# Computer-Assisted Proof of Stability of Traveling Waves in Compressible Gas

Blake Barker, Taylor Paskett, Kevin Zumbrun

Brigham Young University, 2019

## Abstract

We develop a method for proving stability of traveling waves in compressible Navier-Stokes using rigorous numerical verification. We obtain complete error bounds on all computations, including machine truncation error. We explain several novel methods that reduce numerical error in the computer-assisted computations.

## Motivation

The stability of traveling waves is an important field of study. In addition to furthering the study of mathematics in its own right, the study of stability of traveling waves has many practical applications, like model verification. A verified, trusted model can be used in place of expensive experimentation.

## Proving Stability

**Compressible Gas**
In Lagrangian coordinates in one spatial dimension, the Navier-Stokes equations of compressible gas are given by

$$\begin{aligned} v_t - u_t &= 0 \\ u_t + p_x &= \left(\frac{\mu u_x}{v}\right)_x \\ E_t + (pu)_x &= \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa T_x}{v}\right)_x \end{aligned}$$

where $v$ is the specific volume, $u$ is velocity, $T$ is temperature, and total energy, $E$, is given by $E = e + u^2/2$, where $e$ is the internal energy. We linearize this partial differential equation about its traveling wave solution, obtaining the linear *Evans ODE* of the form

$$W'(x) = A(x;\lambda)W(x).$$

**Evans Function**
The Evans function is a type of Wronskian whose zeros correspond to the point spectra of our PDE. Thus, to prove stability we compute the Evans function. In order to compute the Evans function, we must repeatedly solve the Evans ODE. Hence, our method for numerical proof provides rigorous numerical verification of our solution of the Evans ODE.

## Rigorous Numerical Verification

In order to rigorously verify our solution, we will utilize interval arithmetic available in the Python package mpmath. Mpmath utilizes a type of arbitrary-precision floating-point arithmetic which encloses the effects of machine truncation error. This is accomplished by rounding the endpoints of the interval in question so that after any operation, the interval encompasses all possible solutions of the operation. Interval arithmetic allows us to overcome the limits of machine truncation, but leads to problems of its own. If we are not cautious, these enclosures can become much larger than necessary, called the wrapping effect. Because of this wrapping effect, many of the standard, tried-and-true methods for solving the Evans ODE are unusable, leading to intervals that are too large to be useful. Thus, we develop a method which minimizes the wrapping effect and other sources of error.

## Strategy

**Taylor Integrator**
We seek to solve the Evans ODE. In doing so, we let

$$\tilde{A}(x_0 + h;\lambda) = \sum_{k=0}^{N} A_k h^k, \quad 0 < h < 1,$$

be a truncation of the full Taylor expansion of $A(x;\lambda)$. Each coefficient, $A_k$, is matrix-valued.

Let

$$Z_M(x_0 + h) = \sum_{k=0}^{M} B_k h^k,$$

where each $B_k$ is vector-valued and $M \geq N$. Then, we will solve the system $Z_M'(x) = \tilde{A}(x;\lambda)Z_M(x)$ and prove that $Z_M \to W$ as $M, N \to \infty$.

**Recurrence Relationship**
After some manipulation, it can be shown that the coefficients of $Z_M$ follow a recurrence relationship:

$$B_{n+1} = \frac{1}{n+1}\sum_{k=0}^{N} A_k B_{n-k}.$$

This relationship can be used to compute $Z_M$ directly.

## Rigorous Error Bounds

### Proposition

For the sequences $(B_k)_{k=0}^{\infty}$, $(A_k)_{k=0}^{N}$ with recurrence relationship as defined above, if there is $M \geq N$ such that

$$\frac{(N+1)\max_{i \leq N}\|A_i\|_\infty}{M+1} \leq 1,$$

and if

$$\|B_j\| \leq C\|B_{j-1}\|$$

for some $C > 0$ and any $j$ satisfying $M - N \leq j \leq M$, then for all $k > 0$,

$$\|B_{M+1+k}\| \leq C^k\|B_{M-N-1}\|.$$

This proposition helps us bound the remainder of $Z_M$ (call it $R_Z := \sum_{k=M+1}^{\infty} B_k h^k$):

$$\begin{aligned} \left\|\sum_{k=M+1}^{\infty} B_k h^k\right\| &\leq \|B_{M-N-1}\| \sum_{k=0}^{\infty} C^k h^{k+1+M} \\ &= \frac{h^{M+1}}{1-hC}\|B_{M-N-1}\|, \end{aligned}$$

so long as $|hC| < 1$.

Let $Z := Z_\infty$. Then, we can also find a bound on the error, $E(x) = W(x) - Z(x)$, by setting up $E$ as an ODE. Doing so, we obtain

$$\begin{aligned} \|E\|_\infty &\leq \frac{\|(A-\tilde{A})Z\|_\infty}{\|A\|_F}(e^{h\|A\|_F} - 1) \\ &\leq \frac{h^N\|A_N\|_\infty\|Z\|_\infty}{(1-h)\|A\|_F}(e^{h\|A\|_F} - 1), \end{aligned}$$

where $\|\cdot\|_F$ is the Frobenius norm and $\|\cdot\|_\infty$ is the matrix or vector infinity norm.

## Example

Letting $\Gamma = 2/3$, $v_+ = 0.26$, $\mu = 1$, $\nu = 1$, $\lambda = 2i$, $h = 0.1$, we were able to calculate W(x) beginning at L = -18.3, and proceed forward by step size $h$ until reaching $W(0)$. At each $x_i$, we let $W(x_i) = Z_M(x_i) + R_Z(x_i) + E(x_i)$, representing both $R_Z$ and $E$ with intervals that bound them. By the time we reached $W(0)$, the vector $W(0)$ had a maximum interval width of $4.2 \times 10^{-26}$.

### What's next?

- Test more parameters to make sure that our error bounds are sufficiently small for the entire range of valid parameters.
- Use our method to solve the Evans function and prove stability for compressible gas.
- Broaden the method by applying it to other traveling waves.

## Citations

- Barker, B., Zumbrun, K. Numerical Proof of Stability of Viscous Shock Profiles. *Mathematical Models and Methods in Applied Sciences*, **26**(13) (2016)
- Humpherys, J., Lyng, G., Zumbrun, K. Spectral Stability of Ideal Gas Shock Layers. (2008)
- Johansson, F. and others. *mpmath: a Python library for arbitrary-precision floating-point arithmetic (version 1.1.0)*, December 2018. http://mpmath.org/