MATH 213 - Lecture 6: Solving DEs via Laplace Transforms

Basic idea:

Consider a Linear DE with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

with the appropriate number of initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1, \qquad \ldots, \qquad y^{n-1}(0) = y_{n-1}.$$

Taking the Laplace transform of both sides and using the linearity of the Laplace transform gives an equation of the form

$$(s^n + b_{n-1}s^{n-1} + \ldots + b_0 s)Y(s) + G(s, y_0, y_1, \ldots, y_{n-1}) = F(s)$$

In the above $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$ respectively, G is a function of s and the initial conditions, and the $b_i s$ are coefficients that come from taking the Laplace transform of the derivative terms.

Note: One can find the exact form of this equation but I do not want you to memorize this formula so I am not writing the exact form.

We can solve the above for $Y(s)$ and then (in theory) compute the inverse Laplace transform to find $y(t)$.

Picture:

$$\begin{tikzcd}
\boxed{\text{Linear differential Eq (LDE)}} \arrow[r, "\mathscr{L}"] \arrow[d, "\text{Solve?}"'] & \boxed{\text{Algebraic equation (AE)}} \arrow[d, "\text{Solve!}"] \\
\boxed{\text{Solutionto LDE}} & \boxed{\text{Solution to AE}} \arrow[l, "\mathscr{L}^{-1}"']
\end{tikzcd}$$

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Example 1
Solve $y'' = -g$, $y(0) = h_0$, $y'(0) = v_0$. Recall this is the model for a falling ball with initial height h_0 and initial velocity v_0 from the first lecture.
Plot the solution.

$$\mathcal{L}\{y''\} = \mathcal{L}\{-g\}$$ ← one sided

$$s^2 Y(s) - \underbrace{s\,y(0) - y'(0)}_{\text{IC}} = \frac{-g}{s}$$

$$s^2 Y(s) - s\,h_0 - v_0 = \frac{-g}{s}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2}\left[-\frac{g}{s} + s h_0 + v_0\right]$$

↑ Chara. Poly.

$$= \frac{-g}{s^3} + \frac{h_0}{s} + \frac{v_0}{s^2}$$

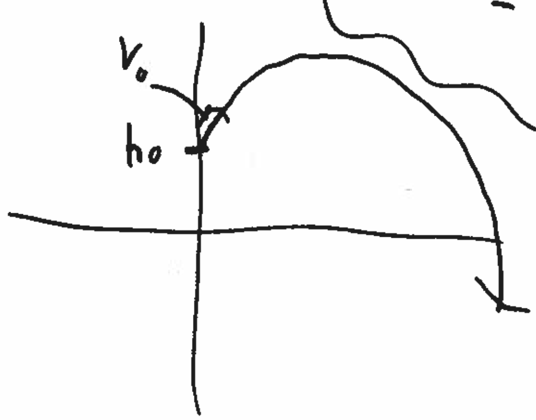
Take $\mathcal{L}^{-1}$:

Use $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$$y(t) = -g\left(\frac{1}{2!}t^2\right) + h_0 \cdot (1) + v_0(t)$$

$$= \boxed{-\frac{g}{2}t^2 + v_0 t + h_0}$$

Plot



Definition 1: Characteristic Polynomial
The **Characteristic Polynomial** of a linear differential equation with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

The polynomial multiplied by $Y(s)$ after you take the Laplace transform. This polynomial will always be $p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$.

Example 2
Solve $y'' + y = e^{-2t}\sin(2t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

$$\mathcal{L}\{y''+y\} = \mathcal{L}\{e^{-2t}\sin 2t\}$$

$$s^2Y(s) - s\,y(0) - y'(0) + Y(s) = \frac{2}{(s+2)^2+4}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2+1}\left[\frac{2}{(s+2)^2+4} + s\cdot\underset{y(0)}{1} + \underset{y'(0)}{0}\right]$$

s^2+1: Char. Poly

$$= \frac{2}{(s^2+1)[(s+2)^2+4]} + \frac{s}{s^2+1}$$

$\boxed{\text{not in Table}}$ $\qquad$ $\boxed{\cos t}$

2 options for first term

PF:

$$\frac{As+B}{s^2+1} + \frac{Cs+D}{(s+2)^2+4}$$

Convo. Thm

$$y(t) = \int_0^t \sin(t-\tau)\cdot \boxed{e^{-2\tau}\sin 2\tau}\, d\tau$$

we do PF but use a nicer form:

$$\frac{2}{(s^2+1)[(s+2)^2+4]} = \frac{As+B}{s^2+1} + \frac{C(s+2)+2D}{(s+2)^2+4}$$

$A\cos t + B\sin t$

$e^{-2t}[C\cos 2t + D\sin 2t]$

For A & B mult. by s^2+1 & let $s=j$:

root of s^2+1

$$\frac{2}{(j+2)^2+4} = Aj + B$$

$$\frac{1}{65}(14-8j) = Aj+B \Rightarrow \underline{A = \frac{-8}{65},\ B = \frac{14}{65}}$$

For C & D mult. by $(s+2)^2+4$ & Let $s = 2j-2$:

root of $(s+2)^2+4$

$$\frac{2}{(2j-2)^2+1} = C[2j-2+2] + 2D \Rightarrow \underline{C = \frac{8}{65},\ D = \frac{1}{65}}$$

$$\frac{1}{65}(2+16j) = 2jC + 2D$$

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so

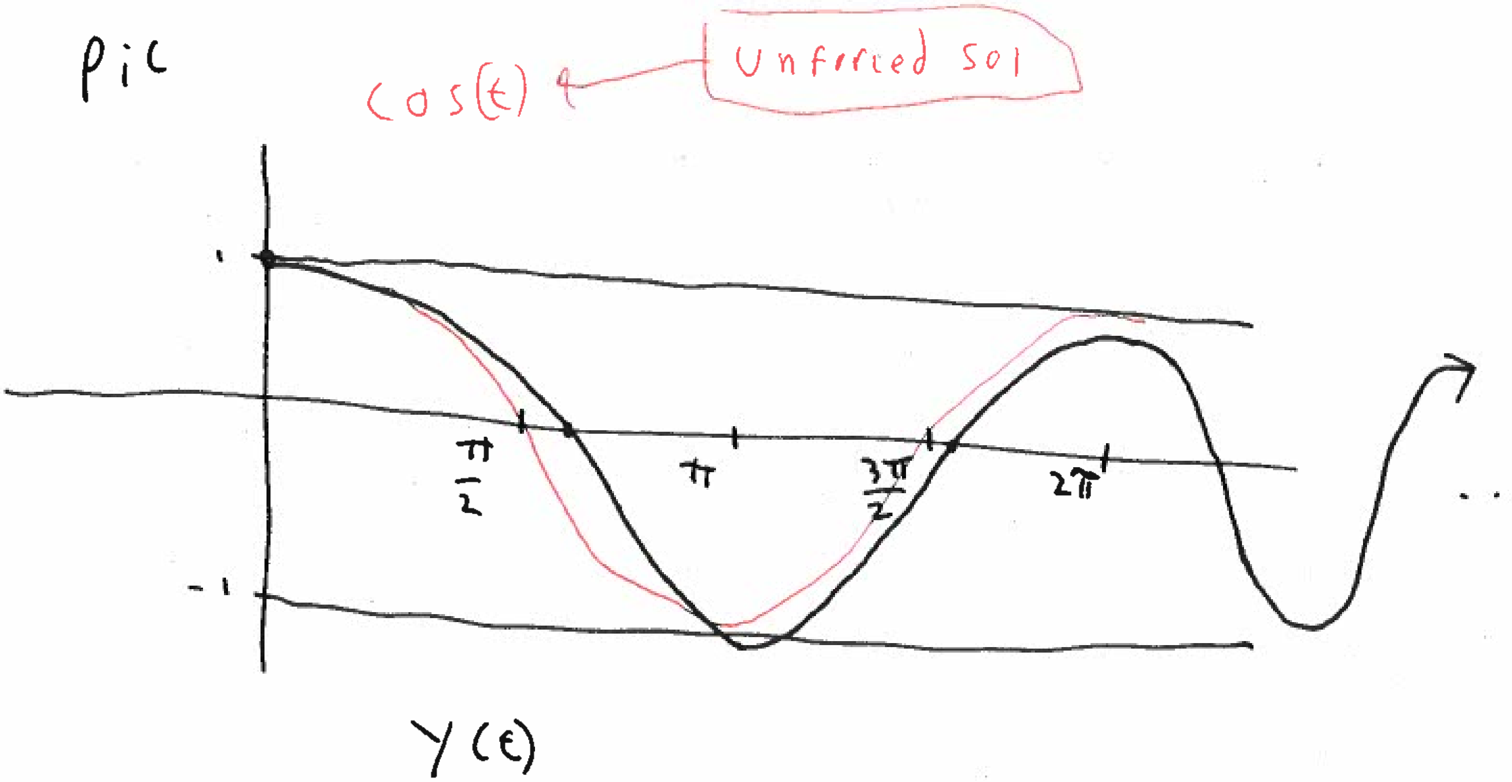
$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{65}\left[\frac{-8s+14}{s^2+1} + \frac{8(s+2)+2\cdot 1}{(s+2)^2+4}\right] + \frac{s}{s^2+1}\right\}$$

$$= \frac{1}{65}\left[-8\cos t + 14\sin t + e^{-2t}[8\cos 2t + \sin 2t]\right]$$

$$+ \cos t$$

$$= \frac{14}{65}\sin t + \frac{57}{65}\cos t + e^{-2t}\left[\frac{8}{65}\cos 2t + \frac{1}{65}\sin 2t\right]$$

pic



Example 3
Solve $y'' + 9y = 2\sin(3t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

$$\mathcal{L}\{y'' + 9y\} = \mathcal{L}\{2\sin 3t\}$$

$$s^2 Y(s) - s\,y(0) - y'(0) + 9Y(s) = 2\,\frac{3}{s^2+9}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{\boxed{s^2+9}}\left[\frac{6}{s^2+9} + s\cdot \underset{y(0)}{1} + \underset{y'(0)}{0}\right]$$

Char. poly

$$= \frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}$$

Kinda like sin but denom. is squared.

cos(3t)

The first term is not really in our table.

We could try to use the fact that $\frac{d}{ds}\left(\frac{1}{s}\right) = \frac{-1}{s^2}$ & the "mult. by t" property to deal with the first term but... not nice!

Convolution!!

$$\frac{6}{(s^2+9)^2} = \frac{2}{3}\cdot\frac{3}{s^2+9}\cdot\frac{3}{s^2+9}$$

sin 3t

so $$\mathcal{L}^{-1}\left\{\frac{6}{(s^2+9)^2}\right\} = \frac{2}{3}\sin 3t * \sin 3t$$

one sided

$$= \frac{2}{3}\int_0^t \sin(3\tau)\sin(3(t-\tau))\,d\tau$$

Options:

1) write as complex exponentials

2) "recall" $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

3) IBP

$$= \frac{1}{3}\int_0^t \cos(-3t+6\tau) - \cos(3t)\,d\tau$$

u-sub

$$= \frac{1}{3}\left[\frac{1}{6}\sin(6\tau-3t) - \tau\cos 3t\right]_0^t$$

$$= \frac{1}{3}\left[\frac{1}{6}\left[\sin(3t) - \sin(\overset{-3t}{\cancel{3t}})\right] - t\cos 3t\right]$$

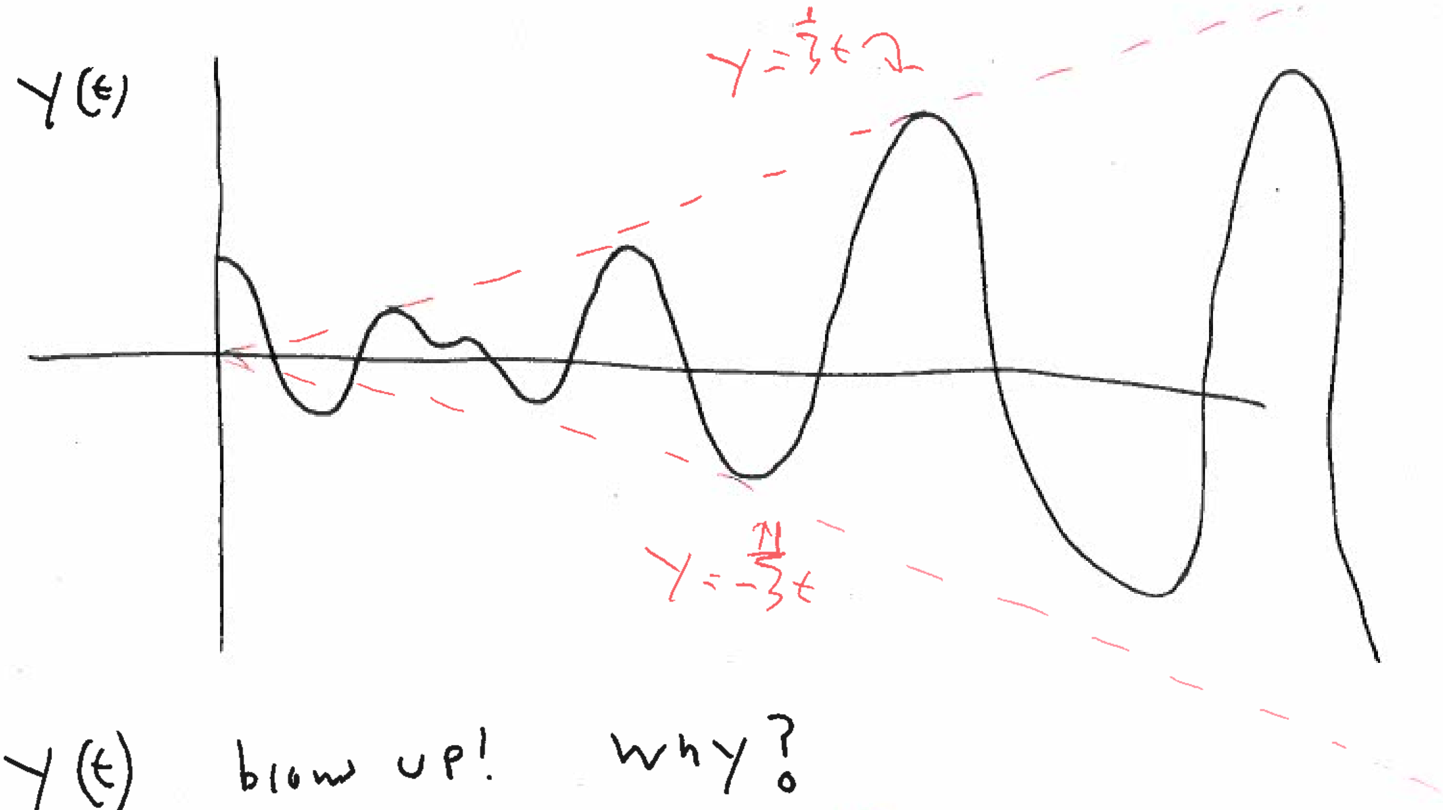
Recall $\sin(-t) = -\sin t$

$$= \frac{1}{3}\left[\frac{1}{3}\sin 3t - t\cos 3t\right]$$

$$= \frac{1}{9}\sin 3t - \frac{t}{3}\cos 3t.$$

$$Y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{6}{(s^2+9)^2} + \frac{s}{s^2+9}\right\}$$

$$= \boxed{\frac{1}{9}\sin(3t) - \frac{t}{3}\cos(3t)} + \cos(3t)$$



$Y(t)$ blow up! why?

$$Y(t) = \underbrace{\cos 3t + \tfrac{1}{9}\sin 3t}_{\text{bounded}} - \frac{t}{3}\cos(3t)$$

not bounded

This term came from $\mathcal{L}^{-1}\left\{\frac{6}{(s^2+9)^2}\right\}$ which came

from $\frac{1}{s^2+9} \cdot \frac{6}{s^2+9}$

$\frac{1}{\text{Char Poly}}$ $\quad$ $\mathcal{L}\{2\sin 3t\}$

Forcing term

the roots $s = \pm 3j$ of the char poly are roots of $\mathcal{L}\{\sin$

This is called <u>resonance</u> & can be bad. i.e. Tacoma bridge...

With this solution in mind note

$$\mathcal{L}\{t\cos wt\} = -\frac{d}{ds}\mathcal{L}\{\cos(wt)\}$$

$$= -\frac{d}{ds}\left(\frac{s}{s^2+w^2}\right)$$

$$= \frac{s^2-w^2}{(s^2+w^2)^2}$$

&

$$\mathcal{L}\{t\sin wt\} = \dots = \frac{2ws}{(s^2+w^2)^2}.$$

So we can guess the dcomp:

$$\frac{6}{(s^2+9)^2} = \underbrace{\frac{A\cdot 6s}{(s^2+9)^2}}_{\boxed{At\sin 3t}} + \underbrace{\frac{B(s^2-3^2)}{(s^2+9)^2}}_{\boxed{Bt\cos 3t}} + \underbrace{\frac{Cs+D}{s^2+9}}_{\boxed{C\cos 3t + D\sin 3t}}$$

For higher powers (i.e. $\frac{A}{(s^2+w^2)^n}$ we add the other needed terms.)

Example 4
Solve $y^{(3)} + 2y' = 2\sin(3t)$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$ and plot the solution.

$$\mathcal{L}\{y''' + 2y'\} = 2\sin(3t)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 2[sY(s) - y(0)] = \frac{6}{s^2+9}$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{\underset{\text{Char. poly.}}{s^3+2s}}\left[\frac{6}{s^2+9} + s^2\cdot \underset{y(0)}{1} + s\cdot \underset{y'(0)}{1} + \underset{y''(0)}{0} + 2\cdot \underset{y(0)}{1}\right]$$

$$= \frac{6}{s(s^2+2)(s^2+9)} + \frac{s^2}{s(s^2+2)} + \frac{s}{s(s^2+2)} + \frac{2}{s(s^2+2)}$$

$$= \underset{\text{Use PF}}{\frac{6}{s(s^2+2)(s^2+9)}} + \underset{\cos\sqrt{2}t}{\frac{s}{s^2+2}} + \underset{\sin\sqrt{2}t}{\frac{1}{s^2+2}} + \underset{\text{Use PF}}{\frac{2}{s(s^2+2)}}$$

$$\frac{6}{s(s^2+2)(s^2+9)} = \frac{A}{s} + \frac{Bs+C}{s^2+2} + \frac{Ds+E}{s^2+9}$$

$$\Rightarrow \quad A = \frac{1}{3}, \quad B = -\frac{3}{7}, \quad D = \frac{2}{21}, \quad C = E = 0$$

$$\frac{2}{s(s^2+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2}$$

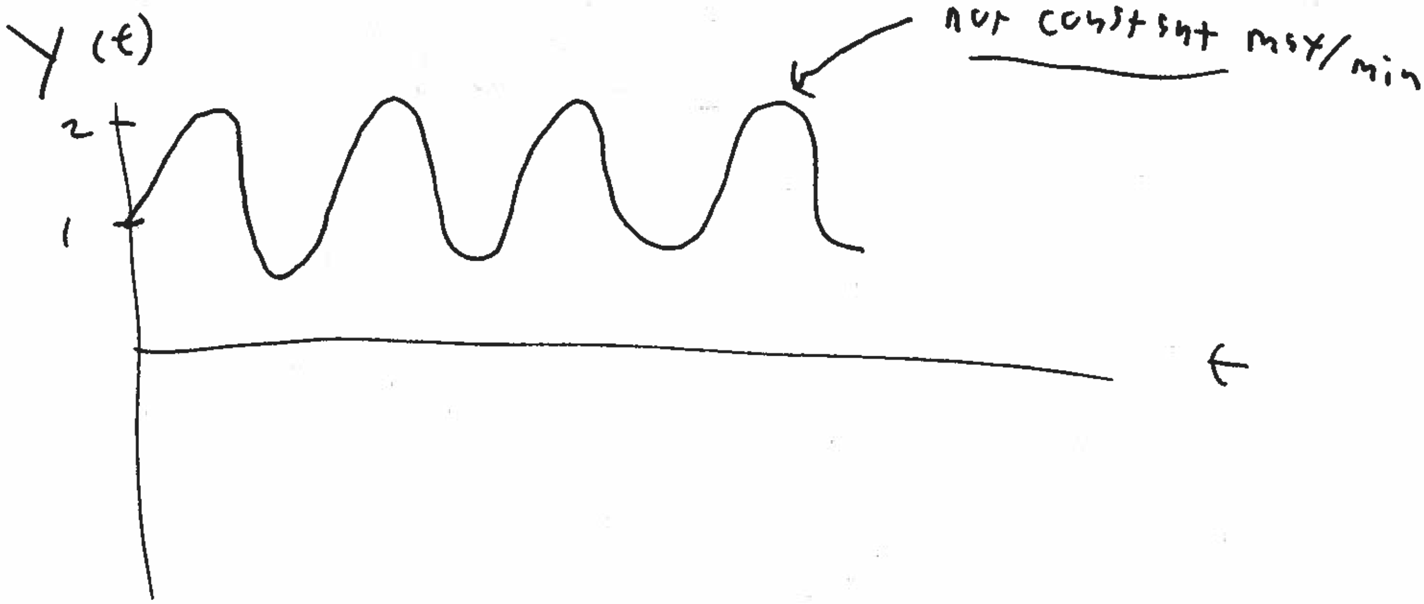
$$\Rightarrow \quad A = 1, \quad B = -1, \quad C = 0.$$

$$Y(s) = \frac{4}{3}\frac{1}{s} + \frac{1}{s^2+2} - \frac{3s}{7(s^2+2)} + \frac{2}{21}\frac{s}{s^2+9}$$

Do algebra to get this

so

$$Y(t) = \frac{4}{3} + \frac{1}{\sqrt{2}}\sin(\sqrt{2}\,t) - \frac{3}{7}\cos(\sqrt{2}\,t) + \frac{2}{21}\cos(3t)$$



Extensions of our method:

Our method also "works" for some Linear ODEs with non-constant coefficient. The problems are 1) evaluating the Laplace transform of $a(t)y^{(n)}(t)$ for some given function $a(t)$ and a unknown function $y^{(n)}$ and 2) we end up with another (but simpler) differential equation to solve. If $a(t) = t^n$ then this is particularly "nice"!

> **Example 5**
> Solve $y'' + 2ty' + y = 0$, $y(0) = 0$, $y'(0) = 1$ and plot the solution.

To take Laplace transform we need to know

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}\mathcal{L}\{y'\}$$

$$= -\frac{d}{ds}\left(sY(s) - y(0)\right)$$

$$= -\left(sY'(s) + Y(s)\right)$$

So

$$\mathcal{L}\{y'' + 2ty' + y\} = \mathcal{L}\{0\}$$

$$s^2Y(s) - s\cancelto{0}{y(0)} - \cancelto{1}{y'(0)} - 2sY'(s) - 2Y(s) + Y(s) = 0$$

or $$2sY'(s) + (1 - s^2)Y(s) = 1$$

or $$Y'(s) + \frac{1-s^2}{2s}Y(s) = \frac{1}{2s}$$

Magic (i.e. integration factors ← (not teaching at least yet))

$$Y(s) = c_1 e^{\frac{s^2}{4} - \frac{\ln s}{2}} - \frac{e^{s^2/4}\int_{\frac{s^2}{4}}^{\infty} t^{-\frac{3}{4}} e^{-t}\,dt}{2\sqrt{2}\sqrt{s}}$$

Taking $\mathcal{L}^{-1}$ gives a solution in terms of special functions

$$y(t) = c_1 e^{-t^2} H_{-\frac{1}{2}}(t) + c_2 \frac{e^{-t^2/2}\sqrt{t}\,\Gamma\left(\frac{3}{4}\right) I_{-\frac{1}{4}}\left(\frac{t^2}{2}\right)}{\sqrt{2}}$$

$$H_{-\frac{1}{2}}(t) = \frac{\Gamma\left(-\frac{1}{2}\right)}{2\pi j} \oint e^{-\tau^2+2\tau t} \sqrt{\tau}\, d\tau$$

$$I_{-\frac{1}{4}}(t) = \frac{1}{2\pi j} \oint e^{\frac{t}{2}\left(\tau+\frac{1}{\tau}\right)} \tau^{-\frac{3}{4}}\, d\tau$$

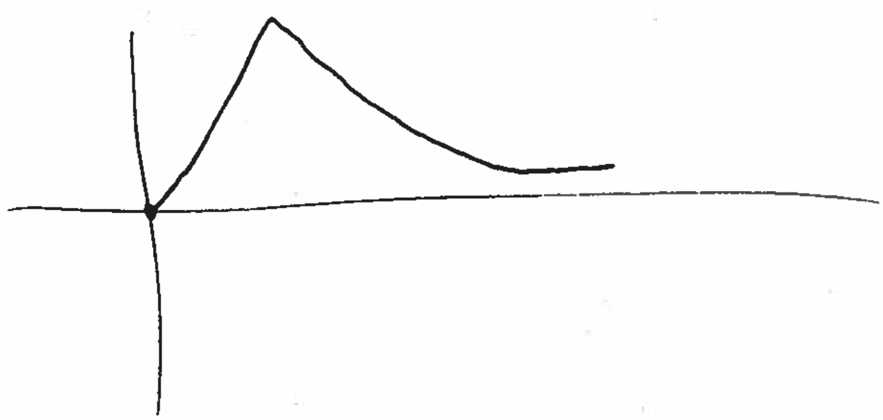
where $\oint$ is a contour containing $(0,0)$

&

$$\Gamma(x) = \int_0^\infty \tau^{x-1} e^{\tau}\, d\tau.$$

& c_1 & c_2 need to be picked to satisfy the ICs.

plot



I Do Not expect you to be able to solve this.

The coefficients need not be complex numbers, they can also be matrices. Consider a DE of the form

$$I\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

where a_i is a $n \times n$ matrix, I is the $n \times n$ identity matrix and the functions $y(t)$ $f(t)$ are n dimensional vectors.

Example 6: Minecraft Chickens
Suppose we have a population of chickens, $C(t)$ and eggs $E(t)$ and suppose that the eggs turn into chickens at a rate α, and the chickens lay eggs at a rate β derive a coupled system of DEs to model the populations of chickens and eggs.

$$\frac{d}{dt}\left(E(t)\right) = \underbrace{\beta\, C(t)}_{\text{making eggs}} - \underbrace{\alpha\, E(t)}_{\text{hatching eggs} \to \text{chickens}}$$

$$\frac{d}{dt}\left(C(t)\right) = \alpha\, E(t)$$

or

$$I\frac{d}{dt}\begin{bmatrix} E \\ C \end{bmatrix} + \underbrace{\begin{bmatrix} \alpha & -\beta \\ -\alpha & 0 \end{bmatrix}}_{a_0}\begin{bmatrix} E \\ C \end{bmatrix} = \vec{0}$$

Looking ahead:

- Many applications of ECE/software engineering involve solving a system of linear DEs (i.e. multivariable systems).
- In control theory, we often want to find initial conditions so that we can "solve for $f(t)$". i.e. pick $f(t)$ and the ICs so that the solution to the DE does something we want.