MATH 213 - Tutorial 3: Using Laplace to solve linear constant coefficient coefficients - Solutions

1. Use the Laplace transform to solve the initial value problem

$$-2y' + y = 0, \qquad y(0) = 1.$$

Verify that your solution works.

Solution: Taking the Laplace transform of the DE gives

$$\begin{aligned}\mathscr{L}\{-2y' + y\} &= \mathscr{L}\{0\}\\ -2(sY(s) - y(0)) + Y(s) &= 0\end{aligned}$$

or

$$Y(s) = \frac{-2y(0)}{1-2s} = \frac{1}{s-1/2}$$

From our Laplace table we have

$$\mathscr{L}^{-1}\left\{\frac{1}{s-1/2}\right\} = e^{1/2t}$$

Hence the solution is $y(t) = e^{1/2t}$.

To check the solution we note that

$$y(0) = e^{1/2 \cdot 0} = e^0 = 1$$

and $y' = \frac{1}{2}e^{1/2t}$ so

$$\begin{aligned}-2y' + y &= -2(\frac{1}{2}e^{1/2t}) + e^{1/2t}\\ &= -e^{1/2t} + e^{1/2t}\\ &= 0\end{aligned}$$

2. Use the Laplace transform to solve the initial value problem

$$y'' - 2y' - 3y = 0, \qquad y(0) = 2, \qquad y'(0) = -1.$$

Verify that your solution works.

Solution: Taking the Laplace transform gives

$$\begin{aligned}\mathscr{L}\{y'' - 2y' - 3y\} &= \mathscr{L}\{0\}\\ \mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} - 3\mathscr{L}\{y\} &= 0\\ \underbrace{s^2Y(s) - sy(0) - y'(0)}_{\mathscr{L}\{y''(t)\}} - 2\underbrace{(sY(s) - y(0))}_{\mathscr{L}\{y'\}} - 3\underbrace{Y(s)}_{\mathscr{L}\{y\}} &= 0\\ (s^2 - 2s - 3)Y(s) &= sy(0) + y'(0) - 2y(0)\\ Y(s) &= \frac{2s-5}{s^2-2s-3}\end{aligned}$$

We now note that $s^2 - 2s - 3 = (s+1)(s+3)$ and make the PF ansatz

$$\frac{2s-5}{s^2-2s-3} = \frac{A}{s+1} + \frac{B}{s-3}.$$

Using the Heaviside coverup method gives

$$A = \frac{2(-1)-5}{-1-3} = \frac{7}{4}$$

and

$$B = \frac{2(3)-5}{3+1} = \frac{1}{4}$$

Hence

$$Y(s) = \frac{7}{4}\left(\frac{1}{s+1}\right) + \frac{1}{4}\left(\frac{1}{s-3}\right).$$

Taking the inverse transform gives

$$\begin{aligned} y(t) &= \mathscr{L}\left\{\frac{7}{4}\left(\frac{1}{s+1}\right) + \frac{1}{4}\left(\frac{1}{s-3}\right)\right\} \\ &= \frac{7}{4}\mathscr{L}\left\{\frac{1}{s+1}\right\} + \frac{1}{4}\mathscr{L}\left\{\frac{1}{s-3}\right\} \\ &= \frac{7}{4}e^{-t} + \frac{1}{4}e^{3t}. \end{aligned}$$

To verify this solution we note that

$$y(0) = \frac{7}{4}e^0 + \frac{1}{4}e^0 = 2.$$

Further

$$y'(t) = -\frac{7}{4}e^{-t} + \frac{3}{4}e^{3t}$$

so

$$y'(0) = -\frac{7}{4}e^0 + \frac{3}{4}e^0 = -1.$$

Finally

$$y''(t) = \frac{7}{4}e^{-t} + \frac{9}{4}e^{3t}$$

and so

$$\begin{aligned} y'' - 2y' - 3y &= \left(\frac{7}{4}e^{-t} + \frac{9}{4}e^{3t}\right) - 2\left(-\frac{7}{4}e^{-t} + \frac{3}{4}e^{3t}\right) - 3\left(\frac{7}{4}e^{-t} + \frac{1}{4}e^{3t}\right) \\ &= \left(\frac{7}{4} + \frac{14}{4} - \frac{21}{4}\right)e^{-t} + \left(\frac{9}{4} - \frac{6}{4} - \frac{3}{4}\right)e^{3t} \\ &= 0 \end{aligned}$$

So our proposed function solves the IVP.

3. Use the Laplace transform to solve the initial value problem

$$y'' + 2y' + 2y = 0, \qquad y(0) = -1, \qquad y'(0) = 2.$$

Verify that your solution works.

Solution: Talking the Laplace transform gives

$$\begin{aligned} \mathscr{L}\{y'' + 2y' + 2y\} &= \mathscr{L}\{0\} \\ \mathscr{L}\{y''\} + 2\mathscr{L}\{y'\} + 2\mathscr{L}\{y\} &= 0 \\ s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 2Y(s) &= 0 \\ Y(s) &= -\frac{s}{s^2+2s+2}. \end{aligned}$$

So find the inverse Laplace transform, we complete the square for the Characteristic polynomial

$$\begin{aligned} s^2+2s+2 &= (s^2+2s+1)-1+2 \\ &= (s+1)^2+1 \end{aligned}$$

Hence

$$Y(s) = -\frac{s}{(s+1)^2+1}$$

This term looks like an exponentionally modulated cos transform but the s is the numerator is not shifted. We thus add and subtract 1 to write

$$\begin{aligned} Y(s) &= -\left(\frac{s+1-1}{(s+1)^2+1}\right) \\ &= -\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1} \end{aligned}$$

Hence

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{-\frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}\right\} \\ &= -\mathscr{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} \\ &= -e^{-t}\mathscr{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + e^{-t}\mathscr{L}^{-1}\left\{\frac{1}{(s)^2+1}\right\} \\ &= -e^{-t}\cos(t) + e^{-t}\sin(t) \end{aligned}$$

To check the solution first note that integrating by parts twice gives

$$\begin{aligned} y'(t) &= 2e^{-t}\cos(t) \\ y''(t) &= -2e^{-t}(\cos(t)+\sin(t)) \end{aligned}$$

and hence

$$\begin{aligned} y(0) &= -e^0\cos(0) + e^0\sin(0) = 1 \\ y'(0) &= 2e^0\cos(0) = 2 \end{aligned}$$

and

$$\begin{aligned} y''+2y'+2y &= -2e^{-t}(\cos(t)+\sin(t)) + 2(2e^{-t}\cos(t)) + 2(-e^{-t}\cos(t)+e^{-t}\sin(t)) \\ &= e^{-t}\left((-2+4-2)\cos(t) + (-2+2)\sin(t)\right) \\ &= 0 \end{aligned}$$

verifying that y does solve the DE.

4. Use the Laplace transform to solve the initial value problem

$$y''-y'-2y = \sin(3t), \qquad y(0)=1, \qquad y'(0)=-1.$$

After taking the Laplace transform but before taking the inverse Laplace transform, argue why the solution will be unbounded.

Solution: Taking the Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = \frac{3}{s^2+9}$$

or

$$\begin{aligned}
Y(s) &= \frac{1}{s^2-s-2}\left(\frac{3}{s^2+9} + sy(0) + y'(0) - y(0)\right) \\
&= \frac{1}{s^2-s-2}\left(\frac{3}{s^2+9} + s - 2\right) \\
&= \frac{3}{(s^2+9)(s-2)(s+1)} + \frac{s-2}{(s-2)(s+1)}
\end{aligned}$$

For the first term we need to use partial fractions:

$$\frac{3}{(s^2+9)(s-2)(s+1)} = \frac{As+3B}{s^2+9} + \frac{C}{s-2} + \frac{D}{s+1}$$

Here we choose to scale B to make computing the sin inverse transform nicer. Using the Heaviside coverup method three times (once with the complex root $3j$) gives

$$\frac{3}{(s^2+9)(s-2)(s+1)} = \frac{1}{130}\left(\frac{3s-3\cdot 11}{s^2+9}\right) + \frac{1}{13}\left(\frac{1}{s-2}\right) - \frac{1}{10}\left(\frac{1}{s+1}\right).$$

Hence simplifying the second term and adding it to the above yields

$$Y(s) = \frac{1}{130}\left(\frac{3s-3\cdot 11}{s^2+9}\right) + \frac{1}{13}\left(\frac{1}{s-2}\right) + \frac{9}{10}\left(\frac{1}{s+1}\right).$$

Examining the poles of these terms, we see that the second last term has a pole in the left half of the complex plane ($z = 2$) and hence the solution will exponentially grow!

Now taking the inverse transform gives

$$\begin{aligned}
y(t) &= \frac{1}{130}\left(3\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} - 11\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\}\right) + \frac{1}{13}\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + \frac{9}{10}\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} \\
&= \frac{1}{130}(3\cos(3t) - 11\sin(3t)) + \frac{1}{13}e^{2t} + \frac{9}{10}e^{-t}
\end{aligned}$$