MATH 213 - Lecture 18: Bode plots

Lecture goals: Understand what the frequency response is and be able to generate and interpret Bode plots.

In lectures 13-17 we learned how to:

- compute the response of a LTI given an input function $f(t)$ by either evaluating $\mathscr{L}^{-1}\{H(s)\} * f(t)$ or computing $\mathscr{L}^{-1}\{H(s)F(s)\}$,
- analyze the general structure of the unit impulse and unit step responses by looking at the poles (and zeros) of the transfer function
- apply simple control systems (P, I, and PI) to control a system to have the behaviour we want it to have (i.e. cruse control problem).
- that the transfer function is the Laplace transform of the system's impulse response.
- that complex exponentials are the eigenfunctions of LTIs

These methods work well for many simple/academic problems but this method requires us either to find all of the poles (or potentially the dominant ones) or to evaluate a convolution integral.

Sometimes it is not possible to find all the poles/roots of the transfer function!

Sometimes we want to work in the time domain rather than the frequency domain.

In these cases, we need to use a different approach that has its own pros and cons.

Deriving the frequency response:

Recall Theorem 4 from lecture 13:

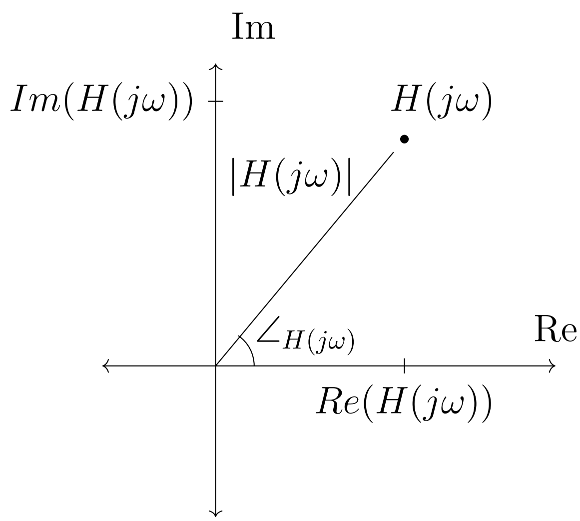
> **Theorem 1: LTI response to an exponential**
> If $S : f \to y$ is a LTI with transfer function $H(s)$ then for any $s \in \mathbb{C}$
> $$e^{st} \xrightarrow{S} H(s)e^{st}.$$

Now if $s = j\omega$ then the above gives $e^{j\omega t} \xrightarrow{S} H(j\omega)e^{j\omega t}$.

$H(j\omega)$ is a complex number so we can "recall" from MATH 115 that we can write it in polar form:

$$H(j\omega) = |H(j\omega)|e^{j\angle_{H(j\omega)}}$$

"Recall" from MATH 115 that this decomposition can be viewed geometrically:



Now we can write the system's response to $e^{j\omega t}$ as

$$\begin{aligned} H(j\omega)e^{j\omega t} &= \underbrace{|H(j\omega)|e^{j\angle_{H(j\omega)}}}_{H(s)} e^{j\omega t} \\ &= |H(j\omega)|e^{(\omega + \angle_{H(j\omega)})jt} \end{aligned}$$

Because of the above, $H(j\omega)$ is called the **frequency response**.

Explicitly, $H(j\omega)$ is the factor we need to scale the input signal $e^{j\omega t}$ by in order to find the system's response of $e^{j\omega t}$.

Theorem 2
If S is a LTI with transfer function $H(s)$, then

$$\sin(\omega t) \xrightarrow{S} |H(j\omega)| \sin(\omega t + \angle_{H(j\omega)})$$

Sketch of proof: Recall that

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}.$$

Use the above identity along with the previous results and then do some algebra.

Observations: The above says that the system response of an LTI to a sin wave of frequency ω:

- has an amplitude scaled by $|H(j\omega)|$
- has the same frequency
- Has a phase shifted by $\angle_{H(j\omega)}$

In the real world signals have a clear starting time (i.e. real world signals are one sided) so the above can't be used for many applications. Lucky for us:

Theorem 3
If S is a stable LTI with transfer function $H(s)$, then as $t \to \infty$

$$\sin(\omega t)u(t) \xrightarrow{S} |H(j\omega)| \sin(\omega t + \angle_{H(j\omega)}).$$

We will skip this proof.

Example 1
Suppose $H(s) = \frac{1}{RCs+1}$ for $RC = 0.01$. Find the magnitude and phase shift for the system response to $0.5\sin(100t)$

Looking ahead: Fourier series/transforms will allow us to decompose functions as a sum/integral of sin and cos waves.

Hence if we know what the LTI does so all complex exponentials, then we can decompose a signal into a sum/integral of complex exponentials, scale and phase shift them, and then add the results together to see the response to the original signal.

We want a nice way to display the scaling factors and phase shifts so we introduce Bode plots.

Bode plots:

- Bode plots are a graphical representation of the frequency response.
- We need two plots: one for $|H(j\omega)|$ in "decibels" (dB) vs $\log_{10}(\omega)$ and one for $\angle H(i\omega)$ vs $\log_{10}(\omega)$
- The aforementioned plots are called the "magnitude" and "phase" curves respectively.

Using these conventions has two benefits:

- it allows curves to be approximated by piecewise lines (we will show this via examples).
- allows plots for complex transfer functions to be build by adding plots from simpler transfer functions.

To see the basic idea behind this approach suppose $H(j\omega) = H_1(j\omega)H_2(j\omega)$ and that we have the Bode plots for H_1 and H_2.

In this case

$$\begin{aligned} H(j\omega) &= |H_1(j\omega)|e^{j\angle_{H_1(j\omega)}}|H_2(j\omega)|e^{j\angle_{H_2(j\omega)}} \\ &= |H_1(j\omega)||H_2(j\omega)|e^{j(\angle_{H_1(j\omega)}+\angle_{H_2(j\omega)})} \end{aligned}$$

The angles are additive so we can simply add the phase curves of H_1 and H_2 to get the phase curve for H.

The magnitudes are not additive but... "recall" that $\log(ab) = \log(a) + \log(b)$ so if we use decibels for the magnitude, then the magnitude curves become additive.

Definition 1: Decibels
$|H(j\omega)|$ in decibels is $20\log_{10}(|H(j\omega)|)$.

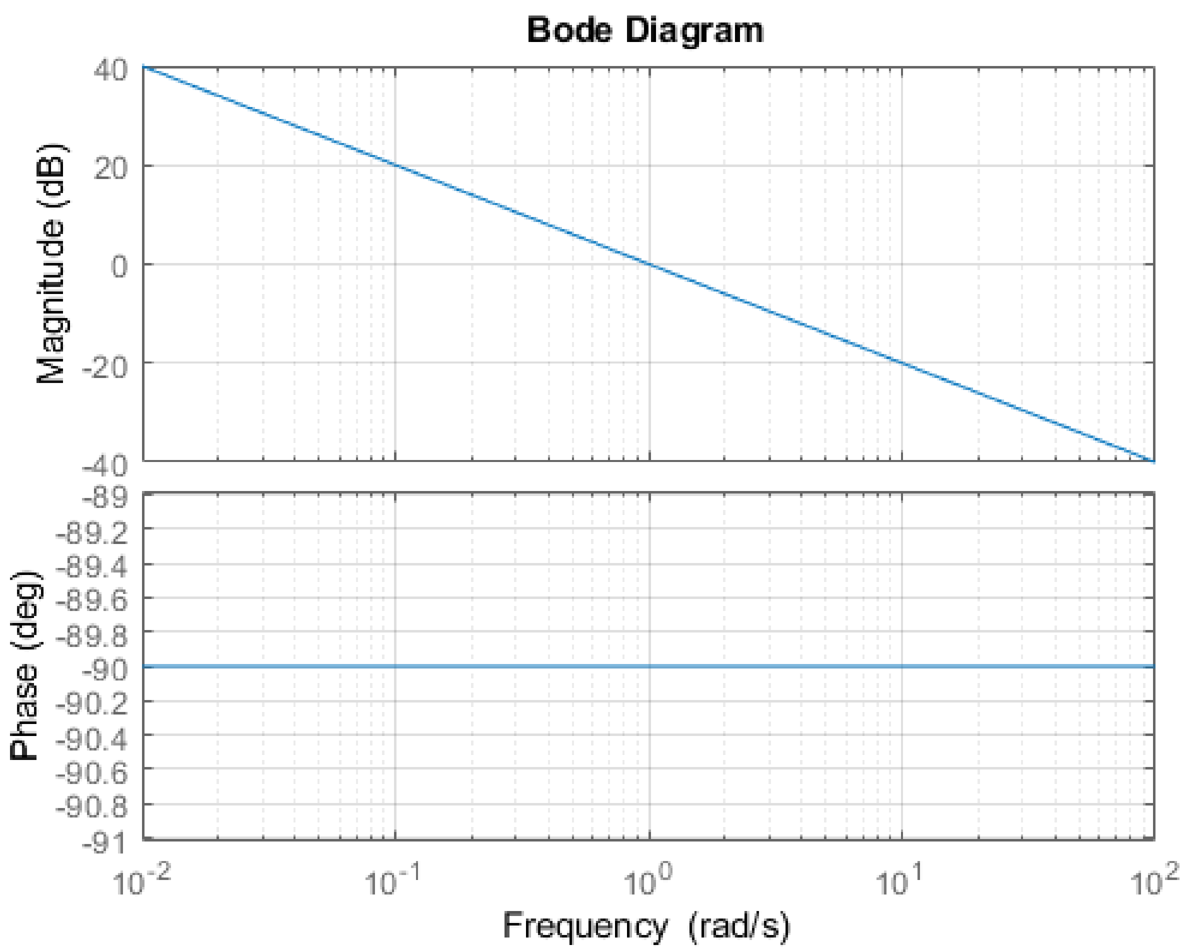
Using decibels for the magnitude: if $H(j\omega) = H_1(j\omega)H_2(j\omega)$ then we have

$$20\log_{10}(|H(j\omega)|) = 20\log_{10}(|H_1(j\omega)|) + 20\log_{10}(|H_2(j\omega)|).$$

So if we use decibels, then we can just add the magnitude curves to find the magnitude curve of $H(j\omega)$.

Bode plot examples:

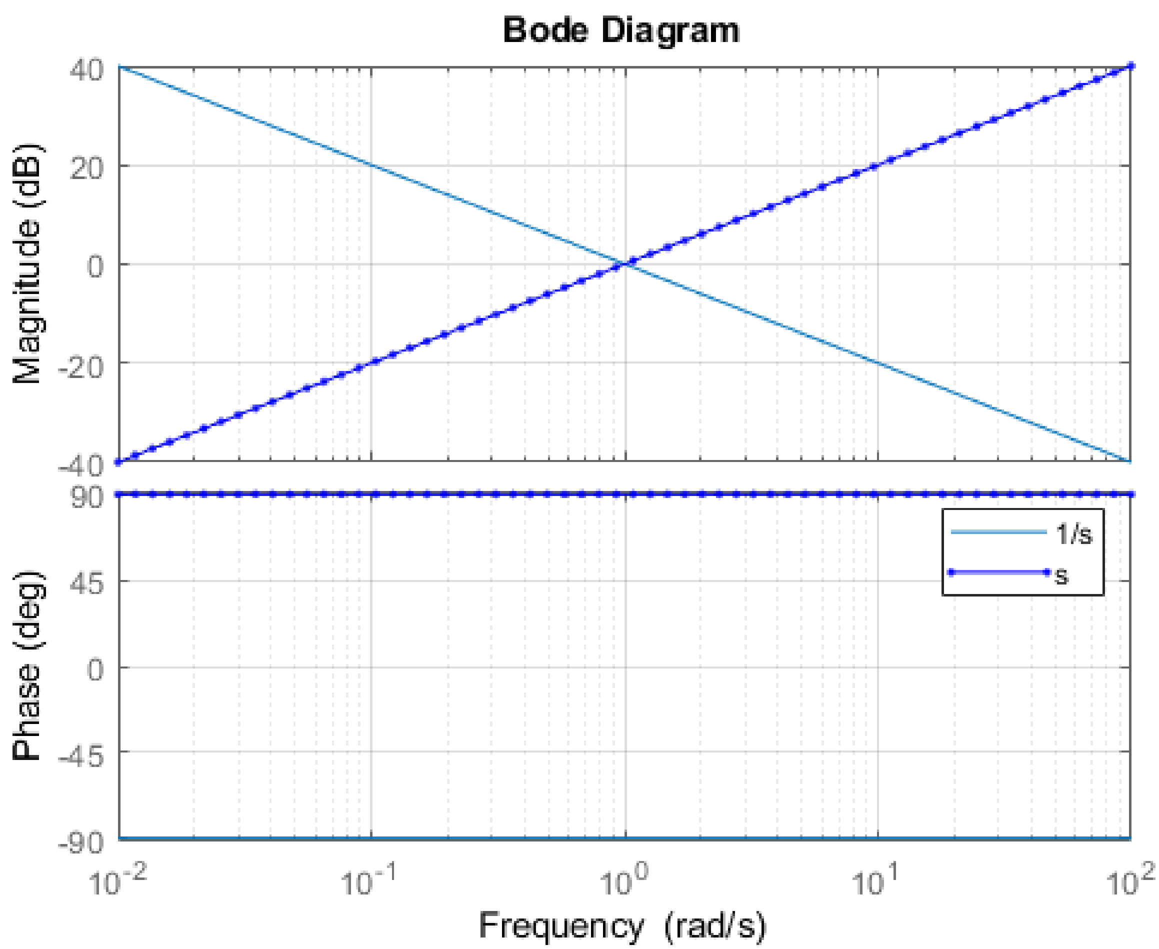
Example 2
Find the Bode plot for the system with transfer function $H(s) = \frac{1}{s}$ and find the system response to $\sin(\omega t)$.



Bode Diagram
Magnitude (dB)
40
20
0
-20
-40
Phase (deg)
-89
-89.2
-89.4
-89.6
-89.8
-90
-90.2
-90.4
-90.6
-90.8
-91
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/s)

Example 3

Find the relation between the Bode plots of $H(s)$ and $G(s) = \frac{1}{H(s)}$. Use this result to find the Bode plot of s given the plot from the previous example.

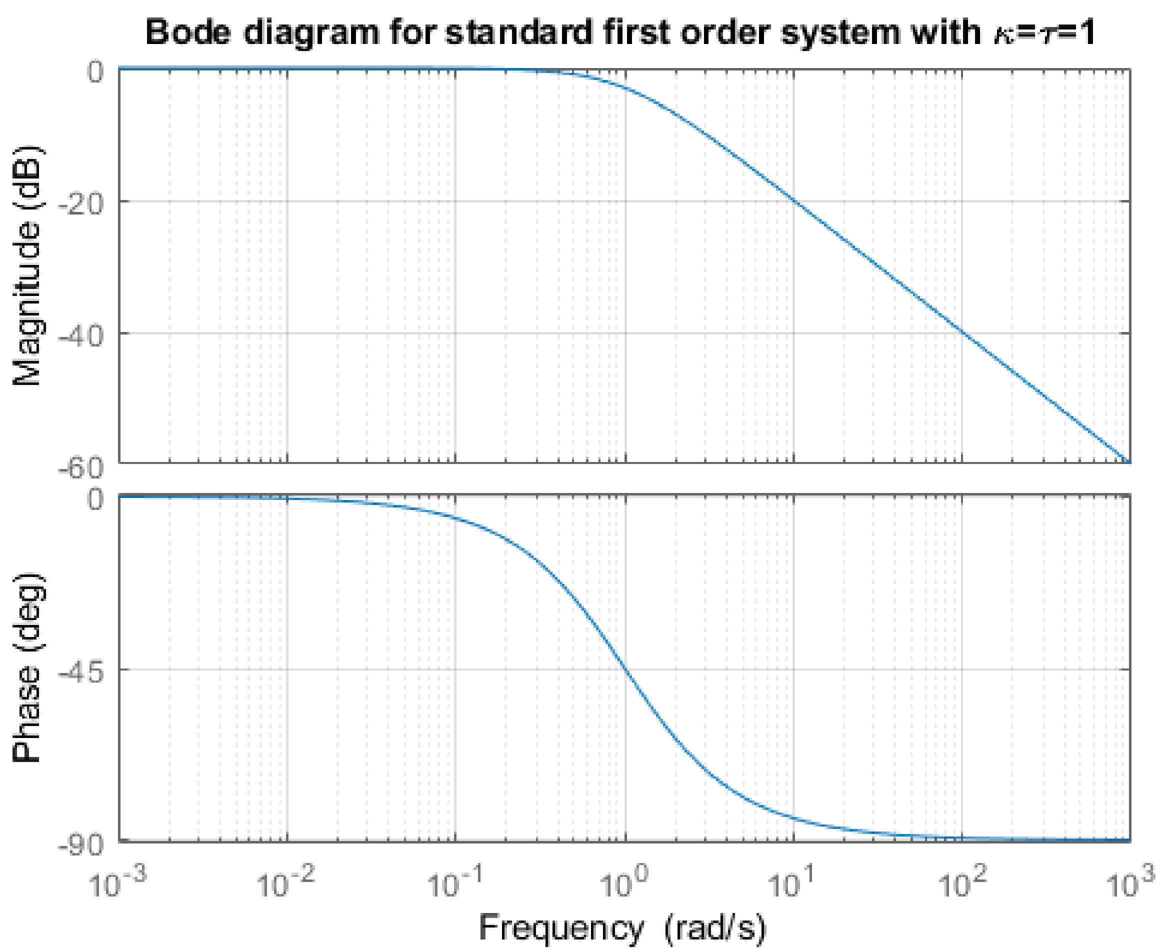


Bode Diagram
Magnitude (dB)
Phase (deg)
Frequency (rad/s)
1/s
s
40
20
0
-20
-40
90
45
0
-45
-90
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

Example 4

Find the Bode plot for the standard first order system:

$$H(s) = \frac{\kappa}{s\tau + 1}, \kappa, \tau > 0.$$



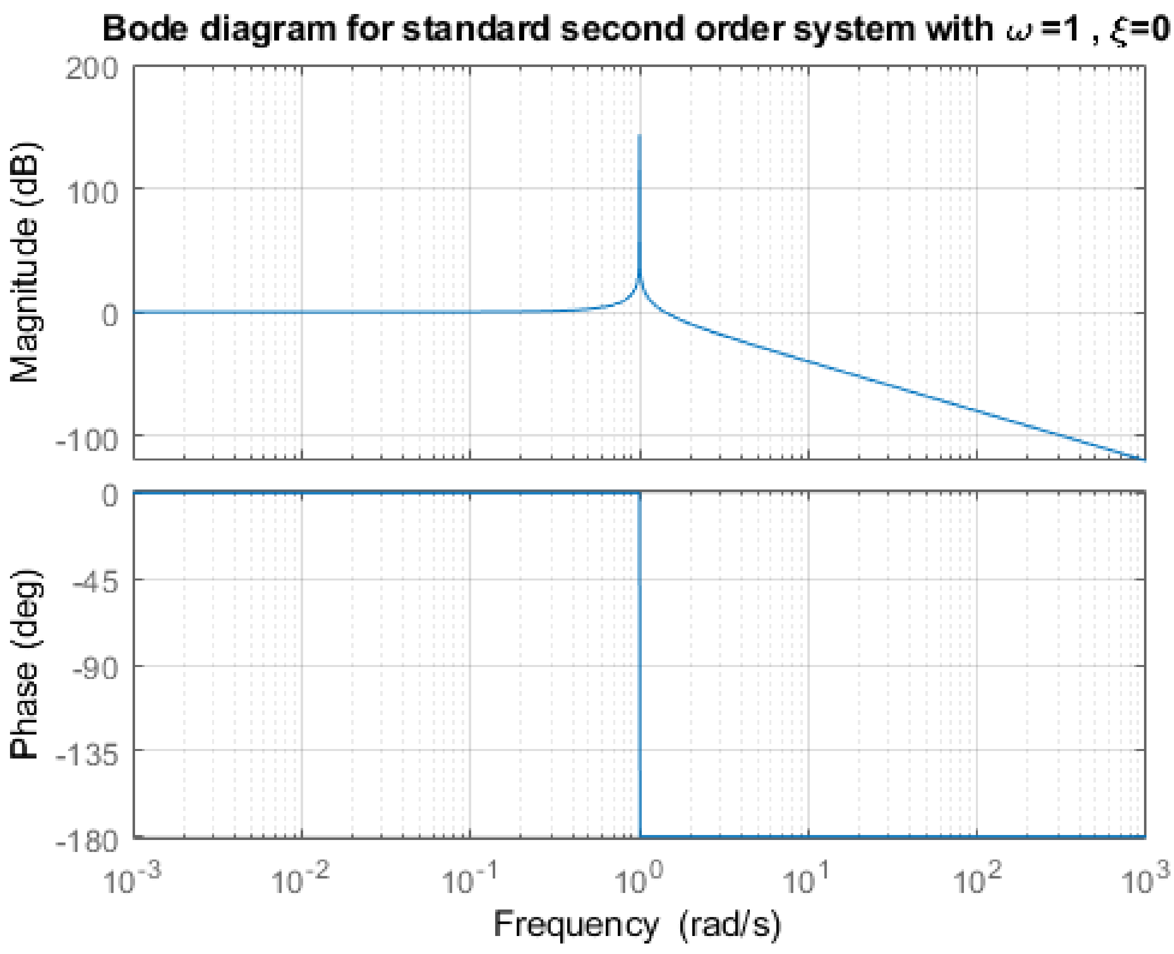
Note that the Bode plot for a system with no poles and a zero is simply this plot flipped.

Example 5

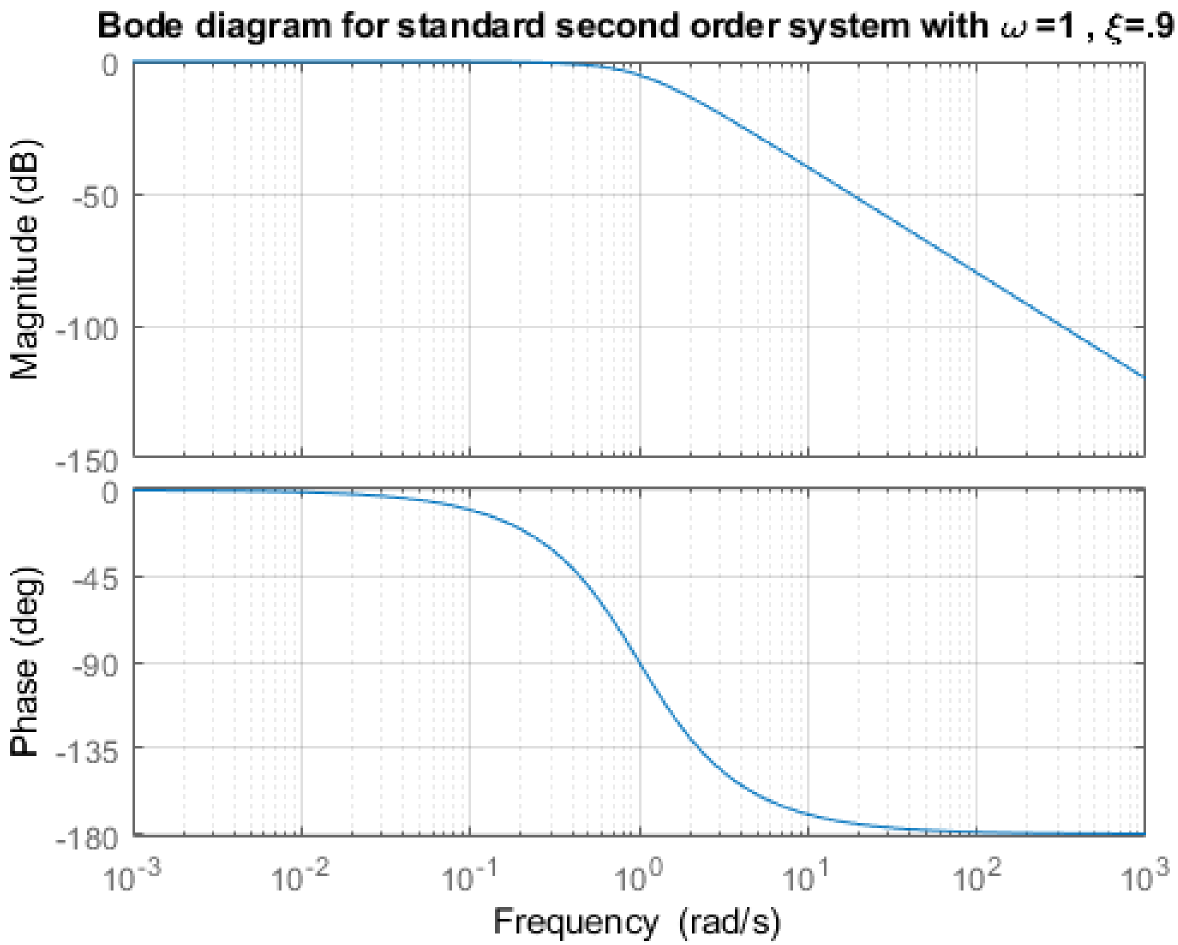
Find the Bode plot for the standard second order system:

$$H(s) = \frac{w^2}{s^2 + 2\xi w s + w^2}, \quad w > 0$$

for cases where $\xi = 0$ and then where $0 < \xi < 1$.



We will use this as out idealized plot for second order systems (that cannot be decomposed into two standard first order systems). Note that the Bode plot for a system with no poles and zeros that are complex-conjugate pairs is simply this plot flipped.



If given a transfer function $H(s) = H_1(s) \cdot H_2(s) \cdots H_k(s)$ then the Bode plot for $H(j\omega)$ is found by

- Finding the magnitude and phase curves for each $H_i(j\omega)$.
- Adding the magnitude and phase curves.