MATH 213 - Tutorial 5: Midterm Review solutions

1. (Easy) Find an expression for the PF decomposition of any function of the form

$$f_n(s) = \frac{1}{(s+1)(s+2)\cdots(s+n)}$$

for $n \geq 1$. You must explicitly find expressions for all the coefficients.

Solution: The PF ansatz is

$$f_n(s) = \frac{A_1}{s+1} + \ldots + \frac{A_n}{s+n}.$$

To find each coefficient A_i we use the Heaviside cover-up method and multiply by $s+i$ and then substitute in $s = -i$. This gives

$$A_i = \frac{1}{(-i+1)(-i+2)\cdots(-i+(i-1))(-i+(i+1))\cdots(-i+n)}$$

More cleanly we can write

$$A_i = \lim_{s \to -i} (s+i) f_n(s).$$

Hence the expression for the PF decomposition is the ansatz we picked where each coefficient is given by your choice of the expressions above.

2. (Medium [hard if you don't use nice tricks]) Find the zero-state solution to

$$y^{(4)} + 2y'' + y = \sin(2x).$$

You may leave your answer as a convolution of two functions of s.

Solution: To find the zero state solution we need to convolve the forcing term with the inverse Laplace transform of the transfer function. The transfer function is

$$T(s) = \frac{1}{s^4 + 2s^2 + 1} \qquad \text{or} \qquad T(s) = \frac{1}{(s^2+1)^2}.$$

To find the inverse transform of this we have some options:

(a) Use the Laplace table to note that this looks like a trigonometric transform but is squared, hence we can use the multiplication be t rule to rewrite $T(s)$ as the derivative of a trig transform potentially plus some other trig terms. This idea is motivated from the example in lecture 6.

(b) Look for a PF decomposition that will give us linear polynomials multiplied by sin/cos terms (this is motivated by the results we saw in lecture 6)

(c) Use the convolution theorem.

In class we showed off options (b) and (c) so to mix it up we will use option (a). Note that the multiplication by t property of the Laplace transform gives us that

$$t(A\cos(t) + B\sin(t)) = \mathscr{L}^{-1}\left\{-\frac{d}{ds}\left(\frac{As+B}{s^2+1}\right)\right\}$$

Direct computations give

$$\frac{d}{ds}\left(\frac{As+B}{s^2+1}\right) = \frac{(As+B)(2s) - A(s^2+1)}{(s^2+1)^2} = \frac{As^2 + 2Bs - A}{(s^2+1)^2}.$$

Hence,

$$t(A\cos(t) + B\sin(t)) = \mathscr{L}^{-1}\left\{\frac{-As^2 - 2Bs + A}{(s^2+1)^2}\right\}$$

Our transfer function cannot be written in this form so we need to also add in the aforementioned unscaled trig terms. Note that

$$C\cos(t) + D\sin(t) = \mathscr{L}^{-1}\left\{\frac{Cs+D}{s^2+1}\right\}.$$

We hence want to decompose $T(s)$ into the form

$$\frac{As^2+2Bs-A}{(s^2+1)^2} + \frac{Cs+D}{s^2+1}.$$

We can do this by either algebraically manipulating $T(s)$ or we can use the methods of PF decomposition. We choose the later since it is more automatic. Hence we want A, B, C and D such that

$$\frac{1}{(s^2+1)^2} = \frac{As^2+2Bs-A}{(s^2+1)^2} + \frac{Cs+D}{s^2+1}$$

Scaling by $(s^2+1)^2$ and using $s = -j$ gives

$$1 = A(j)^2 + 2Bj - A \qquad \text{or} \qquad 1 = 2Bj - 2A$$

so $B = 0$ and $A = -\frac{1}{2}$. Next using $s = 0$ and $s = 1$ along with these results gives the system

$$\begin{aligned} 1 &= \frac{1}{2} + D \\ \frac{1}{4} &= 0 + \frac{C+D}{2} \end{aligned}$$

or $D = \frac{1}{2}$ and $C = 0$. Hence,

$$T(s) = -\frac{1}{2}\frac{s^2-1}{(s^2+1)^2} + \frac{1}{2}\frac{1}{s^2+1}$$

and therefore

$$\mathscr{L}^{-1}\{T(s)\} = -\frac{1}{2}t\cos(t) + \frac{1}{2}\sin(t).$$

The zero-state solution is hence

$$y(t) = \left(-\frac{1}{2}t\cos(t) + \frac{1}{2}\sin(t)\right) * \sin(2t).$$

or

$$y(t) = \int_0^t \left(-\frac{1}{2}\tau\cos(\tau) + \frac{1}{2}\sin(\tau)\right) * \sin(2(t-\tau))d\tau$$

3. (Easy) Compute the convolution of the vectors $\vec{x} = [1, 2, 3, 4]$ and $\vec{y} = [-1, 1]$

Solution: $(\vec{x} * \vec{y})[i] = \sum_j x[j]y[i-j]$. Hence we have

$$\begin{aligned} (\vec{x} * \vec{y})[1] &= 1 \cdot -1 &&= -1 \\ (\vec{x} * \vec{y})[2] &= 2 \cdot -1 + 1 \cdot 1 &&= -1 \\ (\vec{x} * \vec{y})[3] &= 3 \cdot -1 + 2 \cdot 1 &&= -1 \\ (\vec{x} * \vec{y})[4] &= 4 \cdot -1 + 3 \cdot 1 &&= -1 \\ (\vec{x} * \vec{y})[4] &= \qquad\quad 4 \cdot 1 &&= 4 \end{aligned}$$

4. (Easy) Compute the convolution of $f(x) = x^2u(x)$ with

$$g(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 \le x < 2 \\ 0 & else. \end{cases}$$

Solution: We can either shift f or shift g. Since shifting g involves messing with the conditions for the piecewise function and that is kinda annoying to do, I will instead shift f. Hence we use the fact that the functions are one sided to compute

$$\begin{aligned}(f * g)(x) &= \int_0^x g(\xi)f(\xi - x)d\xi \\ &= \int_0^x g(\xi)(\xi - x)^2 d\xi \\ &= \int_0^x g(\xi)(\xi^2 - 2x\xi + x^2)d\xi\end{aligned}$$

There are four cases defined by $g(x)$ and we solve each one in turn:

- $x \le 0$: In this case there is no overlap and hence $(f * g)(x) = 0$.
- $0 < x < 1$: In this case

$$\begin{aligned}(f * g)(x) &= \int_0^x 1(\xi^2 - 2x\xi + x^2)d\xi \\ &= \left(\frac{\xi^3}{3} - x\xi^2 + x^2\xi\right)\Bigg|_{\xi=0}^{\xi=x} \\ &= \frac{x^3}{3}.\end{aligned}$$

- $1 \le x < 2$: In this case

$$\begin{aligned}(f * g)(x) &= \int_0^1 1(\xi^2 - 2x\xi + x^2)d\xi + \int_1^x 2(\xi^2 - 2x\xi + x^2)d\xi \\ &= \left(\frac{\xi^3}{3} - x\xi^2 + x^2\xi\right)\Bigg|_{\xi=0}^{\xi=1} + 2\left(\frac{\xi^3}{3} - x\xi^2 + x^2\xi\right)\Bigg|_{\xi=1}^{\xi=x} \\ &= \left(x^2 - x + \frac{1}{3}\right) \qquad + 2\left(\frac{x^3}{3} - \left(x^2 - x + \frac{1}{3}\right)\right) \\ &= \frac{2x^3}{3} - \left(x^2 - x + \frac{1}{3}\right)\end{aligned}$$

- $2 \le x$: In this case

$$\begin{aligned}(f * g)(x) &= \int_0^1 1(\xi^2 - 2x\xi + x^2)d\xi + \int_1^2 2(\xi^2 - 2x\xi + x^2)d\xi + \int_2^x 0(\xi^2 - 2x\xi + x^2)d\xi \\ &= \left(\frac{\xi^3}{3} - x\xi^2 + x^2\xi\right)\Bigg|_{\xi=0}^{\xi=1} + 2\left(\frac{\xi^3}{3} - x\xi^2 + x^2\xi\right)\Bigg|_{\xi=1}^{\xi=2} \\ &= \left(x^2 - x + \frac{1}{3}\right) \qquad + 2\left(\left(2x^2 - 4x + \frac{8}{3}\right) - \left(x^2 - x + \frac{1}{3}\right)\right) \\ &= 3x^2 - 7x + \frac{15}{3}\end{aligned}$$

Hence

$$(f * g)(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{3} & 0 \le x < 1 \\ \frac{2x^3}{3} - x^2 + x - \frac{1}{3} & 1 \le x < 2 \\ 3x^2 - 7x + \frac{15}{3} & 2 \le x \end{cases}$$

5. (easy) Determine if the function $f(x)$ with Laplace transform is bounded.

$$F(s) = \frac{s^5}{(s+2)(s^2+8s+33)(s+9)(s^2+1)}$$

Solution: Since $F(s)$ is a proper rational function, to determine if f is bounded it is sufficient to examine the locations of the poles.

Noting that (by computing the square)

$$(s+2)(s^2+8s+33)(s+9)(s^2+1) = (s+2)((s+4)^2+17)(s+9)(s^2+1)$$

we can simply read the poles off as $s = -2, -4 \pm j\sqrt{17}, -9, \pm j$. The first 4 poles have real parts less than 0 and hence exponentially decay but the last two roots are on the Im axis. Hence the solution has some decaying terms and some oscillating terms but remains bounded.