1. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} + u_x + u = 0.$$

2. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} - u_{zz} = 0.$$

   Write its general solution.

3. Check that the general solution of the equation

$$\partial_x \left( \frac{xy}{x^2+y^2} \partial_x u \right) + \frac{1}{2} \partial_x \left( \frac{x^2-y^2}{x^2+y^2} \partial_y u \right) + \frac{1}{2} \partial_y \left( \frac{x^2-y^2}{x^2+y^2} \partial_x u \right) - \partial_y \left( \frac{xy}{x^2+y^2} \partial_y u \right) = 0$$

   is given by

$$u(x,y) = F(x^2-y^2) + G(xy).$$

4. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = \theta_H(1-x), \quad u(0,x) = 0, \quad u_t(0,x) = 0, \quad u(t,0) = 0,$$

   where $\theta_H$ is the Heaviside theta function:

$$\theta_H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \\ \frac{1}{2}, & x = 0. \end{cases}$$

   For simplicity, you may consider its solution only for $t > 4$.

5. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = 0, \quad u(0,x) = 0, \quad u_t(0,x) = \theta_H(x-3) - \theta_H(x-2), \quad u(t,0) = 0.$$

   For simplicity, you may again consider only the large time case.

6. Solve the wave equation on the half-line:

$$u_{tt} - u_{xx} = 0, \quad u(0,x) = 0, \quad u_t(0,x) = 0, \quad u(t,0) = \sin t.$$

7. Compute the Fourier transform of the function $f$ on $\mathbb{Z}/(2N)\mathbb{Z}$ defined by

$$f(0) = 0, \quad f(1) = \ldots = f(N-1) = 1, \quad f(N) = 0, \quad f(-1) = \ldots = f(1-N) = -1.$$

8. Compute the Fourier transform of the function $f(\phi)$ on $S^1$ defined by

$$f(\phi) = \left[ \begin{array}{rl} -1, & \phi \in (-\pi, 0) \\ 1, & \phi \in (0, \pi) \end{array} \right.$$

9. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\epsilon|p|}$$

   defined on the real line.

10. Compute the Fourier transform of

$$f_\epsilon(x) = \frac{x}{\pi(x^2+\epsilon^2)}$$

defined on the real line. Compute the integral

$$\int_{-\infty}^{\infty} f_\epsilon(x)dx$$

11. Compute the Fourier transform of

$$f(n) = e^{-\epsilon|n|}$$

defined on $\mathbb{Z}$.

12. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\frac{1}{2}\epsilon p^2}$$

What is the value of its integral from $-\infty$ to $+\infty$?

13. Compute the Fourier transform of the function on $S^1$

$$f_\epsilon(\phi) = \frac{\sinh\epsilon}{\cosh\epsilon - \cos\phi}.$$

Hint: for computing the integral $\int_0^{2\pi}\frac{d\phi}{2\pi}f_\epsilon(\phi)e^{-in\phi}$, introduce the new variable $z = e^{i\phi}$ and find an appropriate way to deform the integration contour.
What is $\int_0^{2\pi} f_\epsilon(\phi)d\phi$?

14. Solve the heat equation

$$u_t = u_{xx}, \quad u(0,x) = e^{-ax^2}$$

15. Compute the Fourier transform of

$$f(x) = xe^{-x^2}$$

and of

$$f(x) = x^2e^{-x^2}$$

16. Find retarded Green function for the equation

$$u''(t) + \gamma u'''(t) = f(t), \quad \gamma > 0.$$

It describes effective radiation friction in the electrodynamics. Compare this Green function with $\gamma = 0$ case.

17. Solve the wave equation

$$u_{tt}(t,\vec{r}) - \Delta u(t,\vec{r}) = \delta(\vec{r}), \quad u(0,\vec{r}) = 0, \quad u(0,\vec{r}) = 0$$

for the following space dimensions:

(a) $d = 1$,

(b) $d = 2$, in this case study the asymptotics of solution for $t \to \infty$ with $\vec{r}$ fixed.

(c) $d = 3$.

18. Solve the heat equation $u_t - u_{xx} = 0$ with different initial conditions

    (a) $u(0, x) = \sin x$,

    (b) $u(0, x) = e^{-x^2} \sin x$.

19. Solve the wave equation

$$u_{tt} - u_{xx} = 0, \quad u_x(t, 0) = u_x(t, \pi) = 0, \quad u(0, x) = \sin^2 x, \quad u_t(0, x) = 0.$$

20. Solve the wave equation

$$\begin{gathered} u_{tt} - u_{xx} - u_{yy} = 0, \\ u(t, 0, y) = u(t, \pi, y) = u(t, x, 0) = u(t, x, \pi) = 0, \\ u(0, x, y) = \sin x \sin y(\cos x - \cos y), \quad u_t(0, x, y) = 0. \end{gathered}$$

21. Solve Laplace equation

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(x, 0) = 0, \quad u(\pi, y) = \sin y, \quad u(x, \pi) = \sin x.$$