# MATH 213 - Lecture 6: Solving DEs via Laplace Transforms

#### Basic idea:

Consider a Linear DE with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0y(t) = f(t)$$

with the appropriate number of initial conditions

$$y(0) = y_0,$$
  $y'(0) = y_1,$  ...,  $y^{n-1}(0) = y_{n-1}.$ 

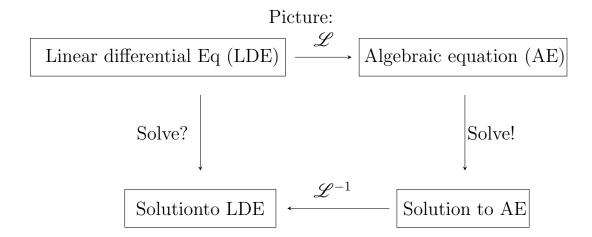
Taking the Laplace transform of both sides and using the linearity of the Laplace transform gives an equation of the form

$$(s^{n} + b_{n-1}s^{n-1} + \ldots + b_{0}s)Y(s) + G(s, y_{0}, y_{1}, \ldots, y_{n-1}) = F(s)$$

In the above Y(s) and F(s) are the Laplace transforms of y(t) and f(t) respectively, G is a function of s and the initial conditions, and the  $b_i s$  are coefficients that come from taking the Laplace transform of the derivative terms.

**Note:** One can find the exact form of this equation but I do not want you to memorize this formula so I am not writing the exact form.

We can solve the above for Y(s) and then (in theory) compute the inverse Laplace transform to find y(t).



#### One-sided Laplace Table:

f(t)	$F(s) = \mathcal{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\operatorname{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\operatorname{Re}(s) > 0$
$3. t^n$	$\frac{n!}{s^{n+1}}$	$\operatorname{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\operatorname{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\operatorname{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	$\operatorname{Re}(s) >  \omega $
7. $\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	$\operatorname{Re}(s) >  \omega $

### Algebraic Properties:

## Example 1

Solve y'' = -g,  $y(0) = h_0$ ,  $y'(0) = v_0$ . Recall this is the model for a falling ball with initial height  $h_0$  and initial velocity  $v_0$  from the first lecture. Plot the solution.

#### **Definition 1: Characteristic Polynomial**

The Characteristic Polynomial of a linear differential equation with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0y(t) = f(t)$$

The polynomial multiplied by Y(s) after you take the Laplace transform. This polynomial will always be  $p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$ .

#### Example 2

Solve  $y'' + y = e^{-2t} \sin(2t)$ , y(0) = 1, y'(0) = 0 and plot the solution.

Example 3 Solve  $y'' + 9y = 2\sin(3t)$ , y(0) = 1, y'(0) = 0 and plot the solution.

Example 4 Solve  $y^{(3)} + 2y' = 2\sin(3t)$ , y(0) = 1, y'(0) = 1, y''(0) = 0 and plot the solution.

#### Extensions of our method:

Our method also "works" for some Linear ODEs with non-constant coefficient. The problems are 1) evaluating the Laplace transform of  $a(t)y^{(n)}(t)$  for some given function a(t) and a unknown function  $y^{(n)}$  and 2) we end up with another (but simpler) differential equation to solve. If  $a(t) = t^n$  then this is particularly "nice"!

Example 5 Solve y'' + 2ty' + y = 0, y(0) = 0, y'(0) = 1 and plot the solution.

The coefficients need not be complex numbers, they can also be matrices. Consider a DE of the form

$$I\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0y(t) = f(t)$$

where  $a_i$  is a  $n \times n$  matrix, I is the  $n \times n$  identity matrix and the functions y(t) f(t) are n dimensional vectors.

#### Example 6: Minecraft Chickens

Suppose we have a population of chickens, C(t) and eggs E(t) and suppose that the eggs turn into chickens at a rate  $\alpha$ , and the chickens lay eggs at a rate  $\beta$  derive a coupled system of DEs to model the populations of chickens and eggs.

### Looking ahead:

- Many applications of ECE/software engineering involve solving a system of linear DEs (i.e. multivariable systems).
- In control theory, we often want to find initial conditions so that we can "solve for f(t)". i.e. pick f(t) and the ICs so that the solution to the DE does something we want.