1 Model with scalar coupling

We consider the action

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{4}V^T_{\mu\nu}\, X\, V^{\mu\nu} - eJ^\mu A_\mu\right] \quad , \quad V_{\mu\nu} = \begin{pmatrix} A_{\mu\nu} \\ B_{\mu\nu} \end{pmatrix} \tag{1}$$

where the kinetic matrix is

$$X = \begin{pmatrix} 1 & \chi\,\epsilon\,(t) \\ \chi\,\epsilon\,(t) & \epsilon^2\,(t) \end{pmatrix} \tag{2}$$

with χ constant, and $\epsilon \ll 1$ during inflation, and $\epsilon = 1$ at the end of inflation.

1.1 Late time magnetic field

After inflation, consider the redefinition

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-\chi^2}} & 0 \\ \frac{-\chi}{\sqrt{1-\chi^2}} & 1 \end{pmatrix} \begin{pmatrix} \tilde{A} \\ \tilde{B} \end{pmatrix} \tag{3}$$

In terms of the new fields, the action is

$$\int d^4x\sqrt{-g}\left[-\frac{1}{4}\tilde{A}_{\mu\nu}\tilde{A}^{\mu\nu} - \frac{1}{4}\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu} - \frac{e}{\sqrt{1-\chi^2}}\tilde{A}^\mu\, J_\mu\right] \tag{4}$$

Therefore, our photon is

$$\tilde{A}_\mu = \sqrt{1-\chi^2}\, A_\mu \tag{5}$$

We learn that, to produce a magnetic field, we need to produce A_μ.

1.2 Helicity decomposition

We use conformal time $ds^2 = a\,(\tau)\,[-d\tau^2 + d\vec{x}^2]$. In the gauge $A_0 = B_0 = 0$, we decompose

$$\begin{aligned} \vec{A} &= \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}}\vec{\epsilon}_\lambda\left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}}\,\hat{A}_\lambda\left(\vec{k}\right) \\ \vec{B} &= \sum_\lambda \int \frac{d^3k}{(2\pi)^{3/2}}\vec{\epsilon}_\lambda\left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}}\,\hat{B}_\lambda\left(\vec{k}\right) \end{aligned} \tag{6}$$

Here $\vec{\epsilon}_\lambda$ are circular polarization vectors satisfying $\vec{k}\cdot\vec{\epsilon}_\pm\left(\vec{k}\right)=0$, $\vec{k}\times\vec{\epsilon}_\pm\left(\vec{k}\right)=\mp ik\vec{\epsilon}_\pm\left(\vec{k}\right)$, $\vec{\epsilon}_\pm\left(-\vec{k}\right)=\vec{\epsilon}_\pm\left(\vec{k}\right)^*$, and normalized according to $\vec{\epsilon}_\lambda\left(\vec{k}\right)^*\cdot\vec{\epsilon}_{\lambda'}\left(\vec{k}\right)=\delta_{\lambda\lambda'}$.

We obtain

$$S=\frac{1}{2}\sum_{\lambda=\pm}\int d\tau\, d^3k\left[V_\lambda^{\dagger\prime}\left(\vec{k}\right)\cdot X\cdot V_\lambda'\left(\vec{k}\right)-k^2V_\lambda^\dagger\left(\vec{k}\right)\cdot X\cdot V_\lambda\left(\vec{k}\right)\right]\quad,\quad V_\lambda\equiv\left(\begin{array}{c}\hat{A}_\lambda\\ \hat{B}_\lambda\end{array}\right)\tag{7}$$

The actions of the two helicities are identical to each other, and decoupled. Therefore, we focus on one of the two actions and we suppressed the suffix λ.

1.3 Large wavelength solutions

In matrix form, we have the equation

$$\left(X\,V'\right)'+k^2\,V=0\tag{8}$$

For any mode, we can consider a time τ_* after which the mode is well outside the horizon, and k can be set to zero. In this case

$$V'\left(\tau\right)=X^{-1}\left(\tau\right)\,X_*\,V_*'\tag{9}$$

or, more explicitly,

$$\begin{aligned}A' &= -\frac{\chi}{1-\chi^2}\left(\chi\,A_*'+\epsilon_*B_*'\right)\frac{\epsilon_*}{\epsilon}+\frac{A_*'+\chi\epsilon_*B_*'}{1-\chi^2}\\ \epsilon_*B' &= -\frac{\chi}{1-\chi^2}\left(\chi\epsilon_*B_*'+A_*'\right)\frac{\epsilon_*}{\epsilon}+\frac{\epsilon_*B_*'+\chi A_*'}{1-\chi^2}\,\frac{\epsilon_*^2}{\epsilon^2}\end{aligned}\tag{10}$$

We now choose

$$\epsilon=\left(\frac{a}{a_0}\right)^n=\left(\frac{\tau_0}{\tau}\right)^n\tag{11}$$

where 0 denotes the end of inflation. The solution is

$$
\begin{aligned}
A(\tau) &= A_* + \tau_* \left\{ -\frac{A'_* + \chi \epsilon_* B'_*}{1-\chi^2} \left(1 - \frac{\tau}{\tau_*}\right) + \frac{\chi \left(\chi A'_* + \epsilon_* B'_*\right)}{(1+n)\left(1-\chi^2\right)} \left[1 - \left(\frac{\tau}{\tau_*}\right)^{n+1}\right] \right\} \\
B(\tau) &= B_* + \tau_* \left(\frac{\tau_*}{\tau_0}\right)^n \left\{ -\frac{\chi \left(A'_* + \chi \epsilon_* B'_*\right)}{1-\chi^2} \left[1 - \left(\frac{\tau}{\tau_*}\right)^{n+1}\right] \right. \\
&\qquad \left. - \frac{\chi A'_* + \epsilon_* B'_*}{(1+2n)\left(1-\chi^2\right)} \left[1 - \left(\frac{\tau}{\tau_*}\right)^{2n+1}\right] \right\}
\end{aligned}
\tag{12}
$$

Let us evaluate these expressions at τ_0 (end of inflation). We are interesting in growing modes for $\tau_0 \to 0$. Disregarding terms that are either constant or $\propto \tau_0$, we have

$$
A \propto \tau_0^{n+1} \quad , \quad B \propto \tau_0^{n+1} \; , \; \tau_0^{-n} \tag{13}
$$

Therefore we have the two possibilities

$$
\begin{aligned}
B &\simeq -\frac{\epsilon_* \tau_* B'_* \left[1 + n - (1+2n)\chi^2\right] - n \chi \tau_* A'_*}{(1+n)(1+2n)\left(1-\chi^2\right)} \left(\frac{a_0}{a_*}\right)^n \quad , \quad A \simeq 0 \quad , \quad n > 0 \\
B &\simeq \frac{\epsilon_* \tau_* B'_* + \chi \tau_* A'_*}{(1+2n)\left(1-\chi^2\right)} \left(\frac{a_0}{a_*}\right)^{-n-1} \quad , \quad A \simeq -\frac{1+2n}{1+n} \chi B \quad , \quad n < -1
\end{aligned}
\tag{14}
$$

For $n > 0$, only B grows in the super-horizon regime, while for $n < -1$ both grow. Unfortunately this option gives $\rho_E \gg \rho_B$, and too much energy in the electric field if one wants $\rho_B \sim$ const during inflation (or only, logarithmically evolving).

2 Pseudo-scalar coupling

Consider instead the model

$$
S = \int d^4x \sqrt{-g} \left[-\frac{1}{4} A_{\mu\nu} A^{\mu\nu} - \frac{\epsilon^2(t)}{4} B_{\mu\nu} B^{\mu\nu} - \frac{\epsilon(t)\,\chi}{2} \frac{\eta^{\mu\nu\alpha\beta}}{\sqrt{-g}} A_{\mu\nu} B_{\alpha\beta} - e J^\mu A_\mu \right] \tag{15}
$$

where χ (assumed constant) is the expectation value of a pseudo-scalar.

2.1 Late time

After inflation $\epsilon = 1$. Redefining

$$\begin{aligned} A^{\mu\nu} &= \frac{1}{\sqrt{1-2\chi^2}}\tilde{A}^{\mu\nu} \\ B^{\mu\nu} &= \tilde{B}^{\mu\nu} - \frac{\chi}{\sqrt{1-2\chi^2}}\eta^{\mu\nu\alpha\beta}\tilde{A}^{\alpha\beta} \end{aligned} \tag{16}$$

we have

$$\int d^4x\sqrt{-g}\left[-\frac{1}{4}\tilde{A}_{\mu\nu}\tilde{A}^{\mu\nu} - \frac{1}{4}\tilde{B}_{\mu\nu}\tilde{B}^{\mu\nu} - \frac{e}{\sqrt{1-2\,\chi^2}}\tilde{A}^{\mu}\,J_{\mu}\right] \tag{17}$$

2.2 Helicity decomposition

We use conformal time $ds^2 = a\left(\tau\right)\left[-d\tau^2 + d\vec{x}^2\right]$. In the gauge $A_0 = B_0 = 0$, we decompose

$$\begin{aligned} \vec{A} &= \sum_{\lambda}\int \frac{d^3k}{\left(2\pi\right)^{3/2}}\vec{\epsilon}_{\lambda}\left(\vec{k}\right)\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{A}_{\lambda}\left(\vec{k}\right) \\ \vec{B} &= \sum_{\lambda}\int \frac{d^3k}{\left(2\pi\right)^{3/2}}\vec{\epsilon}_{\lambda}\left(\vec{k}\right)\mathrm{e}^{i\vec{k}\vec{x}}\,\hat{B}_{\lambda}\left(\vec{k}\right) \end{aligned} \tag{18}$$

Here $\vec{\epsilon}_{\lambda}$ are circular polarization vectors satisfying $\vec{k}\cdot\vec{\epsilon}_{\pm}\left(\vec{k}\right) = 0$, $\vec{k}\times\vec{\epsilon}_{\pm}\left(\vec{k}\right) = \mp ik\vec{\epsilon}_{\pm}\left(\vec{k}\right)$, $\vec{\epsilon}_{\pm}\left(-\vec{k}\right) = \vec{\epsilon}_{\pm}\left(\vec{k}\right)^{*}$, and normalized according to $\vec{\epsilon}_{\lambda}\left(\vec{k}\right)^{*}\cdot\vec{\epsilon}_{\lambda'}\left(\vec{k}\right) = \delta_{\lambda\lambda'}$.

We obtain

$$S = \frac{1}{2}\int d\tau d^3k\sum_{\lambda}\left[\hat{V}_{\lambda}^{\prime\dagger}\begin{pmatrix} 1 & 0 \\ 0 & \epsilon^2 \end{pmatrix}\hat{V}_{\lambda}' - \hat{V}_{\lambda}\begin{pmatrix} k^2 & -2\epsilon'\chi\lambda k \\ -2\epsilon'\chi\lambda k & \epsilon^2k^2 \end{pmatrix}\hat{V}_{\lambda}\right] \tag{19}$$

where $\hat{V}_{\lambda} = \begin{pmatrix} \hat{A}_{\lambda} \\ \hat{B}_{\lambda} \end{pmatrix}$. In terms of the canonical fields

$$\begin{aligned} S &= \frac{1}{2}\int d\tau d^3k\sum_{\lambda}\left[\hat{V}_{c\lambda}^{\prime\dagger}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\hat{V}_{c\lambda}' - \hat{V}_{c\lambda}\begin{pmatrix} k^2 & -2\frac{\epsilon'}{\epsilon}\chi\lambda k \\ -2\frac{\epsilon'}{\epsilon}\chi\lambda k & k^2 - \frac{\epsilon''}{\epsilon} \end{pmatrix}\hat{V}_{c\lambda}\right] \\ \hat{V}_{c\lambda} &\equiv \begin{pmatrix} \hat{A}_{\lambda} \\ \epsilon\,\hat{B}_{\lambda} \end{pmatrix} \equiv \begin{pmatrix} \hat{A}_{\lambda} \\ \hat{B}_{c\lambda} \end{pmatrix} \end{aligned} \tag{20}$$

leading to the equations

$$\hat{B}''_{c\lambda} + \left[k^2 - \frac{\epsilon''}{\epsilon}\right]\hat{B}_{c\lambda} - 2k\lambda\chi\frac{\epsilon'}{\epsilon}\hat{A}_\lambda = 0$$
$$\hat{A}''_\lambda + k^2\hat{A}_\lambda - 2k\lambda\chi\frac{\epsilon'}{\epsilon}\hat{B}_{c\lambda} = 0 \qquad (21)$$

Assuming that A is negligible in the first equation (to be checked) we have

$$B_{c\lambda} = i\frac{\sqrt{\pi}}{2}\sqrt{-\tau}\,H^{(1)}_{n+1/2}\left(-k\tau\right) \qquad (22)$$

This is properly normalized; indeed $B_{c\lambda} \to \frac{(-i)^{n+1}}{\sqrt{2k}}\,\mathrm{e}^{-ik\tau}$ at $-k\tau \to +\infty$. Super horizon, we get

$$\begin{aligned} B_{c\lambda} &\simeq -\frac{i2^{n-\frac{1}{2}}\,\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}\frac{1}{k^{n+\frac{1}{2}}\,\left(-\tau\right)^n} \quad , \quad n > -\frac{1}{2} \\ B_{c\lambda} &\simeq \frac{2^{-n-\frac{3}{2}}\sqrt{\pi}\,\left\{1+i\cot\left[\left(n+\frac{1}{2}\right)\pi\right]\right\}}{\Gamma\left(n+\frac{3}{2}\right)}\frac{1}{k^{-n-\frac{1}{2}}\,\left(-\tau\right)^{-n-1}} \quad , \quad n < -\frac{1}{2} \end{aligned} \qquad (23)$$

To avoid producing too much energy we require $-2 \leq n \leq 2$.

From the second equation at super horizon we then obtain (ignoring factors of order one)

$$\begin{aligned} A_\lambda &\simeq \frac{\lambda\chi}{k^{n-\frac{1}{2}}\,\left(-\tau\right)^{n-1}} \quad , \quad n > -\frac{1}{2} \\ A_\lambda &\simeq \frac{\lambda\chi}{k^{-n-\frac{3}{2}}\,\left(-\tau\right)^{-n-2}} \quad , \quad n < -\frac{1}{2} \end{aligned} \qquad (24)$$

We obtain a scale invariant magnetic field only for $n = -4, 3$, which are forbidden by energy conservation.

3 Electric and magnetic energy for a single field case

Consider the model

$$S = -\frac{1}{4}\int d^4x\,\sqrt{-g}\,I^2\,F^2 \quad , \quad I \propto a^n \propto \frac{1}{\tau^n} \qquad (25)$$

and decompose

$$
\begin{aligned}
\vec{A} &= \sum_{\lambda} \int \frac{d^3k}{(2\pi)^{3/2}} \vec{\epsilon}_{\lambda}\left(\vec{k}\right) \mathrm{e}^{i\vec{k}\vec{x}} \frac{\hat{V}_{\lambda}\left(\vec{k}\right)}{I} \\
\hat{V}_{\lambda}\left(\vec{k}\right) &= V\left(\tau,\, k\right) a_{\lambda}\left(\vec{k}\right) + V^*\left(\tau,\, k\right) a_{\lambda}^{\dagger}\left(-\vec{k}\right)
\end{aligned}
\tag{26}
$$

The mode functions obey

$$
V'' + \left(k^2 - \frac{I''}{I}\right) V = 0 \tag{27}
$$

and the properly normalized solution is

$$
V = \frac{\sqrt{\pi}}{2} \sqrt{-\tau}\, H^{(0)}_{n+\frac{1}{2}}\left(-k\tau\right) \tag{28}
$$

The electromagnetic energy is

$$
\begin{aligned}
\rho_E &= \frac{4\pi I^2}{a^4} \int \frac{dk}{k} k^3 \Big| \left(\frac{V}{I}\right)' \Big|^2 \equiv \int d\ln k\, \rho_{E,k,n} \\
\rho_B &= \frac{4\pi}{a^4} \int \frac{dk}{k} k^5 \Big|V\Big|^2 \equiv \int d\ln k\, \rho_{B,k,n}
\end{aligned}
\tag{29}
$$

so that

$$
\begin{aligned}
\rho_{B,k,n} &= H^4\, \pi^2 \left(-k\,\tau\right)^5 \Big| H^{(0)}_{n+\frac{1}{2}}\left(-k\tau\right) \Big|^2 \\
\rho_{E,k,n} &= H^4\, \pi^2 \left(-k\,\tau\right)^5 \Big| H^{(0)}_{n-\frac{1}{2}}\left(-k\tau\right) \Big|^2
\end{aligned}
\tag{30}
$$

We are interested in the late time / super horizon limit of these quantities. Namely we consider the $\tau \rightarrow 0$ limit (keeping attention to the sign of the index). Notice that

$$
\lim_{x\rightarrow 0} \Big| H^{(0)}_{\alpha}\left(x\right) \Big|^2 = \frac{4^{|\alpha|}\, \Gamma^2\left(|\alpha|\right)}{\pi^2\, x^{-2|\alpha|}} \tag{31}
$$

and we find

$$
\begin{aligned}
\lim_{\tau\rightarrow 0^-} \rho_{B,k,n} &= 2^{2|n+\frac{1}{2}|}\, H^4 \left(\frac{k}{aH}\right)^{5-2|n+\frac{1}{2}|} \Gamma^2\left(|n+\frac{1}{2}|\right) \\
\lim_{\tau\rightarrow 0^-} \rho_{E,k,n} &= \lim_{\tau\rightarrow 0^-} \rho_{B,k,-n}
\end{aligned}
\tag{32}
$$

Therefore we simply evaluate the magnetic energy at the end of inflation

$$\rho_B = H^4 2^{2|n+\frac{1}{2}|}\Gamma^2\left(|n+\frac{1}{2}|\right)\int_{a_{\rm in}/a_{\rm end}}^{1} dx\, x^{4-2|n+\frac{1}{2}|} \tag{33}$$

In the limit of $a_{\rm end} \gg a_{\rm in}$, and disregarding order one factors,

$$\frac{\rho_B}{H^4} \sim \begin{cases} \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-6-2n} & , \quad n < -3 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = -3 \\ 1 & , \quad -3 < n < 2 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = 2 \\ \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-4+2n} & , \quad n > 2 \end{cases} \tag{34}$$

and

$$\frac{\rho_E}{H^4} \sim \begin{cases} \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-4-2n} & , \quad n < -2 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = -2 \\ 1 & , \quad -2 < n < 3 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = 3 \\ \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-6+2n} & , \quad n > 3 \end{cases} \tag{35}$$

Adding them up,

$$\frac{\rho_E + \rho_B}{H^4} \sim \begin{cases} \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-4-2n} & , \quad n < -2 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = -2 \\ 1 & , \quad -2 < n < 2 \\ \ln\frac{a_{\rm end}}{a_{\rm in}} & , \quad n = 2 \\ \left(\frac{a_{\rm end}}{a_{\rm in}}\right)^{-4+2n} & , \quad n > 2 \end{cases} \tag{36}$$