1. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} + u_x + u = 0.$$

2. Find the normal form of the equation

$$u_{xx} + 2u_{xy} + u_{yy} - u_{zz} = 0.$$

   Write its general solution.

3. Check that the general solution of the equation

$$\partial_x \left(\frac{xy}{x^2+y^2} \partial_x u \right) + \frac{1}{2} \partial_x \left(\frac{x^2-y^2}{x^2+y^2} \partial_y u \right) + \frac{1}{2} \partial_y \left(\frac{x^2-y^2}{x^2+y^2} \partial_x u \right) - \partial_y \left(\frac{xy}{x^2+y^2} \partial_y u \right) = 0$$

   is given by

$$u(x,y) = F(x^2 - y^2) + G(xy).$$

4. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = \theta_H(1-x), \quad u(0,x) = 0, \quad u_t(0,x) = 0, \quad u(t,0) = 0,$$

   where θ_H is the Heaviside theta function:

$$\theta_H(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0, \\ \frac{1}{2}, & x = 0. \end{cases}$$

   For simplicity, you may consider its solution only for $t > 4$.

5. Solve the wave equation on the half-line $x > 0$:

$$u_{tt} - u_{xx} = 0, \quad u(0,x) = 0, \quad u_t(0,x) = \theta_H(x-3) - \theta_H(x-2), \quad u(t,0) = 0.$$

   For simplicity, you may again consider only the large time case.

6. Solve the wave equation on the half-line:

$$u_{tt} - u_{xx} = 0, \quad u(0,x) = 0, \quad u_t(0,x) = 0, \quad u(t,0) = \sin t.$$

7. Compute the Fourier transform of the function f on $\mathbb{Z}/(2N)\mathbb{Z}$ defined by

$$f(0) = 0, \quad f(1) = \ldots = f(N-1) = 1, \quad f(N) = 0, \quad f(-1) = \ldots = f(1-N) = -1.$$

8. Compute the Fourier transform of the function $f(\phi)$ on S^1 defined by

$$f(\phi) = \left[\begin{array}{ll} -1, & \phi \in (-\pi, 0) \\ 1, & \phi \in (0, \pi) \end{array} \right.$$

9. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\epsilon|p|}$$

   defined on the real line.

10. Compute the Fourier transform of

$$f_\epsilon(x) = \frac{x}{\pi(x^2+\epsilon^2)}$$

defined on the real line. Compute the integral

$$\int_{-\infty}^{\infty} f_\epsilon(x)dx$$

11. Compute the Fourier transform of

$$f(n) = e^{-\epsilon|n|}$$

defined on $\mathbb{Z}$.

12. Compute the [inverse] Fourier transform of

$$\tilde{f}_\epsilon(p) = e^{-\frac{1}{2}\epsilon p^2}$$

What is the value of its integral from $-\infty$ to $+\infty$?

13. Compute the Fourier transform of the function on S^1

$$f_\epsilon(\phi) = \frac{\sinh \epsilon}{\cosh \epsilon - \cos \phi}.$$

Hint: for computing the integral $\int_0^{2\pi} \frac{d\phi}{2\pi} f_\epsilon(\phi) e^{-in\phi}$, introduce the new variable $z = e^{i\phi}$ and find an appropriate way to deform the integration contour.
What is $\int_0^{2\pi} f_\epsilon(\phi)d\phi$?

14. Solve the heat equation

$$u_t = u_{xx}, \quad u(0,x) = e^{-ax^2}$$

15. Compute the Fourier transform of

$$f(x) = xe^{-x^2}$$

and of

$$f(x) = x^2 e^{-x^2}$$