MATH 213 - Assignment 3 - Solutions

Submit to Crowdmark by 9:00pm EST on Friday, February 16.

Instructions:

1. Answer each question in the space provided or on a separate piece of paper. You may also use typetting software (e.g., Word, TeX) or a writing app (e.g., Notability).
2. All homework problems must be solved independently.
3. For full credit make sure you show **all** intermediate steps. If you have questions regarding showing intermediate steps, feel free to ask me.
4. Scan or photograph your answers.
5. Upload and submit your answers by following the instructions provided in an e-mail sent from Crowdmark to your uWaterloo e-mail address. Make sure to upload each problem in the correct submission area and only upload the relevant work for that problem in the submission area. Failure to do this **will** result in your work not being marked.
6. Close the Crowdmark browser window. Follow your personalized Crowdmark link again to carefully view your submission and ensure it will be accepted for credit. Any pages that are uploaded improperly (sideways, upside down, too dark/light, text cut off, out of order, in the wrong location, etc.) will be given a score of **zero**.

Read before starting the assignment: In some of the problems in this assignment (1bii, 1cii, 2) I permit the use of to aid in some of the more computational aspect of the problems. The reason I am permitting this is to focus on the new material rather than previous material (partial fractions and integration) which you should already know how to do and will be responsible for on the exam. Unless otherwise explicitly stated all computations **must** be done by hand, without the use of outside aid (i.e. peers, google, AI, etc.) and all work must be shown.
Questions:

Applied Problems:

1. (8 marks) Recall the harmonic oscillator considered in example 7 of lecture 2 and example 1 of lecture 8 with spring constant k, dampening coefficient b and an forcing term $f_{ap}(t)$. In lecture 8 we showed that for this system with initial conditions $y(0)$ and $y'(0)$ the Laplace transform of the solution $y(t)$ is given by
$$Y(s) = \frac{F_{ap}(s)}{s^2 + bs + k} + \frac{(s+b)y(0) + y'(0)}{s^2 + bs + k}.$$
In this problem we will use this expression to explore some real world applications of simple harmonic motion in three different regimes the undamped case, the underdamped case and the overdamped case[1].

   (a) Suppose that the UW midnight sun solar car team decided, unwisely, to use an <u>undamped</u> suspension system with spring constant $k = 1$ and dampening constant $b = 0$. In the absence of a forcing term, a spring with these physical properties and the initial conditions $y(0) = 1$ and $y'(0) = 0$ oscillates forever between $y = -1$ and $y = 1$. Using the given $Y(s)$ found in class, find at least two forcing terms so that $f_{ap}(t)$ <u>remains bounded</u> while $y(t)$ is unbounded (i.e. there is a $M \in \mathbb{R}$ such that for all $t \in \mathbb{R}_{\geq 0}$ $|f_{ap}(t)| \leq M$ but $\lim_{t\to\infty} y(t)$ "=" $\pm\infty$).
   You should explicitly compute what your $f_{ap}(t)$ functions are.

   Note: Because such forcing terms sometime exists, for real world design problems one 1) needs to check for such cases and 2) either needs to make sure that these $f_{ap}(t)$ forcing terms <u>never</u> occur naturally or ideally would redesign the system so that no forcing term with finite energy can lead to unbounded solutions. This analysis can be done by simply looking at the locations of the poles!

   (b) Suppose buddy, who has unit mass, is bungee jumping using a cable that can be modelled by a <u>underdamped</u> harmonic oscillator with a spring coefficient of $k = 1$, a dampening factor $b = 1$ and a forcing term $f_{ap}(t) = 0$. At the bottom of buddy's jump the string was stretched by $-2g$ units from equilibrium and had a velocity of 0. These conditions can be represented by the initial conditions $y(0) = -2g$ and $y'(0) = 0$. Here $y = 0$ represents the length of the cord at equilibrium (i.e. the position when buddy is hanging by the cord and is at rest)[2] and $g = 9.81$[3].

      i. Assuming the data in the above statement, solve the IVP for the harmonic motion of the bungee cord. You may use the $Y(s)$ given in lecture.

      ii. The human body can generally withstand a range of g-force levels depending on body positioning and duration of time. Suppose buddy's enjoyable g-force tolerance requires that he only experiences a g-force above 2g (i.e. the 2 times the acceleration of gravity) for less than 0.1 seconds at once and a maximum g-force of $4g$. Note that because of the ever present force of gravity on the surface of the earth, the total g-force Buddy experiences while

[1] There is another case that we do not consider in this question called the critically damped case
[2] We could have alternatively make $y = 0$ be the unstretched length and add a forcing factor of gravity to our DE.
[3] I am using non-dimensional numbers

bungee jumping is given by $\underbrace{y''(t)}_{\text{g-force from cable}} + \underbrace{g}_{\text{g-force from earth}}$ and both positive and negative g-forces are important to check.

Use your result in part i, determine if Buddy's criteria are met.

Comment: For 1b you need to compute the solution $y(t)$ in part i by hand but may use a calculator/numerical methods to compute and plot $y''(t)$ to check the criteria for part ii. Make sure you provide enough information about how you found your results for part ii to justify your results.

(c) Suppose that Jaxton's mom bought an overdamped hydraulic door closer with $k = 1$ and $b = 4$ to prevent Jaxton (her toddler) from slamming his door. Jaxton attempts to slam his door by applying the initial conditions $y(0) = 1$ and $y'(0) = -2$ and the forcing term $f_{ap}(t) = 0$. Here y is the distance from the end of the door and the door frame with $y = 0$ representing a closed door, $y > 0$ representing the door being opened to various degrees and $y < 0$ representing a door that is opened the other way (i.e. the door broke).

i. Use the above information to solve the IVP for the position of the door over time. You may again use the results of $Y(s)$ for the case of simple harmonic motion that we derived in class.

ii. Suppose that in this non-dimensional model rapidly closing a door only causes a loud noise if the position of the door decreases from 0.1 to 0.01 within 1 unit of time. Does the door make a loud noise in this case?

Comment: For 1c you need to compute the solution $y(t)$ in part i by hand but may use a calculator/numerical methods to plot and analyze $y(t)$ to check the criteria for part ii. Make sure you provide enough information about how you found your results for part ii to justify your results.

Solution:

(a) The solution with the given coefficients and ICs is

$$Y(s) = \frac{F_{ap}(s)}{s^2+1} + \frac{s}{s^2+1}.$$

The second term is a cos term and hence oscillates as the question stated but the behaviour of the first term depends on $F_{ap}(s)$. For this term to admit an unbounded solution $y(t)$ it must be the case that $F_{ap}(s)$ forces $Y(s)$ to either have a repeated pole on the imaginary axis or to have a pole in the right part of the plane. In the latter case, $f_{ap}(t)$ itself would be unbounded and hence we need to find a way to have a repeated pole on the imaginary axis. Since the poles Of the transfer function are located at $s = \pm i$, $F_{ap}(s)$ needs to have simple poles at these points in order to resonate. Hence $F_{ap}(s) = \frac{as+b}{s^2+1}$ or $f(t) = a\cos(t) + b\sin(t)$ are the two different functions that have the desired property.

(b)

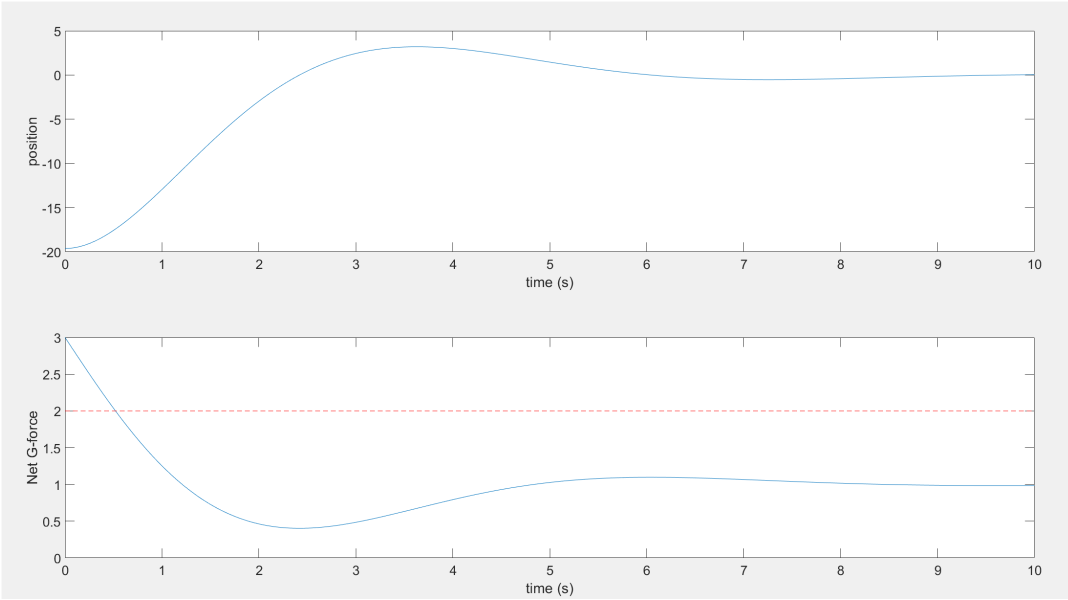
i. The LT of the solution with $k = 1, b = 1$ and the given ICs is

$$Y(s) = \frac{-2g(s+1)}{s^2+s+1} \qquad \text{or} \qquad Y(s) = \frac{-2g(s+0.5+0.5)}{(s+0.5)^2+\frac{3}{4}}$$

The roots are in the LHS of the complex plane so we will have complex exponential solutions. Using the Lapalce table we have

$$y(t) = -2ge^{-\frac{1}{2}t}\left(\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right)$$

ii. Here is a plot of the solution and the G-forces experienced



In the second picture we can see that Bud's net G-force is greater than 2 for around the first 0.5 seconds. Hence Bud did not meet all his his criteria for having a good time. He did however meet the requirement for maximum g-force.
Here is the code to generate this plot is posted to learn.

(c)

i. The Laplace transform on the solution is

$$Y(s) = \frac{(s+4)-2}{s^2+4s+1} \qquad \text{or} \qquad Y(s) = \frac{s+2}{(s+2)^2-3}.$$

The poles of $Y(s)$ are $s = -2 \pm \sqrt{3}$. since both of these real are less than 0, the solution is a sum of decaying exponentials. Doing the needed partial fractions gives us

$$Y(s) = \frac{A}{s-(-2+\sqrt{3})} + \frac{B}{s-(-2-\sqrt{3})}$$

and using the cover-up method gives

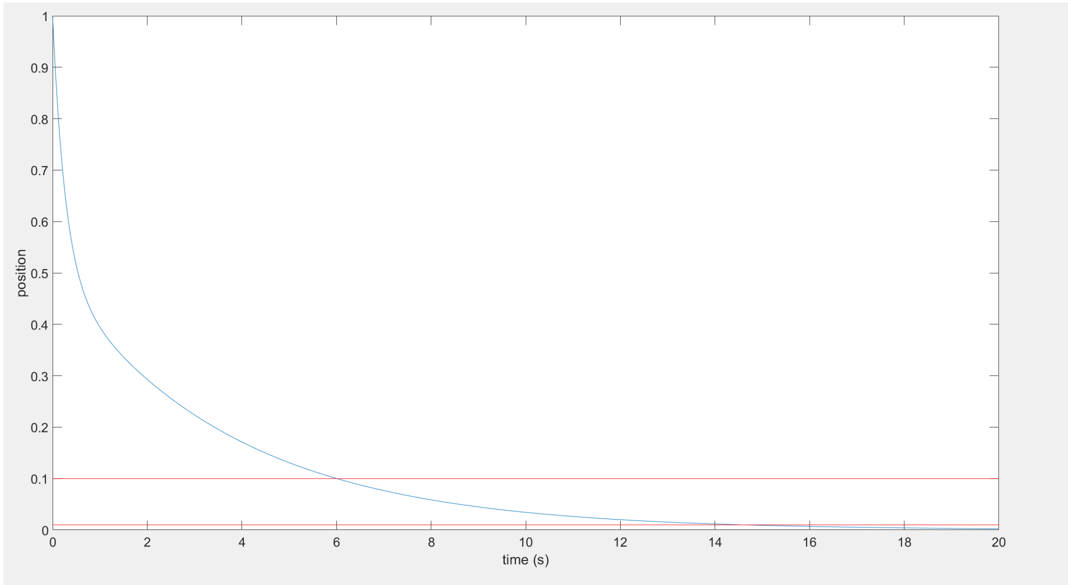
$$A = \frac{(-2+\sqrt{3})+2}{-2+\sqrt{3}-(-2-\sqrt{3})} = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2}$$

and

$$B = \frac{(-2-\sqrt{3})+2}{-2-\sqrt{3}-(-2+\sqrt{3})} = \frac{-\sqrt{3}}{-2\sqrt{3}} = \frac{1}{2}.$$

Hence from our Laplace table the solution is $y(t) = \frac{1}{2}(e^{(-2+\sqrt{3})t} + e^{(-2-\sqrt{3})t})$

ii. Here is a picture of the doors position and the critical values we care about



Here we see that it takes around 8 time units for the door to close from 0.1 units to 0.01 units. Hence the door does not make a loud noise under our chosen conditions.

Computational Problems:

Read before starting problem 2: For problem 2 you must give the correct functional form for any needed partial fraction decompositions **but** can use a calculator to help find the coefficients. Additionally, you may use calculators to compute any needed integrals. You must explicitly state what resource(s) you used and can only use resources to aid in computing the coefficients of the partial fraction decompositions you chose and to help evaluate any integrals you may setup. You must show all steps for how you compute the inverse Laplace transforms.

2. (5 points) Solve the IVP $y^{(4)}(t) - y(t) = \sin(t)$ with the initial conditions $y^{(3)}(0) = 0$, $y''(0) = 0$, $y'(0) = 0$ and $y(0) = 0$.

Solution: Taking the Laplace transform gives

$$s^4Y(s) - s^3y(0) - s^2y'(0) - sy''(0) - y^{(3)}(0) - Y(s) = \frac{1}{s^2+1}$$

Solving for $Y(s)$ gives us

$$Y(s) = \frac{1}{(s^2+1)(s^4-1)} \qquad \text{or} \qquad Y(s) = \frac{1}{(s^2+1)^2(s-1)(s+1)}$$

In the above we used the fact that $s^4 - 1 = (s^2-1)(s^2+1) = (s+1)(s-1)(s^2+1)$. We hence look for a PF decomposition of the form

$$Y(s) = \frac{As+B}{(s^2+1)^2} + \frac{Cs+D}{s^2+1} + \frac{E}{s-1} + \frac{F}{s+1}$$

We now need to find the coefficients:

- To find E and F we use the coverup method by scaling by $s \pm 1$ and using $s = \mp 1$. This gives

$$\frac{1}{((-1)^2+1)^2((-1)-1)} = F \qquad \text{or} \qquad F = -\frac{1}{8}$$

  and

$$\frac{1}{(1^2+1)^2(1+1)} = E \qquad \text{or} \qquad E = \frac{1}{8}$$

- For A an B we scale by $(s^2+1)^2$ and use $s = j$. This gives

$$\frac{1}{(j-1)(j+1)} = Aj + B.$$

  $(j-1)(j+1) = -2$ and hence

$$A = 0 \qquad \text{and} \qquad B = -\frac{1}{2}.$$

- To find C and D we scale by s^2+1 and then use either any complex number other than $\pm j$ or two different real values of s to arrive as a system of equations for C and D. Scaling and using the values of A, B, E and F gives

$$\frac{1}{(s^2+1)(s-1)(s+1)} = -\frac{1}{2}\frac{1}{s^2+1} + (Cs+D) + \frac{1}{8}\frac{s^2+1}{s-1} - \frac{1}{8}\frac{s^2+1}{s+1}.$$

Using $s = 0$ allows us to quickly solve for D. Doing this gives

$$-1 = -\frac{1}{2} + D - \frac{1}{8} - \frac{1}{8} \qquad \text{or} \qquad D = -\frac{1}{4}.$$

For C we use any other value of s. Picking $s = 2$ and using or value for D gives

$$\frac{1}{(5)(1)(3)} = -\frac{1}{2} \cdot \frac{1}{5} + \left(2C - \frac{1}{4}\right) + \frac{1}{8} \cdot \frac{5}{1} - \frac{1}{8} \cdot \frac{5}{3}.$$

or

$$\frac{1}{15} = -\frac{1}{10} + \left(2C - \frac{1}{4}\right) + \frac{5}{8} - \frac{5}{24}$$

or $C = 0$.

Hence

$$Y(s) = -\frac{1}{2}\frac{1}{(s^2+1)^2} - \frac{1}{4}\frac{1}{s^2+1} + \frac{1}{8}\frac{1}{s-1} - \frac{1}{8}\frac{1}{s+1}$$

Alternatively we can just use this. Taking the inverse transforms give us

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin(t) \qquad \text{sin transform}$$

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{s-1}\right\} &= e^t \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} && \text{Exponential Modulation} \\ &= e^t && \text{transform for } \frac{1}{s} \end{aligned}$$

$$\begin{aligned} \mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} &= e^{-t} \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} && \text{Exponential Modulation} \\ &= e^{-t} && \text{transform for } \frac{1}{s} \end{aligned}$$

For the first transform we note that this is not in our transform list but it looks kinda like the derivative of something in our list... Like with lecture 6 example 3 we have two options: compute the convolution of the inverse transform of $\frac{1}{s^2+1}$ with itself or use the PF ansatz given on page 7.2 of lecture 6.

From our Laplace table we have $\mathscr{L}\left\{\frac{1}{s^2+1}\right\} = \sin(t)$ and hence by the convolution theorem

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \int_0^t \sin(\tau)\sin(t-\tau)d\tau$$

We could evaluate this following the steps in lecture 6 page 7 but an online calculator gives

$$\int_0^t \sin(\tau)\sin(t-\tau)d\tau = \frac{1}{2}(\sin(t) - t\cos(t)).$$

Hence the solution is

$$y(t) = -\frac{1}{4}(\sin(t) - t\cos(t)) - \frac{1}{4}\sin(t) + \frac{1}{8}e^t - \frac{1}{8}e^{-t}$$

or

$$y(t) = -\frac{1}{2}\sin(t) + \frac{1}{4}t\cos(t) + \frac{1}{8}e^t - \frac{1}{8}e^{-t}$$

Read before starting questions 3-5: For questions 3-5 all calculations must be done by hand and no outside calculators can be used. Show all your work.

3. In this question we will compare the effect that the initial condition has compared to the delta function.

   (a) (2 marks) Use our revised Laplace table from Lecture 9 to find the zero-input, $Y_{ZIR}(s)$, and zero-state, $Y_{ZSR}(s)$, responses for the IVP

   $$y'' + \alpha y' = a\delta(t) + b\delta'(t), \qquad y(0) = c, \qquad y'(0) = d.$$

   Here α, a, b, c and d are unknown constants.

   (b) (1 mark) Find the relation between a, b and c, d so that $Y_{ZSR}(s) = Y_{ZIR}(s)$.

   Note: This result means that if we allow for the δ function and its derivatives to be used as forcing terms then we can transform all IVPs for linear DEs with constant coefficients to ones where the initial conditions are all zero. Hence, if we know what the zero-input response is for all functions (including $\delta(t)$, $\delta'(t)$, etc), then we also know the zero-state response for all ICs of the standard form.

Solution:

(a) Taking the Laplace transform gives

$$s^2Y(s) - sy(0) - y'(0) + \alpha(-y(0)) = a + bs$$

and hence

$$Y(s) = \underbrace{\frac{bs + a}{s^2 + \alpha s}}_{ZSR} + \underbrace{\frac{y(0)s + y'(0) + \alpha y(0)}{s^2 + \alpha s}}_{ZIR}$$

or by using our ICs

$$Y(s) = \underbrace{\frac{bs + a}{s^2 + \alpha s}}_{ZSR} + \underbrace{\frac{cs + d + \alpha c}{s^2 + \alpha s}}_{ZIR}$$

(b) For ZSR to be equal to the ZIR we need $b = c$ and $a = d + \alpha c$. This means that the solutions to the systems

$$y'' + \alpha y' = 0, \qquad y(0) = c, \qquad y'(0) = d$$

and

$$y'' + \alpha y' = (d + \alpha c)\delta(t) + c\delta'(t), \qquad y(0) = 0, \qquad y'(0) = 0$$

are the same. These results may seem at odds with each other since in one case the ICs are 0 but in the other the ICs are generally non-zero. This along with the fact that engineers like to use the ZSR and ZIR as the basis for the approach to control theory is why we formally need to apply the initial conditions at 0^- (before the system "turns on") as opposed to 0 (when it turns) or 0^+ (right after it turns on). With this convention, the oddness of the above result goes away. Mathematically of course it does not matter which you impose as long as you are consistent with your meanings.

4. While solving some high order DEs, Marmie the cat (shown below and as requested by students in office hours) was working with some Laplace transforms and could not figure out what final values were. Additionally, she was unsure if the initial conditions were properly used (this made her quite tired and hence why she is laying down).

Determine $y(0)$ and $\lim_{t\to\infty} y(t)$ (if they exist) for the various $Y(s)$ functions Marmie found. If the limit as t tends to ∞ does not exist, explain why (i.e. oscillates vs diverges to $\pm\infty$).

(a) (3 marks)

$$Y(s) = \frac{s^{9001}}{(s+2)^{9002} - 1}$$

Hint: In case you forgot refresh how to find the *nth* roots of unity from MATH 115 (PDF available on Learn).

(b) (3 marks)

$$Y(s) = \frac{s^5 + 1}{s^7 + s^5}$$

(c) (4 marks)

$$Y(s) = \frac{s^3}{s^2 + 2s + 1}$$

Solution:

(a) This is of the form so that the Initial value and final value theorems apply. For $y(0)$ we compute

$$\lim_{s\to+\infty} sY(s) = 1.$$

For the final value theorem we need to check the poles. The poles satisfy $(s+2)^{9002} - 1 = 0$ or $s = -2 + (1)^{1/9002}$. From math 115 the 9002 roots of 1 are given by

$$e^{0\pi + \frac{2\pi j n}{9002}}$$

for $n = 0, 1, \ldots, 9001$. All of these roots lie on the complex unit circle so adding -2 to them shifts the roots to the Left hand side of the complex plane (i.e. Re<0). Hence the final value theorem applies and

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0^+} sF(s) = 0.$$

(b) Again this is of the form where the theorems apply. For $y(0)$ we compute

$$\lim_{s\to+\infty} sY(s) = \lim_{s\to+\infty} \frac{s^5+1}{s^6+s^4} = 0.$$

For the Final value theorem we note that $s^7+s^5 = s^5(s^2+1)$ and hence $s=0$ is a pole of order 5. This means that the limit diverges. In particular in this case it is not bounded.

(c) In this case $Y(s)$ is not a proper rational function. Simplifying $Y(s)$ by dividing gives

$$Y(s) = s + \frac{-2s^2-s}{s^2+2s+1} = s-2+\frac{3s+2}{s^2+2s+1}$$

Here the inverse laplace transform of the sum of first two terms are $y(t) = \delta'(t) - 2\delta(t)$ and the poles of the last term are at $s=-1$ (repeated root). Hence while the theorems do not apply to the whole expression, we can apply them to the last term in $Y(s)$ and simply add the contributions of the first two terms. Hence

$$y(0) = \lim_{t\to 0^+} (\delta'(t) - 2\delta(t)) + \lim_{s\to+\infty} s\frac{3s+2}{s^2+2s+1}$$

The first two terms are 0 outside of potentially $t=0$ and hence the limits vanish. For the second term the limit is simply 3 and hence $y(0)=3$.

Similarly for the final values we have

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} (\delta'(t) - 2\delta(t)) + \lim_{s\to 0^+} s\frac{3s+2}{s^2+2s+1}$$

The first limit is trivially 0 and the second limit is also 0 hence the solution is bounded and tends to 0.

5. Your pal Woolipop$^{\text{TM}}$ is working with DE whose transfer function is given by

$$T(s) = \frac{1}{(s+1)(s+5)(s-6)} \qquad \text{or} \qquad \frac{1}{s^3 - 31s - 30}$$

and wants your help to analyze the solution.

a) (1 mark) Find a DE whose transfer function is the one provided.

b) (2 marks) Determine if the solution to the unforced IVP with initial conditions $y(0) = 1$ $y'(0) = -1$ and $y''(0) = 1$ is bounded.

c) (1 bonus mark) Find the least restrictive conditions on the initial conditions $y(0)$, $y'(0)$ and $y''(0)$ so that the inverse Laplace transform of the zero-input response is bounded. i.e. give all possible initial conditions so that the solution is bounded.

Part c is all or none (i.e. no part marks). Make sure you submit in the separate 5c slot on crowdmark.

Solution:

(a) We know that the characteristic polynomial is

$$(s+1)(s+5)(s-6) = s^3 - 31s - 30$$

Hence the DE is

$$y'''(t) - 31y'(t) - 30y(t) = f(t)$$

where $f(t)$ is the forcing term.

(b) Taking the Laplace transform of the unforced problem with these ICs gives us

$$\underbrace{s^3Y(s) - s^2y(0) - sy'(0) - y''(0)}_{y''' \text{ terms}} - 31(\underbrace{sY(s) - y(0)}_{y' \text{ terms}}) - 30Y(s) = 0$$

and solving for $Y(s)$ gives

$$Y(s) = \frac{y(0)s^2 + y'(0)s + y''(0) - 31y(0)}{s^3 - 31s - 30}$$

or with the proposed initial conditions

$$Y(s) = \frac{s^2 - s - 30}{s^3 - 31s - 30}.$$

This is of the form of a function that the final value theorem applies to. Hence the solution is only unbounded if there is a pole in the left hand plane (Re>0) or there is a repeated pole on the imaginary axis. From the factored form of the characteristic polynomial that was given to us in the problem statement, the only possible problematic pole is at $s = 6$. This will be a problem unless $s^2 - s - 30 = 0$ when $s = 6$. This is in fact the case!! Hence the solution is bounded.

In fact the solution is readily found to be $y(t) = e^{-t}$.

(c) The three initial conditions and the functional form of $Y(s)$ allow us to control the coefficients in the numerator of $Y(s)$ independently. Hence we need to find all a, b and c so that the A term the PF decomposition below is equal to 0.

$$\frac{as^2 + bs + c}{s^3 - 31s - 30} = \frac{A}{s-6} + \frac{B}{s+5} + \frac{C}{s+1}$$

where in terms of our initial conditions

$$\begin{aligned} a &= y(0) \\ b &= y'(s) \\ c &= y''(0) - 31y(0) \end{aligned}$$

Using the cover-up method we see that

$$A = \frac{a \cdot 6^2 + b \cdot 6 + c}{(6+1)(6+5)}.$$

Hence we need $36a + 6b + c = 0$ or in terms of our initial conditions we need the following expression to hold

$$5y(0) + 6y'(s) + y''(0) = 0.$$

Our initial conditions in part (b) satisfied this but there are an infinite number of other conditions that work. Explicitly, the vector space of all initial conditions that give bounded solutions is

$$\left\{ \begin{bmatrix} y(0) \\ y'(0) \\ -5y(0) - 6y'(0) \end{bmatrix} \middle| \, y(0), y'(0) \in \mathbb{R} \right\}.$$

If one wanted to find the solutions that these ICs correspond to, they could compute the PF coefficients for B and C under these ICs (or simply fit the ICS using MATH 115 computations). Doing this either of these computations gives

$$y(t; a, b) = -\frac{a+b}{4}e^{-5t} + \frac{5a+b}{4}e^{-t}$$

which confirms that the solutions are bounded, satisfy the ICs we prescribed and solve the unforced DE.