MATH 213 - Lecture 6: Solving DEs via Laplace Transforms

Basic idea:

Consider a Linear DE with <u>constant coefficients</u>

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

with the appropriate number of initial conditions

$$y(0) = y_0, \qquad y'(0) = y_1, \qquad \ldots, \qquad y^{n-1}(0) = y_{n-1}.$$

Taking the Laplace transform of both sides and using the linearity of the Laplace transform gives an equation of the form

$$(s^n + b_{n-1}s^{n-1} + \ldots + b_0 s)Y(s) + G(s, y_0, y_1, \ldots, y_{n-1}) = F(s)$$

In the above $Y(s)$ and $F(s)$ are the Laplace transforms of $y(t)$ and $f(t)$ respectively, G is a function of s and the initial conditions, and the $b_i s$ are coefficients that come from taking the Laplace transform of the derivative terms.

Note: One can find the exact form of this equation but I do not want you to memorize this formula so I am not writing the exact form.

We can solve the above for $Y(s)$ and then (in theory) compute the inverse Laplace transform to find $y(t)$.

Picture:

$$\begin{tikzcd}
\boxed{\text{Linear differential Eq (LDE)}} \arrow[r, "\mathscr{L}"] \arrow[d, "\text{Solve?}"'] & \boxed{\text{Algebraic equation (AE)}} \arrow[d, "\text{Solve!}"] \\
\boxed{\text{Solutionto LDE}} & \boxed{\text{Solution to AE}} \arrow[l, "\mathscr{L}^{-1}"']
\end{tikzcd}$$

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Example 1

Solve $y'' = -g$, $y(0) = h_0$, $y'(0) = v_0$. Recall this is the model for a falling ball with initial height h_0 and initial velocity v_0 from the first lecture.
Plot the solution.

Definition 1: Characteristic Polynomial
The **Characteristic Polynomial** of a linear differential equation with constant coefficients

$$\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

The polynomial multiplied by $Y(s)$ after you take the Laplace transform. This polynomial will always be $p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_0$.

Example 2
Solve $y'' + y = e^{-2t}\sin(2t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

Example 3

Solve $y'' + 9y = 2\sin(3t)$, $y(0) = 1$, $y'(0) = 0$ and plot the solution.

Example 4

Solve $y^{(3)}+2y' = 2\sin(3t)$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = 0$ and plot the solution.

Extensions of our method:

Our method also "works" for some Linear ODEs with non-constant coefficient. The problems are 1) evaluating the Laplace transform of $a(t)y^{(n)}(t)$ for some given function $a(t)$ and a unknown function $y^{(n)}$ and 2) we end up with another (but simpler) differential equation to solve. If $a(t) = t^n$ then this is particularly "nice"!

Example 5

Solve $y'' + 2ty' + y = 0$, $y(0) = 0$, $y'(0) = 1$ and plot the solution.

The coefficients need not be complex numbers, they can also be matrices. Consider a DE of the form

$$I\frac{d^n}{dt^n}y(t) + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0 y(t) = f(t)$$

where a_i is a $n \times n$ matrix, I is the $n \times n$ identity matrix and the functions $y(t)$ $f(t)$ are n dimensional vectors.

Example 6: Minecraft Chickens

Suppose we have a population of chickens, $C(t)$ and eggs $E(t)$ and suppose that the eggs turn into chickens at a rate α, and the chickens lay eggs at a rate β derive a coupled system of DEs to model the populations of chickens and eggs.

Looking ahead:

- Many applications of ECE/software engineering involve solving a system of linear DEs (i.e. multivariable systems).
- In control theory, we often want to find initial conditions so that we can "solve for $f(t)$". i.e. pick $f(t)$ and the ICs so that the solution to the DE does something we want.