MATH 213 - Tutorial 2: Computing Laplace Transforms - Solutions

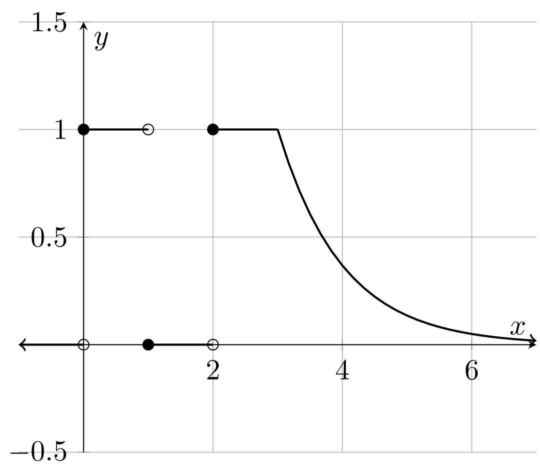
1. Compute the Laplace Transform of

$$f(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t < 1 \\ 0 & 1 \le t < 2 \\ 1 & 2 \le t < 3 \\ e^{-(t-3)} & else \end{cases}$$

   and find the ROC.

   Hint: Be careful with the $s = 0$ case.

   Solution: We simply need to apply the definition and then integrate. Before doing this, we will plot the function to see what the sygnal looks like (this is optional).



   Now we integrate

$$\begin{aligned}
\mathscr{L}(\{f(t)\} &= \int_{-\infty}^{\infty} f(t)e^{-st}dt \\
&= \int_0^1 e^{-st}dt \quad + \int_2^3 e^{-st}dt \qquad + \int_3^\infty e^{-(t-3)}e^{-st}dt \\
&= \left.\frac{e^{-st}}{-s}\right|_0^1 \qquad + \left.\frac{e^{-st}}{-s}\right|_2^3 \qquad + \int_3^\infty e^{-(1+s)t+3}dt \\
&= \frac{-1}{s}\left(e^{-s} - 1\right) - \frac{1}{s}\left(e^{-3s} - e^{-2s}\right) + e^3 \int_3^\infty e^{-(1+s)t}dt \\
&= \frac{-1}{s}\left(e^{-3s} + e^{-s} - 1 - e^{-2s}\right) \qquad + e^3 \left.\frac{e^{-(1+s)t}}{-(1+s)}\right|_3^\infty \\
&= \frac{1 + e^{-2s} - e^{-3s} - e^{-s}}{s} \qquad - \frac{e^3}{1+s}\left(\lim_{t\to\infty} e^{-(1+s)t} - e^{-3(1+s)}\right) \\
&= \frac{1 + e^{-2s} - e^{-3s} - e^{-s}}{s} \qquad + \frac{e^{-3s}}{1+s} - \frac{e^3}{1+s}\lim_{t\to\infty} e^{-(1+s)t}
\end{aligned}$$

For this integral to converge we need $\lim_{t\to\infty} e^{-(1+s)t}$ to be finite. This happens exactly when the real part of the exponent is negative. Hence we need $\text{Re}(-(1+s)) < 0$ or $\text{Re}(1+s) > 0$ or simply $\text{Re}(s) > -1$. In this case the limit is 0.

For this question $s = 0$ is a special case that we must explicitly handle separately. This is because based on the limit argument we just made, the line $\text{Re}(s) = 0$ appears to be in the ROC but the functional form of $\mathscr{L}\{f(t)\}$ that we found above is undefined for $s = 0$.

We hence evaluate the second integral

$$\begin{aligned}
\mathscr{L}\{f(t)\}|_{s=0} &= \int_{-\infty}^{\infty} f(t)e^{-0t}dt \\
&= \int_{-\infty}^{\infty} f(t)dt \\
&= \int_0^1 1dt + \int_2^3 1dt + \int_3^{\infty} e^{-(t-3)}dt \\
&= 1 \qquad + 1 \qquad + e^3 \int_3^{\infty} e^{-t}dt \\
&= 1 \qquad + 1 \qquad - e^3\, e^{-t}\big|_3^{\infty} \\
&= 1 \qquad + 1 \qquad - e^3 \left(\lim_{t\to\infty} e^{-t} - e^{-3}\right) \\
&= 3
\end{aligned}$$

Putting all of this together gives us the final solution

$$\mathscr{L}\{f(t)\} = \begin{cases} \frac{1+e^{-2s}-e^{-3s}-e^{-s}}{s} + \frac{e^{-3s}}{1+s} & \text{Re}(s) > -1 \text{ and } s \neq 0 \\ 3 & s = 0 \\ \infty & else \end{cases}$$

Fun facts: Note that $\lim_{s\to 0} \mathscr{L}\{f(t)\} = 3$. Because of this, in practice one simply writes that $\mathscr{L}\{f(t)\} = \frac{1+e^{-2s}-e^{-3s}-e^{-s}}{s} + \frac{e^{-3s}}{1+s}$ for $\text{Re}(s) > -1$ and in the case where $s = 0$ we take $\mathscr{L}\{f(t)\}$ to be the "continuous continuation" of the function.

The above is the same process you used when you wrote things like

$$\frac{x(x+1)}{x} = x + 1$$

in your previous math classes. The above formally does not hold when $x = 0$ but if you only care about integrating the function, then since the limit of the LHS at $x = 0$ is finite, the point $x = 0$ does not impact the result and hence can be ignored for such computations.

2. Suppose that we know that $\mathscr{L}\{f(t)\} = \frac{s^2}{(s+1)(s+2)}$ and $\mathscr{L}\{g(t)\} = \sin(s)$ where f and g are one-sided functions i.e. for $t < 0$, $f(t) = g(t) = 0$. Use the algebraic properties of the Laplace transform to compute

$$\mathscr{L}\{f(2(t-3) - 2te^t g(t)\}.$$

Solution:

By the linearity property

$$\mathscr{L}\{f(2(t-3) - 2te^t g(t)\} = \mathscr{L}\{f(2(t-3))\} - 2\mathscr{L}\{te^t g(t)\}.$$

For the first part of the transform we note that

$$\begin{aligned}
\mathscr{L}\{f(2(t-3))\} &= \mathscr{L}\{f(2t-6)\} && \text{Algebra} \\
&= \frac{1}{2}\mathscr{L}\{f(t-6)\}|_{s=\frac{s}{2}} && \text{Time-Scaling} \\
&= \frac{1}{2}\mathscr{L}\{f(t-6)u(t-6)\}|_{s=\frac{s}{2}} && f \text{ is one-sided} \\
&= \frac{1}{2}\left(e^{-6s}\mathscr{L}\{f(t)u(t)\}\right)|_{s=\frac{s}{2}} && \text{Time-Shifting} \\
&= \frac{1}{2}e^{-3s}\left(\mathscr{L}\{f(t)\}\right)|_{s=\frac{s}{2}} && \text{Algebra} + f \text{ is one-sided}
\end{aligned}$$

Now using the given transform for $f(t)$ gives us

$$\begin{aligned}
\mathscr{L}\{f(2t-6)\} &= \frac{1}{2}e^{-3s}\left(\frac{s^2}{(s+1)(s+2)}\right)\Bigg|_{s=\frac{s}{2}} \\
&= \frac{1}{2}e^{-3s}\frac{(s/2)^2}{(s/2+1)(s/2+2)} \\
&= \frac{s^2}{2(s+2)(s+4)e^{3s}}
\end{aligned}$$

For the second part of the transform note that

$$\begin{aligned}
\mathscr{L}\{e^t(tg(t))\} &= \mathscr{L}\{tg(t)\}|_{s=s-1} && \text{Exponential Modulation} \\
&= -\frac{d}{ds}\left(\mathscr{L}\{g(t)\}\right)\Bigg|_{s=s-1} && \text{Multiplication by } t
\end{aligned}$$

Now using the given transform for $g(t)$ gives us

$$\begin{aligned}
\mathscr{L}\{e^t(tg(t))\} &= -\frac{d}{ds}\left(\sin(s)\right)\Bigg|_{s=s-1} \\
&= -\cos(s)|_{s=s-1} \\
&= -\cos(s-1).
\end{aligned}$$

Combining these gives

$$\mathscr{L}\{f(2(t-3) - 2te^t g(t)\} = \frac{s^2}{2(s+2)(s+4)e^{3s}} + 2\cos(s-1).$$

3. Use the results from class (i.e. Laplace table) to compute

$$\mathscr{L}^{-1}\left\{\frac{\omega s}{s^3+s}\right\}.$$

Solution: First note that

$$\frac{\omega s}{s^3+s}=\frac{\omega}{s^2+1}$$

By the linearity of the Laplace transform (and hence the inverse transform)

$$\mathscr{L}^{-1}\left\{\frac{\omega s}{s^3+s}\right\}=\omega\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\}.$$

Using the Laplace table we see that $\mathscr{L}\{\sin t\}=\frac{1}{s^2+1}$ and hence

$$\mathscr{L}^{-1}\left\{\frac{\omega s}{s^3+s}\right\}=\omega\sin(t).$$

For the double sided transform we simply multiply by $u(t)$.

4. Use the results from class (i.e. Laplace table) to compute

$$\mathscr{L}^{-1}\left\{\frac{2-s^2}{s^3+2s^2+2s}\right\}.$$

Solution: Note that

$$\frac{2-s^2}{s^3+2s^2+2s}=\frac{2-s^2}{s(s^2+2s+2)}$$

where s^2+2s+2 does not factor in the reals. We hence look for a PF decomposition of the form

$$\frac{A}{s}+\frac{Bs+C}{s^2+2s+2}$$

Using the Heaviside cover-up method (i.e. multiply by s then use $s=0$) gives $A=1$. To solve for B and C, we multiply by $s(s^2+2s+2)$ and then plug in $s=-1+j$ (one of the roots of the inside polynomial). This is a choice but it makes things nicer. Explicitly after doing this we arrive at the equation

$$\underbrace{2-(-1+j)^2}_{2-s^2}=\underbrace{(B(-1+j)+C)(-1+j)}_{(Bs+c)s}$$

or

$$2+2j=-C+(C-2B)j$$

Equating the real and imaginary components gives

$$\begin{aligned}2&=-C\\2&=C-2B\end{aligned}$$

and thus $B=-2$ and $C=-2$. Hence

$$\frac{2-s^2}{s^3+2s^2+2s}=\frac{1}{s}-2\frac{s+1}{s^2+2s+2}.$$

By the Linearity of the inverse Laplace transform, we can simply compute and then add the transforms of the two terms above.

From the Laplace table $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$. Now we need to turn the second term into things that appear in the Laplace table. It looks kinda like the transform of sin/cos but the denominator is not in the correct form. To fix this, we complete the square for the denominator:

$$\begin{aligned} s^2 + 2s + 2 &= s^2 + 2s + 1 - 1 + 2 \\ &= (s+1)^2 + 1. \end{aligned}$$

Thus

$$\frac{s+1}{s^2+2s+2} = \frac{s+1}{(s+1)^2+1}.$$

This looks a lot like the cos transform but with s replaced with $s+1$. To correct for this, we use the Exponential Modulation property to write

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} &= e^{-t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} && \text{Exponential Modulation} \\ &= e^{-t}\cos(t) && \text{Laplace Table} \end{aligned}$$

Putting this together we have

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{2-s^2}{s^3+2s^2+2s}\right\} &= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} \\ &= 1 + e^{-t}\cos(t) \end{aligned}$$

or $(1 + e^{-t}\cos(t))u(t)$ if we take the two sided transform.

For extra practice for computing the forward transform See Paul's notes and for extra practice for computing inverse transforms also see Paul's notes. I **strongly** suggest doing the harder examples in Ex3 on your own before looking at the solutions.