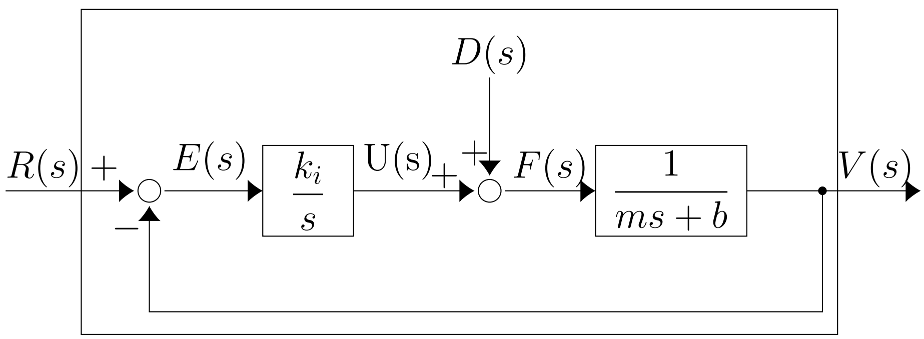
MATH 213 - Lecture 16: Standard 2nd order system, PI controllers and extra poles

Lecture goals: Understand the basics of the standard second order system, Know what a PI controller is.

The system diagram for the integrally controlled car cruise control problem from last lecture



and we found that the transfer function for the controlled system (the big box) is

$$H(s) = \frac{k_i/m}{s^2 + \frac{b}{m}s + \frac{k_i}{m}}.$$

Systems with transfer functions of this form are common so we will introduce and analyze the standard second order system

Definition 1: Standard second order system
The standard second order system has a transfer function given by

$$H(s) = \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}$$

where $\omega > 0$ and $\xi \geq 0$.

Examples of these systems include cruise control with integral control, the harmonic oscillator (with or without damping), and the RLC circuit.

We will now analyze the behaviour of the possible behaviours of the standard second order system. By doing this, we validate the analysis of the cruse control system with an integral controller from the last lecture.

The quadratic formula tells us that the poles of $H(s)$ are

$$s_\pm = -\xi\omega \pm \omega\sqrt{\xi^2 - 1}.$$

There are hence three different forms of the solution based on where the poles lie.

Case 1: Two real distinct pole. In this case $\sqrt{\xi^2-1}$ is real **and** greater than 0 then there are two distinct real valued poles.

Since $\xi > 0$ for the standard second order system, this happens when

$$\xi^2 - 1 > 0 \qquad \text{or} \qquad \xi > 1.$$

In this case we decompose $H(s)$ via PF as

$$H(s) = \frac{A_+}{s - s_+} + \frac{A_-}{s - s_-}$$

for some $A_+, A_- \in \mathbb{R}$.

Thus in this case the standard second order system can be decomposed as a sum of two first order systems!!

To show that these are well behaved first order systems (i.e. remain bounded for bounded inputs), first note that for $\xi > 1$,

$$\xi > \sqrt{\xi^2 - 1}.$$

Hence,

$$\begin{aligned} s_\pm &= -\xi\omega \pm \omega\sqrt{\xi^2 - 1} \\ &= \omega(-\xi \pm \sqrt{\xi^2 - 1}) \\ &< 0. \end{aligned}$$

Thus the poles of the transfer function have negative real parts and therefore any input that is bounded (i.e. has no poles with positive real part) cannot resonate to cause the system response to be unbounded.

For completion, we will find the system's impulse response. Using the coverup method and simplifying gives

$$A_{\pm} = \frac{\pm\omega}{2\sqrt{\xi^2 - 1}}$$

and hence the system's impulse response is

$$h(t) = A_+ e^{s_+ t} + A_- e^{s_- t}$$

which does decay as $t \to \infty$!

Relating this to A3Q1, this is the overdamped case of the harmonic oscillator.

Case 2: One real repeated pole: If $\xi^2 - 1 = 0$, or simply if $\xi = 1$, then there is a single repeated pole at

$$s_{\text{root}} = -\omega.$$

This root has a negative real part and hence the system is again well behaved.

For completion we compute the system's impulse response to examine the solution. In this case

$$H(s) = \frac{\omega^2}{(s - \omega)^2}.$$

Hence the system's impulse response is simply

$$h(t) = \omega^2 t e^{-\omega t}$$

Relating this to A3Q1, this is the critically damped case of the harmonic oscillator.

Case 3: Two complex conjugate poles with a non-zero real component: If $\xi^2 - 1 < 0$, or simply if $0 < \xi < 1$, then we have complex conjugate poles at

$$s_\pm = -\xi\omega \pm \omega j\sqrt{1-\xi^2}$$

where for ease in future computations, we pulled out the negative from inside the square root.

These roots have negative real parts and hence the system is again well behaved.

We now find the system's impulse response. We know that the solution will be decaying sine waves so we prepare the transfer function to use the exponential modulation rule and the sine transform.

$$\begin{aligned}
H(s) &= \frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2} \\
&= \frac{\omega^2}{(s+\xi\omega)^2 + \omega^2 - \xi^2\omega^2} && \leftarrow \text{complete the square or just note the roots} \\
&= \frac{\omega^2}{(s+\xi\omega)^2 + \omega^2(1-\xi^2)} \\
&= \frac{\omega^2}{(s+\xi\omega)^2 + (\omega\sqrt{(1-\xi^2})^2}
\end{aligned}$$

from the LT table we have

$$h(t) = \frac{\omega}{\sqrt{1-\xi^2}} e^{-\xi\omega t} \sin((\omega\sqrt{1-\xi^2})t)$$

Relating this to A3Q1, this is the underdamped case of the harmonic oscillator.

Case 4: Two complex conjugate poles with a zero real component: If $\xi^2 - 1 = 0$, or simply if $\xi = 0$, then we have complex conjugate poles at

$$s_\pm = \pm\omega j$$

These roots have zero real parts and hence the system may not be well behaved is there is resonance!

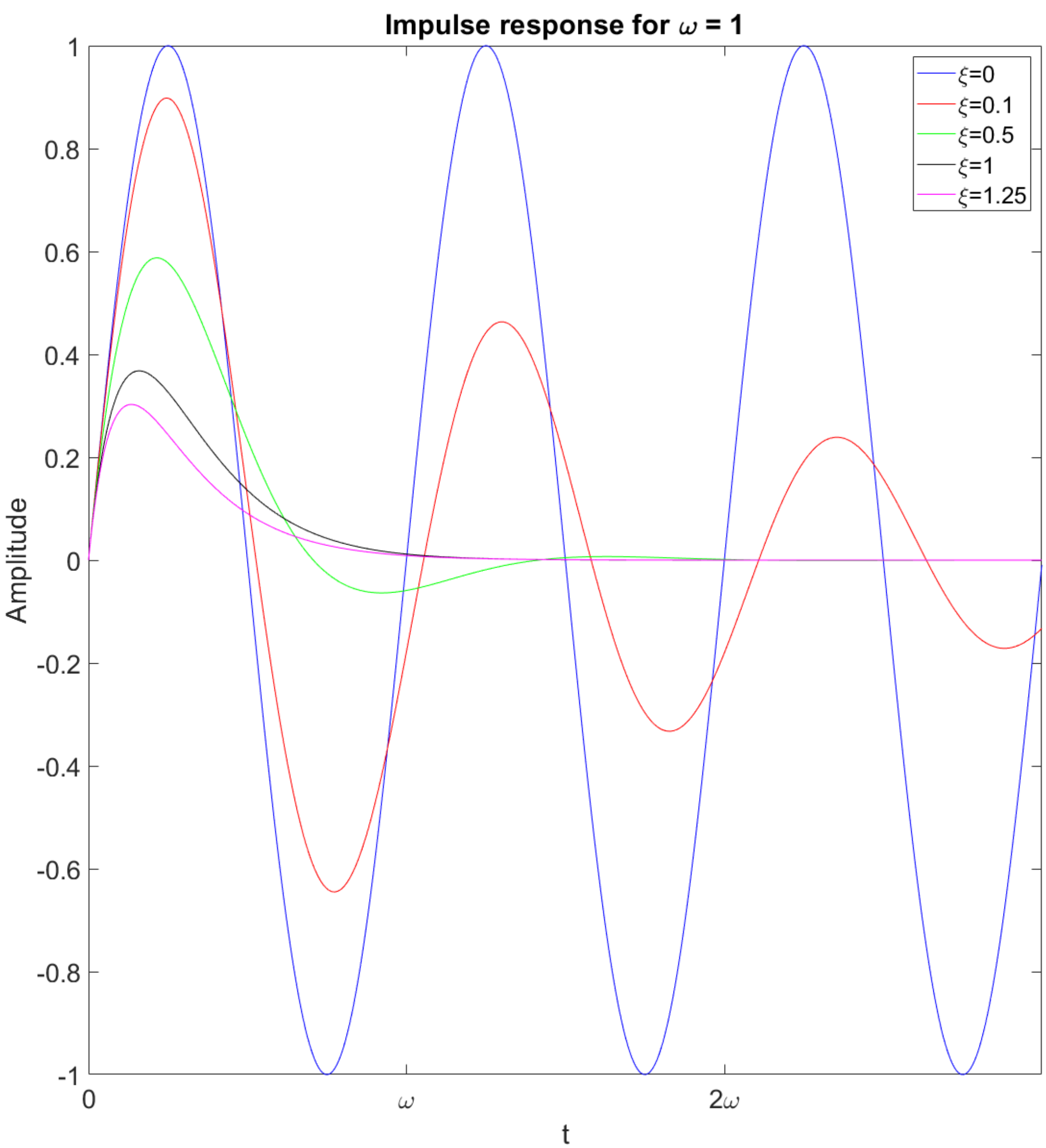
We now find the system's impulse response.

$$H(s) = \frac{\omega^2}{s^2 + \omega^2}$$

so from the LT table we have

$$h(t) = \omega \sin(\omega t)$$

Relating this to A3Q1, this is the undamped case of the harmonic oscillator. We can now plot the impulse response to all these systems. Here is a plot!!

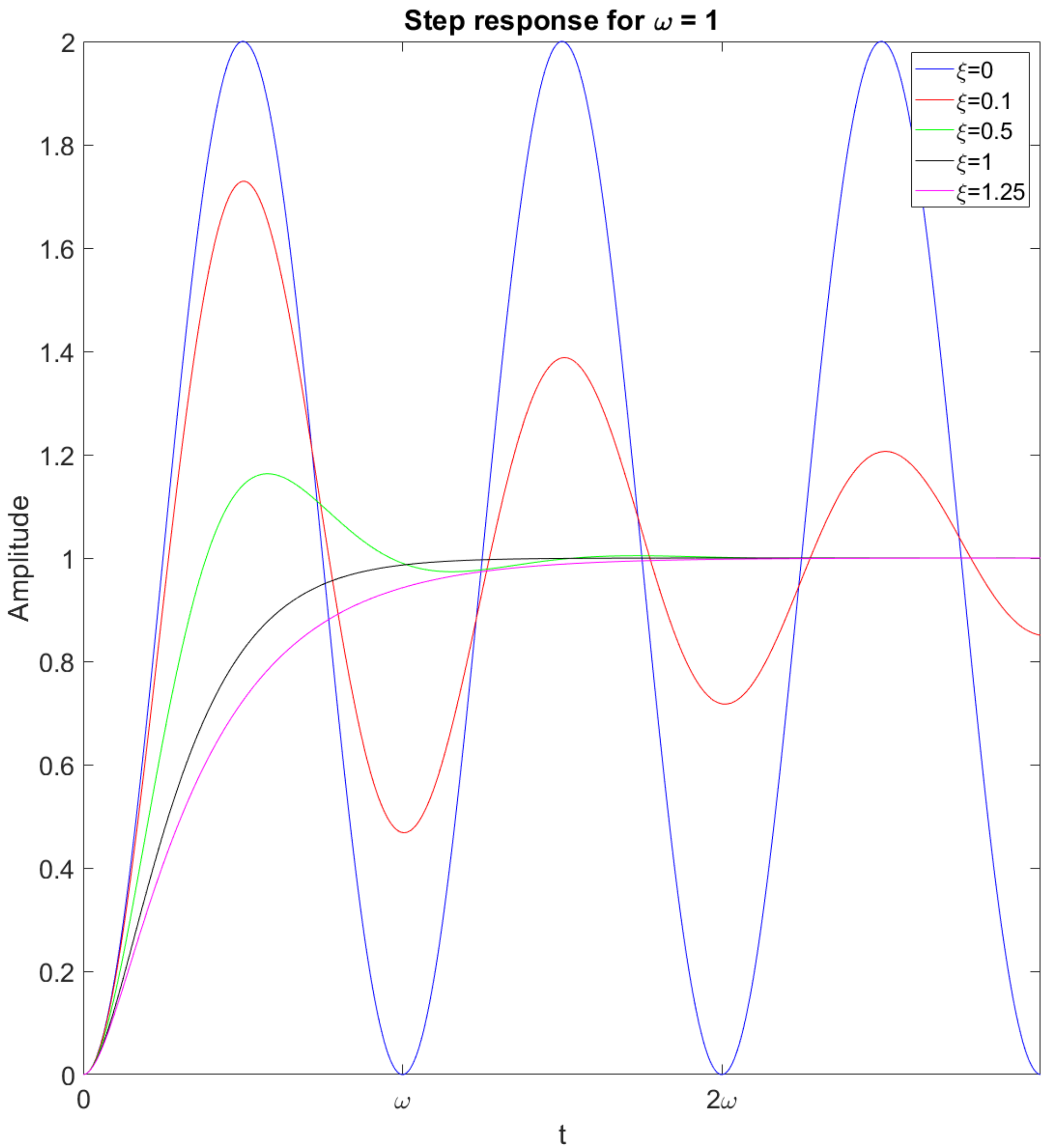


See the L16 .m file for a video!

We can also compute the step responses and plot them, we will not show the computations but the step response for $\xi \geq 0$ other than $\xi \neq 1$ is

$$1 - \frac{1}{\sqrt{1-\xi^2}} e^{-\xi\omega t} \sin\left((\omega\sqrt{1-\xi^2})t + \theta\right)$$

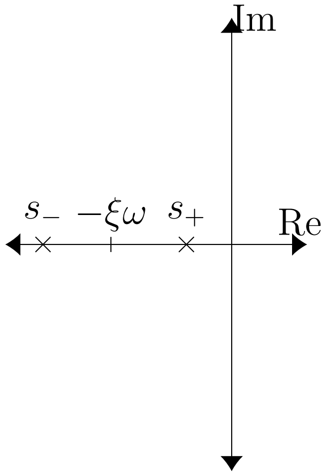
where $\theta = \cos^{-1}\xi$. At $\xi = 1$ we have a t scaled exponential. Here is a plot of some sample solutions



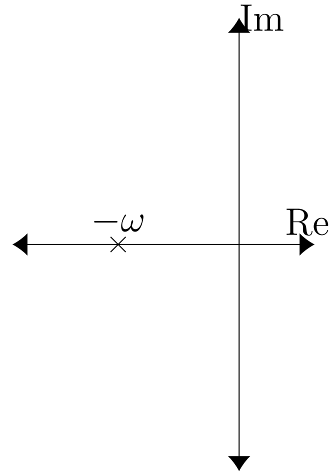
See the L16 .m file for a video!

Pole location summary (poles are marked with an $\times$):

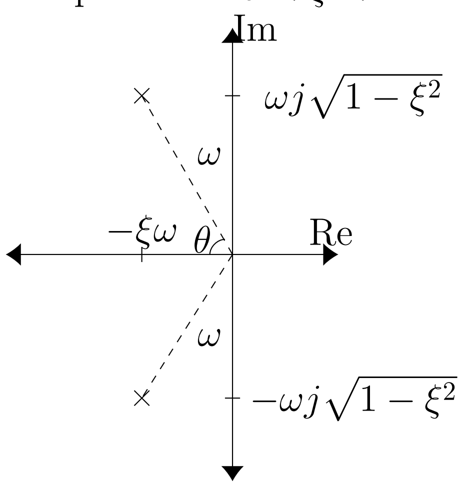
Overdamped case: $\xi > 1$



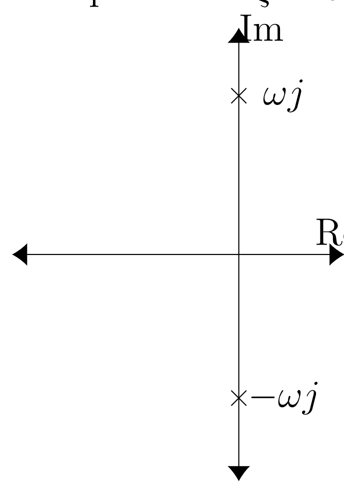
Critically damped case: $\xi = 1$



Underdamped case: $0 < \xi < 1$



Undamped case: $\xi = 0$



In the underdamped cases:

If ω increases the poles move further from the origin and we have a faster response.

If ξ increases, θ decreases and we have more damping.

Matlab video!!! See Lecture 16 .m file for a nice video of the poles and the step response.

Example 1: Cruise control critical dampening

Recall that the transfer function for an integrally controlled car with mass m, drag coefficient b and integral controller constant k_i is

$$H_{RS}(s) = \frac{k_i/m}{s^2 + \frac{b}{m}\ s + \frac{k_i}{m}}.$$

Find k_i such that the curse control system is critically damped and hence adjust to the response $r(t) = v$ for $v \in \mathbb{R}_{\geq 0}$ as fast as possible without experiencing overshot.

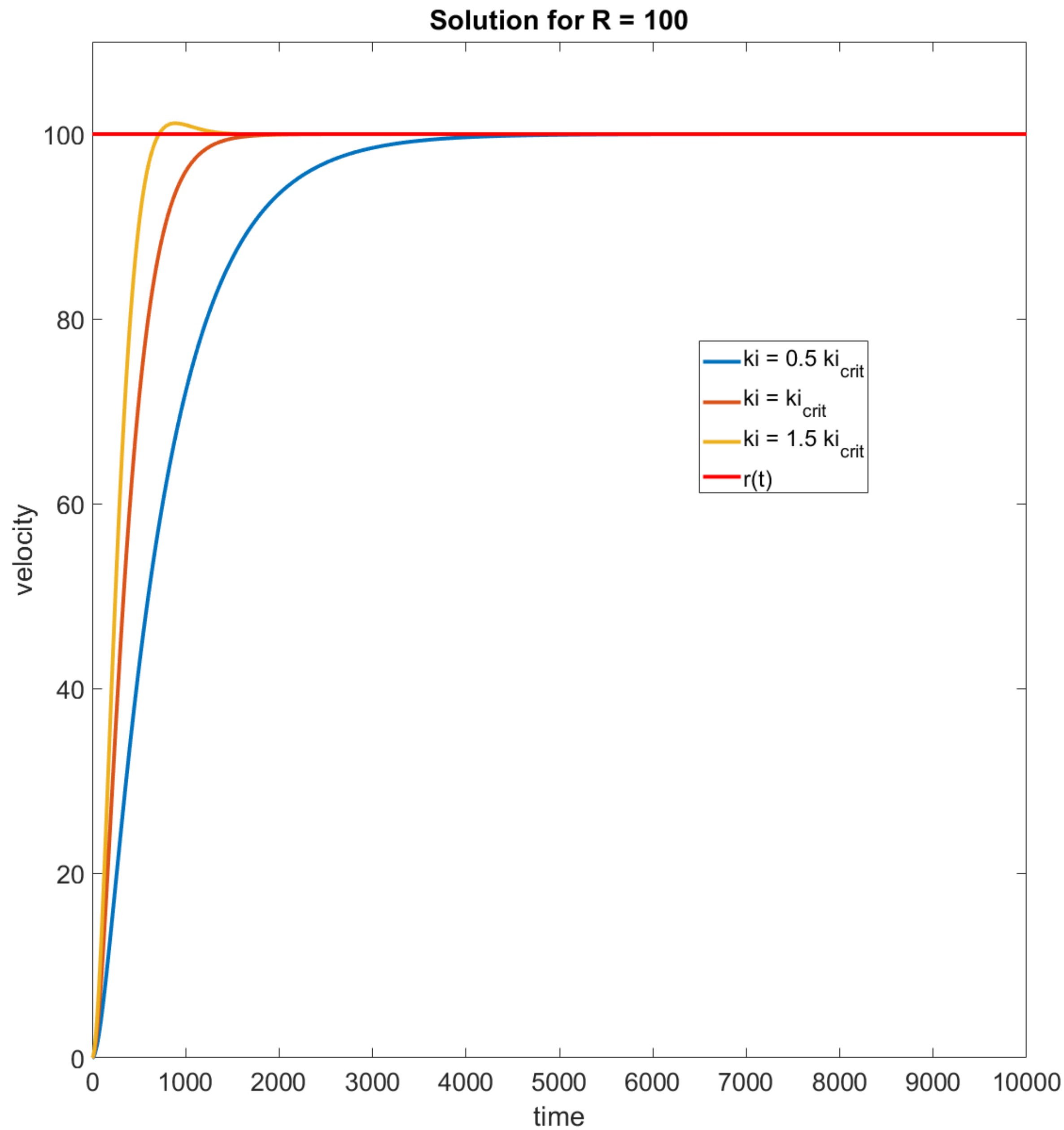
Solution: This is a second order system with $\omega = \sqrt{k_i/m}$ and

$$\begin{aligned} 2\xi\omega &= \frac{b}{m} \\ \xi &= \frac{b}{2m\omega} \\ &= \frac{b}{2\sqrt{mk_i}} \end{aligned}$$

To be critically damped we need $\xi = 1$ and thus k_i needs to satisfy

$$\begin{aligned} 1 &= \frac{b}{2\sqrt{mk_i}} \\ \sqrt{k_i} &= \frac{b}{2\sqrt{m}} \\ k_i &= \frac{b^2}{4m} \end{aligned}$$

Here is a plot of the behaviour of the controlled system's response for values of k_i around this critical value:

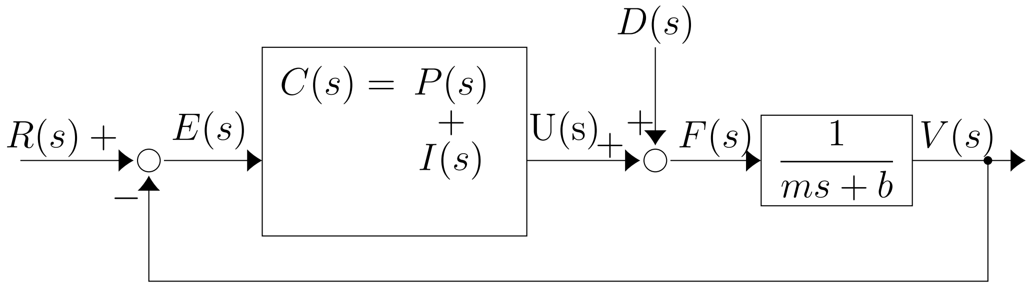


See the second L16 .m file if you want to explore this more!

PI controller:

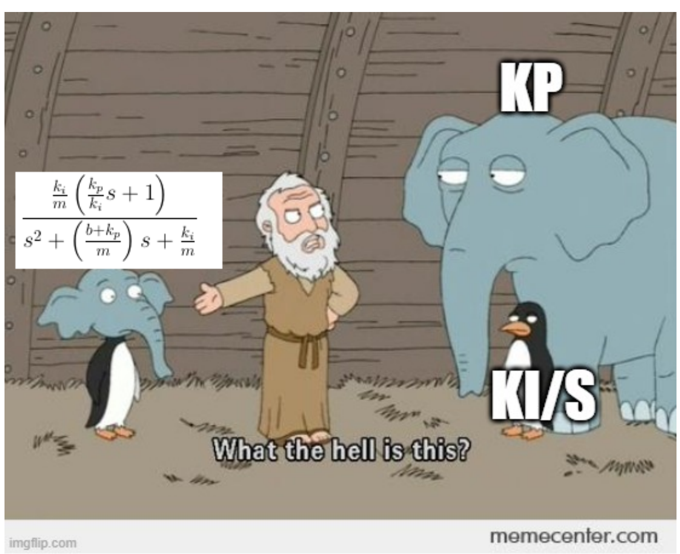
For the cruise control problem, the P controller by itself gets us to a close velocity quickly but misses the exact velocity and the I controller by itself gets us to the exact velocity but is slow to do so.

Let's use them together!!



The transfer function for this system is

$$\begin{aligned} H_{RV}(s) &= \frac{C(S)S(s)}{1 + C(s)S(s)} && \leftarrow \text{ Previous result} \\ &= \frac{\left(k_p + \frac{k_i}{s}\right)\frac{1}{ms+b}}{1 + \left(k_p + \frac{k_i}{s}\right)\frac{1}{ms+b}} \\ &= \frac{\frac{k_i}{m}\left(\frac{k_p}{k_i}s + 1\right)}{s^2 + \left(\frac{b+k_p}{m}\right)s + \frac{k_i}{m}} && \leftarrow \text{ Algebra} \end{aligned}$$



Next lecture we will analyze this transfer function and how its zeros and poles change the system response and then look at more complex transfer functions.