# MATH 213 - Lecture 22: Real Fourier series and convergence

Lecture goals: Know how to find the real Fourier series quickly and know the convergence properties of Fourier series. Get you to fill out the SCP survey.

## $\tau$ periodic functions:

**Definition 1: $\tau$ periodic functions**
A $f$ function defined on $\mathbb{R}$ is $\tau$ **periodic** if for all $t \in \mathbb{R}$

$$f(t) = f(t + \tau).$$

Generally, we pick the smallest value of $\tau$ such that the above holds.

Theorem 3 from L21 holds for $\tau$ periodic functions and we would integrate over one period.

If I ask you to find the Fourier series without telling you a domain for $f$ and $f$ is $\tau$ periodic then you first find out the $\tau$ period of $f$ and do the computation over a period of $f$.

## Fourier sin/cos series:

For a real valued function sometimes we found the Fourier series simplified to a sum of sine waves and sometimes it resulted in a sum of cos waves. We now elaborate on this:

**Definition 2: Even and odd functions**
A function $f$ is **even** if

$$f(-t) = f(t).$$

A function $f$ is **odd** if

$$f(-t) = -f(t)$$

**Example 1**
$\sin(x)$ is an odd function. $\cos(x)$ is an even function.

**Theorem 1**
For a real valued, $\tau$ periodic function $f \in L^2([-\tau/2, \tau/2])$:

- If $f$ is an even then the Fourier series can be simplified to a sum of cos waves.

- If $f$ is an odd then the Fourier series can be simplified to a sum of sin waves.

The sums above are called **Fourier cosine series** and **Fourier sine series** respectively.

For completion we will write Theorem 3 from L21 for sin and cos series.

**Theorem 2: Fourier Sine and Cosine Coefficients**
If $f$ is a real valued function that is in $L^2([-\tau/2, \tau/2])$ then

- If $f$ is even then the Fourier cosine series for $f$ is

$$\sum_{n=0}^{\infty} c_n \cos\left(\frac{2\pi n}{\tau} t\right)$$

where

$$c_n = \begin{cases} \langle f(t), 1 \rangle & n = 0 \\ 2\left\langle f(t), \cos\left(\frac{2\pi n}{\tau} t\right)\right\rangle & n > 0 \end{cases}$$

- If $f$ is odd then the Fourier sine series for $f$ is

$$\sum_{n=1}^{\infty} s_n \sin\left(\frac{2\pi n}{\tau} t\right)$$

where

$$s_n = 2\left\langle f(t), \sin\left(\frac{2\pi n}{\tau} t\right)\right\rangle$$

**Sketch of proof:**

- Start with the coefficients in Theorem 3,
- write the complex exponential in terms of sin/cos waves,

- simplify the integrals by using the properties that
  - the integral of an odd function over $[-a, a]$ for any $a \in \mathbb{R}$ is 0,
  - that $g(-x) + g(x) = 0$ for an odd function $g$ and
  - that $g(-x) + g(x) = 2g(x)$ for an even function $g$.

I strongly suggest not blindly memorizing this but to instead memorize Theorem 3 and then understand how it sill simplify if there are no sin or cos terms.

This is similar to the computations seen in the last lecture

**Theorem 3**
If $f$ is real then we can decompose it into even and odd functions as follows:

$$f_{even}(t) = \frac{f(t) + f(-t)}{2} \qquad \text{and} \qquad f_{odd}(t) = \frac{f(t) - f(-t)}{2}.$$

$f(t) = f_{even}(t) + f_{odd}(t)$.

**Sketch of proof:** Show that $f_{even}$ is even, that $f_{odd}$ is odd and that when summed we get $f$.

**Theorem 4**
Every real valued function in $L^2([-\tau/2, \tau/2])$ admits a real valued Fourier series with some sin and/or cos terms. i.e. the complex form will always simplify to sin and cos terms.

**Sketch of proof:** Since $f$ is real, it can be decomposed into an even and odd part. The even part gives us a cos series and the off part gives us a sin series.

**Example 2**

Find the Fourier series for the $\pi$ periodic version of

$$f(x) = x$$

**Example 3**

Find the Fourier series for the $\pi$ periodic version of

$$f(x) = x^2 + x + 1$$

## Types of Convergence:

In MATH 119 you dealt with the convergence of series of real numbers. Things like

$$\sum_{i=1}^{\infty} 0.9^i.$$

In these cases there is a natural way to define convergence (though this is not actually unique).

With sums of functions like this

$$\sum_{i=0}^{\infty} f_i(x),$$

there are many different ways to define convergence:

**Definition 3: Some Types of Convergence**
If $f_1, f_2, \ldots, f_n \ldots$ is a sequence of $L^2$ functions defined on $[a, b]$ then we say that

- the sequence **converges in the $L^2([a, b])$ norm**, or **converges in the mean** or **converges almost everywhere**, to $f$ if

$$\lim_{n\to\infty} \sqrt{\int_a^b |f_n(x) - f(x)|^2 dx} = 0$$

  tldr; the "average error" goes to 0.

- the sequence **pointwise converges** to $f$, if for any $x \in [a, b]$

$$\lim_{n\to\infty} (f_n(x) - f(x)) = 0$$

  tldr; the error at each point goes to 0.

- The sequence **uniformly converges** to $f$ if

$$\lim_{n\to\infty} \max_{[a,b]} |f_n(x) - f(x)| = 0$$

  If the maximum does not exist then we replace it with the smallest upper bound (called the sup).

  tldr; the maximum error converges to 0.

Note that each of the things we take the limit of are just numbers!
Metanote, the first bullet point can be called by all those names only when we talk about functions on a finite interval (that is the case for this course).

**Example 4**
Determine if the sequence of functions defined by

$$f_n(x) = \begin{cases} 2nx & 0 \leq x \leq \frac{1}{2n} \\ 1 - 2nx & \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & else \end{cases}$$

converges in the $L^2$ norm, pointwise, or uniformly.

If it converges, then what is the limiting function(s) in each case?

## Convergence of Fourier series:

Before talking about the convergence of Fourier series we need to introduce some new definitions

**Definition 4: Piecewise $C^1$**
A function $f$ is **Piecewise** $C^1$ **(PWC1)** on the interval $[a, b]$ if there is a finite partition $a = t_0 < t_1 < \ldots < t_k = b$ such that:

- $f'$ exists on each interval $(t_i, t_{i+1})$,
- $f'$ is continuous on each interval $(t_i, t_{i+1})$,
- $f$ and $f'$ are bounded on each interval $(t_i, t_{i+1})$.

**Example 5**
Determine if the following are continuous, piecewise continuous, differentiable, and/or piecewise $C^1$.

- $f(x) = 1 - |x|$
- $g(x) = 1 - x^{2/3}$
- $h(x) = u(x)(1 - |x|)$
- $\ell(x) = u(x)(1 - x^{2/3})$

**Definition 5: Periodic Extension**

The **periodic extension** of a function $f$ defined on $[a,b]$ is the $b-a$ periodic function $f_p$ such that

- $f_p(t) = f(t)$ for $t \in (a,b)$ where $f(t)$ is continuous.
- $f_p(t) = \frac{f(t^-)+f(t^+)}{2}$ for $t \in (a,b)$ where $f(t)$ is not continuous.
- $f_p(a) = \frac{f(a)+f(b)}{2} = f_p(b)$

**Example 6**

Draw the periodic extension of

$$f(t) = \begin{cases} -1 & -\pi < t < 0 \\ 1 & 0 \le t < \pi \end{cases}$$

**Theorem 5: Convergence of Fourier series**
Let $f_p$ be the periodic extension of a function $f \in L^2([-\tau/2, \tau/2])$.

- The Fourier series of $f$ converges in the $L^2$ norm (also in the mean and almost everywhere) to $f$ (and also $f_p$) on any finite subinterval of $[-\tau/2, \tau/2]$.
- If $f_p$ is piecewise $C^1$ then the Fourier series of $f$ converges pointwise to $f_p$ for all $x \in \mathbb{R}$
- If $f_p$ is piecewise $C^1$ and continuous then the Fourier series of $f$ converges uniformly $f_p$ on any finite interval of $\mathbb{R}$.

The proof is beyond the scope of this course.

**Example 7**
Draw the function that the Fourier series of $f$ defined on $[-\pi, \pi]$

$$f(x) = \frac{x^2}{\pi^2} + \frac{x}{\pi} + 1$$

would converge to. Explain the type of convergence.

**Definition 6: Gibbs Phenomenon**

For a $L^2([a,b])$ function $f$ with periodic extension $f_p$, if $f_p$ is not continuous at some point $t_0$ then truncated Fourier series of $f$ will have growing oscillations near the point $t_0$. This is called **Gibbs Phenomenon**.

These oscillations do not appear in the infinite sum.