Midterm reflection:

q3:

De	system
Linear: no powers of y other than 1	Linear $S(af+bg) = aS(f)+bS(g)$
ie $y''+7y=1$ is <u>Linear</u>	i.e. $y''+7y=1+f(x)$ is <u>non linear</u>!!!

q4: Def of convolution:

$$f*g = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

one sided

Convolution thm:

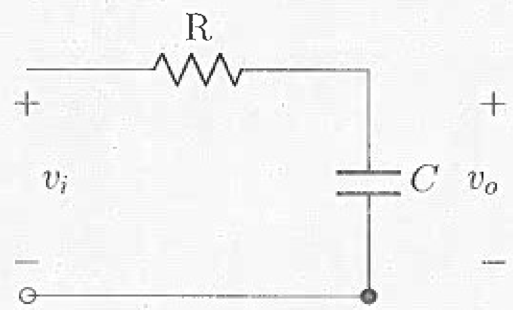
$$f*g = \mathcal{L}^{-1}\{F(s)\, G(s)\}$$

MATH 213 - Lecture 14: Systems Input and Responses

Lecture goals: Understand the connection between our previous material for DEs and systems. Know what the standard 1st order system is

If a system is able to be modelled by a linear DE with constant coefficients, i.e. $D(y(t)) = f(t)$, then the transfer function can be obtained from the differential equation.

Example 1: RC circuit
Find the transfer function for the system $S : v_i \to v_o$ given by the circuit



Verify that the transfer function is the system response of the impulse response.

DE is: $\quad RC\dfrac{dV_o}{dt} + V_o = V_i, \quad V_o(0^-) = 0$

Taking LT gives

$$RC\left[s\bar{V}_o(s) - V_o(0^-)\right] + \bar{V}_o(s) = \bar{V}_i(s)$$

$$\Rightarrow \quad \bar{V}_o(s) = \underbrace{\frac{1}{RCs+1}}_{H(s)} \bar{V}_i(s)$$

$$Y(s) = H(s) \quad F(s)$$

So $\boxed{H(s) = \dfrac{1}{RCs+1}}$

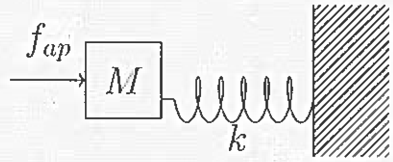
To check Let $V_{in}(t) = \delta(t)$,

$$\Rightarrow \bar{V}_i(s) = 1$$

$$\Rightarrow \bar{V}_o(s) = \frac{1}{sRC+1}$$

✓

Example 2: Linear Harmonic Oscillator
Consider the system $S : f_{ap} \to y$ given by the dampened linear harmonic oscillator



- Find the impulse response.
- Use the impulse response to find the general form of the system's response to a general input $f_ap(t)$

DE is $\quad my'' + ky = f_{ap}$

ZSR is $\quad Y(s) = \frac{1}{ms^2+k} F_{ap}(s)$

so $\quad H(s) = \frac{1}{ms^2+k}$

Impulse response is $\quad h(t) = \mathcal{L}^{-1}\left\{\frac{1}{ms^2+k}\right\}$

$$= \frac{1}{m}\mathcal{L}^{-1}\left\{\frac{1}{s^2+\frac{k}{m}}\right\}$$

$$= \frac{1}{\sqrt{km}}\mathcal{L}^{-1}\left\{\frac{\sqrt{k/m}}{s^2+\frac{k}{m}}\right\}$$

$$= \frac{1}{\sqrt{km}}\sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

"natural frequency"

In general

$$Y(s) = H(s)\, F_{ap}(s) \quad \text{so}$$

$$y(t) = \mathcal{L}^{-1}\{ H(s)\, F_{ap}(s) \}$$

- ~~$F_{ap}(s)$ generally has poles of its own!~~
  - ~~The system's response reflects the poles (and zeros) of $H(s)$ & $F_{ap}(s)$~~

- ~~The poles & zeros of $H(s)$ are particularly important~~

- ~~Note:~~
  - ~~-~~

A few things to note:

- In general $F_ap(s)$ can have poles of its own and thus the system response to will reflect the poles of both the transfer function, $H(s)$, and the LT of the forcing term $F_ap(s)$.
- The effects of the transfer function are present for any input so it is particularly important to understand the effects of the poles (and zeros) of $H(s)$.
- We will mostly look at the cases where the input is a unit impulse, $\delta(t)$, or a unit step, $u(t)$. Recall from A3 Q3 that for a second order DE these terms allow us to effectively set the initial condition at 0^+ of for the system.
  - The response to the unit impulse is $Y(s) = H(s)$.
  - The response to the unit step impulse is $Y(s) = \frac{1}{s} H(s)$. – The step response is the <u>integral</u> of the impulse response (subtle connection to L12...)

  In the "real world" it is often easier to physically generate a unit step function than a unity impulse so point 2 above gives us a nice way to compute transfer functions.

Understanding general system responses:

Recall that <u>all</u> polynomials with real valued coefficients can be factored into a product of linear and quadratic terms. Hence if we want to understand the system response of any system, it is sufficient to understand how first and second order linear systems respond. We will thus explore the standard 1st and 2nd order systems in detail. All other LIT's can be studied by taking a linear combination of the results of the standard systems.

Definition 1: Standard 1st order system
The standard 1st order system has the transfer function

$$H(s) = \frac{\kappa}{s\tau + 1}$$

where $\kappa, \tau > 0$. κ is called the **DC gain** and τ is called the **time constant.**

Example 3

Find and plot the impulse response of the standard 1st order system.

Determine if the standard first order system is causal.

3 Explore the effect of τ on the system's impulse response.
What does the location of the pole tell you about the speed that the system's impulse response decays?

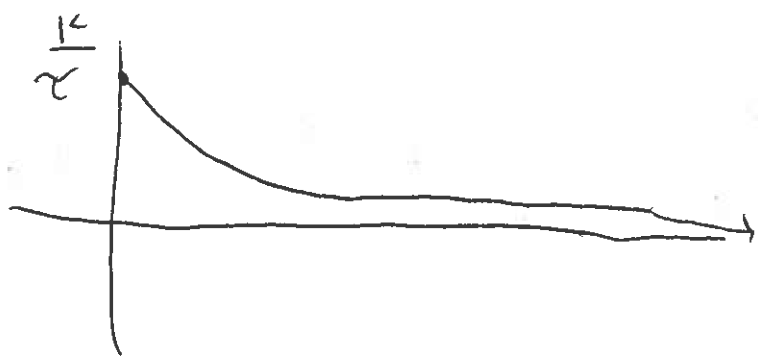
4 Find and plot the system's step response.

5 Explore the effect of κ on the system's step response.
Use the final value theorem to compute the long term behaviour of the system's step response.

$H(s) = \frac{K}{s\tau+1}$ has a pole at $s = -\frac{1}{\tau}$. thus from LT

$$h(t) = \frac{K}{\tau} e^{-t/\tau} U(t)$$

plot:



general solution is:

$$y(t) = h(t) * f(t) = \int_0^t h(\tau) f(t-\tau)\, d\tau$$

so is causal.

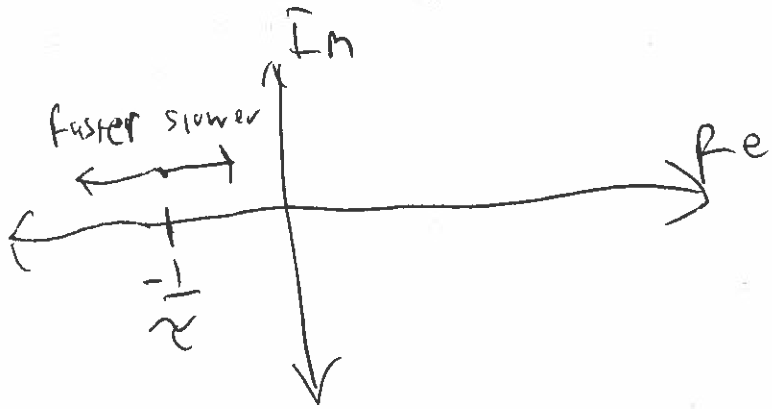
3. Changing τ

1.) Changes the value of $h(0)$ (inversely)

2.) Changes the "growth rate"

i.e τ increases $\rightarrow$ slower decay

τ decreases $\rightarrow$ faster decay



Thus the name "Time constant"

4) Step response is $Y(s) = H(s)\frac{1}{s}$

$$= \frac{K}{s(s\tau+1)}$$

so $\underline{Y(t) = K\left[1-e^{-t/\tau}\right]U(t)}$

plot

5) Changing K

Increases the "Steady State Value" of $y(t)$

explicitly, all poles have $Re < 0$ so

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s\,Y(s)$$

$$= \lim_{s\to 0} \cancel{s}\, H(s)\, \frac{1}{s}$$

$$= \lim_{s\to 0} H(s) = \boxed{H(0)}$$

$$H(0) = \frac{K}{\tau \cdot 0 + 1} = \underline{\underline{K}}$$

In general $H(0)$ is the <u>DC gain</u>

Note that τ again controls how fast we reach "steady state"