# MATH 213 - Assignment 2 - Solutions

**Submit to Crowdmark by 9:00pm EST on Friday, February 2.**

**Instructions:**

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

**Questions:**

1. (5 marks) Use the definition to compute the two-sided Laplace transform of $e^{-\alpha|t|}$ where $\alpha$ is a positive real number (i.e. $\alpha \in \mathbb{R}^+$). Simplify your result as much as possible and explicitly find the ROC with justification.

   For this problem you must show **all** of your work and must justify how you found the ROC.

   **Solution:** By definition of the two-sided Laplace transform and the absolute value function we have

$$\begin{aligned}
\mathscr{L}\{e^{-\alpha|t|}\} &= \int_{-\infty}^{\infty} e^{-\alpha|t|}e^{-st}dt \\
&= \int_{-\infty}^{0} e^{\alpha t}e^{-st}dt + \int_{0}^{\infty} e^{-\alpha t}e^{-st}dt && \text{Def of } |\cdot| \\
&= \int_{-\infty}^{0} e^{(\alpha-s)t}dt + \int_{0}^{\infty} e^{-(\alpha+s)t}dt \\
&= \left.\frac{e^{(\alpha-s)t}}{\alpha-s}\right|_{-\infty}^{0} - \left.\frac{e^{-(\alpha+s)t}}{\alpha+s}\right|_{0}^{\infty} \\
&= \frac{1}{\alpha-s}\left(1-\lim_{t\to-\infty} e^{(\alpha-s)t}\right) - \frac{1}{\alpha+s}\left(\lim_{t\to\infty} e^{-(\alpha+s)t}-1\right).
\end{aligned}$$

   The first limit converges if and only if $\text{Re}(\alpha - s) > 0$ and the second limit converges if and only if $\text{Re}(\alpha + s) > 0$. When these limits converge they converge to 0. Assuming these two conditions are met we have

$$\begin{aligned}
\mathscr{L}\{e^{-\alpha|t|}\} &= \frac{1}{\alpha-s} + \frac{1}{\alpha+s} \\
&= \frac{(\alpha-s)+(\alpha+s)}{(\alpha-s)(\alpha+s)} \\
&= \frac{2\alpha}{\alpha^2-s^2}.
\end{aligned}$$

   To clean up the conditions of convergence to find the ROC note that since $\alpha \in \mathbb{R}^+$ the requirements for convergence are $\alpha >\text{Re}(s)$ and $\alpha >\text{Re}(-s)$. Together these become $-\alpha <\text{Re}(s) < \alpha$.

2. (5 marks) Use the definition to compute $(f * g)(t)$ where

$$f(t) = \begin{cases} \sin(t) & 0 < t < \pi \\ 0 & else \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} 1 & 0 < t < T \\ 0 & else \end{cases}$$

   where $T > \pi$. For this problem you must show **all** of your work.

   **Hint:** You may find that thinking about the convolution geometrically simplifies the process of determining any cases that need to be considered.

   **Solution:** Since $f$ and $g$ are one sided functions, the convolution is given by

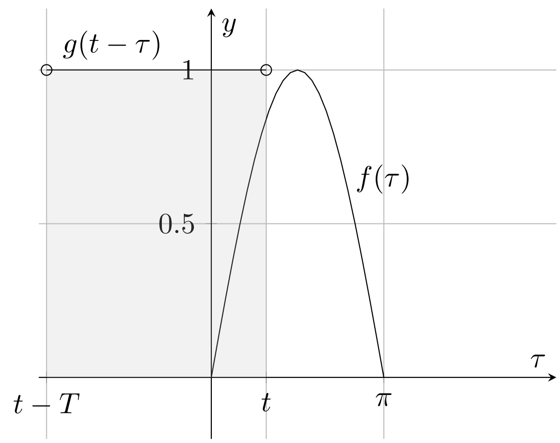
$$(f * g)(t) = \int_{0}^{t} f(\tau)g(t-\tau)d\tau.$$

Since

$$g(t-\tau) = \begin{cases} 1, & 0 < t-\tau < T \\ 0, & else \end{cases} \qquad \text{or} \qquad g(t-\tau) = \begin{cases} 1, & t > \tau > t-T \\ 0, & else \end{cases}.$$

Here are the cases we need to consider:

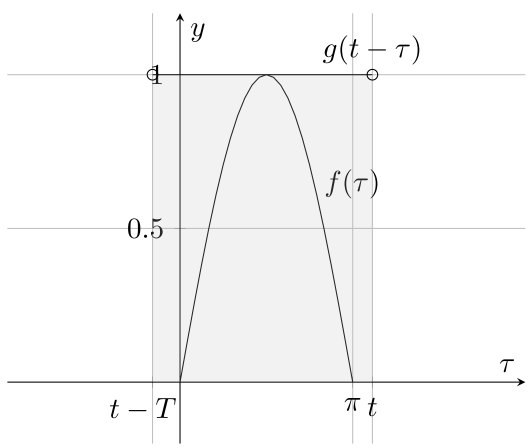
- If $0 < t < \pi$ then since $T > \pi$, $g(t-\tau) = 1$ for $\tau \in [0, t]$. Graphically these cases look like




  and the convolution is

$$(f * g)(t) = \int_0^t \sin(t)dt$$

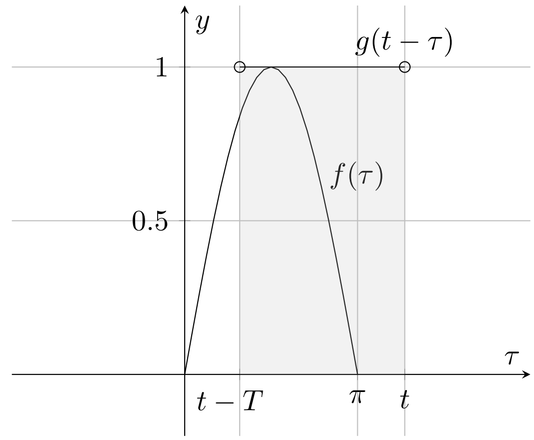
- If $\pi < t < T$ then since $T > \pi$, $g(t-\tau) = 1$ for $\tau \in [0, \pi]$ and $f(t) = 0$ for $\tau > \pi$. Graphically these cases look like




  and the convolution is

$$(f * g)(t) = \int_0^\pi \sin(t)dt$$

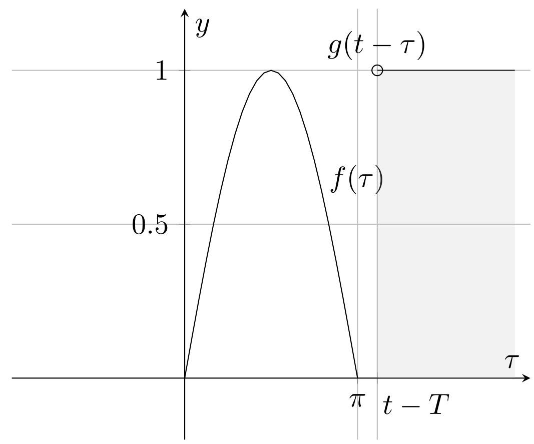
- If $T < t < \pi + T$ then $g(t-\tau) = 1$ for $\tau \in [t, \pi]$ and $f(t) = 0$ for $\tau > \pi$. Graphically these cases look like




and the convolution is

$$(f*g)(t) = \int_t^\pi \sin(t)dt$$

- Finally if $\pi + T < t$ then $g(t-\tau) = 0$ for $\tau \in [1, t]$. Graphically these cases look like




and the convolution is

$$(f*g)(t) = 0$$

Combining the above we have that the convolution is

$$(f*g)(t) = \begin{cases} \int_0^t \sin(\tau)d\tau, & 0 < t < \pi \\ \int_0^\pi \sin(\tau)d\tau, & \pi \le t < T \\ \int_{t-T}^\pi \sin(\tau)d\tau, & T \le t < T+\pi \\ 0, & else \end{cases}$$

Since the integral of $\sin(t)$ is $-\cos(t)$, and $cos(0) = -\cos(\pi) = 1$, direct computations give

$$(f*g)(t) = \begin{cases} 1-\cos(t), & 0 < t < \pi \\ 2, & \pi \le t < T \\ \cos(t-T)+1, & T \le t < T+\pi \\ 0, & else \end{cases}$$

3. (5 marks) Use the one-sided Laplace transform table and partial fractions to compute

$$\mathscr{L}^{-1}\left\{\frac{s^4}{s(2s^2+4)((s+1)^2+9)}\right\}.$$

For this problem you must show **all** of your work and explicitly mention what properties of the Laplace transform you use.

**Pro tip:** Before you blindly try to find a PF decomposition, look at the Laplace table and adjust the form of the PF decomposition to make your life nicer when computing the inverse transforms later.

**Solution:** We first notice that

$$\frac{s^4}{s(2s^2+4)((s+1)^2+9)} = \frac{s^3}{(2s^2+4)((s+1)^2+9)}$$

so the $s=0$ pole does not matter. We now make the PF ansatz:

$$\frac{s^3}{(2s^2+4)((s+1)^2+9)} = \frac{1}{2}\frac{As+\sqrt{2}B}{s^2+2} + \frac{C(s+1)+3D}{(s+1)^2+9}$$

We chose this ansatz because the terms will transform nicely later on.

- To find $A$ and $B$ we multiply by $2(s^2+2)$ and then plug in a root of this polynomial. We pick $\sqrt{2}j$ as the root. This gives

  $$\frac{(\sqrt{2}j)^3}{(\sqrt{2}j+1)^2+9} = A(\sqrt{2}j)+\sqrt{2}B.$$

  Simplifying the RHS gives

  $$\begin{aligned}\frac{(\sqrt{2}j)^3}{(\sqrt{2}j+1)^2+9} &= \frac{-2\sqrt{2}j}{(-2+2\sqrt{2}j+1)+9}\\ &= \frac{-2\sqrt{2}j}{2(4+\sqrt{2}j)}\\ &= \frac{-2\sqrt{2}j(4-\sqrt{2}j)}{2(18)}\\ &= \frac{-4\sqrt{2}j-2}{18}\\ &= -\frac{1}{9}-\frac{2\sqrt{2}j}{9}\end{aligned}$$

  Hence $A=-\frac{2}{9}$ and $B=-\frac{\sqrt{2}}{18}$.
- For $C$ and $D$ we multiply by $(s+1)^2+9$ and then plug in a root. We choose $3j-1$. This gives

  $$\frac{(3j-1)^3}{2(3j-1)^2+4} = C(3j-1+1)+3D.$$

  Note that $(3j-1)^2 = -9-6j+1 = -8-6j$ and $(3j-1)^3 = -24j+8+18+6j = 26-18j$. Hence the RHS simplifies to

  $$\begin{aligned}\frac{(3j-1)^3}{2(3j-1)^2+4} &= \frac{26-18j}{2(-8-6j)+4}\\ &= \frac{26-18j}{-12(1+j)}\\ &= \frac{(26-18j)(1-j)}{-24}\\ &= \frac{8-44j}{-24}\\ &= -\frac{1}{3}+\frac{11}{6}j\end{aligned}$$

The LHS simplifies to $3D + 3Cj$. Hence $D = -\frac{1}{9}$ and $C = \frac{11}{18}$.

Hence,

$$\frac{s^3}{(2s^2+4)((s+1)^2+9)} = \frac{1}{2}\frac{\left(-\frac{2}{9}\right)s - \left(\frac{\sqrt{2}}{18}\right)\sqrt{2}}{s^2+2} + \frac{\left(\frac{11}{18}\right)(s+1) + 3\left(-\frac{1}{9}\right)}{(s+1)^2+9}.$$

By the linearity of the Laplace transform

$$\mathscr{L}\left\{\frac{s^3}{(2s^2+4)((s+1)^2+9)}\right\} = -\frac{1}{9}\mathscr{L}\left\{\frac{s}{s^2+2}\right\} - \frac{\sqrt{2}}{36}\mathscr{L}\left\{\frac{\sqrt{2}}{s^2+2}\right\} + \frac{11}{18}\mathscr{L}\left\{\frac{s+1}{(s+1)^2+9}\right\} - \frac{1}{9}\mathscr{L}\left\{\frac{3}{(s+1)^2+9}\right\}$$

From the Laplace table the first transform is $\cos(\sqrt{2}t)$ and the second transform is $\sin(\sqrt{2}t)$. The nest two transforms can be evaluated as

$$\begin{aligned} \mathscr{L}\left\{\frac{s+1}{(s+1)^2+9}\right\} &= e^{-t}\mathscr{L}\left\{\frac{s}{s^2+9}\right\} && \text{Exp Modulation} \\ &= e^{-t}\cos(3t) && \text{cos transform} \end{aligned}$$

and similarly $\mathscr{L}\left\{\frac{3}{(s+1)^2+9}\right\} = e^{-t}\sin(3t)$. Thus

$$\mathscr{L}\left\{\frac{s^3}{(2s^2+4)((s+1)^2+9)}\right\} = -\frac{1}{9}\cos(\sqrt{2}t) - \frac{\sqrt{2}}{36}\sin(\sqrt{2}t) + \frac{11}{18}e^{-t}\cos(3t) - \frac{1}{9}e^{-t}\sin(3t).$$

4. (4 marks) Use the convolution theorem to compute

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\}$$

where we mean the one sided Laplace Transform. For this problem you must show **all** of your work and explicitly mention what properties of the Laplace transform you use.

**Hint:** For this question you are allowed to freely use any results from Assignment 1 as long as you cite them. If you do this, then make sure you look at the solutions in case you made errors you may have made in your A1.

**Solution:** From the Laplace table

$$\mathscr{L}\left\{\frac{1}{s-1}\right\} = e^t \qquad \text{and} \qquad \mathscr{L}\left\{\frac{s}{s^2+4}\right\} = \cos(2t).$$

Hence the convolution theorem tells us that

$$\mathscr{L}^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\} = \int_0^t e^{t-\tau}\cos(2\tau)d\tau.$$

Since the integral only depends on $\tau$ and cos is an even function, we can let $u = -\tau$ we can rewrite the above as

$$e^t\int_0^t e^{-\tau}\cos(2\tau)d\tau \qquad \text{or} \qquad -e^t\int_0^{-t} e^u\cos(2u)du$$

From A1 we have $\int e^x\cos(2x)dx = \frac{1}{5}\cos(2x)e^x + \frac{2}{5}\sin(2x)e^x + C$. Using this gives us

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s}{(s-1)(s^2+4)}\right\} &= -e^t\left(\frac{1}{5}\cos(2u)e^u + \frac{2}{5}\sin(2u)e^u\right)\Big|_0^{-t}\\ &= -e^t\left(\frac{1}{5}\cos(-2t)e^{-t} + \frac{2}{5}\sin(-2t)e^{-t} - \left(\frac{1}{5}+0\right)\right)\\ &= -\frac{1}{5}\cos(2t) + \frac{2}{5}\sin(2t) + \frac{1}{5}e^t.\end{aligned}$$

5. (5 marks) Use the definition of the one-sided Laplace transform to compute

$$\mathscr{L}\{\sinh(4t)\}$$

where $\sinh(at) = \frac{e^{at}-e^{-at}}{2}$. You **must** explicitly find and state the ROC with justification.

For this problem you must show **all** of your work.

Note: The final solution including ROC is explicitly listed in our Laplace table. Hence <u>all</u> marks for this question are given for the intermediate computations.

**Fun facts (optional):** Students are often curious about what the hyperbolic trig functions "do" and/or why we care about them so I will elaborate here. In class we showed that $\sin(ax) = \frac{e^{ajx}-e^{-ajx}}{2j}$. sinh and cosh typically defined by simply removing the $j$s from these formulas. Geometrically, removing the $j$s changes the underlying geometric curve from the complex unit circle ($e^{jt}$ defines the "unit circle" in $\mathbb{C}$) to a hyperbola. One simple explanation for this is the fact $y = \sinh(t)$ and $x = \cosh(t)$ describe the $x$ and $y$ components on the unit hyperbola ($x^2 - y^2 = 1$). In short sin/cos are to the unit circle what sinh/cosh are to the unit hyperbola. Finally, for a real world application, all cables that are supported by their ends form a shape called a "Catenary". This shape is described by $\cosh(x)$. For details of the DE that gives this result see the "Analysis" section of this wiki page.

**Solution:** Direct computations give

$$
\begin{aligned}
\mathscr{L}\{\sinh(4t)\} &= \int_0^\infty \sinh(4t)e^{-st}dt \\
&= \int_0^\infty \left(\frac{e^{4t}-e^{-4t}}{2}\right)e^{-st}dt \\
&= \frac{1}{2}\int_0^\infty e^{(4-s)t} - e^{-(4+s)t}dt \\
&= \frac{1}{2}\left(\frac{e^{(4-s)t}}{4-s} + \frac{e^{-(4+s)t}}{4+s}\right)\Bigg|_0^\infty \\
&= \frac{1}{2}\left(\lim_{t\to\infty}\left(\frac{e^{(4-s)t}}{4-s} + \frac{e^{-(4+s)t}}{4+s}\right) - \left(\frac{1}{4-s}+\frac{1}{4+s}\right)\right)
\end{aligned}
$$

The limit terms converge given that $\mathrm{Re}(4-s) < 0$ and $\mathrm{Re}(4+s) > 0$ respectively and when this condition is met the limit converges to 0. Hence given that these conditions are met we have

$$
\begin{aligned}
\mathscr{L}\{\sinh(4t)\} &= -\frac{1}{2}\left(\frac{1}{4-s}+\frac{1}{4+s}\right) \\
&= -\frac{1}{2}\frac{8}{16-s^2} \\
&= \frac{4}{s^2-16}
\end{aligned}
$$

To find the ROC note that when the previously mentioned conditions are met we have $4 <\mathrm{Re}(s)$ and $-4 < \mathrm{Re}(s)$. Combining these gives the condition that $\mathrm{Re}(s) > 4$.