We consider the action

$$S = -\frac{1}{4} \int d^4x \sqrt{-g} V_{\mu\nu}^T \, X \, V^{\mu\nu} \quad , \quad V_{\mu\nu} = \begin{pmatrix} A_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \tag{1}$$

and we use conformal time $ds^2 = a(\tau)\left[-d\tau^2 + d\vec{x}^2\right]$. In the gauge $A_0 = B_0 = 0$, we decompose

$$\begin{aligned} \vec{A} &= \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \vec{\epsilon}_\lambda \left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}} \, \hat{A}_\lambda \left(\vec{k}\right) \\ \vec{B} &= \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \vec{\epsilon}_\lambda \left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}} \, \hat{B}_\lambda \left(\vec{k}\right) \end{aligned} \tag{2}$$

Here $\vec{\epsilon}_\lambda$ are circular polarization vectors satisfying $\vec{k} \cdot \vec{\epsilon}_\pm \left(\vec{k}\right) = 0$, $\vec{k} \times \vec{\epsilon}_\pm \left(\vec{k}\right) = \mp i k \vec{\epsilon}_\pm \left(\vec{k}\right)$, $\vec{\epsilon}_\pm \left(-\vec{k}\right) = \vec{\epsilon}_\pm \left(\vec{k}\right)^*$, and normalized according to $\vec{\epsilon}_\lambda \left(\vec{k}\right)^* \cdot \vec{\epsilon}_{\lambda'} \left(\vec{k}\right) = \delta_{\lambda\lambda'}$.

We obtain

$$S = \frac{1}{2} \sum_{\lambda=\pm} \int d\tau \, d^3k \left[V_\lambda^{\dagger'} \left(\vec{k}\right) \cdot X \cdot V_\lambda' \left(\vec{k}\right) - k^2 V_\lambda^\dagger \left(\vec{k}\right) \cdot X \cdot V_\lambda \left(\vec{k}\right) \right] \quad , \quad V_\lambda \equiv \begin{pmatrix} \hat{A}_\lambda \\ \hat{B}_\lambda \end{pmatrix} \tag{3}$$

The actions of the two helicities are identical to each other, and decoupled. Therefore, we focus on one of the two actions and we suppressed the suffix λ. We assume that

$$X(\tau) = O(\tau) \, \Lambda \, \Lambda \, O^T(\tau) \tag{4}$$

where Λ is diagonal and constant, while O is orthogonal. We then define

$$V \equiv O \, \Lambda^{-1} Y \tag{5}$$

and obtain

$$\begin{aligned} S = \frac{1}{2} \int d\tau \, d^3k \quad & \Big[Y^{\dagger'} \, Y' + Y^{\dagger'} \Lambda \, O^T \, O' \Lambda^{-1} Y + Y^\dagger \, \Lambda^{-1} O'^T O \Lambda Y' \\ & + Y^\dagger \left(\Lambda^{-1} O'^T O \Lambda \Lambda O^T O' \Lambda^{-1} - k^2 \right) Y \Big] \end{aligned} \tag{6}$$

To proceed, it is convenient to write the matrices explicitly:

$$O = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \quad , \quad \Lambda = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} \tag{7}$$

Then

$$S = \frac{1}{2}\int d\tau\, d^3k \qquad \Big[Y^{\dagger'}\, Y' + Y^{\dagger'} \begin{pmatrix} 0 & \frac{-b\theta'}{c} \\ \frac{c\theta'}{b} & 0 \end{pmatrix} Y + Y^\dagger \begin{pmatrix} 0 & \frac{c\theta'}{b} \\ -\frac{b\theta'}{c} & 0 \end{pmatrix} Y'$$

$$+ Y^\dagger \begin{pmatrix} -k^2 + \frac{c^2\theta'^2}{b^2} & 0 \\ 0 & -k^2 + \frac{b^2\theta'^2}{c^2} \end{pmatrix} Y \tag{8}$$

Subtracting a total derivative,

$$\begin{aligned} S &= \frac{1}{2}\int d\tau\, d^3k\, \left[Y^{\dagger'}\, Y' + Y^{\dagger'}\, K\, Y - Y^\dagger\, K\, Y' - Y^\dagger\, \Omega^2\, Y \right] \\ K &= \begin{pmatrix} 0 & -\frac{b^2+c^2}{2bc}\,\theta' \\ \frac{b^2+c^2}{2bc}\,\theta' & 0 \end{pmatrix} \\ \Omega^2 &= \begin{pmatrix} k^2 - \frac{c^2}{b^2}\theta'^2 & \frac{-b^2+c^2}{2bc}\theta'' \\ \frac{-b^2+c^2}{2bc}\theta'' & k^2 - \frac{b^2}{c^2}\theta'^2 \end{pmatrix} \end{aligned} \tag{9}$$

We introduce an orthogonal matrix R that satisfies

$$R' = R\, K \tag{10}$$

We further define $\psi = RY$. We obtain

$$\begin{aligned} S &= \frac{1}{2}\int d\tau d^3k \left[\psi'^\dagger\, \psi' - \psi^\dagger\, \tilde{\Omega}^2\, \psi \right] \\ \tilde{\Omega}^2 &= R\, \left(\Omega^2 + K^T\, K \right) R^T \\ \Omega^2 + K^T\, K &= \begin{pmatrix} k^2 + \frac{1}{4}\left(2 + \frac{b^2}{c^2} - \frac{3c^2}{b^2}\right)\theta'^2 & \frac{-b^2+c^2}{2bc}\,\theta'' \\ \frac{-b^2+c^2}{2bc}\,\theta'' & k^2 + \frac{1}{4}\left(2 + \frac{c^2}{b^2} - \frac{3b^2}{c^2}\right)\theta'^2 \end{pmatrix} \\ R &= \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \quad , \quad \zeta = \frac{b^2+c^2}{2bc}\,(\theta - \theta_*) \end{aligned} \tag{11}$$

where θ_* is a constant.

We assume that θ becomes constant after inflation. Therefore R becomes constant. We could choose the integration constant such that $R = 1$ at the end, although this is not required and we will not do so.

To simplify the computation, we assume

$$\theta = H\, t \;\;\Rightarrow\;\; \theta' = aH \;\;,\;\; \theta'' = a^2 H^2 \tag{12}$$

(where we used $\frac{d}{d\tau} = a\frac{d}{dt}$; here H is the Hubble rate $\dot{a}/a$, which we can take as constant during inflation; here a is the scale factor). We further write

$$c = r\, b \tag{13}$$

and without loss of generality we can choose $0 < r < 1$.

We find

$$\begin{aligned} \Omega^2 + K^T K &= k^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{a^2(t)\, H^2}{4} \frac{1-r^2}{r} \begin{pmatrix} \frac{1+3r^2}{r} & -2 \\ -2 & -\frac{3+r^2}{r} \end{pmatrix} \\ R &= \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta & \cos\zeta \end{pmatrix} \quad , \quad \zeta = \frac{1+r^2}{2r}\left(\theta - \theta_*\right) \end{aligned} \tag{14}$$

We introduce the orthogonal matrix C satisfying

$$C^T\, \tilde{\Omega}^2\, C = \text{diag}\left(\omega_1^2,\, \omega_2^2\right) \equiv \omega^2 \tag{15}$$

Explicitly:

$$\begin{aligned} C &= \begin{pmatrix} \cos\left(\zeta - \frac{\rho}{2}\right) & -\sin\left(\zeta - \frac{\rho}{2}\right) \\ \sin\left(\zeta - \frac{\rho}{2}\right) & \cos\left(\zeta - \frac{\rho}{2}\right) \end{pmatrix} \quad , \quad \rho = \arcsin\left(\frac{r}{\sqrt{1+3r^2+r^4}}\right) \\ \omega_1^2 &= k^2 - a^2 H^2\, \frac{1-r^2}{4r}\, \frac{1-r^2-2\sqrt{1+3r^2+r^4}}{r} \\ \omega_2^2 &= k^2 - a^2 H^2\, \frac{1-r^2}{4r}\, \frac{1-r^2+2\sqrt{1+3r^2+r^4}}{r} \end{aligned} \tag{16}$$

Initially $\omega_i \simeq k$. As long as both $\omega_i^2 > 0$, we can perform the usual quantization, starting from

$$\begin{aligned} \psi_i &= C_{ij}\left[h_{jl}\, \hat{a}_l + \text{h.c.}\right] \quad , \quad h = \frac{1}{\sqrt{2\omega}}\left(\alpha + \beta\right)\, P \\ \pi_i &= \dot{\psi}_i = C_{ij}\left[\tilde{h}_{jl}\, \hat{a}_l + \text{h.c.}\right] \quad , \quad \tilde{h} = \frac{-i\omega}{\sqrt{2\omega}}\left(\alpha - \beta\right)\, P \end{aligned} \tag{17}$$

the matric P is an arbitrary constant unitary matrix. We therefore have the initial condition

$$\alpha_{\text{in}} \simeq 1 \quad , \quad \beta_{\text{in}} = 0 \tag{18}$$

provided $k \gg a\, H$ at the initial time.

When one eigenfrequency becomes tachyonic, we can no longer work with the matrices α and β; presumably, the formalism can be extended to that, but we have not done it. There is no reason to do it, since we can work with the matrices h or $\psi = C\,h$ at all times.

It is up to us to choose which variables we evolve; we choose to evolve the matrices ψ, since the evolution equation is particularly simple in these variables, and since $\tilde{\Omega}^2$ is manifestly trivial for $r = 1$ (but one would reach an equivalent conclusion working with h).

We use α and β only to get the initial conditions (18). We then have

$$\begin{aligned} \psi_{ij} &= (Ch)_{ij} \quad , \quad \psi'' + \tilde{\Omega}^2\,\psi = 0 \\ \psi_{\rm in} &= C_{\rm in}\,\frac{1}{\sqrt{2k}}P \quad , \quad \psi'_{\rm in} = -iC_{\rm in}\sqrt{\frac{k}{2}}\,P \end{aligned} \tag{19}$$

Inverting the above diagonalization for $\tilde{\Omega}^2$ we have the explicit expression

$$\begin{aligned} \tilde{\Omega}^2 &= \left(k^2 - a^2H^2\,\frac{\left(1-r^2\right)^2}{4r^2}\right)1 \\ &\quad + a^2H^2\frac{\left(1-r^2\right)\sqrt{1+3r^2+r^4}}{2r^2}\begin{pmatrix} \cos\left(2\zeta-\rho\right) & \sin\left(2\zeta-\rho\right) \\ \sin\left(2\zeta-\rho\right) & -\cos\left(2\zeta-\rho\right) \end{pmatrix} \end{aligned}$$

$$\zeta = \frac{1+r^2}{2r}\left(\theta-\theta_*\right) \quad , \quad \rho = \arcsin\left(\frac{r}{\sqrt{1+3r^2+r^4}}\right) \tag{20}$$

We choose

$$P = {\rm C}_{\rm in}^{-1} \quad , \quad -\frac{1+r^2}{r}\theta_* - \rho = -\frac{1+r^2}{r}\theta_{\rm in} \tag{21}$$

so that everything simplifies into

$$\begin{aligned} &\psi'' + \tilde{\Omega}^2\,\psi = 0 \quad , \quad \psi_{\rm in} = \frac{1}{\sqrt{2k}} \quad , \quad \psi'_{\rm in} = -i\sqrt{\frac{k}{2}} \\ &\tilde{\Omega}^2 = \left(k^2 - a^2H^2\,\frac{\left(1-r^2\right)^2}{4r^2}\right)1 \\ &\qquad\qquad + a^2H^2\frac{\left(1-r^2\right)\sqrt{1+3r^2+r^4}}{2r^2}\begin{pmatrix} \cos\,\hat{\zeta} & \sin\,\hat{\zeta} \\ \sin\,\hat{\zeta} & -\cos\,\hat{\zeta} \end{pmatrix} \\ &\hat{\zeta} = \frac{1+r^2}{r}\left(\theta-\theta_{\rm in}\right) \end{aligned} \tag{22}$$

We evaluate the energy under the assumption that X is diagonal at the end of inflation, namely when O in (7) is the identity. At this moment, we have the decomposition

$$\vec{V}_\alpha = \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}} \vec{\epsilon}_\lambda \left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}} \left[\left(\Lambda^{-1} R^T \psi\left(k\right)\right)_{\alpha\beta} \hat{a}_\beta^{(\lambda)} \left(\vec{k}\right) + \left(\Lambda^{-1} R^T \psi\left(k\right)\right)^*_{\alpha\beta} \hat{a}_\beta^{(\lambda)\dagger} \left(-\vec{k}\right) \right] \tag{23}$$

where $\vec{V}_1 = \vec{A}$ and $\vec{V}_2 = \vec{B}$.

Corresponding to

$$\rho_\alpha = \frac{\Lambda_\alpha^2}{2a^4} \left[V_{0i,\alpha} \, V_{0i,\alpha} + V_{ij,\alpha} \, V_{ij,\alpha} \right] \equiv \rho_{\alpha,\mathrm{electric}} + \rho_{\alpha,\mathrm{magnetic}} \tag{24}$$

We obtain

$$\begin{aligned} \langle \rho_{\alpha,\mathrm{magnetic}} \rangle &= \frac{2}{a^4} \int \frac{d^3k}{(2\pi)^3} \, k^2 \left(R^T \psi \psi^\dagger R^* \right)_{\alpha\alpha} \\ &= \int \frac{dk}{k} \left[\frac{k^5}{\pi^2 a^4} \left(R^T \psi \psi^\dagger R^* \right)_{\alpha\alpha} \right] \equiv \int \frac{dk}{k} P_\alpha \end{aligned} \tag{25}$$

where the α index is not summed over. The expression in square parenthesis is conventionally denoted as the magnetic field power spectrum P_α. It is convenient to define the dimensionless quantity

$$\tilde{\psi} \equiv \sqrt{2k} \, \psi \tag{26}$$

In terms of which the power spectrum is

$$P_\alpha = \frac{1}{2\pi^2} \left(\frac{k}{a} \right)^4 \left(R^T \tilde{\psi} \tilde{\psi}^\dagger R^* \right)_{\alpha\alpha} \quad , \quad (\text{no sum over } \alpha) \tag{27}$$

Moreover, using dimensionless time $\tilde{\tau} = k\,\tau$, and recalling that $a = -\frac{1}{H\tau} = -\frac{k}{H\tilde{\tau}}$, we have

$$\begin{aligned} & \frac{d^2\tilde{\psi}}{d\tilde{\tau}^2} = \tilde{\Omega}^2 \, \tilde{\psi} = 0 \quad , \quad \tilde{\psi}_{\mathrm{in}} = 1 \quad , \quad \psi'_{\mathrm{in}} = -i \\ & \tilde{\Omega}^2 = \left(1 - \frac{1}{\tau^2} \frac{\left(1 - r^2\right)^2}{4r^2} \right) 1 \\ & \qquad\qquad + \frac{1}{\tau^2} \frac{\left(1 - r^2\right) \sqrt{1 + 3r^2 + r^4}}{2r^2} \begin{pmatrix} \cos\hat{\zeta} & \sin\hat{\zeta} \\ \sin\hat{\zeta} & -\cos\hat{\zeta} \end{pmatrix} \\ & \hat{\zeta} = \frac{1 + r^2}{r} \ln \frac{-\tilde{\tau}_{\mathrm{in}}}{-\tilde{\tau}} \end{aligned} \tag{28}$$

Some clarifications:

The quantity k is not physical; it is called "comoving momentum". The physical momentum (the one you measure) is $p = k/a$. The normalization of a determines the normalization of k. It is convenient to normalize $a\left(\tau_{\rm end}\right) = 1$ at the end of inflation. This means

$$\tilde{\tau}_{\rm end} = -\frac{1}{H} \tag{29}$$

For each value of k, we evaluate the numerical equations from $\tilde{\tau}_{\rm in} = k\,\tau_{\rm in}$ to $\tilde{\tau}_{\rm end} = -\frac{k}{H}$. We need the mode to be inside the horizon at the initial time ($k/a_{\rm in} \gg H$) and outside the horizon at the final time ($k/a_{\rm end} = k \ll H$). Since $\frac{k/a}{H} = |k\,\tau| = |\tilde{\tau}|$, we must take $|\tilde{\tau}_{\rm in}| \gg 1$ and $|\tilde{\tau}_{\rm end}| \ll 1$ for all the modes.

We need to verify that the result does not depend on $\tilde{\tau}_{\rm in}$ provided that $|\tilde{\tau}_{\rm in}| \gg 1$.

We need to evaluate the power spectrum at some given $\tau_{\rm end}$; this means different values of $\tilde{\tau}_{\rm end}$ for different values of k.

Each mode is characterized by a given k, or, equivalently, by a given value of $|\tilde{\tau}_{\rm end}| = \frac{k}{H}$. Ideally, we would like to compute the power (27) for modes well outside the horizon, meaning $|\tilde{\tau}_{\rm end}| \sim 10^{-30}$. This is numerically unrealistic. It is ok to perform computations for say $|\tilde{\tau}_{\rm end}| = \{10^{-10},\, 10^{-9},\, \dots,\, 10^{-3}\}$ and see how the plot of P as a function of $\frac{k}{H}$ behaves.

For $r = 1$ we naively obtain the power spectrum

$$P_{\alpha,\,\rm vacuum} = \frac{1}{2\pi^2}\left(\frac{k}{a}\right)^4 \tag{30}$$

This however is the vacuum power spectrum, which one should subtract (it is renormalized away).