19 November 2024

- It is enough to solve 4 problems to get maximal result.
- You have 3 hours.

1. Put to the canonical form
$$u_{xx} + u_{yy} + 4u_{xy} + u_x = 0.$$
2. Solve the wave equation on $\mathbb{R}^{1+1}$:
$$u_{tt} - u_{xx} = 0, \qquad u(0,x) = (1-x^2)\theta_H(1-x^2), \qquad u_t(0,x) = 0.$$
3. Solve the heat equation on $\mathbb{R}^{1+1}$:
$$u_t = u_{xx}, \qquad u(0,x) = x^2 e^{-x^2}.$$
4. Find the Green function of the Poisson equation on a quadrant $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$:
$$u_{xx} + u_{yy} = \delta(x-x_0)\delta(y-y_0), \qquad u(x,0) = u(0,y) = 0, \qquad \lim_{x^2+y^2\to\infty} u(x,y) = 0.$$
5. Find the Fourier transform on $\mathbb{R}$ for the function
$$\theta_H(1-p^2).$$
How smooth is it? What is its behavior at infinity?
6. Solve 3d Laplace equation in the unit ball:
$$\Delta u(r,\theta,\phi) = 0, \qquad u(1,\theta,\phi) = \cos^2\theta$$
7. Solve 1d difference equation
$$u(n+1) = au(n) + f(n), |a| < 1, \qquad f(n) = 0, n < 0, \qquad u(n) = 0, n < 0$$
What is its Green function in coordinate and in the Fourier space?