MATH 213 - Lecture 10: systems and signals introduction continued

Lecture goals: Know what systems and signals are and understand the various properties we generally care about.

Connection between CT systems and DEs:

A DE represents a CT system only when for any signal in the set of input signals of the system, there is a signal in the output class that satisfied the DE.

Example 1

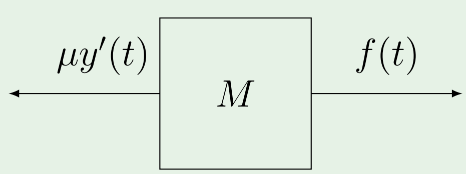
Consider the DE:

$$my''(t) = f(t) - \mu y'(t)$$

where $m, \mu > 0$ are constants and $f(t)$ is some positive valued function.

This DE comes from $ma = F$ applied to some object that experiences drag.

The force diagram is:



Given some forcing term $f(t)$, we can solve the DE to find $y(t)$.

Since this relation exists, we can say that this DE represents some system.

For a concrete example suppose that $f(t)$ is the function denoting the force that is applied to a car when you open the throttle and that μ denotes all drag forces (air resistance, friction, etc) that the car experiences.

In the above case, the DE is a representation of the CT system describing how the car moves based on the amount the accelerator is pressed.

From our work with DEs, if the initial conditions are all 0 then the solution to this DE is

$$y(t) = f(t) * \mathscr{L}^{-1}\left\{\frac{1}{ms^2 + \mu s}\right\}$$

or assuming that $f(t)$ is on-sided

$$y(t) = \int_0^t f(t-\tau)\left(\frac{1}{\mu} - \frac{e^{-\frac{\mu}{m}\tau}}{\mu}\right) d\tau.$$

Thus for any particular parameters and any given forcing term, we can compute the motion of the car system.

To Lecture10_car.m!!

Connection between DT systems and difference equations:

DT systems are represented by difference equations. An example of a difference equation is the mine craft chicken model from assignment 1.

Here is another classic fun difference equation

$$x_{n+1} = rx_n(1-x_n), \qquad r > 0$$

This models the normalized population of bunnies with the constraints that

- there is growth proportional to the population for small populations and
- if the population is close to the environmental limit, 1 in this model, then the population declines.

To Lecture10_magic.m!!
The previous two examples demonstrate that generally speaking we can explore the relation of the input and outputs for many systems by "simply" solving DEs, difference equations etc.

We now examine some more important properties of systems.

Definition 1: Memoryless vs Dynamic systems
A system is **memoryless**, if the instantaneous output value $y(t)$ depends only on the input value $f(t)$ at time t.

A system that is not memoryless is **dynamic**

Example 2

- An ideal amplifier system given by the equation

$$V_{out}(t) = kV_{in}(t)$$

  is memoryless.
- An ideal resistor given by the equation

$$V(t) = Ri(t)$$

  is memoryless.

In the previous two examples, the output signal $y(t)$ at some point t^* is determined by the input $f(t)$ evaluated at t^*

- The harmonic oscillator given by

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$

  is dynamic.

  Here $y = \frac{1}{m}\int_0^t \left(\int_0^\tau f(u)du\right) d\tau$.

Most interesting problems involve systems that are dynamic.

Definition 2: Causality
A system A is **causal** if $y(t) = (Sf)(t)$ only depends on $f(\tau)$ for $\tau \leq t$.

Idea: the output signal y only depends the previous behaviour of $f(t)$ (i.e. not the future values of $f(t)$).

Definition 3: Causality (Alternate Definition)
A system A is **causal** if for all $t \geq 0$ whenever $f_1(\tau) = f_2(\tau)$ for all $\tau \leq t$ and $y_1(t) = (Sf_1)(t)$ and $y_2(t) = (Sf_2)(t)$ then $y_1(\tau) = y_2(\tau)$, for all $\tau \leq t$.

Example 3

- Memoryless systems are causal.
- The harmonic oscillator given by
$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$
is causal.
- The difference equation $y_k = f_{k+2}$ is noncausal.
- Real-time controllers are clausal.
- Most offline signal processing involves working with nonclausal systems.
- The differential equation $y'(t) = f''(t) + f'(t) + f(t)$ for a known but arbitrary forcing term f is considered noncausal.

To see why the last example above is noncausal one needs to notice that $f''(t)$ is defined as a limit of a difference quotient and as such it depends on more future information than $y'(t)$ does.

Definition 4: Multivariable vs Scalar Systems
A system is **multivariable** if is has multiple inputs and/or outputs.

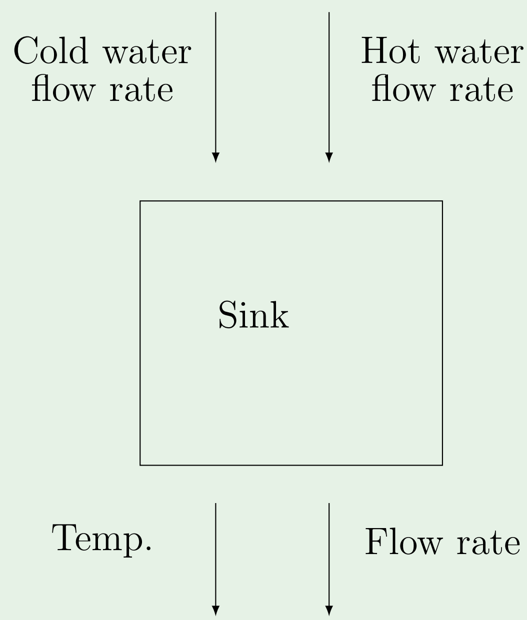
A system is **scalar** if it has a single input and a single output.

Example 4

- The system modelled by the IVP

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$

  is scalar valued.
- Your sink is a multivariable system:



Multivariable systems are harder to control then scalar systems.

We will study scalar systems (but may examine multivariable systems a bit).

Definition 5: Linearity
A system S is **linear** if it has the property that if $f(t)$ and $g(t)$ are input signals and $\alpha, \beta \in \mathbb{C}$ then

$$S(\alpha f_1 + \beta f_2) = \alpha S(f_1) + \beta S(f_2).$$

Example 5
The following are linear systems

- The harmonic oscillator given by

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0$$

- An ideal amplifier system given by the equation

$$V_{out}(t) = kV_{in}(t).$$

- The idealized system that models tsunami wave velocity in a deep ocean of depth D:

$$u_{tt} + \frac{g}{D} u_{xx} = 0$$

The following are nonlinear:

- The pendulum equation for a pendulum of length ℓ

$$m\ell^2 2\theta''(t) = -mg\ell \sin(\theta), \quad y(0) = y_0, \quad y'(0) = y_1$$

- The system with governing equation given by

$$y(t) = f(t) + 1$$

- The idealized system that models tsunami wave velocity near the shore:

$$u_t + uu_x = 0$$

Note if $\theta \ll 1$ then $\sin(\theta) \approx \theta$ and we can approximate the full pendulum by the linear oscillator:

$$m\ell^2 2\theta''(t) + mg\ell\theta = 0, \quad y(0) = y_0, \quad y'(0) = y_1.$$

Linear systems are "easy" to study but tend to have "simpler dynamics" than non-linear systems.

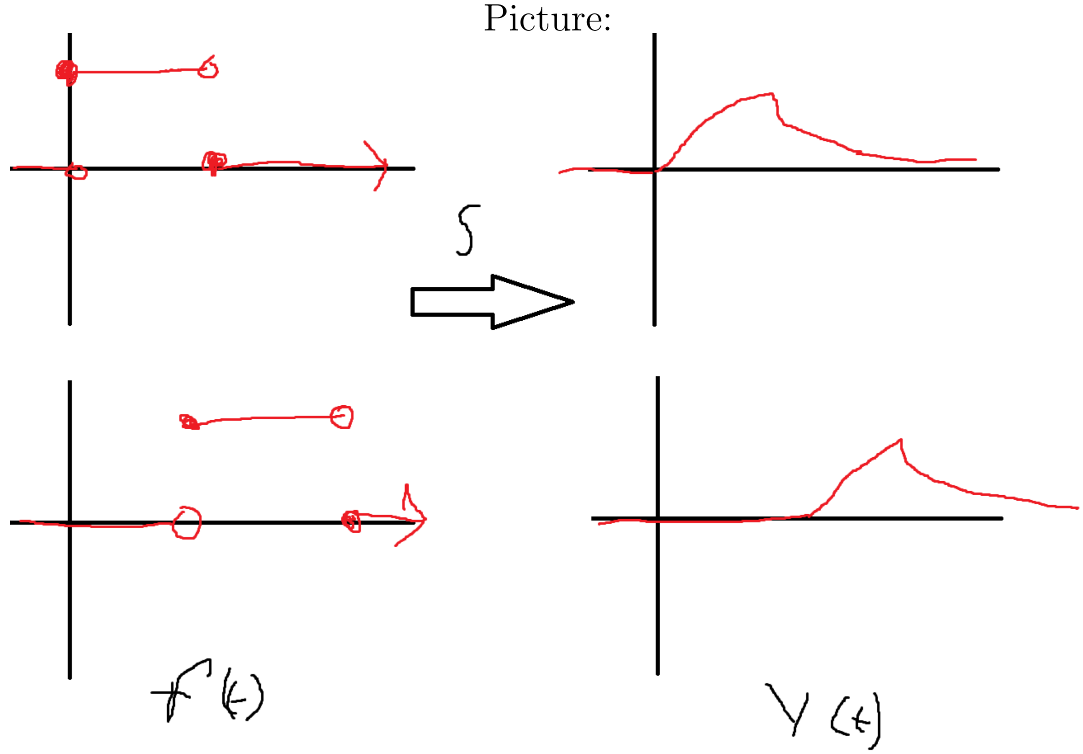
In this course we will focus on linear systems but "most" real world problems are nonlinear.

Let's quickly watch a video of a double pendulum and then move on.

Definition 6: Time-invariance
A system is **time-invariant** if its response does not change with time. Explicitly if $f(t) \xrightarrow{S} y(t)$ then for all $T \in \mathbb{R}$, $f(t-T) \xrightarrow{S} y(t-T)$.

A system that is not time-invariant is called **time-invariant**.

Picture:



Example 6

- Consider the Harmonic oscillator

$$my''(t) = f(t), \quad y(0) = 0, \quad y'(0) = 0, \qquad f(t) \neq 0 \text{ for } t > 0.$$

The solution is
$$y(t) = \frac{1}{m}\int_{-\infty}^{t}\int_{-\infty}^{\tau} f(\theta)d\theta d\tau$$
.

If we shift the input signal by replacing $f(t)$ with $\tilde{f}(t-T)$ then the solution is

$$\begin{aligned}\tilde{y}(t) &= \frac{1}{m}\int_{-\infty}^{t}\int_{-\infty}^{\tau} f(\theta - T)d\theta d\tau \\ &= \frac{1}{m}\int_{-\infty}^{t}\int_{-\infty}^{\tau-T} f(\theta)d\theta d\tau \\ &= \frac{1}{m}\int_{-\infty}^{t-T}\int_{-\infty}^{\tau} f(\theta)d\theta d\tau \\ &= y(t-T).\end{aligned}$$

Hence system is time-invariant.

- Consider the ideal sampler system that takes a continuous signal $f(t)$ and outputs the discrete signal $y_k = f(k), k \in \mathbb{Z}$.

  If $f(t) = \cos(2\pi t)$ then $S(f) = y_k = 1$ for all $k \in \mathbb{Z}$.

  If $f(t) = \cos\left(2\pi t + \frac{\pi}{4}\right)$ or $\sin(2\pi t)$ then $S(f) = y_k = 0$ for all $k \in \mathbb{Z}$.

  Hence the ideal sampler is not time-invariant.

For the rest of this course we will focus on linear time invariant systems of LTSs.

These systems are often modelled by constant-coefficient ODES and hence our work on the Laplace transform can be used to study the system....