MATH 213 - Tutorial 6: Systems 2 Solutions

1. Determine if the system with transfer function given by

$$H(t) = \frac{7s+9}{s^5 - 2s^3 - 3s}$$

is marginally stable (i.e. remains bounded for bounded inputs).

Solution: For the system to be stable, all the poles must be in the LHP with the exception of poles of order 1 on the imaginary line. Note that

$$\begin{aligned} s^5 - 2s^3 - 3s &= s(s^4 - 2s^2 - 3) \\ &= s(s^2 - 3)(s^2 + 1) \end{aligned}$$

The poles are thus $s = 0, \pm j, \pm\sqrt{3}$. The $s = \sqrt{3}$ pole causes the solution to grow without bound so the system is not marginally stable.

2. Consider the unstable system with transfer function given by

$$H(s) = \frac{1}{s-a}$$

for $a > 0$.

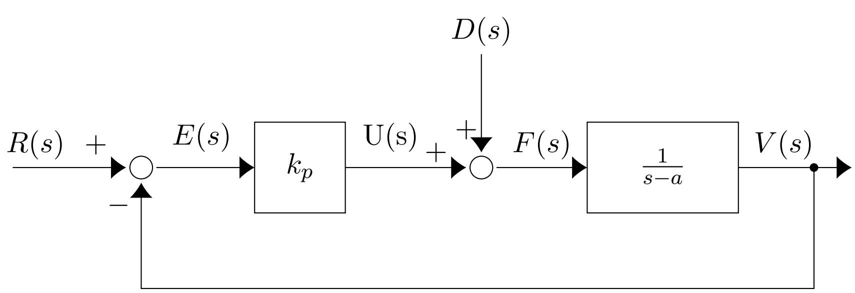
(a) Draw the system diagram for the controlled system.

(b) Find the transfer function for the proportionally controlled system.

(c) Determine a condition on k_p so that the controlled system is stable (all poles have a negative real part).

Solution:

(a) The system diagram is



(b) From L15 we showed that the general form for the transform function of a system of this form is

$$H(s) = \frac{P(s)C(S)}{1 + P(s)C(s)}$$

where $P(s)$ is the transfer function for the uncontrolled system and $C(s)$ is the transfer function for the controller. For our system this gives

$$\begin{aligned} H(s) &= \frac{P(s)C(S)}{1 + P(s)C(s)} \\ &= \frac{\frac{k_p}{s-a}}{1 + \frac{k_p}{s-a}} \\ &= \frac{k_p}{s - a + k_p} \end{aligned}$$

(c) The only pole of the transfer function is located at $s = a - k_p$. For this system to be stable we need this pole to have a negative real component. Hence we need $a - k_p < 0$ or $a < k_p$. Hence, if we make k_p sufficiently large, then the controlled system will be stable.

3. Consider the system with transfer function given by

$$H(s) = \frac{1}{s(s+2)(s+5)((s+1)^2+4)((s+4)^2+9)}.$$

(a) Find the poles of the transfer function.

(b) Use the locations of each pole to identify the contribution to the behaviour of the system's impulse response due to the existence of that pole.

(c) Use the final value theorem to find the final value of the impulse response of this system.

(d) Using the slowest decaying pole(s) to determine an expression that approximated the time for which the impulse response would decay to 1% of its final value. Your expression can be in terms of some coefficients that could be found via partial fraction decomposition.

Solution:

(a) The poles are $s = 0, -2, -5, -1 \pm 2j, -4 \pm 3j$

(b) The pole $s = 0$ is of order 1 and hence it gives us a constant value. The poles $s = -22, -5$ are on the negative real line so they will contribute to some exponential decay. The poles $-1 \pm 2j$ and $-4 \pm 3j$ are on the left hand side of the complex plane and have non-zero real parts so they oscillate but also decay.

(c) $H(s)$ is a strictly proper rational function and the poles all have negative real parts except for the simple pole at $s = 0$ so the FVTtells us that

$$\begin{aligned}
\lim_{t\to\infty} h(t) &= \lim_{s\to 0^+} \frac{s}{s(s+2)(s+5)((s+1)^2+4)((s+4)^2+9)} \\
&= \lim_{s\to 0^+} \frac{1}{(s+2)(s+5)((s+1)^2+4)((s+4)^2+9)} \\
&= \frac{1}{(2)(5)(5)(25)} \\
&= \frac{1}{1250}
\end{aligned}$$

(d) We can write $H(s)$ in the form

$$\frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+5} + \frac{Ds+E}{(s+1)^2+4} + \frac{Fs+G}{(s+4)^2+9}$$

The term $\frac{Ds+E}{(s+1)^2+4}$ contains the poles which decay the slowest and hence as long as the coefficients are comparable in magnitude, this term will approximately determine the decay rate. In the time domain, this term becomes

$$e^{-t}\left(D\cos(2t) + \frac{E}{2}\sin(2t)\right).$$

Since the trig terms are bounded above by 1, we can take $M = 2 \cdot max(|D|, |E/2|)$ to approximately be the upper bound for the trig term. Hence the envelope (function that bounds the magnitude of the trig terms) is approximately

$$Me^{-t}$$

For our solution to decay to roughly 1% of the final value, we need t to be such that $Me^{-t} < \frac{0.01}{1250}$ Hence we need

$$125000M < e^t \qquad \text{or} \qquad t = \ln(125000M).$$

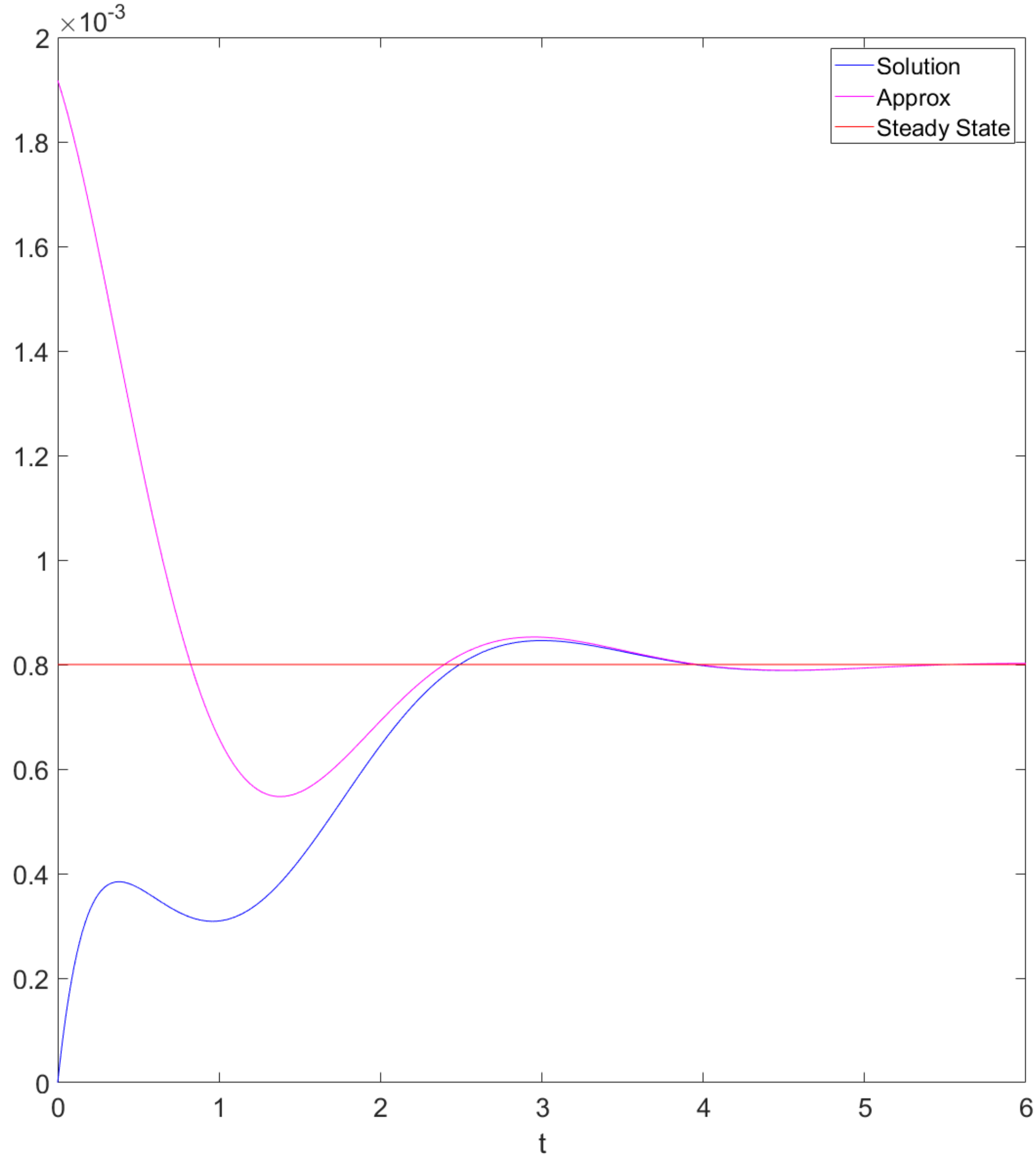
For this case we have the approximate value of $t = 4.9395$.
To check this, note that the PF decomposition gives the coefficients $A = \frac{1}{1250}$, $B = -\frac{1}{390}$, $C = \frac{1}{3000}$, $D = \frac{19}{17000}$, $E = \frac{3}{17000}$, $F = \frac{173}{552500}$, $G = \frac{609}{552500}$. This gives

$$h(t) = A + Be^{-2t} + Ce^{-5t} + e^{-t}(D\cos(2t) + E/2\sin(2t)) + e^{-4t}(F\cos(3t) + G/2\sin(3t))$$

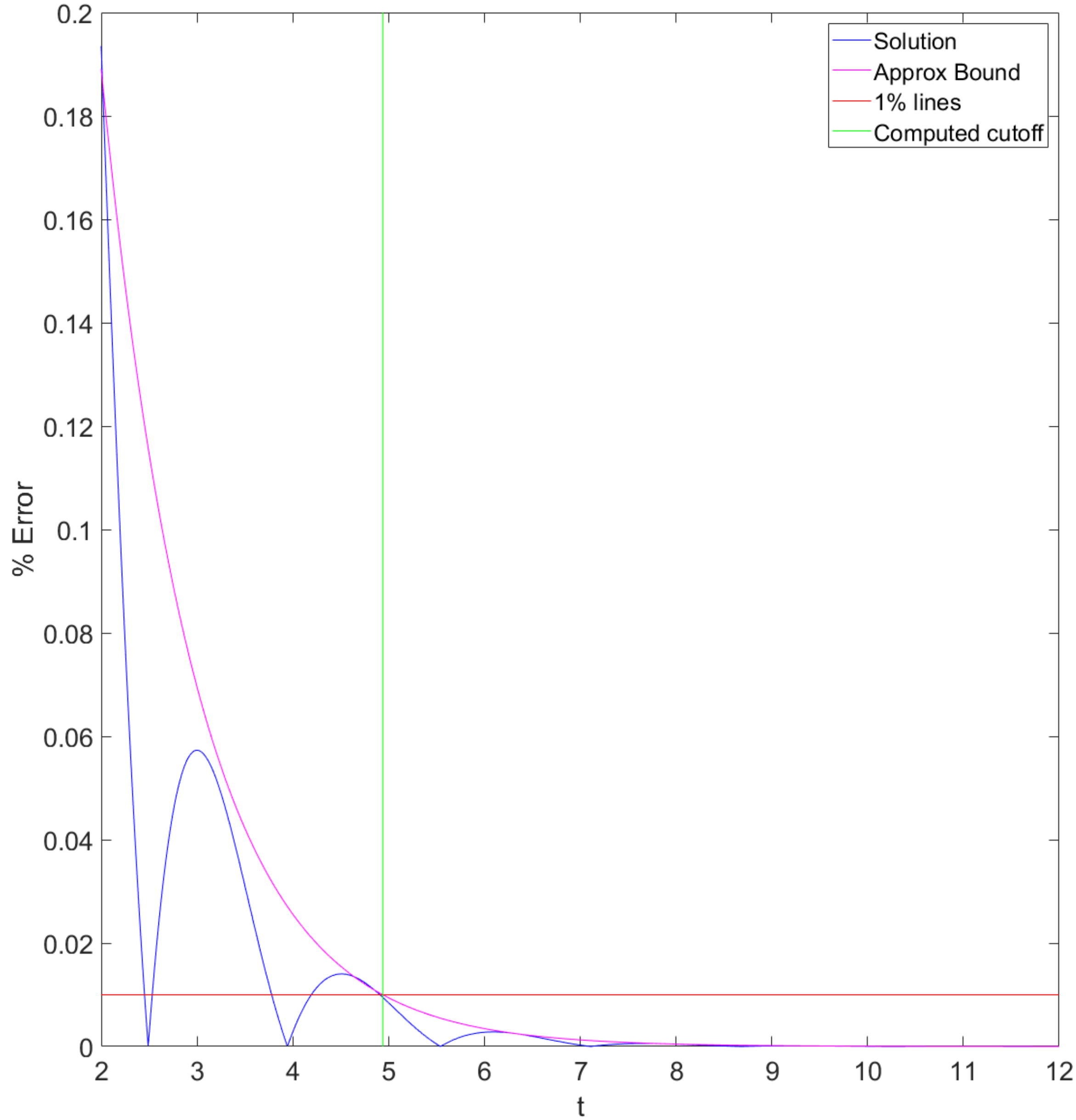
Hence, by using only the slowest decaying terms in our approximation we are effectively dropping the other terms to write

$$h_{approx}(t) = A + e^{-t}(D\cos(2t) + E/2\sin(2t))$$

Plotting the full solution alongside the final value and this approximation gives the plot



Here we can see that our approximation agrees rather well with the exact solution for $t > 3$ or so and hence our approximation of t above should be good. To verify this, we plot the percent error from the steady state for the full solution and our upper bound Me^{-t} along with lines for the 1% error thresholds.



Here we started the plot at t=2 to better show the error. We can see that the magenta line and the blue lines stay within the red 1% error line after the threshold we computed earlier at $t = 4.9395$ (green line). This validates our analysis from before. In general there will be some error in this method but for many applications, this is a sufficient method.