Appendix B: YOUNG TABLEAUX AND IRREDUCIBLE TENSORS OF THE LORENTZ GROUP

To build irreducible tensors of an arbitrary rank one can use Young tableaux. We describe here a recipe how to expand a tensor of a specific rank into its irreducible components. In the text we most extensively exploit rank three tensors, which we here now use as a non-trivial but rather simple example.

For a tensor of rank r one builds all possible numbered Young tableaux consisting of r boxes. For each tableau one builds an irreducible component by (anti)symmetrizing its indices as described below. After that, to make a component truly irreducible, one has to subtract from it all its $g^{\mu\nu}$-traces.

For each numbered diagram, one builds a tensor such that each number corresponds to an index (*e.g.* for $T^{\mu\nu\rho}$, one could identify $1 \to \mu$, $2 \to \nu$, $3 \to \rho$). Indices whose numbers form horizontal rows in the diagram are symmetrized. Indices which form vertical columns are antisymmetrized. Symmetrization always occurs with respect to the name of the index. Antisymmetrization is always done with respect to the position of the index (in this case the number not always corresponds to one and the same index).

As an illustration to what have been said, we build the diagrams for a tensor $T^{\mu\nu\rho}$. One finds four different Young diagrams which can be built out of three boxes:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \tag{B1}$$

The first diagram corresponds to an absolutely symmetric component of the tensor:

$$\begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} \longrightarrow S^{\mu\nu\rho} = T^{(\mu\nu\rho)} = T^{\mu\nu\rho} + T^{\nu\rho\mu} + T^{\rho\mu\nu} + T^{\mu\rho\nu} + T^{\rho\nu\mu} + T^{\nu\mu\rho} \,.$$

The second diagram is the absolutely antisymmetric component:

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \longrightarrow A^{\mu\nu\rho} = T^{[\mu\nu\rho]} = T^{\mu\nu\rho} + T^{\nu\rho\mu} + T^{\rho\mu\nu} - T^{\mu\rho\nu} - T^{\rho\nu\mu} - T^{\nu\mu\rho} \,.$$

The two "corner" diagrams generate, correspondingly,

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array} \longrightarrow T_1^{\mu\nu\rho} = T^{\mu\nu\rho} - T^{\rho\nu\mu} + T^{\nu\mu\rho} - T^{\rho\mu\nu} \,,$$

and

$$\begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \longrightarrow T_2^{\mu\nu\rho} = T^{\mu\nu\rho} - T^{\nu\mu\rho} + T^{\rho\nu\mu} - T^{\nu\rho\mu} \,.$$

To provide a slightly more complicated example of applying the (anti)symmetrization rules we demonstrate a diagram corresponding to an irreducible component of a four-tensor:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \quad \longrightarrow \quad T^{[\mu\nu\rho]\lambda} \; + \; T^{[\lambda\nu\rho]\mu} \; .$$

All four components of (B1) (weighed by appropriate coefficients) sum into the original tensor $T^{\mu\nu\rho}$:

$$T^{\mu\nu\rho} \; = \; \frac{1}{3!} \left(S^{\mu\nu\rho} \; + \; A^{\mu\nu\rho} \; + \; 2T_1^{\mu\nu\rho} \; + 2T_2^{\mu\nu\rho} \right) \; .$$

The last step to perform is subtract from each component all traces obtained by contraction of any two indices which are not antisymmetrized (contraction of antisymmetrized indices is trivial). The solution can be sought by means of a tensor of a rank $r-2$:

$$T_{i\,(\mathrm{irr})}^{\mu\nu\rho\lambda\ldots} \; = \; T_i^{\mu\nu\rho\lambda\ldots} \; - \; a^{\rho\lambda\ldots}g^{\mu\nu} \; + \; a^{\rho\mu\ldots}g^{\lambda\nu} \; + \; \ldots \; , \tag{B2}$$

where $T_i^{\mu\nu\rho\lambda\ldots}$ is the i-the component obtained from the corresponding Young tableau. The traces part in the r.h.s. of (B2) should possess the same symmetries as $T_i^{\mu\nu\rho\lambda\ldots}$ so as to promote these symmetries to the l.h.s. Contracting any two indices in equation (B2) and requiring the result to vanish one can obtain the explicit expression for the trace $a^{\rho\lambda\ldots}$.

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