_{(F)}$, that is, the observable magnetic field.

Although these arguments are heuristic, we do expect them to give the rough idea of the dynamics of the fields and expect that the extra terms in Eq. (4) can be neglected in some sense. Indeed, if the two instances of electrodynamics were completely decoupled, the two fields would develop by themselves. The mixing between them is so weak that hardly it can affect their dynamics on a significant level.

The actual calculations will not rely on this discussion.

* * *

Let us call the undiagonalizeds fields Φ_μ,

$$\Big(F_\mu\,, \quad G_\mu \Big) \quad \equiv \quad \Phi_\mu\,. \tag{5}$$

As we know, at the time when a mode has exited the horizon, the equation of motion for it can be written as

$$\big(\,\mathcal{M}\cdot\Phi'\,\big)' \quad = \quad 0\,, \qquad\qquad \text{or} \qquad\qquad \mathcal{M}\cdot\Phi' \quad = \quad \text{const.} \tag{6}$$

Here we assume that Φ is written in terms of polarizations $\Phi_\pm$.

Fixing some time τ_*, we have

$$\mathcal{M}\cdot\Phi' \quad = \quad \mathcal{M}_*\cdot\Phi'_* \qquad\qquad \text{for that } \tau_*. \tag{7}$$

This leads to the differential equation

$$\Phi' \;=\; \mathcal{M}^{-1}\cdot\mathcal{M}_*\,\Phi'_*\,. \tag{8}$$

The right hand side of this equation is in principle a known function of time, as long as we find the inverse of matrix $\mathcal{M}$. The rest of the right hand side is just a set of constants. We expect the inverse of $\mathcal{M}$ to be expressible as a combination of powers of $a(\tau)$ — that is, as a combination of powers of τ itself.

* * *

Quite generally,

$$\mathcal{M}^{-1} \;=\; \frac{1}{f^2 \,-\, \epsilon^2}\begin{pmatrix} f^2 & -\epsilon \\ -\epsilon & 1 \end{pmatrix}. \tag{9}$$

The denominator can just be expanded,

$$\frac{1}{f^2 \,-\, \epsilon^2} \;=\; \frac{1}{f^2}\left(1 \;+\; \frac{\epsilon^2}{f^2} \;+\; \frac{\epsilon^4}{f^4} \;+\; \ldots\right). \tag{10}$$

This gives the inverse matrix (9) the form of a series in powers of the conformal time.

Using this, we re-write the system of equations (8) in terms of the conformal time. The leading terms in the system generally depend on the relative magnitudes of n and λ. Assuming, for example $\lambda \ll n$, the system, written in terms of polarizations, is

$$\begin{aligned} F'_\lambda \;&=\; \text{const} \;+\; \text{const}\cdot\tau^{2\lambda} \;+\; \ldots \\ G'_\lambda \;&=\; \text{const}\cdot\tau^{n+\lambda} \;+\; \text{const}\cdot\tau^{n+3\lambda} \;+\; \ldots\,. \end{aligned} \tag{11}$$

Integration of these equations gives powers of τ one greater,

$$\begin{aligned} F_\lambda \;&=\; \text{const} \;+\; \text{const}\cdot\tau \;+\; \text{const}\cdot\tau^{2\lambda+1} \\ G_\lambda \;&=\; \text{const} \;+\; \text{const}\cdot\tau^{n+\lambda+1} \;+\; \text{const}\cdot\tau^{n+3\lambda+1}\,. \end{aligned} \tag{12}$$

The initial conditions for these equations are given by the values of the fields at some time τ_* in the past (it should not matter whether it is the same τ_* as before or not)

$$\begin{aligned} F^*_\lambda \;&=\; \text{const} \;+\; \text{const}\cdot\tau_* \;+\; \text{const}\cdot\tau_*^{2\lambda+1} \\ G^*_\lambda \;&=\; \text{const} \;+\; \text{const}\cdot\tau_*^{n+\lambda+1} \;+\; \text{const}\cdot\tau_*^{n+3\lambda+1}\,. \end{aligned} \tag{13}$$

This determines the constants of integration in Eq. (12),

$$
\begin{aligned}
F_\lambda &= F_\lambda^* + \mathrm{const}\cdot(\tau - \tau_*) + \mathrm{const}\cdot(\tau^{n+\lambda+1} - \tau_*^{n+\lambda+1}) \\
G_\lambda &= G_\lambda^* + \mathrm{const}\cdot(\tau^{n+\lambda+1} - \tau_*^{n+\lambda+1}) + \ldots\,.
\end{aligned}
\tag{14}
$$

We presume that the magnetic field was generated *during* the inflation, most certainly, in the end. At time τ_* therefore, the fields were negligible,

$$F_\lambda^* \sim G_\lambda^* \sim 0\,.$$

This allows us to re-write Eq. (14) as

$$
\begin{aligned}
F_\lambda(\tau) &= \mathrm{const}\cdot(\tau - \tau_*) + \mathrm{const}\cdot(\tau^{n+\lambda+1} - \tau_*^{n+\lambda+1}) \\
G_\lambda(\tau) &= \mathrm{const}\cdot(\tau^{n+\lambda+1} - \tau_*^{n+\lambda+1}) + \ldots\,.
\end{aligned}
\tag{15}
$$

Since $\tau \propto e^{Ht}$, we have an exponential approach to the values of the fields at the present time, which are

$$
\begin{aligned}
F_\lambda^{(0)} &\simeq \mathrm{const}\cdot\tau_* + \mathrm{const}\cdot\tau_*^{n+\lambda+1} + \ldots \\
G_\lambda^{(0)} &\simeq \mathrm{const}\cdot\tau_*^{n+\lambda+1} + \ldots\,.
\end{aligned}
\tag{16}
$$

Even though the inflation formally starts at $\tau \to -\infty$, time τ_* can be chosen to be finite, such that the conditions (16) can be satisfied.

We integrate the solution for F_λ over all the modes to obtain the magnetic field

$$\rho_B = \frac{4\pi}{a^4(t_0)} \int\limits_{Ha_i}^{Ha_0} dk\, k^4 \left(F_\lambda^{(0)}\right)^2 \simeq 4\pi\, H^5\, a(t_0) \left(F_\lambda^{(0)}\right)^2 .$$

We believe that the set of constants at our disposal — n, λ, τ_* and the proportionality constants are enough to provide a reasonable fit to the observed constraints on the magnetic field. The criterion of naturalness is that no constant is required to be on the order of 10^{60} or similar.