29 February 2023

- It is enough to solve 4 problems to get maximal result.
- You have 3 hours.

1. Put to the canonical form
$$u_{xy} + u_{yy} + u_y = 0.$$
2. Solve the wave equation on $\mathbb{R}^{1+1}$:
$$u_{tt} - u_{xx} = 0, \qquad u(0,x) = 0, \qquad u_t(0,x) = 4(1-|x|)\theta_H(1-|x|).$$
Draw its solution, for example, for $t = 5$.
3. Solve the heat equation on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $x, t \geq 0$:
$$u_t = u_{xx}, \qquad u(0,x) = xe^{-x^2}, \qquad u(t,0) = 0.$$
4. Find the Green function of the Poisson equation in the upper half plane $y \geq 0$ with Neumann boundary conditions:
$$u_{xx}+u_{yy} = \delta(x-x_0)\delta(y-y_0), \qquad u_y(x,0) = 0, \qquad \lim_{x^2+y^2\to\infty} u(x,y) - \frac{1}{2\pi}\log(x^2+y^2) = 0.$$
5. Find the Fourier transform on $\mathbb{R}$ of the function
$$\frac{1}{1+p^4}.$$
How smooth is it? What is its behavior at infinity?
6. Solve 3d Laplace equation in the unit ball:
$$\Delta u(r,\theta,\phi) = 0, \qquad u(1,\theta,\phi) = \sin\theta\sin\phi.$$
7. Solve 1d difference equation
$$u(n+1) = u(n-1) + u(n), \qquad u(0) = a, \quad u(1) = b.$$