MATH 213 - Tutorial 1: Review of Key Integration Methods - Solutions

1. Evaluate

$$\int \frac{x+2}{x^2(x+1)}dx$$

Solution: For integrals of this form we use partial fractions to simplify the integrand. We first need to find an appropriate form for this problem is

$$\frac{x+2}{x^2(x+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1}.$$

We now need to find A, B and C. There are a few standard options for how to do this:

– Multiply by $x^2(x+1)$, set the coefficients of the polynomials equal to each other and finally solve the resulting linear system for A, B, C. This is generally the last resort option.

– Pick three different values for x and then solve the linear system. This works but still involves solving a MATH 115 problem.

– Try to carefully manipulate the expression and selectively substitute in values for x to quickly isolate for the coefficients. **If there are no repeated roots**, then this method will always work and we can simply remove the various terms in the denominator and then plug in the root of the polynomial in for x. For repeated roots you will generally need to solve a system of equations **but** can reduce the complexity of the system if you try to first find all the coefficients for the higher power terms. See the wiki article for the Heaviside cover-up method for more details

We choose option 3 as it is almost always the best for manual computations. multiplying by x^2 gives

$$\frac{x+2}{x+1} = A + Bx + \frac{Cx^2}{x+1}.$$

Plugging in $x = 0$ gives

$$2 = A.$$

Next we multiply by $x+1$ to get

$$\frac{x+2}{x^2} = (x+1)\left(\frac{A}{x^2} + \frac{B}{x}\right) + C.$$

plugging in $x = -1$ gives

$$1 = C.$$

We now solve for B by picking any value for x that we have not already considered. Generally you should pick a value that makes the algebra nice. Here $x = 1$ is nice but other options of course work. Using $x = 1$ in the original form of the equation (but you could also use the other forms as well) gives

$$\frac{1+2}{1^2(1+1)} = \frac{A}{1^2} + \frac{B}{1} + \frac{C}{1+1} \quad \text{or} \quad B = \frac{3}{2} - A - \frac{C}{2}.$$

Using the previously found values of A and C gives

$$B = \frac{3}{2} - 2 - \frac{1}{2} = -1.$$

Thus

$$\frac{x+2}{x^2(x+1)} = \frac{2}{x^2} - \frac{1}{x} + \frac{1}{x+1}.$$

We can now integrate!

$$\begin{aligned}\int \frac{x+2}{x^2(x+1)}dx &= \int \frac{2}{x^2} - \frac{1}{x} + \frac{1}{x+1}dx \\ &= 2\int \frac{1}{x^2}dx - \int \frac{1}{x}dx + \int \frac{1}{x+1}dx \\ &= 2\left(-\frac{1}{x}\right) - \ln|x| + \ln|x+1| + C \\ &= -\frac{2}{x} + \ln\left|\frac{x+1}{x}\right| + C\end{aligned}$$

2. Evaluate

$$\int \frac{1}{(x^2+1)(x+1)}dx$$

Solution: We have two options for how to decompose this the classical decomposition

$$\frac{1}{(x^2+1)(x+1)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+1}$$

or the complex-valued decomposition

$$\frac{1}{(x+j)(x-j)(x+1)} = \frac{A}{x+j} + \frac{B}{x-j} + \frac{C}{x+1}.$$

We will use the classic version but keep the complex one in mind for later in the course when we talk about poles...

Multiplying by $x+1$ and plugging in $x=-1$ gives

$$\frac{1}{(-1)^2+1} = 0 + C \qquad \text{or} \qquad C = \frac{1}{2}.$$

Now we can't use the cover-up method to find A and B (unless we use the complex valued decomposition) so we now need to pick nice values of x to build a linear system to solve for A and B. Note if we use $x=0$ first then we can solve for B. After that if we use $x=1$ then we can solve for A. Using $x=0$ gives

$$1 = B + C \qquad \text{or} \qquad B = 1 - C = \frac{1}{2}.$$

Now using $x=1$ gives

$$\frac{1}{4} = \frac{A+B}{2} + \frac{C}{2} \qquad \text{or} \qquad A = \frac{1}{2} - C - B = -\frac{1}{2}.$$

Hence

$$\begin{aligned}\frac{1}{(x^2+1)(x+1)} &= \frac{1-x}{2(x^2+1)} + \frac{1}{2(x+1)} \\ &= \frac{1}{2(x^2+1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x+1)}\end{aligned}$$

Integrating gives

$$\begin{aligned}
\int \frac{1}{(x^2+1)(x+1)}dx &= \int \frac{1}{2(x^2+1)} - \frac{x}{2(x^2+1)} + \frac{1}{2(x+1)}dx \\
&= \frac{1}{2}\int \frac{1}{x^2+1}dx - \frac{1}{2}\int \frac{x}{x^2+1}dx + \frac{1}{2}\int \frac{1}{x+1}dx \\
&= \frac{1}{2}\tan^{-1}(x) - \frac{1}{4}\ln|x^2+1| + \frac{1}{2}\ln|x+1| + C.
\end{aligned}$$

Note that the first integral is the standard inverse tan integral and for the second integral we need to use the u-sub $u = x^2+1$, $du = 2xdx$.

For more practice of using partial fractions to evaluate integrals see Paul's Online notes

3. Evaluate

$$\int x^4 e^x dx$$

Pro-tip: Use the tabular method for integration by parts to save some time.

Solution: For an integral of this form we need to use integrating by parts several times until the power of x reduces to 0. We will first show the classical version and then we will show a shortcut called the table method (that you may or may not have seen before).

Letting $u = x^4$ and $dv = e^x dx$ and applying the integration by parts formula (i.e. product rule for integration) gives

$$\int \underbrace{x^4}_{u}\underbrace{e^x dx}_{dv} = \underbrace{x^4}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{4x^3 dx}_{du}$$

Oh no! We need to evaluate $\int 4x^3 e^x dx$. We again use integration by parts:

$$\int \underbrace{4x^3}_{u}\underbrace{e^x dx}_{dv} = \underbrace{4x^3}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{12x^2 dx}_{du}.$$

We now need to evaluate the last integral!

$$\int \underbrace{12x^2}_{u}\underbrace{e^x dx}_{dv} = \underbrace{12x^2}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{24x dx}_{du}.$$

Once more!

$$\int \underbrace{24x}_{u}\underbrace{\cos(x) dx}_{dv} = \underbrace{24x}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{24 dx}_{du}.$$

The last integral is just $24e^x + C$ so we are done! Putting this together gives

$$\begin{aligned}
\int x^4 e^x dx &= x^4e^x - \left(4x^3e^x - \left(12x^2e^x - (24xe^x - 24e^x)\right)\right) + C \\
&= x^4e^x - 4x^3e^x + 12x^2e^x - (24xe^x - 24e^x) + C \\
&= x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x + C
\end{aligned}$$

Fun times!

Notice that 1) the negatives resulting from repeated applications of IBP result in an oscillating negative term and that each successive term as a higher order derivative of u and a higher order antiderivative for e^x (which does not change). This can be exploited to build the "tabular method". In the first row we define u and dv and in subsequent rows we differentiate u and integrate dv. We repeat until the u function becomes 0. Finally, we read off the solution, remembering to oscillate the negative that we get from IBP. Here is the table for this problem:

u	dV	± 1
x^4	e^x	1
$4x^3$	e^x	-1
$12x^2$	e^x	1
$24x$	e^x	-1
24	e^x	1
0	e^x	-1
		1

$$\boxed{\begin{gathered} x^4 e^x \\ -4x^3 e^x \\ +12x^2 e^x \\ -24x e^x \\ +24 e^x \end{gathered}}$$
$$+0 \ldots$$

To read the table multiply the functions along the red lines and then add up all the products. The result gives what we previously found.

4. Evaluate

$$\int e^x \sin(x)dx$$

Solution: This is a classic problem. The table method does not help us a ton here since the exponential and sin functions both have oscillatory derivatives/integrals. Hence we will brute force it.

$$\begin{aligned} \int e^x \sin(x)dx &= e^x \sin(x) - \int -e^x \cos(x)dx \\ &= e^x \sin(x) - (e^x \cos(x) - \int -e^x \sin(x)dx) \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx + C \end{aligned}$$

The integral on the RHS is a scalar multiple of the one on the LHS. Hence we add it to get

$$2\int e^x \sin(x) = e^x \sin(x) - e^x \cos(x)C + \qquad \text{or} \qquad \int e^x \sin(x) = e^x \left(\frac{\sin(x) - \cos(x)}{2}\right) + C.$$

For more practice of using integration by parts to evaluate integrals see Paul's Online notes