MATH 213 - Lecture 8: Zero-input, zero state response, and the delta function

Lecture goals: Understand how poles effect the inverse Laplace transform, what the zero-input and zero-state responses are. Know what the delta function is and its properties are.

Understanding Poles:

The final value theorem is useful but we can actually do a bit better so we will examine and analyze all the various cases of a function with a single pole.
Consider a function $F(s)$ with a single pole of (natural number greater than 1) order n at $a + bi$:

$$f(t) = \frac{1}{(s - (a + bi))^n}.$$

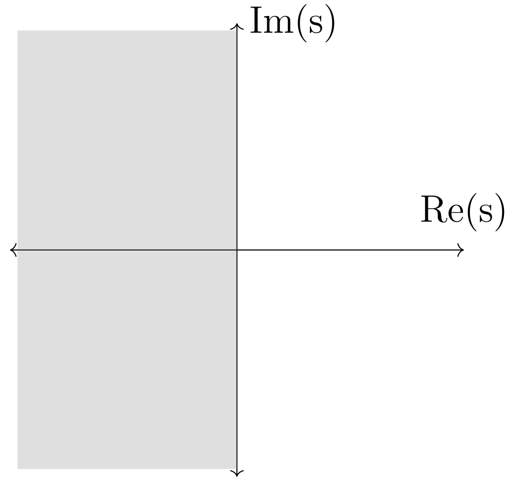
By the Fundamental Theorem of Algebra along with partial fraction decomposition **all** proper rational functions can be decomposed to a sum of functions of this form!

Taking the inverse Laplace transform gives

$$\begin{aligned}
f(t) &= \mathscr{L}^{-1}\left\{F(s)\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{(s-(a+bi))^n}\right\} \\
&= e^{(a+bi)t}\mathscr{L}^{-1}\left\{\frac{1}{s^n}\right\} && \text{Exponential Modulation} \\
&= e^{(a+bi)t}\frac{t^{n-1}}{n!} && t^n \text{ transform}
\end{aligned}$$

Thus **all** functions that have a rational Laplace transform (the ones we have covered thus far), have an inverse transform of the form of a weighted sum of terms of the form of the above.
We hence analyze these functions via case analysis!

$a < 0, b \neq 0$: In this case the pole is in the region shown here (excluding the real and imaginary axes)



In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = e^{at}\frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt))$$

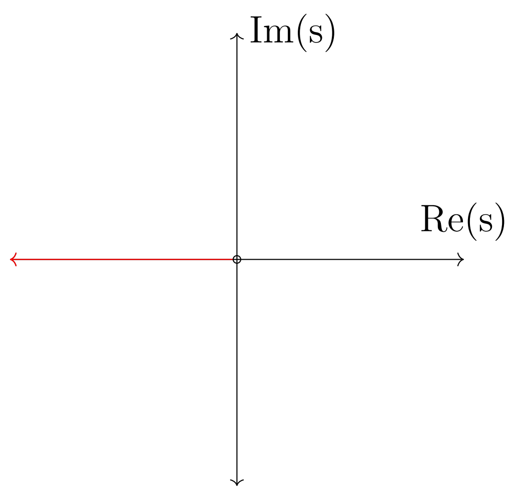
Since $a < 0$ and both t^{n-1} and the trig terms grow slower than e^{at} decays,

$$\lim_{t\to\infty} f(t) = 0.$$

In this case the final value theorem applies and tells us that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s}{(s-(a+bi))^n} = 0.$$

$a < 0, b = 0$: In this case the pole is in the region shown here (excluding the origin)



In these cases

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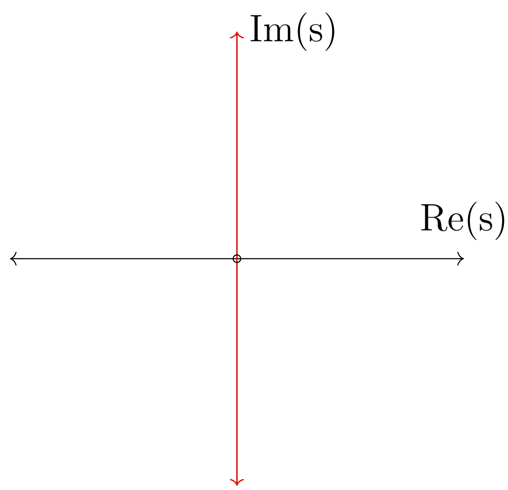
Since $a < 0$ and t^{n-1} grows slower than e^{at} decays,

$$\lim_{t\to\infty} f(t) = 0.$$

In this case the final value theorem applies and tells us that

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} \frac{s}{(s-a)^n} = 0.$$

$a = 0$,$b \neq 0$: In this case the pole is in the region shown here (excluding the origin)



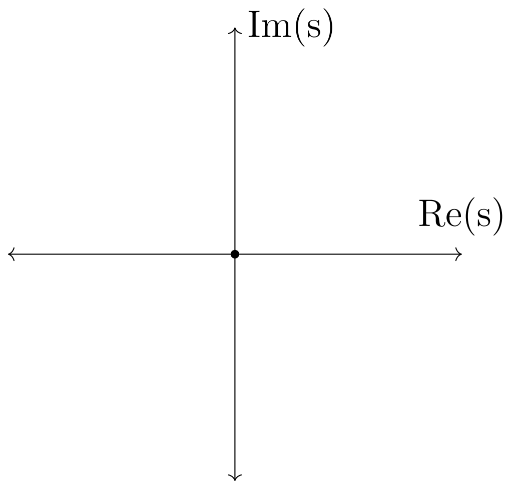
In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = \frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt)))$$

In these cases if $n = 1$ then we have bounded oscillations (and hence $\lim_{t\to\infty} f(t)$ DNE) or we have growing oscillations (and hence the limit also does not exist).

In this case the final value theorem tells us the limit does not exist (since the poles have a real part of 0 and the imaginary part is also non-zero) but does not differentiate between bounded oscillations and unbounded oscillations.

$a = 0, b = 0$: In this case the pole is in the region shown here



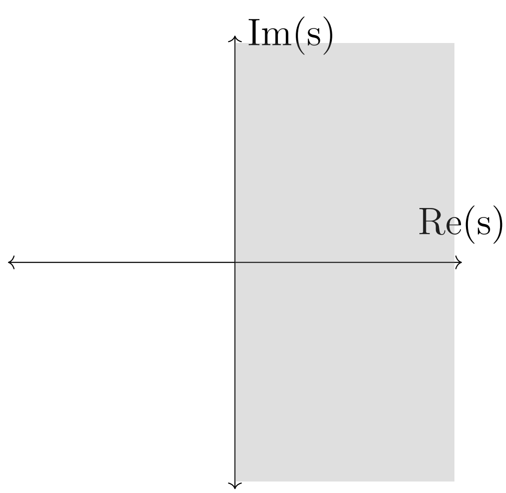
In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = \frac{t^{n-1}}{n!}$$

In these cases when $n = 1$ then we have a constant solution (and hence $\lim_{t\to\infty} f(t) = 1$). Else the solution grows and the limit diverges.

The above is what the final value theorem tells us but it does not explicitly state that the function grows unbounded.

$a > 0, b \in \mathbb{R}$: In this case the pole is in the region shown here (including the real axis but excluding the imaginary axis)



In these cases

$$f(t) = e^{(a+bi)t}\frac{t^{n-1}}{n!} = e^{at}\frac{t^{n-1}}{n!}(\cos(bt) + j\sin(bt))$$

In these cases the exponential term diverges and hence regardless of the values of n or b the transform diverges. This is again what the final value theorem tells us but the details of oscillation or the addition of a power of t are lost.

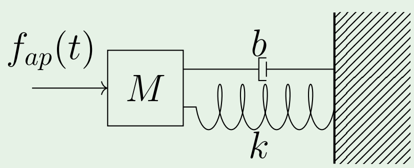
In conclusion the location of the poles tells us about the behaviour of the solution at infinity (via the final value theorem) but also the initial values (via the initial value theorem).

If only... there was a way to <u>control</u> where the poles are and to thus <u>control</u> the behaviour of the solution to a DE we care about...

Enter the ideas of the zero-input response, the zero-state responses and the transfer function.

To motivate the zero-input response, the zero-state response and the transfer function recall the Linear Harmonic Oscillator for Lecture 2:

Example 1: Linear Harmonic Oscillator
The DE that models the spring mass system:



where $f_{ap}(t)$ is an applied force divided by m, $\frac{b}{m}$ is the coefficient of friction and $\frac{k}{m}$ is the spring constant is

$$y'' + by' + ky = f_{ap}(t).$$

Use the Laplace transform method to solve this DE as far as possible (i.e. I am not telling you what $f_{ap}(t)$ is...).

The goal of this problem is to define some new terms "naturally"

Example 2: Primitive Control Example

Assuming that $y(0) = 1$ and $y'(0) = 0$ in the above example (i.e. the spring is held at $y(0) = 1$ with no initial velocity) find conditions on b, k and $F_{ap}(s)$ so that the solution to the DE approaches a finite value as $t \to \infty$.

Definition 1: Zero-State and Zero-Input Responses and the Transfer function

Given a linear DE with constant coefficients of the form

$$\frac{d^n}{dt^n}y + a_{n-1}\frac{d^{n-1}}{dt^{n-1}}y + \ldots + a_0 y = f(t)$$

along with the needed initial conditions $y(0)$, $y'(0)$, …, $y^{(n-1)}(0)$, the Laplace transform of the equation will be of the form

$$Y(s) = \frac{F(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + \frac{g(a_0, \ldots, a_{n-1}, y(0), \ldots, y^{(n-1)}(0), s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

where g is some function of the ICs and coefficients.

- The **zero-input response** is $\frac{g(a_0,\ldots,a_{n-1},y(0),\ldots,y^{(n-1)}(0),s)}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This is the response of the system the DE models to the initial conditions (i.e. when the forcing term is 0).

- The **zero-state response** is $\frac{F(s)}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This is the response of the system the DE models to the forcing term (i.e. when the ICs are all 0).

- The **transfer function** is $\frac{1}{s^n+a_{n-1}s^{n-1}+\ldots+a_0}$. This function determines the zero-input and the effect of the forcing term.

To help understand the role of the transfer function in the relation between the zero input response and the zero-state response, we introduce the idea of **generalized functions**.

In 1927, Physicist Paul Dirac in his "The physical interpretation of the quantum dynamics" paper introduced the function now known as the "Dirac delta" function. ↑ Interesting read for anyone interested in the history of modern mathematics/-physics.

Definition 2: Dirac Delta Function
The $\delta(x)$ function is defined as the "function" (called a special function or a distribution[a]) that satisfies the properties that:

$$\delta(x) = 0 \text{ when } x \neq 0$$
$$\int_{-\infty}^{\infty} \delta(x) = 1$$

[a] Think probability distribution

Of course there is **<u>no</u>** classical function that satisfies the above conditions but we can think of δ as the limit of a collection of functions which we now explore.

Suppose you hit a nail with a hammer and do a total amount of work of 1 unit. If $f(t)$ is the amount of force applied over time then $1 = \int_0^{t_f} f(t)dt$.

If $f(t)$ is constant while in contact the the nail and 0 else, then $f(t) = \begin{cases} \frac{1}{t_f} & 0 \leq t \leq t_f \\ 0 & else \end{cases}$.

How long was the hammer in contact with the nail?

Not long! What if it was 0 seconds? In this case

$$f(t) = \begin{cases} \text{“}\infty\text{”} & t = 0 \\ 0 & else \end{cases}$$

and

$$\int_a^b f(t) = \begin{cases} 1, & a \leq 0 \leq b \\ 0 & else \end{cases}$$

We call this function $\delta(t)$. Mathematically one can make this definition formal but to do so requires a lot of mathematical background.

Theorem 1
If f is a "well-behaved" function defined at t then

$$\begin{aligned}\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau &= \int_{-\infty}^{\infty} f(t)\delta(t-\tau)d\tau \\ &= f(t)\int_{-\infty}^{\infty}\delta(t-\tau)d\tau \\ &= f(t).\end{aligned}$$

The above is the main property of the delta function we need.

Example 3
Compute $\mathscr{L}\{\delta(t)\}$ and find the zero-state response of the harmonic oscillator (Ex 1) to the forcing term $f_{ap}(t) = \delta(t)$.

In general if $Y(s) = \frac{F_{ap}(s)}{P(s)}$ where $F_{ap}(s)$ is the transform of the forcing term and $P(s)$ is the characteristic polynomial, then

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{F_{ap}(s)}{P(s)}\right\} \\ &= f_{ap}(t) * \mathscr{L}\left\{\frac{1}{P(s)}\right\} \\ &= f_{ap}(t) * \mathscr{L}\left\{T(s)\right\} \end{aligned}$$

where $T(s)$ is the system's transfer function.

Hence a system's response to any function $f_{ap}(t)$ is simply the convolution of $f_{ap}(t)$ with the systems response to the delta function – called the system's impulse response.

This is a theorem!

Theorem 2
The zero-state response of a linear DE is the convolution of the input (i.e. forcing term) with the systems impulse response.

For the proof, see the above comments.

With the above results in mind, the solution to any linear DE with constant coefficients can be written as

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left\{\frac{F(s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0} + \frac{g(a_0, \ldots, a_{n-1}, y(0), \ldots, y^{(n-1)}(0), s)}{s^n + a_{n-1}s^{n-1} + \ldots + a_0}\right\} \\ &= f(t) * T(t) + \sum_{k=1}^{M} c_k t^{N_k - 1} e^{\lambda_k t} \end{aligned}$$

where T is the inverse Laplace transform of the transfer function, λ_k are the poles of the zero-input function, N_k is the order of the kth pole, c_k is the coefficient given by the partial fraction decomposition and M is the number of poles (which will always be finite because of the form of the second term).