MATH 213 - Assignment 6 - Solutions

Submit to Crowdmark by 9:00pm EST on Monday, April 8.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Read before starting the assignment: For this assignment you must do all your work independently and without the use of external aids.
Questions:

1. (5 marks) Find the Complex Fourier series for the $\pi/2$ periodic version of

$$f(z) = ze^{2jz}, \qquad z \in \mathbb{C}.$$

Solution: $\tau = \pi/2$ and hence for $n \neq 0$ we have

$$\begin{aligned}
c_n &= \langle xe^{2jx}, e^{\frac{2\pi njx}{\pi/2}} \rangle \\
&= \langle xe^{2jx}, e^{4njx} \rangle \\
&= \frac{1}{\pi/2} \int_{-\pi/4}^{\pi/4} xe^{2jx} e^{-4njx} dx \\
&= \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} xe^{(2-4n)jx} dx \\
&= \frac{2}{\pi} \left(\frac{1}{((2-4n)j)^2} + \left(\frac{x}{(2-4n)j} - \frac{1}{((2-4n)j)^2} \right) e^{(2-4n)jx} \right) \Bigg|_{-\pi/4}^{\pi/4}
\end{aligned}$$

Note that

$$\begin{aligned}
e^{\pm\frac{\pi}{4}(2-4n)j} &= e^{\pm\frac{\pi}{2}j} \cdot e^{\mp n\pi j} \\
&= \pm j \cdot (-1)^n
\end{aligned}$$

and that the constant term will vanish when we compute the limits. Hence

$$\begin{aligned}
c_n &= \frac{2}{\pi} \left(\left(\frac{\pi/4}{(2-4n)j} - \frac{1}{((2-4n)j)^2} \right) (-1)^n j - \left(\frac{-\pi/4}{(2-4n)j} - \frac{1}{((2-4n)j)^2} \right) (-(-1)^n j) \right) \\
&= \frac{2}{\pi} \left(\left(\frac{(-1)^n j\pi/4}{(2-4n)j} - \frac{(-1)^n j}{((2-4n)j)^2} \right) \qquad + \left(\frac{-(-1)^n j\pi/4}{(2-4n)j} - \frac{(-1)^n j}{((2-4n)j)^2} \right) \right) \\
&= \frac{2}{\pi} \left(-\frac{2(-1)^n j}{((2-4n)j)^2} \right) \\
&= -\frac{4(-1)^n j}{\pi(2-4n)^2 j^2} \\
&= -\frac{4(-1)^n}{\pi(2-4n)^2 j} \\
&= \frac{(-1)^n 4j}{\pi(2-4n)^2}
\end{aligned}$$

To compute c_0 we note that there is no signularity in our above work when $n = 0$ or any other value and hence

$$\begin{aligned}
c_0 &= \left\langle xe^{2jx}, 1 \right\rangle \\
&= \frac{(-1)^0 4j}{\pi(2-4 \cdot 0)^2} \\
&= \frac{j}{\pi}
\end{aligned}$$

Thus

$$f(z) = \frac{j}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n 4j}{\pi(2-4n)^2} e^{4njz}$$

or simply

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 4j}{\pi(2-4n)^2} e^{4njz}$$

2. (6 marks) Note: This question is just a glorified: "find the Fourier series" question hidden inside a historically important mathematics/engineering problem.

   In Fourier's 1807 paper titled "Mèmoire sur la propagation de la chaleur dans les corps solides", Fourier introduced a method that is now known as "Fourier Series". In this paper Fourier used the methods we have been exploring to solve the heat equation which models the temperature flow in an object with uniform thermodynamic properties. Subsequently and famously, he used his method to find the general solution to the vibrations of a string modelled by the linear wave equation that we introduced in Lecture 1. This analysis is still used today to study such linear systems (e.g. modelling vibrations in physical objects such as beams, phones, etc, deriving the shapes of the atomic orbitals, determining the composition and temperature of far away stars and planets and modelling the transfer of energy in systems such as CPU cooling systems). In more complex non-linear models (i.e. more realistic models), this analysis is commonly used to explore the failure modes of objects under stress in the "linear regime". This type of analysis tells us how things will break if something goes wrong and subsequently can be used to strengthen system designs. In this problem we will explore how to use Fourier series to solve the problem of the linear 1D vibrating string.

   It can be shown (L1) that the vibrations of an elastic string with length 2π that is clamped at the endpoints and with an initial velocity of 0 can be described by

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2} \qquad \text{for all } -\pi < x < \pi \text{ and } t > 0 \tag{1}$$

$$u(0,t) = u(2\pi,t) = 0 \qquad \text{for all } t > 0 \tag{2}$$

$$u(x,0) = f(x) \qquad \text{for all } -\pi < x < \pi \tag{3}$$

$$u_t(x,0) = 0 \qquad \text{for all } -\pi < x < \pi \tag{4}$$

   Here $u(x,t)$ is the position of the string at any time and $f(x)$ is the initial position of the string. The function

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k \sin(kx)\cos(kt).$$

   solves (1)-(2) with condition (4) and can be made to also satisfy the initial condition (3) if we can find the coefficients α_k.

   Assuming that the initial position of the string is given by $f(x) = x - \frac{x^3}{\pi^2}$, use (3) along with the given form of the proposed solution $u(x,t)$ to find α_k.

   Your α_k **must be** given as a simple expression that does not contain a sum of terms and has been simplified so that all trig terms have been simplified. Make sure to simplify it!

Solution:

Simplifying equation (3) we find that

$$\sum_{k=1}^{\infty} \alpha_k \sin(kx) = f(x).$$

This is just the Fourier sine series for $f(x)$!

To compute the coefficients for the sine series for x we note that $\tau = 2\pi$ and hence

$$\begin{aligned} s_n &= 2\left\langle x, \sin\left(\frac{2\pi n}{2\pi}x\right)\right\rangle \\ &= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin(nx)dx \\ &= \frac{1}{\pi}\left(\frac{\sin(nx)}{n^2} - \frac{x\cos(nx)}{n}\right)\Bigg|_{-\pi}^{\pi} \end{aligned}$$

Note that for integers n, $\sin(\pm\pi n) = 0$ and $\cos(\pm\pi n) = (-1)^n$ Hence

$$\begin{aligned} s_n &= \frac{-1}{\pi}\left(\frac{\pi\cos(n\pi)}{n} - \frac{-\pi\cos(-n\pi)}{n}\right) \\ &= \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Thus

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx).$$

To find the term for x^3 we will integrate twice using Parseval's Theorem to compute any needed constant terms. For the x^2 series we have

$$\begin{aligned} \int_0^x x dx &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\int_0^x \sin(nx)dx \\ \frac{x^2}{2} &= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\left(\frac{-\cos(nx)}{n} - \frac{-1}{n}\right) \\ \frac{x^2}{2} &= \sum_{n=1}^{\infty}\left(\frac{2(-1)^n\cos(nx)}{n^2} + \frac{2(-1)^{n+1}}{n^2}\right) \\ x^2 &= \sum_{n=1}^{\infty} \frac{4(-1)^n\cos(nx)}{n^2} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2} \end{aligned}$$

To compute the series for the constant term we will note that the constant term is the constant term for the Fourier cos series and hence

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2} &= \langle x^2,1\rangle \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}x^2dx \\ &= \frac{1}{\pi}\int_{0}^{\pi}x^2dx \\ &= \frac{1}{\pi}\left.\frac{x^3}{3}\right|_0^{\pi} \\ &= \frac{\pi^2}{3}\end{aligned}$$

Thus

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n\cos(nx)}{n^2}$$

Integrating again gives

$$\begin{aligned}\frac{x^3}{3} &= \int_0^x \frac{\pi^2}{3}dx + \sum_{n=1}^{\infty}\int_0^x \frac{4(-1)^n\cos(nx)}{n^2}dx \\ &= \frac{\pi^2}{3}x + \sum_{n=1}^{\infty}\int_0^x \frac{4(-1)^n\cos(nx)}{n^2}dx \\ &= \frac{\pi^2}{3}x + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^3}\left(\sin(nx)|_0^x\right) \\ &= \frac{\pi^2}{3}x + \sum_{n=1}^{\infty}\frac{4(-1)^n\sin(nx)}{n^3}\end{aligned}$$

Multiplying by 3 and using our previously found series for x gives

$$x^3 = \pi^2\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx) + \sum_{n=1}^{\infty}\frac{12(-1)^n\sin(nx)}{n^3}$$

or

$$x^3 = \sum_{n=1}^{\infty}\left(\frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3}\right)\sin(nx)$$

Putting this together gives

$$\begin{aligned}x - \frac{x^3}{\pi^2} &= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx) - \frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\left(\frac{2\pi^2(-1)^{n+1}}{n} + \frac{12(-1)^n}{n^3}\right)\sin(nx)\right) \\ &= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx) - \left(\sum_{n=1}^{\infty}\left(\frac{2\ (-1)^{n+1}}{n} + \frac{12(-1)^n}{\pi^2 n^3}\right)\sin(nx)\right) \\ &= \sum_{n=1}^{\infty}\frac{12(-1)^{n+1}}{\pi^2 n^3}\sin(nx).\end{aligned}$$

The coefficients are thus,

$$\alpha_k = \frac{12(-1)^{k+1}}{\pi^2 k^3}.$$

3. (3 marks) Determine with some intuitive justification what the pointwise limit of $f_n(x) = x^n$ on $[0, 1]$ is. Does this limit converge uniformly, why or why not?

   Solution: For any fixed $x \in [0, 1)$ as $n \to \infty$ the value becomes closer to 0. Further if $x = 1$ then $x^n = 1$ for all natural numbers n. Hence the pointwise limit must be

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$$

   To check for uniform convergence, we need to find the maximum of $|f_n(x) - f(x)|$ on $[0, 1]$. Clearly the error at $x = 0$ and 1 is 0. Further, for any fixed n the error becomes larger as $x \to 1$. Hence the maximum does not exist and we thus replace it with the smallest upper bound. Explicitly for any fixed finite value of n we have

$$\begin{aligned} \lim_{x \to 1} |f_n(x) - f(x)| &= \lim_{x \to 1} \begin{cases} 0 & x = 0, 1 \\ x^n & else \end{cases} \\ &= 1. \end{aligned}$$

   Since this does not go to 0 as $n \to \infty$, the series does not uniformly converge.

4. (3 marks) In 1872 Weierstrass proved the *super* intuitive result that there exits functions that are continuous on $\mathbb{R}$ but do not have a derivative for at any point in $\mathbb{R}$. One such function is

$$f(x) = \sum_{n=0}^{\infty} 0.9^n \cos(10^n \pi x).$$

   While proving that this function is not differential anywhere is beyond the scope of the class[1], we can prove it is continuous for all of $\mathbb{R}$ by using the classic result that:

   "The limit of a uniformly converging series of continuous functions is continuous".

   Show that the sum converges uniformly to some limit and hence that the limit is continuous by the above theorem.

   You are allowed to plot as many of the terms in the series you wish to plot. This **won't** help for this question but feel free to be curious.

   Hint 1: Do not attempt to find the limit, it is rather horrid... it is actually a fractal...

   Hint 2: To prove the uniform convergence of a series, it is enough to show that the series of the maximum values converge. Explicitly.

   If $\sum_{n=0}^{\infty} \max |f_n|$ converges, then $\sum_{n=0}^{\infty} f_n(x)$ converges.

   This is known as the "Weierstrass M-test". It is a REALLY good idea to use this result for this problem! and you can feel free to use this for any other problems I give you.

   Solution:

   Since cos is absolutely bounded by 1, $|(0.9)^n \cos(10^n \pi x)| \leq 0.9^n$ for all $x \in \mathbb{R}$. Further the series

$$\sum_{n=1}^{\infty} 0.9^n$$

[1] Optionally see this for a short history of the problem and how to prove it!

is a convergent geometric series! Thus by the Weierstrass M-test, the series

$$\sum_{n=1}^{\infty} 0.9^n \cos(10^n \pi x)$$

converges uniformly to some function $f(x)$ on $\mathbb{R}$. Since all terms in the series are continuous, the series converges to a continuous function.

5. (9 marks) For the following functions, determine what type of convergence the Fourier series has. You should not compute the Fourier series themselves.

   a) $f(x) = x^9$ on $(-2, 2)$

   Solution:

   In this case $f_p(x)$ is clearly piecewise C^1 (thus piecewise continuous) but is not continuous as $f(-\pi_+) = -\pi^9 \neq \pi^9 = f(\pi_-)$. Thus the Fourier series for f converges to the mean to f_p on any finite interval and converges pointwise to f_p on $\mathbb{R}$. On the other hand since f_p is not continuous and the terms in the series are, the series does not converge uniformly to f_p.

   b) $f(x) = x^{10} + x^2$ for $(-\pi, \pi)$

   Solution:

   $f_p(x)$ is clearly piecewise C^1 (thus piecewise continuous) and is also continuous as $f(-\pi_+) = f(\pi_-)$. Thus the Fourier series for f converges to the mean to f_p on any finite interval, converges pointwise to f_p on $\mathbb{R}$ and converges uniformly to f_p on any finite interval.

   c) $f(x) = \sqrt{|x|}$ for $(-\pi, \pi)$

   Solution:

   $f_p(x)$ is clearly piecewise continuous. On the other hand the one sided limits of f' are not bounded at 0 and thus f is not piecewise C^1. Thus the Fourier series for f converges to the mean to f_p on any finite interval. Our theorems do not tell us anything about pointwise or uniform convergence.

6. (5 marks) Apply Parseval's Theorem to $\dfrac{-x^2}{4} = \dfrac{-\pi^2}{12} + \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos(nx)$ for $-\pi < x < \pi$, to determine the value of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$.

Solution:

Since

$$\frac{-x^2}{4} = \frac{-\pi^2}{12} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}\cos(nx), \quad -\pi < x < \pi$$

we have

$$f(x) = \frac{-x^2}{4}, \quad a = \frac{-\pi^2}{12}, \quad c_n = \frac{(-1)^{n-1}}{n^2}, \quad S_n = 0, \quad \tau = 2\pi.$$

Applying Parseval's Theorem leads to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\frac{-x^2}{4}\right)^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = |c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 + \sum_{n=-1}^{-\infty} |c_n|^2$$

where

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{(-1)^{n-1}}{2n^2} \quad \Rightarrow \quad |c_n|^2 = |c_{-n}|^2 = \frac{1}{4n^4} \text{ for } n \neq 0$$

$$c_0 = a_0 \quad \Rightarrow \quad |c_0|^2 = |a_0|^2 = \frac{1}{4}\frac{\pi^4}{36}$$

It follows that

$$\underbrace{\frac{1}{\pi}\int_0^\pi \frac{x^4}{16}dx}_{\frac{\pi^4}{5\cdot 16}} = \frac{1}{4}\frac{\pi^4}{36} + 2\sum_{n=1}^{\infty}\frac{1}{4n^4} = \frac{1}{4}\frac{\pi^4}{36} + \sum_{n=1}^{\infty}\frac{1}{2n^4}$$

and rearranging leads to

$$\sum_{n=1}^{\infty}\frac{1}{2n^4} = \frac{\pi^4}{5\cdot 16} - \frac{\pi^4}{4\cdot 36}.$$

Therefore,

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}.$$