Instructions

1. Make sure that your student ID number and UW userid are correctly displayed at the top of this page!
2. No electronic devices are not permitted.
3. Answer the questions in the spaces provided. There is extra space at the end of the exam if you need it.
4. **If you use the extra space page at the end of the exam, clearly indicate so in the problem(s) where you utilize the extra space. Failure to do so will result in the extra work not being marked!**
5. **You must submit all pages, even the last page**
6. This exam is 14 pages long (counting each side of paper as one page), including this cover page, formula sheet and the extra pages. You have 1 hour and 50 minutes to complete this exam.
7. Be sure to take your time during lengthy calculations, when possible.
8. Good luck, not that you need it! **You got this!!!**

Question	Value
1	3 + 1 (bonus)
2	4
3	3
4	5
5	5
6	5
7	6
8	4
9	5
10	6
11	8
Total	55

1. Answer the following questions by **filling in the square corresponding to your answer**.

[1] (a) For <u>all</u> $a \in \mathbb{R}$, if $\mathscr{L}\{f(t)\} = F(\omega)$ then $\mathscr{L}\{f(at)\} = \frac{1}{a}F\left(\frac{\omega}{a}\right)$

   i. ☐ T

   ii. ☐ F

   False!

[1] (b) If f is a piecewise continuous function, then f is also continuous.

   i. ☐ T

   ii. ☐ F

   False!

[1] (c) $\int_{-\infty}^{\infty} e^{i\pi x}\delta(x-1)dx = -1$

   i. ☐ T

   ii. ☐ F

   True

[1] (d) Marmie the cat wants to think you for work on checking her IVPs in assignment 3 by giving you a free point. Do you accept it?

   i. ☐ T

   ii. ☐ F

True

[4] 2. For each DE below state the order of the equation and classify it as PDE or ODE, linear or nonlinear and homogeneous or inhomogeneous.

(a) $y''(t) + y'(t) = \sin(y)$

(b) $u_t(x,t) = -u_x(x,t) + \sin(x)$

Solution:

(a) 2nd order, ODE, nonlinear, homogeneous

(b) 1st order, PE, linear, inhomogeneous

[3] 3. Give an example of a IVP that models a system that is dynamic, linear and time-invariant.

Briefly explain why your IVP models a system with the given properties.

Solution: Any linear DE with constant coefficients that has a forcing term works and all needed initial conditions being zero works. For a DE of the form $D(y) = f$ where f is the forcing term and D is a linear DE with constant coefficients, the ZIR part of the solution will be

$$y(t) = \int_0^t f(\tau)\left(\mathscr{L}^{-1}\{T(s)\}\right)\Big|_{t-\tau}\, d\tau$$

where $T(s)$ is the transfer function.

Clearly the system is dynamic since $y(t)$ depends on all $\tau \leq t$, it is linear since the system response to a linear combination of forcing terms is simply the linear combination of the system responses (because the integral is linear), and it is also time-invariant since if we lag the input, a simple change of variables will show that there is a lag in the system response.

[5] 4. Compute the two-sided Laplace transform of

$$f(t) = \begin{cases} 0 & t < 0 \\ e^t & 0 \le t < 1 \\ e^{50t} & 1 \le t \end{cases}$$

You must explicitly find the ROC with justification.

Solution: Direct computations give

$$\begin{aligned}
\mathscr{L}\{f(t)\} &= \int_{-\infty}^{\infty} f(t)e^{-st}dt \\
&= \int_0^1 e^t e^{-st}dt + \int_1^{\infty} e^{50t}e^{-st}dt \\
&= \int_0^1 e^{(1-s)t}dt + \int_1^{\infty} e^{(50-s)t}dt \\
&= \left.\frac{e^{(1-s)t}}{1-s}\right|_0^1 + \left.\frac{e^{(50-s)t}}{50-s}\right|_1^{\infty} \\
&= \left(\frac{e^{(1-s)}}{1-s} - \frac{1}{1-s}\right) + \left(\lim_{t\to\infty}\frac{e^{(50-s)t}}{50-s} - \frac{e^{50-s}}{50-s}\right)
\end{aligned}$$

The limit above is only finite when $\mathrm{Re}(50-s) < 0$ and in this case the limit is simply 0 hence the ROC is $\mathrm{Re}(50-s) < 0$ or simply $50 < \mathrm{Re}(s)$. The transform is

$$\mathscr{L}\{f(t)\} = \frac{e^{(1-s)} - 1}{1-s} - \frac{e^{50-s}}{50-s}$$

[5] 5. Use the Laplace table to compute the Laplace transform of $te^{-2t}\sin(2t) + 7e^{2t}\cosh(\pi t)$. You do not need to determine the ROC.

Solution: Using the table we compute

$$\begin{aligned}
\mathscr{L}\{te^{-2t}\sin(2t) + 7e^{2t}\cosh(\pi t)\} &= \mathscr{L}\{te^{-2t}\sin(2t)\} + 7\mathscr{L}\{e^{2t}\cosh(\pi t)\} && \text{Linearity}\\
&= \mathscr{L}\{t\sin(2t)\}|_{s=s+2} + 7\mathscr{L}\{\cosh(\pi t)\}|_{s=s-2} && \text{Exp Modulation}\\
&= \mathscr{L}\{t\sin(2t)\}|_{s=s+2} \qquad + 7\frac{s-2}{(s-2)^2-\pi^2} && \text{cosh transform}\\
&= -\frac{d}{ds}\left(\mathscr{L}\{\sin(2t)\}\right)|_{s=s+2} + 7\frac{s-2}{(s-2)^2-\pi^2} && \text{Mult. by } t\\
&= -\frac{d}{ds}\left(\frac{2}{s^2+2^2}\right)\Big|_{s=s+2} \qquad + 7\frac{s-2}{(s-2)^2-\pi^2} && \text{sin transform}\\
&= -\left(-2(s^2+2^2)^{-2}2s\right)\Big|_{s=s+2} + 7\frac{s-2}{(s-2)^2-\pi^2} && \text{Algebra}\\
&= \frac{4(s+2)}{((s+2)^2+4)^2} \qquad + 7\frac{s-2}{(s-2)^2-\pi^2} && \text{Algebra}
\end{aligned}$$

[5] 6. Use the poles to determine if the solution to the DE below is bounded regardless of the initial conditions we choose to apply.

$$y^{(5)} + 3y^{(3)} + 2y' = e^{-t}\sin(2t)$$

Solution: The transfer function is

$$\begin{aligned} T(s) &= \frac{1}{s^5 + 3s^3 + 2s} \\ &= \frac{1}{s(s^4 + 3s^2 + 2)} \\ &= \frac{1}{s(s^2+2)(s^2+1)} \end{aligned}$$

and the Laplace transform of the forcing term is

$$\begin{aligned} F(s) &= \mathscr{L}\{e^{-t}\sin(2t)\} \\ &= \mathscr{L}\{\sin(2t)\}|_{s=s+1} \\ &= \frac{2}{(s+1)^2 + 2^2} \end{aligned}$$

For the solution to remain bounded for all possible ICs we need the poles of $T(s)$ to be in the left hand plane or to be of order 1 if they have a real part of zero AND we also need the poles of $T(s)F(s)$ to have the same property.

The poles of $T(s)$ are 0, $\pm j$, $\pm\sqrt{2}j$ and are all of first order and the poles of $F(s)$ are $-1 \pm 2j$ and are of first order.

None of these poles have positive real parts, the poles with a real part of 0 are are all of order one, and there is no resonance of the poles of F and T with real part of 0 so the solution will remain bounded.

7. For parts (a) and (b) determine if the Initial and Final Value theorems are applicable to the given function. If they are applicable, apply them and explain the result. If a theorem is not directly applicable, explain why.

You must compute any needed limits but do not need to simply arithmetic computations e.g. unsimplified responses like $\frac{2\cdot5\cdot6}{3^2}$ are fine.

[3] (a) $F(s) = \frac{s-1}{s(s+1)(s^2+1)}$

[3] (b) $F(s) = \frac{(s-3)(s-2)^2(s-1)}{s(s-3)(s+6)^3}$

Solution:

(a) F is a strictly proper rational function so it has an inverse Laplace transform and hence the IVT and FVT apply. Further the poles are at $0, \; -1$ and $\pm j$.

The IVT tells us that

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{s-1}{(s+1)(s^2+1)} = 0.$$

Since the poles $\pm j$ have a real part of 0, the FVT tells us that the limit does not exist.

(b) F is a strictly proper rational function so it has an inverse Laplace transform and hence the IVT and FVT apply. Further the poles are at 0 and -6.

The IVT tells us that

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{(s-3)(s-2)^2(s-1)}{(s-3)(s+6)^3} = 1.$$

All poles have a real part of less than 0 except the pole of order 1 at 0 so the FVT tells us the limit exists and is

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{(s-3)(s-2)^2(s-1)}{(s-3)(s+6)^3} = \frac{(-3)(-2)^2(-1)}{(-3)6^3} = -\frac{1}{54}$$

[4] 8. Give the general form for the inverse Laplace transform of

$$F(s) = \frac{1}{s^2(s^2+4)^3(s^2-9)^2}.$$

As an example for what is meant by "general form" the general form for the inverse Laplace transform of $\frac{1}{(s+1)(s^2+4)}$ is $Ae^{-t} + B\sin(2t) + C\cos(2t)$ for some $A, B, c \in \mathbb{R}$. Do not attempt to find the coefficients.

Solution: The standard PF decomposition we need to make is

$$F(s) = \frac{C_1}{s^2} + \frac{C_2}{s} + \frac{C_3s+C_4}{(s^2+4)^3} + \frac{C_5s+C_6}{(s^2+4)^2} + \frac{C_7s+C_8}{s^2+4} + \frac{C_9s+C_{10}}{(s^2-9)^2} + \frac{C_{11}s+C_{12}}{s^2-9}$$

The inverse Laplace transform of the first two terms is $C_1t + C_2$. For the next three terms we note that one and two applications of the multiplication by t rule will transform the Laplace transform of $A\sin(2t)+B\cos(2t)$ into the form of the fourth and third terms respectively. Hence the inverse transform of the three terms that include the s^2+4 term in the denominator will be of the form

$$(D_1 + D_2t + D_3t^2)\sin(2t) + (D_4 + D_5t + D_6t^2)\cos(2t).$$

Finally a similar analysis of the last two terms but replacing the trig terms with their hyperbolic equivalences yields an inverse transform of the form

$$(D_7 + D_8t)\sinh(3t) + (D_9 + D_{10}t)\cosh(3t).$$

Putting this together, gives the general form

$$C_1t+C_2+(D_1+D_2t+D_3t^2)\sin(2t)+(D_4+D_5t+D_6t^2)\cos(2t)+(D_7+D_8t)\sinh(3t)+(D_9+D_{10}t)\cosh(3t)$$

where the Cs and Ds are constants.

[5] 9. Use the convolution theorem to compute $4t^2 * e^{2t}$ where both functions are one-sided.

No marks will be given if you do not use the convolution theorem for this problem.

Hint: The final answer only contains nice numbers.

Solution: The convolution theorem gives us that $4t^2 * e^{2t} = \mathscr{L}^{-1}\{\mathscr{L}\{4t^2\}\mathscr{L}\{e^{2t}\}\}$. From the Laplace table $\mathscr{L}\{4t^2\} = 8/s^3$ and $\mathscr{L}\{e^{2t}\} = 1/(s-2)$. Thus

$$t^2 * e^{2t} = \mathscr{L}^{-1}\left\{\frac{8}{s^3(s-2)}\right\}.$$

We now find the PF decomposition of $8/(s^3(s-2))$. The ansatz is

$$\frac{8}{s^3(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2}$$

If $s = 2$ then we have

$$\frac{8}{2^3} = D \qquad \text{or} \qquad D = 1.$$

If $s = 0$ then we have

$$\frac{8}{-2} = C \qquad \text{or} \qquad C = -4.$$

To get the coefficients for A and B we solve the system of equations we get from letting $s = \pm 1$. The system is

$$\begin{aligned} \frac{8}{1^3(1-2)} &= A + B + C + \frac{D}{1-2} \\ \frac{8}{(-1)^3(-1-2)} &= -A + B - C + \frac{D}{-1-2} \end{aligned}$$

or

$$\begin{aligned} -8 &= A + B - 4 - 1 \\ \frac{8}{3} &= -A + B + 4 - \frac{1}{3} \end{aligned}$$

or

$$\begin{aligned} -3 &= A + B \\ -1 &= -A + B \end{aligned}$$

Adding these equations gives

$$-4 = 2B \qquad \text{or} \qquad B = -2.$$

Thus $A = -1$. Putting this together we have

$$\frac{8}{s^3(s-2)} = -\frac{1}{s} - \frac{2}{s^2} - \frac{4}{s^3} + \frac{1}{s-2}.$$

Thus

$$\begin{aligned} t^2 * e^{2t} &= \mathscr{L}^{-1}\left\{\frac{8}{s^3(s-2)}\right\} \\ &= \mathscr{L}^{-1}\left\{-\frac{1}{s} - \frac{2}{s^2} - \frac{4}{s^3} + \frac{1}{s-2}\right\} \\ &= -\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{2}{s^2}\right\} - \mathscr{L}^{-1}\left\{\frac{4}{s^3}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} \\ &= -1 - 2t - 2t^2 + e^{2t} \end{aligned}$$

where the above function is one sided.

[6] 10. Solve the following IVP:

$$y'' + 2y' + y = \delta(t-1); \quad y(0) = 0, \quad y'(0) = 1.$$

Solution:

We apply the Laplace transform to see that

$$(s^2Y - y(0)s - y'(0)) + 2(sY - y(0)) + Y = e^{-s} \qquad \Longrightarrow \qquad (s^2Y - 1) + 2(sY) + Y = e^{-s}.$$

Solving for Y gives

$$(s+1)^2Y = 1 + e^{-s} \qquad \text{or} \qquad Y = \frac{1}{(s+1)^2} + \frac{e^{-s}}{(s+1)^2}$$

Taking the inverse transform gives

$$\begin{aligned}
y(t) &= \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2}\right\} && \\
&= e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}\Bigg|_{t=t-1} u(t-1) && \text{exp mod. and time shift} \\
&= te^{-t} + \left(e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\}\right)\Bigg|_{t=t-1} u(t-1) && \text{trans. of } t \text{ and exp mod} \\
&= te^{-t} + \left(te^{-t}\right)\Big|_{t=t-1} u(t-1) && \text{trans. of t} \\
&= te^{-t} + (t-1)e^{-(t-1)}u(t-1) && \text{Algebra}
\end{aligned}$$

[8] 11. Find the solution to the following IVP

$$y'' + 6y' - 7y = f(t); \qquad y(0) = 1, \quad y'(0) = -7.$$

You should leave your final answer in a form that includes a convolution integral of two known functions of time and you **must** identify the ZIR and ZSR portions of the solution.

Solution: Taking the Laplace transform gives

$$(s^2Y - y(0)s - y'(0)) + 6(sY - y(0)) - 7Y = F(s) \quad \Longrightarrow \quad (s^2Y - s + 7) + 6(sY - 1) - 7Y = F(s).$$

Solving for Y gives

$$(s^2 + 6s - 7)Y = s - 1 + F(s) \quad \text{or} \quad Y = \frac{s-1}{s^2+6s-7} + \frac{F(s)}{s^2+6s-7} = \frac{1}{s+7} + \frac{F(s)}{s^2+6s-7}$$

Now

$$Y(s) = \frac{1}{s+7} + \frac{F(s)}{s^2+6s-7} = \frac{1}{s+7} + \frac{F(s)}{s^2+6s+3^2-3^2-7} = \frac{1}{s+7} + \frac{F(s)}{(s+3)^2-16}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y(t) &= e^{-7t} + f(t) * \mathscr{L}^{-1}\left\{\frac{1}{(s+3)^2-16}\right\} \\
&= e^{-7t} + f(t) * e^{-3t}\mathscr{L}^{-1}\left\{\frac{1}{s^2-16}\right\} \\
&= e^{-7t} + f(t) * \frac{e^{-3t}\sinh(\sqrt{16}t)}{\sqrt{16}} \\
&= e^{-7t} + f(t) * \frac{e^{-3t}\sinh(4t)}{4} \\
&= e^{-7t} + \frac{1}{4}\int_0^t e^{-3\tau}\sinh(4\tau)f(t-\tau)d\tau
\end{aligned}$$

The ZSR is e^{-7t} and the ZIR is the convolution integral term.

Solution 2: Taking the Laplace transform gives

$$(s^2Y - y(0)s - y'(0)) + 6(sY - y(0)) - 7Y = F(s) \quad \Longrightarrow \quad (s^2Y - s + 7) + 6(sY - 1) - 7Y = F(s).$$

Solving for Y gives

$$(s^2 + 6s - 7)Y = s - 1 + F(s) \quad \text{or} \quad Y = \frac{s-1}{s^2+6s-7} + \frac{F(s)}{s^2+6s-7} = \frac{1}{s+7} + \frac{F(s)}{s^2+6s-7}$$

Now

$$\frac{1}{(s+7)(s-1)} = \frac{A}{s+7} + \frac{B}{s-1}$$

and the Heaviside method gives $s = 1 \implies B = \frac{1}{8}$ and $s = -7 \implies A = -\frac{1}{8}$. Thus

$$Y(s) = \frac{1}{s+7} + \frac{F(s)}{s^2+6s-7} = \frac{1}{s+7} - \frac{F(s)}{8(s+7)} + \frac{F(s)}{8(s-1)}$$

Taking the inverse Laplace transform gives

$$\begin{aligned}
y(t) &= e^{-7t} - \frac{1}{8}f(t) * \mathscr{L}^{-1}\left\{\frac{1}{s+7}\right\} + \frac{1}{8}f(t) * \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} \\
&= e^{-7t} - \frac{1}{8}f(t) * e^{-7t} + \frac{1}{8}f(x) * e^{t} \\
&= e^{-7t} + f(t) * \left(\frac{1}{8}e^{t} - \frac{1}{8}e^{-7t}\right) \\
&= e^{-7t} + \frac{1}{8}\int_0^t \left(e^{\tau} - e^{-7\tau}\right) f(t-\tau)d\tau.
\end{aligned}$$

The ZSR is e^{-7t} and the ZIR is the convolution integral term.

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
8. $\delta^{(n)}(t)$	s^n	$\mathbb{C}$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c} F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s - \alpha)$	Exponential Modulation
$f(t - T)u(t - T)$	$e^{-sT} \mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

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