15 April 2024

- It is enough to solve 4 problems to get maximal result.
- You have 3 hours.

1. Put to the canonical form
$$2u_{xx} + 2u_{xy} + u_{yy} + 2u_x + u = 0.$$

2. Solve the wave equation on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $x, t \geq 0$:
$$u_{tt} - u_{xx} = 0, \qquad u(t,0) = \theta_H(1-t), \qquad u(0,x) = \theta_H(1-x), \qquad u_t(0,x) = 0.$$
Draw its solution, for example, for $t = 5$.

3. Solve the heat equation on $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$, $x, t \geq 0$:
$$u_t = u_{xx}, \qquad u(0,x) = e^{-(x-1)^2}, \qquad u(t,0) = 0.$$

4. Solve Poisson equation with the source in the center of the infinite strip:
$$u_{xx} + u_{yy} = \delta(x)\delta(y), \qquad u(x,1) = u(x,-1) = 0, \qquad \lim_{|x|\to\infty} u(x,y) = 0.$$

5. Find the Fourier transform on $\mathbb{R}$ of the function
$$\frac{1}{(p - \mathrm{i} - 1)(p - \mathrm{i})(p - \mathrm{i} + 1)}.$$
How smooth is it? What is its behavior at infinity?

6. Solve 3d Laplace equation outside the unit ball:
$$\Delta u(r,\theta,\phi) = 0, \qquad u(1,\theta,\phi) = \cos^2\theta, \qquad \lim_{r\to\infty} u(r,\theta,\phi) = 0.$$

7. Solve 1d difference equation
$$u(n+1) = 2u(n-1) - u(n), \qquad u(0) = a, \quad u(1) = b.$$