MATH 213 - Tutorial 8: Bode plots Solutions

1. Draw the Bode plot for
$$T(s) = 100\frac{s+10}{s(s+11)((s+5)^2+100)}$$

Solution: We have a zero which will have an impact at $\omega = 10$, a simple pole that will have an impact at $\omega = 11$ and complex conjugate poles with $\omega = 10$.

For the initial amplitude note that since we have a pole at $s = 0$, we will not have an initial amplitude (like the 1/s diagram from L18) and hence we will pick a small wave number cutoff to start our plot. Since the next points where something of interest happens is $\omega = 10$ we pick $\omega = 0.01$ as our cutoff since this choice will show us the general behaviour for small ω for this omega we have

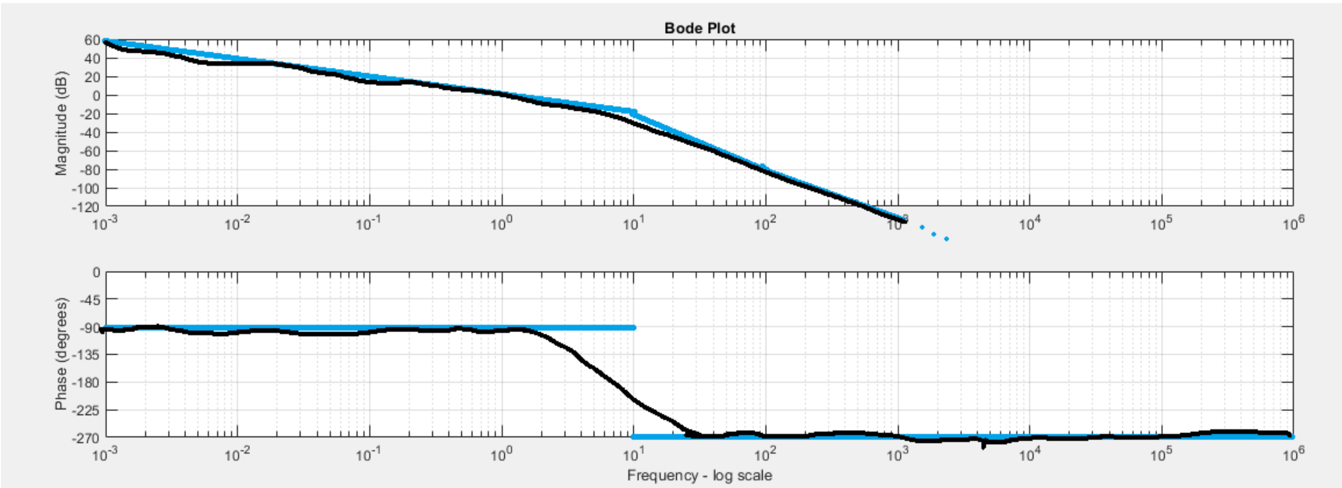
$$\begin{aligned}|T(\omega j)|_{dB} &= 20\log_{10}(100)\left|\frac{\omega j+10}{\omega j(\omega j+11)((\omega j+5)^2+100)}\right|_{dB} \\ &= 20\log_{10}(100) + |\omega j+10|_{dB} - |\omega j|_{dB} - |\omega j+11|_{dB} - \left|(\omega j+5)^2+100\right|_{dB}\end{aligned}$$

For our choice of $\omega = 0.01$ we have
$$\begin{aligned}|T(0.01)|_{dB} &\approx 20\cdot 2 + 20\log_{10}(10) - 20\log_{10}(0.01) - 20\log_{10}(11) - 20\log_{10}(100) \\ &= 20(2+1-(-2)-1-2) \\ &= 40\end{aligned}$$

The initial angle will be -90 since we have the simple pole at $s = 0$.

Now the root at 0 will cause a -20bD/decade drop in amplitude and a constant frequency for “large” ω’s until another pole/zero takes over. This happens at $\omega = 10$. Here we have a pole at 10, a root at 11 and complex conjugate pairs of poles at $\sqrt{125}$. The pole and root will give us no major adjustments due to these effects cancelling. More formally the root kicks in at $\omega \approx 10^1$ and the pole kicks in at $10^{1.04}$ so we will barely see the adjustment caused by these on a log-log plot. On the other hand the complex conjugate poles cause a change in the rate of the amplitude drop of an extra -40 dB/decade causing a net effect of -60dB/decade when adding in the previous effect. Additionally the phase will drop by -180 degrees to a total of -270 degrees. To see the speed of this adjustment note that $(s+5)^2 + 100 = s^2 + 10s + 125$ here $\omega \approx 11$ and hence $2\xi\omega = 10$ so $\xi \approx 0.5$ this will lead to a relatively show adjustment in the phase.

Here is a plot of these features in blue along with approximate curves in black.



2. Draw the Bode plot for

$$T(s) = \frac{10^7(s+0.01)}{((s+56)^2+0.1)((s+12)^2+60)}$$

Solution:

Here there is no pole at 0 and all the poles/zeros have negative real parts so there will be a starting angle of 0 and a starting amplitude of

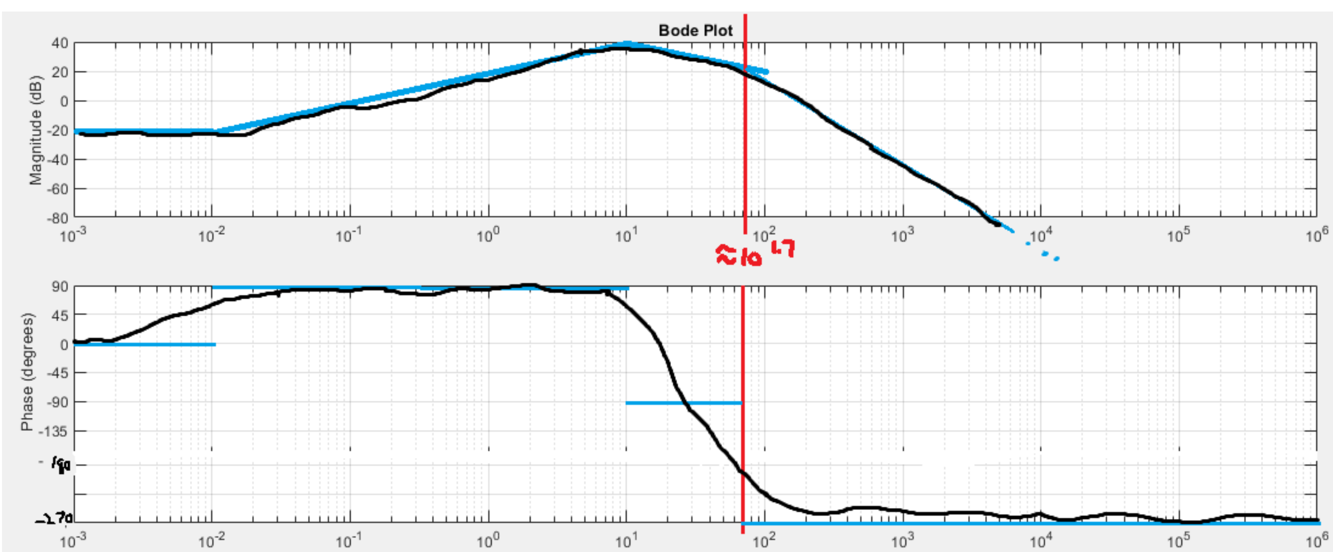
$$\begin{aligned}
|T(0)|_{dB} &= \lim_{\omega\to 0}\left|\frac{10^7(j\omega+0.01)}{(j\omega+56)^2+0.1)((j\omega+12)^2+60)}\right|_{dB} \\
&= \left|\frac{10^7(0.01)}{(56)^2+0.1)((12)^2+60)}\right|_{dB} \\
&= \left|10^7(0.01)\right|_{dB} - \left|(56)^2+0.1\right|_{dB} - \left|(12)^2+60\right|_{dB} \\
&= 20(\log_{10}(10^7(0.02)) - \log_{10}((56)^2+0.1) - \log_{10}((12)^2+60)) \\
&\approx -20
\end{aligned}$$

So we have a starting amplitude of -20 dB

The zero at $\omega = 0.01$ will cause a amplitude gain on $20dB/decade$ and a phase gain of 90 degrees at this point.

Then we have the poles to worry about. These will impact the solution around $\sqrt{56^2+0.1} \approx 56$ and $\sqrt{144+60} \approx 14$ each of these will have an impact of decreasing the amplitude curve by -40 bd/decade and dropping the phase by 180 degrees. This will happen at $10^{1.7}$ and $10^{1.14}$ respectively so we could combine these effects to have a less accurate but close plot but we will draw these effects separately to be a bit more accurate. In terms of the speed of this adjustment, we can compute the ξs to be 0.99 and 0.84. Hence the phase adjustments will be slow in both cases.

Adding up these effects we get the following plot with the blue lines denoting the asymptotic behaviours



3. Draw the Bode plot for

$$T(s) = 100\frac{10s-1}{(s-10)(s+960)}$$

Use the bode plot to argue that the system is unstable.

Solution: We have an initial amplitude of

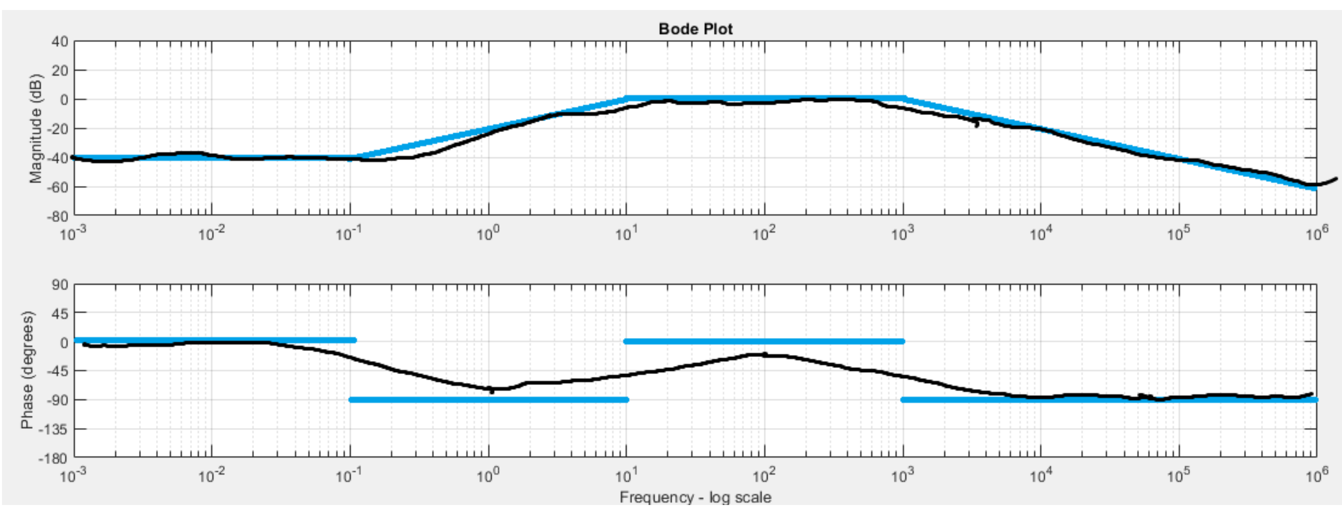
$$\begin{aligned} |T(0)|_{dB} &= \lim_{\omega\to 0}\left|100\frac{10\omega j-1}{(\omega j-10)(\omega j+960)}\right|_{dB} \\ &= \left|\frac{-100}{(-10)(960)}\right|_{dB} \\ &= 20\log_{10}(1/9600) \\ &\approx -40 \end{aligned}$$

Here there is no pole at 0 and we have an even number of poles/zeros with a negative real part so there will be a starting angle of 0

Using the types of analysis we have previously used, there will net be a

- Change to an amplitude gain of 20 dB/decade and adjustment to a phase of -90 degrees because of the zero at 0.1 (note the phase change is opposite of what we has before because of the sign of the zero).
- Change to an amplitude gain of 0 dB/decade and adjustment to a phase of 0 degrees because of the pole at 10 (note the phase change is opposite of what we has before because of the sign of the pole).
- Change to an amplitude drop of 20 dB/decade and adjustment to a phase of -90 degrees because of the pole at 1000.

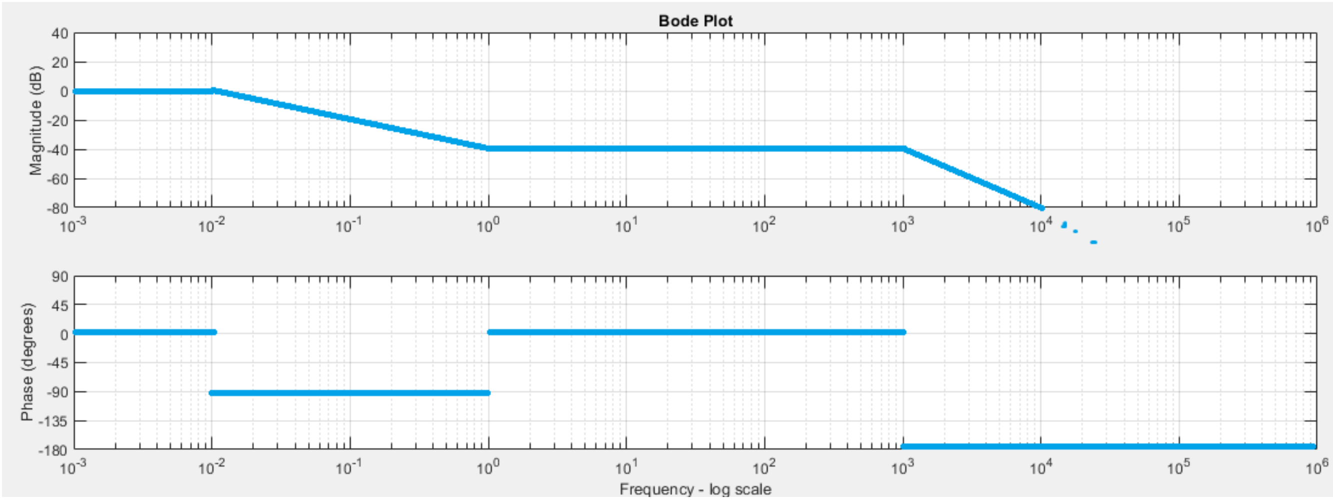
Here is the plot



This is the bode plot for an unstable system since we see that the net amplitude change decreases at 10 while the phase increases indicating a netpole around $s = -10$ which we see from our transfer function as well.

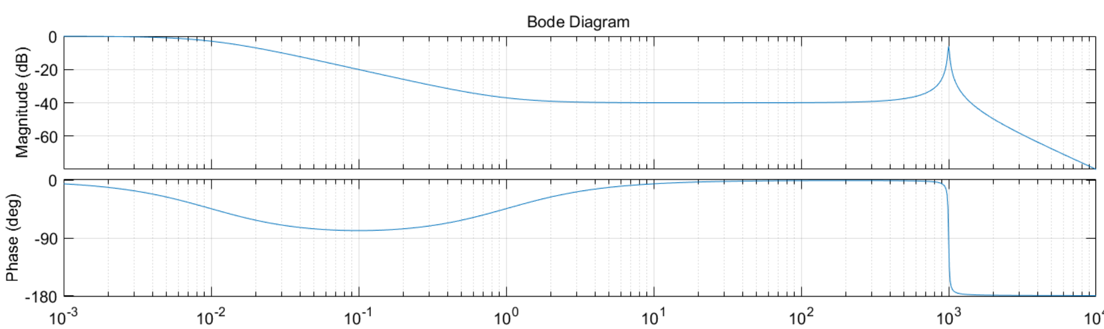
4. Suppose we are controlling the system $T(s) = \frac{10^4}{s+0.01}$ with the controller $C(s) = \frac{s+1}{(s+10)^2+10^6}$ and we know that $T(s)$ and $C(s)$ are BIBO stable transfer functions. Draw the Bode plot for $T(s)C(s)$ and use this to determine if the closed loop system is stable.

   Solution: We have generated a number of Bode plots in this tutorial so we will skip the computational details and simply draw the plot for the asymptotic behaviour of CT



For the closed loop system to be unstable we need $CT = -1$ which translates to the being 0 while the phase is -180. From the plot we see that early on the magnitude is 0 but the phase is also 0. Later on around 10^3, we see that the phase goes to -180 degrees while the magnitude will start to decrease.

This looks good but if the conjugate poles adjust quickly then we can have strong overshoots and might reach a magnitude of 0 while the phase is near -180. Thankfully since the phase before the starting adjustment is 0 and the phase does not overshoot with repeated poles, the system will mathematically be stable. Nevertheless we will check the speed of the adjustment for a ballpark idea of the sensitivity of the system. We thus note that $(s+10)^2 + 10^6 = s^2 + 20s + 10^6 + 100$ so $\omega \approx 10^3$ and $\xi \approx 10^{-2}$ which tells us we will have a super fast adjustment. Plotting the exact curve gives



We see that the closed loop controlled system is stable but one should exercise caution in the case that the parameters have some numerical error/can change in the physical application