QR

It helps solve systems of linear equations Ax= b efficiently, especially when A is not a square matrix or is ill-conditioned.

After decomposing A=QR, solving Ax=b reduces to solving Rx=Q(Transpose)b, which is straightforward because R is triangular.

$\boxed{\text{QR method}}$

$Ax = b \left\{ x = A^{-1}b \left\{ \begin{array}{l} \text{if } A \text{ is ill condition or not square} \\ \text{we will use QR method.} \end{array} \right. \right.$

$\underset{mm}{A} = \underset{mm}{Q}\,\underset{nn}{R}$

orthonormal ← Q ; R → triangular upper

$Q^TQ = QQ^T = I$: orthogonal if square (orthonormal)

* if Q is square $Q^{-1} = Q^T$

instead of solving $Ax = b \rightarrow$ we solve $QRx = b \rightarrow Rx = Q^Tb$

there are many ways to compute this but here we use Gram-schmidt process

<u>construct Q from A</u>

$R(A) = \left\{ \overset{a_1}{[\;\;]}\ \overset{a_2}{[\;\;]} \right\}$ $R(A)$ ← برای متعامد سازی و بردار ترسیم A را یافته

- a_1, a_2 را بگیر ... $q_1 = \frac{a_1}{\|a_1\|}$

- $v_1 = a_1$, $v_2 = a_2 - \frac{\langle v_1, a_2 \rangle}{\|v_1\|^2} v_1$, v_3

$\text{proj}_{v_1}(a_2)$

- نرمال کردن $q_2 = \frac{v_2}{\|v_2\|}$

- در ادامه همه وکتورهای v_1 تا v_n می‌شوند Q

- $R = Q^T A$

E.g

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[\text{برای یافتن فضای برد } R(A)]{\text{بلوکی هم‌پایه}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$
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$R(A) = \left\{ \overset{a_1}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}, \overset{a_2}{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}, \overset{a_3}{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}} \right\}$: دو گونه ابرد

<u>step 1</u>

Normalize $a_1 \Rightarrow v_1 = [\frac{1}{\sqrt3}, \frac{1}{\sqrt3}, \frac{1}{\sqrt3}]$

$V_2 = a_2 - \frac{\langle v_1, a_2 \rangle}{\|v_1\|^2} v_1 \rightarrow \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} - \left(\frac{5}{\sqrt3} \times \begin{bmatrix} \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} \end{bmatrix} \right)$

$v_1 \cdot a_2 = \frac{1}{\sqrt3} + \frac{2}{\sqrt3} + \frac{2}{\sqrt3} = \frac{5}{\sqrt3}$

$\|v_1\|^2 = \sqrt{\frac13 + \frac13 + \frac13} = 1$

$\begin{bmatrix} \frac33 \\ \frac63 \\ \frac63 \end{bmatrix} - \begin{bmatrix} \frac53 \\ \frac53 \\ \frac53 \end{bmatrix} = \begin{bmatrix} -\frac23 \\ \frac13 \\ \frac13 \end{bmatrix}$

$\frac49 + \frac19 + \frac19 = \frac69$

$\rightarrow \frac{\sqrt6}{3} \rightarrow V_2 \Rightarrow$ normalize $\frac{V_2}{\|V_2\|}$

$V_3 = a_3 - \frac{\langle V_2, a_3 \rangle}{\|V_2\|^2} V_2 - \frac{\langle V_1, a_3 \rangle}{\|V_1\|^2} V_1$

$\frac{V_3}{\|V_3\|}$.