MATH 213 - Lecture 20: Signals Intro

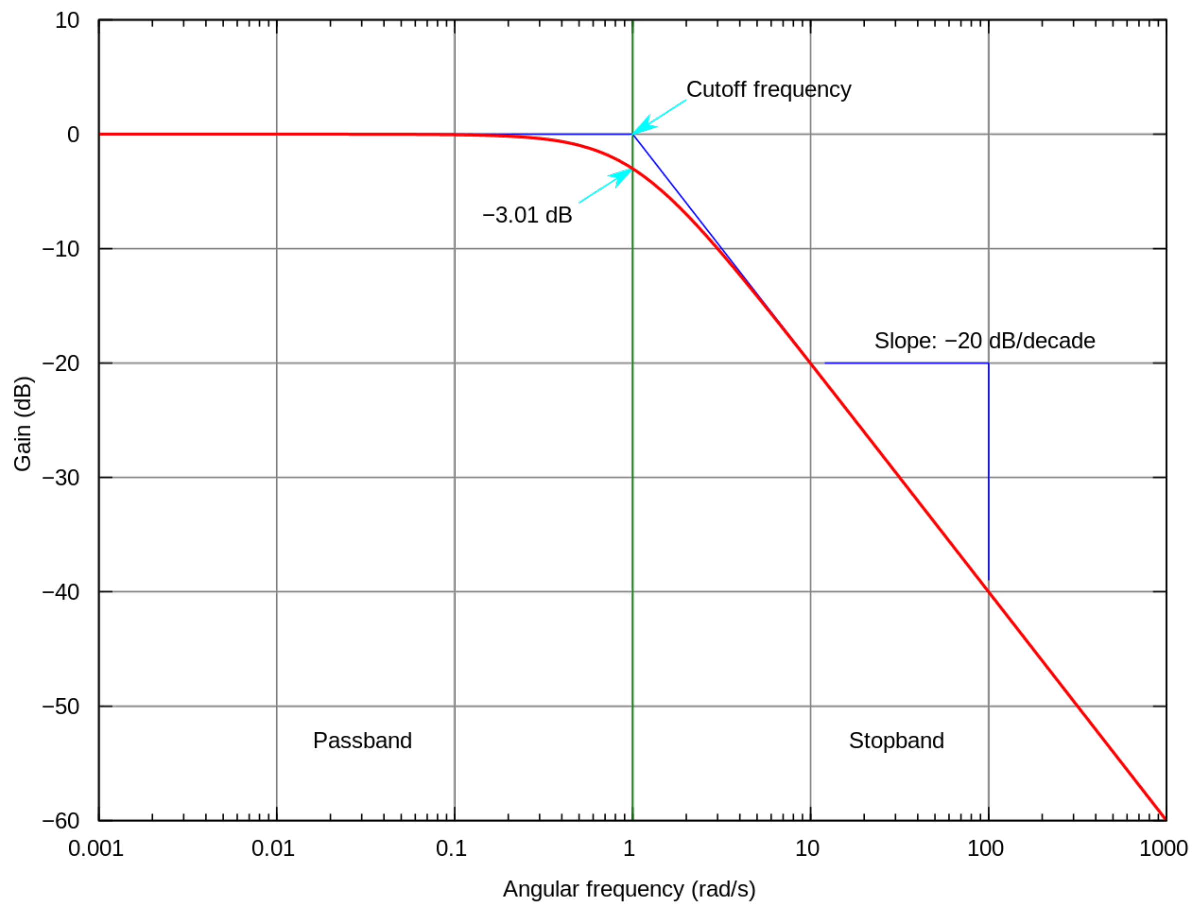
Lecture goals: know the big picture of Fourier series/transforms.

Bode plots allows us to quickly see how a system responds to input signals of the form $\sin(\omega t)$.

This can be used to quickly tell what type of filtering/amplifying the LTI does to the sin wave.

For example a low pass filter is a LTI that removes the “high” frequency waves. Explicitly, they reduce the amplitude of all waves with a frequency larger than some cutoff w.

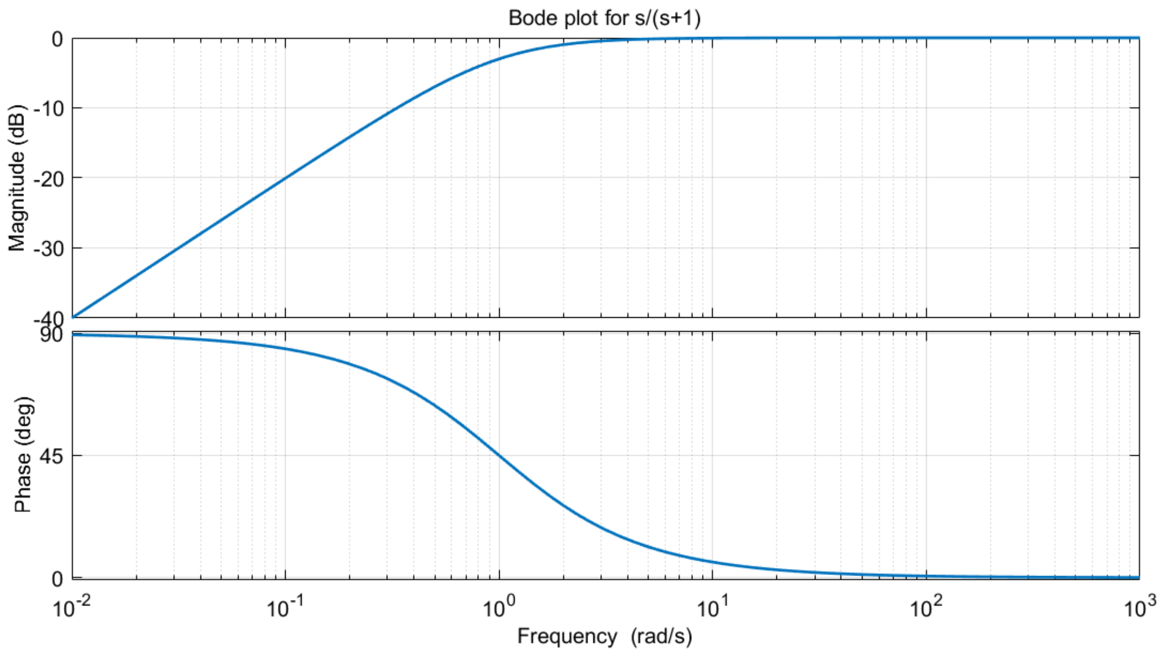
All stable systems with a single pole and no zeros are low pass filters.
Here is the hopefully familiar amplitude plot (stolen from wiki because I liked their annotations):



As another example for a high pass filters we need the amplitude curve to go to 0dB at some cutoff point. The standard high pass filter transfer function is

$$T(s) = \frac{as}{s + \omega_0}$$

for $a, \omega_0 > 0$ and its Bode plot is

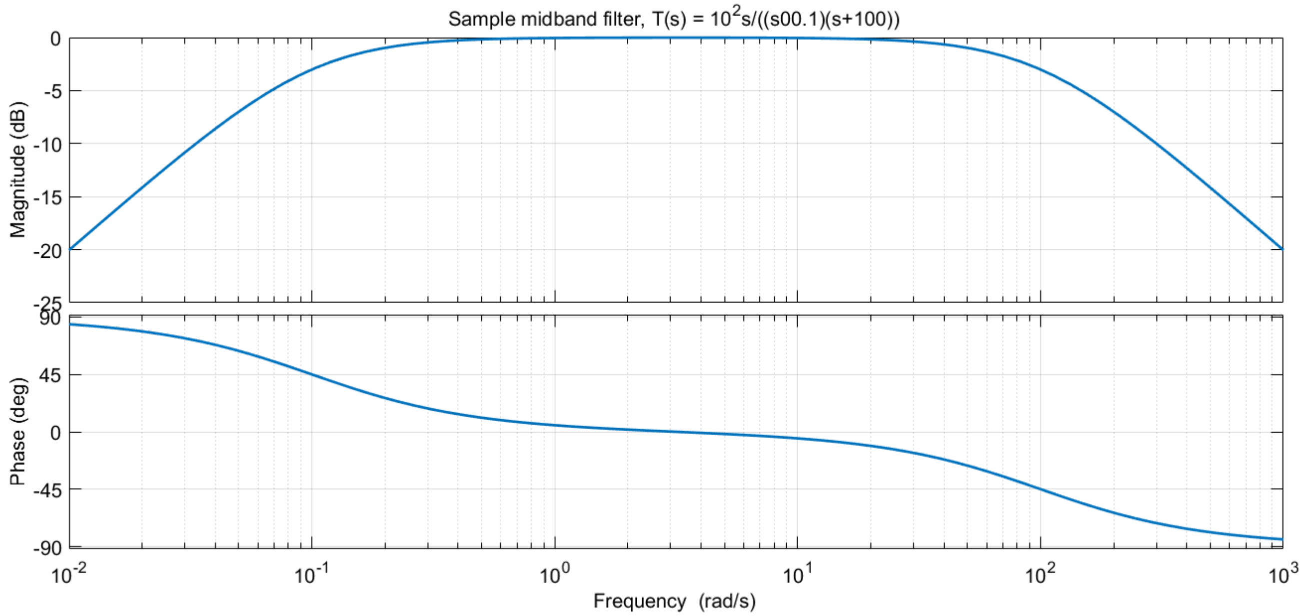


If you wanted stronger drops in the frequency, you could simply add more zeros near 0 and poles to counter them!!!

If you wanted a medium band filter, then you can add another pole later:

$$T(s) = \frac{as}{(s + \omega_0)(s + \omega_1)}$$

for $a, \omega_0, \omega_1 > 0$. Here is a sample Bode plot



This is all fair and good but it only works if we are working with sine waves.

Most functions are not sine waves..... but what if they (i.e. functions we care about) can be written as a linear combination of sine waves?

Fourier series:

Fourier series are obviously useful when working with systems, but they are also useful when working with signals on their own.

We will first work with a classic signal example that was a major problem in the days of the telegraph,

in the early days of landline phones,

in the early days of cell phones and the early Internet,

is the reason why we developed 5G,

and will continue to be a problem for the foreseeable future.

Lecture 23

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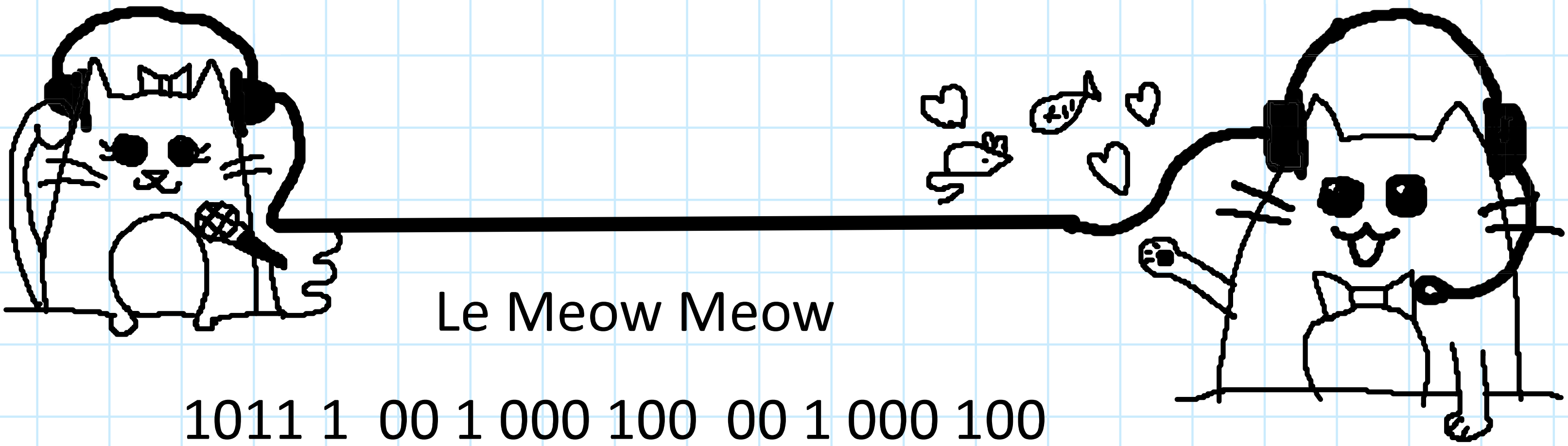
Fourier series!!!!

Suppose you want to send a message down a wire to a cat who lives in a different city.

You can do this via say some binary code!!



Le Meow Meow

1011 1 00 1 000 100 00 1 000 100

Your friend then finds out about your wire and also wants to send a message at the same time.

You say sure because what can possibly go wrong...

P2

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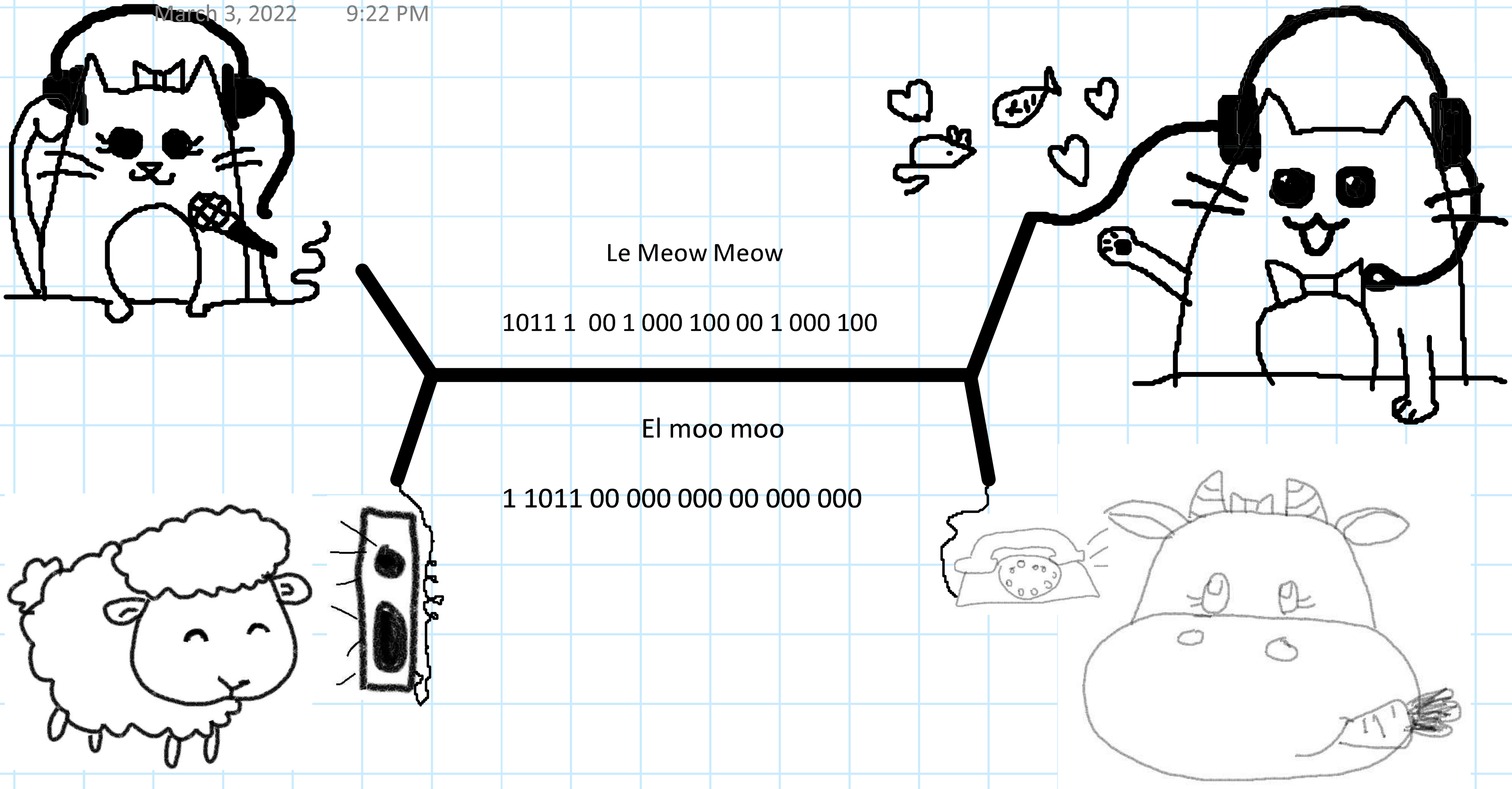

What comes out the other end???

11111101000100001000100

Or

I S T W K O W W in our code......

How can we fix this?

P3

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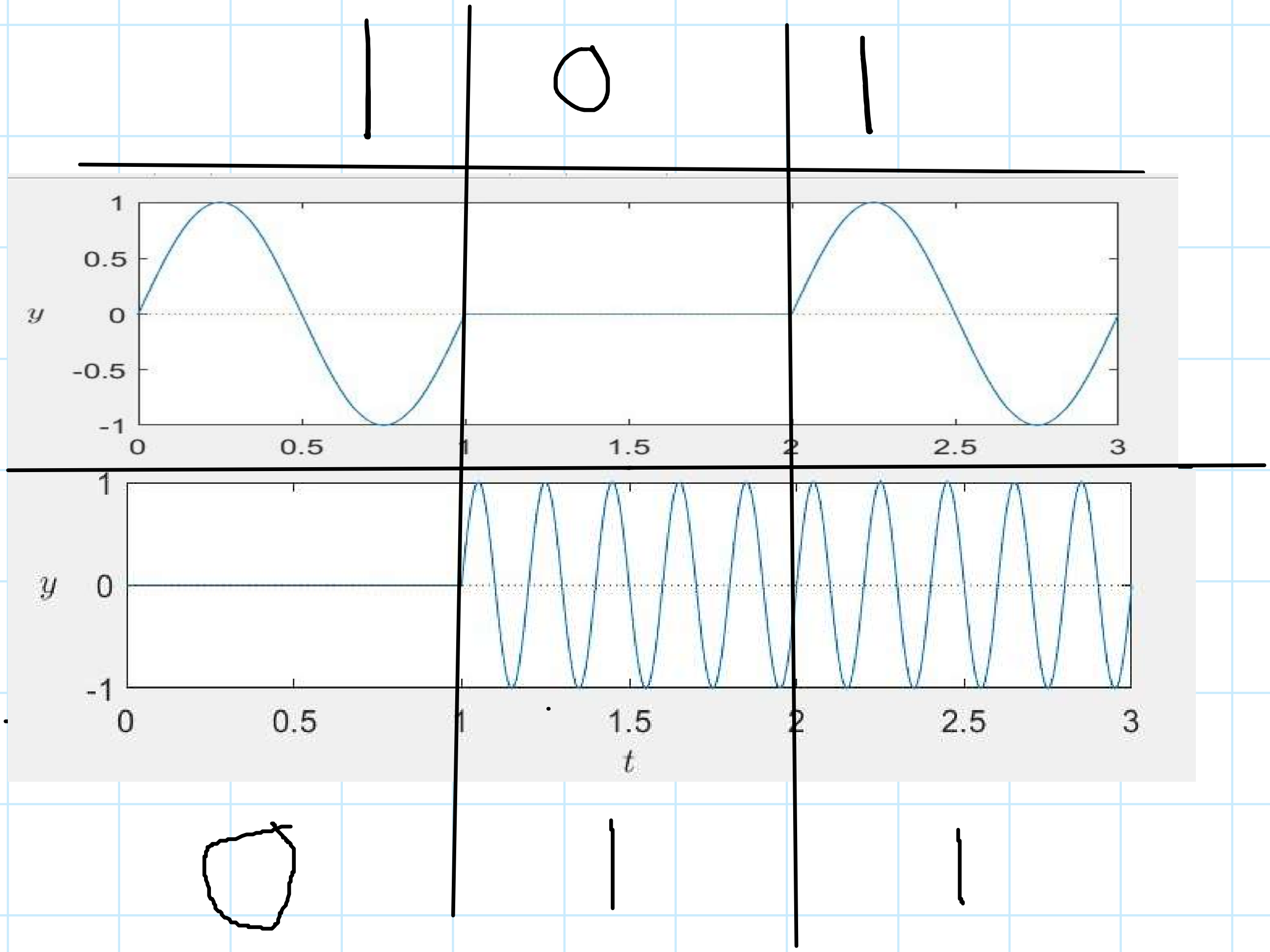
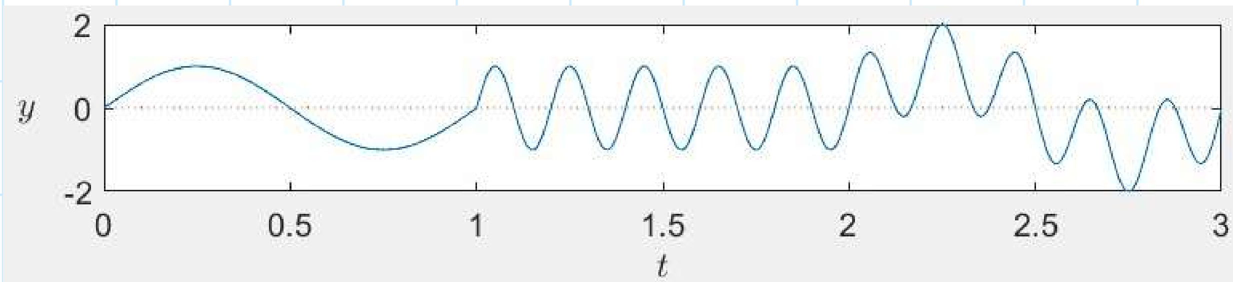
We can use different frequency waves!!!



When we add these together we get



If the recipients know the frequency to use then they can decode their message!!

"Recall" Taylor's theorem from MATH 119:

Theorem 1: Taylor's Theorem

Let $k \geq 1$ be an integer and let f be a real valued function that is differentiable at least k times at some point $a \in \mathbb{R}$. Then there exists a real valued function h_k such that

$$f(x) = \left(\underbrace{\sum_{i=0}^{k} \frac{f^{(i)}(a)}{i!}(x-a)^i}_{\text{Taylor Polynomial}} \right) + h_k(x)(x-a)^{k+1}$$

and $\lim_{x \to a} h_k(x) = 0$.

For infinitely differential functions f we can write

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Examples: $e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$, or $\sin(x) = \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i+1}}{(2i+1)!}$.

Here we used the basis $\{x^n | n \in \{0, 1, 2, \ldots\}\}$ but what if...

somehow... $\{\sin(nx) | n \in \{0, 1, 2, 3, \ldots\}\}$ could be used as a basis for some set of functions?

This would synchronize well with LTIs and it is what we will study for the rest of this course.

MATH 115 orthogonal basis "review" part 2 (see L3 for the version we used for Laplace transforms):

Suppose that there is a orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$.

This means that $\vec{v}_i \cdot \vec{v}_j = a_i \delta[i-j]$ where $a_i \neq 0$.

If we wanted to write $\vec{x} \in \mathbb{R}^n$ in this basis we would need to solve the system of equations

$$\vec{x} = x_1 \vec{v}_1 + \ldots + x_n \vec{v}_n.$$

We can solve this using orthogonality (as in L3). Explicitly to find a_i we take the dot product with $\vec{v}_i$:

$$\begin{aligned}
\vec{v}_i \cdot \vec{x} &= \vec{v}_i \cdot (x_1\vec{v}_1 + \ldots + x_n\vec{v}_n) & \\
&= x_1(\vec{v}_i \cdot \vec{v}_1) + \ldots + x_n(\vec{v}_i \cdot \vec{v}_n) & \text{Dot product is bilinear} \\
&= x_i(\vec{v}_i \cdot \vec{v}_i) & \text{Dot products other that } \vec{v}_i \cdot \vec{v}_i \text{ vanish} \\
&= x_i a_i & \vec{v}_i \cdot \vec{v}_i = a_i
\end{aligned}$$

Thus $x_i = \dfrac{\vec{v}_i \cdot \vec{x}}{a_i}$.

To make this method work for $\{\sin(nx) | n \in \{0, 1, 2, 3, \ldots\}\}$ we **MUST**

- Find a "dot product", called an inner product, for functions such that "$\sin(nx) \cdot \sin(mx) = a_n\delta[n-m]$" and compute the a_ns.
- Define (the different types of) convergence for series of **functions** i.e. $f(x) = \sum_{i=0}^{\infty} a_i \sin(nx)$.

To fft_fun.m

To voice analyzing phone thing!!

In subsequent lectures we will:

- Generalize the dot product to functions
- Learn how to use this new dot product to compute a few different versions of Fourier series (sin, cos, half sin, half cos and complex exponential),
- Talk about convergence and what the Fourier series for $f(x)$ converges to (it is not always $f(x)$!!) and
- Talk about the Fourier transform (which is just the 2-sided Laplace transform when $s = j\omega$).