MATH 213 - Tutorial 10: Fourier series Round 2 - solutions

1. Determine the pointwise limit of $f_n(x) = \frac{\sin(nx)}{n}$. Does the derivative of f_n converge pointwise?

   Solution: Note that for all $x \in \mathbb{R}$.

   $$-\frac{1}{n} \leq \frac{\sin(nx)}{n} \leq \frac{1}{n}$$

   Hence by the squeeze theorem no matter what x is we have

   $$\lim_{n\to\infty} \frac{\sin(nx)}{n} = 0.$$

   so the pointwise limit exists and is 0. Note that our argument shows that this limit is also uniform.

   The derivative of $f_n(x)$ is

   $$f_n'(x) = \cos(nx).$$

   This does not converge pointwise. To see an example of why note that if $x = \pi$ then when n is an integer then

   $$f_n'(\pi) = (-1)^n.$$

   This sequence doesn't converge and hence the more general sequence cannot converge pointwise.

2. Use the Weierstrass M test (introduced in A6 Q4) to prove that

   $$\sum_{n=0}^{\infty} \sin^{2n}(x)$$

   converges uniformly on $[-a, a]$ for a satisfying $0 < a < \pi/2$

   Solution: We need to bound the summand above by some quantity and then show that the sum of those terms vanish. We thus find the maximum of $\sin^{2n}(x)$ on $[-a, a]$. Taking the derivative gives

   $$2n \sin^{2n-1}(x) \cos(x).$$

   This is zero when $x = 0, \pm\pi/2, \pm\pi \ldots$. The only point of these that is in our domain is 0 and at this point each term is 0. Now note that $\sin^{2n}(x) \geq 0$, $\sin^{2n}(x) = \sin^{2n}(-x)$ and if $x_1 > x_2$, $\sin^{2n}(x_1) > \sin^{2n}(x_2)$. Using this together, we have that $\sin^{2n}(x)$ for $x \in [-a, a]$ is bounded by $\sin^{2n}(a)$ i.e. this is the maximum value.

   Now that we have the maximum value, note that since $|\sin(x)| < 1$ for $x \in [-a, a]$, the maximum value is of the form b^{2n} where $b < 1$. Hence by the convergence of the geometric series,

   $$\sum_{n=0}^{\infty} \sin^{2n}(a) < \infty.$$

   We thus conclude by the M-test that the original series converges uniformly on $[-a, a]$ for $0 < a < \pi/2$.

   In this case we can actually say something much stronger (but would need more math than this class covers to prove it):

   $$\sum_{n=0}^{\infty} \sin^{2n}(x) = \frac{1}{\cos^2(x)}$$

3. Suppose you know that
$$x^3 - \pi^2 x = \sum_{n=1}^{\infty} \frac{12(-1)^n}{n^3} \sin(nx).$$
Use Parseval's Theorem to compute $\sum_{n=1}^{\infty} \frac{1}{n^6}$.

Solution:

Note that $f(x) = x^3 - \pi^2 x$, $c_n = 0$, $s_n = \frac{12(-1)^n}{n^3}$ and $\tau = 2\pi$.

The real form of Parseval's theorem thus tells us that
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} (x^3 - \pi^2 x)^2 dx = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{12(-1)^n}{n^3} \right)^2$$
Computing these terms gives
$$\begin{aligned}
\frac{1}{2\pi} \int_{-\pi}^{\pi} (x^3 - \pi^2 x)^2 dx &= \frac{1}{2\pi} \int_{-\pi}^{\pi} x^6 - 2\pi^2 x^4 + \pi^4 x^2 dx \\
&= \frac{1}{\pi} \int_{0}^{\pi} x^6 - 2\pi x^4 + \pi^4 x^2 dx \qquad \text{using evenness} \\
&= \frac{1}{\pi} \left(\frac{1}{7} x^7 - \frac{2\pi^2}{5} x^5 + \frac{\pi^4}{3} x^3 \right) \Big|_0^{\pi} \\
&= \frac{8\pi^6}{105} \\
\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{12(-1)^n}{n^3} \right)^2 &= \frac{1}{2} \sum_{n=1}^{\infty} \frac{144}{n^6} \\
&= 72 \sum_{n=1}^{\infty} \frac{1}{n^6}
\end{aligned}$$
Hence
$$\begin{aligned}
72 \sum_{n=1}^{\infty} \frac{1}{n^6} &= \frac{8\pi^6}{105} \\
\sum_{n=1}^{\infty} \frac{1}{n^6} &= \frac{\pi^6}{945}
\end{aligned}$$

4. Consider $f = x^5 - 80x$ on $[-\pi, \pi]$

   a) Compute $\|f\|_\infty$

   Solution: Since $f(x)$ is C^1 on a closed interval, the L_∞ norm is the same as the maximum of $|f(x)|$. Since $f'(x) = 5x^4 - 80$ the critical points are $x = \pm 2$. Checking the function values at the endpoints and the critical points that lie in $[-\pi, \pi]$ gives

t	$\lvert f(t)\rvert$
$-\pi$	$\lvert -\pi^5 + 80\pi\rvert \approx 54.69$
-2	$\lvert -2^5 + 160\rvert = 128$
2	$\lvert 2^5 - 160\rvert = 128$
π	$\lvert \pi^5 - 80\pi\rvert \approx 54.69$

   Thus $\|f\|_\infty = 128$

b) Compute $\|f\|_2$

Solution:

By definition

$$
\begin{aligned}
\|f\|_2 &= \sqrt{\int_{-\pi}^{\pi} (x^5 - 80x)^2 dx} \\
&= \sqrt{\int_{-\pi}^{\pi} x^{10} - 160x^6 + 6400x^2 dx} \\
&= \sqrt{\frac{x^{11}}{11} - \frac{160x^7}{7} + \frac{6400x^3}{3}\Bigg|_{-\pi}^{\pi}} \\
&= \sqrt{\frac{2\pi^3(21\pi^8 - 5280\pi^4 + 492800)}{231}} \\
&\approx 218.43
\end{aligned}
$$

c) Confirm that $\|f\|_2 \leq \sqrt{b-a}\|f\|_\infty$ holds for this function.

Solution: Trivially,

$$218.48 \leq \sqrt{2\pi} \cdot 128$$

which confirms that Proposition 5.1 holds for this case.

5. For the following functions, determine if the Fourier series will converge pointwise. If it does converge pointwise, draw the periodic extension it converges to for at least 2 periods.

a)

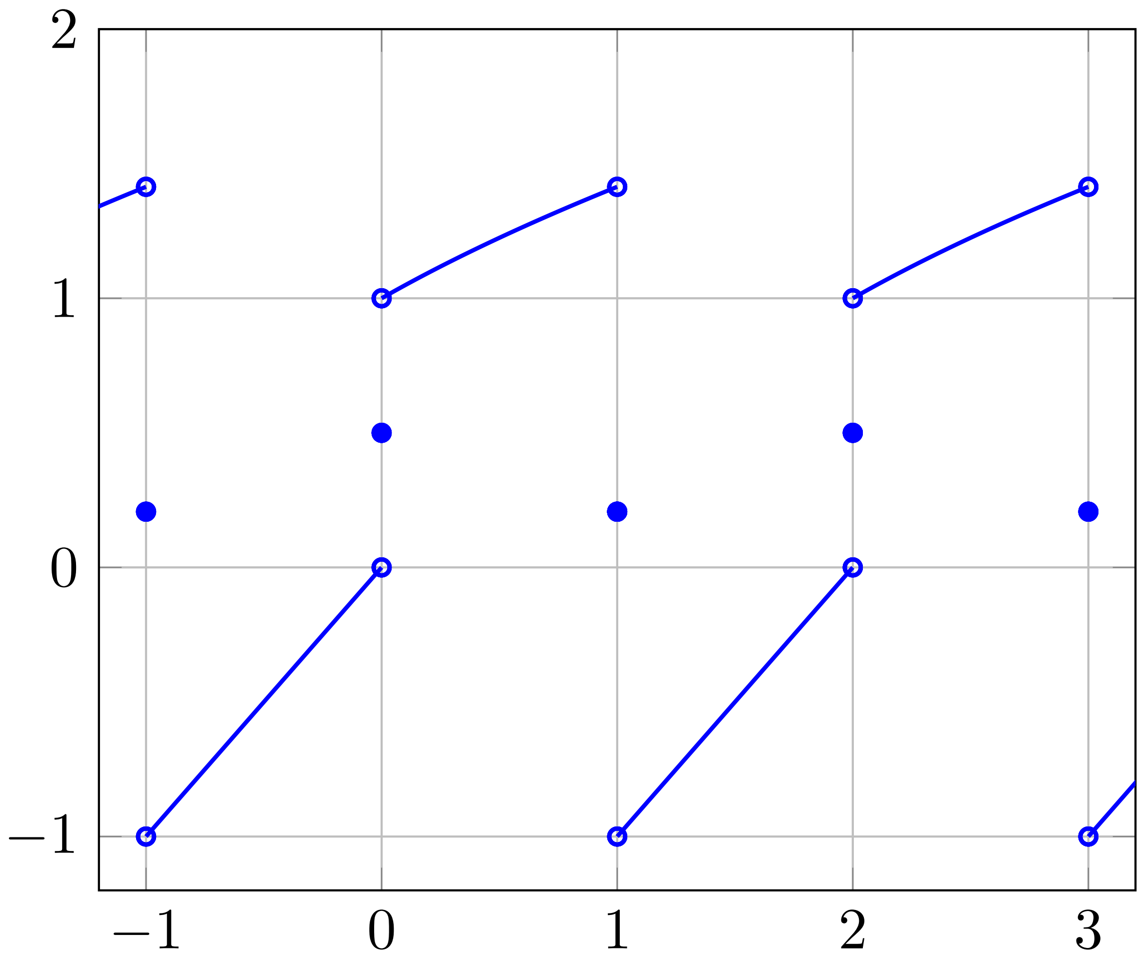
$$
\begin{cases}
x & -1 < x < 0 \\
\sqrt{x+1} & 0 < x < 1
\end{cases}
$$

b)

$$
\begin{cases}
\sin(\pi x) & -1 < x < 0 \\
\frac{x}{x+1} & 0 < x < 1
\end{cases}
$$

Solutions:

In both cases the Fourier series converges pointwise because they are PWC1. Here are the functions they converge to: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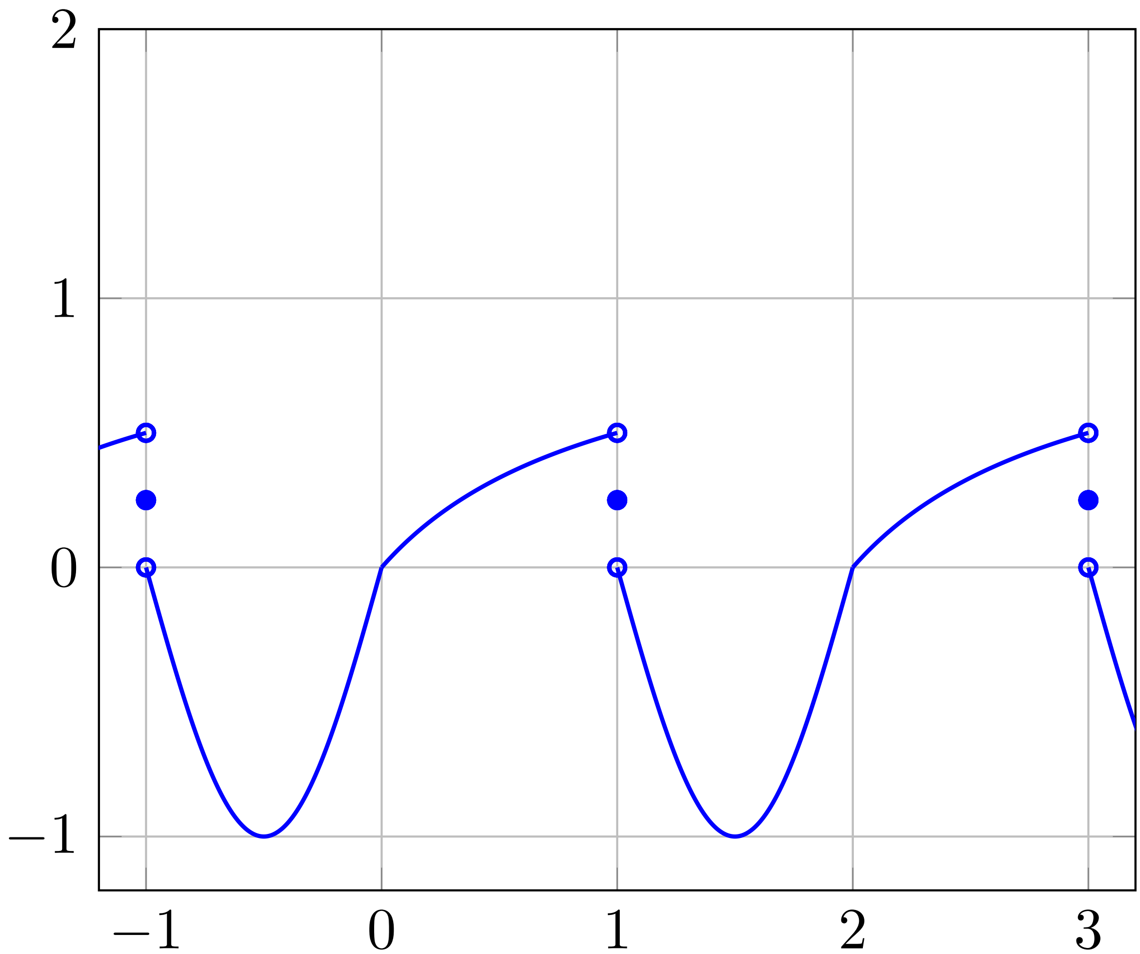
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