1. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} + u_x + u = 0.$$

2. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} - u_{zz} = 0.$$

   Write its general solution.

3. Check that the general solution of the equation

$$\partial_x \left(\frac{xy}{x^2+y^2} \partial_x u \right) + \frac{1}{2} \partial_x \left(\frac{x^2-y^2}{x^2+y^2} \partial_y u \right) + \frac{1}{2} \partial_y \left(\frac{x^2-y^2}{x^2+y^2} \partial_x u \right) - \partial_y \left(\frac{xy}{x^2+y^2} \partial_y u \right) = 0$$

   is given by

$$u(x, y) = F(x^2 - y^2) + G(xy).$$

4. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = \theta_H(1-x), \quad u(0, x) = 0, \quad u_t(0, x) = 0, \quad u(t, 0) = 0,$$

   where θ_H is the Heaviside theta function:

$$\theta_H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \\ \frac{1}{2}, & x = 0. \end{cases}$$

   For simplicity, you may consider its solution only for $t > 4$.

5. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = 0, \quad u(0, x) = 0, \quad u_t(0, x) = \theta_H(x-3) - \theta_H(x-2), \quad u(t, 0) = 0.$$

   For simplicity, you may again consider only the large time case.

6. Solve the wave equation on the half-line:

$$u_{tt} - u_{xx} = 0, \quad u(0, x) = 0, \quad u_t(0, x) = 0, \quad u(t, 0) = \sin t.$$

7. Compute the Fourier transform of the function f on $\mathbb{Z}/(2N)\mathbb{Z}$ defined by

$$f(0) = 0, \quad f(1) = \ldots = f(N-1) = 1, \quad f(N) = 0, \quad f(-1) = \ldots = f(1-N) = -1.$$

8. Compute the Fourier transform of the function $f(\phi)$ on S^1 defined by

$$f(\phi) = \left[\begin{array}{ll} -1, & \phi \in (-\pi, 0) \\ 1, & \phi \in (0, \pi) \end{array} \right.$$

9. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\epsilon|p|}$$

   defined on the real line.

10. Compute the Fourier transform of

$$f_\epsilon(x) = \frac{x}{\pi(x^2 + \epsilon^2)}$$

defined on the real line. Compute the integral

$$\int_{-\infty}^{\infty} f_\epsilon(x)dx$$

11. Compute the Fourier transform of

$$f(n) = e^{-\epsilon|n|}$$

defined on $\mathbb{Z}$.

12. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\frac{1}{2}\epsilon p^2}$$

What is the value of its integral from $-\infty$ to $+\infty$?

13. Compute the Fourier transform of the function on S^1

$$f_\epsilon(\phi) = \frac{\sinh\epsilon}{\cosh\epsilon - \cos\phi}.$$

Hint: for computing the integral $\int_0^{2\pi} \frac{d\phi}{2\pi} f_\epsilon(\phi)e^{-in\phi}$, introduce the new variable $z = e^{i\phi}$ and find an appropriate way to deform the integration contour.
What is $\int_0^{2\pi} f_\epsilon(\phi)d\phi$?

14. Solve the heat equation

$$u_t = u_{xx}, \quad u(0,x) = e^{-ax^2}$$

15. Compute the Fourier transform of

$$f(x) = xe^{-x^2}$$

and of

$$f(x) = x^2e^{-x^2}$$

16. Find retarded Green function for the equation

$$u''(t) + \gamma u'''(t) = f(t), \quad \gamma > 0.$$

It describes effective radiation friction in the electrodynamics. Compare this Green function with $\gamma = 0$ case.

17. Solve the wave equation

$$u_{tt}(t,\vec{r}) - \Delta u(t,\vec{r}) = \delta(\vec{r}), \quad u(0,\vec{r}) = 0, \quad u(0,\vec{r}) = 0$$

for the following space dimensions:

(a) $d = 1$,

(b) $d = 2$, in this case study the asymptotics of solution for $t \to \infty$ with $\vec{r}$ fixed.

(c) $d = 3$.

18. Solve the heat equation $u_t - u_{xx} = 0$ with different initial conditions

 (a) $u(0, x) = \sin x$,

 (b) $u(0, x) = e^{-x^2} \sin x$.

19. Solve the wave equation

$$u_{tt} - u_{xx} = 0, \quad u_x(t, 0) = u_x(t, \pi) = 0, \quad u(0, x) = \sin^2 x, \quad u_t(0, x) = 0.$$

20. Solve the wave equation

$$\begin{gathered} u_{tt} - u_{xx} - u_{yy} = 0, \\ u(t, 0, y) = u(t, \pi, y) = u(t, x, 0) = u(t, x, \pi) = 0, \\ u(0, x, y) = \sin x \sin y(\cos x - \cos y), \quad u_t(0, x, y) = 0. \end{gathered}$$

21. Solve the Laplace equation

$$u_{xx} + u_{yy} = 0, \quad u(0, y) = u(x, 0) = 0, \quad u(\pi, y) = \sin y, \quad u(x, \pi) = \sin x.$$

22. Rewrite in polar coordinates (in the 2d case) and in spherical coordinates (in the 3d case) the following expressions:

 (a) 2d: $(\vec{\nabla} u)^2 = (\partial_x u)^2 + (\partial_y u)^2$

 (b) 3d: $(\vec{\nabla} u)^2 = (\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2$

 (c) 2d: $\Delta u = \partial_{xx} u + \partial_{yy} u$

 (d) 3d: $\Delta u = \partial_{xx} u + \partial_{yy} u + \partial_{zz} u$

 Definitions of coordinates systems:

 - Polar: $x = r \cos \phi$, $y = r \sin \phi$.
 - Spherical: $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.

23. Solve the Laplace equation in the unit disk:

$$\Delta u(r, \phi) = 0, \qquad r \leq 1, \qquad u(1, \phi) = \sin 2\phi.$$

 Express this solution in terms of x, y. Is it obvious why it is a solution?

24. Solve the Laplace equation outside the unit disk:

$$\Delta u(r, \phi) = 0, \qquad r \geq 1, \qquad u(1, \phi) = \cos \phi.$$

25. Find the Green function of the 2d Laplace operator in the following domains with zero boundary conditions:

 (a) $\Im z \in [0, \pi]$ (hint: map it to the upper half-plane using e^z)

 (b) $\arg z \in [0, \pi/a]$ (hint: map it to the upper half-plane using z^a)

 (c) $\Im z \in [0, \pi]$, $\Re z \geq 0$ (hint: map it to the upper half-plane using $\cosh z$)

(d) $\Im z \geq 0$, $|z| \geq 1$ (hint: reduce it to the previous case by taking $\log z$)

(e) $\{z | \Im z \geq 0\} \setminus \{z | \Re z = 0, \Im z \in [0, h]\}$ (hint: map it to the upper half-plane using $\sqrt{z^2 + h^2}$)

26. Compute Legendre polynomials up to $P_3(x)$ by expanding

$$\frac{1}{\sqrt{1 - 2tx + t^2}} = \sum_{n=0}^{\infty} t^n P_n(x).$$

27. Solve the Laplace equation in the unit ball

$$\Delta u(r, \theta, \phi) = 0, \qquad r \leq 1, \qquad u(1, \theta, \phi) = \cos\theta.$$

Rewrite this solution in Cartesian coordinates.

28. Solve the Laplace equation outside the unit ball

$$\Delta u(r, \theta, \phi) = 0, \qquad r \geq 1, \qquad u(1, \theta, \phi) = \cos\theta.$$

29. Solve the Laplace equation inside and outside the unit ball

$$\Delta u(x, y, z) = 0, \qquad u(x, y, z)|_{x^2+y^2+z^2=1} = xy.$$

30. Spherical Bessel functions are defined by $j_l(r) = \sqrt{\frac{1}{2r}} J_{l+\frac{1}{2}}(r)$. Compute $j_1(r)$ and $j_{-1}(r)$. Hint: use the explicit formula for $J_{\frac{1}{2}}(r)$ and the recurrence relation.

31. Compute the Fourier transform of $j_0(r)$ and of $J_0(r)$.

32. Expand as the Fourier series in ϕ:

$$e^{ir\cos\phi} = \sum_{n\in\mathbb{Z}} c_n(r) e^{in\phi}.$$