MATH 213 - Lecture 21: L^2 space, inner product on L^2, and computing Fourier coefficients

Lecture goals: know what the standard inner product in L^2 is and know how to use it to compute Fourier coefficients of τ−periodic functions. Also get you to fill out the SCP survey.

In general properly defining a dot product for functions is a huge issue so we will only work with a special class of functions called "Lebesgue square integrable functions" or L^2 functions which makes things nice:

> **Definition 1: L^2 functions**
> A complex valued function f is in the class $L^2([a,b])$ if
>
> $$\int_a^b |f(x)|^2 dx$$
>
> exists and is finite.
>
> f is in the class L^2 if
>
> $$\int_{-\infty}^{\infty} |f(x)|^2 dx$$
>
> exists and is finite.

L^2 and $L^2([a,b])$ form vector spaces so ideas from MATH 115 can be used (with proper adjustments). Now if f is a member of $L^2([-\tau/2,\tau/2])$ for some fixed τ then our goal is to write

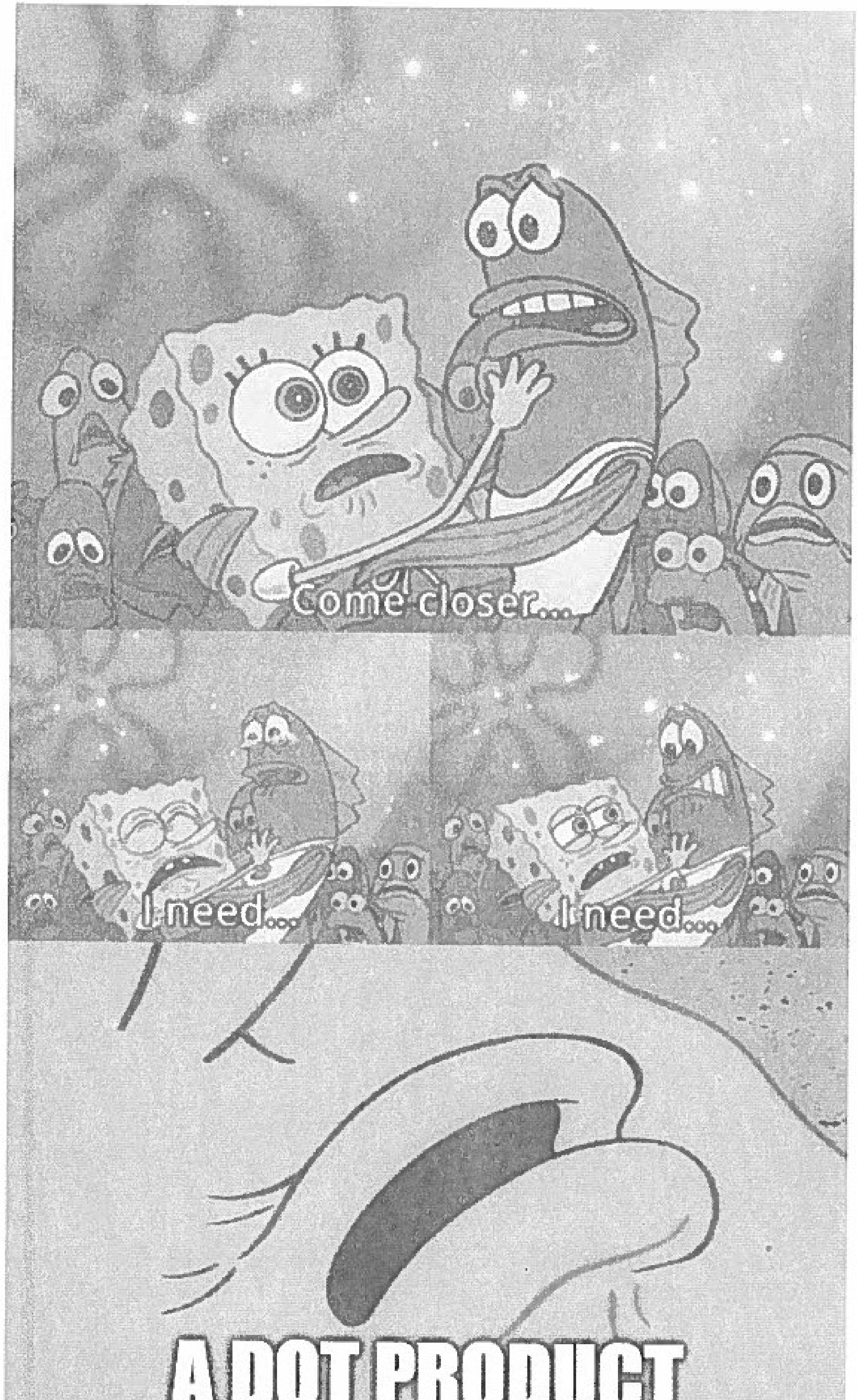
$$f(t) \text{ “}=\text{” } \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau} jt} \qquad \text{for} \qquad t \in [-\tau/2,\tau/2].$$

To do this we need to somehow solve for the c_ns....

By comparison with how we solved

$$\vec{b} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$$

for an orthogonal basis $\{v_1,\ldots,v_n\}$
We need a....


Come closer...
I need...
I need...
A DOT PRODUCT
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Inner product for $L^2([-\tau, \tau])$

Recall that if $\vec{x}, \vec{y} \in \mathbb{C}^n$ then

$$\vec{x} \cdot \vec{y} = x_1\overline{y_1} + x_2\overline{y_2} + \ldots + x_n\overline{y_n}.$$

Now if f and g are complex valued functions, then the "dot product", which will be called an inner product, should follow a similar definition.

The summation becomes integration!!

> **Definition 2: Standard Inner product on $L^2([a, b])$**
> If f and g are complex valued functions in $L^2([a, b])$ then the **standard inner product** is
> $$\langle f, g\rangle = \frac{1}{b-a}\int_a^b f(t)\overline{g(t)}dt.$$

> **Theorem 1: Existence of Inner Product**
> If $f, g \in L^2([a, b])$ then $\langle f, g\rangle$ exists and is finite.

We skip the proof since it needs some real analysis...

> **Theorem 2**
> The set of complex exponentials $\left\{e^{\frac{2\pi n}{\tau}jt} | n \in \{0, \pm 1, \pm 2, \ldots\}\right\}$ is an orthonormal basis for a subspace of $L^2([-\tau/2, \tau/2])$.

Partial proof: We will not prove that the collection is linearly independent but will prove that they are orthonormal.

$$\left\langle e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt}\right\rangle = \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{\frac{2\pi n}{\tau}jt} \cdot e^{-\frac{2\pi m}{\tau}jt} dt$$

$$= \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{\frac{2\pi(n-m)}{\tau}jt} dt$$

$$= \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Now that we have an orthonormal basis for a subset of $L^2([-\tau/2, \tau/2])$, we can project any function in $L^2(-\tau/2, \tau])$ into our basis $\{e^{\frac{2\pi n}{\tau}jt} | n \in \{0, \pm 1, \pm 2, \dots\}\}$ by using our inner product.

Note that since we are doing a projection and the basis may not be (is not...) a basis for $L^2([-\tau/2, \tau/2])$, the result of projecting into this basis may not be equal to the original function in the traditional sense.

> **Definition 3: Fourier Series - Complex Form**
> If $f \in L^2([-\tau/2, \tau/2])$ then the **Fourier series in complex form** of $f(t)$ is $\sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}$ where the c_n are found by projecting f into the basis of complex exponentials.

> **Theorem 3: Fourier Coefficients for Series in Complex Form**
> If $f \in L^2([-\tau/2, \tau/2])$ then the Fourier coefficients c_n of $f(t)$ are
>
> $$c_n = \left\langle f(t), e^{\frac{2\pi n}{\tau}jt} \right\rangle .$$
>
> If f is real valued than $c_n = \overline{c_{-n}}$.

Proof:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}$$

$$\left\langle f(t), e^{\frac{2\pi m}{\tau}jt} \right\rangle = \left\langle \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt} \right\rangle$$

Assuming Convergence "in the mean"

$$= \sum_{n=-\infty}^{\infty} c_n \left\langle e^{\frac{2\pi n}{\tau}jt}, e^{\frac{2\pi m}{\tau}jt} \right\rangle$$

$$= \sum_{n=-\infty}^{\infty} c_n \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

$$= c_m.$$

Compute the things!!

Example 1

Compute the Fourier series of $f : [-0.5, 0.5] \to \mathbb{R}$ defined by

$$f(t) = \sin(2\pi t)$$

If possible simplify the complex exponentials to real valued terms.

Sol: Note $\sin 2\pi t = \dfrac{e^{2\pi j t} - e^{-2\pi j t}}{2j}$

So since $\tau = 1$ we want

$$\sin 2\pi t = \sum_{n=-\infty}^{\infty} c_n e^{2\pi n j t}.$$

Thus

$$c_n = \begin{cases} \frac{1}{2j} & n = 1 \\ -\frac{1}{2j} & n = -1 \\ 0 & \text{else.} \end{cases}$$

Note $c_n = \overline{c_{-n}}$.

In terms of real valued terms the series is

$$\boxed{\sin(2\pi t)}$$

Example 2
Compute the Fourier series of $f : [-\tau/2, \tau/2] \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} -1 & t \in [-\tau/2, 0) \\ 1 & t \in [0, \quad \tau/2] \end{cases}$$

If possible simplify the complex exponentials to real valued terms. Plot $f(t)$ along with several terms of the Fourier series.

Sol:
$$c_n = \left\langle f(t), e^{\frac{2\pi n}{\tau} jt} \right\rangle$$

$$= \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} f(t)\, e^{-\frac{2\pi n}{\tau} jt}\, dt$$

$$= \frac{1}{\tau} \left[\int_{-\frac{\tau}{2}}^{0} -e^{-\frac{2\pi n}{\tau} jt}\, dt + \int_0^{\tau/2} e^{-\frac{2\pi n}{\tau} jt}\, dt \right]$$

if $n \neq 0$

note $\int e^{-\frac{2\pi n}{\tau} jt}\, dt = \frac{-\tau}{2\pi n j} e^{-\frac{2\pi n}{\tau} jt}$

$$= \frac{1}{\tau} \left[\frac{+\tau}{2\pi n j} e^{-\frac{2\pi n}{\tau} jt} \Big|_{-\frac{\tau}{2}}^{0} + \frac{-\tau}{2\pi n j} e^{-\frac{2\pi n}{\tau} jt} \Big|_{0}^{\tau/2} \right]$$

$$= \frac{1}{\tau} \left[\frac{+\tau}{2\pi n j} \left(1 - e^{-\pi n j} \right) + \frac{-\tau}{2\pi n j} \left(e^{+\pi n j t} - 1 \right) \right]$$

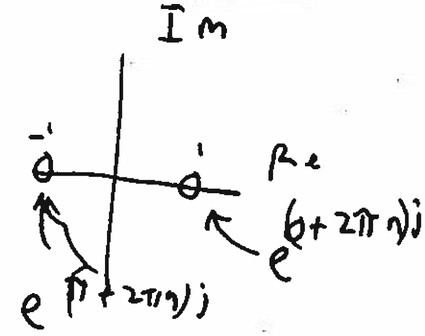
$$= \frac{1}{2\pi n j} \left[-e^{+\pi n j} + 1 - e^{-\pi n j} + 1 \right]$$

$$= \frac{1}{2\pi n j} \left[+2 - \left[e^{\pi n j} + e^{-\pi n j} \right] \right]$$

n is an integer &

$$e^{\pi n j} = (-1)^n$$

$$e^{-\pi n j} = (-1)^n$$



so

$$c_n = \frac{1}{2\pi j}\left[+2 - 2(-1)^n\right]$$

$$= \frac{1}{\pi n j}\left[1 - (-1)^n\right]$$

if $n=0$ then $c_0 = \frac{1}{\tau}\left[\int_{-\frac{\tau}{2}}^{0} -1\,dt + \int_{0}^{\tau/2} 1\,dt\right]$

$$= 0.$$

so

$$c_n = \begin{cases} 0 & n=0 \\ \frac{1}{\pi n j}\left[+1 - (-1)^n\right] & n \neq 0 \end{cases}$$

& $\boxed{f(t) \sim \sum_{n=-\infty}^{\infty} c_n e^{j\frac{2\pi n}{\tau}t}}$

note $\left[c_{-n} e^{-\frac{j2\pi n}{\tau}t} + c_n e^{\frac{j2\pi n}{\tau}t}\right] =$

$$= \frac{-1}{n\pi j}\left[1-(-1)^n\right]e^{-\frac{2\pi n}{\tau}jt} + \frac{1}{n\pi j}\left[1-(-1)^n\right]e^{\frac{2\pi n}{\tau}jn}$$

$$= \frac{1}{n\pi j}\left[1-(-1)^n\right]\cdot\left[e^{\frac{2\pi n}{\tau}jt} - e^{-\frac{2\pi n}{\tau}jt}\right]$$

Recall $\sin t = \frac{e^{jt}-e^{-jt}}{2j}$

$$= \frac{2(1-(-1)^n)}{n\pi}\left[\frac{e^{\frac{2\pi n}{\tau}jt} - e^{-\frac{2\pi n}{\tau}jt}}{2j}\right]$$

$$= \frac{2(1-(-1)^n)}{n\pi}\left[\sin\frac{2\pi n}{\tau}t\right]$$

if n is odd $= \frac{4}{n\pi}$

if n is even $= 0$.

$$f(t) \sim \frac{4}{\pi}\sum_{n=1,3,5\ldots}\frac{1}{n}\sin\left(\frac{2\pi n}{\tau}t\right).$$

Example 3

Compute the Fourier series of $f : [-\pi, \pi] \to \mathbb{R}$ defined by

$$f(t) = \frac{1}{2}(\pi - |t|)$$

If possible simplify the complex exponentials to real valued terms. Plot $f(t)$ along with several terms of the Fourier series.

$\tau = 2\pi$

$$c_n = \langle f(t), e^{\frac{2\pi n j t}{\tau}} \rangle$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}(\pi - |t|)\, e^{-njt}\, dt$$

$$= \frac{1}{4\pi}\left[\int_{-\pi}^{\pi} \pi e^{-njt} - \left[\int_{-\pi}^{0} -t e^{-njt}\, dt + \int_{0}^{\pi} t e^{-njt}\, dt\right]\right]$$

math 119

$$= \frac{1}{4\pi}\left[\pi\left(\frac{e^{-nj\pi}}{-nj} - \frac{e^{+nj\pi}}{-nj}\right) - \left[\frac{-1 + e^{j\pi n}[1 - j\pi n]}{n^2} + \frac{-1 + e^{-j\pi n}[1 + j\pi n]}{n^2}\right]\right]$$

$n \in \mathbb{Z}$

$$= \frac{1}{4\pi}\left[\pi\left(\frac{(-1)^n - (-1)^n}{-nj}\right) - \left[\frac{-2 + (-1)^n[1 - j\pi n] + (-1)^n[1 + j\pi n]}{n^2}\right]\right]$$

0

0

$$= \frac{1}{4\pi}\left[\frac{2 - 2(-1)^n}{n^2}\right] = \frac{1 + (-1)^{n+1}}{2\pi n^2}$$

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(\pi - |t|\right) e^{-0jt}\, dt$$

$$= \cdots = \frac{\pi}{4}$$

so

$$f(t) \sim \frac{\pi}{4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1+(-1)^{n+1}}{2\pi n^2}\, e^{njt}$$

Real version

$$f(t) \sim \frac{\pi}{4} \quad \sum_{n=0}^{\infty} \frac{2}{\pi(2n+1)^2} \cdot \cos\left(2n+1\right)t$$