

Computational methods for imprecise continuous-time birth-death processes: a preliminary study of flipping times

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The Precise Case

Consider a continuous time and finite-state Markov process with state space \mathcal{X} . At any time $t \in [0, +\infty)$, the stochastic matrix of the process P_t is derived from a transition rate matrix Q . For $i, j \in \mathcal{X}$, the element at the i row and j column of Q is denoted by $Q(i, j)$. For the matrix Q , the following properties hold

- (P1) $Q(i, j) \geq 0$ for all $i, j \in \mathcal{X}$ such that $i \neq j$
(P2) $\sum_{j \in \mathcal{X}} Q(i, j) = 0, \forall i \in \mathcal{X}$

A matrix Q is said to be bounded if $Q(i, i) > -\infty$ for all $i \in \mathcal{X}$ or, equivalently, if $\|Q\| < \infty$. Our results hold for various types of norm, but the one we consider is the infinite norm defined by $\|Q\| := \|Q\|_\infty = \max\{|Q(i, j)| : i \in \mathcal{X}\}$.

When Q is bounded, then P_t satisfies the Kolmogorov backward equation

$$\frac{d}{dt}P_t = QP_t. \quad (1)$$

If we let $f_t(i) := E_t(f | X_0 = i)$, with f a real-valued function on the finite state space \mathcal{X} and $i \in \mathcal{X}$ an initial state, then we can rewrite Equation (1) as follows

$$\frac{d}{dt}f_t = Qf_t. \quad (2)$$

Combined with the boundary condition $f_0 = f$, the unique solution of Equation (2) is $f_t = e^{tQ}f$. Instead of considering a time-invariant Q , we can also let Q_t be a function of the time t . In that case, Equation (2) can be rewritten as

$$\frac{d}{dt}f_t = Q_t f_t. \quad (3)$$

which, in general, has no analytical solution.

The Imprecise Case

Set of matrices Instead of a single transition matrix Q , we consider a set of such matrices, denoted by \mathcal{Q} . We assume that each matrix in \mathcal{Q} is bounded and satisfies (P1) and (P2). Let \mathcal{R} be the set of all rate matrices, then for any set $\mathcal{Q} \subseteq \mathcal{R}$ of rate matrices, we let

$$\mathcal{Q}_i := \{Q(i, \cdot) : Q \in \mathcal{Q}\} \text{ for all } i \in \mathcal{X},$$

and we say that \mathcal{Q} has *separately specified rows* if

$$Q \in \mathcal{Q} (\forall i \in \mathcal{X}) Q(i, \cdot) \in \mathcal{Q}_i.$$

We further assume that \mathcal{Q} is the convex hull of a *finite* number of extreme transition rate matrices.

Our Approach At any time $t \in [0, +\infty)$, the only assumption we make about Q_t is that it is an element of \mathcal{Q} . Every such possible choice of non-stationary transition rate matrices will, by (3), result in a—possibly different—solution f_t . Our goal is to calculate exact lower and upper bounds for the set of all these solutions f_t , as denoted by \underline{f}_t and \bar{f}_t . In the recent work of Škulj and with respect to the lower bound, \underline{f}_t is the solution to

$$\frac{d}{dt}\underline{f}_t = \min_{Q \in \mathcal{Q}} Q \underline{f}_t, \text{ with boundary condition } \underline{f}_0 = f. \quad (4)$$

Since \mathcal{Q} is the convex hull of a finite number of extreme transition rate matrices and that the solution to (4) is continuous, there must be time points $0 = t_0 < t_1 < \dots < t_n < t_{n+1} < \dots$ such that, for all $t \in \tau_n := [t_n, t_{n+1}]$, the minimum in (4) is obtained by the same extreme transition rate matrix $Q_{\tau_n} \in \mathcal{Q}$. We call these time points t_n *flipping times*. Equation (4) is then piecewise linear and has the following solution

$$\underline{f}_t = e^{(t-t_n)Q_{\tau_n}} e^{(t_n-t_{n-1})Q_{\tau_{n-1}}} \dots e^{(t_2-t_1)Q_{\tau_1}} e^{t_1 Q_{\tau_0}} f, \text{ for } t \in [t_n, t_{n+1}]. \quad (5)$$

Calculating Lower Expectations We need to find the flipping times t_n and the corresponding extreme transition rate matrices Q_{τ_n} when calculating the lower expectation of a given f on \mathcal{X} . It can be proved that for any pair of matrices Q, Q' in \mathcal{Q} , we have that

$$\text{if } Qf < Q'f, \text{ then } Q_{\tau_0} \neq Q'$$

where $Qf < Q'f$ if $Qf(i) \leq Q'f(i)$ for all $i \in \mathcal{X}$ and $Qf \neq Q'f$. In this way, we can eliminate matrices Q' , which do not yield the minimum expected value of f up to some $t > 0$. Since \mathcal{Q} has separately specified rows, then the matrix Q_{τ_0} is the one that minimises Qf at each row separately. Hence, Q_{τ_0} belongs to the set

$$\mathcal{Q}_{\tau_0} := \{Q \in \mathcal{Q} : Qf(i) \leq Q'f(i), \forall i \in \mathcal{X} \text{ and } \forall Q' \in \mathcal{Q} \setminus \{Q\}\}.$$

In practice, \mathcal{Q}_{τ_0} might not be a singleton and in this case, for any two matrices Q, Q' in \mathcal{Q}_{τ_0} , we have that $Qf = Q'f$. It can be further proved that

$$\text{if } Q^k f < Q'^k f \text{ and } Q'^k f = Q'^{k'} f \text{ for all } k' \in \{1, \dots, k-1\}, \text{ then } Q_{\tau_0} \neq Q' \quad (6)$$

From (6), we understand that if \mathcal{Q}_{τ_0} is not a singleton, for any matrix Q in \mathcal{Q}_{τ_0} we calculate $Q^2 f$ and we compare them, in order to eliminate more matrices. If still the resulted set is not a singleton, from the remaining ones we calculate $Q^3 f$ and so on, till we are left with one matrix, which will be the matrix Q_{τ_0} .

Having found \mathcal{Q}_{τ_0} , then, due to (5), $\underline{f}_{t_1} = e^{t_1 Q_{\tau_0}} f$ and $\tau_0 := [0, t_1]$. The only thing left to find is the flipping time t_1 . If there exists Q_{τ_1} , such that $Q_{\tau_1} \neq Q_{\tau_0}$ and for which we obtain the minimum expected value for some $t > t_1$, then due to continuity, the derivative of the system evaluated at $t = t_1$ should be equal for both Q_{τ_1} and Q_{τ_0} . Therefore, from (3) combined with the boundary condition \underline{f}_{t_1} , we have that

$$Q_{\tau_0} e^{t_1 Q_{\tau_0}} f = Q_{\tau_1} e^{t_1 Q_{\tau_0}} f. \quad (7)$$

We solve (7) with respect to t_1 , for each row separately. At each row i , for Q_{τ_1} , we test all possible extreme matrices from \mathcal{Q}_i . Among the solutions of t_1 , the smallest positive real one is the first flipping time and the corresponding matrix is the matrix Q_{τ_1} . We continue the same procedure till we find no more flipping times.

A "messy" case

Consider the state space $\mathcal{X} := \{0, 1, 2, 3\}$, the following set \mathcal{Q} of bounded matrices

$$\left\{ \begin{pmatrix} -p_i & p_i & 0 & 0 \\ q_j & -q_j & 0 & 0 \\ 0 & 0 & -r & r \\ 0 & 0 & s & -s \end{pmatrix} : a_i \in [\underline{a}, \bar{a}] \text{ and } b_j \in [\underline{b}, \bar{b}] \right\}$$

and a function f of the form $[c, c, f_2, f_3]^T$. In this case, we cannot efficiently identify Q_{τ_0} , because for any $Q, Q' \in \mathcal{Q}$, we have that $Q^k f = Q'^k f$, for all $k \in \mathbb{N}$.

Is there a simple way to check when two different matrices Q, Q' yield the same expected value, without calculating Q^k and Q'^k for all k ?

Imprecise Birth-Death Process

We focus on the case where every state has an interval-valued birth and/or death rate. The transition rate matrices have the following form

$$\begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \mu_i & -(\mu_i + \lambda_i) & \lambda_i & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & 0 & \mu_L & -\mu_L \end{pmatrix}$$

where, for all $i \in \{0, \dots, L-1\}$ and $j \in \{1, \dots, L\}$, $\lambda_i \in [\underline{\lambda}, \bar{\lambda}]$ and $\mu_j \in [\underline{\mu}, \bar{\mu}]$ and $L \in \mathbb{N}$. In this way, we have a set of matrices \mathcal{Q} with finite numbers of extreme points, separately specified rows and which avoids the special case above.

Numerical Results

We calculate the lower expected probability of state 1, $\underline{E}(X_t = 1)$, of an imprecise birth-death chain with state space $\mathcal{X} := \{0, 1, 2, 3\}$ for t approaching infinity. The set of transition rate matrices \mathcal{Q} is derived from the intervals $\lambda_i \in [1, 3]$ and $\mu_j \in [2, 5]$, for all $i \in \{0, \dots, L-1\}$ and $j \in \{1, \dots, L\}$ and the input function is $f = [0, 1, 0, 0]^T$.

Following the procedure described before, we start by finding a matrix Q , such that $Qf < Q'f$ for all $Q' \in \mathcal{X} \setminus Q$. Due to the values of f , there are multiple Q , which minimise Qf . These matrices have the following form

$$Q^* = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 5 & -8 & 3 & 0 \\ 0 & 2 & -(2 + \lambda_2) & \lambda_2 \\ 0 & 0 & \mu_3 & -\mu_3 \end{pmatrix}$$

where, $\lambda_2 \in \{1, 3\}$ and $\mu_3 \in \{2, 5\}$. Let \mathcal{Q}^* be the set containing all the matrices of the above form.

Continuing with the procedure, we check whether there is a matrix Q in \mathcal{Q}^* , such that $Q^2 f < Q'^2 f$ for all $Q' \in \mathcal{Q}^* \setminus \{Q\}$. Indeed, there is such a matrix and therefore we have that

$$Q_{\tau_0} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 5 & -8 & 3 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

for which the flipping time is $t_1 = 0.6403991$ and Q_{τ_1} is

$$Q_{\tau_1} = \begin{pmatrix} -3 & 3 & 0 & 0 \\ 2 & -5 & 3 & 0 \\ 0 & 2 & -5 & 3 \\ 0 & 0 & 2 & -2 \end{pmatrix}$$

For the matrix Q_{τ_1} there is no flipping time and by taking $t \rightarrow \infty$, we have that $\underline{E}(X = 1) = 0.0937540788$.