MATH 213 - Assignment 4 - Solutions

Submit to Crowdmark by 9:00pm EST on Friday, March 15.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Read before starting the assignment: For this assignment you must do all your work independently and without the use of external aids (other than when stated in particular problem subparts of questions).
Questions:

1. (3 marks) Prove that
$$\mathscr{L}\left\{\int_0^t e(\tau)d\tau\right\} = \frac{E(s)}{s}$$
Hints: Use the facts that if $f(t) = \int_0^t e(\tau)d\tau$ then $f'(t) = e(t)$ and $f(0) = 0$. You should use the Laplace transform of the derivative somewhere in your work.

   Solution: Recall that
$$\mathscr{L}\left\{\frac{d}{dt}(f(t))\right\} = s\mathscr{L}\{f(t)\} - f(0)$$
This also holds for $f(t) = \int_0^t e(\tau)d\tau$. Hence
$$\begin{aligned}\mathscr{L}\left\{\frac{d}{dt}\left(\int_0^t e(\tau)d\tau\right)\right\} &= s\mathscr{L}\left\{\int_0^t e(\tau)d\tau\right\} - \int_0^0 e(\tau)d\tau \\ &= s\mathscr{L}\left\{\int_0^t e(\tau)d\tau\right\}\end{aligned}$$
On the other hand $\frac{d}{dt}\left(\int_0^t e(\tau)d\tau\right) = e(t)$ so the above is equivalent to
$$\mathscr{L}\left\{\frac{d}{dt}\left(\int_0^t e(\tau)d\tau\right)\right\} = \mathscr{L}\{e(t)\} = E(s).$$
Hence $E(s) = s\mathscr{L}\left\{\int_0^t e(\tau)d\tau\right\}$ or
$$\mathscr{L}\left\{\int_0^t e(\tau)d\tau\right\} = \frac{E(s)}{s}.$$

2. (10 marks) Consider the harmonic oscillator modelled by the DE
$$y''(t) + 5y'(t) + 6y(t) = f(t)$$
The spring is initially at rest in its rest position but we want it to follow the path given by the function
$$r(t) = \sin(t) + 1.$$
In class, we saw that a proportional controller tends to quickly adjust the system but has problems obtaining an exact constant value and an integral controller tends to act more slowly but asymptotically obtains a specified constant value and can cause overshooting for large k_is. In this problem we will setup a proportional controller to handle the sine forcing term and then an integral controller to handle the constant term. By taking the linear superposition of the results, we can control the full system while having the "best of both worlds".

   (a) Proportional controller:

      i. Find $V(s)$ for the desired input function $r(t) = \sin(t)$ for the closed loop proportionally controlled harmonic oscillator. Make sure to identify the transfer function.

ii. Find an expression for the poles of your transfer in terms of k_p and use this expression to show that for all $k_p > 0$ the poles of the transfer function have negative real parts (and hence exponentially decay).

iii. The response of the controlled harmonic oscillator, $v(t)$, to the input given by $r(t) = \sin(t)$ can be written as the sum of a sinusoidal wave of period 2π (i.e. $A\sin(t) + B\cos(t)$) and some terms that exponentially decay. In this part of the question we will find and examine the sinusoidal terms (i.e. the "steady state solution").

A. Find the coefficients of the sinusoidal terms and give the "steady state" component of the system response.

B. Plot the solution from the previous part for $k_p = 10$, 100 and 1000 over $t = 0$ to 2π. Compare these with the the plot of the desired $r(t)$. For large values of t (i.e. $t \gg 1$ such that the decaying terms have little influence on the system), the system response to $r(t)$ is approximately equal to the functions you plotted. Based on your plots explain whether or not the proportional controller is a good way to control the **long term** behaviour of the spring for this $r(t)$.

(b) Integral controller:

i. Find $V(s)$ for the desired input function $r(t) = 1$ for the closed loop integrally controlled harmonic oscillator. Make sure to identify the transfer function.

ii. Given that the poles of the transfer function are in the left hand plane, the solution of this system, $v(t)$, with the input given by $r(t) = 1$ can be written as the sum of a constant term and some terms that exponentially decay.

A. Use the final value theorem to find the constant term. You may assume that we pick k_i so that the transfer function has no roots that cause problems for the FVT. i.e. do not try to analytically find all the poles.

B. Based on your solution above explain whether or not the integral controller is a good way to control the **long term** behaviour of the spring for this $r(t)$.

(c) (1 bonus all or none) Using some software of your choice, compute the transient terms that we skipped computing in parts (a) and (b) for the case where $k_p = 1000$ and $k_i = 2$.

Note: You can use software for 2c but not for the computations 2a and 2b themselves. You can however validate your solutions from (b) and (c) using the solution you found in this particular case. You also must show proof of how you computed these terms.

Hint: There is a reason why I am not making you do this computation by hand!

Use these terms to plot the sum of the effects of the two controlled systems found in parts (a) and (b) for $t = 0$ to $t = 10\pi$. Do you think this is a well controlled system given that the user wanted to obtain the "correct behaviour" after 5 periods of the input function?

Solution:

From L16 we have

$$V(s) = \underbrace{\frac{P(s)C(s)}{1 + C(s)P(s)}}_{H(s)} R(s)$$

where $R(s) = \mathscr{L}\{r(t)\}$, $P(s)$ is the transfer function for the uncontrolled spring system and $C(s)$ is the transfer function for the controller.

(a) Proportional controller:

i. Using the DE provided, a proportional controller and the results from L16 the transfer function for the controlled system is

$$\begin{aligned} H(s) &= \frac{\frac{k_p}{s^2+5s+6}}{1+\frac{k_p}{s^2+5s+6}} \\ &= \frac{k_p}{s^2+5s+6+k_p} \end{aligned}$$

From the Laplace table

$$R(s) = \frac{1}{s^2+1}.$$

Hence $V(s) = \dfrac{k_p}{(s^2+5s+6+k_p)(s^2+1)}$

ii. We simply need to find the roots of $s^2+5s+6+k_p$ and then show that they all have a negative real part for positive values of k_p. The quadratic formula gives

$$\begin{aligned} s_{\pm} &= \frac{-5 \pm \sqrt{25-4(6+k_p)}}{2} \\ &= \frac{-5 \pm \sqrt{1-4k_p}}{2} \end{aligned}$$

If $k_p > \frac{1}{4}$, then we have complex valued roots with a real part of $-\frac{5}{2}$ so we are good to go there. If $\frac{1}{4} \geq k_p > 0$ then there are real roots and we need to verify that these roots remain negative for positive k_p. To check this we can simply find the value of k_p such that the S_+ root vanishes. Doing this gives

$$\begin{aligned} \frac{-5+\sqrt{1-4k_p}}{2} &= 0 \\ \sqrt{1-4k_p} &= 5 \\ 1-4k_p &= 25 \\ k_p &= -6. \end{aligned}$$

Which is kinda obvious when you think that for $s=0$ to be a root we need k_p to cancel the constant term. Nevertheless, the poles of the transfer function are all in the LHP for positive values of k_p. Note that if $k_p = \frac{1}{4}$ then we will have a repeated real root!

iii.

A. To find the sinusoidal terms we need to find the coefficients A and B for the

$$\frac{As+B}{s^2+1}$$

component of the PF decomposition of $V(s)$. This can be done by using the the cover-up method. Multiplying $V(s)$ by s^2+1 gives

$$\frac{k_p}{s^2+5s+6+k_p} = As + B + (\text{Other PF term}) \cdot (s^2+1)$$

and then using $s = j$ gives

$$
\begin{aligned}
Aj + B &= \frac{k_p}{j^2 + 5j + 6 + k_p} \\
&= \frac{k_p}{5j + 5 + k_p} \\
&= \frac{k_p}{5 + k_p + 5j}\left(\frac{5 + k_p - 5j}{5 + k_p - 5j}\right) \\
&= \frac{5k_p + k_p^2 - 5k_p j}{(5 + k_p)^2 + 25}
\end{aligned}
$$

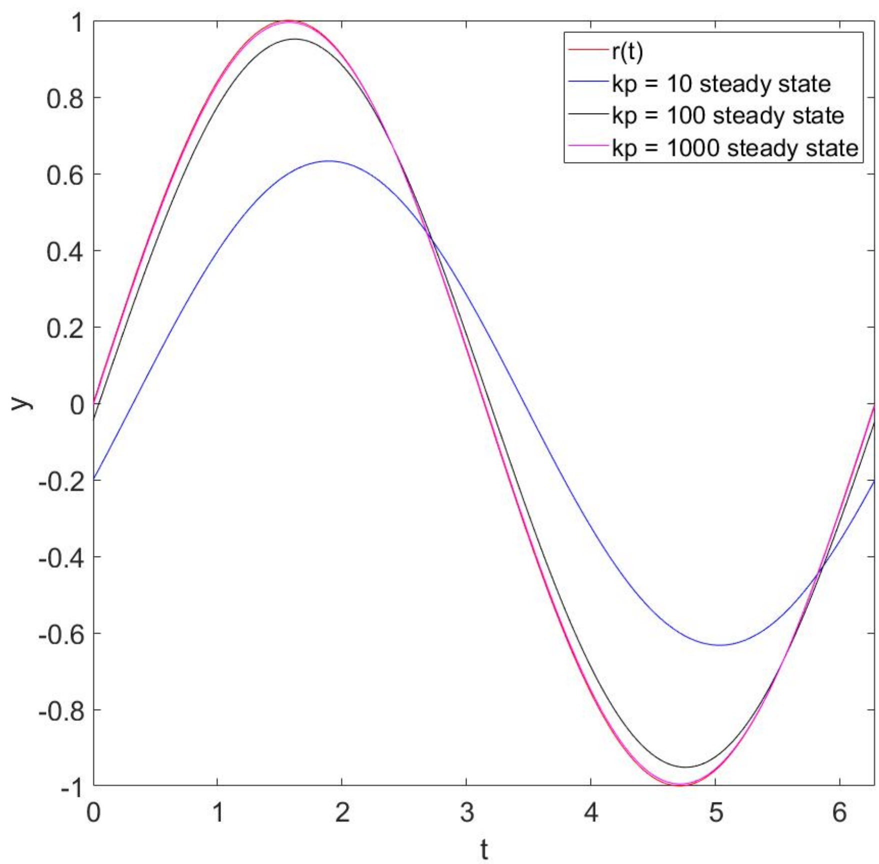
Hence

$$A = \frac{-5k_p}{(5 + k_p)^2 + 25} \qquad \text{and} \qquad B = \frac{5k_p + k_p^2}{(5 + k_p)^2 + 25}$$

and the steady state part of $v(t)$ is

$$\frac{-5k_p}{(5 + k_p)^2 + 25}\cos(t) + \frac{5k_p + k_p^2}{(5 + k_p)^2 + 25}\sin(t).$$

B. Here is a plot of this function for $k_p = 10$, 100 and 1000 along with the function we wanted to achieve for one full period:



One can make arguments for this being a good control system for this case as well as arguments for this being a bad system. Generally, it will depend on how fast you need things to converge to the steady state.

(b) Integral controller:

i. Using the DE provided and a integral controller we have

$$
\begin{aligned}
H(s) &= \frac{\frac{k_i}{s(s^2+5s+6)}}{1+\frac{k_i}{s(s^2+5s+6)}} \\
&= \frac{k_i}{s^3+5s^2+6s+k_i}
\end{aligned}
$$

$R(s) = 1/s$ for this case so

$$
V(s) = \frac{k_i}{s(s^3+5s^2+6s+k_i)}.
$$

ii.

A. The constant term comes from the inverse LT of the $\frac{A}{s}$ term of $V(s)$. Hence to find this term we can use the FVT (assuming the that poles of the transfer function are appropriately placed)and write

$$
\begin{aligned}
A &= \lim_{s\to 0^+} s\frac{k_i}{s(s^3+5s^2+6s+k_p)} \\
&= \lim_{s\to 0^+} \frac{k_p}{k_p} \\
&= 1
\end{aligned}
$$

Hence the constant term is simply 1 which is again just the DC gain(shocked Pikachu)!

B. The steady state part of $v(t)$ is just 1 and hence the asymptotic behaviour of the solution agrees with what we wanted.
Regardless of the value of k_i the integral controller obtains the correct steady state component for the constant part of the forcing term. It is thus a good way to control the long term behaviour of the constant term. We should still keep in mind that it tends to be however rather slow to adjust and can overshoot so these will be design considerations we will need to make if we care about the transient part of the solution.

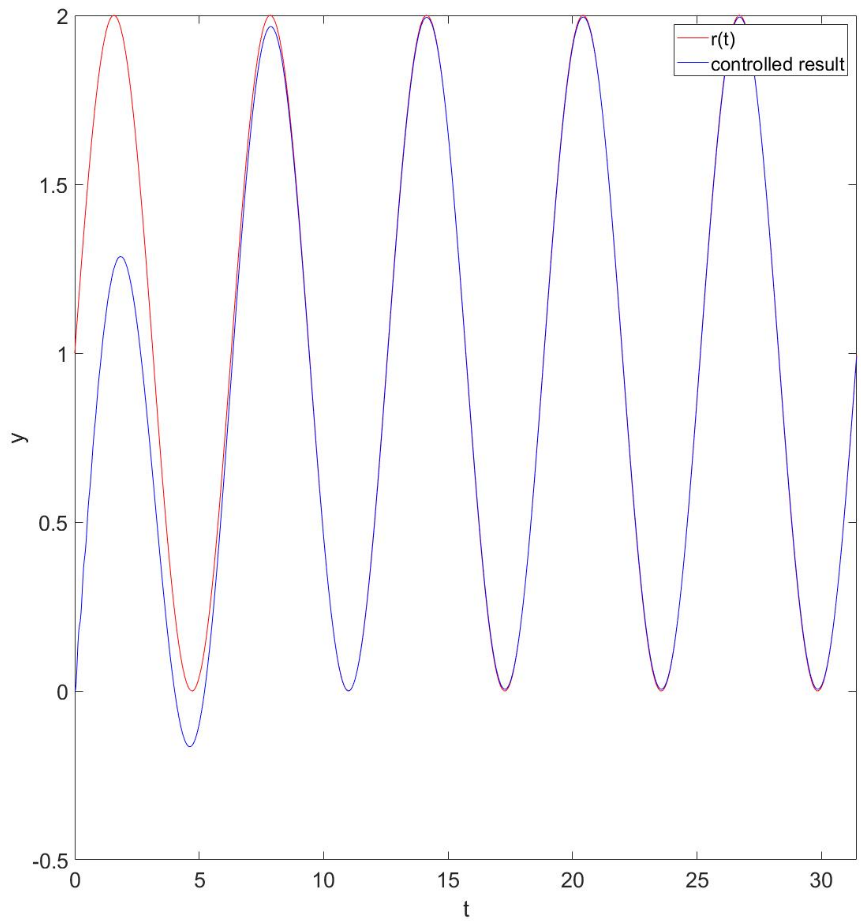
(c) Here is a way to compute the needed terms for the p controller and here is a way to compute the needed terms for the i controller. These computations give the decaying terms of

$$
\frac{1000e^{-5t/2}}{202010\sqrt{3999}}\left(\sqrt{3999}\cos\left(\frac{\sqrt{3999}t}{2}\right) - 397\sin\left(\frac{\sqrt{3999}t}{2}\right)\right)
$$

and

$$
-\frac{(-4e^{(-2-\sqrt{(}2))t} + 3\sqrt{(}2)e^{(-2-\sqrt{(}2))t} + 4e^{(\sqrt{(}2)-2)t} + 3\sqrt{(}2)e^{\sqrt{(}2)-2)t}}{2\sqrt{2}} + 2e^{-t}
$$

Plotting this solution gives:



From the plot we can see that after about the second period, the effects of the decaying terms have basically vanished and we are left with only the error in the asymptotic error due to the p controller. From part a we found that this controller was sufficient (again depending on how close you need) and hence this combination of controllers is sufficient.