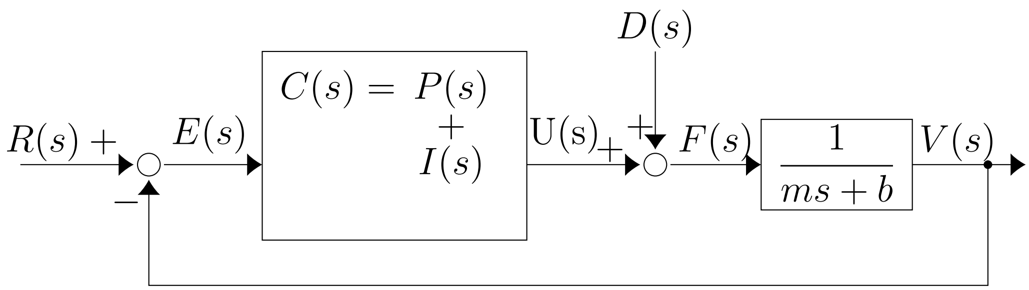
MATH 213 - Lecture 17: PI controllers, zeros, extra poles and stability

Lecture goals: Analyze a PI controller for a first order system, know the effects of the zeros and ("extra") poles of a transfer function.

Recall that the system diagram for a PI controlled car is



and the transfer function for this system is

$$H_{RV}(s) = \frac{\frac{k_i}{m}\left(\frac{k_p}{k_i}s + 1\right)}{s^2 + \left(\frac{b+k_p}{m}\right)s + \frac{k_i}{m}}.$$

Example 1

Analyze the result of using a PI controller to attempt to control the velocity of a car.

We will first simplify the functional form of $H_{RV}(s)$ to something more familiar by noting that it is of the form

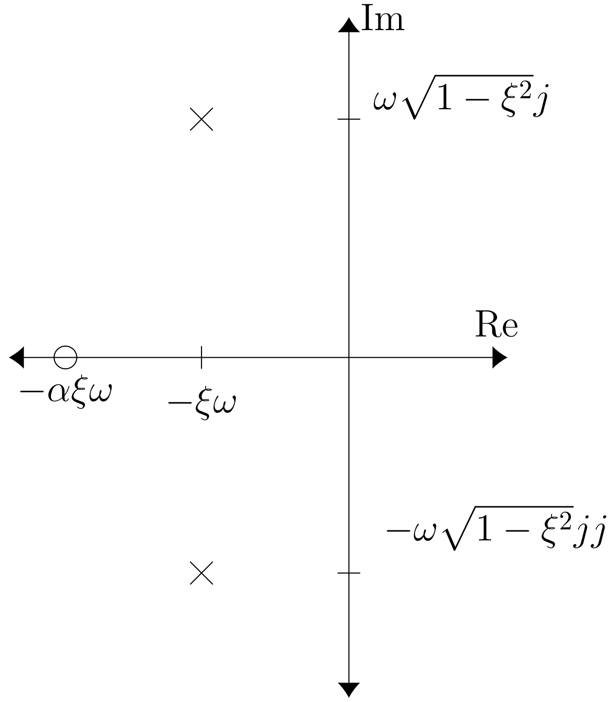
$$H_z(s) = \frac{\omega^2\left(\frac{s}{\alpha\xi\omega} + 1\right)}{s^2 + 2\xi\omega s + \omega^2}$$

where we note that this is almost of the form of the standard second order system but it has a zero.

To make the connection between $H_{RV}(s)$ and $H_z(s)$ note that

$$\omega = \sqrt{\frac{k_i}{m}}, \qquad \xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}, \qquad \text{and} \qquad \alpha = \frac{2k_i m}{k_p(b+k_p)}$$

Since $k_i, k_p, m, b > 0$, the poles and zeros can be plotted in the complex plane as



For the cruise control system we mostly care about the step response, so we will analyze the response to the step function.

$$\begin{aligned}
S(u(t)) &= H_z(s)\cdot\frac{1}{s}\\
&= \left(\frac{\omega^2\frac{s}{\alpha\xi\omega}}{s^2+2\xi\omega s+\omega^2}+\frac{\omega^2}{s^2+2\xi\omega s+\omega^2}\right)\frac{1}{s}\\
&= \frac{1}{\alpha\xi\omega}\cdot\frac{\omega^2}{s^2+2\xi\omega s+\omega^2}+\frac{\omega^2}{(s^2+2\xi\omega s+\omega^2)s}
\end{aligned}$$

This can be decomposed into different responses of the standard second order system

$$S(u(t)) = \frac{1}{\alpha\xi\omega}\cdot\underbrace{\frac{\omega^2}{s^2+2\xi\omega s+\omega^2}}_{\text{Std. 2nd order impulse res.}}+\underbrace{\frac{\omega^2}{(s^2+2\xi\omega s+\omega^2)s}}_{\text{Std. 2nd order step response}}.$$

We know what the standard second order impulse and step responses look like as a function of ω and ξ (L16), so we just need to explore how α changes the linear combination of these responses. Since α controls where the zero is, we are studying the effect of the zero on the response.

We will look at the extreme cases:

- If $\alpha \to \infty$ then $S(u(t)) \to \underbrace{\dfrac{\omega^2}{(s^2 + 2\xi\omega s + \omega^2)s}}_{\text{Std. 2nd order step response}}$.

  In practice, if $\alpha\xi\omega \gtrsim 10$ then the effects of the impulse response term can ignored in many practical applications.

- If $\alpha \to 0$ then $\frac{1}{\alpha\xi\omega} \to \infty$. Thus

$$S(u(t)) \to \text{“}\infty\text{”} \cdot \underbrace{\frac{\omega^2}{s^2 + 2\xi\omega s + \omega^2}}_{\text{Std. 2nd order impulse res.}}.$$

  and we see a lot of overshooting/instability.

In terms of the cruise control example:

- If k_p increases for a fixed value of k_i then
  - $\omega = \sqrt{\frac{k_i}{m}}$ is unaffected.
  - $\xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}$ increases. This leads to a faster system response when $\xi > 1$ and a slower system response with $0 < \xi < 1$. It also leads to more oscillations when $\xi < 1$
  - $\alpha = \frac{2k_i m}{k_p(b+k_p)}$ decreases. This causes the impact of the standard second order impulse response part of the response to increase. i.e. we will see a larger “bump” in the initial velocity.

- If k_i increases for a fixed value of k_p then
  - $\omega = \sqrt{\frac{k_i}{m}}$ increases. This in isolation leads to a faster oscillating response.
  - $\xi = \frac{(b+k_p)\sqrt{\frac{k_i}{m}}}{2k_i}$ decreases. This has the opposite effect of k_p.
  - $\alpha = \frac{2k_i m}{k_p(b+k_p)}$ increases. This has the opposite effect of k_p.

Generally, for this application we care about getting to the steady state velocity quickly and do not want to overshoot. Hence we want no oscillations and we want the impact of the impulse term to not cause overshooting.

Combining the effects above, for **this** system we use k_i to adjust the speed of the response and k_p to control the dampening of the oscillations. The exact values of

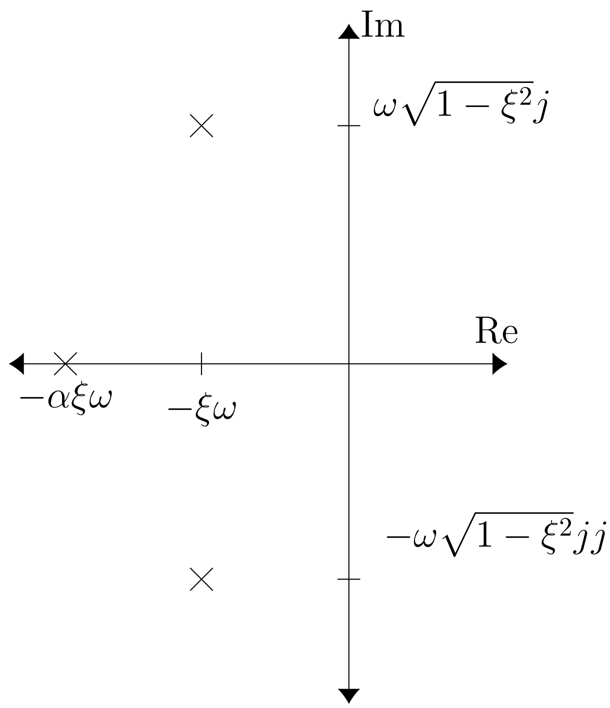
k_i and k_p needed will depend on b and m but generally we want $\xi > 1$ to avoid the oscillations.

See the Lecture17_PI_controller.m for some examples of the controlled velocity.

Adding a pole:

> **Example 2**
> Examine the impulse response of the system with transfer function
>
> $$H(s) = \frac{\omega^2}{\left(\frac{s}{\alpha\xi\omega} + 1\right)(s^2 + 2\xi\omega s + \omega^2)}$$
>
> for $\omega, \xi, \alpha > 0$ in terms of the standard first and second order systems.

Since $k_i, k_p, m, b > 0$, the poles and zeros can be plotted in the complex plane as



Based on the locations of the poles/functional form of $H(s)$ we note that we can decompose $H(s)$ as a linear combination of first and second order systems

$$H(s) = \underbrace{\frac{A}{\frac{s}{\alpha\xi\omega} + 1}}_{\text{Std. 1st order system}} + \underbrace{\frac{Bs + C}{s^2 + 2\xi\omega s + \omega^2}}_{\text{The system we just looked at}}$$

We need to know the relative values of A, B, C (and the decay rates of the system responses) to know what the impulse response looks like.

Evaluating the PF decomposition components gives

$$
\begin{aligned}
A &= \frac{1}{1 - 2\alpha\xi^2 + \alpha^2\xi^2} \\
B &= \frac{-\alpha\xi\omega}{1 - 2\alpha\xi^2 + \alpha^2\xi^2} \\
C &= \frac{\alpha(\alpha - 2)\xi^2\omega^2}{1 - 2\alpha\xi^2 + \alpha^2\xi^2}.
\end{aligned}
$$

α changes the relative weights of the impacts of the first order system and the second order system. What happens as α changes?

We again look at the extreme values:

- If $\alpha \to 0$ then $A \to 1$ and $B, C \to 0$.

  In this case, the impulse response looks like that of a standard first order system given that its pole is closer to the imaginary axis than the complex poles.

  The latter happens when $-\alpha\xi\omega > -\xi\omega$. In practice this happens for $\alpha \lesssim 0.1$ or so.

- If $\alpha \to \infty$ then $A, B \to 0$ and $C \to \omega^2$.

  In this case, the impulse response looks like that of a standard second order system given that the complex poles are closer to the imaginary axis than the real pole.

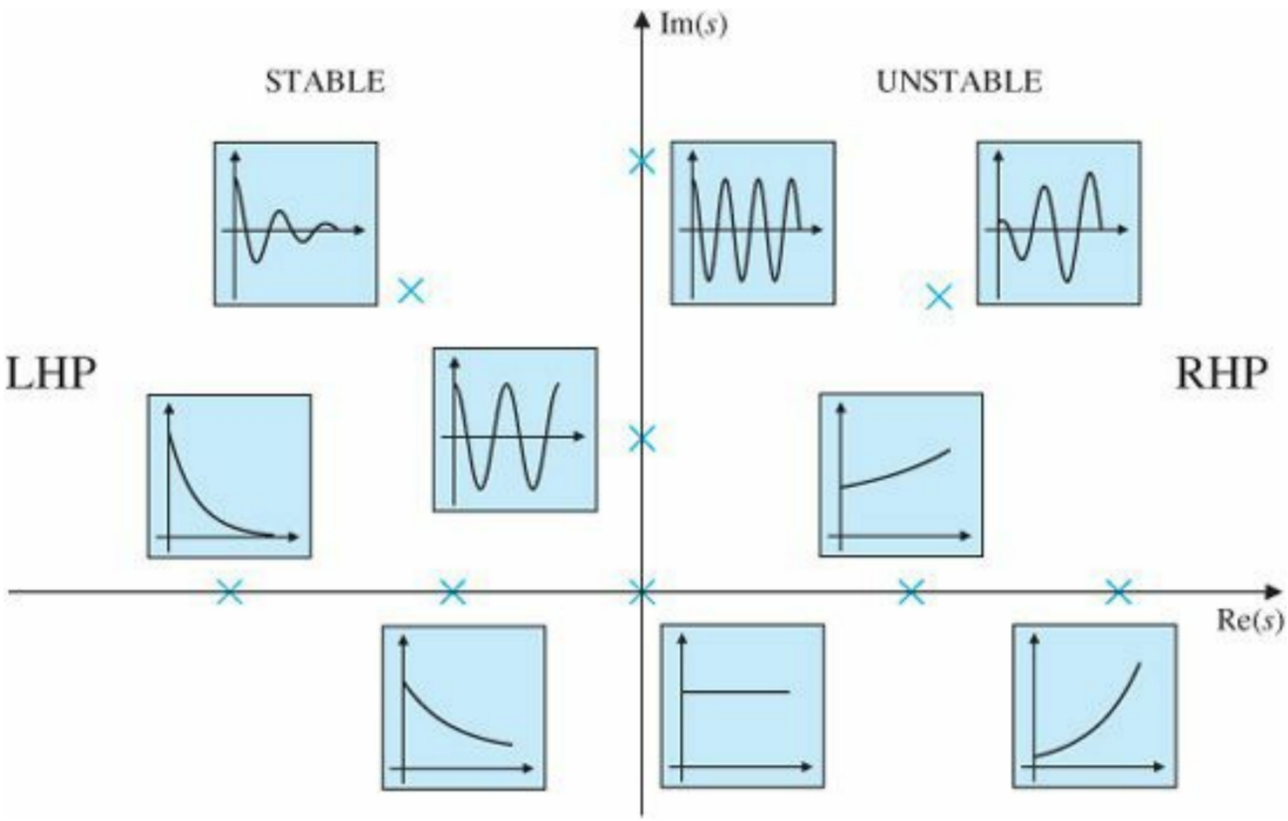
  Assuming the roots are complex valued (i.e $0 < \xi < 1$), the latter happens when $-\alpha\xi\omega < -\xi\omega$. In practice this happens for $\alpha \gtrsim 10$ or so. If the roots are not complex then the second order system decomposed into two first order systems but the result still holds.

See Lecture17_extra_poles.m for pictures showing this.

Definition 1
When a real pole or a complex-conjugate pair of poles are an order of magnitude closer to the imaginary axis than all other poles then we say that they are **dominant**.

Linear Stability:

Recall the general behaviours of the impulse response caused by a pole at some point $a + bj$ for $a, b \in \mathbb{R}$.



From Franklin, Powell and Emami-Naeini,
"Feedback Control of Dynamic Systems" 6th ed.

The impulse response of **ALL** Linear Time Invariant systems is a linear combination of the types of functions shown in the above plot.

Further since $\{\delta(t-\tau) | \tau \in \mathbb{R}\}$ is a basis for the set of all functions, the response of **ANY** LTI to **ANY** function $f(t)$, can be written as a convolution of $f(t)$ with a linear combination of the types of functions shown above.

Definition 2: System Stability
LTI, S, is **stable** if $S(\delta(t))$ decays to 0.

A LTI, S, is **unstable** if $S(\delta(t))$ is unbounded.

A LTI, S is **marginally stable** if $S(\delta(t))$ is bounded but does not decay to 0.

The type of stability mentioned above is often called linear stability.

Definition 3: Transfer function stability
A transfer function is stable if the system it is a transfer function for is stable.
A transfer function is unstable if the system it is a transfer function for is unstable.
A transfer function is marginally stable if the system it is a transfer function for is marginally stable.

Theorem 1: Stability
A transfer function is stable if all poles have a negative real part.

A transfer function is unstable if there is a pole with a positive real part OR there is a second order pole that has a real part of 0.

A transfer function is marginally stable if there are no poles with positive real parts OR second order poles with a real part of 0 and in addition there is at least order 1 pole that has a real part of 0.

Sketch of proof:

If all poles have negative real parts then the impulse response is a sum of exponentials (potentially with oscillations) that decay to 0 and hence the impulse response decays.

If there is any pole with a positive real part, then that component of the impulse response grows. Hence the impulse response will not be bounded.

If the first condition is met, then the system is not unstable and hence the impulse response does not grow without bound. If the second condition is met then there is either a sinusoidal component to the impulse response or a constant component. In either case, these terms will keep the system response from decaying to 0.

Definition 4: Bounded-input, bounded-output (BIBO) stable
A LTI, S, is **bounded-input, bounded-output (BIBO) stable** if $S(f)$ is bounded for all bounded functions f.

Theorem 2
A LTI system with a rational transfer function is BIBO stable if and only if its transfer function is both stable and proper.

Example 3
Build a controller that turns the system with a transfer function

$$H(s) = \frac{1}{s-a}$$

into a **stable** system.

NASA tried this method for a Jupiter rocket (1957) and Atlas 4a rocket (1957) and others... don't do this!

I repeat **NEVER** cancel unstable poles in the above way.

Go to the tutorial if you want to see how to do this properly.