

11 September 2024

- It is enough to solve 4 problems to get maximal result.
- You have 3 hours.

1. Put to the canonical form

$$u_{xx} + u_{yy} + 4u_{xy} + u_x = 0.$$

2. Solve the wave equation on \mathbb{R}^{1+1} :

$$u_{tt} - u_{xx} = 0, \quad u(0, x) = (1 - x^2)\theta_H(1 - x^2), \quad u_t(0, x) = 0.$$

3. Solve the heat equation on \mathbb{R}^{1+1} :

$$u_t = u_{xx}, \quad u(0, x) = x^2 e^{-x^2}.$$

4. Find the Green function of the Poisson equation on a quadrant $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$:

$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0), \quad u(x, 0) = u(0, y) = 0, \quad \lim_{x^2 + y^2 \rightarrow \infty} u(x, y) = 0.$$

5. Find the Fourier transform on \mathbb{R} for the function

$$\theta_H(1 - p^2).$$

How smooth is it? What is its behavior at infinity?

6. Solve 3d Laplace equation in the unit ball:

$$\Delta u(r, \theta, \phi) = 0, \quad u(1, \theta, \phi) = \cos^2 \theta$$

7. Solve 1d difference equation

$$u(n+1) = au(n) + f(n), |a| < 1, \quad f(n) = 0, n < 0, \quad u(n) = 0, n < 0$$

What is its Green function in coordinate and in the Fourier space?