

29 April 2025

- It is enough to solve 4 problems to get the maximal result.
- You have 3 hours.

1. Put to the canonical form

$$u_{xx} + u_{yy} + u_{xy} + u_y = 0.$$

2. Solve the wave equation on \mathbb{R}^{1+1} :

$$u_{tt} - u_{xx} = 0, \quad u(0, x) = e^{-x^2} \cos ax^2, \quad u_t(0, x) = 0.$$

3. Solve the heat equation on \mathbb{R}^{1+1} :

$$u_t = u_{xx}, \quad u(0, x) = e^{-x^2} \cos ax^2, \quad a > 0.$$

4. Find the Green function of the Poisson equation

$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0)$$

in the domain

$$D = \{(x, y) | x \geq y \geq 0\}$$

with the boundary conditions $u(x, y) = 0$ for $(x, y) \in \partial D$.

5. Find the Fourier transform on \mathbb{R} of the function

$$\frac{1}{x^2 + x + 1}.$$

How smooth is it? What is its behavior at infinity? Why?

6. Solve 3d Laplace equation outside the unit ball¹ $r \geq 1$

$$\Delta u(r, \theta, \phi) = 0$$

with the boundary condition $u(1, \theta, \phi) = \cos 2\theta$.

7. Find the Fourier series for the function $f(\phi)$ on S^1

$$f(\phi) = |\phi|, \quad \phi \in [-\pi, \pi], \quad -\pi \sim \pi.$$

What is the behavior of its coefficients at infinity?

¹Spherical coordinates are introduced by the standard formulas, $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$.