MATH 213 - Lecture 7: Rational Functions, Poles, and Initial and Final Value Theorems

Lecture goals: Understand what a pole is and understand the initial and final value theorems.
After computing the Laplace transform of a function we usually end up with a function that is the ratio of polynomials. Such functions have a name.

Definition 1: Rational Function
A **rational function** is a function of the form $\frac{P(x)}{Q(x)}$ where $P(x)$ and $Q(x)$ are polynomials.

In Math 119 we often simplified such functions with no regards to undefined values. As an example we might say

$$\frac{x}{x} = 1.$$

Formally at $x = 0$ this is not true since $\frac{0}{0}$ is undefined but we ignored that and did it anyways. We will make this idea a bit more formal to justify our use of this method for MATH 213.

Definition 2: Regularization
A function f is the **regularization** of a function g if

1. For all x in the domain of g, $f(x) = g(x)$.
2. For all x such that $\lim_{t\to x} g(x)$ exists but $f(x)$ is undefined, $f(x) = \lim_{t\to x} g(x)$.

Example 1
$\frac{s}{s^2+2s+2}$ is the regularization of $\frac{s(s+3)}{(s^2+2s+2)(s+3)}$ and also of $\frac{s}{s^2+2s+2}$.

Theorem 1
If $f(x)$ is the regularization of a function g that has a finite number of discontinuities then $\int f(x)dx = \int g(x)dx$.

When we write $\frac{x}{x} = 1$ we are redefining $=$ to be the equivalence relation "have the same regularization". We did this in MATH 119 because of Theorem 1 above and we will do in in MATH 213 for the same reason.

Back to MATH 213:

Definition 3: Finite Zeros and Finite Poles

A rational function $f(x) = \frac{P(x)}{Q(x)}$ that is its own continuous continuation the roots of $P(x)$ are called the **(finite) zeros of** f and the roots of $Q(x)$ are called the **(finite) poles of** f.

Example 2

Find the finite poles and finite zeros of

$$F(s) = \frac{s}{s^2 + 2s + 2}.$$

Note that in the above example $\lim_{s \to \infty} F(s) = 0$.

Definition 4: Poles and Zeros at Infinity

Given a rational function $f(x) = \frac{P(x)}{Q(x)}$ has a **zero at infinity** if $\lim_{s \to \infty} F(s) = 0$. $f(x)$ is said to have a **pole at infinity** if its reciprocal has a zero at infinity.

Example 3

$$F(s) = \frac{s}{s^2 + 2s + 2}$$

has a zero at infinity and $\frac{1}{F(s)}$ has a pole at infinity.

Convention 1

Unless otherwise stated when we talk about poles and zeros we mean the finite poles and finite zeros.

Definition 5: Proper and Strictly Proper Rational Functions
A rational function is **proper** if the degree of its numerator is less than or equal to that of its denominator. A function is **strictly proper** if the degree of the numerator is strictly less than that of its denominator.

Example 4

$$F(s) = \frac{s}{s^2 + 2s + 2}$$

is both proper and strictly proper.

We care about these ideas because the inverse Laplace transform is a complex valued integral and because of this, we can compute the inverse transform of $F(s)$ by **only** knowing information about the poles of $F(s)$.

This is closely related to how Green's theorem (MATH 119) can be use to compute closed line integrals of the vector field $\vec{F}(x, y)$ by examining the points where $\nabla \times \vec{F}$ is undefined.

While we will not learn how to directly compute inverse Laplace transforms using poles (google residue theorem if interested), we will cover two important results that are related to poles (and more later).

The initial value theorem:

Definition 6: Piecewise Continuous
A function $f(x)$ is **piecewise-continuous on a given finite interval** if

1. f has a finite number of discontinuities in that interval and
2. for each discontinuity x_0 both the left and right hand limits exits (note they can be different values).

A function $g(x)$ is piecewise-continuous if it is piecewise-continuous on all finite intervals.

Example 5

Determine which (if any) of the following are piecewise-continuous:

- $f(x) = 1/x$
- $g(x) = \begin{cases} 1/x & |x| > 1, \\ 0 & else \end{cases}$

Theorem 2: Initial Value theorem

If $f(t)$ is piecewise-continuous and $\int_0^\infty |f(t)|e^{-\alpha t}$ converges for some $\alpha \in \mathbb{R}$ then

$$f(0^+) = \lim_{s \to +\infty} sF(s)$$

Understanding this theorem:

- The above gives us a way of computing the IC of $f(t)$ from $F(s)$ **without** computing the inverse Laplace transform and instead evaluating a limit in the complex plane.
- s is generally complex so the limit is in the complex plane **NOT** the real number line thus:
  - $\text{Re}(s)$ tends to infinity
  - $\text{Im}(s)$ can do anything (stay 0, oscillate, go to infinity, do random things, etc.)
- For many problems we can treat the limit as the standard limit you are used to working with.

Before seeing a proof we examine some examples

Example 6

For the following functions $F(s)$ compute $f(0^+)$ using the initial value theorem and verify the result using the inverse Laplace transform

A. $F(s) = e^{-sT}\frac{1}{s^2}$, $T > 0$

B. $F(s) = \frac{1}{s}$

C. $F(s) = \frac{s}{s^2+\omega^2}$

Proof of the initial value theorem:

Example 7:
Prove the initial value theorem in the special case where $F(s)$ is a proper rational function.

The final value theorem:

Theorem 3: Final Value Theorem
For a function $F(s)$, if

- is a proper rational function
- has the property that all the poles have real parts that are strictly negative with the exception of a single pole (of order 1)[a] at $s = 0$

or if $F(s)$ is the product of a function satisfying the above conditions multiplied by a complex exponential e^{sT}, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

If the poles of a rational function do not satisfy the above condition, then $\lim_{t\to\infty} f(t)$ does not exist.

[a] i.e. $\frac{A}{s}$

Understanding this theorem:

- When it applies, the above gives us a way of computing the "final value" of $f(t)$ from $F(s)$ **without** computing the inverse Laplace transform and instead evaluating a limit in the complex plane.
- Again s is generally complex so the limit is in the complex plane **NOT** the real number line. Limits at finite points in the complex plane are a bit different from infinite limits.
  - Here the formal definition of a limit of a complex valued function $F(z)$ at $x_0 \in \mathbb{C}$ is the constant $L \in \mathbb{C}$ such that for every $\epsilon \in \mathbb{R}^+$ there is a $\delta \in \mathbb{R}^+$ such that if $|z - z_0| < \delta$ then $|F(z) - L| < \epsilon$ where $|\cdot|$ is the standard modulus function.
  - In practice for any problems of computing limits at 0, you can treat the limit as the standard limit you are used to working with.
- Connection to 2D limits from MATH 119.

– Provided that the limits exist, we have

$$\lim_{s\to 0} F(s) = \lim_{(a,b)\to(0,0)} \mathrm{Re}(F(a+bj)) + j \lim_{(a,b)\to(0,0)} \mathrm{Im}(F(a+bj)).$$

Before seeing a proof we examine some examples.

Example 8

For the following functions $F(s)$, use the final value theorem (if applicable) to determine the long time behaviour of the function $f(t)$. Compare this to the result you find from computing the inverse transform.

A. $F(s) = \frac{10}{5s+1}\frac{1}{s}$

B. $F(s) = \frac{1}{s}$

C. $F(s) = \frac{1}{s^2}$

D. $F(s) = \frac{s}{s^2+\omega^2}$, $\omega \in \mathbb{R}^+$

E. $F(s) = \frac{6}{(s^2+9)^2}$ $\leftarrow$ From Ex 3 Lecture 6.

Note: When the final value theorem does not apply, we do not know if the limit diverges or if it oscillates. Hence, we need more math to be able to ascertain if a solution is bounded or not (without computing the inverse transform)! See examples 8C and 8E for unstable systems and D for a stable but non-convergent system.

Proof of the final value theorem: