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Quaternionic Wavefunction

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Abstract

We argue that quaternions form a natural language for description of the wavefunction. In this paper, we tap into the Dirac equation and the Schrödinger equation, all formulated in quaternionic language. We do not exhibit any unknown or unphysical features, as mostly was done previously. Instead, we demonstrate the transparency of the known features and details of the Dirac equation, and derive the Schrödinger equation as its non-relativistic limit. We provide an easy to use grammar for switching between the ordinary spinor language and the description in terms of quaternions.

1 Introduction

The list of literature on the rôle of quaternions in physics is so vast that it is hardly possible to oversee it []. The majority of the literature touches base on the use of quaterions in three-dimensional rotations and Lorentz transformations. Other papers include their applications in Electrodynamics and Quantum Mechanics. The attractive feature of quaternions is that whenever a solution is possible in the quaternionic form, it is much more compact than that in the customary form of Lorentz vectors and tensors.

For the majority of theorists, however, quaternions are seen as a somewhat exotic subject matter, which neither has proven to be exceedingly effective to be adopted permanently, nor has lead to any new phenomena or even a new formalism.

The other drawback, as it is perceived, is a somewhat "strange" mathematical language, often accompanied by strange results following from this language. While from learning Quantum Mechanics and Gauge Theories we are used to non-commutative operators, and non-commutative objects in general, when we see an operator of multiplication "from the right" $|\,\hat{\imath}$ (an example of the so-called "barred" operators) it immediately induces a certain degree of skepsis. That is, mathematically this formalism may be interesting, but physically this seems to be driven away from reality and therefore unnecessary. Furthermore, some treatments on quaternions in Quantum Mechanics predict extra degrees of freedom for fermions, which we obviously do not observe. This is the result of the fact that quaternions are multi-component numbers, with a bit too many components than needed for Quantum Mechanics. These aspects of using quaternions in theoretical physics are enough to discourage the interest in the majority of theorists.

Our goal is to cast a bridge from the regular algebraic language of physics (which is, dominantly based on complex numbers) to the language of quaternions. We argue that quaternions have always been around, and we just neglected to acknowledge them. There is no need or necessity for any new degrees of freedom or new physics to arise. We would like to present a concise dictionary, so that any theorist could connect to and appreciate the quaternionic formalism, which appears to be quite capacious.

The omnipresence of quaternions is easy to observe. We know that Quantum Mechanics is based solely on complex numbers. Complex numbers provide a compact and meaningful way of both formulating and solving quantum-mechanical problems.

With some exceptions, it would be very awkward to split the Schrödinger equation into its real and imanginary parts, and then to attempt to solve the resulting equations. Quantum mechanical operators of momentum, angular momentum are *inherently* complex. That is, to say, that the wavefunction is complex too.

But as soon as relativistic effects are included into Quantum Mechanics, it turns out that particles have *spin*. The way to incorporate spin into the wavefunction is just to make the latter consist of two components, that is, to turn it into a spinor. The spin operator itself is then given by the Pauli matrices. This is where quaternions get involved. The algebra of Pauli matrices is the same as that of quaternionic units, loosely speaking. We argue that there is *no inherent need* to having introduced matrices. Quaternionic units are all that one needs, and implicitly the wavefunction in Quantum Mechanics is quaternionic. A peculiar property of physics is that physical objects are actually described by *complex* quaternions, which are also known as complexified quaternions, and not by the regular quaternions. This does not change the main argument, however.

So, how does one include spin into the wavefunction by means of quaternions? In the literature, quaternionic wavefunction is usually introduced *ad hoc*, and the resulting Dirac or Schrödinger equation then includes non-physical (unobserved) degrees of freedom. We proceed in a very conservative way, essentially expanding on the development originally presented in [*thesis*]: we start out from the Dirac Lagrangian, carefully taking into account the spin degrees of freedom, and then *derive* the Schrödinger equation. Spin naturally appears as a consequence of the fact that the wavefunction is quaternionic. Quaternionic derivations, while a bit unusual for some, are simpler than spinor derivations. This way we argue that quaternionic language is natural for Quantum Mechanics.

It is not just for these reasons that we believe quaternions play a fundamental rôle in physics. There are hints that quaternions are part of the natural language for the entire Standard Model []. We view this work as one of the steps towards the description of the Standard Model in such a language.

- *Getting rid of matrices*
- *Stress the absence of γ-matrices*
- *A note on literature, in particular S. De Leo*

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Let us talk about our notations first, while gradually introducing the subject matter. The reader eager to see the physical results may choose to skip to the next section, occasinally returning here to clarify the notations, when necessary. Here we overview the known facts which are easy to pick up and use, while their proof can be found in the literature. We denote the quaternionic units as

$$\hat{\imath}^2 \;=\; \hat{\jmath}^2 \;=\; \hat{k}^2 \;=\; -1\,, \tag{1.1}$$

and we do not distinguish them from the three-dimensional unit vectors. That is, any three-dimensional vector is a quaternion

$$\vec{a} \;=\; a^1\,\hat{\imath} \;+\; a^2\,\hat{\jmath} \;+\; a^3\,\hat{k}\,. \tag{1.2}$$

Complex quaternions are defined as

$$a \;=\; a^0 \;+\; a^1\hat{\imath} \;+\; a^2\hat{\jmath} \;+\; a^3\hat{k}\,, \tag{1.3}$$ {cq}

where all components

$$a^0 \;=\; b^0 \;+\; i\,b^0\,, \qquad a^1 \;=\; b^1 \;+\; i\,b^1\,, \qquad \ldots, \tag{1.4}$$

are complex numbers. Here i denotes a regular imaginary complex unit, which commutes with $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.

We right away introduce the conjugation operations. We denote the quaternionic conjugation by $\widetilde{a}$,

$$\widetilde{a} \;=\; a^0 \;-\; a^1\hat{\imath} \;-\; a^2\hat{\jmath} \;-\; a^3\hat{k}\,. \tag{1.5}$$

We remember that the quaternionic conjugation switches the order of factors in a product:

$$\widetilde{a\,b} \;=\; \widetilde{b}\,\widetilde{a}\,. \tag{1.6}$$

Since the components are complex numbers, we also have the complex conjugation a^*

$$a^* \;=\; (a^0)^* \;+\; (a^1)^*\,\hat{\imath} \;+\; (a^2)^*\,\hat{\jmath} \;+\; (a^3)^*\,\hat{k}\,. \tag{1.7}$$

It is convenient to introduce the composition of these two conjugations, which we call a *hermitean* conjugation (not without a reason),

$$a^\dagger \;=\; (a^0)^* \;-\; (a^1)^*\,\hat{\imath} \;-\; (a^2)^*\,\hat{\jmath} \;-\; (a^3)^*\,\hat{k}\,. \tag{1.8}$$

By itself it does not give anything new, as it is merely a combination of $\tilde{}$ and $*$ operations, but it is important to have it, as we will see. Hermitean conjugation also interchanges the order of terms in a product, obviously

$$(a\,b)^\dagger \quad = \quad b^\dagger\, a^\dagger\,. \tag{1.9}$$

It is this plentitude of conjugations that give richness to the quaternionic language, when applied to physics.

Now we can define a *true* four-dimensional vector

$$v \quad = \quad v^0 \;+\; i\,(\,v^1\,\hat{\imath} \;+\; v^2\,\hat{\jmath} \;+\; v^3\,\hat{k}\,) \quad \equiv \quad v^0 \;+\; i\,\vec{v} \tag{1.10}$$ {true-vector}

in Minkowsky space. Note that it is precisely the combination $i\,\vec{v}$ that gives a true vector. Without it,

$$v \quad = \quad i\,v^0 \;+\; \vec{v} \tag{1.11}$$ {pseudo-vector

describes a *pseudo*-scalar and a *pseudo*-vector, with respect to inversion. Together, the two objects (1.10) and (1.11) span the entire space of complex quaternions (1.3). In other words, a generic complex quaternion a can be split into two four-vectors. Their time components will represent a true- and a pseudo-scalar, while their three-dimensional parts will represent a true and an axial vector, correspondingly. Notice, how multiplication by the complex i turns a true vector into an axial vector, and the same for scalars.

With four-dimensional vectors in Minkowski space, one has to be careful to always keep in mind whether a vector is contravariant or covariant. Complex conjugation $*$ turns a contravariant vector v into a covariant vector v^*.

$$v^* \quad = \quad v^0 \;-\; i\,\vec{v}\,. \tag{1.12}$$

This actually is equivalent to quaternionic conjugation $v^* \;=\; v^\dagger$.

The derivative operator ∂

$$\partial \quad = \quad \partial^0 \;+\; i\,\vec{\nabla} \tag{1.13}$$

is *by definition* a covariant vector, while obviously

$$\partial^* \quad = \quad \partial^0 \;-\; i\,\vec{\nabla} \tag{1.14}$$

is a contravariant vector.

To make these identifications more meaningful, let us talk about Lorentz transformations. Lorentz transformations are generated by a purely-imaginary quaterionic parameter

$$\Lambda \;=\; \vec{\kappa} \;+\; i\,\vec{\lambda}\,. \tag{1.15}$$

We are not going to explicitly treat it as a vector, so we are not putting a vector sign on this parameter. Parameter $\vec{\kappa}$ generates three-dimensional rotations, while parameter $\vec{\lambda}$ generates boosts. These are "generators" in the sense that the actual finite transformations are performed by the exponent

$$e^{\Lambda}\,. \tag{1.16}$$

It is important to be careful here, as $\vec{\kappa}$ can be interpreted as a three-dimensional rotation only when $\vec{\lambda} \;=\; 0$, and the same for $\vec{\lambda}$ — it can be interpreted as a boost only when $\vec{\kappa} \;=\; 0$. This is because in general $\vec{\kappa}$ and $\vec{\lambda}$ do not commute, and therefore

$$e^{\vec{\kappa} \,+\, i\,\vec{\lambda}} \;\;\neq\;\; e^{\vec{\kappa}} \;\cdot\; e^{i\,\vec{\lambda}}\,, \tag{1.17}$$

where each individual exponent on the right-hand side *i*s treated as a rotation and a boost, correspondingly. Notice that since Λ is purely imaginary, $\widetilde{\Lambda} \;=\; -\,\Lambda$, and therefore

$$\widetilde{(e^{\Lambda})} \;=\; e^{-\Lambda}\,. \tag{1.18}$$

A contravariant vector v transforms under Λ as

$$v \;\;\rightarrow\;\; e^{\Lambda}\, v\, e^{\Lambda^\dagger}\,. \tag{1.19}$$ {contra}

Any covariant vector then should transform the same way that v^* does:

$$v^* \;\;\rightarrow\;\; e^{\Lambda^*}\, v^*\, e^{\widetilde{\Lambda}}\,. \tag{1.20}$$ {co}

As a special case of these, a three-dimensional vector $\vec{v}$ rotates as

$$\vec{v} \;\;\rightarrow\;\; e^{\vec{\kappa}}\, \vec{v}\, e^{-\vec{\kappa}}\,. \tag{1.21}$$ {rotation}

This concludes the basic discussion of vectors for now.

Another way a complex quaternion (1.3) can be split up, is by separating its zeroth component $a^0 \;\equiv\; \phi$ and its vector part $\vec{a}$. The zeroth component ϕ is identified as a complex scalar field. The remaining vector part is then identified as a fieldstrength:

$$\vec{F} \;=\; \vec{B} + i\,\vec{E}\,. \tag{1.22}$$ {fs}

Notice that this agrees with the identifications of (1.10) and (1.11) as polar and axial vectors. While both $\vec{B}$ and $\vec{E}$ are vectors in the three-dimensional sense, one does not view the object (1.22) as a four-dimensional vector in any way. This is an entirely different split up of a complex quaternion.

Both ϕ and $\vec{F}$ transform the same way under Lorentz transformations:

$$p \;\to\; e^{\Lambda}\,p\,e^{\widetilde{\Lambda}}\,. \tag{1.23}$$

For the scalar ϕ this obviously does not do anything, since it is just a complex number:

$$e^{\Lambda}\,\phi\,e^{\widetilde{\Lambda}} \;=\; \phi\,e^{\Lambda}\,e^{\widetilde{\Lambda}} \;=\; \phi\,,$$

so it is indeed a scalar. While for the fieldstrength, this dictates that

$$\vec{F} \;\to\; e^{\Lambda}\,\vec{F}\,e^{\widetilde{\Lambda}}\,. \tag{1.24}$$

Note that for the case of pure rotations, when $\Lambda = \vec{\kappa}$, this agrees with Eq. (1.21), as $\widetilde{\Lambda} \;=\; -\,\Lambda \;=\; -\,\vec{\kappa}$. That is, both the electric and magnetic fields rotate as three-vectors.

Finally, we introduce spinors. Here we give a very brief overview of spinors necessary for Section 2, while a more detailed discussion is postponed until Section 3. We begin with stating that the space of complex quaternions can be split in two halves in a yet another way, namely using chirality projectors, P_L and P_R. A *left-handed* spinor is defined as an arbitrary complex quaternion multiplied by a projector P_L on the right:

$$\psi_L \;=\; a\,P_L\,. \tag{1.25}$$

Here P_L is a complex quaternion

$$P_L \;=\; \frac{1 + i\hat{k}}{2}\,, \tag{1.26}$$

and we call it a projector because

$$P_L^2 \;=\; P_L\,.$$

All accompanying details of these definitions can be found in Section 3. For an ordinary reader ψ_L should be precisely viewed as a left-handed chiral spinor. A convenient basis for ψ_L is formed by elements P_L and $\hat{\jmath}\, P_L$, which are geometrically orthogonal to each other:

$$\psi_L \;=\; \xi_L\, P_L \;+\; \chi_L\, \hat{\jmath}\, P_L\,. \tag{1.27}$$ {lbasis}

Complex numbers ξ_L and χ_L are precisely the "spin-up" and "spin-down" components of ψ_L viewed as a Weyl spinor:

$$\psi_L \;=\; \begin{pmatrix} \xi_L \\ \chi_L \end{pmatrix}. \tag{1.28}$$

Left-handed spinors span a half of the complex quaternion space, while the other half is spanned by the right-handed spinors:

$$\psi_R \;=\; a\, P_R\,, \tag{1.29}$$

where

$$P_R \;=\; \frac{1 - i\hat{k}}{2}\,. \tag{1.30}$$

These two halves are related by complex conjugation $*$, and the projectors are related as

$$P_R \;=\; P_L^*\,, \qquad\qquad P_L \;+\; P_R \;=\; 1\,. \tag{1.31}$$

The basis for ψ_R is similarly given by P_R and $\hat{\jmath}\, P_R$, but the components are identified slightly differently

$$\psi_R \;=\; -\,\xi_R\, \hat{\jmath}\, P_R \;+\; \chi_R\, P_R\,. \tag{1.32}$$ {rbasis}

Defined like so,

$$\psi_R \;=\; \begin{pmatrix} \xi_R \\ \chi_R \end{pmatrix} \tag{1.33}$$

is precisely identified as a right-handed Weyl spinor. The basis (1.27) and especially so (1.32) may seem a bit awkward, but they are convenient for doing algebra. The fact that the projectors are part of the bases allows us to perform various manipulations and conjugations on spinors quite effectively.

Under Lorentz transformations, left spinors by definition transform as

$$\psi_L \quad \to \quad e^{\Lambda}\, \psi_L \,, \tag{1.34}$$

while the right-handed ones transform in the conjugate representation

$$\psi_R \quad \to \quad e^{\Lambda^*}\, \psi_R \,. \tag{1.35}$$

It is interesting to observe and compare how spinors and three-dimensional vectors transform under three dimensional rotaions. Consider a rotation around axis $\hat{a}$ ($\hat{a}^2 \;=\; -1$) through angle α. Vector $\vec{v}$ rotates as

$$\vec{v} \quad \to \quad e^{\alpha\,\hat{a}/2}\, \vec{v}\, e^{-\alpha\,\hat{a}/2} \,, \tag{1.36}$$

while spinors transform as

$$\psi_{L,R} \quad \to \quad e^{\alpha\,\hat{a}/2}\, \psi_{L,R} \,. \tag{1.37}$$

In particular, if we perform a rotation through 2π, then $e^{\pm 2\pi\,\hat{a}/2} \;=\; -1$, and a vector is unchanged, while a spinor changes its sign, as it should be. Of course, this is because of the factor of 1/2 in the exponent (the famous "Rodrigues' two"), which is just the reflection of the fact that SU(2) is a double cover of the rotation group SO(3).

- Parity operator acting on spinors is $1|(-\hat{\imath})$. It sends ψ_L to $i\,\psi_R$ and ψ_R to $i\,\psi_L$. There are other normalizations for parity of course, this is just one example

2 Dirac equation

{section-dirac

Our goal in this section is to construct the Lagrangian for the electron in electromagnetic field, in quaternionic form, and derive the equation of motion — the Dirac equation.

Let us begin with a massless particle, in the absence of the electromagnetic field. The appropriate Lagrangian was given in [*thesis*],

$$\mathcal{L}_{\text{massless}} \quad = \quad \psi_L^\dagger\, i\partial\, \psi_L \quad + \quad \psi_R^\dagger\, i\, \partial^*\, \psi_R \quad + \quad \text{c.c.} \tag{2.1}$$ {L-massless}

Note that Lorentz invariance is manifest here, because

$$\psi_L^\dagger \;\to\; \psi_L^\dagger\, e^{\Lambda^\dagger} \;=\; \psi_L^\dagger\, e^{-\Lambda^*}\,, \qquad \partial \;\to\; e^{\Lambda^*}\,\partial\, e^{-\Lambda}\,, \qquad \psi_L \;\to\; e^{\Lambda}\,\psi_L\,, \tag{2.2}$$

and

$$\psi_R^\dagger \;\to\; \psi_R^\dagger\, e^{\widetilde{\Lambda}} \;=\; \psi_R^\dagger\, e^{-\Lambda}\,, \qquad \partial^* \;\to\; e^{\Lambda}\,\partial^*\, e^{-\Lambda^*}\,, \qquad \psi_R \;\to\; e^{\Lambda^*}\,\psi_R\,. \tag{2.3}$$

Although both $\boldsymbol{\psi_L}$ and $\boldsymbol{\psi_R}$ are parts of a single Dirac spinor

$$\boldsymbol{\psi_D \;=\; \psi_L \;+\; \psi_R\,,} \tag{2.4}$$

it does not seem to be possible to combine the two terms in Eq. (2.1) **into a single term just using $\boldsymbol{\psi_D}$, because of the different representations of the derivative $\boldsymbol{\partial}$ — should think about this.**

We need to make an important remark about conjugating products of spinors, due to the fact that spinors are Grasmann variables. By definition, complex conjugation of two Grassman variable interchanges their order,

$$(\zeta\,\eta)^* \;=\; \eta^*\,\zeta^*\,, \qquad\qquad \text{for Grassmann numbers.} \tag{2.5}$$

If we take two complex quaternions ξ and χ, which are fermions at the same time, complex conjugation cannot change their order, because of their quaternionic content. In that case, the order is preserved, but an extra minus sign appears,

$$(\xi\,\chi)^* \;=\; -\,\xi^*\,\chi^*\,, \qquad \text{for fermionic complex quaternions.} \tag{2.6}$$

If we take a quaternionic conjugate $\widetilde{\xi\,\chi}$, on the other hand, the conjugation will attempt to change their order precisely because of the quaternionic content. Note that the quaternionic algebra *requires* us to interchange the factors, or the result will simply be incorrect. But now because the spinors are fermions, and we are *not* performing a complex conjugation, we get an extra minus sign

$$\widetilde{\xi\,\chi} \;=\; -\,\widetilde{\chi}\,\widetilde{\xi} \qquad \text{for quaternionic spinors.} \tag{2.7}$$

The only kind of conjugation which does not produce a negative sign is hermitean conjugation — this combination changes the order of the spinors in agreement with both complex and quaternionic conjugations,

$$(\xi\,\chi)^\dagger \;=\; \chi^\dagger\,\xi^\dagger\,. \tag{2.8}$$

Let us discuss gauge transformations now, because we need the electron to interact with the electromagnetic field. Gauge transformations just rotate the overall complex phase of a spinor, and so they are defined similarly both for right- and left-handed spinors,

$$\psi_L \;\rightarrow\; e^{i\varphi}\,\psi_L\,, \qquad\qquad \psi_R \;\rightarrow\; e^{i\varphi}\,\psi_R\,. \tag{2.9}$$

Notice that the quaternionic conjugates also transform the same way,

$$\widetilde{\psi}_L \;\rightarrow\; \widetilde{\psi}_L\,e^{i\varphi}\,, \qquad\qquad \widetilde{\psi}_R \;\rightarrow\; \widetilde{\psi}_R\,e^{i\varphi}\,. \tag{2.10}$$

Although $e^{i\varphi}$ certainly commutes with $\widetilde{\psi}_{L,R}$, for convenience we wrote it on the right of the latter. Both the complex conjugates and hermitean conjugates will have the opposite charge,

$$\begin{aligned} \psi_L^* \;&\rightarrow\; \psi_L^*\,e^{-i\varphi}\,, \qquad\qquad & \psi_R^* \;&\rightarrow\; \psi_R^*\,e^{-i\varphi}\,, \\ \psi_L^\dagger \;&\rightarrow\; \psi_L^\dagger\,e^{-i\varphi}\,, \qquad\qquad & \psi_R^\dagger \;&\rightarrow\; \psi_R^\dagger\,e^{-i\varphi}\,. \end{aligned} \tag{2.11}$$

In order to make this transformation local, we define the long derivative,

$$\mathcal{D} \;=\; \partial \;-\; i\,A^*\,. \tag{2.12}$$

The reason that we have to put A^* here instead of just A is because ∂ and $\mathcal{D}$ are covariant vectors. This is just the reflection of the fact that A^μ enters the long derivative with the lower index μ:

$$\mathcal{D}_\mu \;=\; \partial_\mu \;-\; i\,A_\mu\,. \tag{2.13}$$

This long derivative then transforms as

$$\mathcal{D} \;\rightarrow\; e^{i\varphi}\,\mathcal{D}\,e^{-i\varphi}\,, \tag{2.14}$$

meaning that, as usual,

$$A_\mu \;\rightarrow\; A_\mu \;+\; \partial_\mu\,\varphi\,. \tag{2.15}$$

This allows $\mathcal{D}$ to act on ψ_L, so that $\mathcal{D}\,\psi_L$ is again in the fundamental representation of U(1). Now, although ψ_R has the same charge as ψ_L, we cannot act on it with the same derivative, because the product $\mathcal{D}\,\psi_R$ will not transform under Lorentz transformations properly. Remind, that the right-handed and left-handed spinor spaces are in fact related by complex conjugation. This was the reason that we wrote ∂^* in the Lagrangian in Eq. (2.1). One would think that by analogy we should act on ψ_R with $\mathcal{D}^*$ — but that would also be a mistake because it would imply that ψ_R has the opposite charge. In reality, it is the quaternionic conjugate $\widetilde{\mathcal{D}}$ that should be put into the Lagrangian. This conjugation does not change the sign of the electric charge. Overall, the Lagrangian now looks as,

$$\mathcal{L}_{\text{massless}} \;=\; \psi_L^\dagger\, i\mathcal{D}\,\psi_L \;+\; \psi_R^\dagger\, i\,\widetilde{\mathcal{D}}\,\psi_R \;+\; \text{c.c.} \qquad (2.16)$$ {L-gauge}

Before proceeding, let us emphasize the remarkable feature of the Lagrangians (2.1) and (2.16), which we could have done earlier: the absence of γ-matrices (or σ-matrices for that matter). The only residue of the matrix structure of the Dirac's Lagrangian is residing in the fact that the Lagragians have two chiral terms, instead of just one. We will return to this below.

Now we add the mass term. Since this has to be the Dirac mass, it has to flip chirality. The only form that correctly reproduces the mass term is $m\,\hat{\jmath}\,\psi_L^\dagger\,\psi_R$,

$$m\,\hat{\jmath}\,\psi_L^\dagger\,\psi_R \;+\; \text{q.c.} \;+\; \text{c.c} \;=\; m\,\left(\hat{\jmath}\,\psi_L^\dagger\,\psi_R \,+\, \widetilde{\psi}_R\,\psi_L^*\,\hat{\jmath}\right) \;+\; \text{c.c.} \qquad (2.17)$$

We will discuss the occurrence of $\hat{\jmath}$ in this expression in detail in Section 3. Its appearance seems ugly, and we will be able to get rid of it soon, after we discover its meaning. For now we just note that without it the expression would vanish, *i.e.* $\psi_L^\dagger\,\psi_R$ has no real part. Let us also note that this factor of $\hat{\jmath}$ can be moved to the right at our conveniece:

$$m\,\psi_L^\dagger\,\psi_R\,\hat{\jmath} \;+\; \text{q.c.} \;+\; \text{c.c.} \qquad (2.18)$$ {m-term}

This follows from the general cyclic property of quaternion product under the "q.c." sign, analogous to the cyclicity of trace of matrices[1] —

$$a\,b\,c \;+\; \text{q.c.} \;=\; b\,c\,a \;+\; \text{q.c.} \qquad (2.19)$$

The proof is simple — the real part of a product of *two* quaternions cannot depend on their order. From this follows the cyclicity. The fact that we are dealing with complexified quaternions cannot change this property.

[1]In fact, if one chooses to represent quaternions via Pauli matrices, the real part of a quaternion exactly corresponds to the trace of its matrix representation.

As we will see in Section 3, in the product $\psi_R\,\hat{\jmath}$ the factor $\hat{\jmath}$ "elevates" the right-handed spinor ψ_R to the left-handed space (we have used the term "elevates" because conventionally a left-handed spinor is written above right-handed spinor inside the column of a Dirac spinor). Importantly, the factor $\hat{\jmath}$ does not change the spinor's representation (it is obviously still transformed via multiplication by e^{Λ^*} on the left). Instead, the spinor just becomes expandable in the left-handed basis (1.27). We will use this when we define the *standard* representation for spinors below.

Written in terms of the components, the mass term (2.18) gives

$$\psi_L^\dagger\,\psi_R\,\hat{\jmath} \;+\; \text{q.c.} \;+\; \text{c.c.} \;=\; \xi_L^*\,\xi_R \;+\; \chi_L^*\,\chi_R \;+\; \xi_R^*\,\xi_L \;+\; \chi_R^*\,\chi_L\,, \tag{2.20}$$

as it should be for the Dirac mass term.

2.1 Dirac Lagrangian

Now we can derive the Dirac equation. Our starting point is the full Lagrangian

$$\mathcal{L}_{\text{Dirac}} \;=\; \psi_L^\dagger\,i\mathcal{D}\,\psi_L \;+\; \psi_R^\dagger\,i\,\widetilde{\mathcal{D}}\,\psi_R \;-\; \Big(m\,\psi_L^\dagger\,\psi_R\,\hat{\jmath} \;+\; \text{q.c.}\Big) \;+\; \text{c.c.}, \tag{2.21}$$ {L-massive}

where we remember that the mass term has to actually enter with a negative sign, and we consider the electromagnetic field to be fixed.

To derive the equations of motion, we vary the Lagrangian (2.21) with respect to $\psi_L^\dagger$ and $\psi_R^\dagger$. The reader may wonder at this point – how are we going to differentiate this Lagrangian with respect to quaternions, let alone complexified and fermionic? We postpone the formal answer to this question until Section 4. For now, we can just act intuitively, at least when differentiating with respect to $\psi_L^\dagger$. Indeed, in the terms where it is present in Eq. (2.21), it is sitting on the left, and so the left derivative gives

$$i\,\mathcal{D}\,\psi_L \;-\; m\,\psi_R\,\hat{\jmath} \;=\; 0\,. \tag{2.22}$$ {ldirac}

Loosely speaking, for quaternions $\partial\widetilde{q}/\partial q \;\neq\; 0$ — unlike for complex numbers, for which $\partial\bar{z}/\partial z \;=\; 0$. So it seems like there should be more terms on the left-hand side of Eq. (2.22). Why we can act so naively, and why the other terms do not contribute is, again, explained in Section 4. Here we provide an alternative and a more transparent justification, as follows. Let us for a moment pretend that we are dealing with a "quaternionic" Lagrangian

$$\psi_L^\dagger\,i\mathcal{D}\,\psi_L \;+\; m\,\psi_L^\dagger\,\psi_R\,\hat{\jmath} \tag{2.23}$$ {L-quat}

of which we will only want its real part. If we find the extremum of this Lagrangian, it will also extremize its real part. But the extremum of (2.23) is exactly given by Eq. (2.22).

In order to vary with respect to $\psi_R^\dagger$, we just re-write the mass term as

$$\mathcal{L}_{\text{Dirac}} \;\supset\; \Big(m\,\psi_R^\dagger \psi_L\,\hat{\jmath} \;+\; \text{q.c.} \Big) \;+\; \text{c.c.} \tag{2.24}$$

We did not make anything up here, as the first term here was just hidden inside the "q.c." and "c.c." in Eq. (2.21). Notice that this term enters with a *positive* sign.

Now we can vary the Langrangian with respect to $\psi_R^\dagger$, finding

$$i\,\widetilde{\mathcal{D}}\,\psi_R \;+\; m\,\psi_L\,\hat{\jmath} \;=\; 0\,. \tag{2.25}$$ {rdirac}

The two expressions (2.22) and (2.25) are the *quaternionic Dirac equations*.

2.2 Non-relativistic limit

Now let us derive the Schrödinger equation. We do it in the classical way [*BLP*], by taking the large-mass limit. First, we open up the long derivatives in Eqs. (2.22), (2.25),

$$\begin{aligned} i\,\partial_0\,\psi_L \;+\; A_0\,\psi_L \;-\; (\vec{\nabla} \;+\; i\,\vec{A})\,\psi_L \;-\; m\,\psi_R\,\hat{\jmath} \;&=\; 0\,, \\ i\,\partial_0\,\psi_R \;+\; A_0\,\psi_R \;+\; (\vec{\nabla} \;+\; i\,\vec{A})\,\psi_R \;+\; m\,\psi_L\,\hat{\jmath} \;&=\; 0\,. \end{aligned} \tag{2.26}$$ {dirac-split}

We can make one consistency check. For a free particle, the time derivative gives the energy, $i\,\partial_0 \;\to\; E$. If at the same time, it has zero momentum, then $E \;=\; m$, and we find

$$\begin{aligned} m\,\psi_L \;-\; m\,\psi_R\,\hat{\jmath} \;&=\; 0 \qquad \Longrightarrow \qquad \psi_L \;=\; +\,\psi_R\,\hat{\jmath}\,, \\ m\,\psi_R \;+\; m\,\psi_L\,\hat{\jmath} \;&=\; 0 \qquad \Longrightarrow \qquad \psi_R \;=\; -\,\psi_L\,\hat{\jmath}\,. \end{aligned} \tag{2.27}$$ {rest}

The two equations are consistent. What does the equality $\psi_L \;=\; \psi_R\,\hat{\jmath}$ mean? As we already mentioned, and as we shall see in Section 3, multiplying by $\hat{\jmath}$ on the right *rotates* the components of ψ_L and ψ_R into each other. That is, in Dirac spinor language, for a spinor

$$\psi_{\text{chiral}} \;=\; \begin{pmatrix} \psi_L^\alpha \\ \psi_{R\dot{\alpha}} \end{pmatrix} \tag{2.28}$$

this operation turns

$$\psi_L^\alpha \quad \to \quad -\,\psi_{R\dot\alpha}\,, \qquad\qquad \psi_{R\dot\alpha} \quad \to \quad \psi_L^\alpha\,. \tag{2.29}$$

Equation (2.27) then simply implies that the two spinors ψ_L^α and $\psi_{R\dot\alpha}$ are equal. This is correct, as a particle at rest is described by a single two-component spinor.

This is too crude a limit, we want to keep the right-handed and left-handed spinors different to the order $O(1/m)$, and momentum non-zero.

To do that, we, essentially, switch to the standard representation. Let us multiply the second equation in (2.26) by $\hat\jmath$ on the right:

$$\begin{aligned} i\,\partial_0\,\psi_L \;+\; A_0\,\psi_L \;-\; (\vec\nabla \;+\; i\,\vec A)\,\psi_L \;-\; m\,\psi_R\,\hat\jmath \;&=\; 0\,, \\ i\,\partial_0\,\psi_R\,\hat\jmath \;+\; A_0\,\psi_R\,\hat\jmath \;+\; (\vec\nabla \;+\; i\,\vec A)\,\psi_R\,\hat\jmath \;-\; m\,\psi_L \;&=\; 0\,. \end{aligned} \tag{2.30}$$ {dirac-j}

It is natural now to add and subtract these two equations, and introduce the notations

$$\psi_+ \;\equiv\; \zeta \;=\; \frac{\psi_L \,+\, \psi_R\hat\jmath}{\sqrt2}\,, \qquad\qquad \psi_- \;\equiv\; \eta \;=\; \frac{\psi_L \,-\, \psi_R\hat\jmath}{\sqrt2}\,. \tag{2.31}$$

As discussed in Section 3, these are precisely the upper and lower components of a Dirac spinor in the standard representation:

$$\psi_{\rm standard} \;=\; \begin{pmatrix} \zeta \\ \eta \end{pmatrix}. \tag{2.32}$$

In this section, we prefer to call them $\psi_\pm$. For a particle at rest, $\psi_- \;=\; 0$ as it should be. Equations (2.30) now read

$$\begin{aligned} i\,\partial_0\,\psi_+ \;+\; A_0\,\psi_+ \;-\; (\vec\nabla \;+\; i\,\vec A)\,\psi_- \;&=\; m\,\psi_+\,. \\ i\,\partial_0\,\psi_- \;+\; A_0\,\psi_- \;-\; (\vec\nabla \;+\; i\,\vec A)\,\psi_+ \;&=\; -\,m\,\psi_-\,. \end{aligned} \tag{2.33}$$ {dirac-standar

We now get rid of the mass term in the first equation in (2.33) via the redefinition

$$\psi_\pm \;\to\; e^{-imt}\,\psi_\pm\,.$$

The mass term is now gone in that equation:

$$i\,\partial_0\,\psi_+ \;=\; (\vec\nabla \;+\; i\,\vec A)\,\psi_- \;-\; A_0\,\psi_+\,, \tag{2.34}$$

while in the second equation the mass term doubles. Taking the large-mass limit we find

$$2m\,\psi_- \;=\; (\vec\nabla \;+\; i\,\vec A)\,\psi_+\,,$$

and so

$$i\,\partial_0\,\psi_+ \;=\; \frac{1}{2m}\,(\vec\nabla \;+\; i\,\vec A)^2\,\psi_+ \;-\; A_0\,\psi_+\,. \tag{2.35}$$

The final step now is to transform the quaternionic square

$$(\vec\nabla \;+\; i\,\vec A)^2$$

into the usual vector operators — scalar and vector products. Denoting the scalar-product square of a vector by figure brackets { }, one finds

$$(\vec\nabla \;+\; i\,\vec A)^2 \;=\; -\,\left\{\vec\nabla \;+\; i\,\vec A\right\}^2 \;+\; i\,\vec B\,,$$

in the operator form. We thus get

$$i\,\partial_0\,\psi_+ \;=\; -\,\frac{1}{2m}\left\{\vec\nabla \;+\; i\,\vec A\right\}^2\psi_+ \;+\; \frac{i\,\vec B}{2m}\,\psi_+ \;-\; A_0\,\psi_+\,. \tag{2.36}$$ {schrod}

Rescaling $A^\mu \;\to\; e\,A^\mu$, and switching to the Gaussian units, we arrive to

$$i\,\hbar\,\partial_t\,\psi_+ \;=\; \left(\frac{1}{2m}\left\{\vec p \;+\; \frac{e}{c}\,\vec A\right\}^2 \;-\; e\,A_0 \;+\; \frac{ie\hbar}{2mc}\,\vec B\right)\,\psi_+\,, \tag{2.37}$$ {pauli}

which is nothing but the *Pauli equation*. Notice once again that the σ-matrices are absent from equations (2.36) and (2.37). Spin is the intrinsic property of the wavefunction.

The factor of i in the magnetic moment term in Eq. (2.37) has a double rôle. First, it is needed by the correspondence

$$\vec\sigma \;\to\; i\,\hat\imath\,,\;\; i\,\hat\jmath\,,\;\; i\,\hat k\,,$$

because the σ-matrices are hermitean, and the quaternionic units are not. Second (and actually the same), by multiplying the magnetic field $\vec B$, it turns it from an axial vector into a polar vector $i\,\vec B$.

It is precisely the form of equation (2.37) that establishes both the quaternionic form of the Schrödinger equation and the fact that the Dirac Lagrangian (2.21) is correct. Equation (2.37) involves a single two-component spinor ψ_+. As a complex quaternion, ψ_+ only occupies a half of the quaternionic space, and in the most general form it can be written as

$$\psi_+ \;=\; \psi\, P_L\,.$$

In fact this restriction can always be imposed in the end. The Schrödinger equation (2.37) can be solved for an arbitrary ψ, and then projector P_L can be applied. The fact that the set of left-handed spinors forms an ideal guarantees that the projection will be a solution too.

The right-handed projection $\psi\, P_R$ will also be a solution, of course. It may either represent the same solution, or a different one. It can always be "moved" to the left-handed space by multiplying by $\hat{\jmath}$ on the right.

The magnetic moment term in Eq. (2.37) is the only signature of quaternions in that equation. Without it, the equation looks exactly like the ordinary Schrödinger equation,

$$i\,\hbar\,\partial_t\,\psi \;=\; \left(\frac{p^2}{2m} \,+\, V\right)\psi\,, \tag{2.38}$$ {simple-schrod

where we have dropped the gauge field for simplicity. For a complexified quaternion ψ, this equation falls apart into four identical complex equations. So the non-relativistic Quantum Mechanics essentially only needs complex numbers. We stress again that it is the spin of the particle that calls for the quaternionic appearance of the wavefunction.

- Current

3 Spinors

{section-spino

So how do we deduce the bases (1.27) and (1.32) for spinors? In this section we essentially expand on the construction introduced in [*thesis*]. We start from the correspondence

$$\sigma_1\,,\;\; \sigma_2\,,\;\; \sigma_3 \;\;\to\;\; i\,\hat{\imath}\,,\;\; i\,\hat{\jmath}\,,\;\; i\,\hat{k}\,, \tag{3.39}$$

for the Pauli matrices. This is a very natural association, and the only other reasonably alternative choice here would be a different sign on the right-hand side.

Next we build the spin-up and spin-down states, assuming the rest frame of reference. The operator of the canonical z-component of spin should be

$$\frac{\sigma^3}{2} \;\;\to\;\; \frac{i\,\hat{k}}{2}\,,$$

so the up- and down-states should satisfy

$$i\,\hat{k}\,\psi_\uparrow \;=\; +\,\psi_\uparrow \tag{3.40}$$

and

$$i\,\hat{k}\,\psi_\downarrow \;=\; -\,\psi_\downarrow\,. \tag{3.41}$$

Or, in other words,

$$(1\;\mp\;i\,\hat{k})\,\psi_{\uparrow,\downarrow} \;=\; 0\,. \tag{3.42}$$ {updown}

The reason that a product of two complex quaternions can vanish, is because the algebra of complex quaternions does not admit a positive-definite norm. We can still use the usual quaternionic norm,

$$\|\,a\,\|^2 \;=\; a_0^2 \;+\; a_1^2 \;+\; a_2^2 \;+\; a_3^2\,, \tag{3.43}$$ {qnorm}

ignoring the fact that a_0, ... are complex numbers. Such a norm will be multiplicative, but not positive-definite. The reason the product of the two factors in Eq. (3.42) vanishes is because the norm of at least one of those factors vanishes. Indeed, the norm of $1\;\mp\;i\,\hat{k}$ as calculated via Eq. (3.43) is zero.

So how about $\psi_{\uparrow,\downarrow}$? How many solutions can there be? These questions are easily answered by noticing that objects $1\;+\;i\,\hat{k}$ and $1\;-\;i\,\hat{k}$ are, in fact, projectors:

$$\begin{aligned} P_L &= \frac{1\;+\;i\,\hat{k}}{2}\,, & P_L^2 &= P_L\,, \\ P_R &= \frac{1\;-\;i\,\hat{k}}{2}\,, & P_R^2 &= P_R\,, \end{aligned} \tag{3.44}$$

which we judiciously have named them left- and right- projectors. Indeed, by multiplying a quaternion on the left (or equally well, on the right) by P_L, we are obviously performing a linear operation upon the components of that quaternion. The square of such an operation equals the operation itself $\Rightarrow$ the operation is a projection. Obviously the same is true for P_R.

That means that if we run ψ through all values of complex quaternions, the product $P_L\psi$ will span only a portion of the quaternionic space — an ideal. What fraction of the entire algebra does it span? We notice that P_L and P_R are complementary projectors:

$$P_R \;=\; P_L^*\,, \qquad\qquad P_L \;+\; P_R \;=\; 1\,. \tag{3.45}$$

Because they are symmetric, they can only project equal-size subsets of the algebra. And since they are complementary to each other, the union of those subsets must comprise the entire algebra. In other words, P_L and P_R split the algebra in two halves.

Now, equation (3.42) can re-written as,

$$P_{R,L}\,\psi_{\uparrow,\downarrow} \quad = \quad 0\,, \tag{3.46}$$ {PRL}

meaning that the most general $\psi_\uparrow$ must sit in one half of the algebra, while the most general $\psi_\downarrow$ must sit in the other half. Since a generic complex quaternion has four complex components, there are two complex solutions for $\psi_\uparrow$ and as many for $\psi_\downarrow$. We have to stress here that these halves are *not identified* with the left- and right-handed chiral spaces. For this reason we have not given names to these supspaces, other than "spin-up" and "spin-down" spaces.

The easiest solution to (3.46) is using the orthogonality of the projectors:

$$P_R\,P_L \quad = \quad 0\,, \qquad\qquad P_L\,P_R \quad = \quad 0\,. \tag{3.47}$$

So, seemingly P_L could be identified with a spin-up state, and P_R — with the corresponding spin-down state. However, these are states of different chiralities. Indeed, since $P_L \;=\; P_R^*$, such "spinors" transform in the mutually-conjugate representations of the Lorentz group. This fact by puts them into the opposite chirality spaces.

Let us summarize our goal and achivements now. We are looking for four complex states $\psi_{L\uparrow}$, $\psi_{L\downarrow}$, $\psi_{R\uparrow}$ and $\psi_{R\downarrow}$. We have already found

$$\psi_{L\uparrow} \quad = \quad P_L\,, \qquad\qquad \psi_{R\downarrow} \quad = \quad P_R\,. \tag{3.48}$$

It is not difficult to find the other two states. For example, if one wishes to avoid pure guessing, which would perfectly work here too, we know that matrix $-i\sigma^2$ turns a spin-up state into a spin-down state:

$$-i\,\sigma^2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad = \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This exactly corresponds to multiplying by $\hat{\jmath}$ on the left, and we arrive to

$$\psi_{L\downarrow} \quad = \quad \hat{\jmath}\,P_L\,, \qquad\qquad \psi_{R\uparrow} \quad = \quad -\,\hat{\jmath}\,P_R\,. \tag{3.49}$$

So we recap that the *left-handed* spinors can be written as

$$\psi_L \quad = \quad \xi_L \, P_L \quad + \quad \chi_L \, \hat{\jmath} \, P_L \, , \tag{3.50}$$ {lbasis-again}

while the *right-handed* spinors are represented as

$$\psi_R \quad = \quad - \, \xi_R \, \hat{\jmath} \, P_R \quad + \quad \chi_R \, P_R \, . \tag{3.51}$$ {rbasis-again}

A few important comments are in order here. The chiral subspaces are defined here by multiplying arbitrary quaterions by projectors P_L or P_R on the *right*. In other words, $\psi \, P_L$ spans the set of all left-handed spinors, and $\psi \, P_R$ of all right-handed. This way, right multiplication splits the set of quaternions into the two chirality subspaces. Whereas, *left* multiplication splits the quaternions into spin-up and spin-down subspaces, because *e.g.*

$$P_R \, (P_L \, \psi) \quad = \quad 0 \, .$$

It is a trivial fact now that a set of the type $\{ \, P_L \, \psi \, \}$ or $\{ \, \psi \, P_L \, \}$ forms a left (right) ideal, since for an arbitrary quaternion a, the product, say,

$$a \, (\psi \, P_L) \quad = \quad (a \, \psi) \, P_L \, ,$$

resides in the same subspace as $\psi \, P_L$.

The other important remark is about the amount of freedom that we have in defining the bases via Eqs. (3.50), (3.51). We have freedom in choosing the component of the spin to be measurable — for which we chose $\hat{k}$. The other freedom was in parametrizing the spin-down component of ψ_L, for which we chose $\hat{\jmath}$ as an orthogonal direction. Overall, we could have chosen any two unit vectors $\hat{a}$ and $\hat{b}$ in place of $\hat{k}$ and $\hat{\jmath}$, subject to the only restriction

$$\hat{a} \cdot \hat{b} \quad = \quad 0 \, .$$

- Parity
- Cl(2) $\otimes$ Cl(2)

4 Differentiation

{section-diff}

5 Conclusions

Acknowledgments

References

[1] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150].

[2] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [hep-th/0403149].

[3] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].

[4] D. Tong, Annals Phys. **324**, 30 (2009) [arXiv:0809.5060 [hep-th]]; M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170]; K. Konishi, Lect. Notes Phys. **737**, 471 (2008) [arXiv:hep-th/0702102]; M. Shifman and A. Yung, *Supersymmetric Solitons,* (Cambridge University Press, 2009).

[5] E. Witten, Nucl. Phys. B **149**, 285 (1979).

[6] E. Witten, Nucl. Phys. B **403**, 159 (1993) [hep-th/9301042].

[7] A. D'Adda, P. Di Vecchia and M. Lüscher, Nucl. Phys. B **152**, 125 (1979).

[8] M. Shifman and A. Yung, Phys. Rev. D **77**, 125017 (2008) [arXiv:0803.0698 [hep-th]].

[9] K. Hori and C. Vafa, *Mirror symmetry,* arXiv:hep-th/0002222.

[10] E. Frenkel and A. Losev, Commun. Math. Phys. **269**, 39 (2006) [arXiv:hep-th/0505131].

[11] M. Edalati and D. Tong, JHEP **0705**, 005 (2007) [arXiv:hep-th/0703045].

[12] M. Shifman and A. Yung, Phys. Rev. D **77**, 125016 (2008) [arXiv:0803.0158 [hep-th]].

[13] P. A. Bolokhov, M. Shifman and A. Yung, Phys. Rev. D **79**, 085015 (2009) (Erratum: Phys. Rev. D 80, 049902 (2009)) [arXiv:0901.4603 [hep-th]].

[14] P. A. Bolokhov, M. Shifman and A. Yung, Phys. Rev. D **79**, 106001 (2009) (Erratum: Phys. Rev. D 80, 049903 (2009)) [arXiv:0903.1089 [hep-th]].

[15] P. A. Bolokhov, M. Shifman and A. Yung, Phys. Rev. D **81**, 065025 (2010) [arXiv:0907.2715 [hep-th]].

[16] E. Witten, Phys. Rev. D **16**, 2991 (1977); P. Di Vecchia and S. Ferrara, Nucl. Phys. B **130**, 93 (1977).

[17] B. Zumino, Phys. Lett. B **87**, 203 (1979).

[18] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. **116**, 103 (1984).

[19] A. M. Perelomov, Phys. Rept. **174**, 229 (1989).

[20] E. Witten, Nucl. Phys. B **202**, 253 (1982).

[21] L. Alvarez-Gaumé and D. Z. Freedman, Commun. Math. Phys. **91**, 87 (1983); S. J. Gates, Nucl. Phys. B **238**, 349 (1984); S. J. Gates, C. M. Hull and M. Roček, Nucl. Phys. B **248**, 157 (1984).

[22] A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).

[23] A. Ritz, M. Shifman and A. Vainshtein, Phys. Rev. D **66**, 065015 (2002) [arXiv:hep-th/0205083].

[24] J. Wess and J. Bagger, *Supersymmetry and Supergravity*, Second Edition, Princeton University Press, 1992.

[25] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[26] N. Dorey, JHEP **9811**, 005 (1998) [hep-th/9806056].

[27] E. Witten, *Two-dimensional models with (0,2) supersymmetry: Perturbative aspects,* arXiv:hep-th/0504078.

[28] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **73**, 065011 (2006) [hep-th/0512153].

[29] S. R. Coleman, Annals Phys. **101**, 239 (1976).

[30] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71**, 045010 (2005) [hep-th/0412082].

[31] F. Ferrari, JHEP **0205** 044 (2002) [hep-th/0202002].

[32] F. Ferrari, Phys. Lett. **B496** 212 (2000) [hep-th/0003142]; JHEP **0106**, 057 (2001) [hep-th/0102041].

[33] A. D'Adda, A. C. Davis, P. DiVeccia and P. Salamonson, Nucl. Phys. **B222** 45 (1983).

[34] S. Cecotti and C. Vafa, Comm. Math. Phys. **158** 569 (1993) [hep-th/9211097].

[35] A. Hanany and K. Hori, Nucl. Phys. B **513**, 119 (1998) [arXiv:hep-th/9707192].

[36] P. C. Argyres and M. R. Douglas, Nucl. Phys. **B448**, 93 (1995) [arXiv:hep-th/9505062].

[37] P. C. Argyres, M. R. Plesser, N. Seiberg, and E. Witten, Nucl. Phys. **B461**, 71 (1996) [arXiv:hep-th/9511154].

[38] M. Shifman and A. Yung, Rev. Mod. Phys. **79** 1139 (2007) [arXiv:hep-th/0703267].

[39] D. Tong, JHEP **0709**, 022 (2007) [arXiv:hep-th/0703235].

[40] G. Veneziano and S. Yankielowicz, Phys. Lett. B **113**, 231 (1982).

[41] A. Losev and M. Shifman, Phys. Rev. D **68**, 045006 (2003) [arXiv:hep-th/0304003].

[42] M. Shifman, A. Vainshtein and R. Zwicky, J. Phys. A **39**, 13005 (2006) [arXiv:hep-th/0602004].

[43] M. Shifman and A. Yung, Phys. Rev. D **81**, 105022 (2010) [arXiv:0912.3836 [hep-th]].

[44] J. Distler and S. Kachru, Nucl. Phys. B **413**, 213 (1994) [arXiv:hep-th/9309110].

[45] T. Kawai and K. Mohri, Nucl. Phys. B **425**, 191 (1994) [arXiv:hep-th/9402148].

[46] I. V. Melnikov, JHEP **0909**, 118 (2009) [arXiv:0902.3908 [hep-th]].

[47] M. Shifman and A. Yung, Phys. Rev. D **79**, 105006 (2009) [arXiv:0901.4144 [hep-th]].

[48] D. Tong, JHEP **0612**, 051 (2006) [arXiv:hep-th/0610214].

[49] A. Migdal and M. Shifman, Phys. Lett. B **114**, 445 (1982).

[50] A. Kovner and M. A. Shifman, Phys. Rev. D **56**, 2396 (1997) [arXiv:hep-th/9702174].