
MATH 213 - Assignment 4

Submit to Crowdmark by 9:00pm EST on Friday, March 15.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typesetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Read before starting the assignment: For this assignment you must do all your work independently and without the use of external aids (other than when stated in particular problem subparts of questions).

Questions:

1. (3 marks) Prove that

$$\mathcal{L} \left\{ \int_0^t e(\tau) d\tau \right\} = \frac{E(s)}{s}$$

Hints: Use the facts that if $f(t) = \int_0^t e(\tau) d\tau$ then $f'(t) = e(t)$ and $f(0) = 0$. You should use the Laplace transform of the derivative somewhere in your work.

2. (10 marks) Consider the harmonic oscillator modelled by the DE

$$y''(t) + 5y'(t) + 6y(t) = f(t)$$

The spring is initially at rest in its rest position but we want it to follow the path given by the function

$$r(t) = \sin(t) + 1.$$

In class, we saw that a proportional controller tends to quickly adjust the system but has problems obtaining an exact constant value and an integral controller tends to act more slowly but asymptotically obtains a specified constant value and can cause overshooting for large k_i s. In this problem we will setup a proportional controller to handle the sine forcing term and then an integral controller to handle the constant term. By taking the linear superposition of the results, we can control the full system while having the “best of both worlds”.

(a) Proportional controller:

- i. Find $Y(s)$ for the desired input function $r(t) = \sin(t)$ for the closed loop proportionally controlled harmonic oscillator. Make sure to identify the transfer function.
- ii. Find an expression for the poles of your transfer in terms of k_p and use this expression to show that for all $k_p > 0$ the poles of the transfer function have negative real parts (and hence exponentially decay).
- iii. The response of the controlled harmonic oscillator, $v(t)$, to the input given by $r(t) = \sin(t)$ can be written as the sum of a sinusoidal wave of period 2π (i.e. $A \sin(t) + B \cos(t)$) and some terms that exponentially decay. In this part of the question we will find and examine the sinusoidal terms (i.e. the “steady state solution”).
 - A. Find the coefficients of the sinusoidal terms and give the “steady state” component of the system response.
 - B. Plot the solution from the previous part for $k_p = 10, 100$ and 1000 over $t = 0$ to 2π . Compare these with the plot of the desired $r(t)$. For large values of t (i.e. $t \gg 1$ such that the decaying terms have little influence on the system), the system response to $r(t)$ is approximately equal to the functions you plotted. Based on your plots explain whether or not the proportional controller is a good way to control the **long term** behaviour of the spring for this $r(t)$.

(b) Integral controller:

- i. Find $Y(s)$ for the desired input function $r(t) = 1$ for the closed loop integrally controlled harmonic oscillator. Make sure to identify the transfer function.
- ii. Given that the poles of the transfer function are in the left hand plane, the solution of this system, $y(t)$, with the input given by $r(t) = 1$ can be written as the sum of a constant term and some terms that exponentially decay.

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- A. Use the final value theorem to find the constant term. You may assume that we pick k_i so that the transfer function has no roots that cause problems for the FVT. i.e. do not try to analytically find all the poles.
- B. Based on your solution above explain whether or not the integral controller is a good way to control the **long term** behaviour of the spring for this $r(t)$.
- (c) (1 bonus all or none) Using some software of your choice, compute the transient terms that we skipped computing in parts (a) and (b) for the case where $k_p = 1000$ and $k_i = 2$.

Note: You can use software for 2c but not for the computations 2a and 2b themselves. You can however validate your solutions from (b) and (c) using the solution you found in this particular case (i.e. $k_p = 1000$ and $k_i = 2$). You also must show proof of how you computed these terms.

Hint: There is a reason why I am not making you do this computation by hand!

Use these terms to plot the sum of the effects of the two controlled systems found in parts (a) and (b) for $t = 0$ to $t = 10\pi$. Do you think this is a well controlled system given that the user wanted to obtain the “correct behaviour” after 5 periods of the input function?