MATH 213 - Lecture 2: Classifications of DEs and examples of linear DEs

Lecture goals: Understand how to classify DEs (in particular the different types of linear ODEs).

Classifying DEs:

Definition 1: Independent and Dependent Variables and Parameters
The **dependent variable(s)** of a DE are the unknown functions that we want to solve for i.e. $f(x)$, $y(x,t)$, etc.

The **independent variable(s)** of a DE are the variable(s) that the independent variable(s) depend on i.e. x, t, etc.

A **parameter** is a term that is an unknown but is not an independent or dependent variable i.e. a, b, α, β etc.

Example 1
In the following DEs classify all the unknowns as an independent variable, dependent variable or a parameter:

a) $\frac{dy(t)}{dt} = ay(t)$

b) $\frac{d^2y(t)}{dt^2} = -g$

c) $\frac{d^2y(t)}{dt^2} = -g - \mu\frac{dy(t)}{dt}$

d) $\frac{\partial^2 u(x,t)}{\partial t^2} = c^2\frac{\partial^2 u(x,t)}{\partial x^2}$

Definition 2: Order of a DE
The **order** of a DE is the order of the highest derivative.

Example 2
Find the order for the following DEs

a) $\frac{dI}{dt} = R_0 I$.

b) $\frac{\partial^2}{\partial t^2} u(x,t) = \mu \frac{\partial^2}{\partial x^2} u(x,t)$.

c) $y' y^{(n)} + y^2 = 0$ where $n \in \{0, 1, 2, \ldots\}$ is some given number.

Definition 3: ODEs and PDEs

A DE is an **ordinary differential equation (ODE)** if it only contains ordinary derivatives (i.e. no partial derivatives).

A DE is a **partial differential equation (PDE)** if it contains at least one partial derivative of a independent variable.

Example 3

Classify the following DEs as an ODE or a PDE:

a) $\frac{dP}{dt} = aP(1 - bP)(1 + cP)$.

b) $\frac{\partial^2}{\partial t^2}u(x,t) = \mu \frac{\partial^2}{\partial x^2}u(x,t)$.

c) $y' y^{(n)} + y^2 = 0$ where $n \in \{0, 1, 2, \ldots\}$ is some given number.

Definition 4: Linear and nonlinear DEs
A DE that contains no products of terms involving the dependent variable(s) is called **linear**.

If a DE is not linear then it is **nonlinear**.

Example 4
Classify the following DEs as linear or nonlinear

a) $y'' = x^4$

b) $u_t + u_x = 0$

c) $yy'' = 0$

d) $u_t + uu_x = 0$

Definition 5: Homogeneous and Inhomogeneous: DEs
DE where every term depends on a dependent variable is called **homogeneous**.

A DE that is not homogeneous is called **inhomogeneous** or **nonhomogeneous**.

Example 5
Classify the following DEs as homogeneous or inhomogeneous:

a) $a(x)y'' + b(x)y' + c(x)y = 0.$

b) $a(x)y'' + b(x)y' + c(x)y = f(x).$

Linear homogeneous DEs have the property that if f_1 and f_2 both solve the DE then so does $af_1 + bf_2$ for all $a, b \in \mathbb{R}$.

This is the same property that was used to define linearity in MATH 115! i.e. a vector valued function $f : \mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and for all $a, b, \in \mathbb{R}$, $f(a\vec{x} + b\vec{y}) = af(\vec{x}) + bf(\vec{y})$.

The difference is that we now study linear functions applied to the vector space of all sufficiently differentiable functions (e.g. $C^\infty(\mathbb{R})$) instead of vectors in $\mathbb{R}^n$ i.e. there are no matrix representations for linear functions.

This course focuses on linear ODEs of a particular form:

Theorem 1: Linear ODEs with Variable Coefficients
All DEs of the form

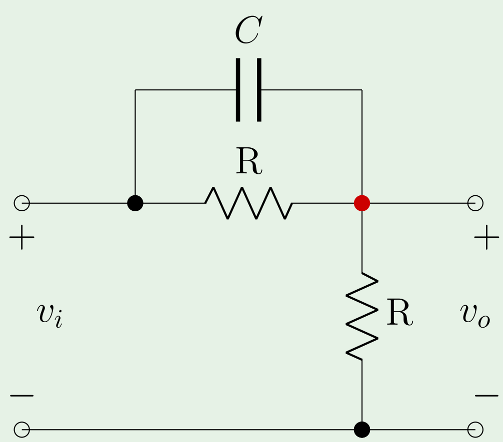
$$\frac{d^n}{dt^n}y(t) + a_{n-1}(t)\frac{d^{n-1}}{dt^{n-1}}y(t) + \ldots + a_0(t)y(t) = f(t)$$

where $n \in \{0, 1, \ldots, n\}$, the $a_i(t)$ functions are real valued, but $y(t)$ and $f(t)$ can be complex valued are linear ODEs.

Here $f(t)$ is called the **forcing term**.

Equations of this form appear in many ECE/CS/SE problems.

Example 6: Circuit Example
Given the RC circuit:



Summing the currents flowing out of the upper-right node (red one) gives the DE:

$$\underbrace{\frac{v_o}{R}}_{\text{Bottom resistor}} + \underbrace{\frac{v_o - v_i}{R}}_{\text{Top Resistor}} + \underbrace{C\frac{d}{dt}(v_o - v_i)}_{\text{Capacitor}} = 0$$

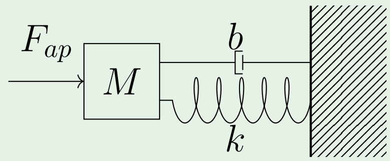
or by rearranging

$$\frac{d}{dt}v_o(t) + \frac{2}{Rc}v_o(t) = \underbrace{\frac{d}{dt}v_i(t) + \frac{1}{Rc}v_i(t)}_{f(t)}$$

This is a linear ODE with a forcing term that is determined by the input voltage.

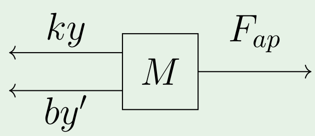
Example 7: Linear Harmonic Oscillator

Consider the spring mass system:



where F is an applied force, b is the coefficient of friction and k is the spring constant.

To find a DE we apply $F = ma$. Here is a force body diagram for the forces on the mass:



$F = Ma$ then becomes

$$F_{ap} - by' - ky = my'' \qquad \text{or} \qquad y'' + \frac{b}{m}y' + \frac{k}{m}y = \frac{1}{m}F_{ap}$$

This is a linear ODE with a forcing term that is determined by F_{ap}.