MATH 213 - Tutorial 6: Systems solutions

1. (Common midterm error) Use the definition of a linear system to formally prove that the system modelled by the the DE
$$y''(t) + 9y'(t) + 8y(t) = f(t)$$
is linear.

Solution: The response of this system to the signal $f(t)$ is
$$y(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot f(t-\tau)d\tau.$$
Thus if $S(f(t)) = y_f(t)$ and $S(g(t)) = y_g(t)$ then
$$\begin{aligned}
S(f(t)+g(t)) &= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot (f+g)(t-\tau)d\tau \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot (f(t-\tau)+g(t-\tau))d\tau \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot f(t-\tau) + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot g(t-\tau)d\tau \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot f(t-\tau)d\tau + \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot g(t-\tau)d\tau \\
&= y_f(t) + y_g(t)
\end{aligned}$$

2. (Common midterm error) Use the definition of a linear system to formally show that the system modelled by the the DE
$$y''(t) + 3y'(t) + 2y(t) = f(t) + 1$$
is nonlinear.

Solution: The response of this system to the signal $f(t)$ is
$$y(t) = \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\}\bigg|_\tau \cdot (f(t-\tau)+1)d\tau.$$
Thus if $S(f(t)) = y_f(t)$ and $S(g(t)) = y_g(t)$ then
$$\begin{aligned}
&S(f(t)+g(t)) \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot (f(t-\tau)+g(t-\tau)+1)d\tau \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot f(t-\tau) + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot g(t-\tau) + \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau d\tau \\
&= \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot f(t-\tau)d\tau + \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau \cdot g(t-\tau)d\tau + \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau d \\
&= y_f(t) + y_g(t) + \int_0^t \mathscr{L}^{-1}\left\{\frac{1}{s^2+9s+8}\right\}\bigg|_\tau d\tau
\end{aligned}$$

This system is linear if and only if the last integral is equal to 0 for all t. Since the inverse Laplace transform of this transfer is a PWC function, the integral is equal to 0 for all t exactly when the integrated is 0. Clearly the inverse Laplace transform is not uniformly 0 so this system is not linear.

3. (Common midterm error) Use the convolution theorem to compute

$$t^2e^t * \sin(t)$$

where both functions are one sided.

Solution: The convolution theorem tells us that

$$t^2e^t * sin(t) = \mathscr{L}^{-1}\left\{\mathscr{L}\{t^2e^t\} \cdot \mathscr{L}\{\sin(t)\}\right\}.$$

We first compute the transforms vie the Laplace table

$$\begin{aligned}
\mathscr{L}\{\sin(t)\} &= \frac{1}{s^2+1} \\
\mathscr{L}\{t^2e^t\} &= \mathscr{L}\{t^2\}\big|_{s-1} \\
&= \mathscr{L}\{t^2\}\big|_{s-1} \\
&= \frac{2}{(s-1)^3}
\end{aligned}$$

Hence

$$t^2e^t * \sin(t) = \mathscr{L}^{-1}\left\{\frac{2}{(s^2+1)(s-1)^3}\right\}.$$

To compute the inverse transform we preform partial fractions

$$\frac{2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^2} + \frac{E}{s-1}$$

Using the Heaviside coverup method to find A, B, and C and then solving a 2x2 system to find D and E gives us

$$\frac{2}{(s^2+1)(s-1)^3} = \frac{-1/2s+1/2}{s^2+1} + \frac{1}{(s-1)^3} - \frac{1}{(s-1)^2} + \frac{1/2}{s-1}$$

Computing the inverse Laplace transforms of each term gives

$$\begin{aligned}
\frac{-1/2s+1/2}{s^2+1} &= \frac{1}{2}(\sin(t)-\cos(t)) \\
\frac{1/2}{s-1} &= \frac{1}{2}e^t \\
-\frac{1}{(s-1)^2} &= \frac{d}{ds}\left(\frac{1}{s-1}\right) && \text{calculus} \\
&= -t\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} && \text{mult by } t \\
&= -te^t \\
\frac{1}{(s-1)^3} &= \frac{1}{2}\frac{d}{ds}\left(\frac{1}{(s-1)^2}\right) && \text{calculus} \\
&= (-1)^2t^2\frac{1}{2}\mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} && \text{mult by } t \\
&= \frac{1}{2}t^2e^t
\end{aligned}$$

Using these gives us

$$t^2e^t * \sin(t) = \frac{1}{2}(\sin(t)-\cos(t)) + \frac{1}{2}e^t - te^t + \frac{1}{2}t^2e^t$$

4. Find the system impulse response of the system modelled by the DE

$$y'''(t) + 3y''(t) + 2y'(t) = f(t)$$

Solution: The system impulse response is simply the inverse Laplace transform of the transfer function. Hence we need to compute

$$\mathscr{L}^{-1}\left\{\frac{1}{s^3 + 3s^2 + 2s}\right\}$$

Factoring gives $s^3 + 3s^2 + 2s = s(s^2 + 3s + 2) = s(s + 2)(s + 1)$ and hence we make the PF ansatz

$$\frac{1}{s^3 + 3s^2 + 2s} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+1}.$$

The coverup method gives

$$\frac{1}{s^3 + 3s^2 + 2s} = \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}.$$

Hence the system's impulse response is

$$h(t) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-t}$$