# MATH 213 - Tutorial 9: Fourier series Solutions

1. a) Compute the Fourier series of $f(x) = x^2$ on $-\pi < x < \pi$.
   b) Draw a picture of the periodic continuation of $f$ on the interval from $-3\pi$ to $3\pi$.
   c) Plot the truncated Fourier series to $N = 8$ (Using some software). Do you see Gibbs phenomena in this case?

**Solution:**

a) First, note that $x^2$ is even and therefore we know The Fourier sin coefficients are zero. The coefficients for this Fourier cosine series are computed directly. First for $n \geq 1$,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos(nx)\, dx, \\
&= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx)\, dx, \\
&= \frac{2}{n\pi}\left[x^2 \sin(nx)\right]_0^{\pi} - \frac{4}{n\pi}\int_0^{\pi} x \sin(nx)\, dx, \\
&= -\frac{4}{n^2\pi}\left(-\left[x\cos(nx)\right]_0^{\pi} + \int_0^{\pi}\cos(nx)\, dx\right), \\
&= -\frac{4}{n^2\pi}\left(-\pi\cos(n\pi) + \frac{1}{n}\left[\sin(nx)\right]_0^{\pi}\right), \\
&= \frac{4}{n^2}\cos(n\pi) \\
&= \frac{(-1)^n 4}{n^2}.
\end{aligned}$$

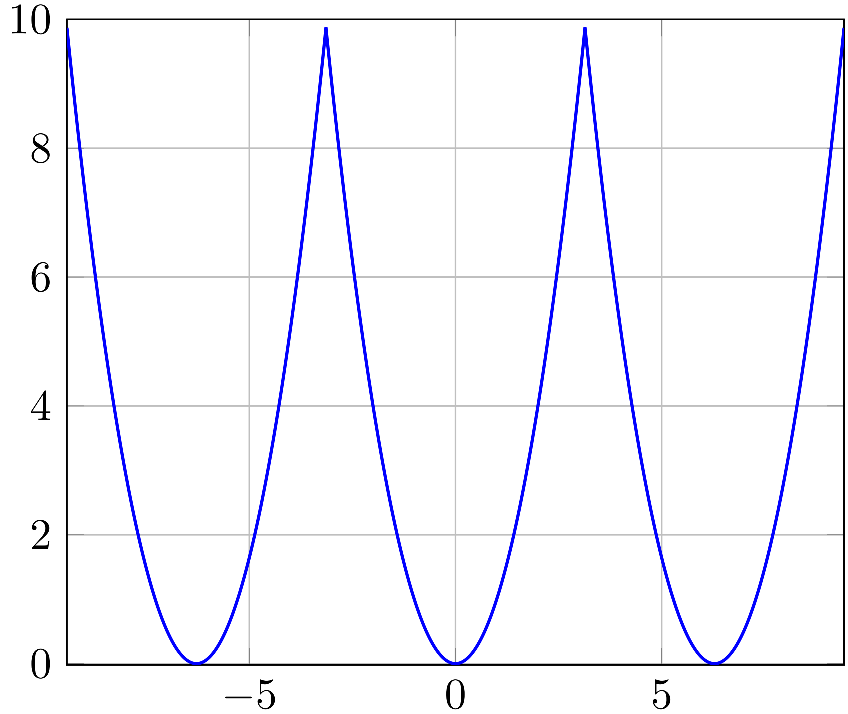
Then for $n = 0$,

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{\pi} x^2\, dx, \\
&= \frac{1}{3\pi}\left[x^3\right]_0^{\pi}, \\
&= \frac{\pi^2}{3}.
\end{aligned}$$

Therefore the Fourier series is

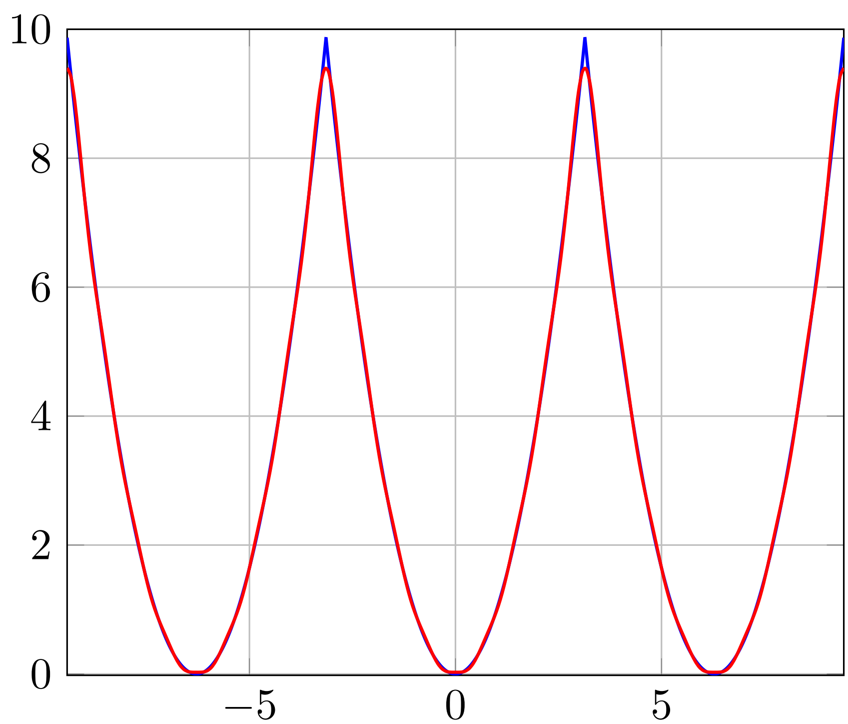
$$\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos(nx)}{n^2}.$$

b)




c) We plot the original plot along side

$$\frac{\pi^2}{3} + 4\sum_{n=1}^{8}(-1)^n \frac{\cos(nx)}{n^2}.$$




2. Recall that a function is $C^1$ if it is **differentiable and its derivative is continuous**.

Marmie found that

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is $C^1$ on $\mathbb{R}$ but wants you yo double check her work. Show that $f(x)$ is $C^1$ on all of $\mathbb{R}$.

Hint 1: Consider two cases, $x \neq 0$ and $x = 0$.

Hint 2: At $x = 0$ you must use the definition of the derivative as the limit of the difference quotient (from calc 1) in order to compute the derivative.

Hint 3: The squeeze theorem is a thing.

**Solution:** If $x \neq 0$ then we can use the product rule to write

$$\begin{aligned} \frac{d}{dx}\left(x^3 \sin\left(\frac{1}{x}\right)\right) &= 3x^2 \sin\left(\frac{1}{x}\right) - x^3 x^{-2} \cos\left(\frac{1}{x}\right) \\ &= 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right). \end{aligned}$$

Clearly this is a well-defined and continuous for all $x \neq 0$ and thus $f$ is $C^1$ on $\mathbb{R}/\{0\}$.

At $x = 0$ by definition we have

$$\begin{aligned} f'(0) &= \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(0)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta x^3 \sin\left(\frac{1}{\Delta x}\right) - 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} \Delta x^2 \sin\left(\frac{1}{\Delta x}\right). \end{aligned}$$

By the squeeze theorem the above limit is 0. Explicitly

$$-(\Delta x)^2 \leq \Delta x^2 \sin\left(\frac{1}{\Delta x}\right) \leq (\Delta x)^2$$

and in the limit the bounding terms go to zero. Thus

$$f'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

By the squeeze theorem $\lim_{x \to 0} f'(x) = 0$ and thus $f$ is $C^1$.

3. Consider the function $g(t) = |\sin(t)|$ on the interval $t \in (0, \pi)$.

   a) Find the complex Fourier series of $g(t)$. Hint: To evaluate the integral, it may be helpful to rewrite $\sin(t)$ in terms of exponential functions by using Euler's formula ($e^{i\theta} = \cos(\theta) + i \sin(\theta)$).

   b) Use this Fourier series along with the assumption that the series converges to $g(t)$ (it does and we will be able to prove it later) to show that

   $$\sum_{n=1}^{\infty} \frac{4}{\pi(4n^2 - 1)} = \frac{2}{\pi}$$

**Solution:**

a) The function $g(t)$ has period $\tau = \pi$ and hence the frequency increment is$\omega_0 = 2\pi/\tau = 2$. The complex Fourier coefficients are computed by

$$c_n = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin(t)|e^{-in\omega_o t}dt.$$

Since $|\sin(t)|$ is periodic with period $\pi$, then we can rewrite $c_n$ as

$$c_n = \frac{1}{\pi}\int_0^{\pi} \sin(t)e^{-in\omega_o t}dt$$

and using Euler's formula, replace $\sin(t)$ as follows

$$c_n = \frac{1}{2i\pi}\int_0^{\pi}(e^{it} - e^{-it})e^{-in2t}dt = \frac{1}{2i\pi}\int_0^{\pi}(e^{-i(2n-1)t} - e^{-i(2n+1)t})dt.$$

Solving the integral leads to

$$\begin{aligned} c_n &= \frac{1}{2i\pi}\left[\frac{-e^{-i(2n-1)t}}{(2n-1)i} + \frac{e^{-i(2n+1)t}}{(2n+1)i}\right]_0^{\pi} \\ &= \frac{1}{2\pi}\left[\frac{e^{-i(2n-1)\pi}}{(2n-1)} - \frac{e^{-i(2n+1)\pi}}{(2n+1)} - \frac{1}{2n-1} + \frac{1}{2n+1}\right]. \end{aligned}$$

From Euler's formula, we have that $e^{\pm i\pi} = \cos(\pm\pi) + i\sin(\pm\pi) = \cos(\pi) = -1$. In a similar manner, $e^{-2in\pi} = 1$. It follows that

$$\begin{aligned} c_n &= \frac{1}{2\pi}\left[\frac{-1}{2n-1} + \frac{1}{2n+1} - \frac{1}{2n-1} + \frac{1}{2n+1}\right] \\ &= \frac{1}{2\pi}\left[\frac{-2}{2n-1} + \frac{2}{2n+1}\right] \\ &= \frac{-2}{\pi(4n^2-1)}. \end{aligned}$$

Therefore, the complex form of the Fourier series of $f(t) = |\sin(t)|$ is

$$\sum_{n=-\infty}^{\infty} \frac{-2e^{2int}}{\pi(4n^2-1)}.$$

b) Since the $\pi$-periodic extension of $|\sin(t)|$ is continuous a theorem we will have later along with our work in part a) implies that

$$\sum_{n=-\infty}^{\infty} \frac{-2e^{2int}}{\pi(4n^2-1)} = |\sin(t)|$$

for all $t \in \mathbb{R}$. Examining this expression when $t = 0$ gives

$$\sum_{n=-\infty}^{\infty} \frac{-2e^{2in0}}{\pi(4n^2-1)} = |\sin(0)| \quad \text{or} \quad \sum_{n=-\infty}^{\infty} \frac{-2}{\pi(4n^2-1)} = 0.$$

Using the symmetry of the summand we rewrite the sum as

$$0 = \sum_{n=-\infty}^{-1}\left(\frac{-2}{\pi(4n^2-1)}\right) + \frac{-2}{\pi(4\cdot 0^2-1)} + \sum_{n=1}^{\infty}\frac{-2}{\pi(4n^2-1)} = \frac{2}{\pi} + 2\sum_{n=1}^{\infty}\frac{-2}{\pi(4n^2-1)}.$$

Thus

$$\sum_{n=1}^{\infty}\frac{4}{\pi(4n^2-1)} = \frac{2}{\pi}$$

as desired.