MATH 213 - Lecture 4: More Laplace Transforms and Properties of Laplace Transforms

Lecture goals: Be able to compute Laplace transforms from the definition, know what the one-sided or unilateral Laplace Transform is and understand some commonly used (and important) properties of the Laplace transform (and be able to prove them if asked).

More Examples:

Example 1
Compute the Laplace transform of $tu(t)$ and find the ROC.

$$f(t) = t\,u(t) = \begin{cases} t & t \geq 0 \\ 0 & \text{else} \end{cases}$$

$$F(s) = \mathcal{L}\{f(t)\}$$

$$= \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} t e^{-st}\, dt$$

Recall:

$(fg)' = f'g + fg'$

so

$\int u\,dv = uv - \int v\,du$

Int by parts:

u	dv	± 1
t	e^{-st}	1
1	$-\frac{1}{s} e^{-st}$	-1
0	$\frac{1}{s^2} e^{-st}$	1

So

$$F(s) = -\frac{t}{s} e^{-st}\Big|_0^\infty - \frac{1}{s^2} e^{-st}\Big|_0^\infty$$

needs $Re(s) > 0$

$$= 0 - \left[0 - \frac{1}{s^2}\right] \quad , \underline{\underline{if}} \quad Re(s) > 0$$

$$\boxed{\text{Note } \lim_{t\to\infty} \frac{t}{e^{st}} \to 0}$$

$$= \boxed{\frac{1}{s^2} \quad , \text{if } Re(s) > 0}$$

Often we care about functions $f(t)$ that are only defined for $t \geq 0$. There is a special transform for that

> **Definition 1: Unilateral Laplace Transform**
> The **Unilateral Laplace Transform** or **One-sided Laplace Transform** of a function $f(t)$ defined only for $t \geq 0$ is defined as
>
> $$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}dt.$$

Recall
$f(0^-) = \lim_{x \to 0^-} f(x)$

Caution: If we use the symbol "$\mathcal{L}$" we mean the two sided transform unless otherwise stated.

> **Example 2**
> Compute the one-sided Laplace transform of $u(t-T)$ for $T > 0$ and find the ROC.

$$F(s) = \mathcal{L}\{u(t-T)\} = \int_{0^-}^{\infty} u(t-T)\,dt = \int_{T}^{\infty} e^{-st}\,dt$$

Note:
$$u(t-T) = \begin{cases} 1 & t \geq T \\ 0 & \text{else} \end{cases}$$

So

$$F(s) = -\frac{e^{-st}}{s}\Bigg|_T^{\infty}$$

$$= 0 - \left(-\frac{e^{-sT}}{s}\right) \quad \text{if } Re(s) > 0$$

$$= \boxed{\frac{e^{-sT}}{s}, \quad \text{if } Re(s) > 0}$$

Example 3
Compute the Laplace transform of $\sin(\omega t)u(t)$ for $\omega \in \mathbb{R}$ and find the ROC.

Hint: Write sin as a sum of complex exponentials.

Option 1: Compute $\int_{-\infty}^{\infty} \sin(\omega t)u(t)e^{-st}\,dt$ via IBP

Option 2:
$$e^{j\omega t} = j\sin\omega t + \cos\omega t$$
$$-\quad e^{-j\omega t} = -j\sin\omega t + \cos\omega t$$
$$e^{j\omega t} - e^{-j\omega t} = 2j\sin\omega t$$
or $\boxed{\sin\omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}}$

Thus
$$\mathcal{L}\{\sin(\omega t)u(t)\} = \mathcal{L}\left\{\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)u(t)\right\}$$
$$= \frac{1}{2j}\mathcal{L}\left\{e^{j\omega t}u(t)\right\} - \frac{1}{2j}\mathcal{L}\left\{e^{-j\omega t}u(t)\right\}$$

$\boxed{L3 \text{ ex } 8} \longrightarrow$
$$= \frac{1}{2j}\frac{1}{s-j\omega} - \frac{1}{2j}\frac{1}{s+j\omega}, \quad \underline{Re(s) > 0}$$
$$= \frac{1}{2j}\left[\frac{s+j\omega - (s-j\omega)}{s^2+\omega^2}\right], \quad Re(s) > 0$$
$$= \boxed{\frac{\omega}{s^2+\omega^2}, \quad Re(s) > 0}$$

Properties of the Laplace Transform:

Theorem 2: Laplace Transform is Linear
Suppose that $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$. Then for all $\alpha, \beta \in \mathbb{C}$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

the the ROC is the intersection of the ROCs for $F(s)$ and $G(s)$.

Proof: Let $f(t)$ & $g(t)$ have LTs $F(s)$ & $G(s)$ defined on S_F & S_G & let $\alpha, \beta \in \mathbb{C}$. We have

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \int_{-\infty}^{\infty} [\alpha f(t) + \beta g(t)] e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} \alpha f(t) e^{-st} + \beta g(t) e^{-st}\, dt$$

$$= \alpha \int_{-\infty}^{\infty} f(t) e^{-st} + \beta \int_{-\infty}^{\infty} g(t) e^{-st}\, dt$$

if integrals converge
i.e. if $s \in S_F \cap S_G$

$$= \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$$

$$= \alpha F(s) + \beta G(s), \text{ if}$$

Theorem 3: Time-Scaling
If $\mathcal{L}\{f(t)\} = F(s)$ then for $c > 0$, $\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$

Proof: Let $F(s) = \mathcal{L}\{f(t)\}$ & let $c > 0$. Then

$$\mathcal{L}\{f(ct)\} = \int_{-\infty}^{\infty} f(ct)\, e^{-st}\, dt$$

Let $u = ct$, $du = c\,dt$

$\Rightarrow t = \frac{u}{c}$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-\frac{s}{c}u}\, \frac{du}{c}$$

$$= \frac{1}{c}\int_{-\infty}^{\infty} f(t)\, e^{-\left(\frac{s}{c}\right)t}\, dt$$

$$= \frac{1}{c}\,\mathcal{L}\{f(t)\}\Big|_{s=\frac{s}{c}}$$

$$= \frac{1}{c}\,F\left(\frac{s}{c}\right)$$

Example 4

for $\omega > 0$

Use the fact that $\mathcal{L}\{\sin(t)u(t)\} = \dfrac{1}{s^2+1}$ to compute $\mathcal{L}\{\sin(\omega t)u(\omega t)\}$ without directly evaluating the integral.

$$\mathcal{L}\{\sin(\omega t)\, u(\omega t)\} = \frac{1}{\omega} \mathcal{L}\{\sin(t)\, u(t)\}\Big|_{s=\frac{s}{\omega}}$$

$$= \frac{1}{\omega} \frac{1}{\left(\frac{s}{\omega}\right)^2 + 1}$$

$$= \frac{1}{\omega} \frac{\omega^2}{s^2+\omega^2}$$

$$= \boxed{\frac{\omega}{s^2+\omega^2}}$$

if $s \in \text{ROC}$

Theorem 4: Exponential Modulation
$\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$.

Proof: Let $F(s) = \mathcal{L}\{f(t)\}$.

$$\mathcal{L}\{e^{\alpha t} f(t)\} = \int_{-\infty}^{\infty} e^{\alpha t} f(t) e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} f(t) e^{-(s-\alpha)t} \, dt$$

$$= \mathcal{L}\{f(t)\}\Big|_{s=s-\alpha}$$

$$= F(s-\alpha)$$

Example 5

Compute $\mathcal{L}\{e^{\alpha t}u(t)\}$ for $\alpha \in \mathbb{R}$ without directly evaluating the integral.

Recall from L3 ex 7 that $\mathcal{L}\{u(t)\} = \begin{cases} \frac{1}{s}, & Re(s) > 0 \\ \infty, & \text{else} \end{cases}$

thus

$$\mathcal{L}\{e^{\alpha t}u(t)\} = \left.\begin{cases} \frac{1}{s} & Re(s) > 0 \\ \infty & \text{else} \end{cases}\right|_{s = s - \alpha}$$

$$= \boxed{\begin{cases} \frac{1}{s-\alpha} & Re(s) > \alpha \\ \infty & \text{else} \end{cases}}$$

L3 ex 8

Theorem 5: Time-Shifting
If $F(s) = \mathscr{L}\{f(t)u(t)\}$ and $g(t) = f(t-T)u(t-T)$ then

$$G(s) = e^{-sT}F(s).$$

proof: Let $F(s) = \mathcal{L}\{f(t)\,u(t)\}$ & $g(t) = f(t-T)\,u(t-T)$.

Then

$$G(s) = \mathcal{L}\{g(t)\}$$

$$= \mathcal{L}\{f(t-T)\,u(t-T)\}$$

$$= \int_{-\infty}^{\infty} f(t-T)\,u(t-T)\,e^{-st}\,dt$$

$$r = t-T \Rightarrow dr = dt, \quad t = r+T$$

$$= \int_{-\infty}^{\infty} f(r)\,u(r)\,e^{-s(r+T)}\,dr$$

$$= \int_{-\infty}^{\infty} f(r)\,u(r)\,e^{-sr}\,e^{-sT}\,dr$$

$$= e^{-sT}\int_{-\infty}^{\infty} f(t)\,u(t)\,e^{-st}\,dt \quad \leftarrow \text{Changed var}$$

$$= e^{-sT}\,\mathcal{L}\{f(t)\,u(t)\}$$

$$= \boxed{e^{-sT}\,F(s)}$$

Example 6
Evaluate $\mathscr{L}\{u(t-T)\}$ without directly evaluating the integral.

L3 Ex 7 shows $\mathcal{L}\{u(t)\} = \begin{cases} \frac{1}{s}, & Re(s) > 0 \\ \infty, & \text{else} \end{cases}$

So $\mathcal{L}\{u(t-T)\} = e^{-sT}\frac{1}{s}, \quad Re(s) > 0$

Theorem 6: Multiplication by t
If $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{tf(t)\} = -\frac{d}{ds}F(s)$.

Proof: Let $F(s) = \mathcal{L}\{f(t)\}$ then

$$\frac{d}{ds}F(s) = \frac{d}{ds}\left(\mathcal{L}\{f(t)\}\right)$$

$$= \frac{d}{ds}\int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

Leibnitz rule

$$= \int_{-\infty}^{\infty} \frac{\partial}{\partial s}\left(f(t)\, e^{-st}\right) dt$$

$$= \int_{-\infty}^{\infty} -t\, f(t)\, e^{-st}\, dt$$

$$= -\int_{-\infty}^{\infty} t\, f(t)\, e^{-st}\, dt$$

$$= -\mathcal{L}\{t f(t)\}.$$

So

$$\boxed{\mathcal{L}\{t f(t)\} = -\frac{d}{ds}F(s)}$$

Example 7
Compute $\mathscr{L}\{tu(t)\}$ without directly computing the integral.

$$\mathcal{L}\{t\,u(t)\} = -\frac{d}{ds}\left(\mathcal{L}\{u(t)\}\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right)$$

$$= -\left(-\frac{1}{s^2}\right)$$

$$= \frac{1}{s^2}$$

Example 8: Foreshadowing
Use integration by parts to evaluate $\mathscr{L}\{f'(t)\}$ where we mean the one-sided transform for a "sufficiently nice" function $f(t)$.

$$\mathcal{L}\{f'(t)\} = \int_{0^-}^{\infty} f'(t)\, e^{-st}\, dt$$

$$u = e^{-st} \qquad dv = f'(t)\,dt$$

$$du = -s e^{-st}\, dt \qquad v = f(t)$$

$$= f(t)\, e^{-st} \Big|_{0^-}^{\infty} - \int_{0^-}^{\infty} f(t)\left(-s e^{-st}\right) dt$$

$$= \left[0 - f(0^-)\right] + s \int_{0^-}^{\infty} f(t)\, e^{-st}\, dt$$

Re(s) big enough

$$= -f(0^-) + s\,\mathcal{L}\{f(t)\}$$

$$= s\,F(s) - f(0^-) \longleftarrow \star$$

VERY

important

Theorem 7: Laplace Transform of a Derivative/Integral
Let $f(t)$ be such that there is a real value α such that the integral

$$\int_{0^-}^{\infty} |f(t)|e^{-\alpha t}dt$$

converge **and** such that there exists a function $f'(t)$ such that for $t \geq 0$

$$f(t) = f(0^-) + \int_{0^-}^{\infty} f'(\tau)d\tau$$

and there exists a real value β such that

$$\int_{0^-}^{\infty} |f'(t)|e^{-\beta t}dt$$

converges. In this case

$$F(s) = \frac{1}{s}f(0^-) + \frac{1}{s}\mathscr{L}\{f'(t)\}$$

or in other-words

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0^-)$$

The proof for this is basically in the previous example. This theorem is how we will solve linear DEs!