CIVE2470: Water Engineering and Geotechnics
Water Engineering : Pipes and Pipe Networks

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Contents

1 Fluid Flow in Pipes

We will be looking here at the flow of real fluid in pipes - *real* meaning a fluid that possesses viscosity hence losses energy due to friction as fluid particles interact with one another and the pipe wall. Recall from Level 1 that the shear stress induced in a fluid flowing near a boundary is given by Newton's law of viscosity:

$$\tau \propto \frac{du}{dy}$$

This tells us that the shear stress, τ, in a fluid is proportional to the velocity gradient - the rate of change of velocity across the fluid path. For a *Newtonian* fluid we can write:

$$\tau = \mu \frac{du}{dy}$$

where the constant of proportionality, μ is known as the coefficient of *dynamic viscosity* (or simply viscosity).

Recall also that flow can be classified into one of two types, **laminar** or **turbulent** flow (with a small transitional region between these two). The non-dimensional number, the Reynolds number, Re, is used to determine which type of flow occurs:

$$\text{Re} = \frac{\rho u d}{\mu}$$

For a pipe

Laminar flow:	$\text{Re} > 2000$
Transitional flow:	$2000 < \text{Re} > 4000$
Turbulent flow:	$\text{Re} > 4000$

It is important to determine the flow type as this governs how the amount of energy lost to friction relates to the velocity of the flow. And hence how much energy must be used to move the fluid.

Flow in pipes is usually turbulent some common exceptions are oils of high viscosity and blood flow. In turbulent flow random fluctuating movements of the fluid particles are superimposed on the main flow - these movements are unpredictable. No complete theory is available to analyze turbulent flow as it is essentially a stochastic process (unlike laminar flow where good theory exists.) Most of what is known about turbulent flow has been obtained from experiments with pipes. It is convenient to study it in this form and also the pipe flow problem has significant commercial importance.

We shall cover sufficient to be able to predict the energy degradation (loss) in a pipe line. Any more than this and a detailed knowledge and investigation of boundary layers is required. Note that pipes which are not completely full and under pressure e.g. sewers are not treated by the theory presented here. They are essentially the same as open channels which will be covered elsewhere in this module.

1.1 Analysis of pipelines

To analyses the flow in a pipe line we will use Bernoulli's equation. The Bernoulli equation was introduced in the Level 1 module, and as a reminder it is presented again here:

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 = H = \text{Constant} \tag{1.1}$$

Which is written linking conditions at point 1 to conditions at point 1 in a flow. H is the total head which does not change. When applied to a pipeline we must also take into account any losses (or gains) in energy along the flow length.

Consider a pipeline as shown below linking two reservoirs A and B with a pump followed by a pipe that expands before reaching the downstream reservoir.

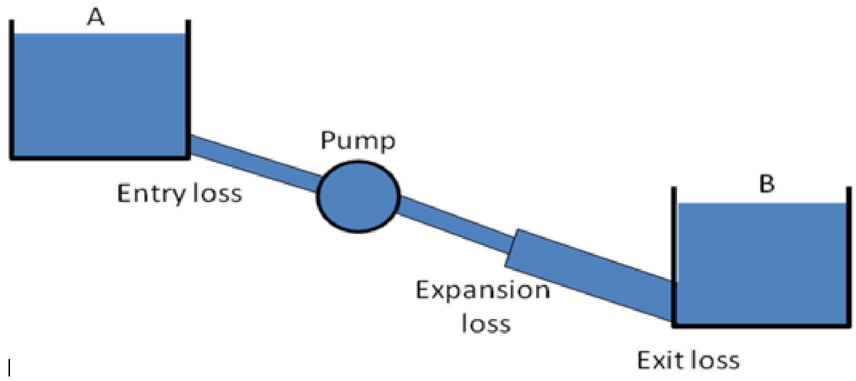

Figure 1.1: General components on a pipe line

Each term in the Bernoulli equation is energy per unit weight and has dimensionf of length, and thus units of m. We refer to this as *head*. This energy loss is then referred to a *head loss*. Examining the head losses, there will be a loss of head as the fluid flows into the pipe, the *Entry Loss* ($h_{L\,entry}$), then the pump puts energy into the fluid in terms of increasing the pressure head (h_{pump}). As the pipe expands there is an expansion loss ($h_{L\,expansion}$), then a second expansion loss, labeled Exit loss ($h_{L\,exit}$), as the fluid leaves the pipe into the reservoir. Along the whole length of the all the pipes there is a loss due to pipe friction (h_f).

The Bernoulli equation linking reservoir A with B would be written thus:

$$\frac{p_A}{\rho g} + \frac{u_A^2}{2g} + z_A + h_{\text{pump}} = \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + h_{L\,\text{entry}} + h_{L\,\text{expansion}} + h_{L\,\text{exit}} + h_f \tag{1.2}$$

(Where p_A, u_A, z_A are the pressure, velocity and height (elevation) of the surface of reservoir A. The corresponding terms are the same for B)

This is the general equation we use to solve for flow in a pipeline. The difficult part is the determination of the head loss terms in this equation. The following sections describe how these are quantified.

Before continuing it is useful to note that the above general equation can be quickly simplified to leave only expressions for driving head and head losses. Remember that the points A and

B are surfaces of reservoirs - they moves very slowly compared to the flow in the pipe so we can say $u_A = u_B \approx 0$. Also the pressure is atmospheric, $p_A = p_B = p_{Atmospheric}$. $z_A - z_B$ is the height difference between the two reservoir surfaces. So

$$(z_A - z_B) + h_{\text{pump}} = h_{L\text{ entry}} + h_{L\text{ expansion}} + h_{L\text{ exit}} + h_f \tag{1.3}$$

Which is the usual form we end up solving.

1.2 Pressure loss due to friction in a pipeline.

Consider a cylindrical element of incompressible fluid flowing in the pipe, as shown

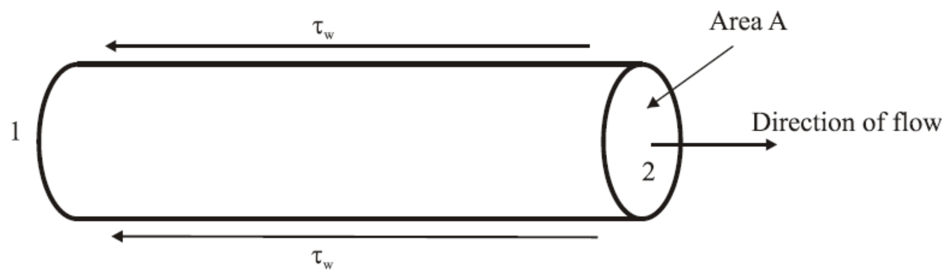

Figure 1.2: Element of fluid in a pipe

The pressure at the upstream end, 1, is p, and at the downstream end, 2, the pressure has fallen by Δp to $(p - \Delta p)$.

The driving force due to pressure (F = Pressure × Area) can then be written

$$\text{driving force} = \text{pressure force at 1} - \text{pressure force at 2}$$

$$pA - (p - \Delta p)A = \Delta pA = \Delta p \frac{\pi d^2}{4}$$

The retarding force is that due to the shear stress by the walls

$$\begin{aligned} \text{retarding force} &= \text{shear stress} \times \text{area over which acts} \\ &= \tau_w \times \text{area of wall} \\ &= \tau_w \pi d L \end{aligned}$$

As the flow is in equilibrium,

$$\begin{aligned} \text{driving force} &= \text{retarding force} \\ \Delta p \frac{\pi d^2}{4} &= \tau_w \pi d L \\ \Delta p &= \frac{\tau_w 4L}{d} \end{aligned} \tag{1.4}$$

Giving an expression for pressure loss in a pipe in terms of the pipe diameter and the shear stress at the wall on the pipe.

The shear stress will vary with velocity of flow and hence with Re. Many experiments have been done with various fluids measuring the pressure loss at various Reynolds numbers. These results plotted to show a graph of the relationship between pressure loss and Re look similar to the figure below:

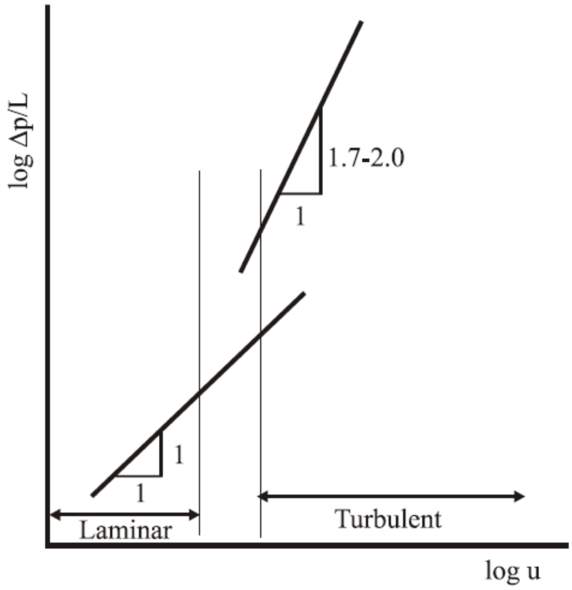

Figure 1.3: Relationship between velocity and pressure loss in pipes

This graph shows that the relationship between pressure loss and velocity can be expressed as

$$\begin{array}{ll} \text{Laminar flow:} & \Delta p \propto u \\ \text{Turbulent flow:} & \Delta p \propto u^a \end{array}$$

where $1.7 \leq a \leq 2.0$

As these are empirical relationships, they help in determining the pressure loss but not in finding the magnitude of the shear stress at the wall τ_w on a particular fluid. If we knew τ_w we could then use it to give a general equation to predict the pressure loss.

1.3 Pressure loss during laminar flow in a pipe

In general the shear stress τ_w is almost impossible to measure. But for laminar flow it is possible to calculate a theoretical value for a given velocity, fluid and pipe dimension. (As this was covered in the Level 1 module, only the result is presented here.) The pressure loss in a pipe with laminar flow is given by the Hagen-Poiseuille equation:

$$\Delta p = \frac{32\mu L u}{d^2}$$

or in terms of head

$$h_f = \frac{32\mu L u}{\rho g d^2} \tag{1.5}$$

Where h_f is known as the *head-loss due to friction.*

(Remember the velocity, u, is *mean* velocity.)

1.4 Pressure loss during turbulent flow in a pipe

In this derivation we will consider a general bounded flow - fluid flowing in a channel - we will then apply this to pipe flow. In general it is most common in engineering to have $Re > 2000$ i.e. turbulent flow - in both closed (pipes and ducts) and open (rivers and channels). However analytical expressions are not available so empirical relationships are required (those derived from experimental measurements). Consider the element of fluid, shown in 1.4 below, flowing in a channel, it has length L and with wetted perimeter P. The flow is steady and uniform so that acceleration is zero and the flow area at sections 1 and 2 is equal to A.

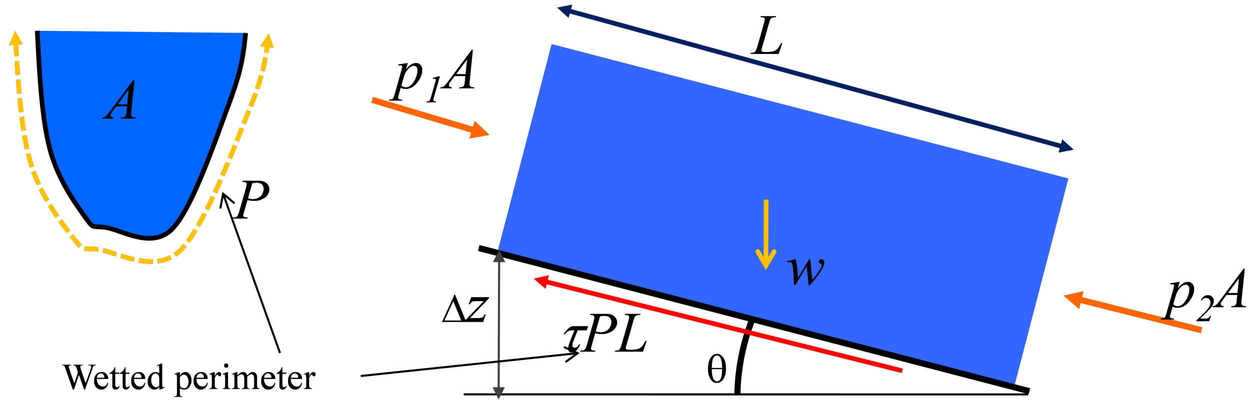

Figure 1.4: Element of fluid in a channel flowing with uniform flow

$$p_1 A - p_2 A - \tau_w LP + W \sin\theta = 0$$

writing the weight term as $\rho g AL$ and $\sin\theta = -\Delta z/L$ gives

$$A(p_1 - p_2) - \tau_w LP - \rho g A \Delta z = 0$$

this can be rearranged to give

$$\frac{[(p_1 - p_2) - \rho g \Delta z]}{L} - \tau_w \frac{P}{A} = 0$$

where the first term represents the piezometric head loss of the length L or (writing piezometric head p^*)

$$\tau_w = R\frac{dp^*}{dx} \tag{1.6}$$

where $R = A/P$ is known as the *hydraulic radius.*

Writing piezometric head loss as $p* = \rho g h_f$, then shear stress per unit length is expressed as

$$\tau_w = R\frac{dp^*}{dx} = R\frac{\rho g h_f}{L}$$

So we now have a relationship of shear stress at the wall to the rate of change in piezometric pressure. To make use of this equation an empirical factor must be introduced. This is usually in the form of a **friction factor** f, and written

$$\tau_w = f\frac{\rho u^2}{2}$$

where u is the mean flow velocity.

Hence

$$\frac{dp^*}{dx} = f\frac{\rho u^2}{2R} = \frac{\rho g h_f}{L}$$

So, for a general bounded flow, head loss due to friction can be written

$$h_f = \frac{fLu^2}{2gR} \tag{1.7}$$

More specifically, for a circular pipe, $R = A/P = (\pi d^2/4/(\pi d) = d/4$ giving

$$h_f = \frac{4fLu^2}{2gd} \tag{1.8}$$

We introduce another term for friction factore λ wheer $\lambda = 4f$, to give the following equivalent head loss equation

$$h_f = \frac{\lambda L u^2}{2gd} \tag{1.9}$$

Both equations 1.8 and 1.9 are form of the **Darcy-Weisbach equation** for head loss in circular pipes. (Often referred to simply as the **Darcy equation**)

This equation is equivalent to the Hagen-Poiseuille equation for laminar flow with the exception of the empirical friction factor f (or λ) introduced.

It is sometimes useful to write the Darcy equation in terms of discharge Q, (using $Q = Au$)

$$u = \frac{Q}{\pi d^2/4}$$

$$h_f = \frac{64fLQ^2}{2g\pi^2 d^5} = \frac{fLQ^2}{3.03d^5} = \frac{\lambda LQ^2}{12.1d^5} \tag{1.10}$$

Or with a 1% error

$$h_f = \frac{fLQ^2}{3d^5} = \frac{\lambda LQ^2}{12.1d^5} \tag{1.11}$$

NOTE On Friction Factor value

The f value shown above is different to that used in North American practice. Their relationship is

$$f_{American} = 4f$$

Commonly λ is used in place of f. For clarity:

$$\lambda = f_{American} = 4f$$

Consequently great care must be taken when choosing the value of f with attention taken to the source of that value. If you look for "darcy friction factor" on Wikipedia it uses the $f_{American}$, but does not use this terminology, just f. To be certain you must go back to check the original definition of f in the Darcy equation, Eqn 1.8 or 1.9.

1.4.1 Hazen-Williams equation

In North American practice it is most common **not** to use the Darcy-Weisbach formula but to use an empirical formula call the **Hazen-Williams** equation. I will quote it here to give you the knowledge of this equation - that is in common use but we shall not be using it further in this course.

Units are very important to note. The usual form is to express the pressure loss in *pounds per square inch* (psi) which results in this form of the equation:

$$P_d = \frac{4.52L}{C^{1.85}d^{4.87}}Q^{1.85}$$

where L = length in feet, Q = flow in US gallons per min, d = diameter in inches.

The equation can be re-written in SI units and in a form similar to the Darcy-Weisbach equation giving head loss in m:

$$h_f = \frac{10.67L}{C^{1.85}d^{4.87}}Q^{1.85} \tag{1.12}$$

where L is in m, Q is in m^3/s, d is in m.

The value of C is found by looking in tables. Typical values are:

Pipe Material	C
Cast Iron	100
Concrete	100
Copper	130
Steel	90
Polyethylene	140

1.4.2 A general Head loss formula

All of the friction formula can be written in the general form:

$$h_f = kQ^n$$

Where for the Darcy-Weisbach formula $n = 2$, and the Hazen-Williams $n = 1.85$.

1.5 Choice of friction factor f and λ

The value of f must be chosen with care or else the head loss will not be calculated correctly. Assessment of the physics governing the value of friction in a fluid has led to the following relationships

1. $h_f \propto L$
2. $h_f \propto v^2$
3. $h_f \propto 1/d$
4. h_f depends on surface roughness of pipes
5. h_f depends on fluid density and viscosity

6. h_f is independent of pressure

Consequently f cannot be a constant if it is to give correct head loss values from the Darcy equation. An expression that gives f based on fluid properties and the flow conditions is required.

1.5.1 The value of f and λ for Laminar flow

As mentioned above, the equation derived for head loss in turbulent flow is equivalent to that derived for laminar flow - the only difference being the empirical f (or λ). Equating the two expressions for head loss allows us to derive an expression of f (or λ) that allows the Darcy equation to be applied to laminar flow.

Equating the Hagen-Poiseuille and Darcy-Weisbach equations gives:

$$\begin{aligned} \frac{32\mu L u}{\rho g d^2} &= \frac{4fLu^2}{2gd} & \qquad \frac{32\mu L u}{\rho g d^2} &= \frac{\lambda L u^2}{2gd} \\ f &= \frac{16\mu}{\rho u d} & \lambda &= \frac{64\mu}{\rho u d} \\ f &= \frac{16}{\mathrm{Re}} & \lambda &= \frac{64}{\mathrm{Re}} \end{aligned} \tag{1.13}$$

1.5.2 Blasius equation for f and λ

Blasius, in 1913, was the first to give an accurate empirical expression for f for turbulent flow in smooth pipes, that is:

$$f = \frac{0.079}{\mathrm{Re}^{0.25}} \qquad \lambda = \frac{0.316}{\mathrm{Re}^{0.25}} \tag{1.14}$$

This expression is fairly accurate, giving head losses $\pm 5\%$ of actual values for Re up to 100000.

1.5.3 Nikuradse

Nikuradse made a great contribution to the theory of pipe flow by differentiating between rough and smooth pipes. A rough pipe is one where the mean height of roughness is greater than the thickness of the laminar sub-layer. Nikuradse artificially roughened pipe by coating them with sand. He defined a relative roughness value k_s/d (mean height of roughness over pipe diameter) and produced graphs of f (or λ) against Re for a range of relative roughness 1/30 to 1/1014.

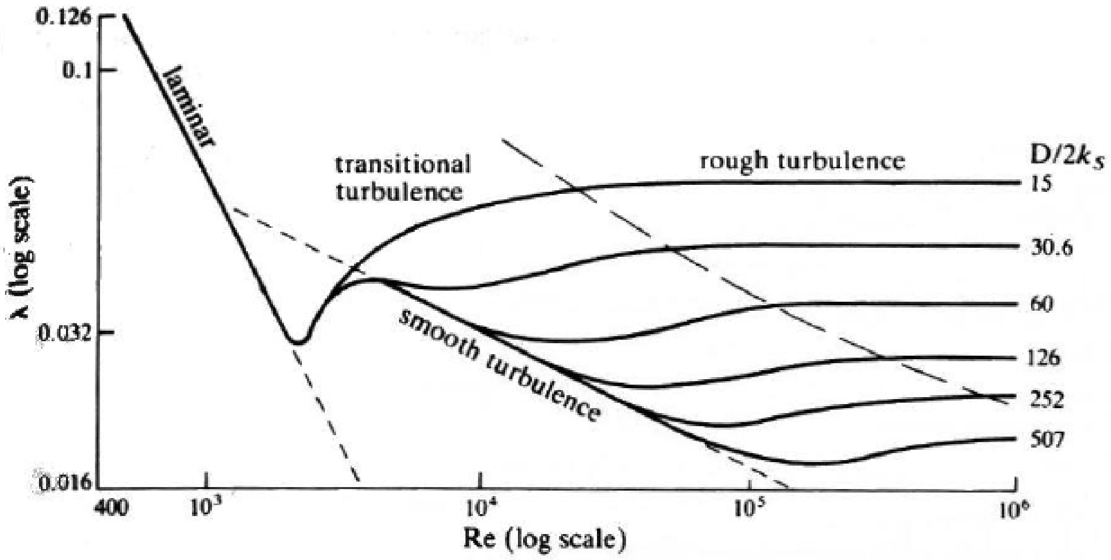

Figure 1.5: Regions on plot of Nikurades's data

A number of distinct regions can be identified on the diagram.

The regions which can be identified are:

1. **Laminar flow** ($f = 16/\mathrm{Re}$ or $\lambda = 64/\mathrm{Re}$)
2. **Transition from laminar to turbulent**. An unstable region between Re = 2000 and 4000. Pipe flow normally lies outside this region
3. **Smooth turbulent** ($f = \frac{0.079}{\mathrm{Re}^{0.25}}$, $\lambda = \frac{0.316}{\mathrm{Re}^{0.25}}$)
   The limiting line of turbulent flow. All values of relative roughness tend toward this as Re decreases.
4. **Transitional turbulent**.
   The region which f (or λ) varies with both Re and relative roughness. Most pipes lie in this region.
5. **Rough turbulent**. f (or λ) remains constant for a given relative roughness. It is independent of Re.

The reasons why these regions exist:

Laminar flow: Surface roughness has no influence on the shear stress in the fluid.
Smooth and Transitional Turbulence: The laminar sub-layer covers and *smooths* the rough surface with a thin laminar region. This means that the main body of the turbulent flow is unaffected by the roughness.
Rough turbulence: The laminar sub-layer is much less than the height of the roughness so the boundary affects the whole of the turbulent flow.

1.5.4 Hydraulically rough and smooth pipes.

a. In the short entry length of the pipe the flow will be laminar but this will, a short distance downstream, give way to fully developed turbulent flow and a laminar sub-layer.

b. In the laminar sub-layer is thick enough it will protect the turbulent flow from the roughness of the boundary and the pipe would be hydraulically smooth.

c. If the laminar sub-layer is thinner than the height of roughness, then the roughness protrudes through and the pipe is hydraulically rough.

d. The laminar sub-layer decreases in thickness with increasing Re. Therefore surface may be hydraulically smooth for low flows but hydraulically rough at high flows.

e. If the height of roughness is large the flow will be completely turbulent and f (or λ) will be unaffected by Re. i.e. if k/d is large then f (or λ) remains constant.

1.5.5 Colebrook-White equation for f (or λ)

Colebrook and White did a large number of experiments on commercial pipes and they also brought together some important theoretical work by von Karman and Prandtl. This work resulted in an equation attributed to them as the Colebrook-White equation:

$$\frac{1}{\sqrt{f}} = -4\log_{10}\left(\frac{k_s}{3.71d} + \frac{1.26}{\mathrm{Re}\sqrt{f}}\right) \quad , \qquad \frac{1}{\sqrt{\lambda}} = -2\log_{10}\left(\frac{k_s}{3.71d} + \frac{2.51}{\mathrm{Re}\sqrt{\lambda}}\right) \tag{1.15}$$

It is applicable to the whole of the turbulent region for commercial pipes and uses an effective roughness value (k_s) obtained experimentally for all commercial pipes. Note a particular difficulty with this equation. f (or λ) appears on both sides in a square root term and so cannot be calculated easily.

Trial and error methods used to be the only way to get f (or λ) once k_s Re and d are known. (In the 1940s when calculations were done by slide rule this was a time consuming task.) Nowadays it is relatively trivial to solve the equation on a programmable calculator or spreadsheet or phone app. (There is also an explicit solution method - not given here, but a web search would demonstrate this.)

Moody made a useful contribution to help, he plotted f (or λ) against Re for commercial pipes - see the figure below. This figure has become known as the *Moody Diagram* (or sometimes the *Stanton Diagram*). [Note that the version of the Moody diagram shown uses λ for friction factor. The shape of the diagram will not change if f were used instead.]

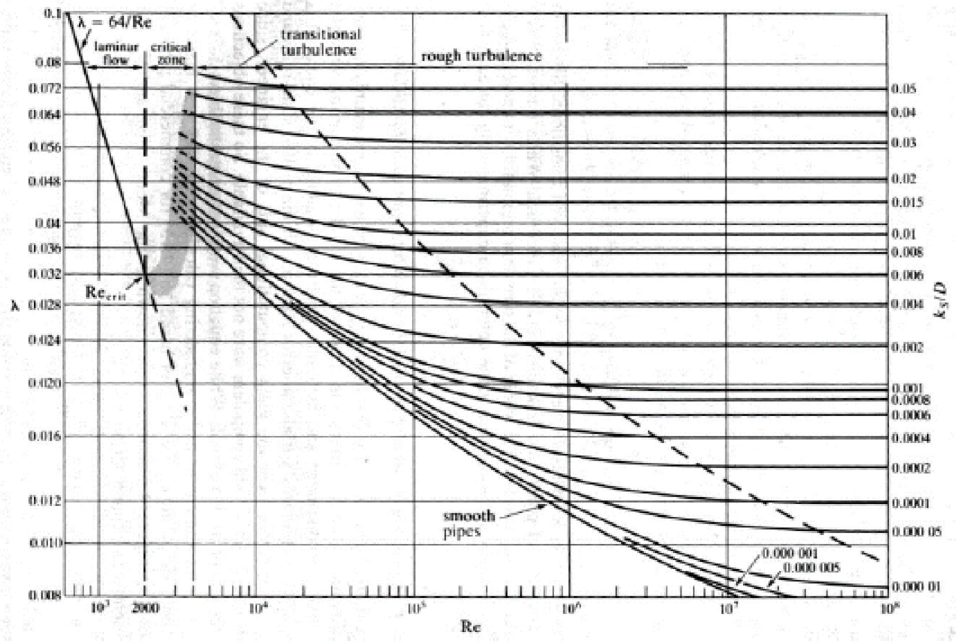

Figure 1.6: Moody Diagram.

Moody also developed an equation based on the Colebrook-White equation that made it simpler to calculate f (or λ) :

$$f = 0.001375 \left[1 + \left(20000 \frac{k_s}{d} + \frac{10^6}{\mathrm{Re}} \right)^{1/3} \right] \quad , \qquad \lambda = 0.0055 \left[1 + \left(20000 \frac{k_s}{d} + \frac{10^6}{\mathrm{Re}} \right)^{1/3} \right] \tag{1.16}$$

This equation of Moody gives f correct to $\pm 5\%$ for $4 \times 10^3 < \mathrm{Re} < 1 \times 10^7$ and for $k_s/d < 0.01$.

Barr presented an alternative explicit equation for f in 1975

$$\frac{1}{\sqrt{f}} = -4 \log_{10} \left[\frac{k_s}{3.71d} + \frac{5.1286}{\mathrm{Re}^{0.89}} \right] \quad , \qquad \frac{1}{\sqrt{\lambda}} = -2 \log_{10} \left[\frac{k_s}{3.71d} + \frac{5.1286}{\mathrm{Re}^{0.89}} \right] \tag{1.17}$$

or

$$f = 1 \Bigg/ \left(-4 \log_{10} \left[\frac{k_s}{3.71d} + \frac{5.1286}{\mathrm{Re}^{0.89}} \right] \right)^2 \quad , \qquad \lambda = 1 \Bigg/ \left(-2 \log_{10} \left[\frac{k_s}{3.71d} + \frac{5.1286}{\mathrm{Re}^{0.89}} \right] \right)^2 \tag{1.18}$$

Here the last term of the Colebrook-White equation has been replaced with $5.1286/\mathrm{Re}^{\,0.89}$ which provides more accurate results for $\mathrm{Re} > 10^5$. Many other formulas exist to approximate the solutions of λ. The Wikipedi page on *darcy friction factor* identifies a long list.

The problem with these formulas still remains that these contain a dependence on k_s. What value of k_s should be used for any particular pipe? Fortunately pipe manufactures provide values and typical values can often be taken similar to those in table 1.1 below.

Pipe Material	$k_s(mm)$
Brass, copper, glass, Perspex	0.003
Asbestos cement	0.03
Wrought iron	0.06
Galvanised iron	0.15
Plastic	0.03
Bitumen-lined ductile iron	0.03
Spun concrete lined ductile iron	0.03
Slimed concrete sewer	6.0

Table 1.1: Typical k_s values

1.6 Local Head Losses

In addition to head loss due to friction there are always head losses in pipe lines due to bends, junctions, valves etc. (See notes from Level 1 for a discussion of energy losses in flowing fluids.) For completeness of analysis these should be taken into account. In practice, in long pipe lines of several kilometres the effect of local head losses may be negligible. For short pipeline the losses may be greater than those for friction.

A general theory for local losses is not possible; however rough turbulent flow is usually assumed which gives the simple formula

$$h_L = k_L \frac{u^2}{2g} \tag{1.19}$$

Where h_L is the local head loss and k_L is a constant for a particular fitting (valve or junction etc.)

For the cases of sudden contraction (e.g. flowing out of a tank into a pipe) of a sudden enlargement (e.g. flowing from a pipe into a tank) then a theoretical value of k_L can be derived. For junctions bend etc. k_L must be obtained experimentally.

1.6.1 Losses at Sudden Enlargement

Consider the flow in the sudden enlargement, shown in 1.7 below, fluid flows from section 1 to section 2. The velocity must reduce and so the pressure increases (this follows from Bernoulli). At position 1' turbulent eddies occur which give rise to the local head loss.

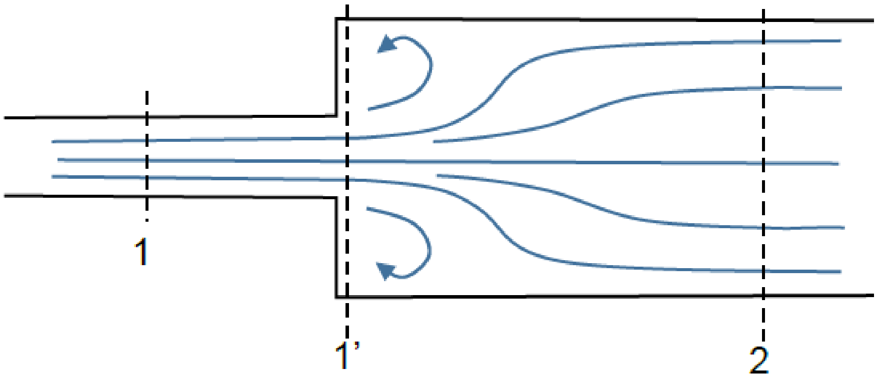

Figure 1.7: Sudden Expansion

Making the assumption that the pressure at the annular area $A_2 - A_1$ is equal to the pressure in the smaller pipe p_1. If we apply the momentum equation between positions 1' (just inside the larger pipe) and 2 to give:

$$p_1 A_2 - p_2 A_2 = \rho Q(u_2 - u_1)$$

Now use the continuity equation to remove Q. (i.e. substitute $Q = A_2 u_2$)

$$p_1 A_2 - p_2 A_2 = \rho A_2 u_2 (u_2 - u_1)$$

Rearranging and dividing by g gives

$$\frac{p_2 - p_1}{\rho g} = \frac{u_2}{g}(u_1 - u_2) \tag{1.20}$$

Now apply the Bernoulli equation from point 1 to 2, with the head loss term h_L

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} = \frac{p_2}{\rho g} + \frac{u_2^2}{g} + h_L$$

And rearranging gives

$$h_L = \frac{u_1^2 - u_2^2}{2g} - \frac{p_2 - p_1}{\rho g} \tag{1.21}$$

Combining Equations 1.20 and 1.21 gives

$$h_L = \frac{u_1^2 - u_2^2}{2g} - \frac{u_2}{g}(u_1 - u_2)$$

$$h_L = \frac{(u_1 - u_2)^2}{2g} \tag{1.22}$$

Substituting again for the continuity equation to get an expression involving the two areas, (i.e. $u_2 = u_1 A_1 / A_2$) gives

$$h_L = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{u_1^2}{2g} \tag{1.23}$$

Comparing this with Equation 1.19 gives k_L

$$k_L = \left(1 - \frac{A_1}{A_2}\right)^2 \tag{1.24}$$

When a pipe expands in to a large tank $A_1 \ll A_2$ i.e. $A_1/A_2 = 0$ so $k_L = 1$. That is, the head loss is equal to the velocity head just before the expansion into the tank - i.e. for entry into a large tank the theoretical energy loss is equal to *all of the velocity head* in the pipe:

$$h_{L\ \text{(entry to large tank)}} = \frac{u_1^2}{2g}$$

1.6.2 Losses at Sudden Contraction

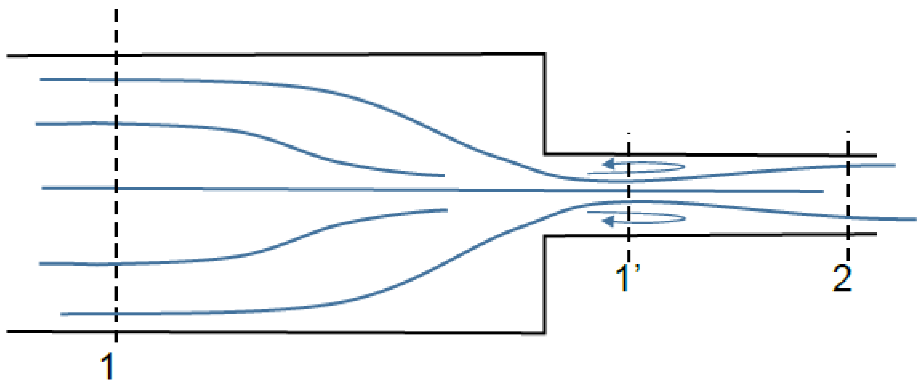

Figure 1.8: Sudden Contraction

In a sudden contraction, flow contracts from point 1 to point 1', forming a vena contraction. Following the same analysis for a sudden expansion and taking equation 1.22, we get the following for a sudden contraction:

$$k_{L\ \text{contraction}} = \left(\frac{A_2}{A_c} - 1\right)^2 \tag{1.25}$$

Where A_c represents the area of flow at the vena contraction.

You will notice that in this expression the cross sectional area A_1 does not appear, however the area A_c does depend on the ration A_2/A_1. From experiments on coaxial pipe it has been found that the following values are suitable

A_2/A_1	k_L contraction
0.00	0.50
0.04	0.45
0.16	0.38
0.36	0.28
0.64	0.14
1.00	0.00

We see from this table, that when A_1 is very much larger than A_2, the ratio is approximately 0. This an example of a fluid flowing from a tank into a pipe i.e. an entry loss, and $k_L = 0.5$. Also when the ratio is 1, both pipe are equal so there is no energy loss and $k_L = 0$.

If we have a protruding pipe then that will increase energy loss at entry, if smoothed it will reduce the loss. Here are two examples with indication of appropriate k_L values:

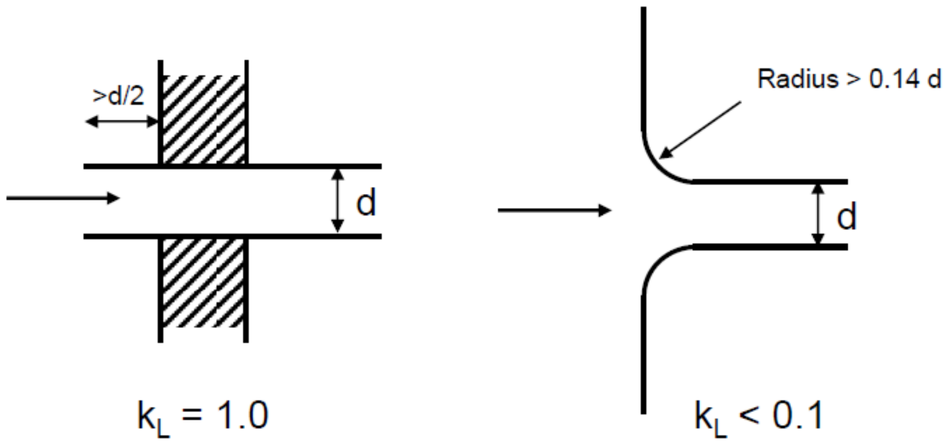

Figure 1.9: Exit type and k_l values

1.6.3 Other Local Losses

Large losses in energy in energy usually occur only where flow expands. The mechanism at work in these situations is that as velocity decreases (by continuity) so pressure must increase (by Bernoulli).

When the pressure increases in the direction of fluid outside the boundary layer has enough momentum to overcome this pressure that is trying to push it backwards. The fluid within the boundary layer has so little momentum that it will very quickly be brought to rest, and possibly reversed in direction. If this reversal occurs it lifts the boundary layer away from the surface as shown in Figure 1.10. This phenomenon is known as boundary layer separation.

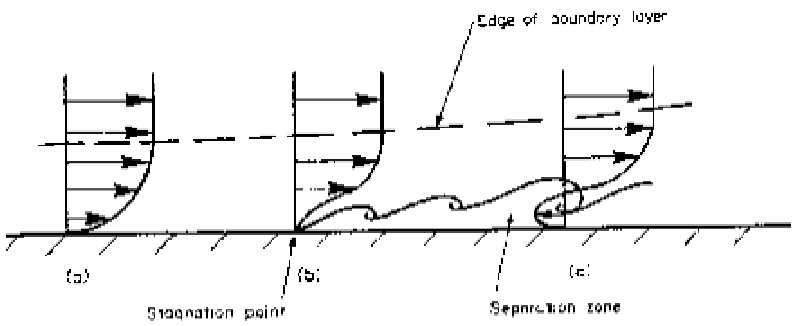

Figure 1.10: Boundary layer separation

At the edge of the separated boundary layer, where the velocities change direction, a line of vortices occur (known as a vortex sheet). This happens because fluid to either side is moving in the opposite direction. This boundary layer separation and increase in the turbulence because of the vortices results in very large energy losses in the flow. These separating / divergent flows are inherently unstable and far more energy is lost than in parallel or convergent flow.

Some common situation where significant head losses occur in pipe are shown in figure 1.11

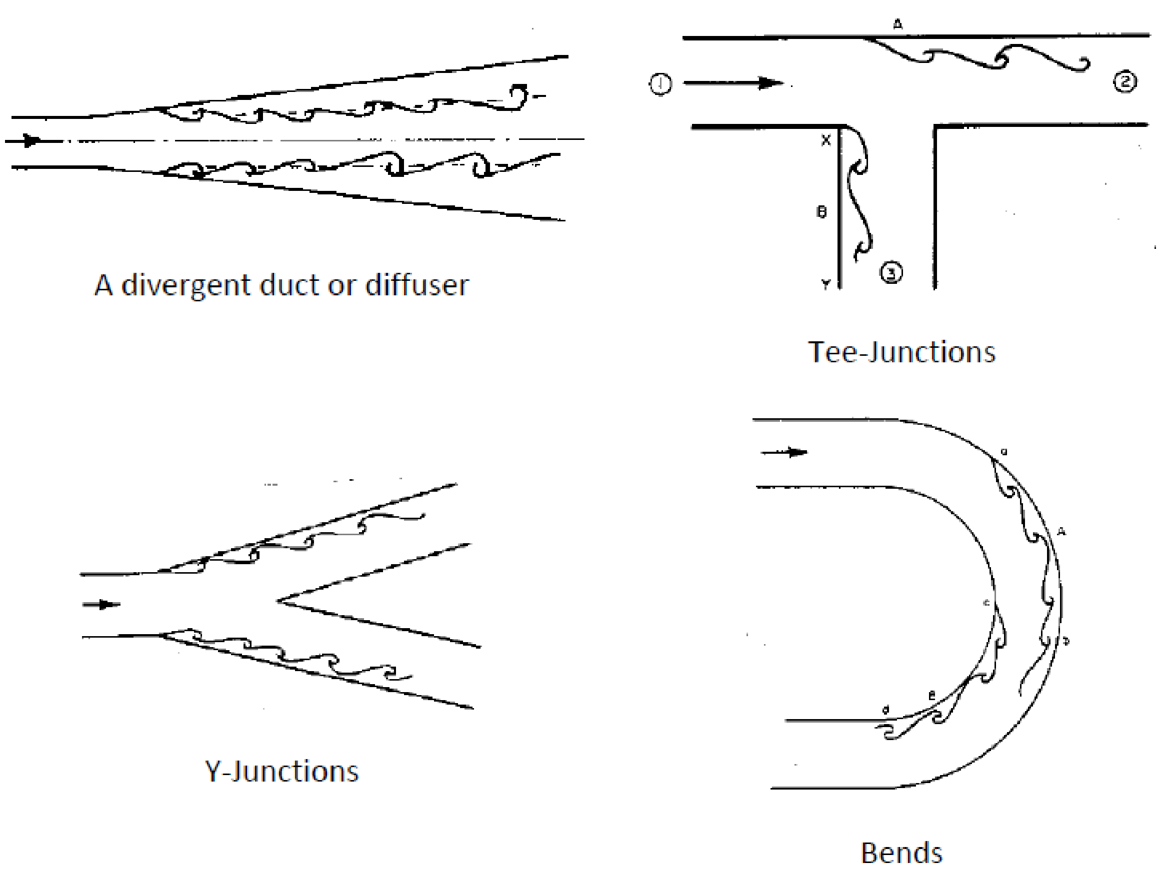

Figure 1.11: Local losses in pipe flow

The values of k_L for these common situations are shown in Table 1.2. It gives value that are used in practice.

Type of local loss	Practical k_L values
Bellmouth entry	0.10
Sharp entry	0.5
Sharp exit	1.0
90 bend	0.4
90 tees	
In-line flow	0.4
Branch to line	1.5
Gate value (open)	0.25

Table 1.2: Practical k_L values

1.7 Pipeline Analysis

As discussed at the start of these notes for analysis of flow in pipelines we will use the Bernoulli equation.

Bernoulli's equation is a statement of conservation of energy per unit weight along a streamline, by this principle the total energy per unit weight in the system does not change. Thus the total head does not change. So the Bernoulli equation can be written

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H = \text{constant}$$

or

$$\begin{array}{c}\text{Pressure}\\ \text{energy per}\\ \text{unit weight}\end{array} + \begin{array}{c}\text{Kinetic}\\ \text{energy per}\\ \text{unit weight}\end{array} + \begin{array}{c}\text{Potential}\\ \text{energy per}\\ \text{unit weight}\end{array} = \begin{array}{c}\text{Total}\\ \text{energy per}\\ \text{unit weight}\end{array}$$

As all of these elements of the equation have units of length, they are often referred to as the following:

$$\begin{aligned} \text{pressure head} &= \frac{p}{\rho g} \\ \text{velocity head} &= \frac{u^2}{2g} \\ \text{potential head} &= z \\ \text{total head} &= H \end{aligned}$$

In this form, Bernoulli's equation has some restrictions in its applicability, they are:

- Flow is steady;
- Density is constant (i.e. fluid is incompressible);
- Friction losses are negligible.
- The equation relates the states at two points along a single streamline.

Applying the equation between two points an including, entry, expansion, exit and friction losses, we have (similarly to equation 1.2):

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = H = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h_{L\,\text{entry}} + h_{L\,\text{expansion}} + h_{L\,\text{exit}} + h_f \tag{1.26}$$

Below we will see how these can be viewed graphically, then we will solve some typical problems for pipelines and their various losses.

1.8 Pressure Head, Velocity Head, Potential Head and Total Head in a Pipeline.

By looking at the example of the reservoir with which feeds a pipe we will see how these different heads relate to each other. Consider the reservoir below feeding a pipe of constant diameter that rises (in reality it may have to pass over a hill) before falling to its final level.

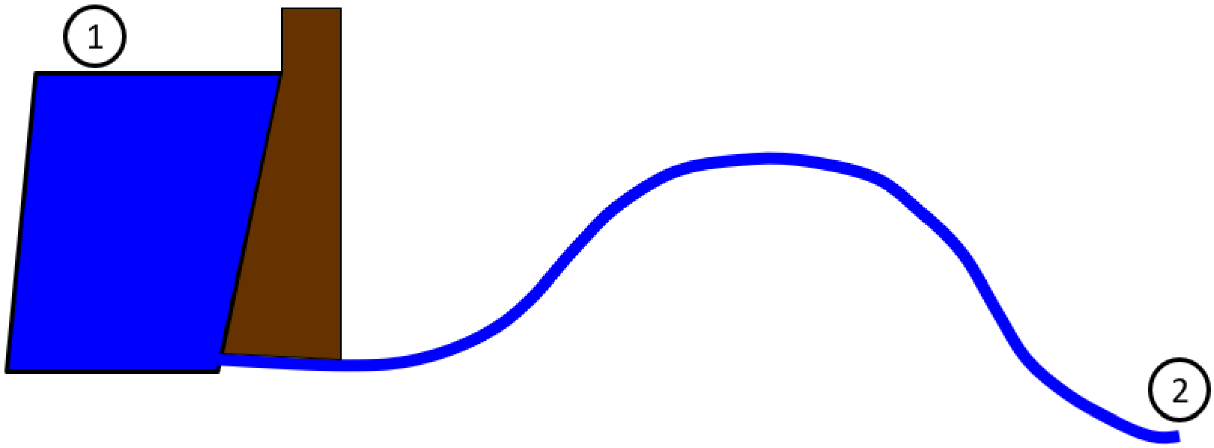

Figure 1.12: Reservoir feeding a pipe

To analyses the flow in the pipe we apply the Bernoulli equation along a streamline from point 1 on the surface of the reservoir to point 2 at the outlet nozzle of the pipe. And we know that the total energy per unit weight or the total head does not change - it is constant - along a streamline. But what is this value of this constant? We have the Bernoulli equation

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = H = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2$$

We can calculate the total head, H, at the reservoir, $p_1 = 0$ as this is atmospheric and atmospheric gauge pressure is zero, the surface is moving very slowly compared to that in the pipe so $u_1 = 0$, so all we are left with is *total head* $= H = z_1$ the elevation of the reservoir.

A useful method of analysing the flow is to show the pressures graphically on the same diagram as the pipe and reservoir. In the figure 1.13 the total head line is shown. If we attached piezometers at points along the pipe, what would be their levels when the pipe nozzle was closed? (Piezometers, as you will remember, are simply open ended vertical tubes filled with the same liquid whose pressure they are measuring).

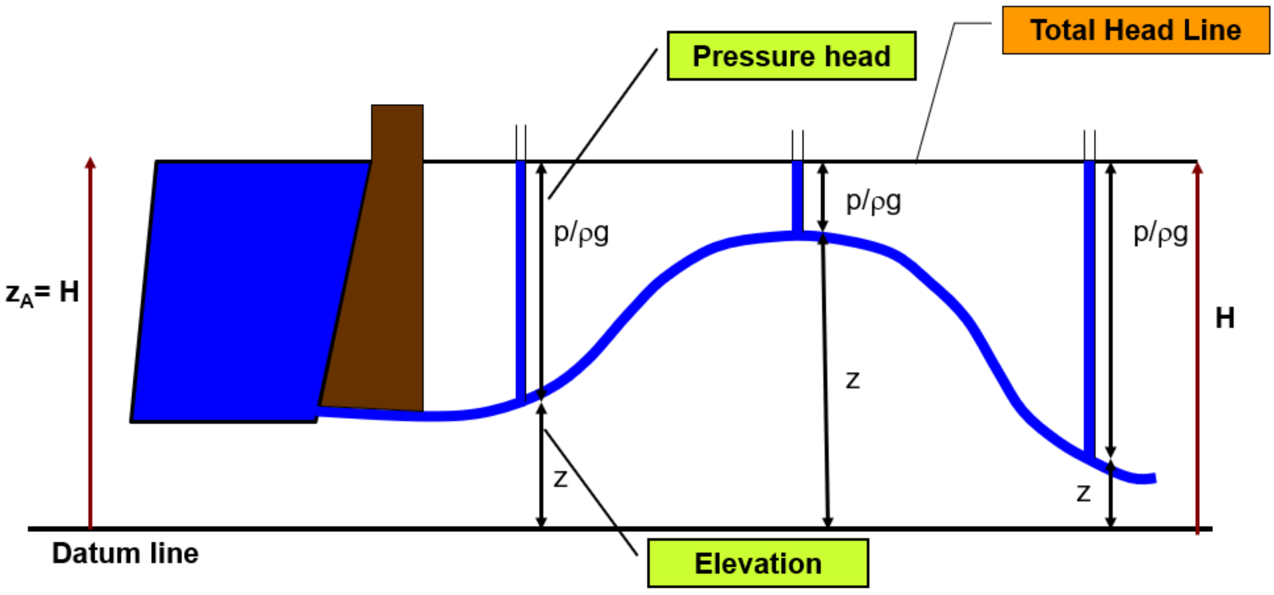

Figure 1.13: Piezometer levels with zero flow in the main pipe

As you can see in figure 1.13, with zero velocity all of the levels in the piezometers are equal and the same as the total head line. At each point on the line, when $u = 0$, the Bernoulli equation says

$$\frac{p}{\rho g} + z = H$$

The level in the piezometer is the pressure head and its value is given by $\frac{p}{\rho g}$.

What would happen to the levels in the piezometers (pressure heads) if the water was flowing with velocity $= u$ We know from earlier examples that as velocity increases so pressure falls.

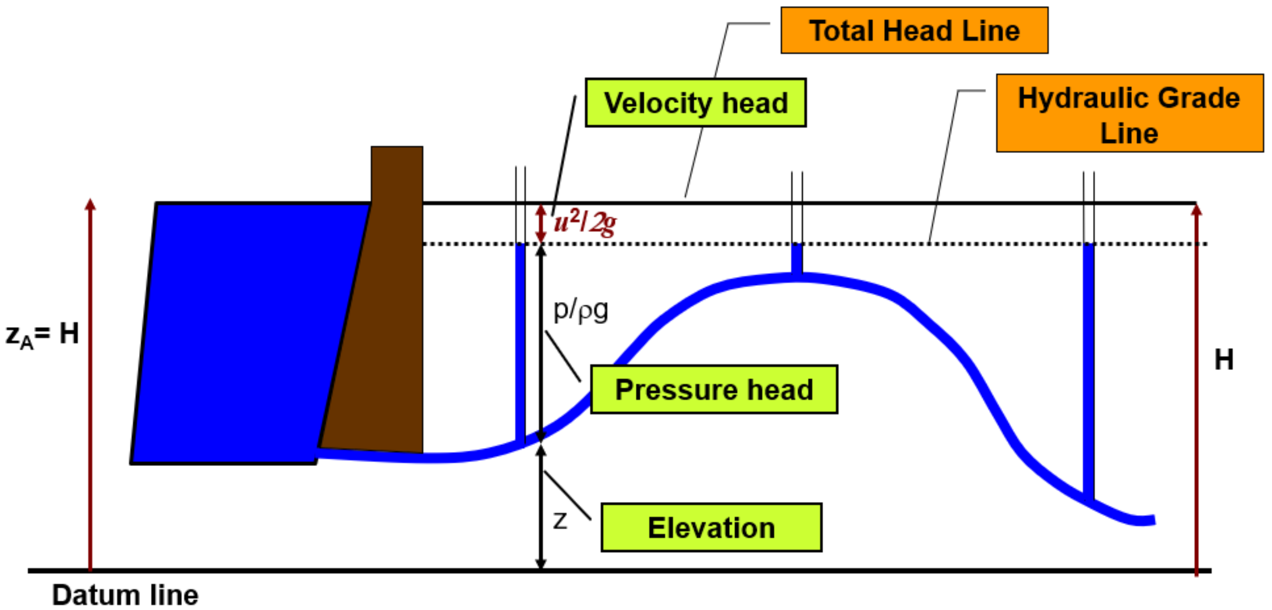

Figure 1.14: Piezometer levels when fluid is flowing

$$\frac{p}{\rho g} + \frac{u^2}{2g} + z = H$$

We see in this figure that the levels have reduced by an amount equal to the velocity head, $\frac{u^2}{2g}$. Now as the pipe is of constant diameter we know that the velocity is constant along the pipe so the velocity head is constant and represented graphically by the horizontal line shown. (This line is known as the *hydraulic grade line*).

What would happen if the pipe were not of constant diameter? Look at the figure 1.15 where the pipe from the example above is replaced by a pipe of three sections with the middle section of larger diameter

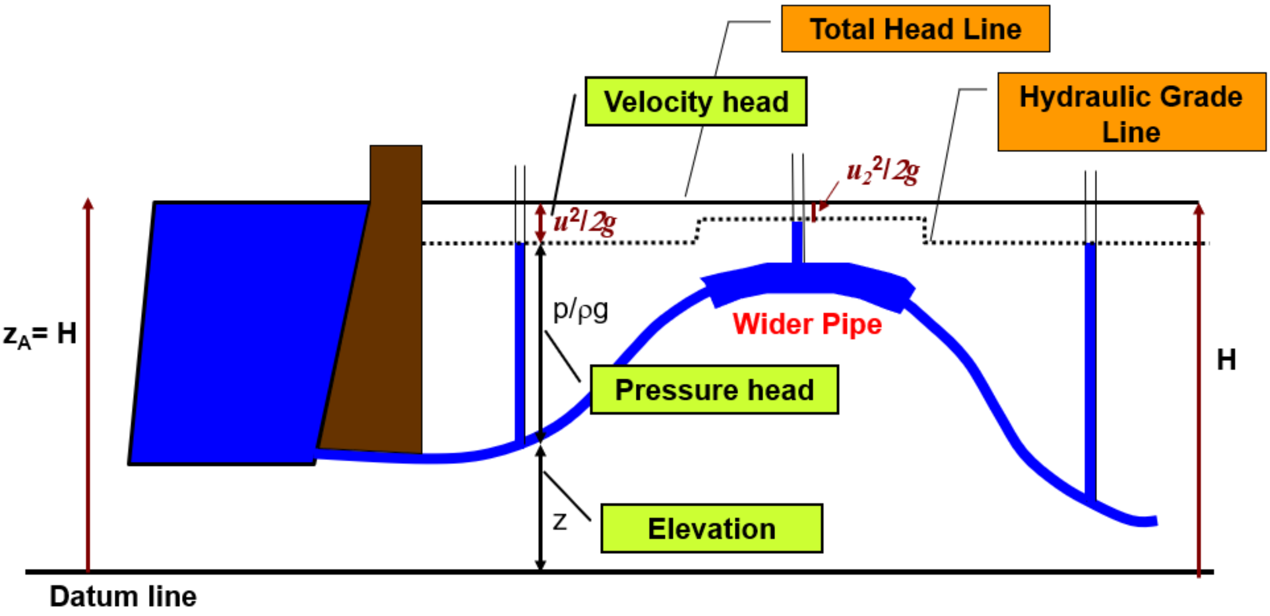

Figure 1.15: Piezometer levels and velocity heads with fluid flowing in varying diameter pipes

The velocity head at each point is now different. This is because the velocity is different at each point. By considering continuity we know that the velocity is different because the diameter of the pipe is different. Which pipe has the greatest diameter?

It is Pipe 2, because the velocity, and hence the velocity head, is the smallest.

This graphical representation has the advantage that we can see at a glance the pressures in the system. For example, where along the whole line is the lowest pressure head? It is where the hydraulic grade line is nearest to the pipe elevation i.e. at the highest point of the pipe.

1.9 Flow in pipes with losses due to friction

In a real pipe line there are energy losses due to friction - these must be taken into account as they can be very significant. How would the pressure and hydraulic grade lines change with friction? Going back to the constant diameter pipe, we would have a pressure situation like this shown in figure 1.16:

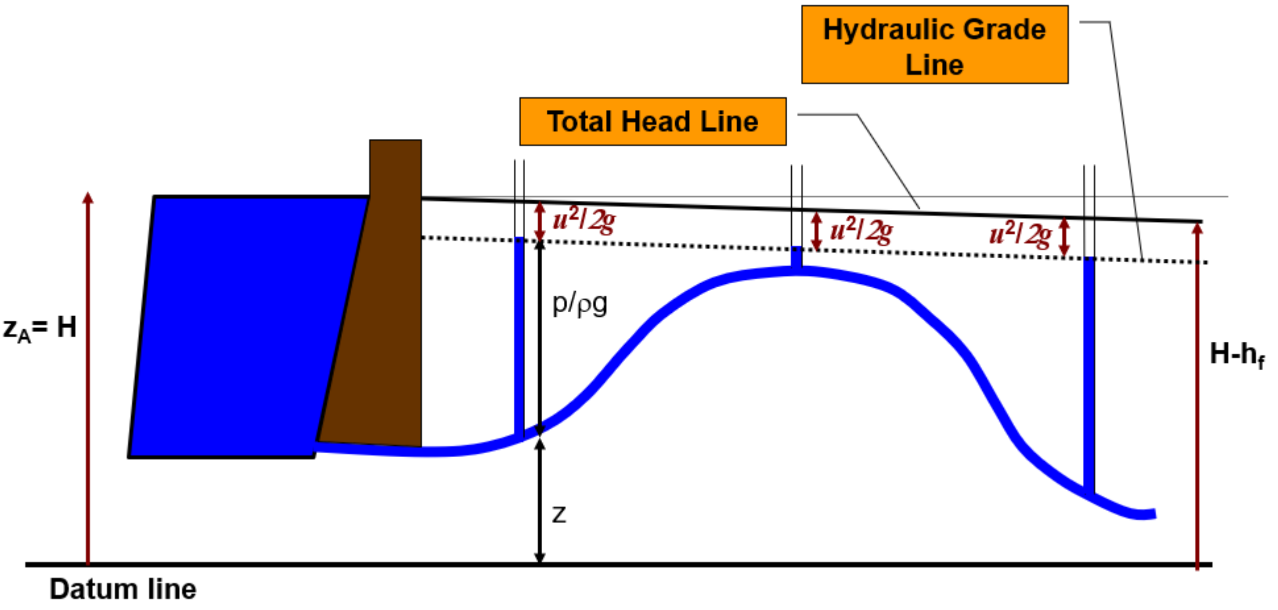

Figure 1.16: Hydraulic Grade line and Total head lines for a constant diameter pipe with friction

How can the total head be changing? We have said that the total head - or total energy per unit weight - is constant. We are considering energy conservation, so if we allow for an amount of energy to be lost due to friction the total head will change. Equation 1.22 is the Bernoulli equation as applied to a pipe line with the energy loss due to friction written as a head and given the symbol h_f (the head loss due to friction) and the local energy losses written as a head, h_L (the local head loss).

$$\frac{p_1}{\rho g} + \frac{u_1^2}{2g} + z_1 = H = \frac{p_2}{\rho g} + \frac{u_2^2}{2g} + z_2 + h_f + h_L \tag{1.27}$$

1.10 Reservoir and Pipe Example

Consider the example of a reservoir feeding a pipe, as shown in figure 1.17.

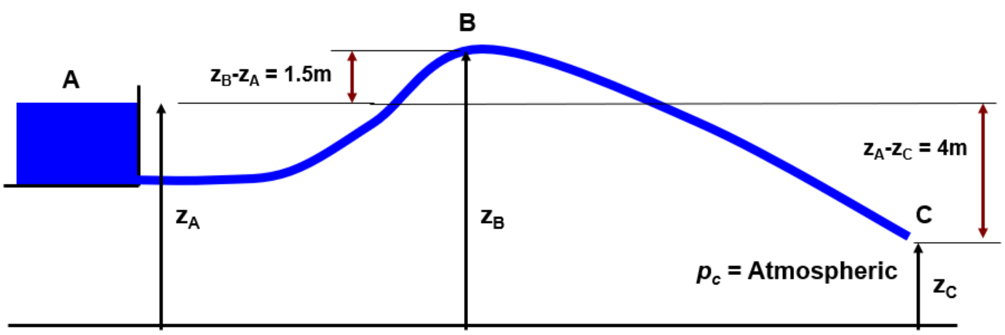

Figure 1.17: Reservoir feeding a pipe example

The pipe diameter is 100mm and has length 15m and feeds directly into the atmosphere at point C, 4.0m below the surface of the reservoir (i.e. $z_a - z_c = 4.0m$). The highest point on the pipe is a B which is 1.5m above the surface of the reservoir (i.e. $z_b - z_a = 1.5m$) and 5 m along the pipe measured from the reservoir. Assume the entrance and exit to the pipe to be sharp and the value of friction factor λ to be 0.08.

Calculate **a)** velocity of water leaving the pipe at point C, **b)** pressure in the pipe at point B.

a)
We use the Bernoulli equation with appropriate losses from point A to C and for entry loss $k_L = 0.5$ and exit loss $k_L = 1.0$.

For the local losses from Table 1.2 for a sharp entry $k_L = 0.5$ and for the sharp exit as it opens in to the atmosphere with no contraction there are no losses, so

$$h_L = 0.5\frac{u^2}{2g}$$

Friction losses are given by the Darcy equation

$$h_f = \frac{\lambda L u^2}{2gd}$$

Pressure at A and C are both atmospheric, u_A is very small so can be set to zero, giving

$$z_A = z_c + \frac{u^2}{2g} + \frac{\lambda L u^2}{2gd} + 0.5\frac{u^2}{2g}$$

$$z_A - z_c = \frac{u^2}{2g}\left(1 + \frac{\lambda L}{d} + 0.5\right)$$

Substitute in the numbers from the question

$$\begin{aligned} 4 &= \frac{u^2}{2 \times 9.8}\left(1.5 + \frac{0.08 \times 15.0}{0.1}\right) \\ u &= 2.41 m/s \end{aligned}$$

b) To find the pressure at B apply Bernoulli from point A to B using the velocity calculated above. The length of the pipe is $L_1 = 5m$:

$$\begin{aligned} z_A &= \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + \frac{\lambda L_1 u^2}{2gd} + 0.5\frac{u^2}{2g} \\ \text{as } u_B &= u \\ z_A - z_B &= \frac{p_B}{\rho g} + \frac{u^2}{2g}\left(1.0 + \frac{\lambda L_1}{d} + 0.5\right) \\ -1.5 &= \frac{p_B}{\rho g} + \frac{2.41^2}{2 \times 9.8}\left(1.5 + \frac{0.08 \times 5.0}{0.1}\right) \\ \frac{p_B}{\rho g} &= 3.13m \\ p_B &= 3.13 \times 1000 \times 9.8 = -30.7 \times 10^3 N/m^2 \end{aligned}$$

Note this is negative which means that pressure is $30.7kN/m^2$ below atmospheric pressure, and *pressure head* is $3.13m$ below atmospheric pressure head.

1.11 Pipes in series

When pipes of different diameters are connected end to end to form a pipe line, they are said to be in series. The total loss of energy (or head) will be the sum of the losses in each pipe plus local losses at connections.

1.11.1 Pipes in Series Example

Consider the two reservoirs shown in figure 1.18, connected by a single pipe that changes diameter over its length. The surfaces of the two reservoirs have a difference in level of 9m. The pipe has a diameter of 200mm for the first 15m (from A to C) then a diameter of 250mm for the remaining 45m (from C to B).

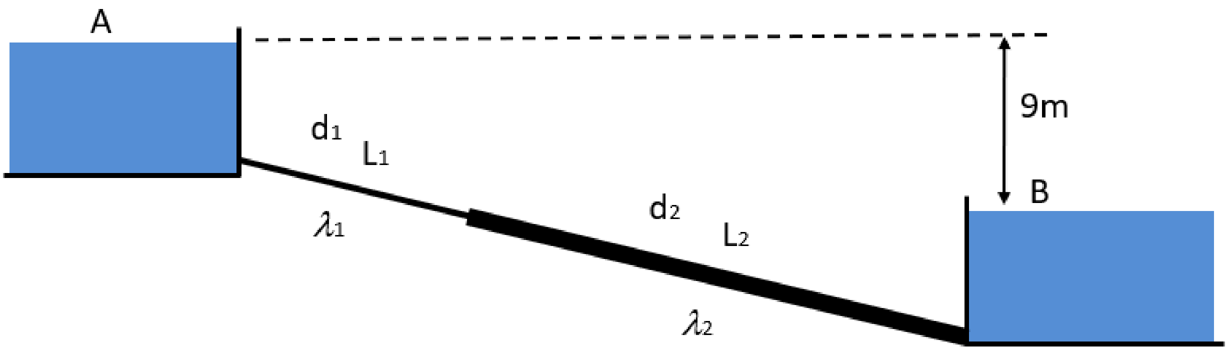

Figure 1.18: Pipes in series: Increasing diameter pipes example

For the entrance use $k_L = 0.5$ and the exit $k_L = 1.0$. The join at C is sudden. For both pipes use $\lambda = 0.04$.

Total head loss for the system H = height difference of reservoirs.

We will use this notation:

- h_{f1} = head loss for 200mm diameter section of pipe
- h_{f2} = head loss for 250mm diameter section of pipe
- $h_{L\text{ entry}}$ = head loss at entry point
- $h_{L\text{ join}}$ = head loss at join of the two pipes
- $h_{L\text{ exit}}$ = head loss at exit point

So

$$H = h_{f1} + h_{f2} + h_{L\text{ entry}} + h_{L\text{ join}} + h_{L\text{ exit}} = 9m \tag{1.28}$$

All losses are, in terms of Q:

$$h_{f1} = \frac{\lambda L_1 Q^2}{12.1 d_1^5} = \frac{0.04 \times 15.0 \times Q^2}{12.1 \times 0.2^5} = 154.96Q^2$$

$$h_{f2} = \frac{\lambda L_2 Q^2}{12.1 d_2^5} = \frac{0.04 \times 45.0 \times Q^2}{12.1 \times 0.25^5} = 152.33Q^2$$

$$h_{L\text{ entry}} = 0.5\frac{u^2}{2g} = 0.5\frac{1}{2g}\left(\frac{4Q}{\pi 0.2^2}\right)^2 = 0.5 \times 0.0826 \times \frac{Q^2}{d_1^4} = 25.85Q^2$$

$$h_{L\text{ exit}} = 1.0\frac{u^2}{2g} = 1.0\frac{1}{2g}\left(\frac{4Q}{\pi d_2^2}\right)^2 = 0.5 \times 0.0826 \times \frac{Q^2}{0.25^4} = 21.17Q^2$$

From equation 1.22

$$h_{L\text{ join}} = \frac{(u_1 - u_2)^2}{2g} = \frac{\left(\left[\frac{4Q}{\pi d_1^2}\right]^2 + \left[\frac{4Q}{\pi d_2^2}\right]\right)^2}{2g} = 6.70Q^2$$

An alternative way to clauclate $h_{L\text{ join}}$ would be to use equation 1.23 and 1.24:

$$h_{L\text{ join}} = k_{L\text{ entry}}\left(\frac{u_1^2}{2g}\right)^2$$

$$k_{L\text{ entry}} = \left(1 - \frac{A_1}{A_2}\right)^2 = \left(1 - \frac{\pi d_1^2/4}{\pi d_2^2/4}\right)^2 = 0.13$$

$$h_{L\text{ join}} = k_{L\text{ entry}}\frac{1}{2g}\left(\frac{Q}{A_1^2}\right)^2 = 6.70Q^2$$

Substitute these into equation 1.28 and solve for Q, to give $Q = 0.158m^3/s$

1.12 Pipes in parallel

When two or more pipes in parallel connect two reservoirs, as shown in Figure 1.19, for example, then the fluid may flow down any of the available pipes at possible different rates. But **the head difference over each pipe will always be the same**. The total volume flow rate will be the sum of the flow in each pipe.

The analysis can be carried out by simply treating each pipe individually and summing flow rates at the end.

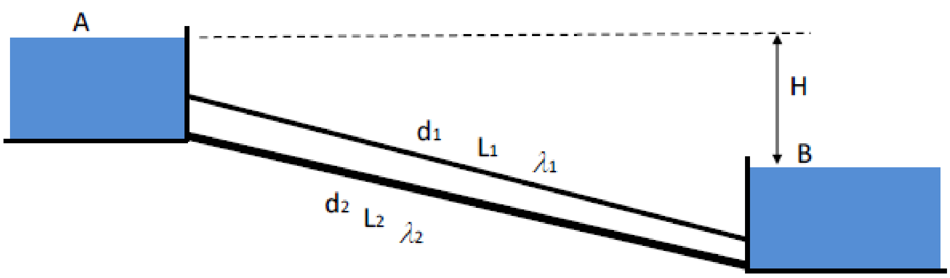

Figure 1.19: Pipes in Parallel example

1.12.1 Pipes in Parallel Example

Two pipes connect two reservoirs (A and B) whose surface height difference is 10m. Pipe 1 has diameter $50mm$ and length $100m$. Pipe 2 has diameter $100mm$ and length $100m$. Both pipes have entry loss coefficients of $k_L = 0.5$ and exit loss coefficients of kL=1.0 and friction factor $\lambda = 0.032$. Calculate: a) rate of flow for each pipe b) the diameter D of a pipe 100m long that could replace the two pipes and provide the same flow.

a) Apply Bernoulli to each pipe separately. For pipe 1:

$$\frac{p_A}{\rho g} + \frac{u_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + 0.5\frac{u_1^2}{2g} + \frac{\lambda_1 L_1 u_1^2}{2gd_1} + 1.0\frac{u_1^2}{2g}$$

p_A and p_B are both atmospheric pressure, and as the reservoir surfaces move slowly (compared to the velocity in the pipe) u_A and u_B are negligible, so

$$\begin{aligned} z_A - z_B &= \left(0.5 + \frac{\lambda_1 L_1}{2gd_1} + 1.0\right)\frac{u_1^2}{2g} \\ 10 &= \left(1.5 + \frac{0.032 \times 100}{0.05}\right)\frac{u_1^2}{2 \times 9.8} \\ u_1 &= 1.731m/s \end{aligned}$$

And flow rate is given by

$$Q_1 = u_1\frac{\pi d_1^2}{4} = 0.0034m^3/s$$

For pipe 2:

$$\frac{p_A}{\rho g} + \frac{u_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + 0.5\frac{u_2^2}{2g} + \frac{\lambda_2 L_2 u_2^2}{2gd_2} + 1.0\frac{u_2^2}{2g}$$

Again, p_A and p_B are atmospheric, and as the reservoir surfaces move slowly (compared to the velocity in the pipe) u_A and u_B are negligible, so

$$\begin{aligned} z_A - z_B &= \left(0.5 + \frac{\lambda_2 L_2}{2gd_2} + 1.0\right)\frac{u_2^2}{2g} \\ 10 &= \left(1.5 + \frac{0.032 \times 100}{0.05}\right)\frac{u_2^2}{2 \times 9.8} \\ u_2 &= 2.42m/s \end{aligned}$$

And flow rate is given by

$$Q_2 = u_2\frac{\pi d_2^2}{4} = 0.0190m^3/s$$

b)
Replacing the pipe, we need $Q = Q1 + Q2 = 0.0034 + 0.0190 = 0.0224m3/s$ For this pipe, diameter D, velocity u , and making the same assumptions about entry/exit losses, we have:

$$\frac{p_A}{\rho g} + \frac{u_A^2}{2g} + z_A = \frac{p_B}{\rho g} + \frac{u_B^2}{2g} + z_B + 0.5\frac{u_2^2}{2g} + \frac{\lambda L u^2}{2gD} + 1.0\frac{u_2^2}{2g}$$

$$z_A - z_B = \left(0.5 + \frac{\lambda L}{2gD} + 1.0\right)\frac{u^2}{2g}$$
$$10 = \left(1.5 + \frac{0.032 \times 100}{D}\right)\frac{u^2}{2 \times 9.8}$$
$$196.2 = \left(1.5 + \frac{3.2}{D}\right)u^2$$

The velocity can be replaced by Q usinge.

$$Q = Au = \frac{\pi D^2}{4}u$$
$$u = \frac{4Q}{\pi D^2} = \frac{0.02852}{D^2}$$

So

$$196.2 = \left(1.5 + \frac{3.2}{D}\right)\left(\frac{0.02852}{D^2}\right)^2$$
$$0 = 241212D^5 - 1.5D - 3.2$$

which must be solved iteratively.

An approximate answer can be obtained by dropping the second term:

$$0 = 241212D^5 - 3.2$$
$$D = \sqrt[5]{\frac{3.2}{241212}}$$
$$D = 0.1058m$$

Another stratergy would be to iterate on the function:

$$f(D) = 241212D^5 - 1.5D - .32$$

With a good startig point being:

$$f(0.1058) = -0.161$$

Using a trial an error method we could increase our D slightly to 0.107,for which

$$f(0.107) = 0.022$$

i.e. the solution is between $0.107m$ and $0.1058m$ but $0.107m$ is sufficiently accurate.

1.13 Branched pipes

If pipes connect three reservoirs, as shown in Figure 1.20, then the problem becomes more complex. One of the problems is that it is sometimes difficult to decide which direction fluid will flow. In practice solutions are now done by computer techniques that can determine flow direction, however it is useful to examine the techniques necessary to solve this problem.

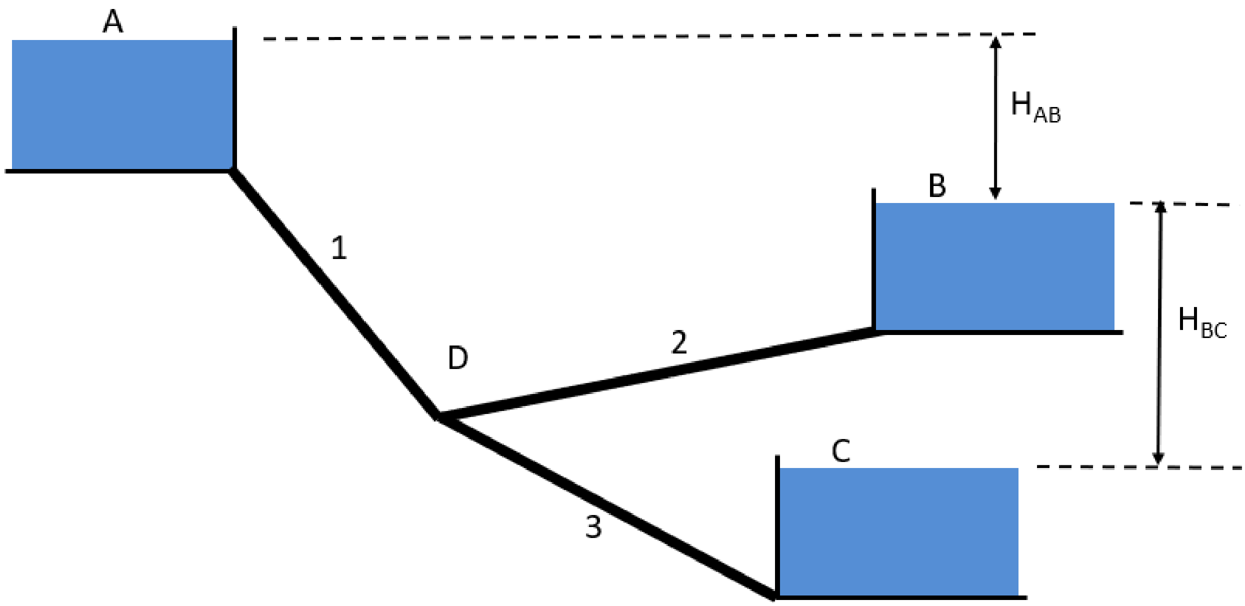

Figure 1.20: The three reservoir problem

For these problems it is best to use the Darcy equation expressed in terms of discharge Q i.e. equation 1.11.

$$h_f = \frac{fLQ^2}{3d^5} = \frac{\lambda LQ^2}{12.1d^5} \tag{1.29}$$

When three or more pipes meet at a junction then the following basic principles apply:

1. The continuity equation must be obeyed i.e. total flow into the junction must equal total flow out of the junction;
2. At that point there can only be one value of head, and
3. Darcy?s equation must be satisfied for each pipe.

It is usual to ignore minor losses (entry and exit losses) as practical hand calculations become impossible - fortunately they are often negligible.

One problem still to be resolved is that however we calculate friction it will always produce a positive drop - when in reality head loss is in the direction of flow. The direction of flow is often obvious, but when it is not, a direction has to be assumed. If the wrong assumption is made then no physically possible solution will be obtained.

In figure 1.20 above the heads at the reservoir are known but the head at the junction D is not. Neither are any of the pipe flows known. The flow in pipes 1 and 2 are obviously from A to D and D to C respectively. If one assumes that the flow in pipe 2 is from D to B then the following relationships could be written:

$$\begin{aligned} z_a - h_D &= h_{f1} \\ H_D - z_b &= h_{f2} \\ H_D - z_c &= h_{f3} \\ Q_1 &= Q_2 + Q_3 \end{aligned}$$

The h_f expressions are functions of Q, so we have 4 equations with four unknowns, h_D, Q_1, Q_2 and Q_3 which we must solve simultaneously.

The algebraic solution is rather tedious so a trial and error method is usually recommended. For example this procedure usually converges to a solution quickly:

1. estimate a value of the head at the junction,h_D
2. substitute this into the first three equations to get an estimate for Q for each pipe.
3. check to see if continuity is (or is not) satisfied from the fourth equation
4. if the flow into the junction is too high choose a larger h_D and vice versa.
5. return to step 2

If the direction of the flow in pipe 2 was wrongly assumed then no solution will be found. If you have made this mistake then switch the direction to obtain these four equations

Looking at these two sets of equations we can see that they are identical if $h_D = z_b$. This suggests that a good starting value for the iteration is z_b then the direction of flow will become clear at the first iteration.

1.13.1 Example of Branched Pipe - The Three Reservoir Problem

Water flows from reservoir A through pipe 1, diameter $d_1 = 120mm$, length $L_1 = 120m$, to junction D from which the two pipes leave, pipe 2, diameter $d_2 = 75mm$, length $L_2 = 60m$ goes to reservoir B, and pipe 3, diameter $d_3 = 60mm$, length $L_3 = 40m$ goes to reservoir C. Reservoir B is 16m below reservoir A, and reservoir C is 24m below reservoir A. All pipes have $\lambda = 0.04$. (Ignore and entry and exit losses.)

We know the flow is from A to D and from D to C but are never quite sure which way the flow is along the other pipe - either D to B or B to D. We first must assume one direction. If that is not correct there will not be a sensible solution. To keep the notation from above we can write $z_a = 24$, $z_b = 16$ and $z_c = 0$.

For flow A to D

$$z_a - h_D = h_{f1}$$

$$24 - h_D = \frac{\lambda_1 L_1 Q_1^2}{12.1 d_1^5} = 15942Q^2$$

Assume flow is D to B

$$h_D - z_b = h_{f2}$$

$$h_D - 8 = \frac{\lambda_2 L_2 Q_2^2}{12.1 d_2^5} = 83583Q^2$$

For flow is D to C

$$h_D - z_c = h_{f3}$$

$$h_D - 0 = \frac{\lambda_3 L_3 Q_3^2}{12.1 d_3^5} = 170050Q^2$$

The final equation is continuity, which for this chosen direction D to B is

$$Q_1 = Q_2 + Q_3$$

Now it is a matter of systematically testing values of h_D until continuity is satisfied. This is best done in a table. And it is usually best to initially take $h_D = z_a$ then reduce its value, or procede in a bi-sector type iteration, until the error in continuity is sufficiently small:

h_D trial	Q_1	Q_2	Q_3	$Q_2 + Q_3$	error $= Q_1 - (Q_2 + Q_3)$
24.00	0.0000	0.0098	0.0119	0.0217	-0.0217
20.00	0.0158	0.0069	0.0108	0.0178	-0.0019
18.00	0.0194	0.0049	0.0103	0.0152	0.0042
19.00	0.0177	0.0060	0.0106	0.0166	0.0011
19.50	0.0168	0.0065	0.0107	0.0172	-0.0004
19.40	0.0170	0.0064	0.0107	0.0171	-0.0001
19.36	0.0171	0.0063	0.0107	0.0170	0.0000

The solution is that the head at the junction is $19.36m$, which gives $Q_1 = 0.017m^3/s$, $Q_2 = 0.006m^3/s$ and $Q_3 = 0.0107m^3/s$. This gives a continuity error of < 0.01 litres/s. This is very small. Practically $19.50m$ which gives a continuity error of 0.4 litres/s would be sufficient.

Had we guessed that the flow was from B to D, the second equation would have been

$$z_b - h_D = h_{f2}$$

$$8 - h_D = \frac{\lambda_2 L_2 Q_2^2}{12.1 d_2^5} = 83583Q^2$$

and continuity would have been

$$Q_1 + Q_2 = Q_3$$

If you then attempted to solve this you would soon see that there is no practical solution (due to requirement of having to take square-root of a negative number).

Alternative Method - Reformulate the equation for h_f to incorporate direction of flow.

In the above method it is necessary to choose the direction of flow and hope that it is correct. It is possible to reformulate the h_f expression to incorporate a direction of flow. We know that head loss is in the direction of flow, so we can write our equation such that if the head drops the flow is positive, if the head increases flow is negative. To do this we can introduce the idea of a sign of the flow using the convention that and increase in head is negative. So the *sign* will take the value of +1 for positive flow and -1 for negative.

We can use this formulation for head loss between points A and B:

$$h_A - h_B = sign(h_f) h_f$$

Which, for when heads are known could be expressed

$$h_A - h_B = \left[\frac{h_A - h_B}{|h_A - h_B|}\right]\frac{\lambda L Q^2}{12.1 d^5}$$

i.e. writing $sign(h_f) = \left[\frac{h_A - h_B}{|h_A - h_B|}\right]$

Or, for which the Q is known then, it could be expressed

$$h_A - h_B = \frac{\lambda L Q|Q|}{12.1 d^5}$$

And to solve for Q

$$Q = sign(h_f)\left[sign(h_f)\left((h_A - h_B)\frac{12.1 d^5}{\lambda L}\right)\right]^{1/2}$$

(The sign is there twice to avoid requiring the square root of a negative number.)

We can choose ANY initial direction of flow in any pipe and the final sign of the Q will tell us the direction of the flow. i.e. positive then the direction of flow is the direction we chose, negative and the direction of flow is the opposite direction we chose.

So, in this example, let's take all flows flowing towards the junction. We know that at least one is not towards the junction, so the solution will give us a negative flow: and summing up the Q values will give zero, thus continuity conserved.

For flow A to D

$$z_a - h_D = h_{f1}$$

$$24 - h_D = sign(h_{f1})\frac{\lambda_1 L_1 Q_1|Q_1|}{12.1 d_1^5} = sign(h_{f1})15942Q^2$$

Assume flow is D to B

$$z_b - h_D = h_{f2}$$

$$8 - h_D = sign(h_{f2})\frac{\lambda_2 L_2 Q_2|Q_2|}{12.1 d_2^5} = sign(h_{f2})83583Q^2$$

For flow is D to C

$$z_c - h_D = h_{f3}$$

$$h_D - 0 = sign(h_{f3})\frac{\lambda_3 L_3 Q_3|Q_3|}{12.1 d_3^5} = sign(h_{f3})170050Q^2$$

The final equation is continuity, which for any direction of flow is

$$Q_1 + Q_2 + Q_3 = 0$$

Applying these equations to the solution of the above problem, results in the table of iterations below:

h_D trial	$sign(h_{f1})$	Q_1	$sign(h_{f2})$	Q_2	$sign(h_{f3})$	Q_3	error $= Q_1 + Q_2 + Q_3$
24.00	0	0.0000	-1	-0.0098	-1	-0.0119	-0.0217
20.00	1	0.0158	-1	-0.0069	-1	-0.0108	-0.0019
18.00	1	0.0194	-1	-0.0049	-1	-0.0103	0.0042
19.00	1	0.0177	-1	-0.0060	-1	-0.0106	0.0011
19.50	1	0.0168	-1	-0.0065	-1	-0.0107	-0.0004
19.40	1	0.0170	-1	-0.0064	-1	-0.0107	-0.0001
19.36	1	0.0171	-1	-0.0063	-1	-0.0107	0.0000

1.14 Other Pipe Flow Examples

1.14.1 Adding a parallel pipe example

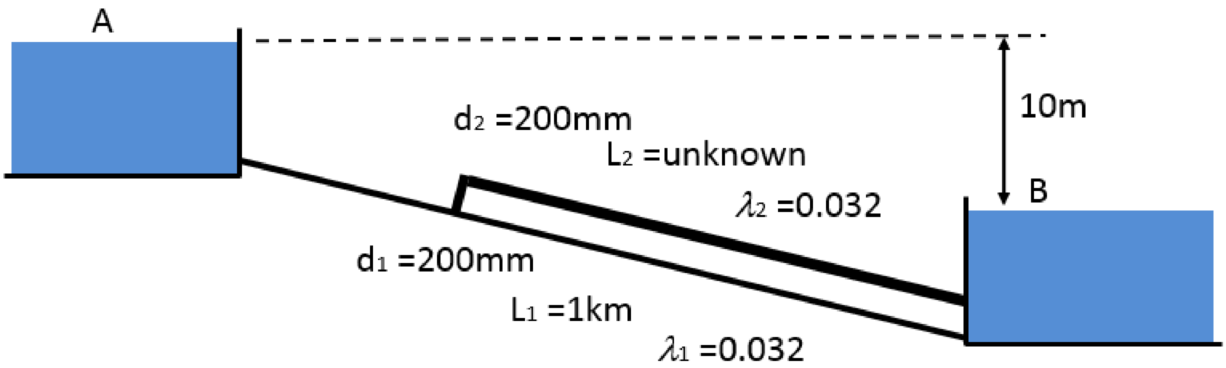

Figure 1.21: Adding parallel pipe example

A pipe joins two reservoirs whose head difference is 10m. The pipe is 0.2m diameter, 1000m in length and has a λ value of 0.032.

a) What is the flow in the pipeline?

b) It is required to increase the flow to the downstream reservoir by 30%. This is to be done adding a second pipe of the same diameter that connects at some point along the old pipe and runs down to the lower reservoir. Assuming the diameter and the friction factor are the same as the old pipe, how long should the new pipe be?

a)

$$h_f = \frac{\lambda L Q^2}{12.1 d^5}$$

$$10 = \frac{0.032 \times 1000 Q^2}{12.1 \times 0.2^5}$$
$$Q = 0.0347 m^3/s$$
$$Q = 34.7 \text{ litres/s}$$

b)

$$H = 10 = h_{f1} + h_{f2}$$
$$10 = h_{f1} + h_{f3}$$

i.e.

$$h_{f2} = h_{f3}$$
$$\frac{\lambda_2 L_2 Q_2^2}{12.1 d_2^5} = \frac{\lambda_3 L_3 Q_3^2}{12.1 d_3^5}$$

as the pipes 2 and 3 have the same λ, same length and the same diameter then $Q_2 = Q_3$.
By continuity $Q_1 = Q_2 + Q_3 = 2Q_2 = 2Q_3$
So

$$Q_2 = \frac{Q_1}{2}$$

and

$$L_2 = 1000 - L_1$$

Then

$$10 = h_{f1} + h_{f2}$$
$$10 = \frac{\lambda_1 L_1 Q_1^2}{12.1 d_1^5} + \frac{\lambda_2 L_2 Q_2^2}{12.1 d_2^5}$$
$$10 = \frac{\lambda_1 L_1 Q_1^2}{12.1 d_1^5} + \frac{\lambda_2 (1000 - L_1)(Q_1/2)^2}{12.1 d_2^5}$$

As $\lambda_1 = \lambda_2$ and $d_1 = d_2$

$$10 = \frac{\lambda_1 Q_1^2}{12.1 d_1^5}\left(L_1 + \frac{(1000 - L_1)}{4}\right)$$

The new Q_1 is to be 30% greater than before so $Q_1 = 1.3 \times 0.0346 = 0.045 m^3/s$
Solve for L to give

$$L_1 = 456.7m$$

$$L_2 = 1000 - 456.7 = 543.2m$$

2 Fluid Flow in Pipe Networks

We are now going to look at how to determine flows and pressures pipe networks, in particular **water distribution networks**. Piped water distribution systems are widespread. In a city there may be millions of pipes and interconnects that need to be accounted for if the flow and pressures in the system are required. These networks connect large storage reservoirs via water treatment plant to houses and businesses. On the way the water will encounter pumps, valves, smaller storage reservoirs, flow measuring devices ... etc. Water demand is quite variable - both between day and night, but also from season to season, the supply - reservoir volumes and levels - on the other hand are relatively stable. To maintain sufficient water and pressure for the varying demand means a distribution system must include small reservoirs to maintain constant pressure lower down the system and the whole systems must be flexible - capable of switching supply to and from different sources. It is usually aimed to keep water pressure at a maximum of 70m head and minimum of 20m head. These levels ensure that demand can be met, yet are not too high so that too much leakage would occur.

2.1 Analysis of pipe networks.

The understanding of the network flows and pressures are essential if any improvement or changes are to be made to an existing system. Traditionally this analysis has split the networks in to manageable sections - one perhaps that can be separated by reservoirs or connections of relatively constant pressure or flow. Then subsequent analysis of this smaller sub-network would be done by hand. Nowadays this is more likely to be done on a computer. We will see in this section of the module how the hand calculations are performed - then how computer simulations differ. This approach will demonstrate how the governing equations are used and what solution techniques are necessary. A clearer physical understanding of the problem - and appreciation of sensible solutions will results from undertaking these hand calculations. There are essentially two types of network to analyse (although most are in reality a combinations of both) they are a looped system and a branched system as shown.

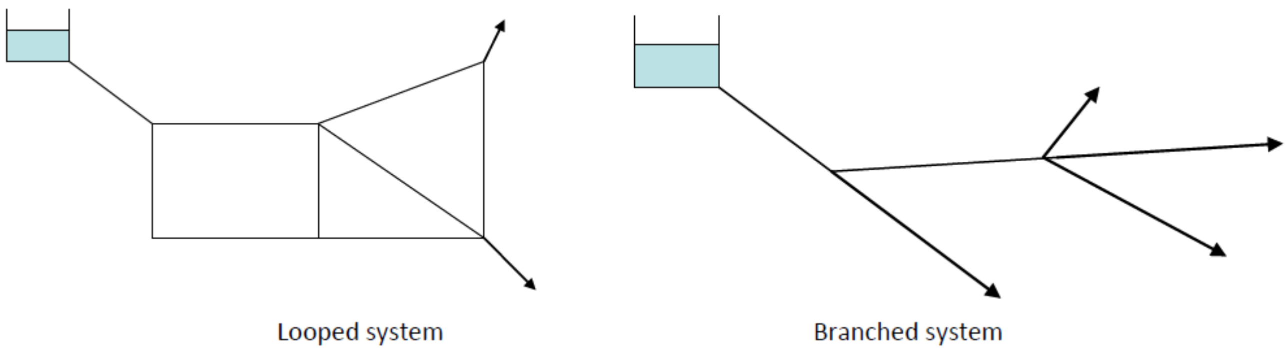

Figure 2.1: Looped and Branched network systems

2.2 Equations of flow

The equations of flow in either of these systems are no more than what you have seen before: Bernoulli, Head loss i.e. Darcy-Weisbach and local loss terms plus continuity at junctions.

We will neglect local losses in this initial analysis, but they can be easily added later once the basic ideas are understood.

textbfContinuity At each junction the sum of the flow into the junction is zero. For a junction of n pipes whose flow in each pipe is q_i then

$$\sum_{i=1}^{n} q_i = 0$$

Note how this implies a direction and a sign convention e.g. flow into the junction is positive, out is negative.

The second equation we have already seen is that for head loss in a pipe i.e. Darcy-Weisbach, equation 1.11:

$$h_f = \frac{fLQ^2}{3d^5} = \frac{\lambda LQ^2}{12.1d^5} \tag{2.1}$$

We will assume a constant f or λ now, but it is simple to include this as a variable dependent on the flow and be calculated by an appropriate equation e.g. Colebrook-White at each step of the solution. Assuming constant f or λ it is convenient to gather the (constant) terms other than Q on the right-hand-side and label them k. So which ever friction value we are using we get a form of equation:

$$h_f = kQ^2 \tag{2.2}$$

Continuity at a junction and the head loss for each pipe are the basic equations we have available.

Energy conservation: Head loss round any loop is zero. For any 'loop' in a system then the sum of the head losses around that loop is zero. For a loop such as that shown here, with 4 pipes ($m = 4$), then

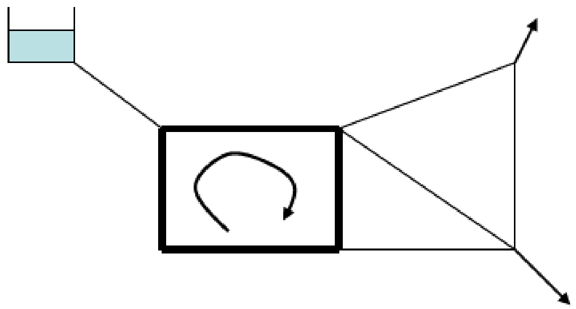

Figure 2.2: Loop in a newtork

Note again how this implies a direction - head is lost in the direction of flow. Care must be taken to write the equations such that h_f is calculated according to the direction of flow in the pipe.

To do this we will normally write the head loss equation as

$$h_f = kQ|Q| \tag{2.3}$$

(where $|Q|$ means the absolute value of Q) this ensures negative flows give a negative h_f. Another way of stating this energy conversation is that head loss between any two junctions (not necessarily adjacent ones) is the same whatever the path taken between the junctions.

In summary for any network we can write out our continuity equations for each junction and head losses for each pipe and we will have a large number of non-linear equations to solve. Their solution is not trivial. A systematic approach must be defined to solving them. There are several methods that have been proposed. We will look at a few.

2.3 Looped Network Analysis: Head Balance, or Hardy Cross method.

This method is the most well know and was developed around 1935 and has been in use ever since. The method can be undertaken by hand and has now also been programmed into software for easy of solution.

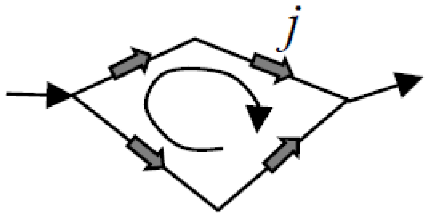

Figure 2.3: Loop of Hardy Cross method

Assuming the head loss equation for a pipe, $h_f = kQ^2$, where Q is the *true* solution. We can write Q as

$$Q = q_j + \delta q$$

Where q_j is the *estimate* of the flow in the jth pipe in a loop and δq is a correction required to make it the true value.

So the true head loss in pipe j will be

$$H_{f\,j} = k(q_j + \delta q)^2$$

Expanding the term in the brackets using the binomial theorem gives:

$$H_{f\,j} = kq_j^2 \left(1 + 2\frac{\delta q}{q_j} + \frac{2(2-1)}{2!} \left(\frac{\delta q}{q_j} \right)^2 + ... \right)$$

As δq is very small compared to q, the third term in the brackets will be small and we can ignore it, giving

$$H_{f\,j} = kq_j^2 \left(1 + 2\delta q / q_j\right)$$

For any loop the sum of the heads will be zero

$$\sum_{j=1}^{m} H_{f\,j} = 0 = \sum_{j=1}^{m} kq_j^2 + \delta q \sum_{j=1}^{m} \left(kq_j^2 / q_j\right)$$

or, substituting in for the estimate of head loss, $h_f = kq^2$

$$0 = \sum_{j=1}^{m} h_{f\,j} + \delta q \sum_{j=1}^{m} \left(h_{f\,j} / q_j\right)$$

We can rearrange this to get an expression for the required update δq

$$\delta q = -\frac{\sum_{j=1}^{m} h_{f\,j}}{2\sum_{j=1}^{m}\left(h_{f\,j}/q_j\right)} \tag{2.4}$$

Note: *Had* we chosen the the Hazen-Williams equation 1.12 as our starting point for representation of head loss i.e. $h_f = kQ^{1.85}$ our update equation in the Hardy-Cross method would be:

$$\delta q = -\frac{\sum_{j=1}^{m} h_{f\,j}}{1.85\sum_{j=1}^{m}\left(h_{f\,j}/q_j\right)} \tag{2.5}$$

2.3.1 An example of Hardy Cross for a single loop:

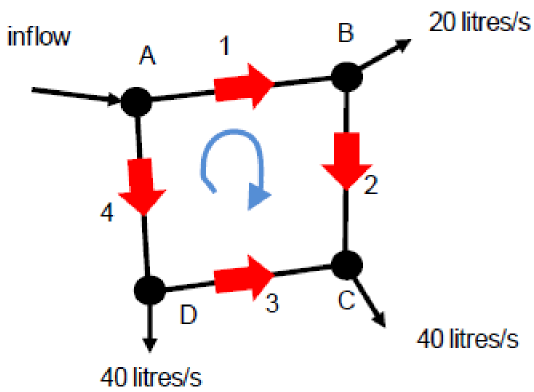

Figure 2.4: Loop of Hardy Cross method example

All pipes of length $1km$, diameter $300mm$, and constant friction factor $\lambda = 0.017$.

a) Calculate the flow in each pipe.
b) Calculate the head at each node if the head at $A = 70m$

Steps:

1. Calculate inflow to A from continuity of network
2. Choose a direction of flow for positive (it doesn't matter if it is wrong, the answer will simply be negative)
3. Identify loops in the system (in this case there is only one)
4. Estimate flows in each pipe so that continuity is satisfied at each junction
5. Perform Hardy Cross method for each loop to calculate δq
6. Add this δq to each estimated flow in each pipe
7. Go back to 5, repeat Hardy Cross method. Keep repeating until δq is sufficiently small

From Continuity of the whole system, flow in = flow out. Inflow to junction $A = 20 + 40 + 40 = 100 litres/s$ The directions chosen are drawn on figure 2.4.

The loop is indicated on figure 2.4. Note the direction of flow relative to the loop. For pipe 3 and pipe 4 it is in the opposite direction to the clock-wise loop, so head losses will be negative for that loop.

You are free to choose any set of initial flows as long as they **satisfy continuity** at each node. In this example selecting a flow of 70 litres/s in pipe 1 (A-B) then the others follow:

Pipes	1	2	3	4
Q (litres/s)	**70**	50	-10	30

Note how in pipe 3 the flow is negative; this is in the opposite direction to the direction of flow chosen (the thick arrow on the diagram). To perform the Hardy Cross step you should organise all calculations into a table. The first step for this network would give:

Pipe	k	$q(m^3/s)$	q^2	h_f	h_f/q	δq	q_{new}
1	578.2	0.070	0.0049	2.833	40.47	-0.020625	0.04938
2	578.2	0.050	0.0025	1.445	28.91	-0.020625	0.02938
3	578.2	0.010	0.0001	0.058	5.78	-0.020625	-0.01063
4	578.2	-0.030	-0.0009	-0.520	17.3	5 -0.0206	25 -0.05063
			sum =	3.816	92.51		

Note that $k = \frac{\lambda L}{12.1 d^5}$ and that q is negative if the chosen direction of flow relative to the loop direction.

Calculating $q^2 \equiv q|q| = q \times abs(q)$ conveniently gives the correct sign for the head loss h_f.

The correction of flow $\delta q = -0.0206 m^3/s$ is added to each estimated flow in the loop to give a new flow, as shown in the last column. This correction, 20 litres/s, is quite large, around 20% of inflow, so a second iteration step of Hardy Cross should be performed. The table would look like this:

Pipe	k	$q(m^3/s)$	q^2	h_f	h_f/q	δq	q_{new}
1	578.2	0.049	0.0024	1.410	28.55	-0.00223	0.04714
2	578.2	0.029	0.0009	0.499	16.98	-0.00223	0.02714
3	578.2	-0.011	-0.0001	-0.065	6.14	-0.00223	-0.01286
4	578.2	-0.051	-0.0026	-1.482	29.27	-0.00223	-0.05286
			sum=	0.361	80.94		

The correction this time is 2.2 litres/s, which is much smaller than the previous step. So we can stop here. Had this been larger we would simply carry on performing Hardy Cross steps until the correction reduced to a sufficiently small value. If it does not reduce it is likely that a mistake has been made.

b) Head calculation.

To calculate the heads at each node we need to calculate he head losses n each pipe with these final flows:

Pipe	k	q (m3/s)	q2	hf
1	578.2	0.0471	0.0022	1.285
2	578.2	0.0271	0.0007	0.426
3	578.2	-0.0129	-0.0002	-0.096
4	578.2	-0.0529	-0.0028	-1.615

The known head at A is 70.0m so the head at each node can be calculated:

Node	Head (m)	Head (m)
A	70.000	70.000
B	$H_A - h_{f1}$	68.715
C	$H_B - h_{f2}$	68.289
D	$H_C - h_{f3}$	68.385
A	$H_D - h_{f4}$	70.000

A check can be made to see if the head at A is that started with.

2.3.2 Further Examples

1. Perform the calculation on the network in the example above using flows at the junctions: $Q_B = 30$ litres/s, $Q_C = -50$ litres/s and $Q_D = 100$ litres/s. Use a value of $\lambda = 0.001$.

2. Repeat question 1 using the Moody formula to calculate λ with $k_s = 0.03mm$.

   The Moody formula is given by:

$$\lambda = 0.0055 \left[1 + \left(20000 \frac{k_s}{d} + \frac{10^6}{\text{Re}} \right)^{1/3} \right]$$

   (The calculation is much the same as the previous examples, but you will need more columns in the table to take account of the variable λ.)

3. Use the hardy Cross method to calculate the flow in each pipe in the network below that contains two loops. Assuming a head of 100m at junction A, what would be the head at the other 5 junctions.

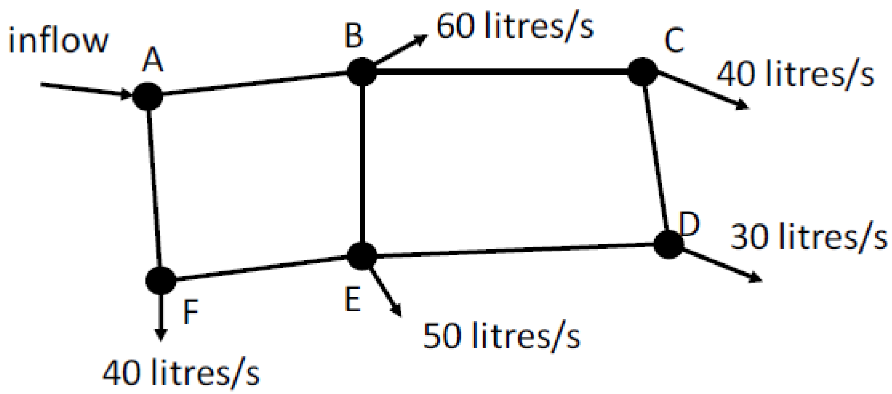

Figure 2.5: Two loop network exampel

Use a constant value of $\lambda = 0.018$ for all pipes.

Pipes	AB	BC	CD	DE	EF	AF	BE
length (m)	600	600	200	600	600	200	200
Diameter (mm)	250	150	100	150	150	200	100

Note: The solution is performed as described in the example of a single loop above, but there are two loops. The pipe BE will be in both loops, but it will be in the opposite direction for each loop. This means that the Δq must be added during one loop update and subtracted during the other.

2.4 Branched Network Analysis - Quantity balance

This quantity balance analysis takes the continuity equation for each junction $\sum_{i=1}^{n}$ (where i is the index of pipe joining at that jinction, and n the number of pipes joining). The head loss equations for each pipe are $h_f = k_i q_i^2$ and these are combined to give a set of equations in terms of head losses only. It may be applied to loops or branches where the external heads are known and the heads within the network are required. (From these heads the flows can, of course, be calculated.)

In this technique the head is corrected at each node by the formula:

$$\delta H = \frac{2\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n}\left(q_i / h_{f\,i}\right)} \tag{2.6}$$

Note how the summations in this formula are summations of the pipes connecting at a junction. The δH is applied to that particular junction only. (Though not worked through here, the formula is derived in a similar way to Hardy Cross)

Steps:

1. Assume a value of head (H_j) at each junction
2. Calculate the flow, q_i, in each pipe using the Darcy-Weisbach equation i.e. for this pipe

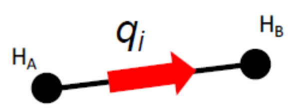

Figure 2.6: Single pipe of branched network

$$h_f = H_A - H_B = \frac{\lambda L Q_i^2}{12.1 d^5} = k c q_i^2$$

$$q_i = \left(\frac{H_A - H_B}{k}\right)^{1/2}$$

Note that care must be taken with this formula. $(H_A - H_B)$ must be positive else no solution is possible. To avoid this problem it could be written as

$$q_i = sign(H_A - H_B)\left(\frac{|H_A - H_B|}{k}\right)^{1/2}$$

In this form the sign of the flow will indicate the direction of the flow: A to B if q_i is positive, B to A if negative

3. If $\sum_{i=1}^{n} q_i = 0$ for each junction then the solution is correct
4. If $\sum_{i=1}^{n} q_i \neq 0$ for each junction calculate the correction to the heads using equation 2.6:

$$\delta H = \frac{2\sum_{i=1}^{n} q_i}{\sum_{i=1}^{n} (q_i/h_{f\,i})}$$

2.4.1 An example of the quantity balance method

This method works very well for the three reservoir problem we saw earlier. This iteration scheme works efficiently and is applicable to networks of many more pipes and reservoirs. Consider this network with the heights and pipe data given below:

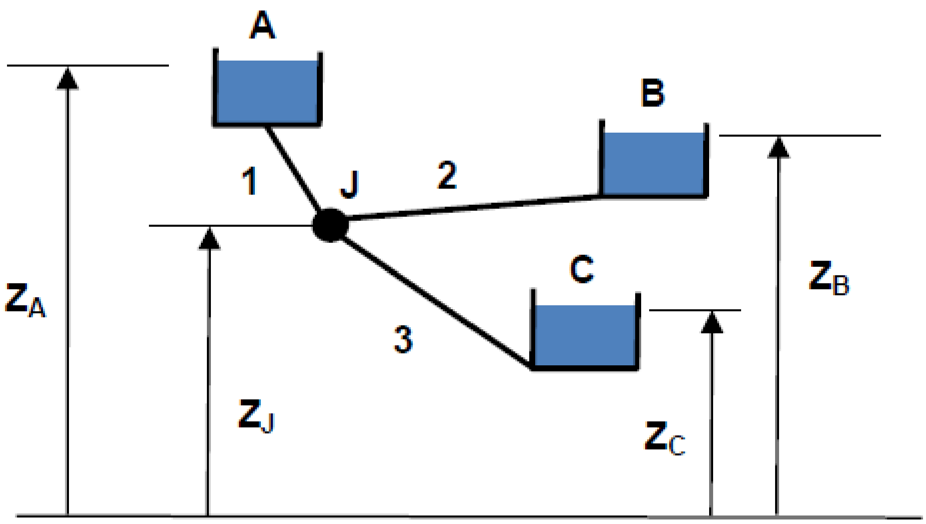

Figure 2.7: Three reservoir, quantity balance example

Node	Elevation (m)
Z_A	800
Z_B	780
Z_C	700
Z_J	720

Pipe	Length (m)	Diameter (mm)	λ
1	5000	300	0.015
2	2000	150	0.015
3	4000	350	0.015

An initial value H_J is required. You could choose any suitable value, anything between z_A and z_C would be normally be fine. In this case as we know the height of J is 720m, we should choose a H just above this, say 750m.

Care must again be taken with the direction of flow. A sign convention must be adopted. e.g. If the flow into the junction it is positive, and away from the junction negative.

Using $H_J = 750m$, the first step a table of calculation may look like

Pipe	h_{fi}	k	q_i	q_i/h_{fi}
1 (A-J)	50	2550.8	0.1400	0.0028
2 (B-J)	30	32649.7	0.0303	0.0010
3 (C-J)	-50	944.1	-0.2301	0.0046
		sum =	-0.0598	0.0084

Thus δH would be $\delta H = \frac{2\times(-0.0598)}{0.0084} = -14.22m$.

And the next estimate for H_J becomes: $H_J = 740 - 14.22 = 735.78m$.

The then next iteration step should be performed:

Pipe	h_{fi}	k	q_i	q_i/h_{fi}
1 (A-J)	64.218	2550.8	0.1587	0.0025
2 (B-J)	44.218	32649.7	0.0368	0.0008
3 (C-J)	-35.782	944.1	-0.1947	0.0054
		sum =	0.0008	0.0087

Thus δH would be $\delta H = \frac{2\times(-0.0008)}{0.0087} = 0.18m$.

And the next estimate for H_J becomes: $H_J = 735.78 + 0.18 = 735.96m$.

As this δH correction is small at 18cm, we can stop iterating. (If you were to perform one more iteration the next δH would be 0.01mm; definitely not worth the extra calculation!)

Notice that with this method you do not have to calculate a balanced flow to start with (i.e. continuity at nodes is not necessary.) You are reasonably free to choose you own initial values of unknown heads - if you were to choose values wildly different from the solution the system may not converge to an answer.

2.4.2 Further examples

1. Perform the calculation on the network in the example above to calculate the head at the junction and the flow in each pipe. Use $z_A = 1000m$, $z_B = 500m$, $z_C = 100m$ and a constant value of $\lambda = 0.01$ for each pipe?
2. Repeat the question using the Barr formula to calculate λ with $k_s = 0.03mm$. The Barr formula is given by:

$$\frac{1}{\sqrt{\lambda}} = -2\log_{10}\left[\frac{k_s}{3.71d} + \frac{5.1286}{\mathrm{Re}^{0.89}}\right]$$

3. For the network of 5 reservoirs and two junctions shown calculate the flow in each pipe and the head at each junction. Assume a constant $\lambda = 0.015$.

Pipe	AB	BC	BD	BE	EF	EG
Length (m)	10000	3000	3000	6000	3000	3000
Diameter (mm)	500	300	300	400	300	300

Level	(m)
z_A	150
z_C	100
z_D	80
z_F	75
z_G	60

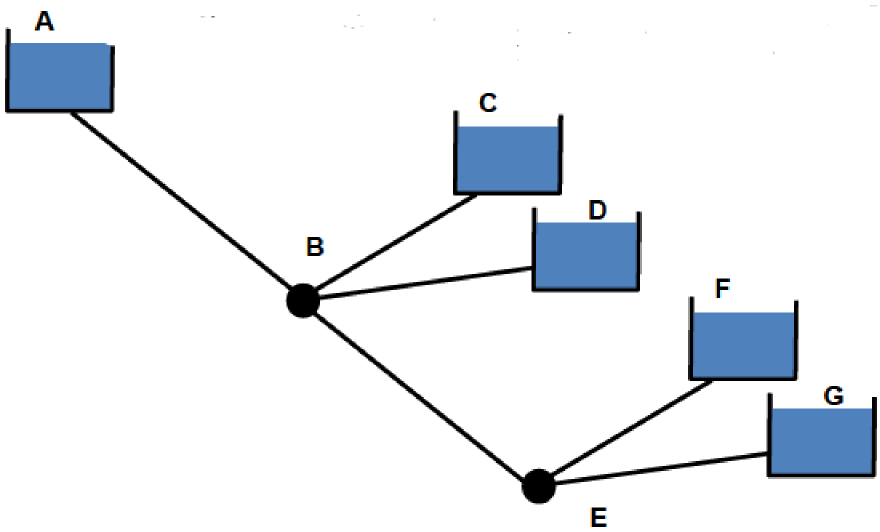

Figure 2.8: Five reservoir problem example

2.5 Matrix Solution Methods

We have seen the quantity balance method that combined continuity and the head loss equation for each pipe. In a similar way it is possible to write out all continuity equations and head loss equations for any network and solve them as a set of (non-linear) simultaneous equations. This solution method is not usually possible by hand but relatively straight forward for a computer when the equations are written in matrix form.

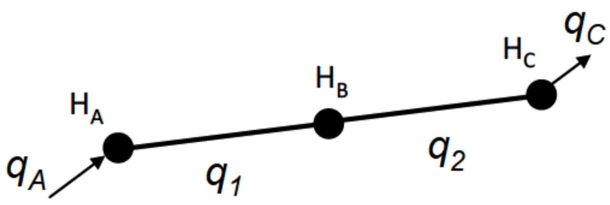

Figure 2.9: A simple, two-pipe network

However, because the equations are non-linear it is still an iterative process requiring the matrix solution of the large set of equations and at each step the flows and heads are updated and the solution recalculated until the updates are small.

Consider the simple two pipe network shown. It consists of 2 unknown flows in the pipes and 3 unknown heads at the junctions.

From Darcy-Weisbach we have for each pipe:

$$H_A - H_B = kq_1^2$$
$$H_B - H_C = kq_2^2$$

or

$$kq_1^2 - H_A + H_B = 0$$
$$kq_2^2 - H_B + H_C = 0$$

In matrix form we could write these as

$$\begin{pmatrix} kq_1 & 0 & -1 & 1 & 0 \\ 0 & kq_2 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ H_A \\ H_B \\ H_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

There are three continuity equations that can be written, one for each node:

$$q_A = q_1 \qquad q_1 = q_2 \qquad q_2 = q_C$$

Incorporating these into the same matrix equation gives:

$$\begin{pmatrix} kq_1 & 0 & -1 & 1 & 0 \\ 0 & kq_2 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ H_A \\ H_B \\ H_C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ q_A \\ 0 \\ q_C \end{pmatrix}$$

This formulation and matrix forms the basis of the Gradient Method. The **Gradient Method** further manipulates this matrix to get it into a form for easier solution with computer based matrix solvers.

Actually the set of equations / matrix as it stands cannot be solved. There must be in any network solution a fixed head. We saw this in the head calculation of the hand calculations above. In this case we need to adjust the above matrix, fixing one head which results in one equation being removed.

Assume we know the head H_B. This can not be in the vector of solutions as it is no longer a variable. The equations that includes this could be written

$$kq_1^2 - H_A = -H_B \qquad\qquad kq_2^2 + H_C = H_B$$

And replacing these in the matrix we get:

$$\begin{pmatrix} kq_1 & 0 & -1 & 0 \\ 0 & kq_2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ H_A \\ H_C \end{pmatrix} = \begin{pmatrix} -H_B \\ H_B \\ q_A \\ q_C \end{pmatrix}$$

This method can easily be expanded to include local losses and pumps, valves etc. The resulting matrix can be solved quite easily for large complex networks. For example a 1000 pipe network would only take a few seconds to solve with the appropriate matrix solver.

Commercial programs for network analysis use this method - some still use the Hardy-Cross method - but the Gradient Method seems more robust and more suited to computer implementation than Hardy-Cross.

A popular solver for complex pipe networks is EPANET
(http://www.epa.gov/nrmrl/wswrd/dw/epanet.html)

This is available free and is very comprehensive in the range of features it will solve for on large water networks. The basis of the solution method in this software is the Gradient Method.

3 Pumps and Turbines - Fluid Machines

Fluid machines either take energy from a fluid and convert it into mechanical energy or vice versa. Machines which take energy from a fluid are called **turbines** and machines which give energy to a fluid are called **pumps** and **fans**. From a theoretical viewpoint there is no difference between the two, in practice there is also a great deal of similarity.

Both pumps and turbines can be split into two distinct groups, **positive displacement** and **rotodynamic**

Positive displacement

- reciprocating pumps - often used for temporary site drainage
- reciprocating engines - gas, petrol, diesel
- gear pumps - two intermeshing gear wheels, often used for the oil pump
- rotor/stator pumps - a double helix stator with a single helix rotor inside it - used for pumping food, concrete, sewage and many other fluids

The behaviour of such pumps is not readily amenable to analytical treatment by the methods of fluid mechanics.

Rotodynamic - Centrifugal, mixed flow and axial flow pumps

- centrifugal, mixed flow and axial flow pumps - centrifugal pumps are used for high head, low flow situations and axial flow pumps for low head high flow e.g. land drainage
- radial, mixed flow and axial flow turbines - radial flow turbines, often referred to as Francis turbines are used for high head hydroelectric sites and axial flow turbines for low head sites e.g. tidal barrage schemes
- Pelton wheels - for very high heads and low flows, called an impulse turbine
- Banki and Turgo wheels - other, less common forms of high head impulse turbines

These notes will concentrate on rotodynamic pumps and how to select such a pump for use on a given pipeline. There are several reasons why it is useful for us to have some idea of how pumps work and the different types of pumps available, for example:-

- matching pumps to pipelines i.e. which is the most suitable size and type of pump for any given pipeline
- to understand the relationship between head and flow in a pump
- to determine the power requirements of pumps

3.1 Flow through a centrifugal pump

Consider the figure below which show a cross section through a centrifugal pump. Fluid flows along a pipe straight into the centre of the impeller - the eye of the impeller. It is turned through 90 degrees and passes up through the spinning impeller where it receives energy from the spinning impeller. This increases the velocity (and hence kinetic energy) and the pressure in the fluid. It leaves the impeller through the outer edges but is guided by the casing of the pump - the volute which expands around the impeller - converting kinetic energy into potential energy (head) in the process. The design of the volute is very important as this is where large energy losses may occur (25

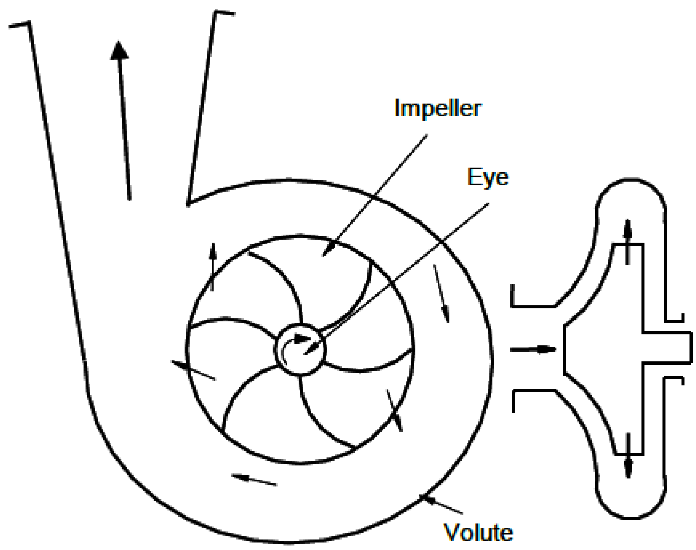

Figure 3.1: Pump impeller and volute

The pump impeller itself has blades to push the fluid round and guide it to the outer edge. These blades may take different angles - they may be *forward facing* (pointing in the direction of the spinning impeller), *radial* (straight) or *backward facing*. Each gives a different characteristic for the pump. For pumping water backward facing blades are usually preferred. Although they produce a lower head for a particular size and flow they have a lower loss in the volute and much greater efficiency. Where for water the usual aim is to increase head, for air it is to increase velocity, with little pressure change, in which case forward facing blades are preferred.

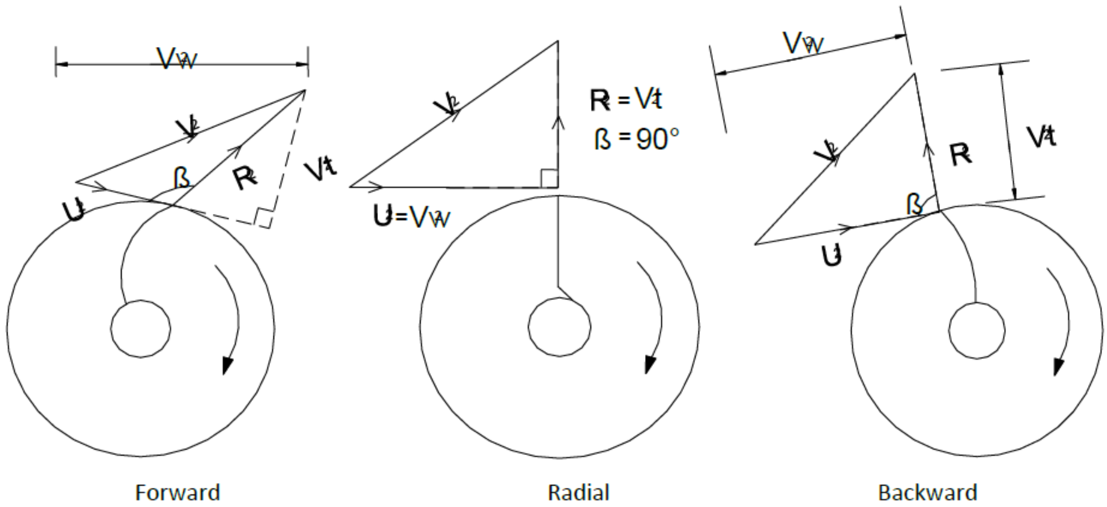

Figure 3.2: Pump impeller blade angles and the resulting exit velocity diagrams

3.2 Definitions of Head for a pump on a pipeline

It is important that we define what we mean be *head* in relation to a pump and the losses on the pipeline it is connected to. Consider the pump in the figure below, it is pumping water from a lower tank (on the *suction* side of the pump) along a pipeline, to a higher tank on the *delivery* side of the pump.

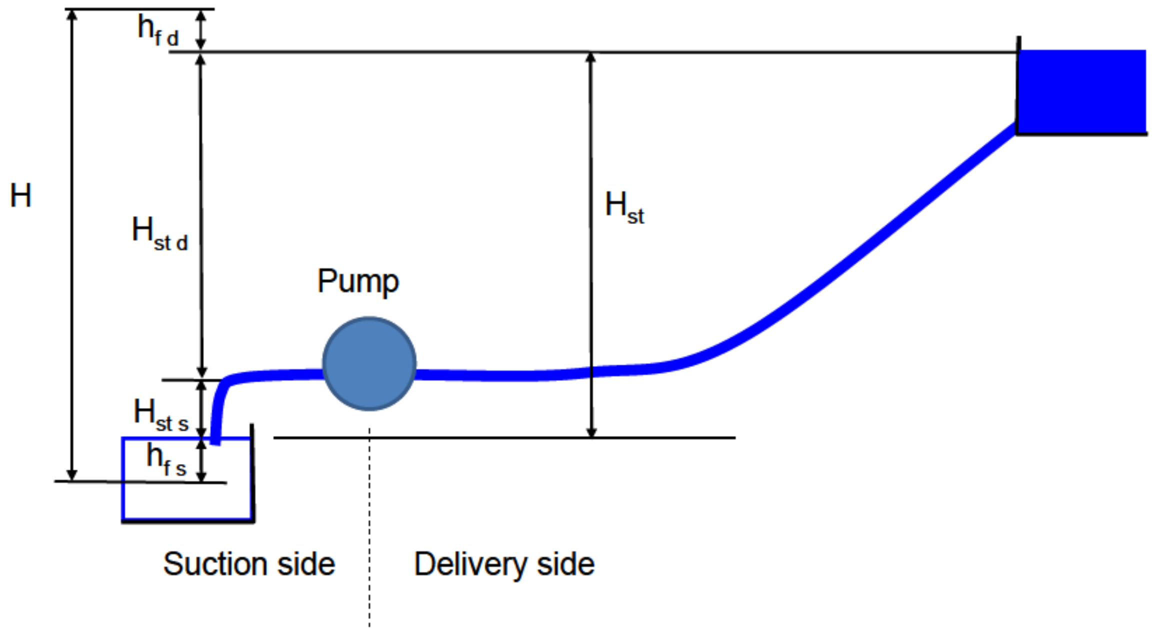

Figure 3.3: Pump in the context of the pipeline

The pump in figure 3.3 is being asked to move water from the left tank to the right tank. The difference in height is known as the static head, H_{static} (or h_{st}). We have a static head on the left the suction side and the right delivery side the sum of which is the total static head:

$$H_{static} = H_{static\ suction} + H_{static\ delivery}$$

This is the head over which we know we must increase the pressure by to pump water. It does not change whatever the flow rate (assuming the tank levels do not change).

Often the pump is so close to the suction level that we can ignore the static suction head.

As we increase the flow in the pipeline we get friction and other head losses as we have seen earlier. In the suction pipe inlet we will get local entry losses and, assuming a short pipe, some small friction losses - shown as hf s in the figure. Similarly, on the delivery side we get friction losses along the whole of the pipe and a small exit loss plus any other losses along the pipe. This is shown as hf d on the figure. All of these losses increase with flow rate, for example:

$$h_{f\ suction} = k_{entry}\frac{v^2}{2g} + \frac{\lambda L_s Q^2}{12.1 d_s^5} \qquad h_{f\ delivery} = \frac{\lambda L_d Q^2}{12.1 d_d^5} + k_{exit}\frac{v^2}{2g}$$

Thus the total head H that a pump must produce to move water is

$$H = H_{static} + h_{f\ suction} + h_{f\ delivery}$$

Applying the Bernoulli equation we can get an expression for pressure on the suction and delivery sides.

If the pressure and velocity at pump on the suction side are p_s and v_s. And on the delivery side p_d and v_d and the pump is at level z_p. then

$$\frac{p_1}{\rho g} + \frac{v_1^2}{2g} + z_1 = \frac{p_s}{\rho g} + \frac{v_s^2}{2g} + z_p + h_{f\ s}$$

p_1 is atmospheric and v_1 is small compared to the pipe velocity, and $z_p - z_1 = H_{st\ suction}$

$$-\frac{p_s}{\rho g} = H_{st\ suction} + \frac{v_s^2}{2g} + h_{f\ s}$$

On the delivery side

$$\frac{p_d}{\rho g} + \frac{v_d^2}{2g} + z_p = \frac{p_2}{\rho g} + \frac{v_2^2}{2g} + z_2 + h_{f\ d}$$

p_2 is atmospheric and v_2 is small compared to the pipe velocity, and $z_2 - z_p = H_{st\ delivery}$

$$-\frac{p_d}{\rho g} = H_{st\ delivery} + \frac{v_d^2}{2g} + h_{f\ d}$$

There is a physical limit on the lowest value of the suction side pressure. When this pressure gets too low the water will begin to boil - turns to vapour. (This pressure is know as teh *vapour pressure* and at 20^oC it is about -7.7m head). At this point the pump will cease to work. In practice the pump gets very noisy - with a sound like it is pumping gravel. This situation is called *cavitation*. It can cause major damage to the impeller blades. As the pressure increases the vapour bubble collapse with enormous pressures that can cause damage - pitting - of the metal surface. Over time this may be sufficient to break the blades.

3.3 Pump Equations

It can be shown through consideration of the radial and tangential velocities of the impeller calculation of the rate of change of momentum together with geometrical properties of the impeller (e.g. diameter and angle of blades) that the following equation governs the relationship between head produced by a pump and the flow passing through it:

$$H = AN^2 + BNQ^2 - CQ \tag{3.1}$$

Where, N is the speed (e.g. revs/min) and A, B and C are constants for teh specific pump.

This is a parabolic relationship between H and Q. Notice (at a given N) how the head produced is greater than zero when the flow is zero and depending on B and C, H may rise first before falling as Q increases.

As power is given by ρgQH then the power produced by a pump has the following relationship to H:

$$P = \rho gQH = \rho gQ(AN^2 + BNQ^2 - CQ) \tag{3.2}$$

The overall efficiency η of a pump is defined as the ratio of the power produced to the power input to the motor shaft, P_i.

$$\eta = \frac{\rho gQH}{P_i} \tag{3.3}$$

This equation allow us to calculate the power required for a given pump of known efficiency pumping Q against a known H.

3.4 Pump and Pipeline Characteristic Curves

Pump Curve:
The equation of the pump performance is:

$$H = AN^2 + BNQ^2 - CQ$$

Some typical values for A, B and C at a given speed, N are shown in the table below. Notice how they differ slightly for forward, radial and backward facing blades.

	A	B	C	N
Backward	2.38×10^{-5}	0.069	5000	1450
Radial	2.37×10^{-5}	0.1	4000	1280
Forward	2.34×10^{-5}	0.4	3000	740

Plotting these on a graph gives the characteristic curve for each pump. Increasing the speed would simply increase the height of the line (head produced)

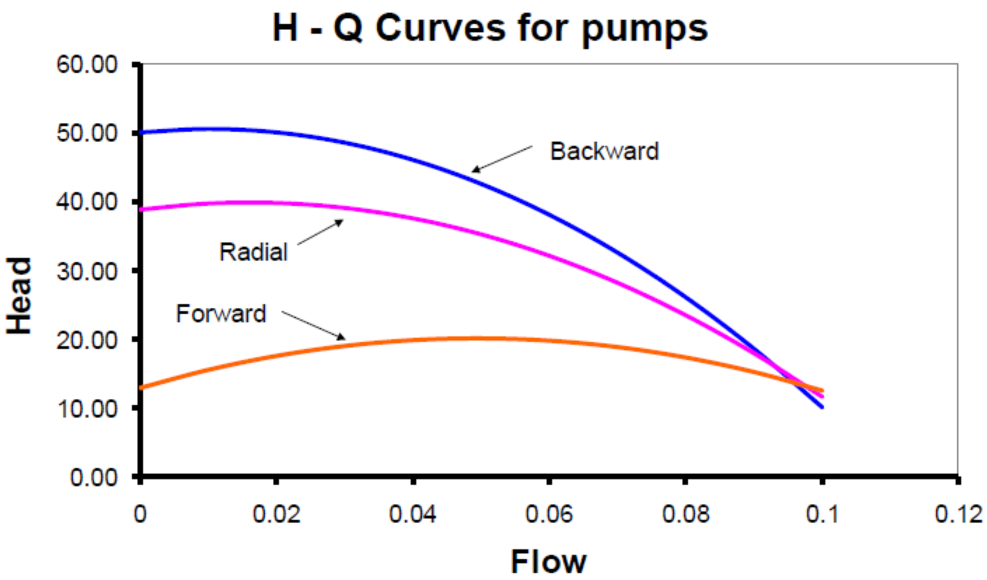

Figure 3.4: Pump curves for difference blade angles

Pipeline Curve:
For any pipeline we can plot a similar characteristic curve of $H = h_f + h_{entry} + h_{exit}$... which for our pump situation is written $H = H_{st} + h_{f\ s} + h_{f\ d}$. All terms on the right hand side are functions of Q and would give something like that shown below.

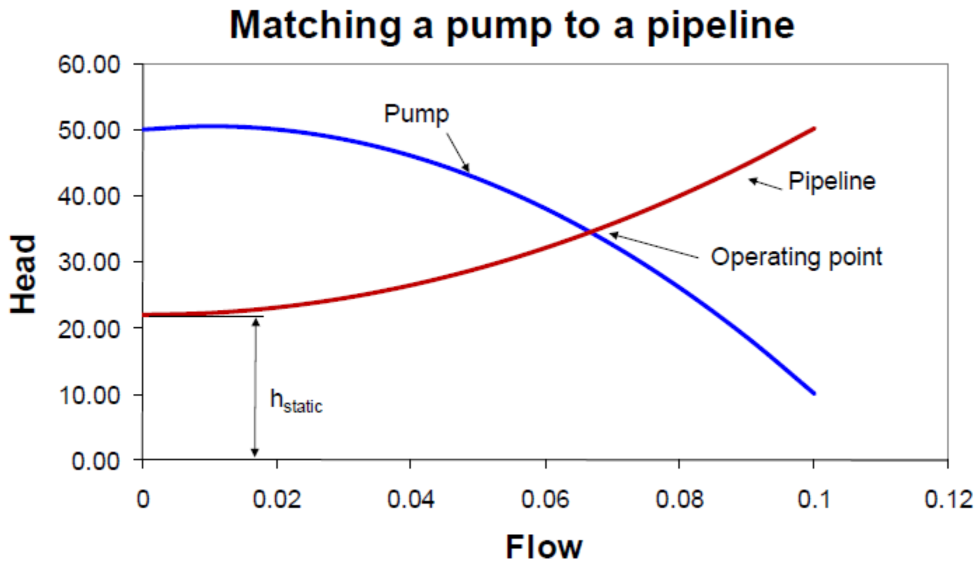

Figure 3.5: Pump curve and Pipeline curve

Note that the pipeline head *loss* increases as flow increases, and that it starts equal to the value of the static head. If on the same graph the pump curve is plotted then the point at which these two lines cross is the point at which the pump would operate on this particular pipeline. We can thus using this graphical method easily identify the flow that would result.

Although this method identifies the flow, it is important to calculate the power required for this - as power will be used throughout the life of the project - and hence very costly - we need to aim for the pump to be at its most efficient at this operating point. Plotting the efficiency against Q on the same graph would give something like the figure below.

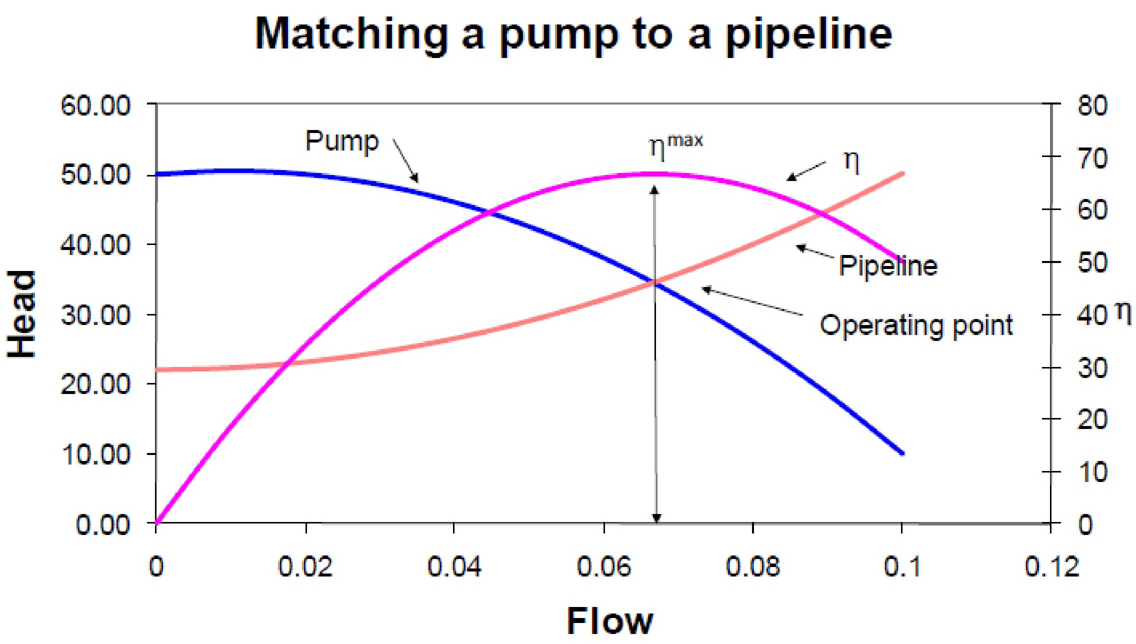

Figure 3.6: Pump, Pipe line and Efficiency Curves

In this figure the efficiency of the pump is at its maximum at the operating point of the pump/pipeline system. Should the efficiency maximum not coincide, then a different pump should be selected.

3.5 Pump selection

Matching a pump top a pipeline is a mixture of hydraulics, site engineering and economics. By considering the Darcy-Weisbach expression for head loss it is readily seen that a small increase in diameter will cause a significant reduction in head loss hence a reduction in pump size, this reduces both capital and running costs of the pump but may increase the capital and construction costs of the pipeline. A further restriction may be that the retention time of the fluid in the pipe may be limited, for example when pumping sewage, clearly this restricts the diameter of the pipe. When all these factors have been considered a pipe diameter is chosen which enables a friction head to be calculated, the total head is then determined by adding this to the static head, this is the head that the pipe has to overcome.

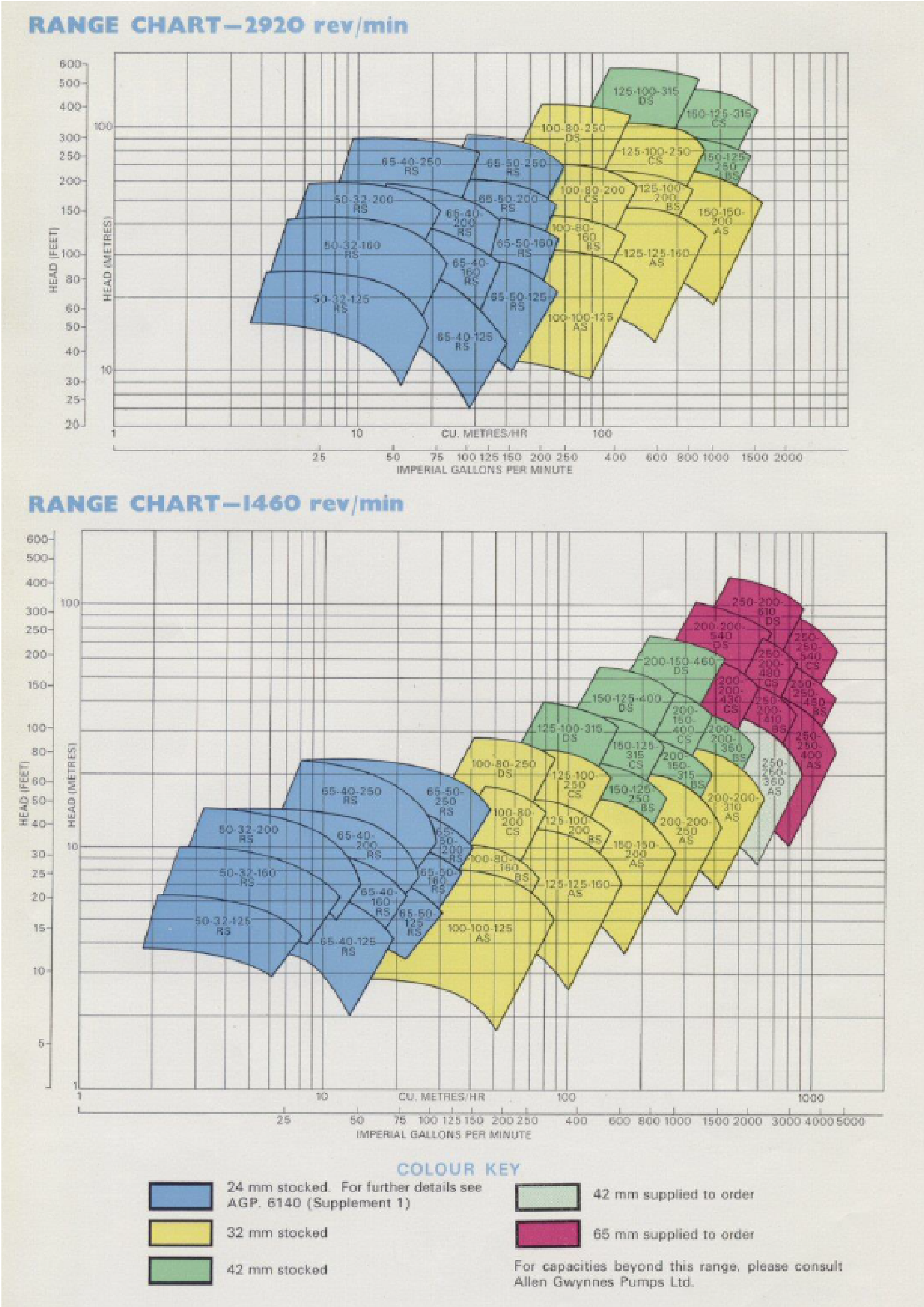

Figure 3.7: Comercial pupm curve charts

Pumps come in a family of different sizes but all geometrically similar. For each pump in the series the manufacturer will supply the $H-Q$ curve for a specific pump speed or more correctly the part of the curve around the point of maximum efficiency. The point where this intersects the $H-Q$ curve for the pipeline gives the flow.

Pump manufacturers produce charts for use in selecting the most appropriate pump. They put the efficient part of the $H-Q$ for all the pumps in an homologous series onto one graph. An example of this is shown below. The short length of curve is that part of the curve under the highest point of the efficiency curve, this enables the user, once the $H-Q$ for the pipeline is known to read off the model number of the pump most suitable for the job.

If one pump cannot produce sufficient head then two or more pumps may be used in series; for the great majority of pipelines this would not be considered a good arrangement, it would be better to specify a larger pump. However such an arrangement is frequently used in deep boreholes.

If the pump cannot produce sufficient flow then two or more pumps can be used in parallel. This arrangement is very common is water supply and sewerage pumping stations. In fact in these designs it is usual to have several pumps running in parallel, with the pumps cutting in sequentially as the flow increases. When pumps are running in parallel it is essential to fit reflux (non return) valves so that one pump does not drive another pump as a turbine.

3.5.1 An example of matching a pump to a pipeline

A pump it required to lift water $40m$ to a header tank through a $50mm$ diameter pipe which is $100m$ long. Assuming a constant friction coefficient $\lambda = 0.02$ and ignoring local losses what would be the flow in the pipeline if the $H-Q$ characteristic curve for the pump is as given below?

$$H = 45 + 25Q - 500Q^2$$

This is the arrangement:

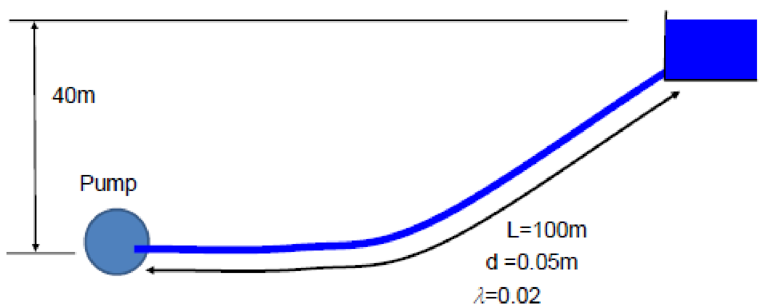

Figure 3.8: Pump and pipeline example

We have the pump equation, we need a pipeline $H-Q$ equation. In this case as local losses are to be ignored, this is simply the Darcy-Weisbach equation

$$H = H_{st} + h_{f\ d} = H_{st} + \frac{\lambda L Q^2}{12.1 d^5}$$

Q	H pump	H pipe
0.0000	45.00	40.00
0.0005	45.01	40.13
0.0010	45.02	40.53
0.0015	45.04	41.19
0.0020	45.05	42.12
0.0025	45.06	43.31
0.0030	45.07	44.76
0.0035	45.08	46.48
0.0040	45.09	48.46

We could plot these two equation on the same graph:

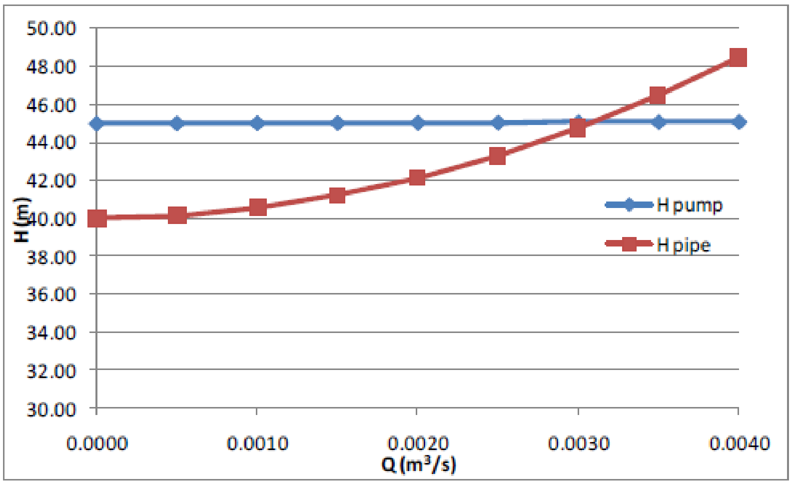

Figure 3.9: Pump and pipeline curves for example

And see that the approximate solution is $0.0031m^3/s$.

To get a more accurate solution we can equate the pump and the pipeline heads

$$45 + 25Q - 500Q^2 = 40 + \frac{0.02 \times 100Q^2}{12.1 \times 0.05^5}$$

And solve for Q. This is simply a quadratic. Rewrite it in the form $aQ^2 + bQ + c = 0$ and solve using the quadratic formula $Q = \frac{-b+\sqrt{b^2-4ac}}{2a}$. This gives $Q = 0.00309m^3/s$.

3.6 Further examples

1. Repeat the example above use the pump equation $H = 100 + 20Q - 1500Q^2$ and a static head of 90m. What will be the flow in the pipeline? What would be the flow if the pipe diameter was doubled?
2. In the question above assume there is an orifice plate just downstream of the pump with a head loss characteristic of $4\frac{v_p^2}{2g}$, where v_p is the velocity in the pipe. What would be the flow in the pipe?.

3. A pump has the characteristics given in the table below. It is to be used on a pipe 100m long, diameter 200 mm to raising water 30m. The pipe can be assumed to have a constant λ of 0.015.

Q (litres/s)	H (m)	η (%)
0	58	-
100	56	35
200	52	55
300	45	65
400	36	65
500	24	57
600	6	32

   (a) Draw a H/Q graph of both the pump curve and the system curve and hence estimate the discharge in the pipeline.
   (b) Approximating the appropriate portion of the pump curve to a straight line equate this with the pipeline curve to calculate the discharge in the pipeline.
   (c) Is the pump operating at its most efficient. If it is not what changes could be made to correct this?

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