

GCD: Greatest Common Divisor

HCF: Highest Common Factor.

$\text{GCD}(A, B) = \text{Greatest } \underline{\text{no.}} \text{ that divides}$   
both  $A \neq B$

$$\begin{aligned}\gcd(a, b) &= x \\ a \cdot x &= 0 \\ b \cdot x &= 0\end{aligned}$$

$$\gcd\left(\frac{15}{\downarrow}, \frac{-25}{\downarrow}\right) = \underline{\underline{5}}$$

Diagram showing the Euclidean algorithm for  $\gcd(15, -25)$ . The numbers 15 and -25 are written above a horizontal line. Below the line, their remainders after division by 5 are shown: 1 and -5. A vertical arrow points down from each number to its remainder. The remainder -5 is then divided by 5, resulting in 1, which is circled in green. A vertical arrow points down from -5 to 1. Finally, 1 is divided by 5, resulting in 0, which is circled in green. A vertical arrow points down from 1 to 0.

$$\gcd\left(\frac{12}{\downarrow}, \frac{30}{\downarrow}\right) = 6$$

Diagram showing the Euclidean algorithm for  $\gcd(12, 30)$ . The numbers 12 and 30 are written above a horizontal line. Below the line, their remainders after division by 6 are shown: 0 and 0. Both remainders are circled in green. A vertical arrow points down from each number to its remainder. The remainders 0 and 0 are then divided by 6, resulting in 0, which is circled in green. A vertical arrow points down from 0 to 0.

$$\gcd\left(\frac{10}{\downarrow}, \frac{-25}{\downarrow}\right) = 5$$

Diagram showing the Euclidean algorithm for  $\gcd(10, -25)$ . The numbers 10 and -25 are written above a horizontal line. Below the line, their remainders after division by 5 are shown: 0 and -5. A vertical arrow points down from each number to its remainder. The remainder -5 is then divided by 5, resulting in 1, which is circled in green. A vertical arrow points down from -5 to 1. Finally, 1 is divided by 5, resulting in 0, which is circled in green. A vertical arrow points down from 1 to 0.

Can 0 ever be GCD of 2 no's ??  
NO

$$\gcd\left(\frac{0}{\downarrow}, \frac{8}{\downarrow}\right) = 8$$

Diagram showing the Euclidean algorithm for  $\gcd(0, 8)$ . The numbers 0 and 8 are written above a horizontal line. Below the line, their remainders after division by 8 are shown: 0 and 0. Both remainders are circled in green. A vertical arrow points down from each number to its remainder. The remainders 0 and 0 are then divided by 8, resulting in 0, which is circled in green. A vertical arrow points down from 0 to 0.

$$\gcd\left(\frac{0}{\downarrow}, \frac{0}{\downarrow}\right) : \text{Not defined}$$



## Properties

$$1) \gcd(a, b) = \gcd(|a|, |b|)$$

$$2) \gcd(a, b) = \gcd(b, a) \Rightarrow \text{Commutative}$$

$$3) \gcd(0, x) = |x|$$

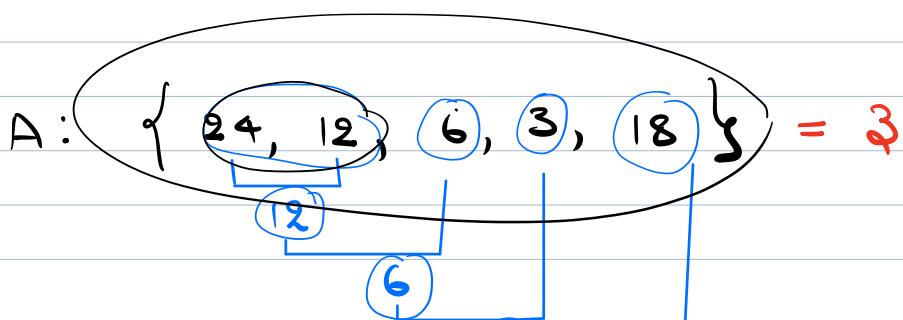
$$\begin{matrix} \gcd(0, -10) \\ \downarrow \\ 10 \end{matrix}$$

$$4) \gcd(a, b, c) = \gcd(\gcd(a, b), c)$$

$$\quad\quad\quad \gcd(\gcd(a, c), b)$$

$$\quad\quad\quad \gcd(\gcd(b, c), a)$$

$\Rightarrow$  Associative



A diagram consisting of two circles, each containing the number '3'. A horizontal line connects the top of the first circle to the left side of the second circle. An arrow points from the second circle to the right. Below the second circle, the word 'Ans' is written and underlined.

$$\text{def } \gcd(a, b, c) = x$$

$$\text{gcd}(a, b, c, d) = \frac{1}{\text{lcm}} \leq x$$

$$\text{gcd} \left( \frac{12}{2}, \frac{24}{2}, \frac{18}{2} \right) = \boxed{6} - \boxed{3}$$

The diagram illustrates the Euclidean algorithm for finding the GCD of 18 and 24. It shows the division steps:

$$\begin{array}{r} 24 \\ \times 1 \\ \hline 24 \\ - 18 \\ \hline 6 \\ \end{array}$$

The remainders 6 and 0 are circled in green.

$$\text{gcd}(12, 24, 18, 36) = \underline{\underline{6}}$$

$$\text{g.c.d}(6, d) \leq \underline{\underline{6}}$$

$$\text{g} \text{cd}(\underline{12}, \underline{24}, \underline{18}, \underline{36}, \underline{5}) = \underline{\underline{1}}$$

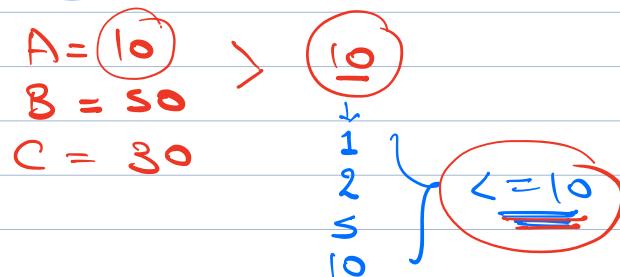
⇒ Adding a no. to a list of no's can never increase the GCD

GCD either remains same or decreases.

Given 2 nos' A & B.  
Find GCD of A & B.

Sol<sup>n</sup>

$$\text{GCD}(A, B) \leq \min(A, B)$$



$\Rightarrow$  Find the factors of  $\min(A, B)$  &

Check if it is a factor of both the  
no's

ans = 1;

for (i = 1; i \* i <=  $\min(A, B)$ ; i++) {

fact1 = i;

fact2 =  $\min(A, B) / i$ ;

if (A % fact1 == 0 && B % fact1 == 0) {

ans =  $\max(\text{ans}, \text{fact1})$ ;

}

if (A % fact2 == 0 && B % fact2 == 0) {

ans =  $\max(\text{ans}, \text{fact2})$ ;

}

↳

(i more)

$$36 = \overbrace{1 \times 36}^{\cdot} + \overbrace{2 \times 18}^{\cdot} + \overbrace{3 \times 12}^{\cdot} + \overbrace{4 \times 9}^{\cdot} + \overbrace{6 \times 6}^{\cdot}$$

$$T.C. = \sum_{i=1}^n i \leq \min(A, B)$$

$$i^2 \leq \min(A, B)$$

$$i \leq \text{sgt}(\min(A, B))$$

$$T.C. = \text{sqrt}(\min(A, B));$$

$$\text{if } \gcd(A, B) = g \quad \underline{\underline{B \geq A}} \quad \underline{\underline{g \leq A}}$$

$$\underline{A} = \underline{\int x \tau_1}$$

$$\underline{B} = g x \equiv k_2$$

where  $K_1 \neq K_2$

are co prime.

$$\text{gcd} \left( \underbrace{\underline{1s}, \underline{2s}}_{\sim}, \underline{3s} \right) = \underline{s}$$

$$\begin{array}{r} 15 \\ - 25 \\ \hline \end{array} = \begin{array}{r} 5 \\ - 5 \\ \hline \end{array} \times \begin{array}{r} 3 \\ - 3 \\ \hline \end{array}$$

$$B - A = gK_2 - gK_1$$

$$= g(K_2 - K_1)$$

$$\mathcal{B} - \mathcal{A} = g \times (\mathcal{K}_2 - \mathcal{K}_1)$$

$(B-A)$  is also divisible by  $g$ .

$$\gcd(A, B-A) = g$$

$$2s - 1s = s \times s - s \times 3$$

$$\underline{10} = \underline{\underline{s}}(s-3)$$

Proof

$$\gcd(a, b) = \underline{\underline{g}} \quad | \quad \gcd(a, b-a) = \underline{\underline{x}}$$

To prove  $g = x$

$$a \cdot \cancel{g} = 0$$

$$b \cdot \cancel{g} = 0$$

$$\begin{aligned} a &= g k_1 \quad \text{--- (1)} \\ b &= g k_2 \quad \text{--- (2)} \end{aligned}$$

Subtract (1) from (2)

$$b - a = g (k_2 - k_1)$$

$$(b-a) \cdot \cancel{g} = 0$$

$g$  is a factor of  
 $a, b, b-a$

$$a \cdot \cancel{x} = 0$$

$$(b-a) \cdot \cancel{x} = 0$$

$$\begin{aligned} a &= t_1 \times x \quad \text{--- (3)} \\ (b-a) &= t_2 \times x \quad \text{--- (4)} \end{aligned}$$

Add (3) and (4)

$$b = x (t_1 + t_2)$$

↓  
 $x$  is also a factor  
of  
 $a, b, b-a$

$$\gcd(a, b) = g$$

$x$  is also a factor of  $a \neq b$

$$x \leq g$$

$$\gcd(a, b-a) = x$$

$$g \text{ is also a factor of } a \neq b-a$$

$$g \leq x$$

$$x = g$$

$$B > A$$

$$\gcd(A, B) = \gcd(A, B-A)$$

$$\gcd(\underline{\underline{6}}, \underline{\underline{8}})$$

$$\gcd(6, 8-6=2) \Rightarrow$$

$$\gcd(\underline{\underline{6}}, \underline{\underline{2}})$$

$$\gcd(\underline{\underline{2}}, \underline{\underline{6}})$$

$$\gcd(\underline{\underline{2}}, \underline{\underline{4}})$$

$$\gcd(\underline{\underline{2}}, \underline{\underline{2}})$$

$$\gcd(\underline{\underline{2}}, \underline{\underline{0}}) = \underline{\underline{2}}$$

$$\gcd(\underline{\underline{0}}, \underline{\underline{2}}) = \underline{\underline{2}}$$

## Code

(a)  $\leq b$

```
int gcd ( a , b ) {  
    if ( a == 0 ) { return b ; }  
    if ( a > b ) {  
        swap ( a , b );  
    }  
    return gcd ( a , b - a ); →  
}
```

$$\underline{gcd(2, 5)} = 1$$

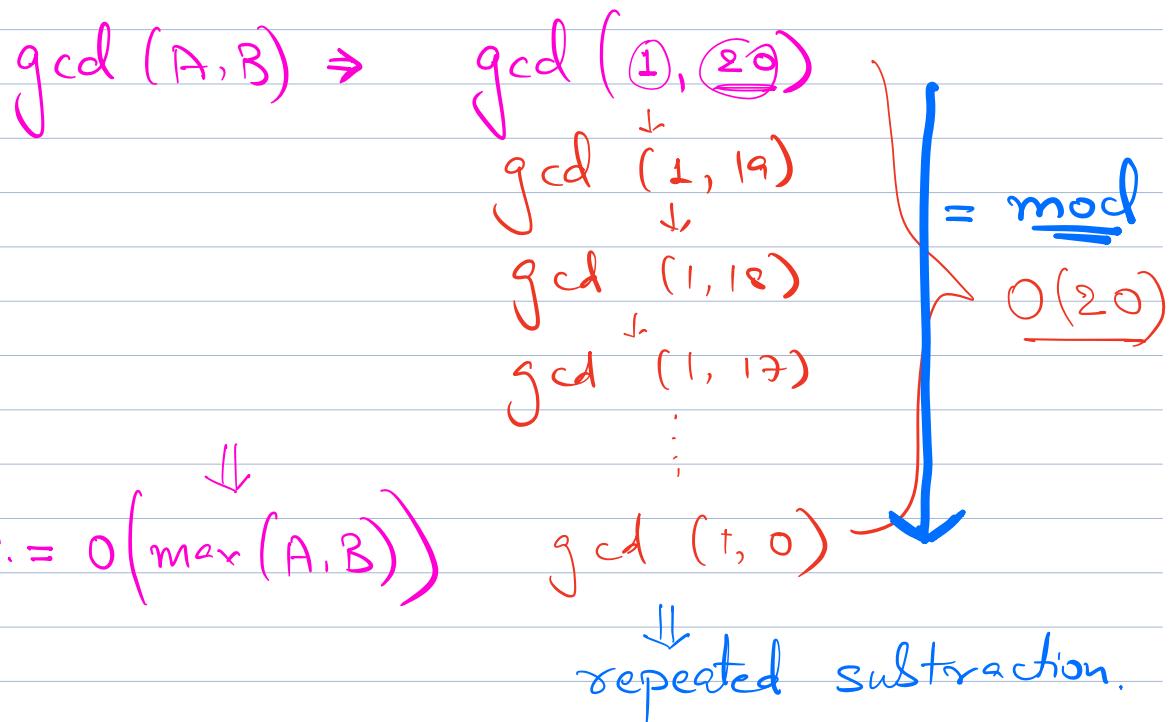
$$gcd(2, 3)$$

$$gcd(2, 1)$$

$$gcd(1, 1)$$

$$gcd(1, 0)$$

$$gcd(0, 1) \Rightarrow 1$$



Eg       $\text{gcd}(60, 200)$        $B-A \Rightarrow B \cdot \frac{1}{A}$   
 $\downarrow$   
 $60, \quad 200 - 60$   
 $= 140$   
 $60, \quad 140 - 60$   
 $= 80$   
 $\underline{60}$   
 $80 - 60$   
 $= 20$   
 $= \underline{\underline{200}} - \underline{60} \times 3$   
 $\underline{\underline{20}} \quad \underline{60}$   
 $= 200 \cdot 1 \cdot 60$

$$20 \quad 60 - 20 \\ 40$$

$$20 \quad 40 - 20 \\ = 20$$

$$20 \quad 20 - 20 \\ = 0 \Rightarrow \underline{\underline{60}} - \underline{20} \times 3 \\ \underline{\underline{60}} \cdot 1 \cdot 20$$

$$= 0, \quad \begin{array}{c} 20 \\ \text{Ans} \end{array}$$

$$\left( \begin{array}{l} \text{gcd}(a, b) = \text{gcd}(a, b-a) \\ \text{gcd}(a, b) = \text{gcd}(a, \frac{b \% a}{}) \end{array} \right) \downarrow \downarrow [0, a-1]$$

$$\begin{array}{l} \text{gcd}(60, 200) \\ \text{gcd}(60, 200 \% 60 = 20) \\ \text{A} > \text{B} \\ \text{A} \underline{\leq} \text{B} \end{array} > \begin{array}{l} 60 \% 20 = 0 \\ = 60 \end{array}$$

$$\text{gcd}(a, b) = \text{gcd}\left(\frac{b \% a}{}, a\right)$$

Code

```
int gcd (a, b) {
    if (a == 0) return b;
    return gcd(b \% a, a);
```

b

$\Rightarrow$  Euclidean Algo for GCD

$$\begin{aligned} \gcd(23, 15) &\Rightarrow 1 \\ \gcd\left(\frac{23}{15}, 15\right) & \quad \text{A} > \text{B} \\ \gcd\left(\frac{23 \cdot 1.15}{15}, 15\right) & \quad \text{A} < \text{B} \\ \gcd\left(\frac{23 \cdot 1.15}{8}, 15\right) & \\ \gcd\left(\frac{15 \cdot 1.8}{7}, 8\right) & \\ \gcd\left(\frac{8 \cdot 1.7}{1}, 7\right) & \\ \gcd\left(\frac{7 \cdot 1.1}{0}, 1\right) & \\ \gcd(0, 1) &\Rightarrow 1 \text{ } \underline{\text{Ans}} \end{aligned}$$

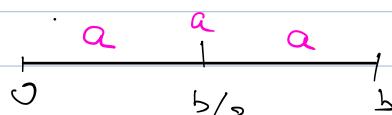
T.C.

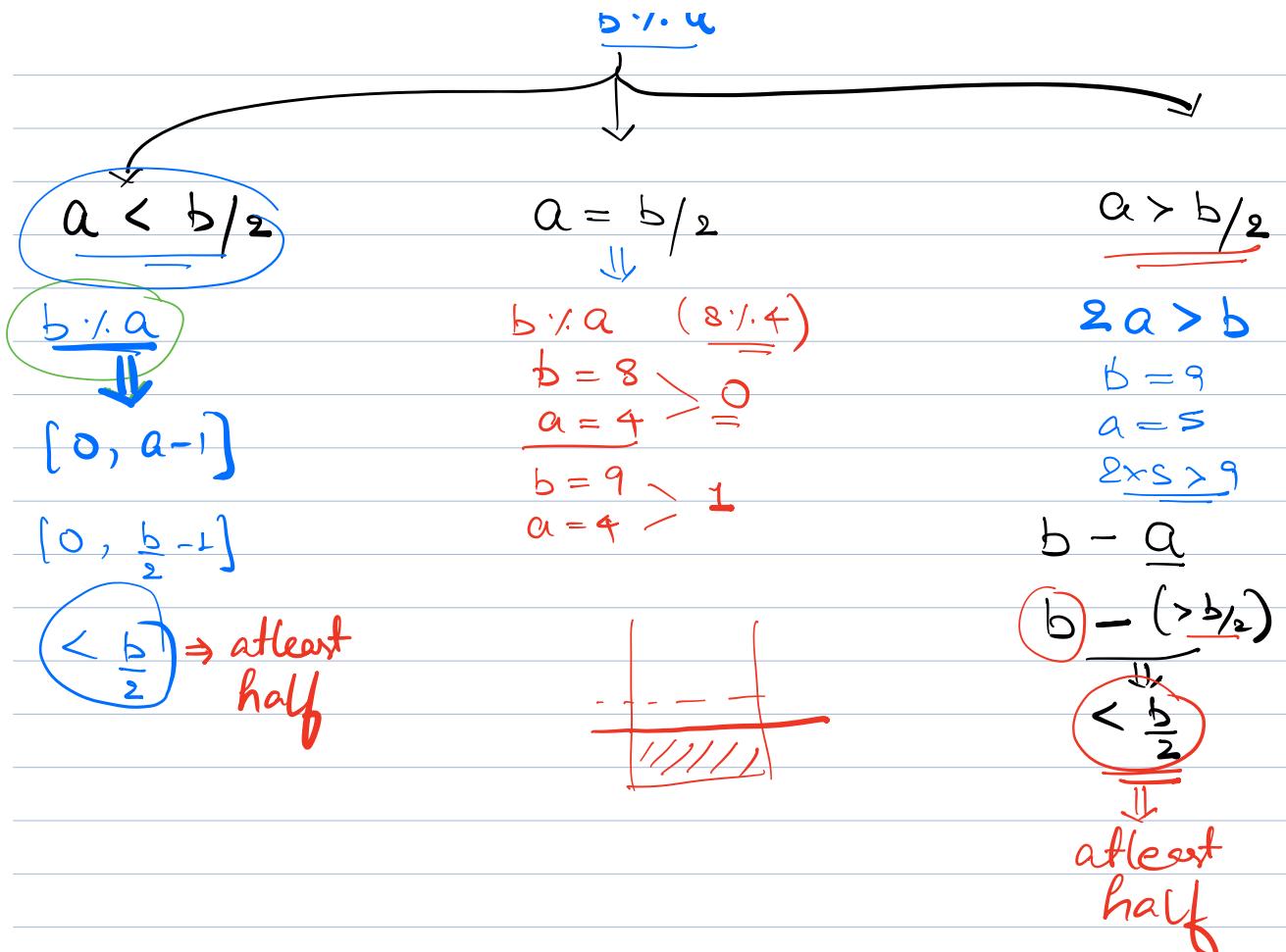
$$N \rightarrow N-1 \rightarrow N-2 \dots \dots 1 \Rightarrow O(N)$$

$$N \rightarrow N/2 \rightarrow N/4 \rightarrow N/8 \dots \dots 1 \Rightarrow O(\log N)$$

Upper bound

$$a, b, \dots$$





$N \rightarrow N/2$

$$\text{T.C.} = \log \left( \max \left( A, B \right) \right)$$

8

$$8/2 = 4$$

$\cancel{> 4} \Rightarrow \frac{5}{1}, \frac{6}{2}, \frac{7}{1}, \frac{8}{0}$



Subsequence  $\times \Rightarrow$  Subsequences & subsets

~~= - 1 → + - -~~  
A: { ~~0 1 2 3 4 5 6 7~~  
3, 2, 1, 6, 4, 8, 10, 9 }

A sequence generated by deleting 0 or more elements from the array.

Order should be same as A

{ 3, 1, 6, 8, 9 }

{ 8, 1, 2 } X

{ 4, 10 } ✓

Given an array. Return true if there exists a subsequence with  $\text{GCD} = 1$

Eg  $\Rightarrow$  A: { 4, 6, 3, 8 }  $\Rightarrow$  True

$$\Rightarrow \{ 4, 3 \} = \boxed{\frac{1}{1}}$$

$$\Rightarrow \{ 3, 8 \}$$

2) A: { 2, 4, 6, 8 }  $\Rightarrow$  false

A:  $\{3, 6, 9\} \Rightarrow \text{false}$

A:  $\{3, 4, 8\} \Rightarrow \text{true}$   
 $\{3, 8\}$

$\{9, 8\}$   $\Rightarrow \text{true}$

Sol<sup>n</sup>

If there exists a subsequence  
with  $\text{GCD} = 1$ ,

$\text{GCD}$  of the entire array =  $\underline{\underline{1}}$

1) Find  $\text{GCD}$  of entire array

2) If ( $\text{GCD} == 1$ ) { return True; }  
else { return false; }

Code

$g = \text{gcd}(A[0], A[1]);$

for ( $i = 2$ ;  $i < N$ ;  $i++$ ) { //  $O(N)$

$g = \text{gcd}(g, A[i])$ ; //  $\log(\max)$

}

if ( $g == 1$ ) { return True; }

else { return False; }

$O(N \times \log(\max(A[i])))$

Q

Given an array  $A$

~~trick question~~ Delete min no., s.t. gcd of  $A$  becomes 1

If it is possible. return true

If not return false

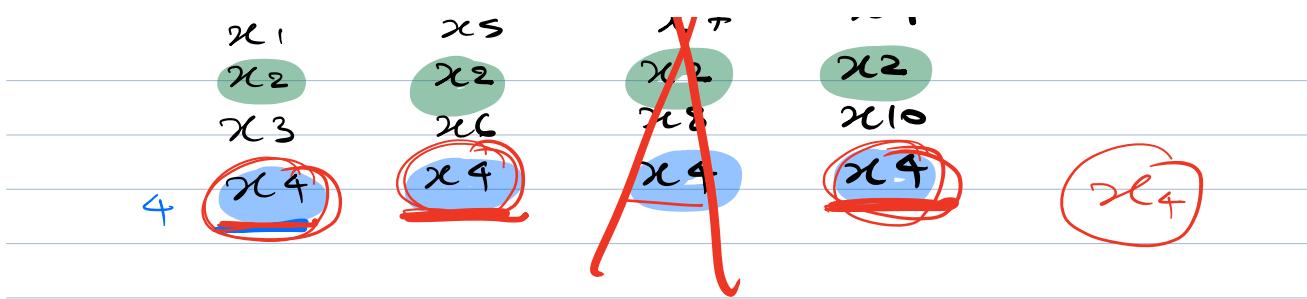
$A: \{6, 10, 15, 25, 24, 18\} \Rightarrow \boxed{\frac{1}{1}}$

Ans =  $\boxed{0}$

$A: \{6, 18, 24, \cancel{36}\} \Rightarrow \boxed{6}$

false

a      b      c      d  
↓      ↓  
 $x_1 > x_2$



Given an integer array.

~~Goal~~ Delete one element s.t. the GCD is maximised.

Eg A: {9, 18, 49, 12, 30}  $\Rightarrow$  1

Delete	GCD
9	1
18	1
49	3
12	1
30	1

A: {3, 16, 18}  $\Rightarrow$  1

$$\begin{aligned} 3 &\Rightarrow \underline{2} \\ 16 &\Rightarrow \underline{3} \end{aligned}$$

~~Sol~~ BF  $\Rightarrow$  Delete each element & check

index = 0

gcdmax = 0;

for (i = 0; i < N; i++) {

// i<sup>th</sup> element i am deleting

g = 0;

for (j = 0; j < N; j++) {

if (i != j) {  
g = gcd(g, A[j]);

}

}

if (g > gcdmax) {

gcdmax = g;

index = i;

}

}

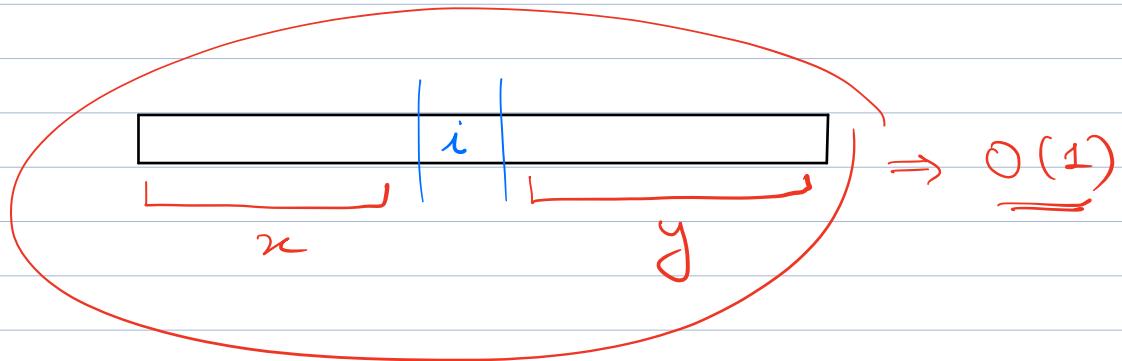
T.C = O(N^2 × log(max))

max of  
array

||,

H.W.

$$O(N \times \log(\max(A[i])))$$



$O(N)$  extra space allowed

Doubt

$$D^{n(B)}$$

$(A^B \cdot P) \Rightarrow$  fast power

$$A^P \equiv A \bmod P$$

B!

$n_{C_R}$

$A^B! \% (10^9 + 7)$

$s[i] : 1 \quad 4 \quad 10 \quad 16 \quad 25$

$e[i] : 3 \quad 8 \quad 14 \quad 20 \quad 31$

