MATH 213 - Lecture 12: Fun with Foreshadowing

In the synthesis problem we are given $y(t)$ and want to find $f(t)$. Later we will cover how to do this in mathematical detail but... how do we in principle do this??? We will look at an example!

Example 1
Consider the DE

$$my'' = f(t) - \mu y'(t)$$

that comes from $ma = F$ applied to a car with mass m and drag coefficient mu.

Solve the DE for $y(t)$ given $f(t)$. You can leave your solution as a convolution integral.

Solution: The solution is

$$y(t) = \int_0^t f(\tau)\mathscr{L}^{-1}\left\{\frac{1}{ms^2+\mu s}\right\}\bigg|_{t-\tau} d\tau$$

To compute the inverse Laplace transform note that

$$\begin{aligned}\frac{1}{ms^2+\mu s} &= \frac{1}{s\,(ms+\mu)} \\ &= \frac{1}{\mu s} - \frac{m}{\mu(ms+\mu)} \\ &= \frac{1}{\mu s} - \frac{1}{\mu\left(s+\frac{\mu}{m}\right)}\end{aligned}$$

Hence,

$$\mathscr{L}^{-1}\left\{\frac{1}{ms^2+\mu s}\right\} = \frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu t}{m}}.$$

Combining this with the previous result gives

$$y(t) = \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau$$

Now suppose that we want to find $f(t)$ such that $y(t)$ is given by a particular provided function. In the next few examples, we will derive a P controller that is commonly used to solve this problem in real time.

Example 2
Suppose someone tells you that $f(t)$ is a forcing term that might result in the $y(t)$ you want. Find the expression for the error between the system's response to this forcing term and and actual function $\tilde{y}(t)$ that you wanted.

The response of the system to $f(t)$ is given by

$$y(t) = \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau$$

and hence the error is

$$\tilde{y}(t) - \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau$$

Example 3
Formulate an optimization problem that if solved, would solve the synthesis problem of finding $f(t)$ so that the system response is some given $\tilde{y}(t)$.

If the error is 0 for all values of t, then we found the solution to the synthesis problem!

We thus want to pick $f(t)$ (ideally that is smooth) such that the error is as close to 0 as possible. Hence we want to solve the optimization problem

$$\min_{f(t)}\left|\tilde{y}(t) - \int_0^t f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right)d\tau\right|$$

In the above there are a few different ways to define what we mean by min. Two common ones are

- Minimize the maximum value over all t i.e. the L_∞ norm
- Minimize the average value i.e. the L_1 norm

Example 4
In many real world problems in computing, time is discrete and one knows $y(\tau)$ and $f(\tau)$ for $\tau \leq T$.

In these cases for the synthesis problem, one wants to pick $f(T)$ so that $y(T+\Delta t)$ is approximately $\tilde{y}(T+\Delta t)$ where Δt is the discreet increment in time where we know $y(t)$.

Use the error term above to find an expression for $f(T)$ that will approximately minimize the error at $T+\Delta t$.

The (signed) error at $T+\Delta t$ is given by

$$\begin{aligned} E(T+\Delta t) &= \tilde{y}(T+\Delta t) - \int_0^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \\ &= E(T) + \tilde{y}(T+\Delta t) - \tilde{y}(T) - \int_T^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \end{aligned}$$

Assuming $\tilde{y}(t)$ is continuous and Δt is small, we can write

$$\tilde{y}(T+\Delta t) - \tilde{y}(T) \approx 0$$

and

$$\int_T^{T+\Delta t} f(\tau)\left(\frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}}\right) d\tau \approx f(T)\int_T^{T+\Delta t} \frac{1}{\mu} - \frac{1}{\mu}e^{-\frac{\mu(t-\tau)}{m}} d\tau = k_p f(T)$$

where k_p is some number (formally it is the integral but we don't want to compute it in practice...). Thus

$$E(T+\Delta t) \approx E(T) - k_p f(T).$$

If we want $E(T+\Delta t) = 0$ then we can pick $f(T) \approx \frac{E(T)}{k_p}$ which we will write as $f(T) = k_p E(T)$ where k_p is a non-zero constant.

Example 5

Use the previous results to write an algorithm that takes a function $\tilde{y}(t)$ and computes $f(t)$ that produces a $y(t)$ that approximates $\tilde{y}(t)$.

Here is some psudo code

```
Define y_want(t), kp, the error at t=a and
the convolution kernal, G, for the given DE.
for t=0:Dt:b
        f(t+Dt) = kp*E(t)
        y(t+Dt) = num_int(f(u)*G(t-u),u,0,t+Dt)
        E(t+Dt)= y_want(T+Dt)-y(t+Dt)
end
```

In the above num_int is some numerical method that integrates.
Lecture12_basic_PID.m implements a method that uses the above psudo code with extra controls for the integral and derivative of the error.