MATH 213 - Assignment 5 - Solutions

Submit to Crowdmark by 9:00pm EST on Friday, March 22.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Read before starting the assignment: For this assignment you must do all your work independently and without the use of external aids.
Questions:

1. (5 marks) In tutorial we showed that a proportional controller could be used to stabilize a system. It is also the case that such a controller can cause a BIBO stable system to become unstable! Suppose we are applying a proportional controller to a BIBO stable system with transfer function

$$T(s) = \frac{s+a}{s^2+bs+c}, \qquad a,b,c \in \mathbb{R}.$$

Find and analyze the poles of the transfer functions of the controlled and uncontrolled systems to find conditions on k_p and potentially the coefficients a, b and/or c such that the proportionally controlled system is **not** BIBO stable while uncontrolled system **is** BIBO stable.

Solution: For the uncontrolled system to be BIBO stable, we need a, b and c to be such that

$$Re\left(\frac{-b \pm \sqrt{b^2-4c}}{2}\right) < 0 \qquad \text{or to have one root with } Re \geq 0 \text{ and for } s+a \text{ to cancel that root.}$$

We do case analysis on b (which controls the position of the vortex of the quadratic form) to find the stability conditions:

- If $b > 0$ then:
  - The system is stable if $b^2 - 4c < b^2$ which simplifies to $c > 0$.
  - The complementary case is $c \leq 0$ and in this case we need $a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$ to remove the $+$ root and maintain stability.
- If $b = 0$ then the poles are

$$s_\pm = \frac{\pm\sqrt{-4c}}{2} \qquad \text{or} \qquad s_\pm = \pm\sqrt{-c}$$

  so we need $c < 0$ and $a = -\sqrt{-c}$.
- If $b < 0$ then $-b > 0$ so for stability we at a minimum need the poles to be real and given real roots we then need

$$\underbrace{-b}_{\text{Positive number}} \quad -\sqrt{b^2-4c} < 0$$

  and $a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$ to both hold. The condition on a is needed to cancel the positive pole that can't be removed by adjusting c. The inline condition gives us that $c < 0$.

In summary the conditions for stability are (where we combined the $b = 0$ and $b \leq 0$ cases)

- $b > 0, c > 0, a \in \mathbb{R}$
- $b > 0, c \leq 0, a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$
- $b \leq 0, c < 0, a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$

Now the transfer function for the proportionally controlled system is

$$\begin{aligned} T_c(s) &= \frac{k_p \frac{s+a}{s^2+bs+c}}{1 + k_p \frac{s+a}{s^2+bs+c}} \\ &= \frac{k_p(s+a)}{s^2+bs+c+k_p(s+a)} \\ &= \frac{k_p(s+a)}{s^2+(b+k_p)s+c+k_pa} \end{aligned}$$

The poles of the controlled system are

$$s_{\pm} = \frac{-(b+k_p) \pm \sqrt{(b+k_p)^2 - 4(c+k_p a)}}{2}$$

and the zero is still $s = -a$. For instability we either need both roots to have positive real parts or to have one root with a positive real part and for the zero to not cancel it. We will examine the previous cases to find the conditions on k_p.

- If $b > 0, c > 0, a \in \mathbb{R}$:

  In this case a, b, and c are given constants and we have control of $k_p > 0$. Now since $b + k_p > 0$, for instability we just need the square root term to be larger in magnitude than $b + k_p$ to push the pole into the right hand plane. Hence we need

  $$(b+k_p)^2 - 4(c+k_p a) > (b+k_p)^2 \quad \text{or} \quad -4(c+k_p a) > 0$$

  If $a \geq 0$ then this condition never holds and if $a < 0$ then we solve the inequality to see that

  $$k_p > -\frac{c}{a}.$$

  Since the vortex of this parabola is at $-0.5(b+k_p) < 0$, the other root will be negative and a can sometimes remove the pole. Hence the new system is unstable if:

  $$a < 0,\ k_p > -\frac{c}{a}, \qquad \text{and} \qquad a \neq -s_+$$

  where s_+ is the unstable pole for the given a, b, c and k_p values. In particular in this situation, most values of k_p that satisfy the above inequality will cause an instability

- If $b > 0, c \leq 0, a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$ then like the previous case there will at most be one real root with real part ≥ 0. Thus we will still have the same condition that

  $$-4(c + k_p a) > 0$$

  but now $a < 0$ and is fixed while $c < 0$. Hence the above inequality always holds. Note that we cannot pick a to remove the pole since it is already been used to remove a pole in the uncontrolled system. We now need to check if this $-a$ is also equal to the unstable pole in the controlled system. This happens when

  $$\begin{aligned} -a &= \frac{-(b+k_p) + \sqrt{(b+k_p)^2 - 4(c+k_p a)}}{2} \\ -2a + b + k_p &= \sqrt{(b+k_p)^2 - 4(c+k_p a)} \\ (-2a + b + k_p)^2 &= (b+k_p)^2 - 4(c+k_p a) \\ 4a^2 - 4a(b+k_p) + (b+k_p)^2 &= (b+k_p)^2 - 4(c+k_p a) \\ 4a^2 - 4ab &= -4c \\ a^2 - ab + c &= 0 \end{aligned}$$

  The solutions to this are *gasp* the roots we found before. Hence in this case, the controlled system is stable by pole cancellation!

- If $b \leq 0, c < 0, a = -\left(\frac{-b+\sqrt{b^2-4c}}{2}\right)$ then like in the above case, a cannot be freely used to remove the new pole since it will only cancel the poles if it happens by happenstance. In this case both a and c are negative so $-4(c+k_p a) > 0$ and hence the square root term is real and greater than $b+k_p$. Hence we will have one positive real root and one negative real root. The computations to show that the $-a$ was one of the poles of the new system did not rely on the sign of b. Hence, this will remain the case in this case. The controlled system is therefore also stable in these cases

In summery for the controlled system to be stable and the uncontrolled system to be unstable we need

$$b > 0, c > 0, a < 0, k_p > -\frac{c}{a} \qquad \text{and} \qquad a \neq \frac{-(b+k_p) + \sqrt{(b+k_p)^2 - 4(c+k_p a)}}{2}$$

2. (4 marks) Consider the system with transfer function

$$H(s) = \frac{(s+20)^6}{((s+1)((s+20)^2+9)(s+2)((s+21)^2+16)(s+46)^2}.$$

Find all of the poles of $H(s)$. Identify with justification the dominate pole(s) and then use these poles to find $h_{approx}(t)$ such that $h_{approx}(t) \approx h(t)$.

Solution: The plots of $H(s)$ are:

$$-1, \qquad -20 \pm 3j, \qquad -2, \qquad -21 \pm 4j, \qquad \text{and} \qquad -46$$

The poles -1 and -2 are of the same magnitude and dominate all the other poles. Hence we just need to find the contributions of these two poles. We thus need to find A and B such that

$$H(s) = \frac{A}{s+1} + \frac{B}{s+2} + \text{other non-dominant terms}$$

Using the coverup method gives

$$A = \frac{(-1+20)^6}{((-1+20)^2+9)(-1+2)((-1+21)^2+16)(-1+46)^2} = \frac{19^6}{311688000}$$

and

$$B = \frac{(-2+20)^6}{(-2+1)((-2+20)^2+9)((-2+21)^2+16)(-2+46)^2} = \frac{18^6}{243047376}$$

Hence

$$h_{approx}(t) = \frac{19^6}{311688000}e^{-t} - \frac{18^6}{243047376}e^{-2t}$$

3. (9 marks) Manually construct the Bode plot of

$$H(s) = (s+0.1) \cdot \frac{10^6}{((s+1)^2+100)} \cdot \frac{1}{s+1000}$$

for $\log_{10}(\omega)$ ranging from -2 to 6.

Note: I am **not** looking for an exact plot. You need to plot the asymptotic behaviours (blue/red lines I drew in class), approximately correct timescales for the adjustments to the asymptotics, and show how you use this to construct the approximate curves. You need to show all your work but can use the results from class where applicable.

For ease of plotting, a blank Bode plot is included at the end of this assignment.

Solution: For the initial magnitude, we have

$$\begin{aligned}
H(0) &= (0+0.1)\cdot\frac{10^6}{((0+1)^2+100)}\cdot\frac{1}{0+1000} \\
&\approx (10^{-1})\cdot\frac{10^6}{10^2}\cdot\frac{1}{10^3} \\
&\approx 10^{6-1-2-3} \\
&= 1
\end{aligned}$$

Hence $|H(0)|_{dB} = 0$. Further the initial phase is

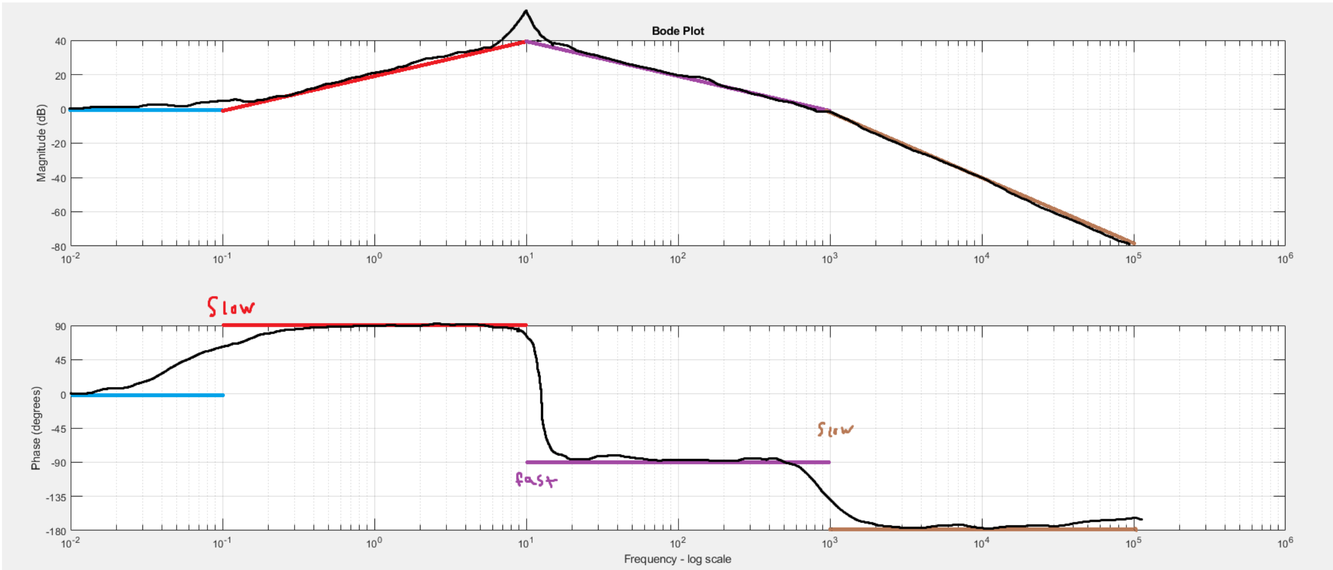
$$\begin{aligned}
\angle H(j\omega) &= \angle\left((j\omega+0.1)\cdot\frac{10^6}{((j\omega+1)^2+100)}\cdot\frac{1}{j\omega+1000}\right) \\
&= \angle(10^6(j\omega+0.1)) - \angle((j\omega+1)^2+100) + \angle(1) - \angle(j\omega+1000) \\
&\overset{\text{as } \omega\to 0}{\approx} 90^\circ - 90^\circ + 90^\circ - 90^\circ \\
&= 0^\circ
\end{aligned}$$

There is a zero of order 1 at 10^{-1}. Hence before this zero both the phase and magnitude will approximately be constant (blue lines). After this zero, the amplitude will have a gain of 20dB/Decade and the phase will slowly increase 90° (red lines).

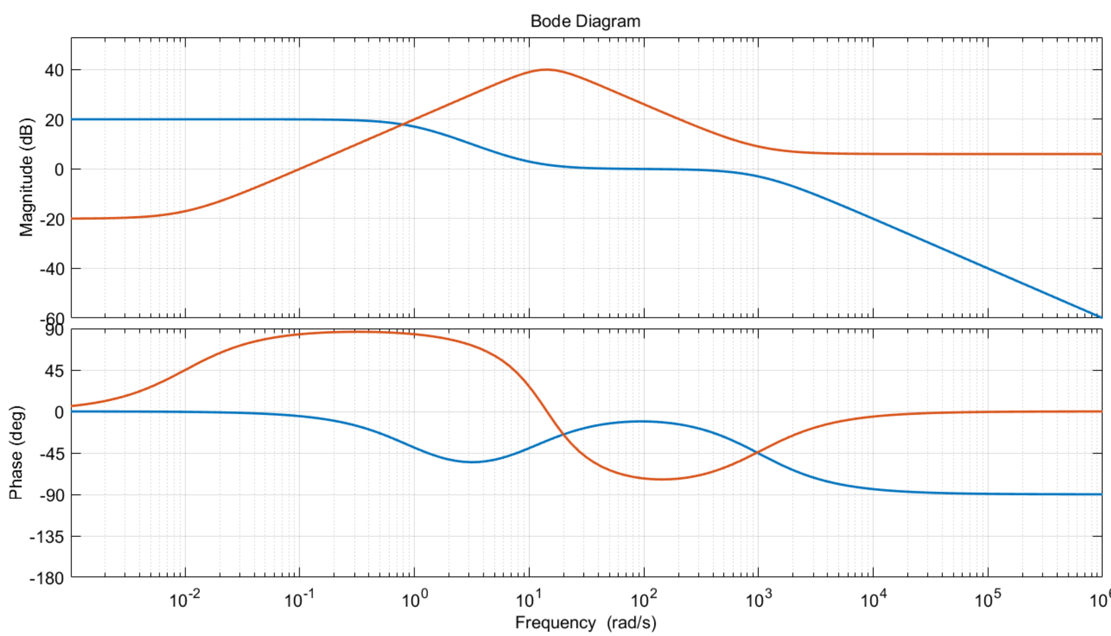
Next we have (stable)complex conjugate poles of order 1 at $w = 10$. Comparing this to the standard system shows that $\omega = \sqrt{101}$ and $\xi = \frac{2}{2\cdot\sqrt{101}} \approx 0.0995$. This will cause a drop in the amplitude by 40dB/Decade making the net drop 20dB/Decade (i.e. $20 - 40 = 20$). In terms of the phase, we will see a fast (because of the small ξ) decrease of -180° moving us to -90° (purple lines).

Finally we have a simple (stable) pole at 10^3. This causes an additional drop in the magnitude of 20dB/decade making the new total drop per decade 40dB. The phase will also slowly drop by 90° (brown line).

Here is the plot



4. Marmie the engineering cat generated the following Bode plots:



(a) (8 marks)Use the curves to approximate the transfer functions $H_{red}(s)$ and $H_{blue}(s)$ from the bode plots for the two curves Marmie gave you. Your transfer function is **not** expected to be perfect but should be good enough to be in the ballpark of being correct.

(b) (4 marks) Use either your transfer functions or the given plots to draw an approximate Bode plot for $H_{red}(s)H_{blue}(s)$. For ease of plotting, a blank Bode plot is included at the end of this assignment.

(c) (2 marks) Based on the Bode plot for the blue curve can you conclude that the system with transfer function $H_{blue}(s)$ is unstable?

(d) (2 marks) Suppose that $H_{red}(s)$ is the transfer function for a proposed controller of a system with transfer function $H_{blue}(S)$. Use your plot in part (b) determine if the closed loop controlled system is stable. Explain why or why not.

Solution:

(a) For the red curve:

- There is a constant magnitude of -20dB and phase of 0° around 0. Thus the net scaling of $H_{red}(s)$ is of the order of magnitude $20\log_{10}(|H_{red}(0)) = -20$ or $H_{red}(0) = 10^{-1}$.
- At $\omega = 10^{-2}$ we see a gain of 20dB/decade and a slow phase shift to 90°. This tells us that there is net a zero of order 1 at this point. Hence $H(s)$ contains a term like $A(s + 0.01)$.
- At $\omega = 10^1$ we see a change to a drop of 20dB/Decade (net a -40dB/decade change) and a slow phase drop of 180°. This tells us that there are net two stable poles at this point. They could be 2 real poles or a complex conjugate pair of poles with ξ "near" 1. Hence $H_{red}(s)$ contains a term like $\frac{A}{(s+1)^2+10}$. where 1 could be replaces with other values.
- At $\omega = 10^3$ we see a change to a gain 20dB/decade and a slow phase increase of 90°. This tells us that there is net a zero around this frequency. Hence $H_{red}(s)$ contains a term like $A(s + 1000)$.

Putting the above together $H_{red}(s)$ looks like

$$H_{red}(s) = A\frac{(s + 0.01)(s + 1000)}{((s + 1)^2 + 10^2)}$$

where A is such that $H_{red}(0) \approx 0.1$. We thus need

$$\begin{gathered} A\frac{10^{-2} \cdot 10^3}{10^2} \approx 10^{-1} \\ A = 1 \end{gathered}$$

Hence $H_{red}(s)$ looks like

$$H_{red}(s) = \frac{(s + 0.01)(s + 1000)}{((s + 1)^2 + 10^2)}$$

For the blue curve:

- A constant magnitude of 20dB and phase of 0° around 0. Thus the net scaling of $H_{blue}(s)$ is of the order of magnitude $20\log_{10}(|H_{blue}(0)) = 20$ or $H_{blue}(0) = 10^1$.
- At $\omega = 10^0$ we see a drop of 20dB/decade and a slow phase shift drop. This tells us that there is net a pole of order 1 at this point. Hence $H(s)$ contains a term like $\frac{A}{(s+1)}$.
- At $\omega = 10^1$ we see a change to a no drop the magnitude (net a +20dB/decade change) and a slow phase increase. This tells us that there is a net of a single zero at this point. Hence $H_{blue}(s)$ contains a term like $A(s + 10)$.
- At $\omega = 10^3$ we see a change to a change to a drop of -20dB/decade and a slow phase decrease of 90°. This tells us that there is net a stable pole around this frequency. Hence $H_{blue}(s)$ contains a term like $\frac{A}{s+1000}$.

Putting this together gives that the transfer function looks like

$$H_{blue}(s) = A\frac{s + 10}{(s + 1)(s + 1000)}.$$

Matching the starting magnitude gives

$$\begin{gathered} A\frac{1}{10^2} \approx 10^1 \\ A \approx 10^3. \end{gathered}$$

Thus

$$H_{blue}(s) = \frac{10^3(s + 10)}{(s + 1)(s + 1000)}$$

These transfer functions are not exactly what I used to plot the curves but are close.

(b) Using the results from part (a) we have that

$$
\begin{aligned}
H(s) &= H_{red}(s)H_{blue}(s) \\
&= \left(\frac{(s+0.01)(s+1000)}{((s+1)^2+10^2)}\right)\left(\frac{10^3(s+10)}{(s+1)(s+1000)}\right) \\
&= \frac{10^3(s+0.01)(s+10)}{(s+1)((s+1)^2+10^2)}
\end{aligned}
$$

To plot this we note that from the plots the initial magnitudes were 20 and -20 dB and hence $|H(0)|_{dB} \approx 0$. Similarly the angles cancel to give us $\angle H(0) \approx 0$.

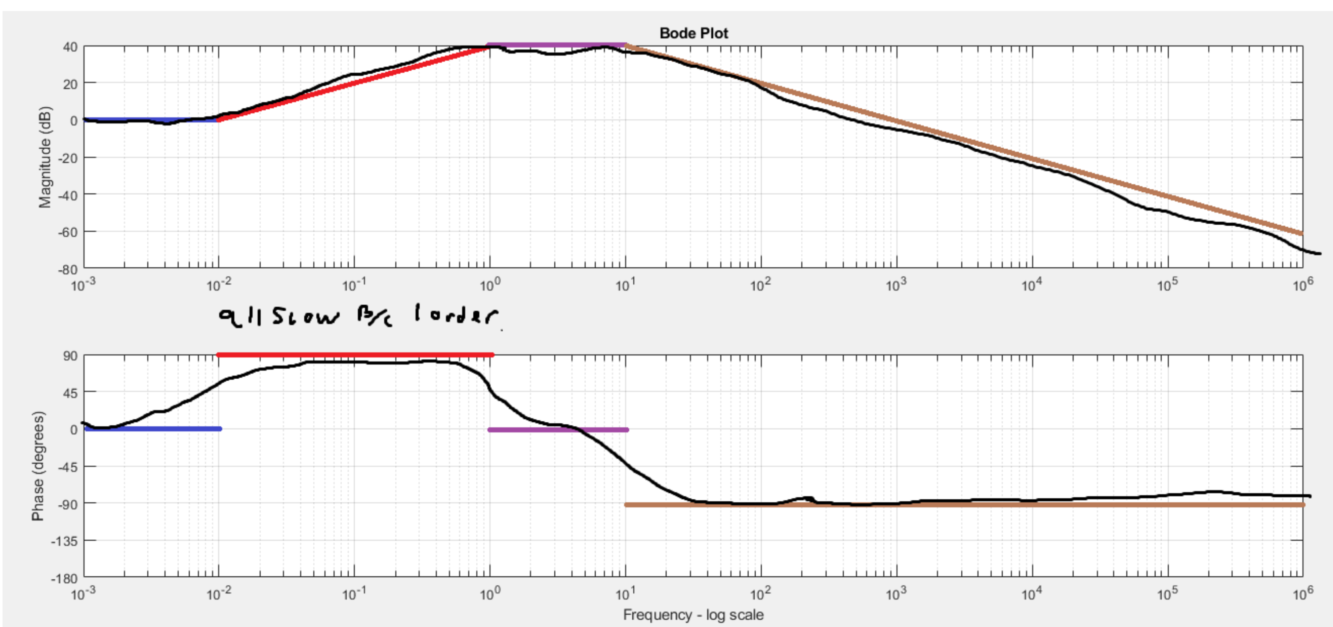
We have zeros at 10^{-2} and 10 so at these points there will be a exta dB/decade gain of 20 and a phase increase by 90°. We also have a pair of stable complex conjugate poles at $w = 10$ so there will be a gain of 40 dB/decade and a phase drop of $-180°$ at that point and a stable single pole at $\omega = 1$ which will cause a gain of 20 dB/decade and a phase drop of $-90°$.

Net:

- The magnitude and phase are held constant at 0 in their respective units (blue lines) until $\omega = 10^{-2}$. At this point there is a 20 dB/decade gain in magnitude and the phase slowly moves towards 90° (red lines)
- Then at $\omega = 1$, we have a 20 dB/decade drop in magnitude (net 0 dB/decade) and the phase moves towards 0° (purple lines).
- Then at $\omega = 10$, the effects of the poles and zeros fight each other and we have a net change of a 20 dB/decade drop (-20 dB/Decade net) in magnitude and the phase drops $-90°$(brown lines).

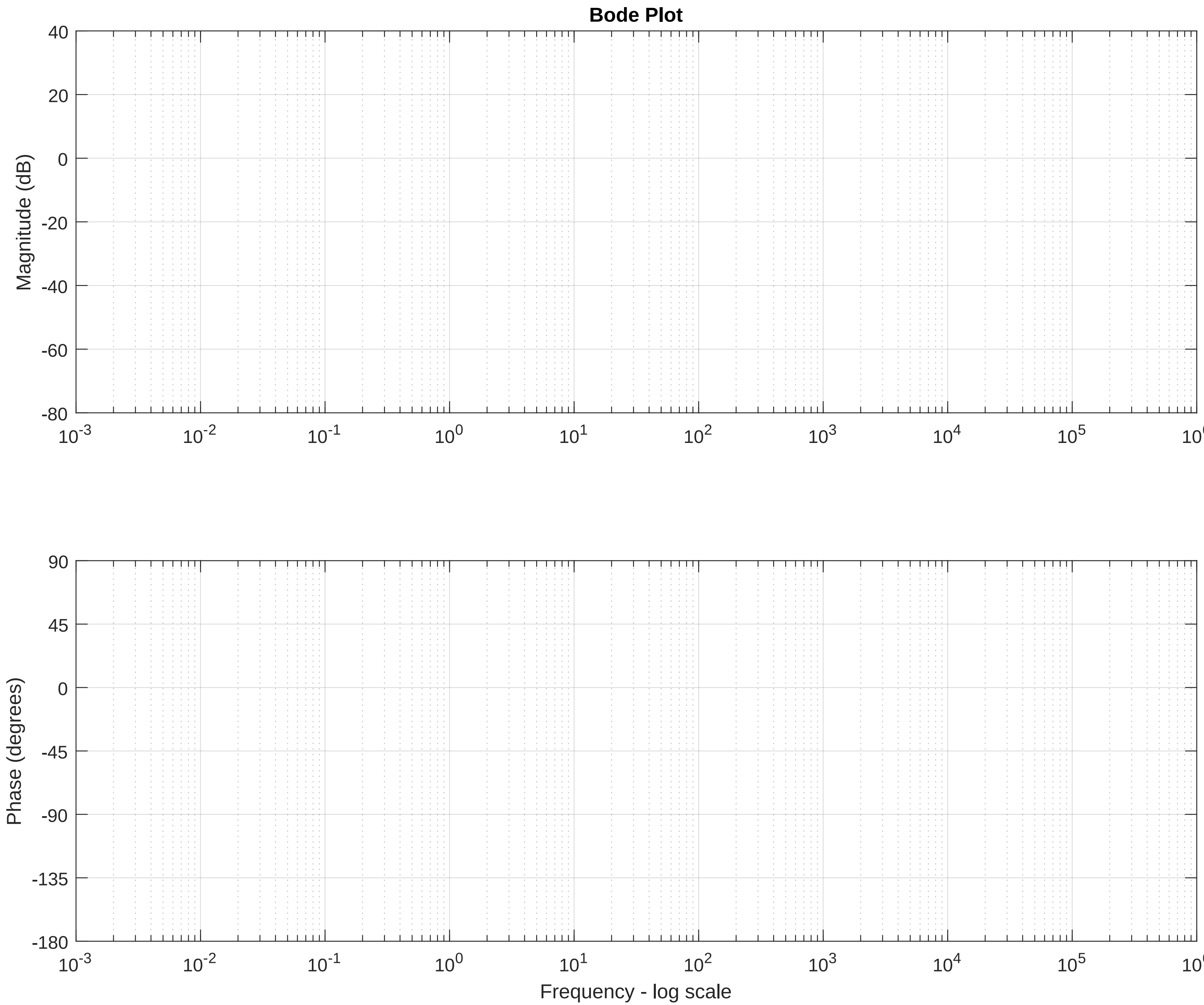
All the net poles are first order so the adjustment will be slow.

Here is a plot:



(c) In the Bode pole for H_{blue}, we never see the magnitude stat to drop while the phase adjusts upwardly. Hence we can't conclude that the system has an unstable pole. Also note that we an also not conclude stability since we could have poles/zeros that nearly cancel.

(d) In our plot from part (b) the magnitude does go to zero but the phase is never nea r $-180°$. Hence the closed loop system will be stable given that that H_{blue} and H_{red} are stable.


Bode Plot
Magnitude (dB)
40
20
0
-20
-40
-60
-80
Phase (degrees)
90
45
0
-45
-90
-135
-180
10^-3 10^-2 10^-1 10^0 10^1 10^2 10^3 10^4 10^5 10^6
Frequency - log scale