18 September 2025

- It is enough to solve 4 problems to get the maximal result.
- You have 3 hours.

1. Put to the canonical form
$$u_{xx} + 4u_{yy} - 4u_{xy} + u_y = 0.$$
2. Solve the wave equation on $\mathbb{R}^{1+1}$:
$$u_{tt} - u_{xx} = 0, \qquad u(0,x) = e^{-\frac{x^2}{2}} \sin\frac{ax}{2}, \qquad u_t(0,x) = 0.$$
3. Solve the heat equation on $\mathbb{R}^{1+1}$:
$$u_t = u_{xx}, \qquad u(0,x) = e^{-\frac{x^2}{2}} \sin\frac{ax}{2}, \qquad a > 0.$$
4. Find the Green function of the Poisson equation
$$u_{xx} + u_{yy} = \delta(x - x_0)\delta(y - y_0)$$
in the domain
$$D = \{(x,y) | x \le 0, y \le 0\}$$
with the boundary conditions $u(x,y) = 0$ for $(x,y) \in \partial D$.
5. Find the Fourier transform on $\mathbb{R}$ of the function
$$\frac{1}{x^4 + 1}.$$
How smooth is it? What is its behavior at infinity? Why?
6. Solve 3d Laplace equation outside the unit ball[1] $r \ge 1$
$$\Delta u(r, \theta, \phi) = 0$$
with the boundary condition $u(1, \theta, \phi) = \cos 2\theta$.
7. Find the Fourier series for the function $f(\phi)$ on S^1
$$f(\phi) = \phi^2, \quad \phi \in [-\pi, \pi], \quad -\pi \sim \pi.$$
What is the behavior of its coefficients at infinity?

[1]Spherical coordinates are introduced by the standard formulas, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$.