MATH 213 - Assignment 1 - Solutions

Submit to Crowdmark by 9:00pm EST on Friday, January 26.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Applied problem:

1. In class we talked about using differential equations to model the physical world. To do this we assumed that the independent variables, space and/or time, were continuous. While this is an ok assumption to make when working with hardware, for many software applications the independent variables are discrete. This is because CPUs/GPUs compute using discrete temporal cycles and memory is also discrete and finite. Thus, for some applications one wants to replace differential equations with their discrete analogue known as **difference equations.** In this question we will briefly explore the connection between these two deeply related concepts.

   Recall the model for exponential population growth we discussed in lecture 1 Example 4. As mentioned above, in computer simulations time is discrete and thus to model the population growth of chickens in Minecraft™ we replace the continuous model with the following discrete analogue

$$y_{n+1} = ay_n, \qquad\qquad y_0 = 2. \tag{1}$$

   Here $a \in \mathbb{R}_{>0}$, y_n is the population of chickens at time t_n where n is the number of in game ticks[1] and we assume we start with 2 chickens[2].

   (a) (2 marks) Find a solution to the discrete IVP (1). Explicitly, find some function y_n that satisfies the recurrence relation $y_{n+1} = ay_n$ along with the initial condition $y_0 = 2$. Your solution should depend on n, a and the initial condition.

   Here I am not asking you to formally prove that your solution holds[3], but should show your work for how you found your proposed solution.

   Hint: Try examining y_n for several values of n and look for a pattern.

   (b) (2 marks) Now suppose that $t_n = nh$ where $h = 50$ ms[4] and that $a = (1.5)^{1/39\,000}$[5].
   Using your solution from part (a) determine the number of chickens after 4 hours.
   Round your solution to the nearest integer and show your work.

   (c) (3 marks) Convert the difference equation to a corresponding differential equation.

   Hint (follow these steps):

   i. Subtract y_n from both sides of the difference equation and then divide both sides of the equation by the symbol h.
   ii. Introduce a continuous function $y_{cont}(t)$ such that if $t_n = nh$ then $y_n = y_{cont}(t_n)$ and $y_{n+1} = y_{cont}(t_n + h)$.
   iii. Replace the terms in the discrete equation with the continuous analogues from ii.
   iv. Look for terms that approximate derivatives and replace them with the derivatives.

   (d) (2 marks) Use the solution we found in class to solve your DE from part (c) for $y_{cont}(t)$ using the IC $y_{cont}(0) = 2$.
   Compare your new prediction for the number of chickens at 4 hours to the result you found in part (b). Is it a good approximation? Why or why not?

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[1] Number of discrete temporal updates to the game.
[2] Named Foghorn Leghorn and Lady Cluck.
[3] This would require a formal proof by mathematical induction.
[4] This corresponds to 20 UPS (updates per second) which is the default value used in Minecraft.
[5] This corresponds to each pair of chickens laying an egg every $\underbrace{7.5}_{\text{average egg time}} + \underbrace{5}_{\text{Egg laying refraction period}} + \underbrace{20}_{\text{Grow up time}}$ minutes which is an approximation for what happens in Minecraft. For the Minecraft players out there, we need a different model to correctly predict the chicken population but we currently lack the needed math to do this.

Solution:

(a) We manually compute the first few values of y_n and look for a pattern. Computing y_n for $n = 0, 1, 2, 3$ gives

$$\begin{aligned} y_0 &= 2 \\ y_1 &= ay_0 = 2a \\ y_2 &= ay_1 = 2a^2 \\ y_3 &= ay_2 = 2a^3 \end{aligned}$$

The pattern suggests that $y_n = 2a^n$ is the solution.

(b) We need to find what value of n is needed for t to be equal to 4 hours. We first convert the time to ms:

$$4 \text{ h} \frac{60 \text{ min}}{1 \text{ h}} \frac{60 \text{ s}}{1 \text{ min}} \frac{1000 \text{ ms}}{1 \text{ s}} = 14\,400\,000 \text{ ms}$$

hence the number of ticks n in an hour is

$$n = \frac{t_n}{h} = \frac{14\,400\,000 \text{ ms}}{50 \text{ ms}} = 288\,000$$

Hence there are $y_{288\,000} = 2 \cdot ((1.5)^{1/39\,000})^{288000} = 39.9389$ or 40 chickens.

(c) Following the suggested steps. We see the following

$$\begin{aligned}
y_{n+1} &= ay_n && \text{Difference equation} \\
y_{n+1} - y_n &= ay_n - y_n && \text{Subtracting } y_n \\
\frac{y_{n+1} - y_n}{h} &= \frac{ay_n - y_n}{h} && \text{Dividing by } h \\
\frac{y_{n+1} - y_n}{h} &= \left(\frac{a-1}{h}\right) y_n && \text{Algebra} \\
\frac{y_{cont}(t_n + h) - y_{cont}(t_n)}{h} &= \left(\frac{a-1}{h}\right) y_{cont}(t_n) && \text{Replacing } y_n \text{ with } y_{cont}(t_n) \\
y'_{cont}(t_n) &= \left(\frac{a-1}{h}\right) y_{cont}(t_n) && \text{RHS is approximately } y'_{cont}(t_n)
\end{aligned}$$

(d) Using the formula from class, the solution is $y_{cont}(t_n) = 2e^{\left(\frac{a-1}{h}\right)t_n}$. At 4 hours,

$$t_n = 288\,000h = 14\,400\,000$$

so the prediction is

$$y_{cont}(14\,400\,000) = 2e^{\left(\frac{a-1}{h}\right)14\,400\,000} \approx 39.9389$$

or 40 chickens. This is a good approximation. In fact for this value of h the approximation remaining good (under 0.4%) for all values of time such that the number of chickens can be represented by a standard IEEE double. See the provided Assignment_1.m code and change the bool to false to see this result.

Computational Problems:

2. (12 marks) Integration review!

Solve the following differential equations/evaluate the integral. To receive credit you **must** show all your work. Using a calculator/CAS to evaluate the integrals or "check" your answers is an academic integrity violation (and will not help you on the exams...).

Protip: To check your answers for (a)-(e) you can differentiate your proposed solutions to see if it works.

(a) $y' = xe^{x^2}$

(b) $(1+x^6)y' = x^5$

(c) $y' = e^x \cos(2x)$

(d) $y' = \dfrac{x}{x^3 - x}$

(e) $y' = \sin^3(x)\cos^3(x)$

(f) $\int_0^\pi |\cos(x)|dx$

Hint for (e): Think about some common trig identities.

Solution: For parts (a)-(e) we solve for y' and then integrate with respect to x to turn the ODEs into integrals and then simply evaluate the integrals.

(a) Let $u = x^2$. It follows that $du = 2xdx$ or $xdx = 1/2du$ and thus

$$\begin{aligned} y &= \int xe^{x^2}dx \\ &= \int \frac{1}{2}e^u du \\ &= \frac{1}{2}e^u + C \\ &= \frac{1}{2}e^{x^2} + C. \end{aligned}$$

(b) Let $u = 1 + x^6$ it follows that $du = 6x^5dx$. Thus

$$\begin{aligned} y &= \int \frac{x^5}{1+x^6}dx \\ &= \frac{1}{6}\int \frac{du}{u} \\ &= \frac{1}{6}\ln|u| + C \\ &= \frac{1}{6}\ln(1+x^6) + C \end{aligned}$$

(c) We use integration by parts with $u = \cos(2x)$ and $dv = e^x dx$. Direct computations give $du = -2\sin(2x)dx$ and $v = e^x$. Thus

$$\begin{aligned} y &= \int e^x \cos(2x)dx \\ &= \underbrace{\cos(2x)}_{u}\underbrace{e^x}_{v} - \int \underbrace{e^x}_{v}\underbrace{(-2\sin(2x))dx}_{du} \\ &= \cos(2x)e^x + 2\int e^x \sin(2x)dx. \end{aligned}$$

We integrate by parts again with $u = \sin(2x)$ and $dv = e^x dx$. Direct computations give $du = 2\cos(2x)dx$ and $v = e^x$. Thus,

$$\begin{aligned} y = \int e^x\cos(2x)dx &= \cos(2x)e^x + 2\left(\sin(2x)e^x - \int 2e^x\cos(2x)dx\right) \\ &= \cos(2x)e^x + 2\sin(2x)e^x - 4\int e^x \cos(2x)dx. \end{aligned}$$

Solving the above for $\int e^x \cos(2x)dx$ gives

$$
\begin{aligned}
y &= \int e^x \cos(2x)dx \\
&= \frac{1}{5}\cos(2x)e^x + \frac{2}{5}\sin(2x)e^x + C.
\end{aligned}
$$

(d) Note that

$$\frac{x}{x^3 - x} = \frac{1}{x^2 - 1} = \frac{1}{(x+1)(x-1)}.$$

We thus look for a partial fraction decomposition of the form

$$\frac{1}{x^2 - 1} = \frac{A}{x+1} + \frac{B}{x-1}.$$

Multiplying by $x^2 - 1$ gives

$$1 = A(x-1) + B(x+1).$$

Using $x = 1$ gives $1 = 2B$ or $B = 0.5$, and using $x = -1$ gives $1 = -2A$ or $A = -0.5$. Thus

$$\frac{1}{x^2 - 1} = \frac{-1}{2(x+1)} + \frac{1}{2(x-1)}$$

and so

$$
\begin{aligned}
y &= \int \frac{x}{x^3 - x}dx \\
&= \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)}dx \\
&= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.
\end{aligned}
$$

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&= \int \frac{1}{2(x-1)} - \frac{1}{2(x+1)}dx \\
&= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.
\end{aligned}
$$

(e) We have

$$\begin{aligned}
y &= \int \sin^3(x)\cos^3(x)dx \\
&= \int \sin(x)(1-\cos(x)^2)\cos^3(x)dx \\
&= \int \sin(x)(\cos^3(x)-\cos^5(x))dx \\
&= \int \sin(x)\cos^3(x)dx - \int \sin(x)\cos^5(x)dx.
\end{aligned}$$

Letting $u=\cos(x)$ we have $du=-\sin(x)dx$ or $\sin(x)dx=-du$ and thus

$$\begin{aligned}
y &= \int \sin(x)\cos^3(x)dx - \int \sin(x)\cos^5(x)dx \\
&= -\int u^3du + \int u^5du \\
&= -\frac{u^4}{4}+\frac{u^6}{6}+C \\
&= \frac{\cos^6(x)}{6}-\frac{\cos^4(x)}{4}+C.
\end{aligned}$$

(f) To evaluate this integral we first notice that on the interval $[0,\frac{\pi}{2}]$, $\cos(x)\geq 0$ and on the interval $[\frac{\pi}{2},\pi]$, $\cos(x)\leq 0$. Thus

$$|\cos(x)| = \begin{cases} \cos(x) & x\in[0,\frac{\pi}{2}] \\ -\cos(x) & x\in(\frac{\pi}{2},\pi] \end{cases}$$

and thus

$$\begin{aligned}
\int_0^\pi |\cos(x)|dx &= \int_0^{\frac{\pi}{2}} \cos(x)dx - \int_{\frac{\pi}{2}}^{\pi} \cos(x)dx \\
&= \sin(x)|_0^{\frac{\pi}{2}} - \sin(x)|_{\frac{\pi}{2}}^{\pi} \\
&= \sin(\pi/2)-\sin(0)-(\sin\pi-\sin\pi/2) \\
&= 2.
\end{aligned}$$

3. (6 marks) Fill out the in the following table with the various properties of the DEs

DE	Dep. Var(s)	Indep. Var(s)	Linear/ Nonlinear	Order	ODE/ PDE	Homogeneous/ Nonhomogeneous
a) $\frac{dm}{dt} = -2t^2$						
b) $\frac{\partial f}{\partial t} = f(x,t)\frac{\partial f}{\partial x}$						
c) $-9.81\frac{d^2 f}{dt^2} + 10\frac{df}{dt} + 1 = 0$						
d) $\left(\frac{dy}{dx}\right)^2 + x^3 = 0$						
e) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial x^2}$						

Solution:

DE	Dep. Var(s)	Indep. Var(s)	Linear/ Nonlinear	Order	ODE/ PDE	Homogeneous/ Nonhomogeneous
a) $\frac{dm}{dt} = -2t^2$	m	t	Linear	1st	ODE	NonHomogen.
b) $\frac{\partial f}{\partial t} = f(x,t)\frac{\partial f}{\partial x}$	f	t, x	Nonlinear	1st	PDE	Homogen.
c) $-9.81\frac{d^2 f}{dt^2} + 10\frac{df}{dt} + 1 = 0$	f	t	Linear	2nd	ODE	Nonhomogen.
d) $\left(\frac{dy}{dx}\right)^2 + x^3 = 0$	y	x	Nonlinear	1st	ODE	Nonhomogen.
e) $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial^2 v}{\partial x^2}$	u, v	t, x	Nonlinear	2nd	PDE	Homogen.