1 Model

We consider the action

$$S=\int d^4x\sqrt{-g}\left[-\frac{1}{4}V^T_{\mu\nu}\,X\,V^{\mu\nu}-eJ^\mu A_\mu\right]\quad,\quad V_{\mu\nu}=\left(\begin{array}{c}A_{\mu\nu}\\B_{\mu\nu}\end{array}\right)\tag{1}$$

where the kinetic matrix is

$$X=\left(\begin{array}{cc}1&\chi\,\epsilon\,(t)\\\chi\,\epsilon\,(t)&\epsilon^2\,(t)\end{array}\right)\tag{2}$$

with χ constant, and $\epsilon \ll 1$ during inflation, and $\epsilon = 1$ at the end of inflation.

2 Late time magnetic field

After inflation, consider the redefinition

$$\left(\begin{array}{c}A\\B\end{array}\right)=\left(\begin{array}{cc}\frac{1}{\sqrt{1-\chi^2}}&0\\\frac{-\chi}{\sqrt{1-\chi^2}}&1\end{array}\right)\left(\begin{array}{c}\tilde{A}\\\tilde{B}\end{array}\right)\tag{3}$$

In terms of the new fields, the action is

$$\int d^4x\sqrt{-g}\left[-\frac{1}{4}\tilde{A}_{\mu\nu}\tilde{A}^{\mu\nu}-\frac{1}{4}\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu}-\frac{e}{\sqrt{1-\chi^2}}\tilde{A}^\mu\,J_\mu\right]\tag{4}$$

Therefore, our photon is

$$\tilde{A}_\mu=\sqrt{1-\chi^2}\,A_\mu\tag{5}$$

We learn that, to produce a magnetic field, we need to produce A_μ.

3 Helicity decomposition

We use conformal time $ds^2 = a\,(\tau)\,[-d\tau^2 + d\vec{x}^2]$. In the gauge $A_0 = B_0 = 0$, we decompose

$$\begin{aligned}\vec{A}&=\sum_\lambda\int\frac{d^3k}{(2\pi)^{3/2}}\vec{\epsilon}_\lambda\left(\vec{k}\right)\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{A}_\lambda\left(\vec{k}\right)\\\vec{B}&=\sum_\lambda\int\frac{d^3k}{(2\pi)^{3/2}}\vec{\epsilon}_\lambda\left(\vec{k}\right)\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{B}_\lambda\left(\vec{k}\right)\end{aligned}\tag{6}$$

Here $\vec{\epsilon}_\lambda$ are circular polarization vectors satisfying $\vec{k}\cdot\vec{\epsilon}_\pm\left(\vec{k}\right) = 0$, $\vec{k}\times\vec{\epsilon}_\pm\left(\vec{k}\right) = \mp ik\vec{\epsilon}_\pm\left(\vec{k}\right)$, $\vec{\epsilon}_\pm\left(-\vec{k}\right) = \vec{\epsilon}_\pm\left(\vec{k}\right)^*$, and normalized according to $\vec{\epsilon}_\lambda\left(\vec{k}\right)^*\cdot\vec{\epsilon}_{\lambda'}\left(\vec{k}\right) = \delta_{\lambda\lambda'}$.

We obtain

$$S = \frac{1}{2}\sum_{\lambda=\pm}\int d\tau\, d^3k\left[V_\lambda^{\dagger\prime}\left(\vec{k}\right)\cdot X\cdot V_\lambda'\left(\vec{k}\right) - k^2 V_\lambda^\dagger\left(\vec{k}\right)\cdot X\cdot V_\lambda\left(\vec{k}\right)\right] \quad , \quad V_\lambda \equiv \begin{pmatrix}\hat{A}_\lambda\\ \hat{B}_\lambda\end{pmatrix} \tag{7}$$

The actions of the two helicities are identical to each other, and decoupled. Therefore, we focus on one of the two actions and we suppressed the suffix λ.

4 Large wavelength approximation

In matrix form, we have the equation

$$\left(X\,V'\right)' + k^2\,V = 0 \tag{8}$$

For any mode, we can consider a time τ_* after which the mode is well outside the horizon, and k can be set to zero. In this case

$$V'\left(\tau\right) = X^{-1}\left(\tau\right)\,X_*\,V_*' \tag{9}$$

or, more explicitly,

$$\begin{aligned} A' &= -\frac{\chi}{1-\chi^2}\left(\chi\,A_*' + \epsilon_* B_*'\right)\frac{\epsilon_*}{\epsilon} + \frac{A_*' + \chi\epsilon_* B_*'}{1-\chi^2} \\ \epsilon_* B' &= -\frac{\chi}{1-\chi^2}\left(\chi\epsilon_* B_*' + A_*'\right)\frac{\epsilon_*}{\epsilon} + \frac{\epsilon_* B_*' + \chi A_*'}{1-\chi^2}\,\frac{\epsilon_*^2}{\epsilon^2} \end{aligned} \tag{10}$$

We now choose

$$\epsilon = \left(\frac{a}{a_0}\right)^n = \left(\frac{\tau_0}{\tau}\right)^n \tag{11}$$

where 0 denotes the end of inflation. The solution is

$$
\begin{aligned}
A(\tau) &= A_* + \tau_* \left\{ -\frac{A'_* + \chi\epsilon_* B'_*}{1-\chi^2}\left(1 - \frac{\tau}{\tau_*}\right) + \frac{\chi\left(\chi A'_* + \epsilon_* B'_*\right)}{(1+n)\left(1-\chi^2\right)}\left[1 - \left(\frac{\tau}{\tau_*}\right)^{n+1}\right]\right\} \\
B(\tau) &= B_* + \tau_* \left(\frac{\tau_*}{\tau_0}\right)^n \left\{ -\frac{\chi\left(A'_* + \chi\epsilon_* B'_*\right)}{1-\chi^2}\left[1 - \left(\frac{\tau}{\tau_*}\right)^{n+1}\right] \right. \\
&\qquad \left. - \frac{\chi A'_* + \epsilon_* B'_*}{(1+2n)\left(1-\chi^2\right)}\left[1 - \left(\frac{\tau}{\tau_*}\right)^{2n+1}\right]\right\}
\end{aligned} \tag{12}
$$

Let us evaluate these expressions at τ_0 (end of inflation). We are interesting in growing modes for $\tau_0 \to 0$. Disregarding terms that are wither constant or $\propto \tau_0$, we have

$$
A \propto \tau_0^{n+1} \quad , \quad B \propto \tau_0^{n+1} \, , \, \tau_0^{-n} \tag{13}
$$

Therefore we have the two possibilities

$$
\begin{aligned}
&B \simeq -\frac{\epsilon_* \tau_* B'_* \left[1 + n - (1+2n)\,\chi^2\right] - n\chi\tau_* A'_*}{(1+n)\,(1+2n)\left(1-\chi^2\right)}\left(\frac{a_0}{a_*}\right)^n \quad , \quad A \simeq 0 \quad , \quad n > 0 \\
&B \simeq \frac{\epsilon_* \tau_* B'_* + \chi\tau_* A'_*}{(1+2n)\left(1-\chi^2\right)}\left(\frac{a_0}{a_*}\right)^{-n-1} \quad , \quad A \simeq -\frac{1+2n}{1+n}\,\chi\, B \quad , \quad n < -1
\end{aligned} \tag{14}
$$

For $n > 0$, only B grows in the super-horizon regime, while for $n < -1$ both grow. Unfortunately this option gives $\rho_E \gg \rho_B$, and too much energy in the electric field if one wants $\rho_B \sim$ const during inflation (or only, logarithmically evolving).