MATH 213 - Lecture 23: Differentiating and integrating Fourier series, Persivalle and Dirichlet theorems and Fourier transform

Lecture goals: Know when you can term-wise integrate and differentiate Fourier series, what the Persivalle and Dirichlet theorems state and how to use Persivalle's theorem. Get you to fill out the SCP survey.

Integrating and differentiating Fourier series:

For general series of piecewise convergent functions

$$f(x) = \sum_{n=-\infty}^{\infty} f_n(x)$$

it is **NOT** true that

$$f'(x) = \sum_{n=-\infty}^{\infty} f_n'(x)$$

or that

$$\int f(x)dx = \sum_{n=-\infty}^{\infty} \int f_n(x)dx.$$

So when can we do this for Fourier series?

Theorem 1: Term-by-term integration of Fourier series
The Fourier series of a PWC1 $\tau-$periodic L^2 function $f(t)$ can be term-by-term integrated to give a convergent series that **pointwise converges** (and sometimes uniformly) over any finite interval.

Explicitly if f_p is in L^2 and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi njt}{\tau}}$$

then

$$\int_a^t f_p(t)dt = \sum_{n=-\infty}^{\infty} c_n \int_a^t e^{\frac{2\pi njt}{\tau}} dt$$

and the convergence is at least pointwise.

Note 1: It is because of this theorem, that we can guarantee the existence of the Fourier coefficients derived in L21.
Note 2: If the Fourier series has a non-trivial constant a_0 term then when integrated we obtain $a_0 t$ which means that the RHS might not be a Fourier series.

Theorem 2: Term-by-term differentiation of Fourier series
Let f

- be a PWC1 $\tau-$periodic function,
- be a continuous function and
- satisfy $f(-\tau/2) = f(\tau/2)$.

If the above conditions are met then the series for f can be term-by-term differentiated to give a pointwise convergent series that **pointwise converges** to $f_p'(t)$ on for all t such that $f_p''(t)$ exists

Explicitly if f satisfied the above and

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi njt}{\tau}}$$

then

$$f_p'(t)dt = \sum_{n=-\infty}^{\infty} c_n \frac{d}{dt} e^{\frac{2\pi njt}{\tau}}$$

and the convergence is pointwise for all points where $f_p''(t)$ exists.

Example 1
In Lecture 22 we showed that Fourier series for the π periodic version of x^2 is

$$\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(2nx)$$

and the Fourier series for the π periodic version of x is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx)$$

Use these facts to find the Fourier series for the π periodic version of x^3 and then x^4.

$$x^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(2nx)$$

Converges in L^2

$$\int_0^x x^2\,dx = \int_0^x \frac{\pi^2}{12}\,dx + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \int_0^x \cos(2nx)\,dx$$

$$\frac{x^3}{3} = \frac{\pi^2}{12} x + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{\sin(2nx)}{2n}$$

$$= \frac{\pi^2}{12} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2nx) + \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^3} \sin(2nx)$$

$$= \sum_{n=1}^{\infty} \left[\frac{\pi^2}{12}\frac{(-1)^{n+1}}{n} + \frac{(-1)^n}{2n^3}\right] \sin(2nx)$$

$$\Rightarrow \quad x^3 = \sum_{n=1}^{\infty} \left[\frac{\pi^2(-1)^{n+1}}{4n} + \frac{3(-1)^n}{2n^3}\right] \sin(2nx)$$

$$\int_0^x x^3\,dx = \sum_{n=1}^{\infty} c_n \int_0^x \sin(2nx)\,dx$$

conv in L^2

$$\frac{x^4}{4} = \sum_{n=1}^{\infty} c_n \left[-\frac{\cos 2nx}{2n} - -\frac{\cos(2n0)}{2n} \right]$$

$$= \sum_{n=1}^{\infty} \frac{-c_n}{2n}\cos(2nx) + \frac{c_n}{2n}$$

$$x^4 = \sum_{n=1}^{\infty} \frac{-4c_n}{2n}\cos(2nx) + \underbrace{\sum_{n=1}^{\infty} \frac{4c_n}{2n}}_{\text{constant term}}$$

$$c_0 = \sum_{n=1}^{\infty} \frac{2c_n}{n} = \langle x^4, 1 \rangle$$

$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^4\,dx$$

$$= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x^4 = \frac{2}{\pi}\cdot\frac{\pi^5}{5\cdot 2^5}$$

$$= \boxed{\frac{\pi^4}{80}}$$

Thus,

$$x^4 = \sum_{n=1}^{\infty} \left[\frac{\pi^2 (-1)^n}{2n^2} + \frac{12(-1)^{n+1}}{4n^4} \right] \cos(2nx) + \frac{\pi^4}{80}.$$

Recall that (although I did not previously name it in L22 Theorem 5):

> **Theorem 3: Dirichlet's theorem**
> If f is PWC1 and τ periodic then the Fourier series of $f(t)$ converges pointwise to $f_p(t)$.

If we define the error of a truncated Fourier series to be

$$e_N(t) = f_p(t) - \sum_{n=-N}^{N} c_n e^{\frac{2\pi n j t}{\tau}}$$

Then the above theorem tells us that for PWC1 τ periodic functions f,

$$\lim_{N\to\infty} \langle e_N, e_N \rangle = 0$$

i.e. the average error goes to zero in the limit. In other words:

> **Theorem 4: Parseval's theorem**
> If f is $L^2[-\tau/2, \tau/2]$ and τ periodic function then
>
> $$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2$$

The term $\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt$ is the average energy of the wave and hence

- this gives us a way to compute the energy of a wave from the Fourier coefficients (i.e. using fft),
- to determine how many terms we need to include in a truncation of the Fourier series to approximate the original signal to some given power,
- or to derive many useful formulas for computing sums...

Before giving an example, here is the real valued version of Parseval's theorem:

> **Theorem 5: Parseval's theorem**
> If f is a real valued PWC1 and τ periodic function then
>
> $$\frac{1}{\tau}\int_{-\tau/2}^{\tau/2} |f(t)|^2 dt = c_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (c_n^2 + s_n^2)$$
>
> where c_n and s_n are the Fourier sine and cosine coefficients.

Example 2

Recall from L21 that on $t \in [-\tau/2, \tau/2]$ and for $\tau > 0$,

$$\begin{cases} -1 & t \in (-\tau/2, 0) \\ 0 & else \\ 1 & t \in (0, -\tau/2) \end{cases} = \frac{4}{\pi} \sum_{n=1,\cancel{2},3,\ldots} \frac{1}{n} \sin\left(\frac{2\pi n}{\tau} t\right)$$

pointwise. Use the above with Perceval's theorem to find an infinite series that evaluates to π.

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\tau/2} 1 \, dt = 0 + \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{4}{\pi}\right)^2 \cdot \left(\frac{1}{(2n+1)}\right)^2$$

$$1 = \frac{16}{2\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{2n+1}\right)^2$$

$$\Rightarrow \pi^2 = 8 \sum_{n=1}^{\infty} \left(\frac{1}{2n+1}\right)^2$$

$$\Rightarrow \pi = 2\sqrt{2} \sqrt{\sum_{n=1}^{\infty} \left(\frac{1}{2n+1}\right)^2}$$

Fourier transform (tldr;):

Recall that the two sided Laplace transform was defined by

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$$

given that the integral converged.

When working with (causal) DEs, we used the one sided version of the above.

In our work with Fourier series, we required that the functions be τ periodic. If this is not the case, then our methods do not work!!

What do we do with these cases? Recall

$$f_\tau(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n t j}{\tau}}; \qquad c_n = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f_\tau(t)e^{-\frac{2\pi n t j}{\tau}}\,dt$$

If $\tau \to \infty$ then n is not allowed to be continuous so c_n becomes a function of $\mathbb{R}$ and the discrete sum becomes an integral. Further we would need to do some regularization to deal with the averaging in c_n. Doing this defining ω to be the limiting values of $\frac{2\pi n}{\tau}$ (and some complex analysis to deal with the regularization) gives:

$$f_\infty(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\omega t j}\,d\omega \qquad F(\omega) = \int_{-\infty}^{\infty} f_\infty(t)e^{-\omega t j}\,dt.$$

$F(\omega)$ is called the Fourier transform of f_∞ and we write $F(\omega) = \mathcal{F}\{f_\infty\}$ explicitly:

Definition 1: Fourier Transform
If $f \in L^1$ then the Fourier transform of f is

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-\omega t j}\,dt$$

and the inverse Fourier transform of $F(\omega)$ is

$$f(t) = \mathcal{F}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{\omega t j}\,d\omega.$$

Note that the Fourier transform/inverse transform is just the two-sided Laplace transform in the case that $s = \omega j$.

Example 3
Compute the Fourier transform of

$$f(t) = \begin{cases} 1 & |t| \leq \frac{1}{2} \\ 0 & else \end{cases}$$

$$F(w) = \mathcal{F}\{f(t)\}$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-wjt}\, dt$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-wjt}\, dt$$

$$= \left. \frac{e^{-wjt}}{-wj} \right|_{-\frac{1}{2}}^{\frac{1}{2}}$$

$$= \frac{e^{-\frac{wj}{2}} - e^{\frac{wj}{2}}}{-wj}$$

Looks kinda like

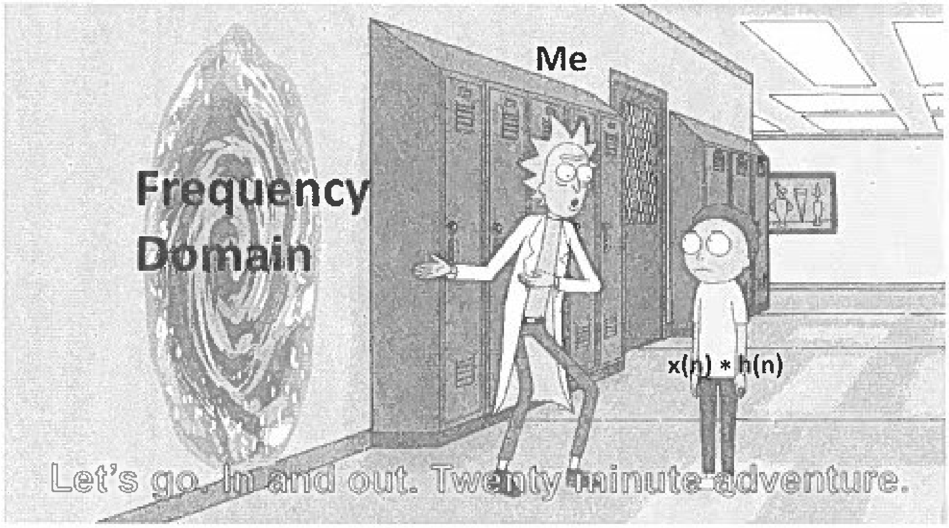
$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

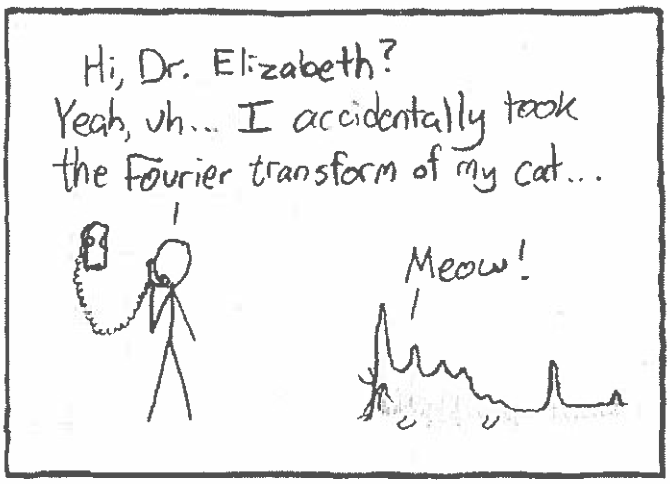
$$= \frac{2}{w} \cdot \frac{e^{\frac{wj}{2}} - e^{-\frac{wj}{2}}}{2j}$$

$$= \frac{2}{w} \sin\left(\frac{w}{2}\right)$$

$$= \frac{\sin\left(\frac{w}{2}\right)}{w/2} \equiv \operatorname{sinc}\left(\frac{w}{2}\right)$$

When you don't want to do a convolution





Well, here at last, dear friends, on this page comes the end of our MATH 213 Journey. Go in peace! I will not say: do not weep; for not all tears are an evil.