MATH 213 - Lecture 4: More Laplace Transforms and Properties of Laplace Transforms

Lecture goals: Be able to compute Laplace transforms from the definition, know what the one-sided or unilateral Laplace Transform is and understand some commonly used (and important) properties of the Laplace transform (and be able to prove them if asked).

More Examples:

Example 1
Compute the Laplace transform of $tu(t)$ and find the ROC.

Often we care about functions $f(t)$ that are only defined for $t \geq 0$. There is a special transform for that

Definition 1: Unilateral Laplace Transform
The **Unilateral Laplace Transform** or **One-sided Laplace Transform** of a function $f(t)$ defined only for $t \geq 0$ is defined as

$$F(s) = \mathscr{L}\{f(t)\} = \int_{0^+}^{\infty} f(t)e^{-st}dt.$$

Caution: If we use the symbol "$\mathscr{L}$" we mean the two sided transform unless otherwise stated.

Example 2
Compute the one-sided Laplace transform of $u(t-T)$ for $T > 0$ and find the ROC.

Example 3
Compute the Laplace transform of $\sin(\omega t)u(t)$ for $\omega \in \mathbb{R}$ and find the ROC.

Hint: Write sin as a sum of complex exponentials.

Properties of the Laplace Transform:

Theorem 1

For any function $f(t)$, the one-sided Laplace transform will always converge given that there is some sufficiently large $s \in \mathbb{R}$ such that

$$\int_0^\infty f(t)e^{-st}dt$$

exists

Theorem 2: Laplace Transform is Linear

Suppose that $f(t)$ and $g(t)$ have Laplace transforms $F(s)$ and $G(s)$. Then for all $\alpha, \beta \in \mathbb{C}$

$$\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$$

the the ROC is the intersection of the ROCs for $F(s)$ and $G(s)$.

Theorem 3: Time-Scaling
If $\mathscr{L}\{f(t)\} = F(s)$ then for $c > 0$, $\mathscr{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$

Example 4

Use the fact that $\mathscr{L}\left\{\sin(t)u(t)\right\} = \dfrac{1}{s^2+1}$ to compute $\mathscr{L}\left\{\sin(\omega t)u(\omega t)\right\}$ for positive real ω without directly evaluating the integral.

Theorem 4: Exponential Modulation
$\mathscr{L}\{e^{\alpha t}f(t)\} = F(s-\alpha)$.

Example 5

Compute $\mathscr{L}\{e^{\alpha t}u(t)\}$ for $\alpha \in \mathbb{R}$ without directly evaluating the integral.

Theorem 5: Time-Shifting

If $F(s) = \mathscr{L}\{f(t)u(t)\}$ and $g(t) = f(t-T)u(t-T)$ then

$$G(s) = e^{-sT}F(s).$$

Example 6

Evaluate $\mathscr{L}\{u(t-T)\}$ without directly evaluating the integral.

Theorem 6: Multiplication by t
If $\mathscr{L}\{f(t)\} = F(s)$ then $\mathscr{L}\{tf(t)\} = -\frac{d}{ds}F(s)$.

Example 7

Compute $\mathscr{L}\{tu(t)\}$ without directly computing the integral.

Example 8: Foreshadowing
Use integration by parts to evaluate $\mathscr{L}\{f'(t)\}$ where we mean the one-sided transform for a "sufficiently nice" function $f(t)$.

Theorem 7: Laplace Transform of a Derivative/Integral
Let $f(t)$ be such that there is a real value α such that the integral

$$\int_{0^+}^{\infty} |f(t)|e^{-\alpha t}dt$$

converge **and** such that there exists a function $f'(t)$ such that for $t \geq 0$

$$f(t) = f(0^+) + \int_{0^+}^{\infty} f'(\tau)d\tau$$

and there exists a real value β such that

$$\int_{0^+}^{\infty} |f'(t)|e^{-\beta t}dt$$

converges. In this case

$$F(s) = \frac{1}{s}f(0^+) + \frac{1}{s}\mathscr{L}\{f'(t)\}$$

or in other-words

$$\mathscr{L}\{f'(t)\} = sF(s) - f(0^+)$$

The proof for this is basically in the previous example. This theorem is how we will solve linear DEs!