MATH 213 - Lecture 11: Midterm Review

Example 1
Let S be the system such that $(Sy)(t) = f(t)$ is described by

$$y^{(6)}(t) + y^{(4)}(t) + 6y(t) = f(t)$$

with all the needed ICs given by 0. Classify the system in terms of:

- Continuous time Vs discrete time
- Memoryless vs dynamic $\longrightarrow$ depends on all $\tau \leq t$
- Causal vs non-causal $\longrightarrow$ does not depend on $\tau > t$
- Multivariable vs Scalar
- Linear vs nonlinear
- Time-invariant vs time-variant $\longrightarrow$ $f(t-T) \xrightarrow{S} y(t-T)$

Sol. is $y(t) = f(t) * \mathcal{L}^{-1}\{T(s)\}$

$$= \int_0^t f(\tau)\, \mathcal{L}^{-1}\left\{\frac{1}{s^6+s^4+6}\right\}\Big|_{t-\tau} d\tau$$

Example 2
Compute $\mathcal{L}\{\cos(\omega t)u(t)\}$ from the definition of the Laplace transform. State the ROC with justification.

Recall $e^{wtj} = \cos wt + j \sin wt$

and $e^{-wtj} = \cos wt - j \sin wt$

So $\cos wt = \frac{1}{2}\left(e^{wtj} + e^{-wtj}\right)$

Thus

$$\mathcal{L}\{\cos w \underset{u(t)}{t}\} = \int_0^\infty \cos wt \, e^{-st} \, dt$$

$$= \frac{1}{2} \int_0^\infty \left(e^{wtj} + e^{-wtj}\right) e^{-st} \, dt$$

$$= \frac{1}{2} \left[\int_0^\infty e^{(-s+wj)t} + e^{(-s-wj)t} \, dt\right]$$

$u = (-s+wj)t \qquad v = (-s-wj)t$

$$= \frac{1}{2} \left[\int_{t=0}^{t=\infty} e^u \frac{du}{-s+wj} + \int_{t=0}^{t=\infty} e^v \frac{dv}{-s-wj}\right]$$

$$= \frac{1}{2} \left[\frac{1}{-s+wj}\left(\lim_{t\to\infty} e^{(-s+wj)t} - 1\right) + \frac{1}{-s-wj}\left(\lim_{t\to\infty} e^{(-s-wj)t} - 1\right)\right]$$

if $Re(-s+wj) < 0$

& $Re(-s-wj) < 0$

or simply

$Re(-s) < 0$

or $\boxed{Re(s) > 0}$

ROC

$$= \frac{1}{2}\left[\frac{-1}{-s+wj} + \frac{1}{s+wj}\right]$$

$$= \frac{1}{2}\,\frac{s-wj+s+wj}{s^2+w^2}$$

$$= \boxed{\frac{s}{s^2+w^2}}$$

Example 3
Find the partial fraction decomposition of

$$F(s) = \frac{s^2+1}{(s-1)(s+2)(s-5)(s+7)(s-9)}.$$

You may leave the coefficients in an unsimplified form. i.e. things like $\frac{1}{8\cdot 9+2\cdot(-4)}$ are ok.

PF ansatz:

$$F(s) = \frac{A}{s-1} + \frac{B}{s+2} + \frac{C}{s-5} + \frac{D}{s+7} + \frac{E}{s-9}$$

By cover up

$$A = \frac{1^2+1}{(1+2)(1-5)(1+7)(1-9)} = \frac{2}{3\cdot(-4)\cdot 8\cdot(-8)}$$

$$B = \frac{5}{(-3)\cdot(-7)\cdot 5\cdot(-11)}, \quad C = \frac{26}{4\cdot 7\cdot 12\cdot(-4)}$$

$$D = \frac{50}{(-8)\cdot(-5)\cdot(-12)\cdot(-15)}, \quad E = \frac{82}{8\cdot 11\cdot 4\cdot 16}$$

Example 4
Solve the IVP:

$$y^{(4)} + y^{(3)} = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 1$$

Taking $\mathcal{L}$:

$$s^4 Y(s) - s^3 \cancel{y(0)} - s^2 \cancel{y'(0)} - s\cancel{y''(0)} - \cancel{y'''(0)}$$
$$+ s^3 Y(s) - s^2 \cancel{y(0)} - s\cancel{y'(0)} - \cancel{y''(0)} = 0$$

So $Y(s) = \dfrac{1}{s^4 + s^3}$ or $\dfrac{1}{s^3(s+1)}$

Options PF or convolution

PF: $Y(s) = \dfrac{A}{s^3} + \dfrac{B}{s^2} + \dfrac{C}{s} + \dfrac{D}{s+1}$ ← need to solve a 2x2 system in the best case

$\underline{\text{Conv.}}$ $\quad y(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^3(s+1)}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{s^3}\right\} * \mathcal{L}^{-1}\left\{\dfrac{1}{s+1}\right\}$

$\boxed{\text{Table}} \longrightarrow \quad = \left[\dfrac{1}{2!} t^2\right] * e^{-t}$

$$= \frac{1}{2}\int_0^t \tau^2 e^{-(t-\tau)}\, d\tau$$

$$= \frac{e^{-t}}{2}\int_0^t \tau^2 e^{\tau}\, d\tau$$

u	dv	± 1
τ^2	e^{τ}	$+1$
2τ	e^{τ}	-1
2	e^{τ}	$+1$
0	e^{τ}	-1
		$+1$

$$Y(t) = \frac{e^{-t}}{2}\left[(\tau^2 - 2\tau + 2)\, e^{\tau}\right]_0^t$$

$$= \frac{e^{-t}}{2}\left[(t^2 - 2t + 2)\, e^{t} - 2\right]$$

$$= \boxed{\frac{t^2}{2} - t + 1 - e^{-t}}$$

Example 5
Solve the IVP:

$$y^{(4)} - y^{(2)} = t^2, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 0, \quad y'''(0) = 0$$

You may leave the final solution as a convolution integral.

$\mathcal{L}$ gives:

$$s^4 Y(s) - s^3 \cancel{y(0)} - s^2 \cancel{y'(0)} - s \cancel{y''(0)} - \cancel{y'''(0)} - [s^2 Y(s) - s\cancel{y(0)} - \cancel{y'(0)}] = \frac{2}{s^3}$$

Table

0 0 0 0

so $Y(s) = \dfrac{2}{s^3[s^4 - s^2]}$ or $\dfrac{2}{s^5[s^2-1]}$ or $\dfrac{2}{s^5[s-1][s+1]}$

PF is rough b/c of the $\frac{1}{s^5}$ term. 4x4 system

$$Y(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^5[s-1][s+1]}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{[s-1][s+1]}\right\}$$

$$\mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\} = \frac{2}{4!}\mathcal{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{2}{4!}t^4$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2(s-1)} + \frac{-1}{2(s+1)}\right\}$$

$$= \frac{1}{2}e^t - \frac{1}{2}e^{-t}$$

$$y(t) = \left[\frac{2}{4!}\, t^4\right] * \left[\frac{1}{2}\left[e^{t} - e^{-t}\right]\right]$$

$$= \boxed{\frac{1}{4!}\int_0^t \tau^4\left[e^{t-\tau} - e^{-(t-\tau)}\right] d\tau}$$

Example 6
Suppose that the system S has the transfer function

$$T(s) = \frac{1}{(s+2)(s+1)(s^2+1)}.$$

Determine if the zero-input response component of the solution is bounded for all possible initial conditions of the system.

What conditions must be imposed on the poles of the forcing term to ensure that the zero-state response component of the solution is bounded.

ZIR is of the form

$$\frac{as^3+bs^2+cs+d}{(s+2)(s+1)(s^2+1)}$$

This will give a bounded ZIR component of the solution that is bounded $\forall$ ICs iff

all poles have $Re \leq 0$ and if $Re = 0$ then the order of the pole is 1.

The only poles of $F(s)$ are potentially $s = -2, -1, \pm j$. These satisfy the needed property so ZIR is bounded in time.

ZSR is of the form

$$\frac{F(s)}{(s+2)(s+1)(s^2+1)}$$

Need same condition as mentioned above. The poles of $F(s)$ must have $Re \leq 0$, can't be $\pm j$, + any pole of $F(s)$ w/ $Re=0$ must be first order.

Example 7
Write the inverse Laplace transform of

$$\frac{1}{(s^2+1)^3}$$

as a real valued integral.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \left[\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}\right]$$

$$= \sin(t) * [\sin(t) * \sin t]$$

$$= \sin(t) * \int_0^t \sin(\tau)\sin(t-\tau)\,d\tau$$

$$\boxed{= \int_0^t \sin(u)\left[\int_0^{t-u} \sin\tau \sin(t-u-\tau)\,d\tau\right]du}$$