MATH 213 - Lecture 21: L^2 space, inner product on L^2, and computing Fourier coefficients

Lecture goals: know what the standard inner product in L^2 is and know how to use it to compute Fourier coefficients of $\tau-$periodic functions. Also get you to fill out the SCP survey.

In general properly defining a dot product for functions is a huge issue so we will only work with a special class of functions called "Lebesgue square integrable functions" or L^2 functions which makes things nice:

Definition 1: L^2 functions
A complex valued function f is in the class $L^2([a, b]])$ if

$$\int_a^b |f(x)|^2 dx$$

exists and is finite.

f is in the class L^2 if

$$\int_{-\infty}^{\infty} |f(x)|^2 dx$$

exists and is finite.

L^2 and $L^2([a, b])$ form vector spaces so ideas from MATH 115 can be used (with proper adjustments). Now if f is a member of $L^2([-\tau/2, \tau/2])$ for some fixed τ then our goal is to write

$$f(t) \text{``}=\text{''} \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau} jt} \qquad \text{for} \qquad t \in [-\tau/2, \tau/2].$$

To do this we need to somehow solve for the c_ns....

By comparison with how we solved

$$\vec{b} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

for an orthogonal basis $\{v_1, \ldots, v_n\}$
We need a....


Come closer...
I need...
I need...
A DOT PRODUCT
imgflip.com

Inner product for $L^2([-\tau, \tau])$

Recall that if $\vec{x}, \vec{y} \in \mathbb{C}^n$ then

$$\vec{x} \cdot \vec{y} = x_1 \overline{y_1} + x_2 \overline{y_2} + \ldots + x_n \overline{y_n}.$$

Now if f and g are complex valued functions, then the "dot product", which will be called an inner product, should follow a similar definition.

The summation becomes integration!!

Definition 2: Standard Inner product on $L^2([a, b])$
If f and g are complex valued functions in $L^2([a, b])$ then the **standard inner product** is

$$\langle f, g \rangle = \frac{1}{b-a} \int_a^b f(t) \overline{g(t)} dt.$$

Theorem 1: Existence of Inner Product
If $f, g \in L^2([a, b])$ then $\langle f, g \rangle$ exists and is finite.

We skip the proof since it needs some real analysis...

Theorem 2
The set of complex exponentials $\left\{ e^{\frac{2\pi n}{\tau} jt} | n \in \{0, \pm 1, \pm 2, \ldots\} \right\}$ is an orthonormal basis for a <u>subspace</u> of $L^2([-\tau/2, \tau/2])$.

Partial proof: We will not prove that the collection is linearly independent but will prove that they are orthonormal.

Now that we have an orthonormal basis for a subset of $L^2([-\tau/2, \tau/2])$, we can project any function in $L^2(-\tau/2, \tau])$ into our basis $\{e^{\frac{2\pi n}{\tau}jt} | n \in \{0, \pm 1, \pm 2, \dots\}\}$ by using our inner product.

Note that since we are doing a projection and the basis may not be (is not...) a basis for $L^2([-\tau/2, \tau/2])$, the result of projecting into this basis may not be equal to the original function in the traditional sense.

Definition 3: Fourier Series - Complex Form
If $f \in L^2([-\tau/2, \tau/2])$ then the **Fourier series in complex form** of $f(t)$ is $\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}$ where the c_n are found by projecting f into the basis of complex exponentials.

Theorem 3: Fourier Coefficients for Series in Complex Form
If $f \in L^2([-\tau/2, \tau/2])$ then the Fourier coefficients c_n of $f(t)$ are

$$c_n = \left\langle f(t), e^{\frac{2\pi n}{\tau}jt} \right\rangle.$$

If f is real valued than $c_n = \overline{c_{-n}}$.

Proof:

Compute the things!!

Example 1

Compute the Fourier series of $f : [-0.5, 0.5] \to \mathbb{R}$ defined by

$$f(t) = \sin(2\pi t)$$

If possible simplify the complex exponentials to real valued terms.

Example 2

Compute the Fourier series of $f : [-\tau/2, \tau/2] \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} -1 & t \in [-\tau/2, 0) \\ 1 & t \in [0, \quad \tau/2] \end{cases}$$

If possible simplify the complex exponentials to real valued terms.
Plot $f(t)$ along with several terms of the Fourier series.

Example 3

Compute the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{2}(\pi - |t|)$$

If possible simplify the complex exponentials to real valued terms.
Plot $f(t)$ along with several terms of the Fourier series.