MATH 213 - Lecture 5: More Laplace Transforms and Convolution

Lecture goals: Know how to use a Laplace Transform table and the properties of the Laplace transform to evaluate Laplace Transforms and "simple" inverse Laplace Transforms. Know what the convolution operator is and its connection to Laplace Transforms.

Motivation for why we care about Laplace:

Example 1

Use the Laplace Transform to solve the IVP $y' = \sin(t)$, $y(0) = 0$.

One-sided Laplace Table:

$f(t)$	$F(s) = \mathscr{L}\{f\}(s)$	ROC
1. 1	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
2. t	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$
3. t^n	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$
4. $\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
5. $\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\mathrm{Re}(s) > 0$
6. $\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$
7. $\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$	$\mathrm{Re}(s) > \lvert\omega\rvert$

Algebraic Properties:

$\alpha f(t) + \beta g(t)$	$\alpha F(s) + \beta G(s)$	Linearity
$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$	Time Scaling
$e^{\alpha t} f(t)$	$F(s-\alpha)$	Exponential Modulation
$f(t-T)u(t-T)$	$e^{-sT}\mathscr{L}\{f(t)u(t)\}$	Time-Shifting
$t^n f(t)$	$(-1)^n F^{(n)}(s)$	Multiplication by t^n
$f'(t)$	$sF(s) - f(0)$	
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$	
$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0)$	

Example 2

Use the Laplace table to compute $\mathscr{L}^{-1}\left\{\frac{1}{s-3}+\frac{s}{s^2-4}\right\}$

Example 3

Use the Laplace table to compute $\mathscr{L}\{e^2t - \sin(4t) + 6t^3\}$

Convolution:

Before solving DEs we introduce one more super useful property of the Laplace Transform by first introducing convolution.

To motivate the continuous convolution operator we start with some discrete cases.

2D Image filtering:

Consider the image of a random internet cat:

Suppose we wanted to blur the image. We could do this by replacing each pixel with a weighted average of the nearby pixels

$$
\begin{array}{ccccc}
\text{Image} & & \text{Filter} & & \text{Filtered Image} \\
\begin{array}{|c|c|c|c|c|c|}
\hline
a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} & a_{5,1} & a_{6,1} \\ \hline
a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} & a_{5,2} & a_{6,2} \\ \hline
a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} & a_{5,3} & a_{6,3} \\ \hline
a_{1,4} & a_{2,4} & a_{3,4} & a_{4,4} & a_{5,4} & a_{6,4} \\ \hline
a_{1,5} & a_{2,5} & a_{3,5} & a_{4,5} & a_{5,5} & a_{6,5} \\ \hline
a_{1,6} & a_{2,6} & a_{3,6} & a_{4,6} & a_{5,6} & a_{6,6} \\ \hline
\end{array}
& * &
\begin{array}{|c|c|c|}
\hline
w_{1,1} & w_{2,1} & w_{3,1} \\ \hline
w_{1,2} & w_{2,2} & w_{3,2} \\ \hline
w_{1,3} & w_{2,3} & w_{3,3} \\ \hline
\end{array}
& = &
\begin{array}{|c|c|c|c|c|c|}
\hline
b_{1,1} & b_{2,1} & b_{3,1} & b_{4,1} & b_{5,1} & b_{6,1} \\ \hline
b_{1,2} & b_{2,2} & b_{3,2} & b_{4,2} & b_{5,2} & b_{6,2} \\ \hline
b_{1,3} & b_{2,3} & b_{3,3} & b_{4,3} & b_{5,3} & b_{6,3} \\ \hline
b_{1,4} & b_{2,4} & b_{3,4} & \boxed{b_{4,4}} & b_{5,4} & b_{6,4} \\ \hline
b_{1,5} & b_{2,5} & b_{3,5} & b_{4,5} & b_{5,5} & b_{6,5} \\ \hline
b_{1,6} & b_{2,6} & b_{3,6} & b_{4,6} & b_{5,6} & b_{6,6} \\ \hline
\end{array}
\end{array}
$$

$$
\begin{aligned}
b_{4,4} &= a_{3,3}w_{1,1} + a_{4,3}w_{2,1} + a_{5,3}w_{3,1} \\
&\quad + a_{3,4}w_{1,2} + a_{4,4}w_{2,2} + a_{5,4}w_{3,2} \\
&\quad + a_{3,5}w_{1,3} + a_{4,5}w_{2,3} + a_{5,5}w_{3,3} \\
b_{i,j} &= \sum_{n=1}^{N} \sum_{m=1}^{M} a_{i-m,j-n} w_{m,n}
\end{aligned}
$$

This process of filtering the image is called **discrete convolution.** If we use a Gaussian filter matrix i.e. something of the form

$$
W = \frac{1}{16} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}
$$

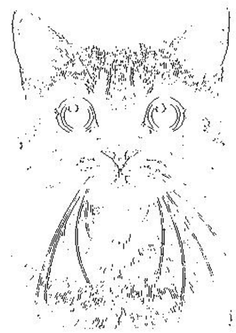
then the filtered cat image becomes

If we use a Sobel filter matrix i.e. something of the form

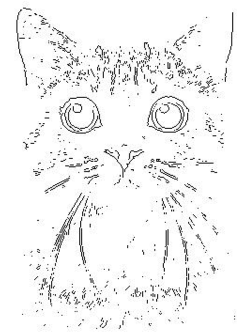
$$
\begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix}
$$

or the norm of the results then the aforementioned filtered cat images are summed becomes

or



or



Polynomial Multiplication:

Suppose that $p(x) = a_0 + a_1 x + \ldots + a_m x^m$ and $q(x) = b_0 + b_1 x + \ldots + b_n x^n$. The product would be

$$p(x)q(x) = a_0 b_0 + (a_1 b_0 + a_0 b_1)x + (a_2 b_0 + a_1 b_1 + a_0 b_2)x^2 + \ldots + a_m b_n a^{n+m}$$

Note that the coefficient of x^i is just $\displaystyle\sum_{\substack{j \text{ s.t.} \\ j \in \{0,\ldots,m\} \\ i-j \in \{0,\ldots,n\}}} a_j b_{i-j}$. This has the same functional form as the filtering example (but is in 1D) and is hence a 1D convolution of the vectors $[a_0, \ldots, a_m]$ and $[b_0, \ldots, b_n]$.

Continuous Convolution:

Motivated by the previous discrete applications we define a continuous version of the convolution.

> **Definition 1: Convolution**
> The **convolution** of functions $f(t)$ and $g(t)$, denoted by $(f * g)(t)$ is the integral
> $$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau) d\tau$$
> given that the integral converges.

Note: Similar to the discrete convolutions, this is used to apply filters and to do continuous analogues of the polynomial multiplication trick...

The second comment is useful for Laplace Transforms and for solving DEs!

Before we continue, we define the idea of an integral transform.

Definition 2: Integral Transform

An **integral transform** is a functional (function on a set of functions) that can be written in the form

$$\mathscr{I}\{f(t)\} = \int_{-\infty}^{\infty} f(t)K(t,s)dt.$$

The function $K(t,u)$ is called the **kernel** of the integral transform.

Example 4

The Laplace Transform is the integral transform with kernel $K(t,s) = e^{-st}$.

The convolution of f with some fixed function $g(t)$ is an integral transform with the kernel $K(t,s) = g(t-s)$.

Theorem 1: Convolution Properties

A. The convolution operator is commutative $(f * g)(t) = (g * f)(t)$.

B. If $f(t)$ and $g(t)$ are one-sided functions (i.e. $f(t) = g(t) = 0$ for $t < 0$) then

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

and hence the convolution is also one sided.

Example 5

Convolve the function $f(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & else \end{cases}$ with $g(t) = e^{-t}u(t)$.

Plot the result.

Theorem 2: Convolution Theorem
If there exist $\alpha, \beta \in \mathbb{R}$ such that the integrals

$$\int_{-\infty}^{\infty} |f(t)|e^{-\alpha t}dt \qquad \text{and} \qquad \int_{-\infty}^{\infty} |g(t)|e^{-\beta t}dt$$

converge then,

$$\mathscr{L}\{(f * g)(t)\} = F(s)G(s).$$

This theorem states that the Laplace Transform of a convolution is the product of the transforms! A direct result of this allows us to "quickly" compute inverse Laplace transforms:

$$\mathscr{L}^{-1}\{F(s)G(s)\} = (f * g)(t).$$

Note: for real valued functions f and g, the convolution is a real valued integral (i.e. not a contour integral in the complex plane)!!

Example 6

Use the convolution theorem to compute $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)(s-2)}\right\}$.