1. Solve wave equation $u_{tt} - u_{xx} = 0$ on $\mathbb{R}$ with the following initial conditions

$$u_t(0,x) = |1-x^2|\theta_H(1-x^2), \qquad u(0,x) = 0,$$

   where $\theta_H(x) = 0$ for $x \le 0$, $\theta_H(x) = 1$ for $x > 0$.

   Plot this solution for some values of t.

2. Solve wave equation $u_{tt} - u_{xx} = 0$ on $\mathbb{R}_{>0}$ with the following initial and boundary conditions

$$u_t(0,x) = 0, \qquad u(0,x) = \sin^2(2\pi x)\theta_H(1-|x-2|), \quad u(t,0) = \frac{1}{2}\sin^2(2\pi t)\theta_H(1-|t-2|)$$

   Plot this solution for some values of t.

3. Find the Fourier transform of the function $\frac{1}{e^{2\pi i x}-1/2}$ on a circle $x \sim x+1$.

4. Find the Fourier transform of the function $\frac{1}{\cos(2\pi x)-\cosh(2\pi a)}$ on a circle $x \sim x+1$.

5. Find the Fourier transform of the function $\frac{1}{x^2+1}$ on $\mathbb{R}$.

6. Find the Fourier transform of the function $\frac{\sin x}{x^2+1}$ on $\mathbb{R}$.

7. Find the Fourier transform of the function $\frac{\sin x}{x}$ on $\mathbb{R}$.

8. Solve the equation

$$u_{tt} - 2u_{xx} - u_{tx} = 0, \qquad u_t(0,x) = 0, \qquad u(0,x) = (1-x^2)\theta_H(1-x^2).$$

9. Find the Fourier transform of $\frac{1}{\cosh x}$.

   - One option is to consider the difference of the two integral, over $\mathbb{R}$ and over $\mathbb{R}+2\pi i$: first to compare it with the original integral, and second compute by residues.
   - Another option is to close the contour and compute it as a sum of residues.

10. Solve the equation

$$u_{tt} - u_{xx} = f(t,x), \qquad u_t(0,x) = 0, \qquad u(0,x) = 0, \qquad f(t,x) = |1-|x||\theta_H(1-|x|)\theta_H(t(1-t)).$$

    Plot this solution, e.g., for $t = 10$.

11. Solve the equation

$$u_t - u_{xx} = 0, \qquad u(0,x) = e^{-(x-a)^2}$$

12. Solve the equation

$$u_t - u_{xx} = 0, \qquad u(0,x) = \sin x$$

13. Solve the equation

$$u_t - u_{xx} = 0, \quad x \ge 0, \qquad |u(-\infty,x)| < \infty, \qquad |u(t,+\infty)| < \infty, \quad u(t,0) = \sin t$$

14. Solve the equation

$$u_{tt} - u_{xx} - u_{yy} = \delta(x)\delta(y)\theta_H(t(1-t)), \quad t > 0, \qquad u(0,x,y) = u_t(0,x,y) = 0.$$

15. Find a formula for $\mathrm{P}\frac{1}{f(x)}$, analogous to $\delta(f(x)) = \sum\limits_{f(x_n)=0} \frac{1}{|f'(x_n)|}\delta(x - x_n)$.

16. Solve the equation

$$u_t(t, \vec{r}) - \Delta u(t, \vec{r}) = 0, \quad \vec{r} \in \mathbb{R}^d, \qquad u(0, \vec{r}) = \vec{r} e^{-(\vec{r}-\vec{r}_0)^2}.$$

17. Solve the equation

$$\Delta u(x, y) = 0, \quad x^2 + y^2 \leq 1, \quad u(\cos\phi, \sin\phi) = \sin 2\phi.$$

18. Solve the equation

$$\partial_t u(t, x) - u_{xx}(t, x) = \sin x \sin t, \quad u(0, x) = 0$$

in the limit $t \to \infty$.

19. Solve the equation

$$\Delta u(x, y, z) = 0, \quad x^2 + y^2 + z^2 \leq 1, \quad u(\sin\theta\cos\phi, \sin\theta\cos\phi, \cos\theta) = \cos 2\theta.$$

20. Solve the equation

$$\Delta u(x, y) = \delta(x - x_0)\delta(y - y_0), \quad \arg(x + iy) \in (0, \pi/3), \quad u(r\cos\frac{\pi}{3}, r\sin\frac{\pi}{3}) = u(r, 0) = 0.$$

21. Expand each component of the vector

$$(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)P_1(\cos\theta)$$

in the basis of spherical functions.

22. Find solutions of the Sturm-Liouville problem

$$u''(x) = -\lambda u(x), \quad u(0) = 0, \quad u'(1) = 0.$$

Check that corresponding $u_n(x)$ form a complete system (expand an arbitrary function in this basis and check that the inverse transformation reproduces it).

23. Find radially symmetric solution of the Helmholtz equation

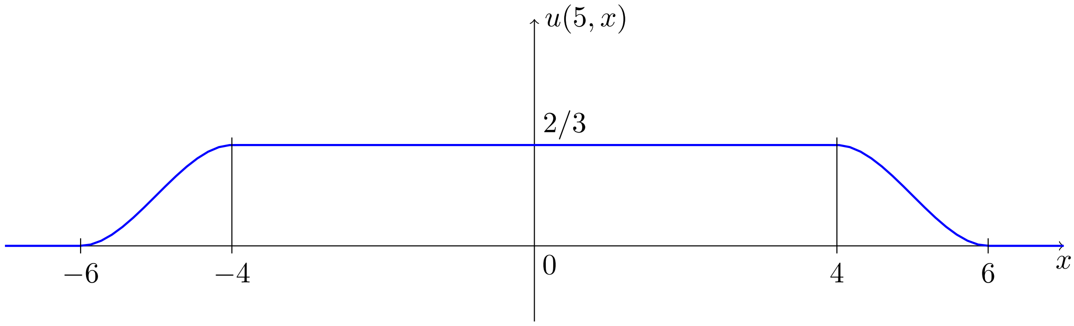
$$\Delta u(r) + u(r) = 0,$$

regular at $r = 0$, in two- and three-dimensional space.

Answers

1. $u(t,x) = \frac{1}{2}(\phi(x+t) - \phi(x-t))$, where $\phi(x) = \int_0^x (1-y^2)dy = x - x^3/3, |x| < 1$, and $\phi(x) = 2/3\,\mathrm{sign}\,x, |x| > 1$.

   The plot for $t = 5$:



2. $u(t,x) = \frac{1}{2}\sin^2(2\pi(x-t))\theta_H(1-|x-t-2|) + \frac{1}{2}\sin^2(2\pi(x+t))\theta_H(1-|x+t-2|)$
3. $\frac{1}{e^{2\pi i x}-1/2} = \sum_{n=0}^{\infty} 2^{-n} e^{-2\pi i n x}$.
4. $\frac{1}{\cos 2\pi x - \cosh 2\pi a} = -\sum_{n\in\mathbb{Z}} \frac{e^{2\pi i n x - 2\pi a|n|}}{\sinh 2\pi a}$
5. $\frac{1}{x^2+1} = \int_{-\infty}^{\infty} \frac{1}{2} e^{-|p|+ipx} dp$
6. $\frac{\sin x}{x^2+1} = \int_{-\infty}^{\infty} \frac{i}{4}\left(e^{-|p+1|} - e^{-|p-1|}\right) e^{ipx} dp$
7. $\frac{\sin x}{x} = \int \frac{1}{2}\left(\theta_H(p+1) - \theta_H(p-1)\right) e^{ipx} dx$
8. $u(t,x) = \frac{1}{3}\phi(x+2t) + \frac{2}{3}\phi(x-t), \quad \phi(x) = (1-x^2)\theta_H(1-x^2)$
9. $\frac{1}{\cosh x} = \int_{-\infty}^{\infty} \frac{e^{ipx}}{2\cosh\frac{\pi p}{2}} dp$
10. $u(t,x) = \frac{1}{2}\int_0^{\min(t,1)} dt' \int_{x-t+t'}^{x+t-t'} dx'(1-|x'|)\theta_H(1-|x'|)$,

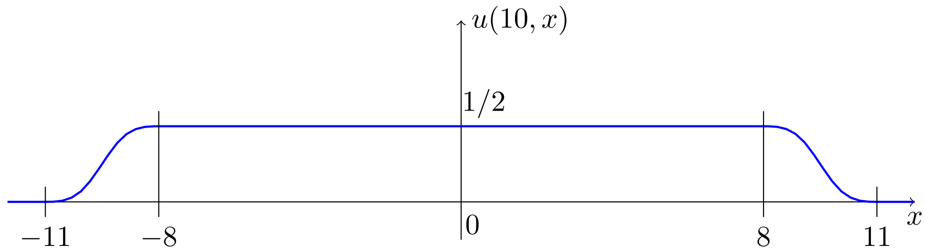
    $t > 1$: $u(t,x) = \frac{1}{2}\int_0^1 dt'\left(\phi(x+t-t') - \phi(x-t+t')\right) =$

    $= \frac{1}{2}\left(-\Phi(x+t-1) + \Phi(x+t) - \Phi(x-t+1) + \Phi(x-t)\right)$,

    where $\phi(x) = \int_0^x dy(1-|y|)\theta_H(1-|y|) = \mathrm{sign}(y)\max\left(|y| - \frac{1}{2}y^2, \frac{1}{2}\right)$

    and $\Phi(x) = \int_0^x dy\phi(y) = \max(\frac{1}{2}x^2 - \frac{1}{6}|x|^3, \frac{1}{2}|x| - \frac{1}{6})$

    The plot for $t = 10$:



11. $u(t,x) = \frac{e^{-\frac{(x-a)^2}{1+4t}}}{\sqrt{1+4t}}$

12. $u(t,x) = e^{-t}\sin x$

13. $u(t,x) = e^{-x/\sqrt{2}}\sin\left(t - x/\sqrt{2}\right)$

14. $u(t,x,y) = \phi(t,\sqrt{x^2+y^2}) - \phi(t-1,\sqrt{x^2+y^2})$, where $\phi(t,r) = \theta_H(t-r)\log\frac{\sqrt{t^2-r^2}+t}{r}$

15. $\mathrm{P}\frac{1}{f(x)} = \left(\frac{1}{f(x)} - \sum_i \frac{1}{f'(x_i)(x-x_i)}\right) + \sum_i \frac{1}{f'(x_i)}\mathrm{P}\frac{1}{x-x_i}$

16. $u(t,\vec{r}) = \frac{e^{-(\vec{r}-\vec{r}_0)^2}(\vec{r}+4t\vec{r}_0)}{(1+4t)^{3/2}}$

17. $u(x,y) = 2xy$

18. $u(t,x) = \frac{1}{2}\sin x\,(\sin t - \cos t)$

19. $u(x,y,z) = \frac{1}{3}\left(4z^2 - 2x^2 - 2y^2 - 1\right)$

20. $u(z,\bar{z}) = \frac{1}{4\pi}\log\left|\frac{(z^3-z_0^3)}{(z^3-\bar{z}^3)}\right|$, where $z = x+iy$.

21. $\frac{1}{3}\,(Y_{2,1}(\theta,\phi) + Y_{2,-1}(\theta,\phi), i(Y_{2,1}(\theta,\phi) - Y_{2,-1}(\theta,\phi)), 2Y_{2,0}(\theta,\phi)+1)$, where we fixed for simplicity $Y_{2,\pm1}(\theta,\phi) = P_2^1(\cos\theta)e^{\pm i\phi}$, $Y_{2,0} = P_2(\cos\theta)$

22. $f_n(x) = \sqrt{2}\sin\pi(n+1/2)x,\ (f_n,f_m) = \delta_{n,m},\ \tilde{u}_n = \int_0^1 f_n(x)u(x),\ u(x) = \sum_{n=1}^\infty \tilde{u}_n f_n(x)$, $\sum_{n=0}^\infty e^{-\epsilon n} f_n(x)f_n(y) = \delta_\epsilon(x,y)$. $\delta_0(x,y) = 0$ for $x \neq y$. $\delta_\epsilon(x,x+\delta) \sim \frac{\epsilon}{\pi^2\delta^2+\epsilon^2}$, so $\delta_\epsilon(x,y) \to \delta(x-y)$.

23. d=2: $u(r) = J_0(r)$, d=3: $u(r) = \frac{\sin r}{r}$.