MATH 213 - Lecture 15: Basic Control Theory - Cruise Control

Lecture goals: Understand the basics of closed control loops, what proportional and integral control systems are, their benefits/limitations and how to set them up to control a system.

Consider the simple system that models the velocity of a vehicle:

$$mv'(t) + bv(t) = f(t)$$

where $v(t)$ is the velocity of the vehicle, m is the mass, b is the coefficient of drag and $f(t)$ is the velocity forcing term which is a combination of the effects of the engine/brakes as well as the effects of the road (hills etc).

It is nice to split up the input into terms we can control (brake/gas) and terms we can't (road conditions):

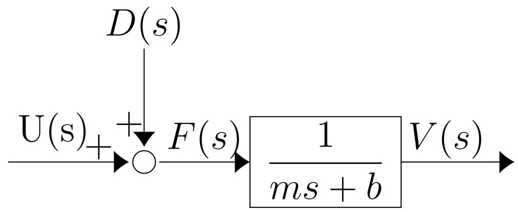
$$f(t) = \underbrace{u(t)}_{\text{Control input}} + \underbrace{d(t)}_{\text{Disturbance input}}$$

Goal: Find $u(t)$ so that the vehicle velocity $v(t)$ follows some given reference input, $r(t)$.

Solution: We use a "controller" to adjust $u(t)$ to obtain our desired velocity. We do this is frequency space.

We examine two options.

Open loop controller:



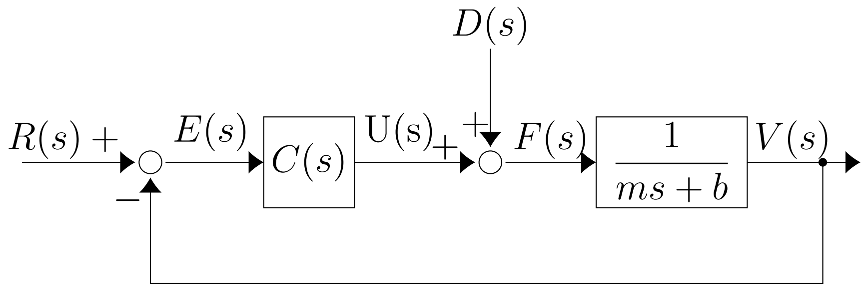
Here:

- $U(s)$ is the Laplace transform of the control input.
- $D(s)$ is the Laplace transform of the disturbance input.
- $F(s)$ is the Laplace transform of the total forcing term.
- $V(s)$ is the Laplace transform of the resulting velocity.

These controllers only work well if we know the disturbance the system undergoes, if the system always needs a constant known output or if the user can make adjustments as needed.

Some applications include washer/dryers for clothing, toasters, watering systems, step motors etc but open loop systems do not work well for our application.

Closed loop controller:



Here:

- $U(s)$, $D(s)$, $F(S)$ and $V(s)$ remain unchanged from before but...

- We added a controller with transfer function $C(S)$ to dynamically adjust $U(s)$ based on feedback.

- $R(s)$ is the Laplace transform of the **desired** velocity function $v(t)$.

- $E(s)$ is the Laplace transform of the error of our velocity $r(t) - v(t)$. This is the input into the control system!

New Goal: Find a transfer function $C(s)$ such that $E(s) = 0$ and hence we have the desired velocity i.e. $V(s) = R(s)$.

Proportional control:

Idea: Let $u(t) = k_p e(t)$ for $k_p \in \mathbb{R}_{>0}$. In this case

- if $e(t) > 0$ then we "hit the gas" to speed up,
- if $e(t) < 0$ we "hit the breaks" to slow down and
- if $e(t) = 0$ we do nothing.

We will explore how this works in the case where $D(s) = 0$.

If we use this controller then $U(s) = \underbrace{k_p}_{\text{Transfer function } C(s)} E(s)$.

We have

$$V(s) = \frac{1}{ms+b}F(s) \qquad \text{System response to } f(t) \text{ is } H(s)F(s).$$

$$= \frac{1}{ms+b}U(s) \qquad \text{System input is } F(s) = D(s) + U(s) \text{ and } D(s) = 0$$

$$= \frac{1}{ms+b}k_pE(s) \qquad \text{Controller output is } U(s) = k_pE(s)$$

$$= \frac{1}{ms+b}k_p(R(s) - V(s)) \qquad E(s) = R(s) - V(s)$$

Solve for $V(s)$:

$$\left(1 + \frac{1}{ms+b}k_p\right)V(s) = \frac{1}{ms+b}k_pR(s)$$

$$V(s) = \frac{\frac{k_p}{ms+b}}{1+\frac{k_p}{ms+b}}R(s)$$

Let's write this in standard form:

$$\begin{aligned}
V(s) &= \frac{\frac{k_p}{ms+b}}{1+\frac{k_p}{ms+b}}R(s) \\
&= \frac{k_p}{\left(1+\frac{k_p}{ms+b}\right)(ms+b)}R(s) \\
&= \frac{k_p}{ms+b+k_p}R(s) \\
&= \frac{k_p/(b+k_p)}{[m/(b+k_p)]s+1}R(s)
\end{aligned}$$

The transfer function for the controlled system is

$$\frac{k_p/(b+k_p)}{[m/(b+k_p)]s+1}$$

which is the transfer function for a first order system with a DC gain of

$$\kappa = \frac{k_p}{b+k_p}$$

and a time constant of

$$\tau = \frac{m}{b + \kappa_p}.$$

This transfer function looks similar to the transfer function for the uncontrolled car:

$$\frac{k_p/(b+k_p)}{m/(b+k_p)s+1} \qquad \text{vs} \qquad \frac{1/b}{(m/b)s+1}$$

but we can now adjust the DC gain and time constants by changing k_p.

We will call this transfer function for the controlled system

$$H_{RV}(s) = \frac{k_p/(b+k_p)}{m/(b+k_p)s+1}$$

RV because it controls based on R and V.

If we wanted to account for a disturbance term, we can simply find a transfer function $H_{DV}(s)$ such that $V(s) = H_{DV}(s)D(s)$.

By linear superposition (all the systems are linear) the net response would be

$$V(s) = H_{RV}(s)R(s) + H_{DV}(s)D(s).$$

Now, we want to pick k_p appropriately so that the system does what we want.

Changing k_p lets us

- Adjust the time constant:

$$\begin{array}{cc} \text{Uncontrolled } \tau & \text{Controlled } \tau \\ \dfrac{m}{b} & \dfrac{m}{b+k_p} \end{array}$$

  For larger values of k_p, τ becomes smaller.

  Recall that a smaller τ leads to faster decay to the "final" value of the system (which always exists for the standard first order system).

- Adjust the DC gain:

$$\begin{array}{cc} \text{Uncontrolled } \kappa & \text{Controlled } \kappa \\ \dfrac{1}{b} & \dfrac{k_p}{b+k_p} \end{array}$$

by changing k_p we can make κ obtain any value between 0 and 1.

Example 1

Suppose we want to use a proportional controller to set a speed of 50 (i.e. $r(t) = 50$) in a vehicle with a mass of m, a coefficient of drag of b and no outside disturbances $d(t) = 0$.

Find the function $V(S)$ for the controlled system in terms of the k_p constant.

Use the FVT to explore and analyze the effects that changing k_p has on the "final" velocity.

Find $v(t)$ and plot $v(t)$ for several values of k_p to explore this effect in the time domain.

To see the effects of k_p in the time domain run Lecture15_pcontroller.m

Major limitation: If $e(t)$ is ever 0 then $u(t) = k_p r(t) = 0$.

Thus in the absence of any disturbance to the system (i.e. $d(t) = 0$), the input into the system $f(t)$ also vanishes.

Thus there is no system input and hence $v(t)$ will drop (since $b > 0$).

In general p controllers, can never achieve the perfect asymptotic velocity!

While in this example, we can get as close to the desired velocity as we want, in for higher order systems p controllers tend to overshoot and potentially become unstable for large values of k_p.

Hence we introduce

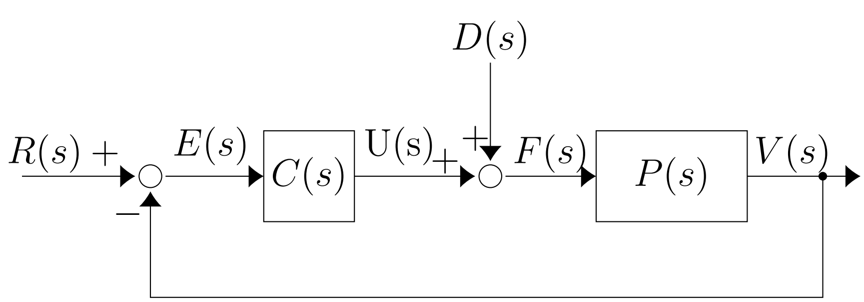
Integral controllers:

Let the error term be proportional to the integral of the error:

$$c(t) = k_i \int_0^t e(\tau) d\tau$$

This will go to zero exactly when the average error is 0.

It can hence (in principle) bring the asymptotic error exactly to 0!
We hence repeat the same analysis to this new controller.



The transfer function for the integral term is $C(s) = \frac{k_i}{s}$ (Hw 4 Q1).
Hence

$$\begin{aligned} V(s) &= \underbrace{P(s)}_{\text{Car transfer function}} \underbrace{C(s)}_{\text{Controller transfer function}} E(s) \\ &= P(s)C(s)(R(s) - V(S)) \end{aligned}$$

so

$$V(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} R(s).$$

Thus the new controlled system has a transfer function of

$$\begin{aligned} H(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} &= \frac{\frac{1}{ms+b} \frac{k_i}{s}}{1 + \frac{1}{ms+b} \frac{k_i}{s}} \\ &= \frac{k_i/m}{s^2 + \frac{b}{m}s + \frac{k_i}{m}} \qquad \leftarrow \text{Do some Algebra} \end{aligned}$$

This is the transfer function for a second order system so we will delay its full analysis until we cover the standard second order system in the next lecture but the DC gain is

$$H(0) = \frac{k_i/m}{0^2 + 0 \cdot \frac{b}{m} + \frac{k_i}{m}} = 1.$$

Now let's play with this system a bit more.

Example 2

Suppose we want to use an integral controller to set a speed of 50 (i.e. $r(t) = 50$) in a vehicle with a mass of m, a coefficient of drag of b and no outside disturbances $d(t) = 0$.

Find the function $V(s)$ for the controlled system in terms of the k_i constant.

Use the FVT to explore and analyze the effects that changing k_i has on the "final" velocity.

Find $v(t)$ and plot $v(t)$ for several values of k_i to explore this effect in the time domain.

To see the effects of k_p in the time domain run Lecture15_pcontroller.m
Major Limitations:

- Since we adjust based on the integral the error, integral controllers do strongly quickly adjust to changes. i.e. they "adjust the average"
- The above leads to the potential to overshooting the goal velocity. This is also im part because the solutions generally with come in conjugate pairs and hence the solution will have sinusoidal terms that will decay to 0.