MATH 213 - Tutorial 4: Using Laplace to solve DEs and analyzing poles - Solutions

1. Compute the following limits without the use of L'Hospital's rule

   (a) $\lim_{x\to\infty} \frac{4x^4+1}{x^4-2x+1}$

   (b) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

   Solution:

   (a) The limit is in an indeterminate form. If we used L'Hospital's rule then we would compute the ratios of the derivatives up to the second derivatives and then we would find that the limit is 4. To deal with the limit from scratch we first need to divide the polynomials. This gives

   $$\frac{4x^4+1}{x^4-2x+1} = 4 + \frac{8x-3}{x^4-2x+1}$$

   Now in the limit 4 stays as 4 but the other term vanishes. Hence the limit is 4.

   Note in homework etc I will not make you do these computations without using L'Hospital's rule but the intuition for regularization is useful to have.

   (b) The limit is in an indeterminate form. If we used Hospital's rule then we would compute the ratios of the first derivatives to see that the limit is 2. Without using this we simply factor and cancel the $(x-1)$ term. This gives

   $$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} x+1 = 2.$$

2. Johnathan computed that the transfer function for the system he is building is given by

   $$T(s) = \frac{1}{s^3+2s^2+3s}.$$

   Suppose that Johnathan starts his system with the initial conditions $y(0) = 1$, $y'(0) = 1$ and $y''(0) = 1$.

   (a) What is the zero-input response of the system?

   (b) Without solving for $y(t)$, determine if Johnathan's system has a bounded solution. If the system has a bounded solution find $\lim_{t\to\infty} y(t)$ without computing $y(t)$.

   Solution:

   (a) The transfer function is the inverse of the characteristic polynomial. Hence the DE that has this transfer function is

   $$y''' + 2y'' + 3y' = f(t)$$

   where $f(t)$ is the input of the system (which wr could set to 0 for this problem). Taking the Laplace transform of this gives

   $$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 2(s^2Y(s) - sy(0) - y(0)) + 3(sY(s) - y(0)) = F(s)$$

   so with our ICs

   $$Y(s) = \frac{F(s)}{s^3+2s^2+3s} + \frac{s^2+3s+2}{s^3+2s^2+3s}.$$

   The zero-input response is hence

   $$Y(s) = \frac{s^2+3s+2}{s^3+2s^2+3s}.$$

(b) Since $s^3 + 2s^2 + 3s = s((s+1)^2 + 2)$ the potential poles are 0 and $-1 \pm \sqrt{2}i$. The first pole is a pole of order 1 at 0 so its contribution to the solution is bounded and the other two poles have real parts less than 0 so they are also bounded. Hence the solution remains bounded!

Applying the final value theorem gives

$$\begin{aligned}\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sY(s) \\ &= \lim_{s\to 0} \frac{s^2+3s+2}{s^2+2s+2} \\ &= 1\end{aligned}$$

3. Find the zeros and poles of

$$F(s) = \frac{s^2+s}{s^6+2s^4+s^2}$$

and use the poles to write $f(t)$ as a linear combination of appropriate terms.

Here you do not need to find the actual coefficients for the terms in $f(t)$.

Solution: We first need to regularize F. This gives us

$$\begin{aligned}F(s) &= \frac{s(s+1)}{s^2(s^4+2s^2+1)} \\ &= \frac{s+1}{s(s^2+1)^2}\end{aligned}$$

The roots are $s = -1$ and the poles are $s = 0$ and $s = \pm j$. Using the form of the poles and the fact that the complex roots are of order 2 (repeated twice), we know the solution looks like

$$f(t) = A + B\cos(t) + C\sin(t) + Dt\cos(t) + Et\sin(t)$$

4. Recall the Minecraft chicken problem from A1Q1 and lecture 6 example 6. In class we discussed some of the limitations of the model we considered in A1Q1 and replaced the model with the system of equations

$$\begin{aligned}E'(t) &= -\alpha E(t) + \beta C(t) \\ C'(t) &= \quad \alpha E(t)\end{aligned}$$

or in matrix form

$$\begin{bmatrix} E' \\ C' \end{bmatrix} - \begin{bmatrix} -\alpha & \beta \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} E \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\alpha, \beta > 0$ are the rates of egg hatching and egg laying.

Use the Laplace transform to first find the Laplace transforms of E and/or C under the initial conditions $E(0) = 0$ and $C(0) = 2$ and then to find the solutions in the case where $\alpha = 1$ and $\beta = 2$.

In this problem we will show two standard ways of using the Laplace transform to solve this linear system of ODEs with constant coefficients and one novel way that will not be covered in tutorial but will be uploaded to Learn.

Solution 1: The DEs together are called a **coupled** system of equations. This is because the dependent variables depend on each other via their governing equations and hence are "coupled". In this solution we will uncouple the system and solve the resulting second-order DE via Laplace.

Taking the derivative the E' equation gives

$$E''(t) = -\alpha E'(t) + \beta C'(t)$$

Now using the second equation to eliminate C' gives

$$E''(t) = -\alpha E'(t) + \alpha\beta E(t).$$

To solve this system we need two initial conditions on E, namely $E(0)$ and $E'(0)$ but we only have conditions on $E(0)$ and $C(0)$. We hence use the DE for E' to find the needed initial condition as follows

$$\begin{aligned} E'(0) &= -\alpha E(0) + \beta C(0) \\ &= -\alpha \cdot 0 + \beta \cdot 2 \\ &= 2\beta \end{aligned}$$

Now we have a standard DE problem that we can solve via Laplace:

$$E''(t) + \alpha E'(t) - \alpha\beta E(t) = 0, \qquad E(0) = 0, \qquad E'(0) = 2\beta$$

The problem is unforced and hence the zero-input response will only be present in the solution. Using results from the harmonic oscillator problem covered in example 1 of lecture 8 the Laplace transform of the solution $E(t)$ is given by

$$E_L(s) = \frac{2\beta}{s^2 + \alpha s - \alpha\beta}.$$

In general the poles are $s_\pm = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\alpha\beta}}{2}$ and one can analyze the solution in the various parameter regions but we now use $\alpha = 1$ and $\beta = 2$. In this case factoring gives

$$E_L(s) = \frac{4}{s^2 + s - 2} = \frac{4}{(s+2)(s-1)}$$

One of these roots is in the right half of the complex plane so we know the solution will exponentially grow! Computing the PF decomposition gives

$$E_L(s) = -\frac{4/3}{s+2} + \frac{4/3}{s-1}$$

and hence taking the inverse transform gives

$$E(t) = -\frac{4}{3}e^{-2t} + \frac{4}{3}e^{t}.$$

Now to find $C(t)$ we simply integrate the second equation with the now known forcing term $\alpha E(t)$. Doing this gives

$$\begin{aligned} C(t) &= \int \alpha E(t) dt \\ &= \int -\frac{4}{3}e^{-2t} + \frac{4}{3}e^{t} dt \\ &= \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t} + Const. \end{aligned}$$

To find the constant we use the IC $C(0) = 2$ to see that $Const. = 0$ and hence the solutions to this system are

$$\begin{aligned} E(t) &= -\frac{4}{3}e^{-2t} + \frac{4}{3}e^{t} \\ C(t) &= \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t}. \end{aligned}$$

Solution 2: Another option is to take the Laplace transform of the system of equations, solve the MATH 115 problem for the transforms and then to compute the inverse transforms. Taking the Laplace transforms of the governing equations gives

$$\begin{aligned} sE_L(s) - E(0) &= -\alpha E_L(s) + \beta C_L(s) \\ sC_L(s) - C(0) &= \quad \alpha E_L(s) \end{aligned}$$

or

$$\begin{bmatrix} s+\alpha & -\beta \\ -\alpha & s \end{bmatrix} \begin{bmatrix} E_L(s) \\ C_L(s) \end{bmatrix} = \begin{bmatrix} E(0) \\ C(0) \end{bmatrix}$$

or with our particular ICs and values for α and β

$$\begin{bmatrix} s+1 & -2 \\ -1 & s \end{bmatrix} \begin{bmatrix} E_L(s) \\ C_L(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

Oh look! The matrix above looks rather like the matrix we would use to find the eigenvalues of $\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$... Anyways using one of the MATH 115 methods (Gaussian elimination, method of elimination, etc see the posted 115 notes for a refresher if needed) we see that

$$\begin{aligned} E_L(s) &= \frac{4}{s^2+s-2} \\ C_L(s) &= \frac{2(s+1)}{s^2+s-2} \end{aligned}$$

Note that the denominators are the characteristic polynomial of the matrix $\begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix}$ with the eigenvalue replaced with the negative of the usual convention (hence the name in MATH 213)!

We already have shown how to compute the inverse transform of $E_L(s)$ so we will skip to $C_L(s)$. PF decomposition gives

$$\frac{2(s+1)}{s^2+s-2} = \frac{2/3}{s+2} + \frac{4/3}{s-1}$$

Hence using the Laplace table gives

$$C(t) = \frac{2}{3}e^{-2t} + \frac{4}{3}e^{t}.$$

These are the same results as in solution 1.

Solution 3 (Not expected to know for now but used in practice): The system we want to solve can be written as

$$\frac{d}{dt}\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} E(t) \\ C(t) \end{bmatrix}$$

with initial condition

$$\begin{bmatrix} E(0) \\ C(0) \end{bmatrix}.$$

This looks like an equation we have already solved $\frac{d}{dt}f(t) = af(t),\ f(0) = b$. In the later case the solution was $f(t) = be^a t$. Why not expand the definition of the exponential to be defined for a matrix! In this case the solution "should" be

$$\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = \underbrace{e^{\begin{bmatrix} -\alpha & \beta \\ \alpha & 0 \end{bmatrix} t}}_{\text{Some matrix}} \begin{bmatrix} E(0) \\ C(0) \end{bmatrix}.$$

But how can one do this? Well recall from MATH 119 that

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots$$

The RHS is defined with t is a square matrix!! Now using the values of α, β and the ICs would give us

$$\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = e^{\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} t} \begin{bmatrix} 0 \\ 2 \end{bmatrix}.$$

The problem becomes computing

$$\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}^n$$

The typical trick for this is to use the Eigendecomposition of the matrix. "Recall" from MATH 115 that this is writing $A = QDQ^{-1}$ where D is a diagonal matrix of eigenvalues and Q is the matrix with columns given by the eigenvectors. Solving the eigenvalue problem (you can review how to do this from the 115 notes) gives the eigenvalues of *gasp* -2 and 1 with corresponding eigenvectors of $\langle -2, 1\rangle$ and $\langle 1, 1\rangle$. Hence

$$\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$$

and therefore

$$\begin{aligned}
\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix}^n &= \left(\begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \right)^n \\
&= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}^n \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
e^{\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} t} &= \sum_{n=0}^{\infty} \left(\frac{t^n}{n!} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (-2)^n & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \right) \\
&= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \sum_{n=0}^{\infty} \left(\frac{t^n}{n!} \begin{bmatrix} (-2)^n & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \sum_{n=0}^{\infty} \left(\begin{bmatrix} \frac{(-2t)^n}{n!} & 0 \\ 0 & \frac{t^n}{n!} \end{bmatrix} \right) \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} & 0 \\ 0 & \sum_{n=0}^{\infty} \frac{t^n}{n!} \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}
\end{aligned}$$

The two infinite series on the inside are simply exponential functions evaluated at $-2t$ and t !! Hence

$$\begin{aligned}
e^{\begin{bmatrix} -1 & 2 \\ 1 & 0 \end{bmatrix} t} &= \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \\
&= \frac{1}{3} \begin{bmatrix} e^t + 2e^{-2t} & 2e^t - 2e^{-2t} \\ e^t - e^{-2t} & 2e^t + e^{-2t} \end{bmatrix}.
\end{aligned}$$

Using this computation the solution is

$$\begin{bmatrix} E(t) \\ C(t) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} e^t + 2e^{-2t} & 2e^t - 2e^{-2t} \\ e^t - e^{-2t} & 2e^t + e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4e^t - 4e^{-2t} \\ 4e^t + 2e^{-2t} \end{bmatrix}$$

or equating the vector components

$$\begin{aligned} E(t) &= \frac{4}{3}e^t - \frac{4}{3}e^{-2t} \\ C(t) &= \frac{4}{3}e^t + \frac{2}{3}e^{-2t} \end{aligned}$$

Which is what we found before!

Lesson: In practice, given a linear system with constant coefficients that has a “nice” matrix A one can solve the DE by simply:

- Finding the diagonalization of A i.e. D and Q such that $A = QDQ^{-1}$.
- Computing $Qe^D Q^{-1}$
- Multiplying the vector of ICs by this matrix.

If we only care about determining if the solution will grow, then we only need to check if any eigenvalues have a real part greater than 0.

There are many packages (LAPACK being a commonly used one) that will find the full diagonalization of a matrix for small matrices ($< 100 \times 100$) and will find eigenvalues with largest real part for large systems.