MATH 213 - Lecture 1: Introduction to Differential Equations (DEs)

Lecture goals: To understand how some DEs are derived, the importance of boundary conditions/initial conditions and how to solve some simple DEs.

What are DEs and where do they come from?:

In 1687 Isaac Newton published the now famous equation $F = ma$ in his book *Philosophiae Naturalis Principia Mathematica* now commonly known as the *Principia Mathematica.* In $F = ma$, F denotes the net force being applied to an object, m denoted the mass of the object and a is the acceleration of the object. This simple principle gives rise to many DEs.

Example 1: DE for the motion of the height of a ball
Consider a ball of mass m being influenced by only the force of gravity. Use $F = ma$ to find equations for the vertical height of the ball.

$$-mg = m\frac{d^2y}{dt^2}$$

Example 2: Solving a simple DE
Solve the DE you found in Example 1 to find an expression for the height of the ball as a function of time.

$$\frac{d^2y}{dt^2} = -g \Rightarrow \frac{dy}{dt} = -gt + c_1$$

$$\Rightarrow \boxed{y = -\frac{g}{2}t^2 + c_1t + c_2}$$

Example 3: Initial Conditions (ICs)
The solution in Example 2 is not unique. Determine the extra information you need to find the exact height of the ball as a function of time.
This information is known as the **initial conditions** or more generally (and depending on context) as the **boundary conditions**.

We need $\left.\frac{dy}{dt}\right|_{t=t_0}$ & $y(t_0)$

i.e. initial speed & height.

Example 4: Constant Growth
Suppose you have 1 E. coli bacteria (in Minecraft) at time $t = 0$ and it is known that each E. coli continuously splits to produce 5 new bacterium (we allow for fractional numbers of E. coli).

a. Find a DE for the number of E. coli as a function of time, $f(t)$.

b. State the initial condition(s).

c. Solve the **Initial Value Boundary Problem** (IVBP) found in parts a-b.

a) $\frac{df}{dt} = 5f$

Each E. coli makes 5 new ones

b) $f(0) = 1$

@ $t=0$ we have 1 E. coli

c) $f' = 5f \Rightarrow \frac{f'}{f} = 5 \Rightarrow \int \frac{f'}{f}\,dt = \int 5\,dt$

$\Rightarrow \ln|f| + C_1 = 5t + C_2$

$\Rightarrow \ln|f| = 5t + C_3$ $\quad \boxed{C_3 = C_2 - C_1}$

$\Rightarrow f = e^{5t + C_3}$

$\Rightarrow f = Ce^{5t}$ $\quad \boxed{C = e^{C_3}}$

$f(0) = Ce^0 = 1$ IC

$\Rightarrow C = 1$

So $\boxed{f(t) = e^{5t}}$

There is a slight problem with the previous model: In the real world, growth is not limitless!

But the number of E. coli in our previous model imply that there is a c~~ourse~~ase of...

which does not exist!!
To correct for this we need to include factors that limit the population growth so that the population remains bounded over time!

Example 5: Limited growth
Consider the new model for our E. coli population:

$$\frac{df}{dt} = af - bf^2$$

where $a, b \in \mathbb{R}^+$ are constants.

a. Suppose that $0 < f(0) \ll \frac{a}{b}$.
Without solving the above DE, find the "maximum" value for $f(t)$.

b. Solve the DE for $f(t)$.

a) $f(t)$ is at a max when $f'(t)=0$.

What is $f'(t)$?

$f'(t) = af - bf^2$ so max requires $f'=0$ (or undefin

$f'=0 \Rightarrow af - bf^2 = 0 \Rightarrow f(a-bf) = 0 \Rightarrow f=0$ or $f=\frac{a}{b}$

Since $a, b \in R^+$ ~~&~~ $0 < f(0) << \frac{a}{b}$, max is $\boxed{\frac{a}{b}}$

Note if $0 < f < \frac{a}{b}$ then $f' = af - bf^2 > 0$

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b) $\frac{df}{dt} = af - bf^2 \Rightarrow \frac{f'}{af - bf^2} = 1$

$$\Rightarrow \int \frac{df}{af - bf^2} = \int dt$$

How to integrate? Partial fractions!

$$\frac{1}{f(a-bf)} = \frac{A}{f} + \frac{B}{a-bf}$$ Find A & B

Option 1: S1: Multiply by $f(a-bf)$: $1 = A(a-bf) + Bf$

Bad

S2: Setup matrix: $1 = Aa$

$0 = -Ab + B$

S3: Solve $\boxed{A = \frac{1}{a}, \quad B = \frac{b}{a}}$

Option 2°: Plug in singular values $f=0$, $f=\boxed{a/b}$.

Better

if $f=0$ then eq is:

$$\frac{1}{a} = A + 0 \qquad \text{so } A = \frac{1}{a}$$

if $f=\boxed{a/b}$ then eq is:

$$\frac{1}{\boxed{a/b}} = 0 + B \qquad \text{so } B = \boxed{b/a}$$

Option 3°: Heaviside method: remove f to find A.

Best but depends on problem

$$\frac{1}{a} = A \quad \& \quad \frac{b}{a} = B$$

Now $\int dt = \int \frac{df}{af-bf^2} \Rightarrow t + C_1 = \int \frac{1}{af}\,df + \int \frac{b}{a}\frac{1}{a-bf}\,df$

$$= \frac{1}{a}\ln|f| + \frac{-b}{ab}\ln|a-bf| + C$$

$$\Rightarrow t + C_1 = \frac{1}{a}\ln\left|\frac{f}{a-bf}\right| + C_2$$

$$\Rightarrow at + C_3 = \ln\left|\frac{f}{a-bf}\right| \qquad \boxed{C_3 = aC_1 - C_2}$$

$$\Rightarrow \frac{f}{a-bf} = C_4 e^{at} \qquad C_4 = e^{C_3}$$

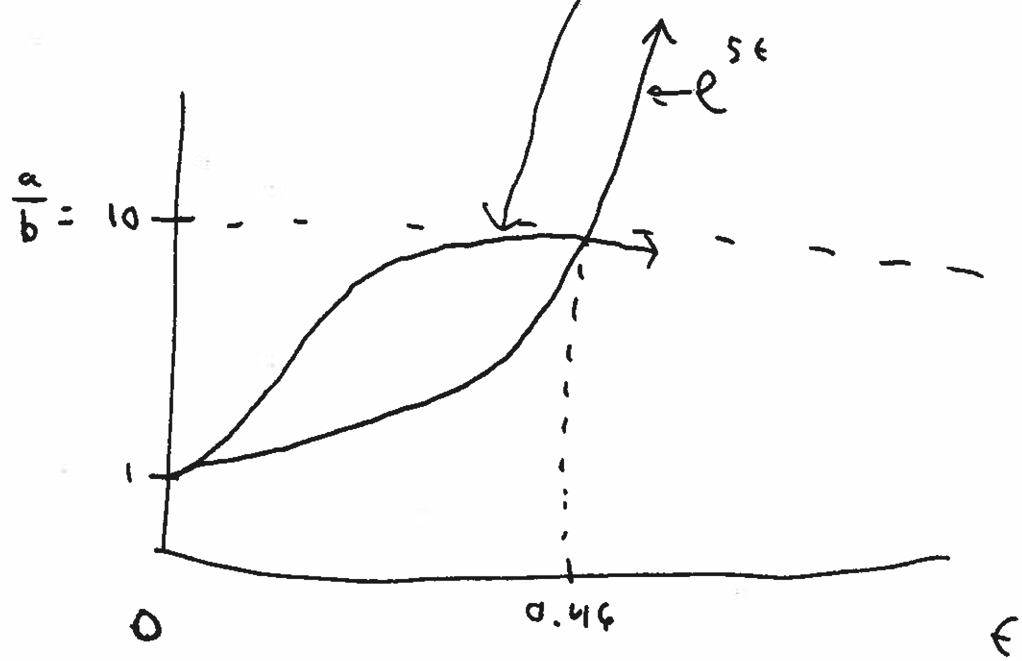
$$\Rightarrow f = C_4 e^{at}(a-bf)$$

$$\Rightarrow f = \frac{a}{b}\frac{1}{1+Ce^{-at}}$$

If $f(0)=1$ then $c = \frac{a}{b} - 1$ &

$$f(t) = \frac{a}{b} \frac{1}{1+\left(\frac{a}{b}-1\right) e^{-at}}$$

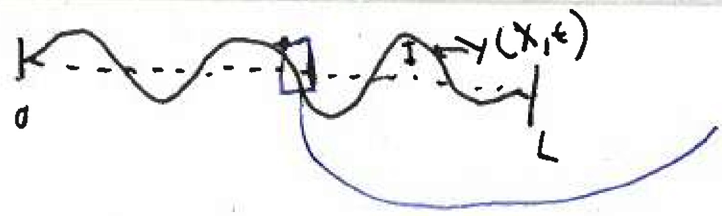
Now if $a = 50$ & $b = 5$ then the solutions to ex 4 & 5 look like

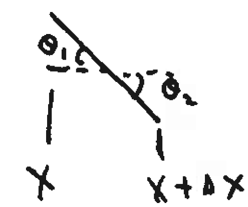


Sometimes things change over space and time leading to a **Partial Differential Equation** (PDE).

> **Example 6: Linear Wave Equation**
> Consider a string of length L, constant density ρ and under uniform tension T.
>
> a. Assuming the motion of the string is "small", find an equation for the perturbations, $y(x,t)$ of the string.
>
> b. Show that $C_1 \sin\left(k\left(x - t\sqrt{\frac{T}{\rho}}\right)\right) + C_2 \sin\left(k\left(x + t\sqrt{\frac{T}{\rho}}\right)\right)$ solves the equation from part a.
>
> c. Suppose our string is clamped at $x = 0, L$ so that $y(0,t) = y(L,t) = 0$. What limits do these boundary conditions impose of the solution given in part b?

a)





$$\underbrace{m}_{\rho\,\Delta x}\ \underbrace{a}_{\frac{\partial^2 y}{\partial t^2}} = F$$

approx

$$\rho\,\Delta x \frac{\partial^2 y}{\partial t^2} = T\sin\theta_1 - T\sin\theta_2 \qquad T := \text{constant tension}$$

$$\approx T\tan\theta_1 - T\tan\theta_2$$

$$\approx T\left.\frac{\partial y}{\partial x}\right|_{x} - T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}$$

or

$$\rho\frac{\partial^2 y}{\partial t^2} \approx T\,\frac{\left.\frac{\partial y}{\partial x}\right|_{x} - \left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}}{\Delta x}$$

In Limit

$$\rho\, y_{tt} = T\, y_{xx} \quad \text{or} \quad y_{tt} = \frac{T}{\rho}\, y_{xx}$$

b) if $Y_s = C_1 \sin\left(k\left(x - t\sqrt{\frac{T}{\rho}}\right)\right) + C_2 \underbrace{\sin\left(k\left(x + t\sqrt{\frac{T}{\rho}}\right)\right)}$,

Then $$\frac{\partial^2 Y_s}{\partial t^2} = -C_1 k^2 \frac{T}{\rho} \sin\left(k\left(x - t\sqrt{\frac{T}{\rho}}\right)\right) - C_2 k^2 \frac{T}{\rho}$$

& $$\frac{\partial^2 Y_s}{\partial x^2} = -C_1 k^2 \sin\left(k\left(x - t\sqrt{\frac{T}{\rho}}\right)\right) - C_2 k^2 \sin\left(k\left(x + t\sqrt{\frac{T}{\rho}}\right)\right)$$

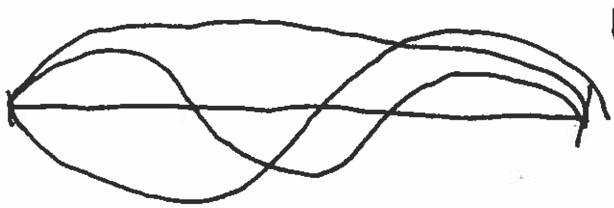
so Y_s solves PDE!

c) $Y_s(0,t) = 0 \Rightarrow C_1 = C_2$

$Y_s(L,t) = 0 \Rightarrow 0 = C_1 \sin\left(k\left(L - t\sqrt{\frac{T}{\rho}}\right)\right) + C_1 \sin\left(k\left(L + t\sqrt{\frac{T}{\rho}}\right)\right)$

$\Rightarrow kL = n\pi$ for $n \in \mathbb{Z}$

or $k = \frac{n\pi}{L}$



$n = 1, 2, 3$