MATH 213 - Lecture 19: Bode plots 2

Lecture goals: Refine our ability to quickly draw bode plots, approximate the transfer function from bode plots, and determine stability of the system/closed loops using Bode plots.

Summary of main results from Lecture 18:

- The "starting value" of the magnitude curve is $|H(0)|_{dB}$ and the "starting value" of the phase curve is $\angle H(0)$.

  This will be not exist (i.e. the curve is unbounded) if there is a pole with a real value of 0.

- If the transfer function has an order one **stable** pole at a point then the magnitude curve will experience an extra decrease of -20 dB/Decade and the phase curve will experience a drop of -90° (see L18 Ex 4).

  These phase adjustment will be rather "slow" taking an order of magnitude (before and after the pole) to adjust.

- If the transfer function has complex conjugate poles that are stable then the magnitude curve will experience an extra decrease of -40 dB/Decade and the phase curve will experience a gain of -180°. The location of this adjustment is given by ω where ω is the term in the standard second order system (see L18 Ex 5).

  The speed of adjustment is related to ξ:

  - $\xi \approx 0$ causes fast adjustments and overshooting in the amplitude plot for nearby ωs.
  - $\xi \approx 1$ causes slow adjustments and undershooting in the amplitude plot for nearby ωs.

- If we have zeros of order 1 or of order 2, at some point then the magnitude and phase curves experience the opposite effects as listed in the above two points.

What if we have an unstable pole?

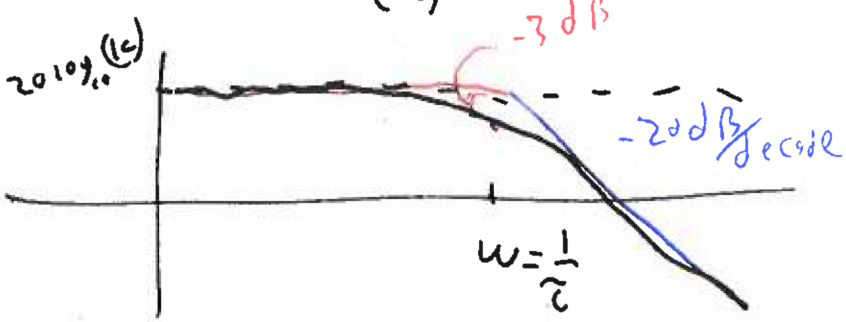
In this case Bode plots do not make sense to apply as they assume there is a steady state solution but... for completion

Example 1
Find the Bode plot for the "standard" unstable linear system

$$H(s) = \frac{K}{s\tau - 1}, K, \tau > 0$$

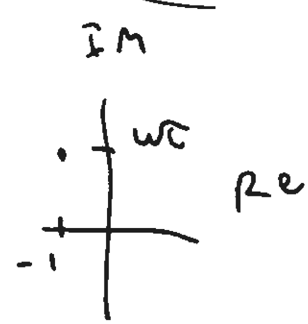
$$|H(j\omega)| = \frac{K}{|j\omega\tau - 1|}$$

$$= \frac{K}{\sqrt{1+(\omega\tau)^2}} \longleftarrow \text{same as std. first order sys.}$$



$$\angle H(j\omega) = \angle(K) - \angle(j\omega\tau - 1)$$

$$= 0 - \angle(j\omega\tau - 1)$$

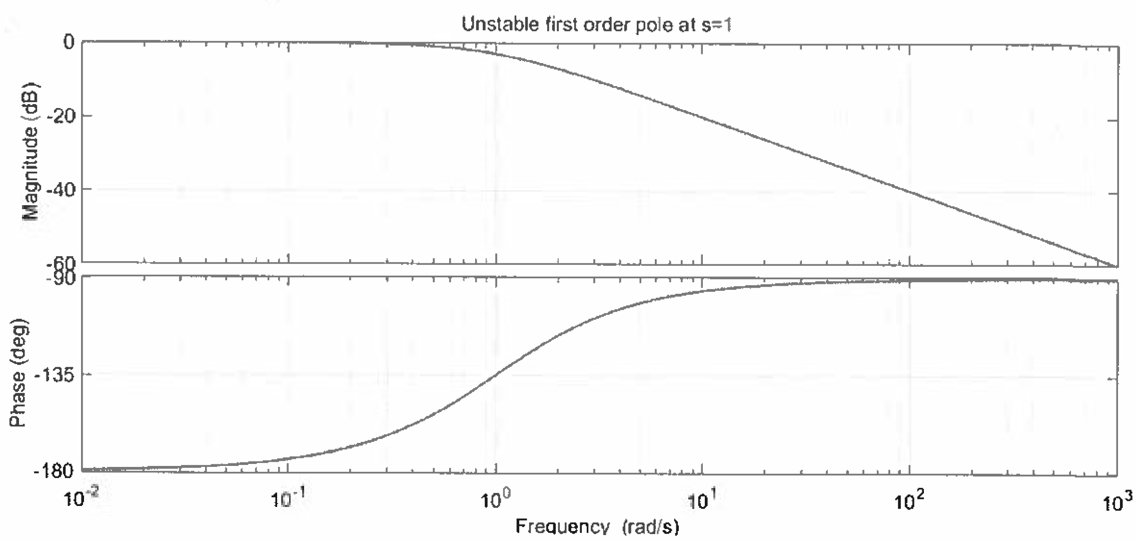


- if $\omega \ll \frac{1}{\tau}$

$$\angle H(j\omega) \approx -180^\circ$$

- if $\omega = \frac{1}{\tau}$, $\angle H(j\omega) = -135^\circ$

- if $\omega \gg \frac{1}{\tau}$, $\angle H(j\omega) \approx -90^\circ$



Similar result for second order systems but they are a bit more complex to analyze and often there are better ways to test stability so we skip the analysis.
In general if you see a decrease in the slope of the asymptotic amplitude curve and an increase in the phase, then you can conclude that the system is unstable (and hence a Bode plot should have never been made). You can not conclude anything about stability of the system modelled by the transfer function used to generate the plot though.

To quickly generate plots, we can use the results derived from lectures 18 and 19 by either adding the curves or using the relation of the poles/zeros and the curves to sketch asymptotic lines. The latter is often nicer.

Example 2
Sketch the Bode plot for

$$T(s) = \left(\frac{0.01}{s + 0.01}\right)\left(\frac{s + 100}{s + 10}\right)$$

$$|T(0)| = \lim_{\omega \to 0} \left|\frac{0.01}{\omega j + 0.01}\right| \left|\frac{\omega j + 100}{\omega j + 10}\right|$$

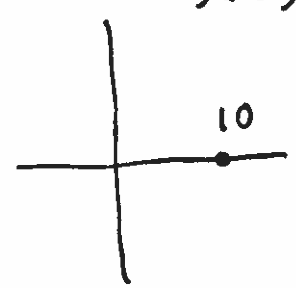
$$= \lim_{\omega \to 0} \frac{0.01}{\sqrt{(0.01)^2 + \omega^2}} \frac{\sqrt{\omega^2 + 100^2}}{\sqrt{\omega^2 + 10^2}}$$

$$= \frac{0.01}{0.01} \frac{100}{10} = 10$$

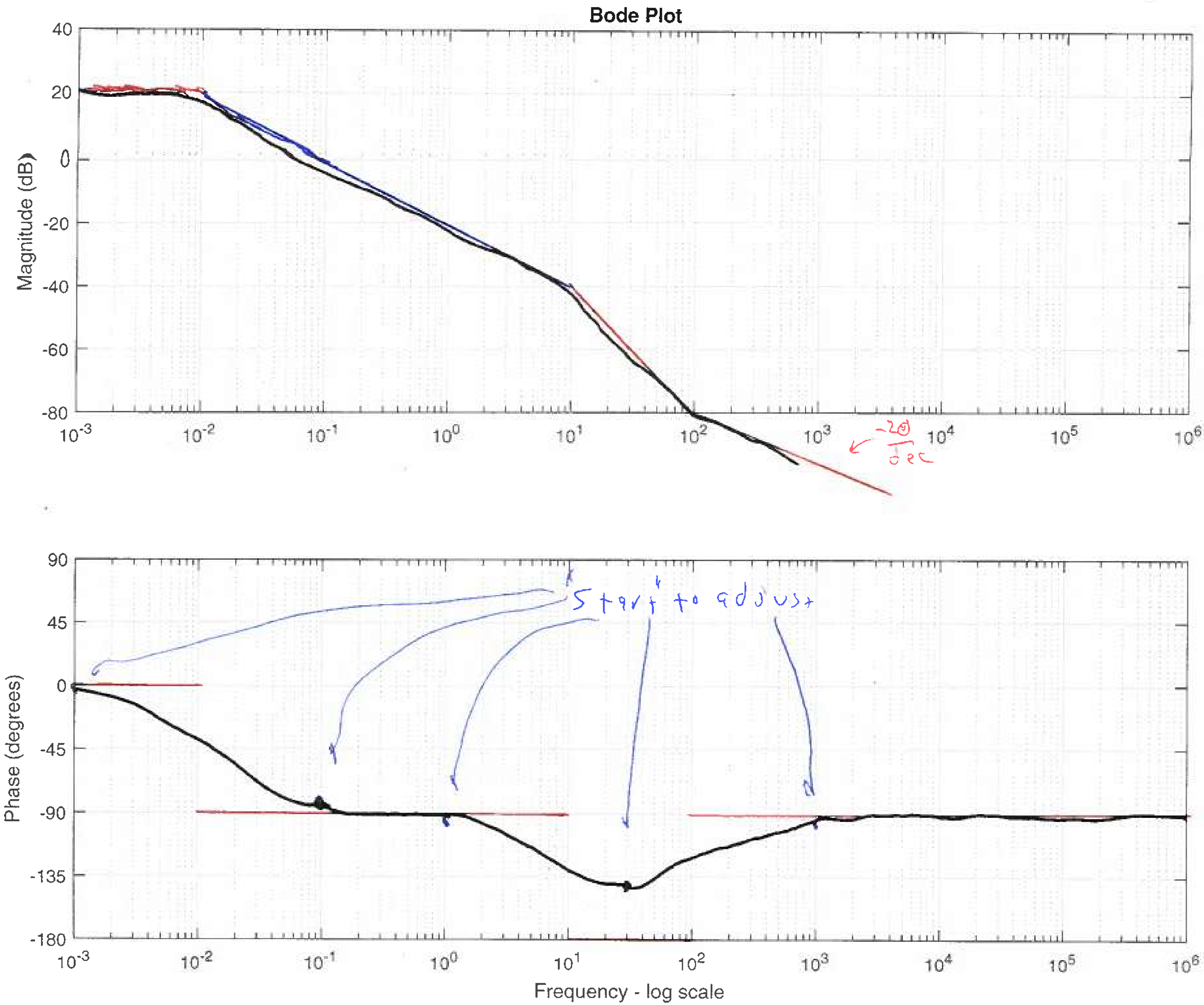
so $|T(0)|_{dB} = 20 \log_{10}(10) = 20$

$\angle(T(0)) = 0^\circ$

s = 0 gives



Poles/roots @ $\omega = 10^{-2}, 10^{1}, 10^{2}$.



Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
-80
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
10^{5}
10^{6}
$\frac{-20}{dec}$
Phase (degrees)
90
45
0
-45
-90
-135
-180
Start to adjust
Frequency - log scale

$0 \le w \lesssim 10^{-2}$ $^{0}_{0}$

amp ~~Phase~~ = 20 dB

Phase = 0°

$10^{-2} \lesssim w \lesssim 10^{1}$ $^{0}_{0}$

amp slope -20 dB/decade

phase adj to -90°

$10^{1} \lesssim w \lesssim 10^{2}$ $^{0}_{0}$

amp slope -40 dB/decade

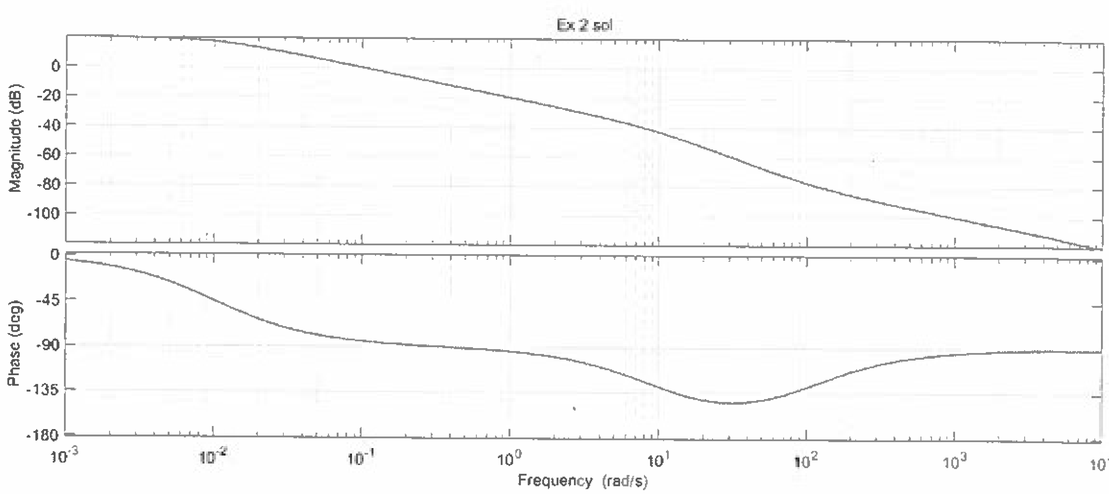
Phase adj to -180°

$10^{2} \lesssim w$ $^{0}_{0}$

amp slope -20 dB/decade

Phase adj to -90°

all poles/roots are first order so phase adjs are slow



Example 3
Sketch the Bode plot for

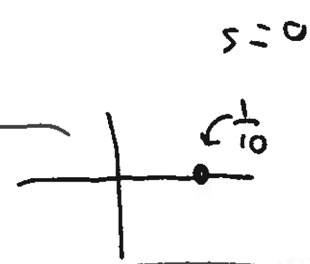
$$T(s) = \left(\frac{s+1}{s+1.01}\right)\left(\frac{s+10}{(s+3)^2+100}\right)$$

$$|T(0)| = \frac{1}{1.01}\ \frac{10}{9+100}$$

$$\approx \frac{1}{10}$$

$$|T(0)|_{dB} = -20$$

$$\angle(T(0)) = 0^\circ$$



Poles/zeros @ $\omega = +1, +1.01, +10, +\sqrt{109}$

cancel

cancel to "1st order pole"

$0 \le \omega \lesssim 10^1$ amp $= -20$

phase $= 0^\circ$

amp slope $= -20$ dB/decade

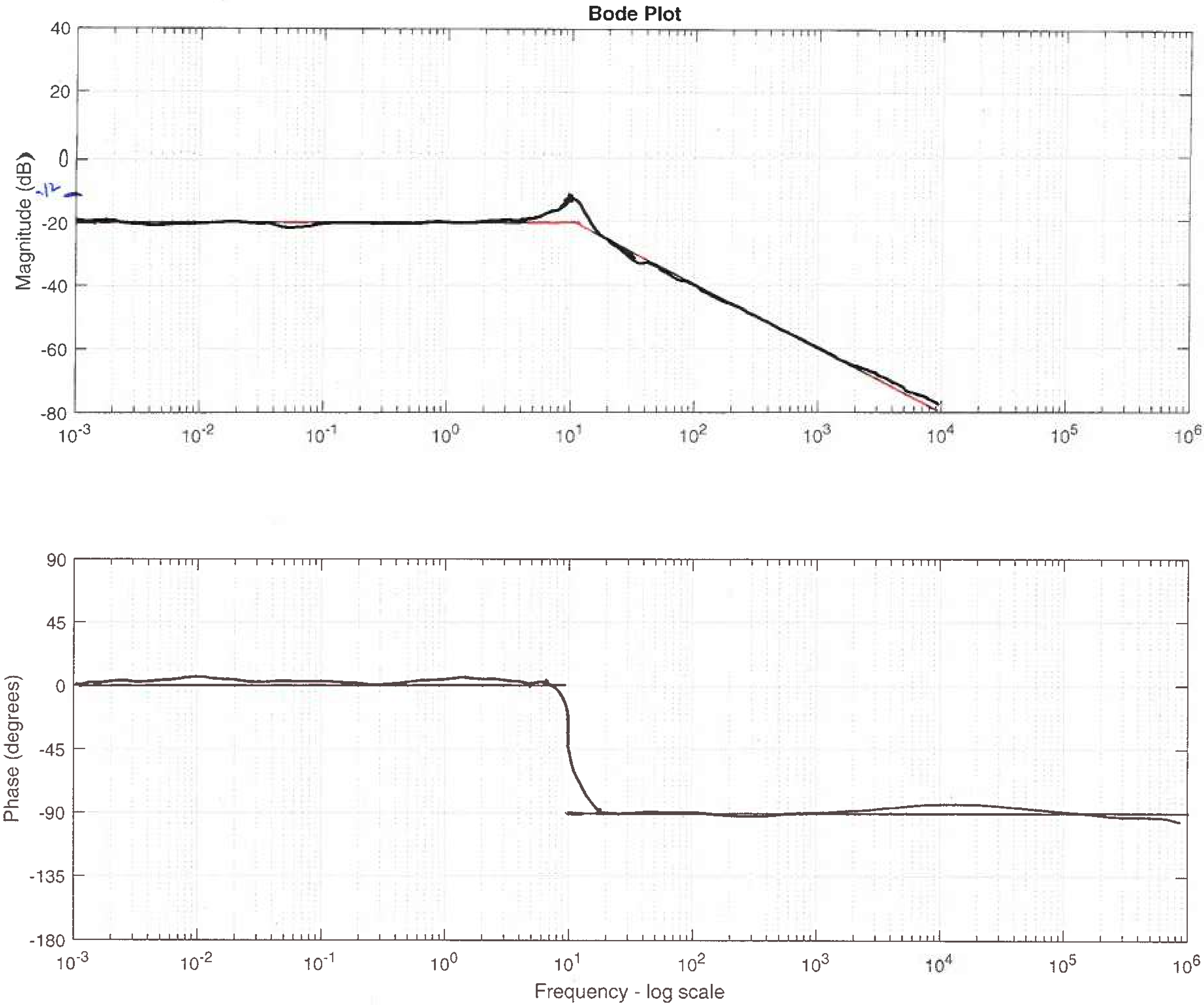
$10^1 \lesssim \omega$

Phase adj. to -90°

To see the speed/overshoot note

$$(s+3)^2 + 100 = s^2 + 6s + 109$$

$$\Rightarrow \omega = \sqrt{109} \qquad \zeta = \frac{6}{2\sqrt{109}} \ll 1$$

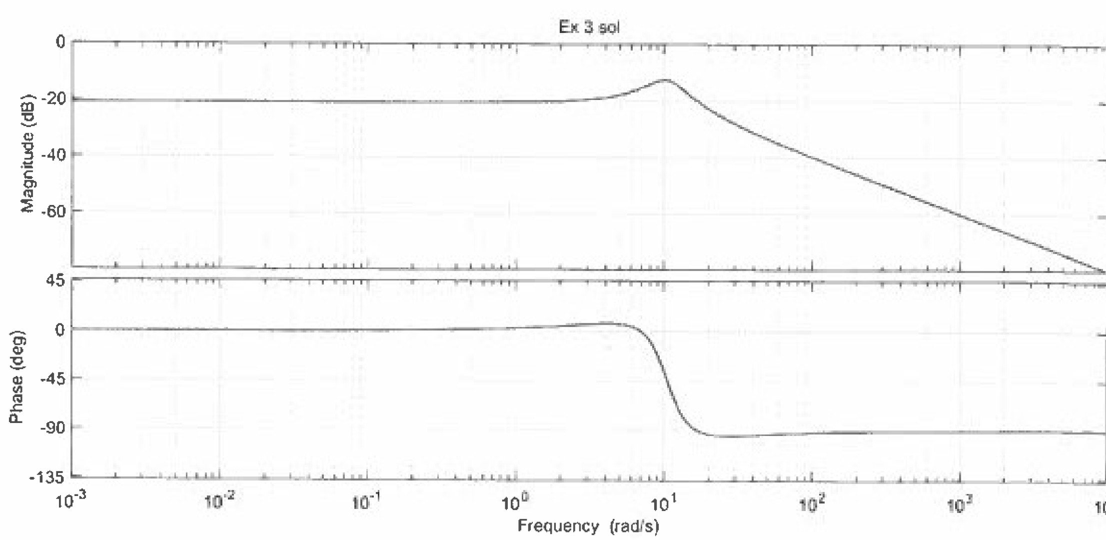

Bode Plot
Magnitude (dB)
40
20
0
-12
-20
-40
-60
-80
Phase (degrees)
90
45
0
-45
-90
-135
-180
10-3
10-2
10-1
100
101
102
103
104
105
106
Frequency - log scale

Fast adjustment!! Also Overshoot!

$$|T(10w)| = \frac{\sqrt{10^2+1}}{\sqrt{10^2+(1.01)^2}} \cdot \frac{\sqrt{(10)^2+(10)^2}}{\sqrt{(109-100)^2+(60)^2}}$$

$$\approx \frac{\sqrt{200}}{\sqrt{60^2}} \approx 0.2$$

$$|T(10w)|_{dB} \approx -12$$



Example 4
Sketch the Bode plot for

$$T(s) = \left(\frac{s-1}{(s+1)^2}\right)\left(\frac{10^5(s+0.01)}{(s+10)^2+1000}\right)$$

$$|T(0)| \approx 1 \cdot \frac{10^5 \cdot 10^{-2}}{10^3}$$

$$\approx 1$$

$$|T(0)|_{dB} \approx 0$$

$$\angle(T(0)) \approx \angle(-1) - \angle(1) + \angle(10^3) - \angle(1000+100)$$

$$= -180^\circ - 0^\circ + 0^\circ - 0^\circ$$

$$= -180^\circ \qquad (180^\circ \text{ is also correct}).$$

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poles/zero @ $\omega = 10^{-2}, \; 1, \; \sqrt{1000+100} \approx 3 \cdot 10^1$

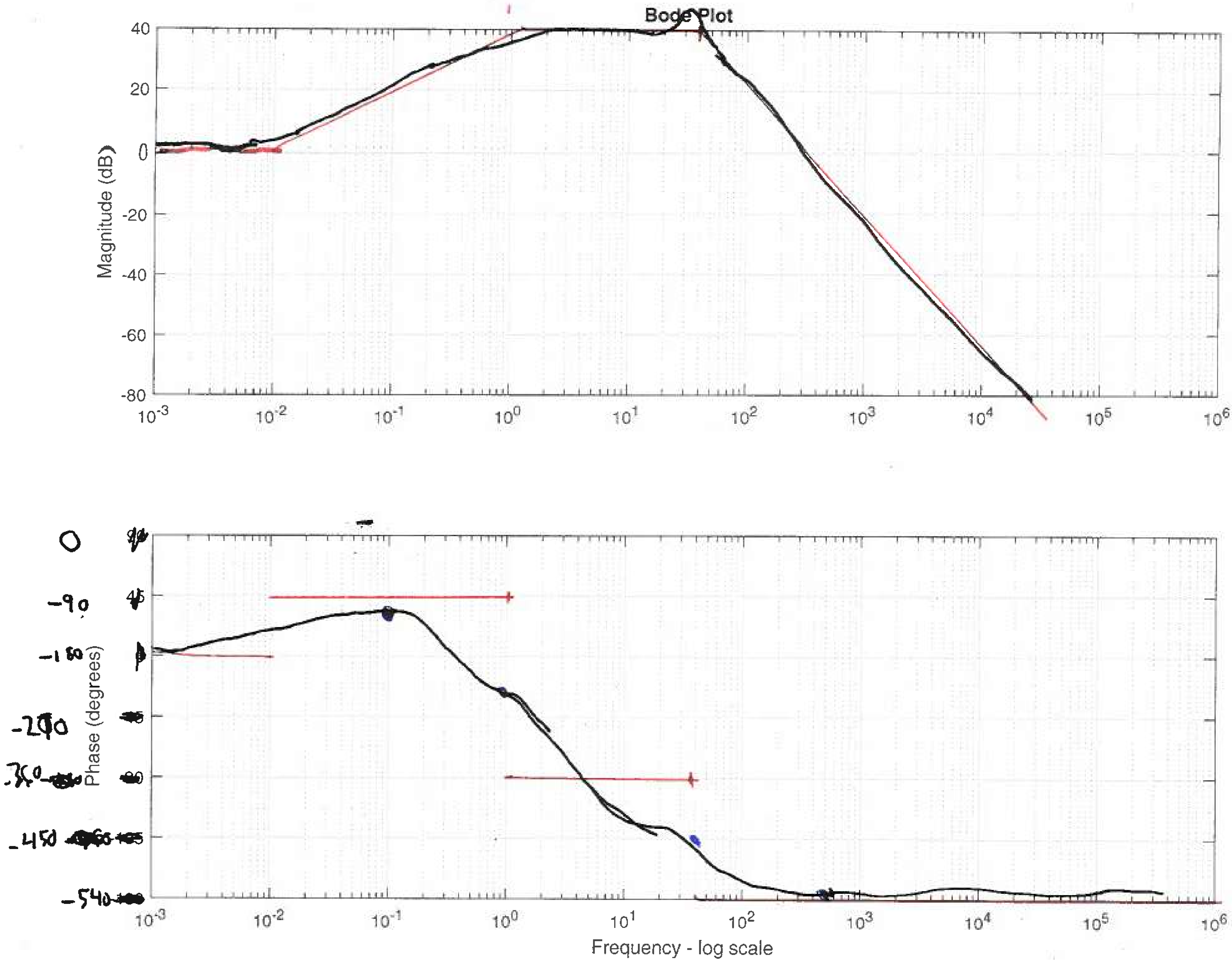
1 zeros w/ -Re, 2 poles; conj. pair

$0 \le \omega \lesssim 10^{-2}$ $\quad A = 0$

$Ph = -180^\circ$

$10^{-2} \lesssim \omega \lesssim 1 \cdot 10^0$ $\quad \frac{d}{d\omega}(A) = +20 \; dB/dec$

$Ph = -90^\circ \longleftarrow +90^\circ$



Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
-80
Phase (degrees)
0
-90
-180
-270
-360
-450
-540
10^-3
10^-2
10^-1
10^0
10^1
10^2
10^3
10^4
10^5
10^6
Frequency - log scale

prev +20

Poles −40

zero +20

$\overline{}$ 0

$10^0 \lesssim w \lesssim 3 \cdot 10^1$: $\frac{d}{dw}(A) = 0$ ←

$Ph = -360^\circ$ ← zero w/ Re<0 gives −90°

2x Poles −180°

$3 \cdot 10^1 \lesssim w$: $\frac{d}{dw}(A) = -40 \text{ dB/dec}$

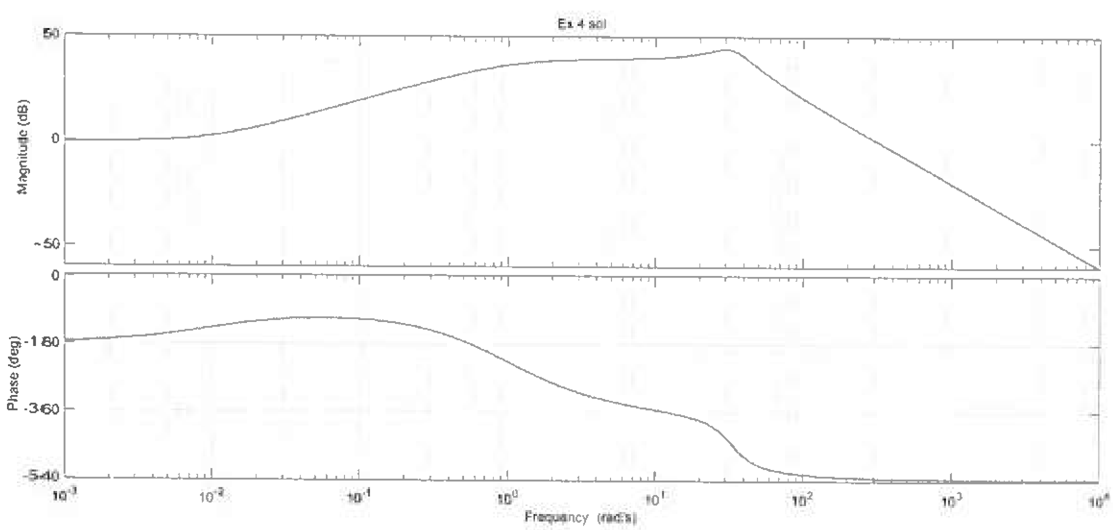
$Ph = -540^\circ$

$$(s+10)^2 + 10^3 = s^2 + 20s + 10^3 + 100$$

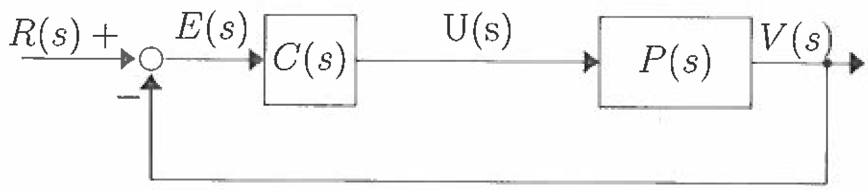
$w \approx 3 \cdot 10$, $\xi = \frac{20}{2 \cdot 30} \approx 0.33$

↑

Moderate speed



Recall that for a closed loop system with system diagram given by:



The transfer function for this system is

$$\frac{P(s)C(s)}{1+P(s)C(s)}.$$

Hence the closed system will be stable when both $C(s)$ and $P(s)$ are stable and when

$$1+P(s)C(s) \neq 0.$$

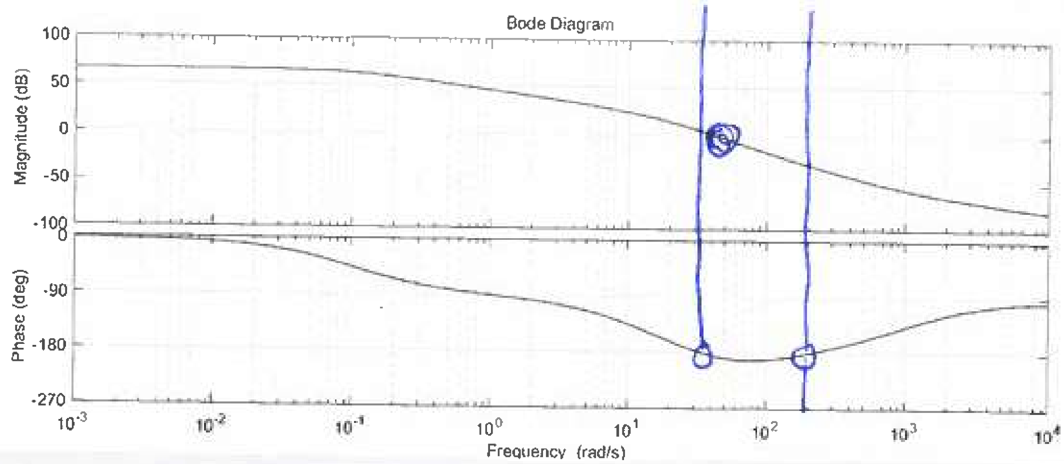
Hence if $C(s)$ and $P(s)$ are both stable, then the closed loop control system is **unstable** when

$$\begin{aligned} P(s)C(s) &= -1 \\ &= 1 \cdot e^{i\pi}. \end{aligned}$$

This happens when the amplitude curve is $|1|_{dB} \approx 0$ while the phase is either 180° or -180° (or some other equivalent angle).

In general to talk about stability we will not look for these exact conditions but will instead look at the points where either the amplitude or the phase is critical and then look at how far we are from stability. We generally require some threshold to be "far enough" from being unstable.

> **Example 5**
> Determine if the closed system with Bode diagram given by



is stable.

formally stable but close to unstable!!

Zoom in

In pratice this would be called unstable BC/ of how close it is