MATH 213 - Lecture 13: Systems Input and Responses

Lecture goals: Understand the impulse response of an LTI, the convolution property of LTIs and the response of an LIT to exponential inputs.

Recap of some DE results:

Recall that for a linear DE with constant coefficients the zero state response is

$$Y(s) = H(s)F(s)$$

where $H(s)$ is the transfer function of the DE and $F(s)$ is the Laplace transform of the forcing term.

If the forcing term is the delta impulse function (i.e. $f(t) = \delta(t)$), then $F(s) = 1$ and thus

$$Y(s) = H(t) \cdot 1 = H(t).$$

Thus we also call

$$h(t) = \mathscr{L}^{-1}\{H(s)\}$$

the **impulse response of the** DE

Note that this is simply the zero-state response to a unit impulse.

Thus when we write

$$Y(s) = H(s)F(s)$$

the transfer function, $H(s)$, is always just the Laplace transform of the impulse response. Finally by the convolution theorem the zero-state solution is simply

$$y(t) = (h * f)(t).$$

Impulse response of a LTI:

The above holds for Linear DEs with constant coefficients but what about more general LTIs. In general these might not be able to be modelled by a DE... so how do we find their responses to general inputs.

$A\vec{f} = \vec{y}$ as an ex

Theorem 1
The response of an LTI system is the convolution of the input with the system's impulse response.

Proving this in the CT case is a bit messy so we will show that the result holds for the DT case and then make the connection to the CT case.

First we need to define some DT analogues of CT objects we have worked with.

Definition 1: Kronecker delta
The **Kronecker delta** function $\delta[t]$ is defined as the function from $\mathbb{Z}$ to $\{0, 1\}$ given by

$$\delta[t] = \begin{cases} 1 & t = 0 \\ 0 & \text{else} \end{cases}$$

Theorem 2
For all functions $f : \mathbb{Z} \to \mathbb{C}$

$$f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\delta[\tau - t]$$

$$f[t] = \begin{cases} \vdots & \\ f[-1] & t=-1 \\ f[0] & t=0 \\ f[1] & t=1 \\ \vdots & \end{cases} = \sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[\tau-t]$$

Theorem 2 shows how to decompose every function f into a linear combination of impulse functions (i.e. $\{\delta[\tau - t] | \tau \in \mathbb{Z}\}$ is a basis for all DT functions.)

Proof of Th 1: Let $\delta[t-\tau] \xrightarrow{S} h[t,\tau]$

we do not know what this looks like

For any $f: \mathbb{Z} \to \mathbb{C}$, $f[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[t-\tau]$.

Thus the system response to f is

$$y[t] = S(f[t])$$
$$= S\left(\sum_{\tau=-\infty}^{\infty} f[\tau]\,\delta[t-\tau]\right)$$
$$= \sum_{\tau=-\infty}^{\infty} f[\tau]\, S(\delta[t-\tau])$$

S is linear & $f[\tau]$ is const WRT t.

$$= \sum_{\tau=-\infty}^{\infty} f[\tau]\, h[t,\tau]$$

$\boxed{\text{first line in proof}}$

This is not a convolution! We need $h[t,\tau] = \text{``}h[t-\tau]\text{''}$

We use time-invariance to prove this.

Note $\delta[t] \xrightarrow{S} h[t,0]$

so $\delta[t-\tau] \xrightarrow{S} h[t-\tau, 0]$ $\leftarrow$ Time-invariance

but we also have

$\delta[t-\tau] \xrightarrow{S} h[t,\tau]$

So we must have $h[t,\tau] = h[t-\tau, 0]$

call $h[t-\tau, 0]$, $h[t-\tau]$.

Thus,

$$y[t] = \sum_{\tau=-\infty}^{\infty} f[\tau]\, h[t-\tau]$$

$$= (f * h)[t]$$

so Thm holds for DT LTIs

For CT LTIs if

$$f \xrightarrow{S} y$$

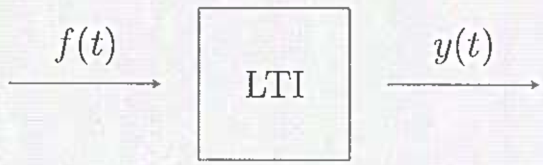
then

$$y(t) = \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, d\tau$$

where $h(t)$ is the system's impulse response.

Idea: Do the same computations w/ cont. t. functions & δ

Theorem 3: More explicit version of Th 1
Given the LTI



$$y(t) = (h * f)(t)$$

where $h(t)$ is the **impulse response of the system**.

Theorem 4: LTI response to an exponential
If $S : f \to y$ is a LTI then for any $s \in \mathbb{C}$

$$e^{st} \to H(s)e^{st}$$

where $H(s)$ is the LT of the system's impulse response and is called the **system's transfer function.**

Note this means that complex exponentials are the eigenfunctions of LTIs and the transfer function tells you the eigenvalues of the system! (like with DEs)

Proof: we only know the system is LTI and nothing else!

Suppose $S : f \longrightarrow y$ is LTI and suppose for some $s \in \mathbb{C}$

$$e^{st} \xrightarrow{S} y(t)$$ want to know

S is time-inv. so

$$e^{s(t-T)} \xrightarrow{S} y(t-T)$$

BUT $e^{s(t-T)} = e^{-sT}e^{st}$ so by linearity

$$e^{s(t-T)} = e^{-sT}e^{st} \xrightarrow{S} e^{-sT}y(t)$$

Hence $\forall\, T \in \mathbb{R}$, $y(t-T) = e^{-sT}y(t)$

If we pick $T = t$ then we have

$$y(0) = e^{-st} y(t)$$

or $y(t) = y(0)\, e^{+st}$

Hence $e^{st} \xrightarrow{S} y(0)\, e^{st}$ or $\underbrace{S(e^{st}) = y(0)\, e^{st}}$

eigenfunction equati

What is $y(0)$?

By Th 3:

$$y(t) = (h * e^{st})(t)$$

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$$

eigenvalues

TLDR; In general the response of a LTI system to an input $f(t)$ is given by

$$y(t) = (h * f)(t)$$

and since the eigenfunctions of LTIs are complex exponentials, we can decompose $f(t)$ into the eigenbasis of complex exponentials (like we did with DEs).

In the frequency domain we have

$$Y(s) = H(s)F(s).$$

Hence, if we want to analyze the system we can replace convolution with multiplication and simply analyze the system based on its eigenfunction decomposition, poles, etc. (like we did with DEs).