MATH 213 - Lecture 3: Introduction to the Laplace Transform

Lecture goals: Understand the definition of the Laplace transform, the basic idea of frequency space and optionally the motivating linear algebra ideas behind the Laplace transform.

One cannot find exact solutions to many linear ODEs (more on that later), but when the $a_i(t)$'s in Theorem 1 from lecture 2 are constant and $f(t)$ is "sufficiently nice" we can!

Laplace Transform Motivation:

Definition 1: Differential operator
A **differential operator** is a special type of function (called a **functional**) that accepts a function and returns another function <u>and</u> only consists of taking derivatives, multiplying by functions, and adding other differential operators.

Example 1
The following are differential operators

- $\frac{d}{dx}$
- $\frac{d^2}{dx^2} + \frac{d}{dx}$
- $u\frac{\partial}{\partial x}$

Example 2: Differential operators
<u>All</u> ODEs are of the form

$$D(y(x)) = g(x)$$

where y is the independent variable, x is the dependent vairable, $g(x)$ is the forcing term and D is a differential operator.

Example 3: Connection to MATH 115
In MATH 115 you studied functionals called linear transformations!!!

> We often asked you to solve equations of the from
>
> $$A\vec{x} = \vec{b}$$
>
> where $A \in \mathbb{R}^{n\times n}$ and $\vec{x}, \vec{b} \in \mathbb{R}^n$.
>
> The linear function $f(\vec{x}) = A\vec{x}$ is a functional!

Notice that $A\vec{x} = \vec{b}$ looks quite similar to $Dy = g$ given that D is linear...
This is the basis of the Laplace transform approach to solving ODEs.

MATH 115 "review" time!!!!

Suppose that the $n \times n$ matrix A admits a basis of orthonormal eigenvectors.

Explicitly, there is a basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that there are $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $A\vec{v}_i = \lambda_i v_i$ <u>and</u> $\vec{v}_i \cdot \vec{v}_j = 0$ when $i \neq j$ <u>and</u> $\vec{v}_i \cdot \vec{v}_i = 1$ for all $i \in \{1, \ldots n\}$.

In this case we can solve the equation $A\vec{x} = \vec{b}$ by:

1. Writing $\vec{b}$ as $b_1\vec{v}_1 + \ldots + b_n\vec{v}_n$.
2. Supposing that $\vec{x} = x_1\vec{v}_1 + \ldots + x_n\vec{v}_n$ (we can do this as $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis).
3. Noting that

$$\begin{aligned}
b_1\vec{v}_1 + \ldots + b_n\vec{v}_n &= \vec{b} && \text{def of } \vec{b} \\
&= A\vec{x} && \text{Eq. we want to solve} \\
&= A(x_1\vec{v}_1 + \ldots + x_n\vec{v}_n) && \text{def of } \vec{x} \\
&= x_1 A\vec{v}_1 + \ldots + x_n A\vec{v}_n && A \text{ is a matrix} \\
&= x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n && \text{Def. of eigenvalue/vector}
\end{aligned}$$

4. Now we can solve for x_i by taking the dot product of both sides of the equation with $\vec{v}_i$. Explicitly

$$\begin{aligned}
b_1\vec{v}_1 + \ldots + b_n\vec{v}_n &= x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n && \text{above} \\
v_i \cdot (b_1\vec{v}_1 + \ldots + b_n\vec{v}_n) &= v_i \cdot (x_1\lambda_1\vec{v}_1 + \ldots + x_n\lambda_n\vec{v}_n) && \text{dot with } \vec{v}_i \\
0 + \ldots + 0 + b_i(v_i \cdot v_i) + 0 + \ldots 0 &= 0 + \ldots + 0 + \lambda_i x_i(v_i \cdot v_i) + 0 + \ldots 0 && \text{simplify} \\
b_i &= \lambda_i x_i && \text{simplify}
\end{aligned}$$

so $x_i = \frac{b_i}{\lambda_i}$ for all $i \in \{1, \ldots, n\}$.

5. Thus $\vec{x} = \frac{b_1}{\lambda_1}\vec{v}_1 + \ldots + \frac{b_n}{\lambda_n}\vec{v}_n$!

There is no need to invert the matrix if we know a orthogonal basis of eigenvectors!!!

Profound idea: To solve the DE $Dy = g$ why not find an orthogonal basis of **eigenfunctions** ("eigenvectors" but functions) of D and use the same process???

Problem: How do we find an orthonormal eigenbasis for D???

What does that even mean?

We don't but.... Luckily for us if D is "nice" then we can use the eigenfunctions of $\frac{d}{dx}$ then things tend to work out (because of integration by parts)...
more on that later.

Example 4
Find the eigenfunctions of the $\frac{d}{dx}$ operator. i.e. solve $\frac{d}{dx}f = \lambda f$ for all $\lambda \in \mathbb{R}$.

$$\frac{d}{dx} f = \lambda f \underset{\text{if } f \neq 0}{\Rightarrow} \frac{f'}{f} = \lambda \Rightarrow \int \frac{df}{f} = \int \lambda \, dt$$

$$\Rightarrow \ln|f| = \begin{cases} \lambda t + C, & \lambda \neq 0 \\ C_1 & \lambda = 0 \end{cases}$$

$$\Rightarrow \ldots \boxed{f = Ce^{\lambda t}}$$

Proposed "basis" of eigenfunctions is $\{e^{\lambda t} / \lambda \in \mathbb{R}\}$

$\boxed{\text{We ignore } f = 0}$

While $\{e^{\lambda x} | \lambda \in \mathbb{R}\}$ does not form an orthonormal basis for the set of all differential functions (whatever that means...), it does form a basis for *something* **and** can be used to solve $\frac{d}{dx}y = f$. To do this:

1. The finite sum is replaced with an integral!
2. The dot product is replaced with a general **inner product** (more on that later).
3. The details for how to invert the system change...

It turns out that we can use this idea for all equations of the form $Dy = f$ where D is "sufficiently nice"!!!

In practice, this process turns the DE $Dy = f$ into an algebraic equation! Notice:

$$\underbrace{\frac{d}{dx}y}_{\text{calculus}} = \underbrace{\lambda y}_{Algebra}$$

We try to find solutions to $Dy=f$ of the form

$$y(t) = \underbrace{\int_{?_1}^{?_2} \underbrace{Y(s)}_{\boxed{\text{need to find}}} \underbrace{e^{st}}_{\boxed{\text{eigen function}}} \, ds}_{\text{"Linear combination" of eigenfunctions of } \frac{d}{dx}}$$

Laplace Transform:

Definition 2: Laplace Transform
Given a function $f(t)$ the **Laplace transform** of $f(t)$ denoted by $F(s)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-st}dt$$

given that the integral exists.

Here s is a complex valued scalar called the "**frequency**".

Question: Why is s called the frequency?

Example 5: Pure imaginary case
Suppose that $s = j\omega$ where $j = \sqrt{-1}$. In this case

$$\begin{aligned} e^{-st} &= e^{-j\omega t} \\ &= \cos(\omega t) - j\sin(\omega t) \end{aligned}$$

Recall from MATH 115 that
$e^{jat} = j\sin(at) + \cos(a$

ω is the frequency of the sinusoidal waves!

Example 6: Complex case
Suppose that $s = \sigma + j\omega$ where $j = \sqrt{-1}$. In this case

$$\begin{aligned} e^{-st} &= e^{-\sigma t - j\omega t} \\ &= e^{-\sigma t}e^{-j\omega t} \\ &= e^{-\sigma t}(\cos(\omega t) - j\sin(\omega t)) \end{aligned}$$

ω is still the frequency of the sinusoidal waves but the sigma controls the decay or growth of the amplitude of the waves

Definition 3: Unit Step Function or Heaviside Function
The **unit step function** or **heaviside function** is the function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & else \end{cases}$$

Example 7: $\mathcal{L}\{u(t)\}$
compute the Laplace transform of the heaviside function and determine when it exists.

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt$$

$$\boxed{\text{Note: } \quad u(t)\, e^{-st} = \begin{cases} e^{-st} & t \geq 0 \\ 0 & \text{else.} \end{cases}}$$

Thus

$$\mathcal{L}\{u(t)\} = \int_{0}^{\infty} e^{-st}\, dt$$

Let $u = -st$
$du = -s\,dt \Rightarrow dt = \frac{-du}{s}$

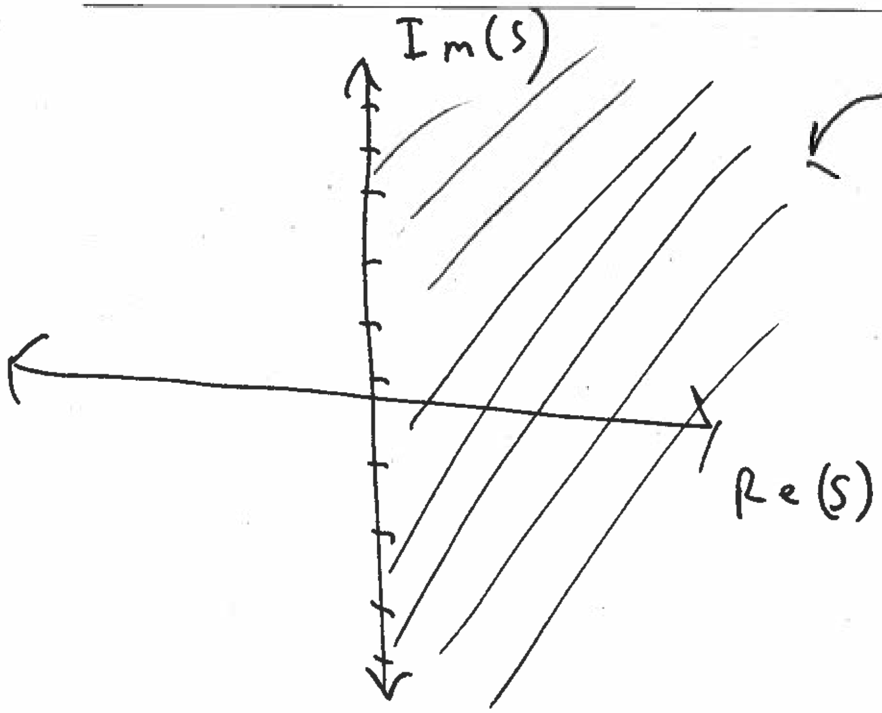
$$= \int_0^{\infty} e^{u} \, \frac{-du}{s}$$

$$= \frac{-1}{s} e^{-st} \Big|_{t=0}^{t=\infty}$$

$$= \frac{-1}{s}\left[\lim_{t\to\infty} e^{-st} - 1\right]$$

$$\boxed{\text{note } e^{-st} \to 0 \text{ iff } \underline{\underline{\operatorname{Re}(s) > 0}} \text{ else } \to \infty}$$

$$= \begin{cases} \frac{1}{s}, & \operatorname{Re}(s) > 0 \\ \infty & \text{else} \end{cases}$$



Region of convergence (ROC) for $\mathcal{L}\{u(t)\}$

Example 8
Compute the Laplace transform of

$$f(t) = e^{\alpha t}u(t) = \begin{cases} e^{\alpha t} & t \geq 0 \\ 0 & else \end{cases}$$

and determine when it exists.

$$\mathcal{L}\{f(t)\} = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{\alpha t}\, e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{(\alpha - s)t}\, dt$$

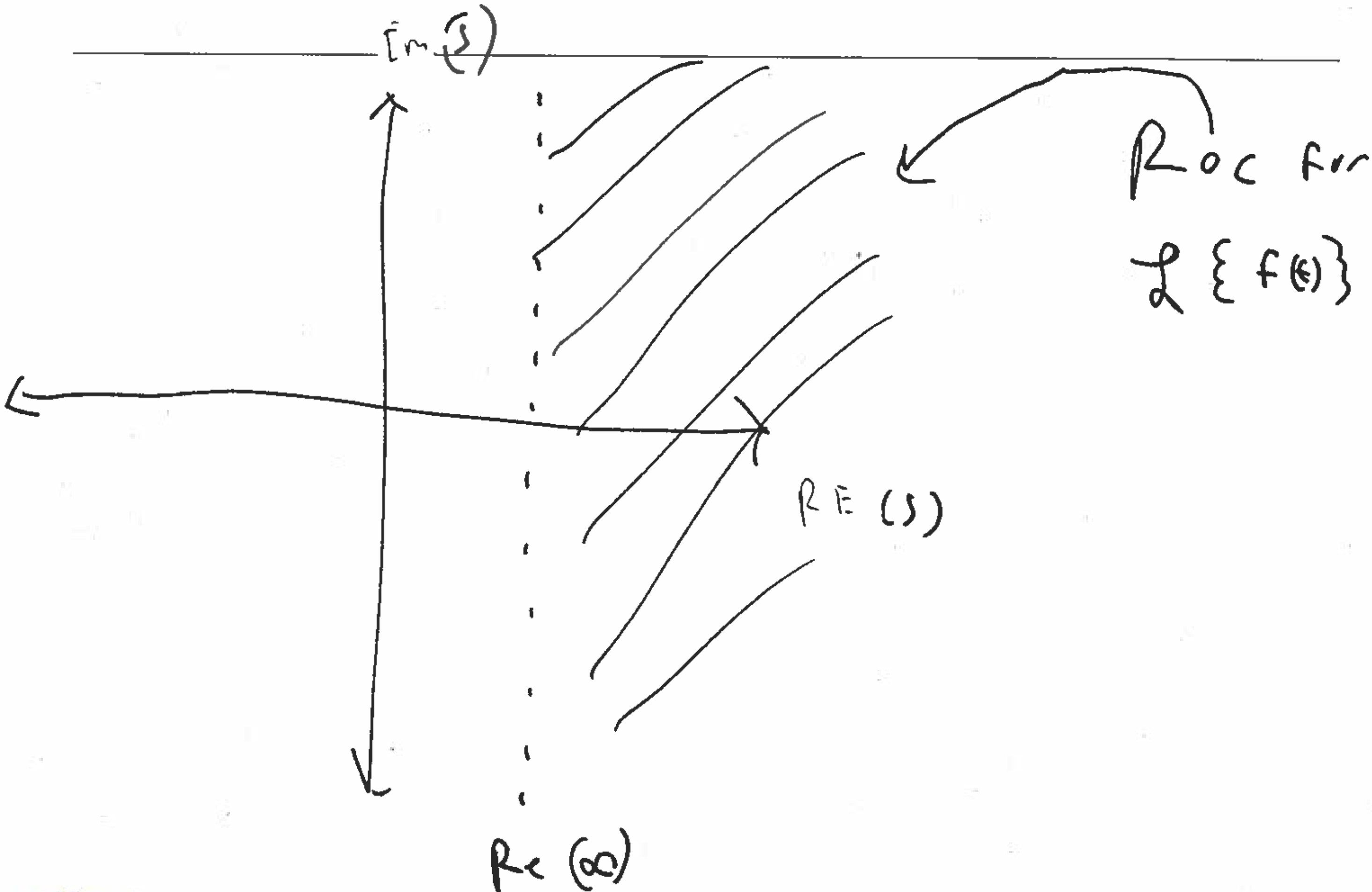
$$= \int_{0}^{\infty} e^{-(s-\alpha)t}\, dt \qquad u = -(s-\alpha)t$$

$$du = -(s-\alpha)\, dt$$

$$= \int_{0}^{\infty} e^{ut}\, \frac{-1}{s-\alpha}\, dt$$

$$= \frac{-1}{s-\alpha}\left[\lim_{t\to\infty} e^{-(s-\alpha)t} - 1\right] \qquad \begin{cases} 0 & Re(s-\alpha) \\ \infty & else \end{cases}$$

$$= \begin{cases} \dfrac{1}{s-\alpha} & Re(s) > Re(\alpha) \\ \infty & else \end{cases}$$



Note: multiplying by $e^{\alpha t}$ Shifts ROC

d S

Insight: This is because $e^{\alpha t}$ is an eig. function of d_x

This is how we write $f(t)$ as a LC of e^{st}

Definition 4: Inverse Laplace Transform
Given a "nice" function $f(t)$ and its Laplace transform $F(s)$, we have

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

where σ is such the vertical line lies in the radius of convergence of the Laplace Transform, $F(s)$.

Note that

- The above is a contour integral in the complex plane!
- The above is "sum of exponentials" I talked about!!

We have not really taught you how to evaluate these... so we will use partial-fraction decomposition and look up the inverse transforms in a table.

Example 9
Suppose $F(s) = \frac{1}{s(s+10)}$ for $Re(s) > 10$. Find $f(t)$

Step 1: Partial fractions!!

see tut 1 for more examples!

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$s=0 \Rightarrow \frac{1}{0+10} = A + 0\frac{B}{0+10} \Rightarrow A = \frac{1}{10}$$

$$s=-10 \Rightarrow \frac{1}{-10} = 0 \cdot \frac{A}{-10} + B \Rightarrow B = \frac{-1}{10}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1/10}{s} + \frac{-1/10}{s+10}\right\}$$

Inverse Laplace

Linear

$$= \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{10}\mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= \frac{1}{10} u(t) - \frac{1}{10} e^{-10t} u(t) = \frac{u(t)}{10}\left[1 - e^{-10t}\right]$$

Ex. 7 Ex 8