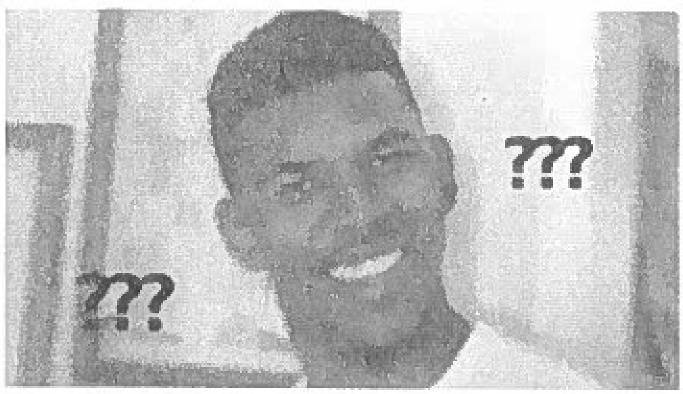
MATH 213 - Lecture 9: delta function derivatives and intro to systems and signals

Lecture goals: Understand how to differentiate the delta function. Know what systems and signals are.

Derivatives of $\delta(t)$:

It turns out that it is useful to take the derivatives of $\delta(t)$.

In MATH 119 we told you that we can't differentiate functions at discontinuities so?



Recall that $\delta(t)$ is not a function in the classic sense but is instead a distribution, a collection of functions that have a common property when we take a limit[1]. Hence to differentiate such functions, we need to redefine what it is to be differentiable.

We will not do this formally but will instead play with delta until we magically arrive at the result.

We will start with the derivative of a simpler function.

[1]Definition for MATH 213. For a formal definition you can read wiki ← clickable.

Example 1: Derivative of the Heaviside function

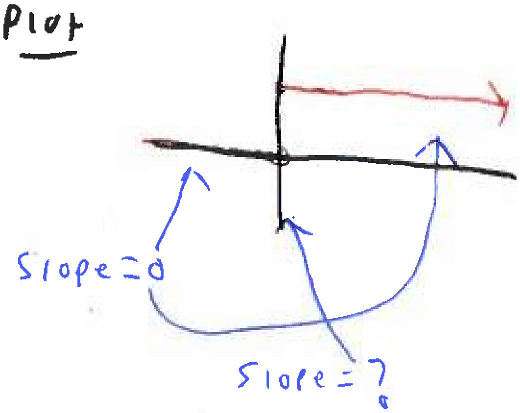
Find a "function" $f(t)$ such that $\int_{-\infty}^{t} f(t)dt = u(t)$ where $u(t)$ is the usual notation for the Heaviside function.

Such a function $f(t)$ would be the "derivative" of the Heaviside function since mechanically differentiating both sides of the previous equation under the assumption that $\lim_{t\to\infty} f(t) = Const.$ gives

$$f(t) = u'(t)$$

Method 1: need $\int_{-\infty}^{t} f(t)\,dt = \begin{cases} 0 & t < 0 \\ 1 & \text{else} \end{cases}$

Plot



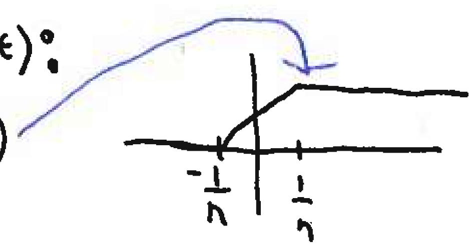
Why not use "∞"? ← Yolo

Hence $f(t) = \delta(t)$.

Note we defined δ to do what we need f to do

Method 2 (better way): Approximate $u(t)$:

$$u(t) = \lim_{n\to\infty} f_n(t)$$



Hence $u'(t) = \lim_{n\to\infty} f_n'(t)$

$$f_n'(t) = \begin{cases} \text{Und} & |t| = \frac{1}{n} \\ 0 & |t| > \frac{1}{n} \\ \frac{n}{2} & |t| < \frac{1}{n} \end{cases}$$

So

$$U'(t) = \begin{cases} 0 & t \neq 0 \\ \text{"}\infty\text{"} & t = 0 \end{cases}$$

$$= \delta(t)$$

Example 2
Use the fact that $\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$ for "nice" functions f to find an expression for $\int_{-\infty}^{\infty} f(t)\delta'(t)dt$ where $\delta'(t)$ is the derivative of $\delta(t)$.

This gives a definition for the derivative of $\delta(t)$ by defining what it does to a **test function** $f(t)$. Namely

$$\delta'[f] = \begin{cases} \text{what we find in this example} & t = 0 \\ 0 & else \end{cases}.$$

In the above

Notation 1
$\delta'[f]$ denotes $\int_{-\infty}^{t} f(t)\delta'(t)dt$ which is the effect of δ' on the test function f.

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = f(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\,f'(t)\,dt$$

$u = f(t) \qquad dv = \delta'(t)dt$

$du = f'(t)dt \qquad v = \delta(t)$

$\delta(\pm\infty) = 0$

δ property

$$= 0 - f'(0)$$

$$= -f'(0)$$

also holds for 1-sided $\mathcal{L}$

Hence,

$$\delta'[f](t) = \begin{cases} -f'(0) & t = 0 \\ 0 & else. \end{cases}$$

Graphically we can view $\delta'(x)$ by looking at some particular functions in its collection.

The Gaussian distribution centered at 0:

$$G(x;\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\dfrac{x^2}{2\sigma^2}}$$

is one of the collections of functions that is in $\delta(x)$. Explicitly as $\sigma \to 0^+$, the $G(x;\sigma)$ does what the delta function does when integrated:

$$\lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} G(x,\sigma) f(x) = f(0).$$

See matlab file for a numerical example of the above statement.

We thus can write

$$\delta(x) = \lim_{\sigma\to 0^+} G(x,\sigma)$$

$\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} G(x,\sigma)d$

where the limit is taken in the "weak" sense.

Now since $G(x;\sigma)$ acts like $\delta(x)$, $G_x(x;\sigma)$ will act like $\delta'(x)$:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x)dx &= \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{d}{dx} G(x;\sigma)dx \\ &= \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} \frac{-xG(x;\sigma)}{\sigma^2} dx\end{aligned}$$

See matlab file for a numerical example of the above statement.

We thus can write

$$\delta'(x) = \lim_{\sigma\to 0^+} G_x(x,\sigma)$$

$\int_{-\infty}^{\infty} \delta'(x)dx = \lim_{\sigma\to 0^+} \int_{-\infty}^{\infty} G_x(x,\sigma)$

where the limit is taken in the "weak" sense.

In general $\delta^{(n)}[f] = (-1)^n f^{(n)}(0)$.

Example 3
Compute the Laplace transform of $\delta'(t)$.

$$\mathcal{L}\{\delta'(t)\} = \int_{-\infty}^{\infty} \delta'(t) e^{-st}\, dt$$

$$= -\frac{d}{ds}\left(e^{-st}\right)\Big|_{t=0}$$

$$= s e^{-st}\Big|_{t=0}$$

$$= s.$$

Also holds for 1-sided δ

Note: in L8 ex2. we wanted $F(s)$ s.t.

$F(s) = -s - b$.

we now know $f(t) = -\delta'(t) - b\,\delta(t)$!!!

Updated One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\text{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\text{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\text{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\text{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\text{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\text{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\text{Re}(s) > \lvert\omega\rvert$
8. $\delta^{(n)}(t)$	s^n	$\mathbb{C}$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$(f * g)(t)$	$F(s)G(s)$	Convolution Theorem
$f^{(n)}(t)$	$s^n F(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Systems and Signals:

Given a linear differential equations and some specified initial conditions:

$$\frac{d^n}{dt^n}y(t)+a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t)+\ldots+a_0(t)y(t)=f(t), \quad y(0)=b_0,\ldots,y^{(n-1)}(0)=b_{n-1}.$$

There are two classic problems one considers in engineering

Definition 1: Analysis and Synthesis

- **Analysis:** Given a forcing term $f(t)$ solve for $y(t)$.
- **Synthesis:** Given a $y(t)$, or some conditions on $y(t)$, find a forcing term $f(t)$ such that $y(t)$ solves the differential equation.

We have thus far focused on the analysis problem, which is the easier of the two, but the synthesis problem is generally more useful in engineering design.

To solve the synthesis problem we need a good understanding of how $f(t)$ relates to $y(t)$. Think about the spring problem from L8 Ex 2 as an example.

Definition 2: Signals

A **signal** is a complex-valued function of an independent real variable t.

- t usually (but not always) represents time.
- If the domain of the function is $\mathbb{R}$ or some interval of $\mathbb{R}$[a] then the signal is said to be a **continuous-time (CT)** signal.
- If the domain of the function is $\mathbb{Z}$, $\mathbb{Q}$ or other discrete subset of $\mathbb{R}$[b] then the signal is said to be a **discrete-time (DT)** signal.

[a] In general any uncountable subset of $\mathbb{R}$

[b] In general any countable subset of $\mathbb{R}$

Example 4

Physical variable such as voltages, currents, displacements, forces, etc. are continuous time.

Quantities such as those seen in economics (houses, money etc.), digital hardware, anything stored digitally are are discrete time.

Definition 3: Systems
Mathematically, a **system** is a map from a space of input signals to a space of output signals.

Notation 2
If S represents some system then $f \xrightarrow{S} y$ or $y(t) = (Sf)(t)$ or simply $y = Sf$ indicated that $y(t)$ is the **response** (solution) of the system S to the input signal $f(t)$.

The above is similar to systems of equations from 115 and the connection to matrices and the "input" and "output" vectors.

Explicitly, the system of equations

$$\begin{aligned} s_{11}f_1 + \ldots + s_{1n}f_n &= y_1 \\ &\vdots \\ s_{m1}f_1 + \ldots + s_{mn}f_n &= y_m \end{aligned}$$

can be written as

$$S\vec{f} = \vec{y}.$$

Here one can ask the analysis question of solving for $\vec{y}$ ("easy") or the synthesis question of solving for $\vec{f}$ ("hard") and generally wants to know the relation between the response and input.

Further, the matrix S is the map between the vector space of input functions (all valid $\vec{f}$s) and the vector space of output functions (all valid $\vec{y}$s).

Definition 4
A system S is continuous-time (CT) if both its input and output signals are CT.
A system S is discrete-time (FT) if both its input and output signals are DT.
S is a hybrid system if ti contains both CT and DT signals.