# Computational methods for imprecise continuous-time birth-death processes: a preliminary study of flipping times

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## The Precise Case

Consider a continuous time and finite-state Markov process with state space $\mathcal{X}$. At any time $t \in [0,+\infty)$, the stochastic matrix of the process $P_t$ is derived from a transition rate matrix $Q$. For $i,j \in \mathcal{X}$, the element at the $i$ row and $j$ column of $Q$ (or $P_t$) is denoted by $Q(i,j)$ (or $P_t(i,j)$). For the matrix $Q$, the following properties hold

(P1) $Q(i,j) \geq 0$ for all $i,j \in \mathcal{X}$ such that $i \neq j$
(P2) $\sum_{j\in\mathcal{X}} Q(i,j) = 0, \forall i \in \mathcal{X}$

A matrix $Q$ is said to be bounded if $Q(i,i) > -\infty$ for all $i \in \mathcal{X}$ or, equivalently, if $\|Q\| < \infty$. Our results hold for various types of norm, but the one we consider is the infinite norm defined by $\|Q\| := \|Q\|_\infty = \max\{\sum_{j\in\mathcal{X}}|Q(i,j)| : x \in \mathcal{X}\}$.

When $Q$ is bounded, then $P_t$ satisfies the Kolmogorov backward equation

$$\frac{d}{dt}P_t = QP_t. \qquad (1)$$

If we let $f_t(i) := E_t(f|X_0 = i)$, with $f$ a real-valued function on the finite state space $\mathcal{X}$ and $i \in \mathcal{X}$ an initial state, then we can rewrite Equation (1) as follows

$$\frac{d}{dt}f_t = Qf_t. \qquad (2)$$

Combined with the boundary condition $f_0 = f$, the unique solution of Equation (2) is $f_t = e^{tQ}f$. Instead of considering a time-invariant $Q$, we can also let $Q_t$ be a function of the time $t$. In that case, Equation (2) can be rewritten as

$$\frac{d}{dt}f_t = Q_t f_t. \qquad (3)$$

which, in general, has no analytical solution.

## A "messy" case

Consider the state space $\mathcal{X} := \{0,1,2,3\}$, the following set $\mathcal{Q}$ of bounded matrices

$$\left\{ \begin{pmatrix} -p_i & p_i & 0 & 0 \\ q_j & -q_j & 0 & 0 \\ 0 & 0 & -r & r \\ 0 & 0 & s & -s \end{pmatrix} : a_i \in [\underline{a},\overline{a}] \text{ and } b_j \in [\underline{b},\overline{b}] \right\}$$

and a function $f$ of the form $[c,c,f_2,f_3]^T$. In this case, we cannot efficiently identify $Q_{\tau_0}$, because for any $Q,Q' \in \mathcal{Q}$, we have that $Q^k f = Q'^k f$, for all $k \in \mathbb{N}$.

Is there a simple way to check when two different matrices $Q,Q'$ yield the same expected value, without calculating $Q^k$ and $Q'^k$ for all $k$?

## Imprecise Birth-Death Process

We focus on the case where every state has an interval-valued birth and/or death rate. The transition rate matrices have the following form

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mu_i & -(\mu_i+\lambda_i) & \lambda_i & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \mu_L & -\mu_L \end{pmatrix}$$

where, for all $i \in \{0,\ldots,L-1\}$ and $j \in \{1,\ldots,L\}$, $\lambda_i \in [\underline{\lambda},\overline{\lambda}]$ and $\mu_j \in [\underline{\mu},\overline{\mu}]$ and $L \in \mathbb{N}$. In this way, we have a set of matrices $\mathcal{Q}$ with finite numbers of extreme points, separately specified rows and which avoids the special case above.

## The Imprecise Case

**Set of matrices** Instead of a single transition matrix $Q$, we consider a set of such matrices, denoted by $\mathcal{Q}$. We assume that each matrix in $\mathcal{Q}$ is bounded and satisfies (P1) and (P2). Let $\mathcal{R}$ be the set of all rate matrices, then for any set $\mathcal{Q} \subseteq \mathcal{R}$ of rate matrices, we let

$$\mathcal{Q}_i := \{Q(i,\cdot) : Q \in \mathcal{Q}\} \text{ for all } i \in \mathcal{X},$$

and we say that $\mathcal{Q}$ has *separately specified rows* if

$$Q \in \mathcal{Q} (\forall i \in \mathcal{X})\ Q(i,\cdot) \in \mathcal{Q}_i.$$

We further assume that $\mathcal{Q}$ is the convex hull of a *finite* number of extreme transition rate matrices.

**Our Approach** At any time $t \in [0,+\infty)$, the only assumption we make about $Q_t$ is that it is an element of $\mathcal{Q}$. Every such possible choice of non-stationary transition rate matrices will, by (3), result in a—possibly different—solution $f_t$. Our goal is to calculate exact lower and upper bounds for the set of all these solutions $f_t$, as denoted by $\underline{f}_t$ and $\overline{f}_t$. In the recent work of Škulj and with respect to the lower bound, $\underline{f}_t$ is the solution to

$$\frac{d}{dt}\underline{f}_t = \min_{Q\in\mathcal{Q}} Q\underline{f}_t, \text{ with boundary condition } \underline{f}_0 = f. \qquad (4)$$

Since $\mathcal{Q}$ is the convex hull of a finite number of extreme transition rate matrices and that the solution to (4) is continuous, there must be time points $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots$ such that, for all $t \in \tau_n := [t_n, t_{n+1}]$, the minimum in (4) is obtained by the same extreme transition rate matrix $Q_{\tau_n} \in \mathcal{Q}$. We call these time points $t_n$ *flipping times*. Equation (4) is then piecewise linear and has the following solution

$$\underline{f}_t = e^{(t-t_n)Q_{\tau_n}} e^{(t_n-t_{n-1})Q_{\tau_{n-1}}} \ldots e^{(t_2-t_1)Q_{\tau_1}} e^{t_1 Q_{\tau_0}} f, \text{ for } t \in [t_n, t_{n+1}]. \qquad (5)$$

**Calculating Lower Expectations** We need to find the flipping times $t_n$ and the corresponding extreme transition rate matrices $Q_{\tau_n}$ when calculating the lower expectation of a given $f$ on $\mathcal{X}$. It is known that

$$\frac{\partial}{\partial t}[e^{tQ}f]\Big|_{t=0} = Qf \Rightarrow (\forall\varepsilon > 0)(\exists\delta > 0)(\forall t \in (0,\delta))\|e^{tQ}f - f - tQf\|_\infty < t\varepsilon \qquad (6)$$

and it can be further proved that for any pair of matrices $Q,Q'$ in $\mathcal{Q}$

$$\text{if } Qf < Q'f, \text{ then } (\exists\delta > 0)(\forall t \in (0,\delta))\ e^{tQ}f < e^{tQ'}f,$$

where $Qf < Q'f$ if $Qf(i) \leq Q'f(i)$ for all $i \in \mathcal{X}$ and $Qf \neq Q'f$. In order to find $Q_{\tau_0}$ in (5), we need to find a $Q$ such that $Qf < Q'f$, for all $Q' \in \mathcal{Q} \setminus \{Q\}$. Since $\mathcal{Q}$ has separately specified rows, we can identify $Q_{\tau_0}$ by minimising $Qf$ at each row separately. Hence, $Q_{\tau_0}$ belongs to the set $\mathcal{Q}_{\tau_0} := \{Q \in \mathcal{Q} : Qf(i) \leq Q'f(i), \forall i \in \mathcal{X} \text{ and } \forall Q' \in \mathcal{Q} \setminus \{Q\}\}$.

In practice, $\mathcal{Q}_{\tau_0}$ might not be a singleton and in this case, for any two matrices $Q,Q'$ in $\mathcal{Q}_{\tau_0}$, we have that $Qf = Q'f$. Since we know that $\frac{\partial^k}{\partial^k t}[e^{tQ}f]\Big|_{t=0} = Q^k f$, it can be proved that

$$\text{if } Q^k f < Q'^k f \text{ and } Q^{k'} f = Q'^{k'} f \text{ for all } k' \in \{1,\ldots,k-1\}, \text{ then } (\exists\delta > 0)(\forall t \in (0,\delta))\ e^{tQ}f < e^{tQ'}f \qquad (7)$$

Therefore, if $\mathcal{Q}_{\tau_0}$ is not a singleton, then $Q_{\tau_0}$ is a single matrix $Q$ of $\mathcal{Q}_{\tau_0}$, for which (7) holds.

Having found $Q_{\tau_0}$, then $\underline{f}_{t_1} = e^{t_1 Q_{\tau_0}} f$ and $\tau_0 := [t_1, 0]$ due to (5). In order to find the corresponding, if any, flipping time $t_1$, we take the derivative of $e^{tQ}\underline{f}_{t_1}$.

$$\frac{\partial}{\partial t}[e^{tQ}\underline{f}_{t_1}]\Big|_{t=0} = Q\underline{f}_{t_1} = Qe^{t_1 Q_{\tau_0}} f$$

and due to continuity, it holds that

$$Q_{\tau_0} e^{t_1 Q_{\tau_0}} f = Q e^{t_1 Q_{\tau_0}} f. \qquad (8)$$

We solve (8) for each $i \in \mathcal{X}$ separately, since $\mathcal{Q}$ has separately specified rows and the smallest positive real solution of $t_1$ is the first flipping time and the corresponding matrix $Q$ is the matrix $Q_{\tau_1}$. We continue the same procedure till we find no more flipping times.

## Numerical Results

We calculate the lower expected probability of state 1, $\underline{E}(X_t = 1)$, of an imprecise birth-death chain with state space $\mathcal{X} := \{0,1,2,3\}$ for $t$ approaching infinity. The set of transition rate matrices $\mathcal{Q}$ is derived from the intervals $\lambda_i \in [1,3]$ and $\mu_j \in [2,5]$, for all $i \in \{0,\ldots,L-1\}$ and $j \in \{1,\ldots,L\}$ and the input function is $f = [0,1,0,0]^T$.

Following the procedure described before, we start by finding a matrix $Q$, such that $Qf < Q'f$ for all $Q' \in \mathcal{X} \setminus Q$. Due to the values of $f$, any matrix $Q'$ of the following form minimises $Q'f$

$$Q' = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 5 & -8 & 3 & 0 \\ 0 & 2 & -(2+\lambda_2) & \lambda_2 \\ 0 & 0 & \mu_3 & -\mu_3 \end{pmatrix}$$

where, $\lambda_2 \in \{1,3\}$ and $\mu_3 \in \{2,5\}$, as we can consider only the extreme points of the respective rate intervals.

Continuing with the procedure, we check whether there is a matrix $Q$, such that $Q^2 f < Q'^2 f$ for all $Q' \in \mathcal{X} \setminus Q$. Indeed, there is such a matrix and therefore we have that

$$Q_{\tau_0} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 5 & -8 & 3 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

for which the flipping time is $t_1 = 0.6403991$ and $Q_{\tau_1}$ is

$$Q_{\tau_1} = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 2 & -5 & 3 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

For the matrix $Q_{\tau_1}$ there is no flipping time and by taking $t \to \infty$, we have that $\underline{E}(X = 1) = 0.0937540788$.