# Convergence of Continuous-Time Imprecise Markov Chains

*Jasper De Bock*

**Abstract**

*** We provide necessary and sufficient conditions for the unique convergence of a continuous-time imprecise Markov chain to a stationary distribution. ***

Potential journals:
Journal of Differential Equations (IF:1.68)
Journal of Mathematical Analysis and Applications (IF:1.12)

## 1 Introduction

*** For now, this introduction is simply a copy-paste from the poster; I still need to clean this up ***

Consider the set of all the continuous-time non-stationary Markov chains with finite state space $\mathcal{X}$ of which the transition rate matrix $Q_t$ is a function of time such that $Q_t \in \mathcal{Q}$, where $\mathcal{Q}$ is a closed convex set of transition rate matrices that has *separately specified rows*, meaning that

$$Q \in \mathcal{Q} \Leftrightarrow (\forall x \in \mathcal{X})\ Q(x,*) \in \mathcal{Q}_x$$

where, for all $x \in \mathcal{X}$, $\mathcal{Q}_x$ is a closed convex set of row vectors. We call such a set of Markov chains a *continuous-time imprecise Markov chain*.

Fix any $t > 0$. Then for all $f \in \mathbb{R}^{\mathcal{X}}$ and $x \in \mathcal{X}$, the expected value $E_t(f|X_0 = x)$ of $f$ at time $t$, conditional on $X_0 = x$, ranges over a closed interval of which we will denote the lower bound by $\underline{T}_t(f|x)$. For all $x \in \mathcal{X}$, $\underline{T}_t(\cdot|x)$ is a *coherent lower prevision* on $\mathbb{R}^{\mathcal{X}}$. The corresponding *lower transition operator* $\underline{T}_t : \mathbb{R}^{\mathcal{X}} \to \mathbb{R}^{\mathcal{X}}$ is defined by

$$\underline{T}_t f(x) := \underline{T}_t(f|x) \text{ for all } x \in \mathcal{X}.$$

By a recent result of Škulj [3], $\underline{f}_t := \underline{T}_t f$ is the solution to the differential equation

$$\frac{d}{dt}\underline{f}_t = \underline{Q}\,\underline{f}_t$$

with initial condition $\underline{f}_0 = f$, where for all $h \in \mathbb{R}^{\mathcal{X}}$:

$$\underline{Q}h(x) := \min_{Q \in \mathcal{Q}} \sum_{y \in \mathcal{X}} Q(x,y)h(y) \text{ for all } x \in \mathcal{X}.$$

We study the limit behaviour of $\underline{T}_t$. In particular, we provide necessary and sufficient conditions for $\mathcal{Q}$ to be *Perron-Frobenius-like (PF)*, meaning that there is some $\underline{P}_\infty : \mathbb{R}^\mathcal{X} \to \mathbb{R}$ such that, for all $x \in \mathcal{X}$:

$$\lim_{t\to+\infty} \underline{T}_t f(x) = \underline{P}_\infty f \text{ for all } f \in \mathbb{R}^\mathcal{X},$$

or, equivalently, for $\underline{T}_t(\cdot|x)$ to converge to a stationary distribution $\underline{P}_\infty$ that does not depend on $x$.

Our main result is that the following four conditions are equivalent:

1. $\mathcal{Q}$ is PF,
2. $\underline{T}_t$ is PF for some $t > 0$,
3. $\underline{T}_t$ is PF for all $t > 0$,
4. $\mathcal{Q}$ is regularly absorbing,

where (i) for any $t > 0$, we say that $\underline{T}_t$ is PF if the discrete-time imprecise Markov chain that has $\underline{T}_t$ as its lower transition operator is PF, in the sense that, for all $f \in \mathbb{R}^\mathcal{X}$, $\lim_{n\to\infty} \underline{T}_t^n f$ exists and is constant and (ii) '*regularly absorbing*' is a qualitative property of $\mathcal{Q}$ that is fully determined by the signs of the *upper transition rates to singletons* $\overline{Q}(x,y) := \max_{Q\in\mathcal{Q}} Q(x,y)$ and the *lower transition rates to sets* $\underline{Q}(x,A) := \min_{Q\in\mathcal{Q}} \sum_{y\in A} Q(x,y)$, for $x,y \in \mathcal{X}$, $x \neq y$ and $A \subset \mathcal{X} \setminus \{x\}$.

## 2 Preliminaries

Consider some finite *state space* $\mathcal{X}$. Let $\mathcal{L}(\mathcal{X})$ be the set of all real-valued functions on $\mathcal{X}$. For any $S \in \mathcal{X}$, let $\mathbb{I}_S \in \mathcal{L}(\mathcal{X})$ be the indicator of $S$, defined by $\mathbb{I}_S(x) := 1$ if $x \in S$ and $\mathbb{I}_s(x) := 0$ otherwise. If $S$ is a singleton $\{x\}$, we also write $\mathbb{I}_x$ instead of $\mathbb{I}_{\{x\}}$. We use $I$ to denote the indentity map that maps any $f \in \mathcal{L}(\mathcal{X})$ to itself. $\mathbb{N}$ is the set of natural numbers without zero and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

For any $f \in \mathcal{L}(\mathcal{X})$, we let $\|f\| := \|f\|_\infty := \max\{|f(x)| : x \in \mathcal{X}\}$ be the maximum norm. For any operator $A$ from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ that is non-negatively homogeneous, meaning that

$$A(\lambda f) = \lambda A(f) \text{ for all } f \in \mathcal{L}(\mathcal{X}) \text{ and all } \lambda \geq 0,$$

we consider the induced operator norm

$$\|A\| := \sup\{\|Af\| : f \in \mathcal{L}(\mathcal{X}), \|f\| = 1\}. \tag{1}$$

Not only do these norms satisfy the usual defining properties of a norm, they also satisfy the following additional properties; see Appendix A.1 for a proof. For all $f \in \mathcal{L}(\mathcal{X})$ and all maps $A,B$ from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ that are non-negatively homogeneous:

N1: $\|Af\| \leq \|A\| \|f\|$

N2: $\|AB\| \leq \|A\| \|B\|$

## 3 Lower transition operators

**Definition 1** (Lower transition operator)**.** *A lower transition operator $\underline{T}$ is a map from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ such that for all $f, g \in \mathcal{L}(\mathcal{X})$ and $\lambda \geq 0$:*

L1: $\underline{T}f \geq \min f$;

L2: $\underline{T}(f+g) \geq \underline{T}(f) + \underline{T}(g)$; [subadditivity]

L3: $\underline{T}(\lambda f) = \lambda \underline{T}(f)$. [non-negative homogeneity]

*The corresponding upper transition operator $\overline{T}$ is defined by*

$$\overline{T}f := -\underline{T}(-f) \text{ for all } f \in \mathcal{L}(\mathcal{X}). \tag{2}$$

For every lower transition operator $\underline{T}$ and any $x \in \mathcal{X}$, the operator $\underline{T}(\cdot)(x)$ is a coherent lower prevision [5, 4]: a subadditive, non-negatively homogeneous map from $\mathcal{L}(\mathcal{X})$ to $\mathbb{R}$ that dominates the min-operator. Therefore, lower transition operators are basically just finite vectors of coherent lower previsions. As a direct consequence, the following properties are implied by the corresponding versions for coherent lower previsions; see Reference [5, 2.6.1]. For all $f, g, f_n \in \mathcal{L}(\mathcal{X})$ and $\mu \in \mathbb{R}$:

L4: $\min f \leq \underline{T}f \leq \overline{T}f \leq \max f$;

L5: $\underline{T}(f+\mu) = \underline{T}(f) + \mu$;

L6: $f \geq g \Rightarrow \underline{T}(f) \geq \underline{T}(g)$ and $\overline{T}(f) \geq \overline{T}(g)$;

L7: $|\underline{T}f - \underline{T}g| \leq \overline{T}(|f-g|)$;

L8: $f_n \to f \Rightarrow \underline{T}f_n \to \underline{T}f$.

As a rather straightforward consequence of L4 and L7, we also find that

L7: $\|\underline{T}\| \leq 1$;

L8: $\|\underline{T}f - \underline{T}g\| \leq \|f-g\|$;

L9: $\|\underline{T}A - \underline{T}B\| \leq \|A-B\|$,

with $A$ and $B$ non-negatively homogeneous maps from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$; see Appendix A.2 for a proof. Finally, as this next result establishes, a sequence of lower transition operators convergences pointwise if and only if it converges with respect to the operator norm.

**Proposition 1.** *For any lower transition operator $\underline{T}$ and any sequence $\underline{T}_n$, $n \in \mathbb{N}$, of lower transition operators:*

$$\underline{T}_n \to \underline{T} \Leftrightarrow \forall f \in \mathcal{L}(\mathcal{X}) \colon \underline{T}_n f \to \underline{T}f.$$

## 4 Ergodicity for lower transition operators

*** Ergodicity introduceren en motiveren, vertellen dat de terminologie soms ook voor andere dingen gebruikt wordt, en dat deze notie van ergodiciteit soms ook wel Perron-Frobenius-like gedrag genoemd wordt. ***

**Definition 2** (Ergodic lower transition operator)**.** *A lower transition operator $\underline{T}$ is ergodic if, for all $f \in \mathcal{L}(\mathcal{X})$, $\lim_{n\to\infty} \underline{T}^n f$ exists and is a constant function.*

Similarly, the corresponding upper transition operator $\overline{T}$ is said to be ergodic if, for all $f \in \mathcal{L}(\mathcal{X})$, $\lim_{n\to\infty} \overline{T}^n f$ exists and is a constant function. It follows from Equation (2) that both notions are equivalent: $\underline{T}$ is ergodic if and only if $\overline{T}$ is.

Hermans and De Cooman characterised ergodicity in Reference [1], showing that a lower transition operator is ergodic if and only if it is regularly absorbing; see Proposition 2 further on. The following definition of a regularly absorbing lower transition operator is an equivalent but slightly simplified version of theirs; the equivalence is established in Proposition 3.

**Definition 3** (Regularly absorbing lower transition operator)**.** *A lower transition operator $\underline{T}$ is regularly absorbing if it satisfies the following two conditions:*

$$\mathcal{X}_{\mathrm{RA}} := \{x \in \mathcal{X} \colon (\exists n \in \mathbb{N})\ \min \overline{T}^n \mathbb{I}_x > 0\} \neq \emptyset$$

*and*

$$(\forall x \in \mathcal{X} \setminus \mathcal{X}_{\mathrm{RA}})(\exists n \in \mathbb{N})\ \underline{T}^n \mathbb{I}_{\mathcal{X}_{\mathrm{RA}}}(x) > 0.$$

*The first condition is called top class regularity and the second condition is called top class absorption.*

**Proposition 2** ([1, Proposition 3])**.** *A lower transition operator $\underline{T}$ is ergodic if and only if it is regularly absorbing.*

**Proposition 3.** *A lower transition operator $\underline{T}$ is regularly absorbing if and only if*

$$\mathcal{X}'_{\mathrm{RA}} := \{x \in \mathcal{X} \colon (\exists n \in \mathbb{N})(\forall k \geq n)\ \min \overline{T}^k \mathbb{I}_x > 0\} \neq \emptyset$$

*and*

$$(\forall x \in \mathcal{X} \setminus \mathcal{X}'_{\mathrm{RA}})(\exists n \in \mathbb{N})\ \overline{T}^n \mathbb{I}_{\mathcal{X} \setminus \mathcal{X}'_{\mathrm{RA}}}(x) < 1.$$

*The set $\mathcal{X}'_{\mathrm{RA}}$ is furthermore equal to the set $\mathcal{X}_{\mathrm{RA}}$ that was used in Definition 3.*

If a lower transition operator satisfies Definition 3 with $n = 1$, we call it 1-step absorbing.

**Definition 4** (1-step absorbing lower transition operator)**.** *A lower transition operator $\underline{T}$ is 1-step absorbing if it satisfies the following two conditions:*

$$\mathcal{X}_{1\mathrm{A}} := \{x \in \mathcal{X} \colon \min \overline{T} \mathbb{I}_x > 0\} \neq \emptyset$$

*and*

$$(\forall x \in \mathcal{X} \setminus \mathcal{X}_{1\mathrm{A}})\ \underline{T} \mathbb{I}_{\mathcal{X}_{1\mathrm{A}}}(x) > 0.$$

*** clean this up a bit *** Since $\mathcal{X}_{1\mathrm{A}}$ is clearly subset of $\mathcal{X}_{\mathrm{RA}}$, it follows from L6 that every 1-step absorbing lower transition operator is regularly absorbing and therefore also ergodic because of Proposition 2. The converse implication does not hold generally. However, as we will show further on, for the particular lower transition operators that we are interested in in this paper, Definitions 3 and 4 will be equivalent; see Proposition 11 further on.

## 5 Lower transition rate operators

**Definition 5** (Lower transition rate operator)**.** *A lower transition rate operator $\underline{Q}$ is a map from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$ such that for all $f,g \in \mathcal{L}(\mathcal{X})$, $\lambda \geq 0$, $\mu \in \mathbb{R}$ and $x,y \in \mathcal{X}$:*

R1: $\underline{Q}(\mu) = 0$;

R2: $\underline{Q}(f+g) \geq \underline{Q}(f) + \underline{Q}(g)$;

R3: $\underline{Q}(\lambda f) = \lambda \underline{Q}(f)$;

R4: $x \neq y \ \Rightarrow\ \underline{Q}(\mathbb{I}_y)(x) \geq 0$.

*The corresponding upper transition operator $\overline{Q}$ is defined by*

$$\overline{Q}f := -\underline{Q}(-f) \text{ for all } f \in \mathcal{L}(\mathcal{X}). \tag{3}$$

As a rather straightforward consequence of this definition, a lower transition rate operator also satisfies the following properties; see Appendix A.4 for a proof. For all $f \in \mathcal{L}(\mathcal{X})$, $\mu \in \mathbb{R}$ and $x \in \mathcal{X}$:

R5: $\underline{Q}(f) \leq \overline{Q}(f)$;

R6: $\underline{Q}(f+\mu) = \underline{Q}(f)$;

R7: $\overline{Q}(\mathbb{I}_x)(x) \leq 0$;

R8: $2\|f\|\underline{Q}(\mathbb{I}_x)(x) \leq (f(x) - \min f)\underline{Q}(\mathbb{I}_x)(x) \leq \underline{Q}(f)(x)$;

R9: $\left\|\underline{Q}\right\| \leq 2\max_{x\in\mathcal{X}} \left|\underline{Q}(\mathbb{I}_x)(x)\right|$.

Lower transition rate operators are very closely related to lower transition operators: they can be derived from each other. The following two results make this explicit.

**Proposition 4.** *Let $\underline{Q}$ be a lower transition rate operator. Then for all $0 \leq \Delta \leq {}^1/\|\underline{Q}\|$, $I + \Delta\underline{Q}$ is a lower transition operator.*

**Proposition 5.** *Let $\underline{T}$ be a lower transition operator. Then for all $\Delta > 0$, $\underline{Q} := {}^1/\Delta(\underline{T} - I)$ is a lower transition rate operator.*

Because of this connection, we can use results for lower transition operators to obtain similar results for lower transition rate operators. The following properties can for example be derived from L8, L8 and L9 respectively; see Appendix A.4 for a proof. For all $f_n, f, g \in \mathcal{L}(\mathcal{X})$:

R10: $f_n \to f \ \Rightarrow\ \underline{Q}f_n \to \underline{Q}f$;

R11: $\left\|\underline{Q}f - \underline{Q}g\right\| \leq 2\left\|\underline{Q}\right\|\|f-g\|$;

R12: $\left\|\underline{Q}A - \underline{Q}B\right\| \leq 2\left\|\underline{Q}\right\|\|A-B\|$,

where $A$ and $B$ can be any two non-negatively homogeneous maps from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$. Similarly, the following result can be derived from Proposition 1.

**Proposition 6.** *For any lower transition rate operator $\underline{Q}$ and any sequence $\underline{Q}_n$, $n \in \mathbb{N}$, of lower transition rate operators:*

$$\underline{Q}_n \to \underline{Q} \ \Leftrightarrow\ \forall f \in \mathcal{L}(\mathcal{X})\colon \underline{Q}_n f \to \underline{Q}f.$$

## 6 The differential equation of interest

Let $\underline{Q}$ be an arbitrary lower transition rate operator. For any $t \geq 0$, we then let $\underline{T}_t$ be a map from $\mathcal{L}(\mathcal{X})$ to $\mathcal{L}(\mathcal{X})$, defined for all $f \in \mathcal{L}(\mathcal{X})$ by the differential equation

$$\frac{d}{dt}\underline{T}_t f = \underline{Q}\underline{T}_t f \text{ for all } t \geq 0 \tag{4}$$

and the boundary condition $\underline{T}_0 f := f$. This definition is justified by a recent result of Škulj [3], who showed that the above differential equation has a unique solution for all $t \geq 0$.

This unique solution furthermore satisfies a number of convenient properties. First of all, as a direct consequence of its definition, we find that

$$\underline{T}_{t_1+t_2} = \underline{T}_{t_1}\underline{T}_{t_2} \text{ for all } t_1, t_2 \geq 0. \tag{5}$$

Secondly, as already suggested by our notation, $\underline{T}_t$ is a lower transition operator.

**Proposition 7.** *Let $\underline{Q}$ be a lower transition rate operator. Then for all $t \geq 0$, $\underline{T}_t$ is a lower transition operator.*

Thirdly, as the following result establishes, we do not need to consider the above differential equation for every $f \in \mathcal{L}(\mathcal{X})$ separately. Instead, we can apply it to the operator $\underline{T}_t$ itself.

**Proposition 8.** *Let $\underline{Q}$ be a lower transition rate operator. Then $\underline{T}_0 = I$ and*

$$\frac{d}{dt}\underline{T}_t = \underline{Q}\underline{T}_t \text{ for all } t \geq 0, \tag{6}$$

*where the derivative is taken with respect to the operator norm.*

**Proposition 9.** *Let $\underline{Q}$ be a lower transition rate operator. Then*

$$\underline{T}_t = \lim_{n\to\infty}(I + \frac{t}{n}\underline{Q})^n$$

*for all $t \geq 0$.*

In Appendices A.5 and A.6, we state and prove some additional technical properties of $\underline{T}_t$. Since we only need them in our proofs, we do not state them in the main text. Nevertheless, some of these properties—especially Proposition 22—may be of independent interest to the reader.

## 7 Ergodicity for lower transition rate operators

**Definition 6** (Ergodic lower transition rate operator)**.** *A lower transition rate operator $\underline{Q}$ is ergodic if, for all $f \in \mathcal{L}(\mathcal{X})$, $\lim_{t\to\infty}\underline{T}_t f$ exists and is a constant function.*

**Proposition 10.** *Let $\underline{Q}$ be a lower transition rate operator. Then for any $t > 0$, $\underline{Q}$ is ergodic if and only if $\underline{T}_t$ is ergodic.*

**Proposition 11.** *Let $\underline{Q}$ be a lower transition rate operator. Then for any $t \geq 0$, $\underline{T}_t$ is regularly absorbing if and only if it is 1-step absorbing.*

**Definition 7** (Lower reachability). *For any $x, y \in \mathcal{X}$, we say that $x$ is upper reachable from $y$ if there is some sequence $y = x_0, \dots, x_n = x$ such that, for all $k \in \{1, \dots, n\}$:*

$$x_k \neq x_{k-1} \text{ and } \overline{Q}(\mathbb{I}_{x_k})(x_{k-1}) > 0.$$

**Definition 8.** *(Upper reachability) For any $x \in \mathcal{X}$ and $A \subseteq \mathcal{X}$, we say that $A$ is lower reachable from $x$ if $x \in A_n$, where $A_k$, $k \in \mathbb{N}_0$, is the nested sequence defined by $A_0 := A$ and*

$$A_{k+1} := A_k \cup \{y \in \mathcal{X} \setminus A_k : \underline{Q}(\mathbb{I}_{A_k})(y) > 0\} \text{ for all } k \in \mathbb{N}_0,$$

*and where $n$ is the first index such that $A_n = A_{n+1}$.*

**Proposition 12.** *Let $\underline{Q}$ be a lower transition rate operator. Then for any $t > 0$ and any $x, y \in \mathcal{X}$: $\overline{T}_t \mathbb{I}_x(y) > 0$ if and only if $x$ is upper reachable from $y$.*

*Proof.* First assume that $\overline{T}_t \mathbb{I}_x(y) > 0$. It then follows from Proposition 9 and Equation (3) that there is some $n \in \mathbb{N}$ such that $n \geq t \left\|\underline{Q}\right\|$ and

$$\Big((I + \frac{t}{n}\overline{Q})^n \mathbb{I}_x\Big)(y) > 0. \tag{7}$$

Let $\Delta := t/n \leq 1/\|\underline{Q}\|$ and, for all $z \in \mathcal{X}$ and $w \in \mathcal{X}$, let $c(w,z) := \big((I + \Delta\overline{Q})\mathbb{I}_z\big)(w)$. Then for all $z \in \mathcal{X}$, we have that

$$(I + \Delta\overline{Q})\mathbb{I}_z = \sum_{w \in \mathcal{X}} \mathbb{I}_w \cdot \big((I + \Delta\overline{Q})\mathbb{I}_z\big)(w) = \sum_{w \in \mathcal{X}} c(w,z)\mathbb{I}_w,$$

where, because of Proposition 4 and L4, we know that $c(w,z) \geq 0$. Hence, for all $x_n \in \mathcal{X}$, it follows from Equation (3) and R2 that

$$\begin{aligned}(I + \Delta\overline{Q})^n \mathbb{I}_{x_n} = (I + \Delta\overline{Q})^{n-1}(I + \Delta\overline{Q})\mathbb{I}_{x_n} &= (I + \Delta\overline{Q})^{n-1} \sum_{x_{n-1} \in \mathcal{X}} c(x_{n-1}, x_n)\mathbb{I}_{x_{n-1}} \\ &\leq \sum_{x_{n-1} \in \mathcal{X}} c(x_{n-1}, x_n)(I + \Delta\overline{Q})^{n-1}\mathbb{I}_{x_{n-1}}\end{aligned}$$

and, by continuing in this way, that

$$(I + \Delta\overline{Q})^n \mathbb{I}_{x_n} \leq \sum_{x_{n-1} \in \mathcal{X}} c(x_{n-1}, x_n) \sum_{x_{n-2} \in \mathcal{X}} c(x_{n-2}, x_{n-1}) \cdots \sum_{x_1 \in \mathcal{X}} c(x_1, x_2)(I + \Delta\overline{Q})\mathbb{I}_{x_1}.$$

Therefore, for all $x_n \in \mathcal{X}$ and $x_0 \in \mathcal{X}$, we find that

$$\big((I + \Delta\overline{Q})^n \mathbb{I}_{x_n}\big)(x_0) \leq \sum_{x_{n-1} \in \mathcal{X}} c(x_{n-1}, x_n) \sum_{x_{n-2} \in \mathcal{X}} c(x_{n-2}, x_{n-1}) \cdots \sum_{x_1 \in \mathcal{X}} c(x_1, x_2) c(x_0, x_1).$$

Hence, if we let $x_0 := y$ and $x_n := x$, it follows from Equation (7) that

$$\sum_{x_{n-1} \in \mathcal{X}} c(x_{n-1}, x_n) \sum_{x_{n-2} \in \mathcal{X}} c(x_{n-2}, x_{n-1}) \cdots \sum_{x_1 \in \mathcal{X}} c(x_1, x_2) c(x_0, x_1) > 0.$$

This implies that there is some sequence $y = x_0, x_1, \dots, x_n = x$ such that

$$c(x_{n-1}, x_n) c(x_{n-2}, x_{n-1}) \cdots c(x_1, x_2) c(x_0, x_1) > 0.$$

Since each of the factors in this product is non-negative, it follows that $c(x_{k-1}, x_k) > 0$ for all $k \in \{1, \dots, n\}$. Therefore, for any $k \in \{1, \dots, n\}$ such that $x_k \neq x_{k-1}$, it follows that

$$\begin{aligned}\overline{Q}(\mathbb{I}_{x_k})(x_{k-1}) &= \frac{1}{\Delta}\Big(\mathbb{I}_{x_k}(x_{k-1}) + \Delta\overline{Q}(\mathbb{I}_{x_k})(x_{k-1})\Big)\\ &= \frac{1}{\Delta}\Big((\mathbb{I}_{x_k} + \Delta\overline{Q}(\mathbb{I}_{x_k}))(x_{k-1})\Big)\\ &= \frac{1}{\Delta}\Big(((I+\Delta\overline{Q})\mathbb{I}_{x_k})(x_{k-1})\Big) = \frac{1}{\Delta}c(x_{k-1}, x_k) > 0.\end{aligned}$$

If $x_k \neq x_{k-1}$ for all $k \in \{1, \dots, n\}$, this implies that $x$ is upper reachable from $y$. Otherwise, let $x'_0, \dots, x'_m$ be a new sequence, obtained by removing from $x_0, \dots, x_n$ those elements $x_k$ for which $x_k = x_{k-1}$; $n - m$ is the number of elements that is removed. Then $x'_0 = y$, $x'_m = x$ and, for all $k \in \{1, \dots, m\}$, we have that $x'_k \neq x'_{k-1}$ and $\overline{Q}(\mathbb{I}_{x'_k})(x'_{k-1}) > 0$. Therefore, $x$ is upper reachable from $y$.

Conversely, assume that there is some sequence $y = x_0, x_1, \dots, x_n = x$ such that, for all $k \in \{1, \dots, n\}$, $x_k \neq x_{k-1}$ and $\overline{Q}\mathbb{I}_{x_k}(x_{k-1}) > 0$. If $n = 0$, then $x = y$ and therefore, it follows from Corollary 23 that $\overline{T}_t\mathbb{I}_x(y) > 0$. Hence, for the remainder of this proof, we may assume that $n \geq 1$. Fix any $k \in \{1, \dots, n\}$. We then have that $\overline{T}_0\mathbb{I}_{x_k}(x_{k-1}) = \mathbb{I}_{x_k}(x_{k-1}) = 0$ and

$$\begin{aligned}\frac{d}{ds}\overline{T}_s\mathbb{I}_{x_k}(x_{k-1})\Big|_{s=0} &= -\frac{d}{ds}\underline{T}_s(-\mathbb{I}_{x_k})(x_{k-1})\Big|_{s=0}\\ &= -\underline{Q}(\underline{T}_0(-\mathbb{I}_{x_k}))(x_{k-1}) = -\underline{Q}(-\mathbb{I}_{x_k})(x_{k-1}) = \overline{Q}(\mathbb{I}_{x_k})(x_{k-1}) > 0,\end{aligned}$$

where the first equality follows from Equation (2), the second equality follows from Equation (4) and the last equality follows from Equation (3). Therefore, there is some $\varepsilon_k > 0$ such that $\overline{T}_{\varepsilon_k}\mathbb{I}_{x_k}(x_{k-1}) > 0$. Consequently, if we let $c_k := \overline{T}_{\varepsilon_k}\mathbb{I}_{x_k}(x_{k-1}) > 0$, then because it follows from Proposition 7 and L4 that $\overline{T}_{\varepsilon_k}\mathbb{I}_{x_k} \geq 0$, we have that $\overline{T}_{\varepsilon_k}\mathbb{I}_{x_k} \geq c_k\mathbb{I}_{x_{k-1}}$. Let $\varepsilon := \sum_{k=1}^n \varepsilon_k > 0$. Then

$$\overline{T}_\varepsilon\mathbb{I}_{x_n} = \overline{T}_{\varepsilon_1}\cdots\overline{T}_{\varepsilon_{n-1}}\overline{T}_{\varepsilon_n}\mathbb{I}_{x_n} \geq c_n\overline{T}_{\varepsilon_1}\cdots\overline{T}_{\varepsilon_{n-1}}\mathbb{I}_{x_{n-1}} \geq \dots \geq \prod_{k=1}^n c_k\mathbb{I}_{x_0},$$

where the equality follows from Equation (5) and the inequalities follow from Proposition 7, L3 and L6. Therefore, we find that

$$\overline{T}_\varepsilon\mathbb{I}_x(y) = \overline{T}_\varepsilon\mathbb{I}_{x_n}(x_0) \geq \prod_{k=1}^n c_k\mathbb{I}_{x_0}(x_0) = \prod_{k=1}^n c_k > 0,$$

which implies that $\overline{T}_t\mathbb{I}_x(y) > 0$ because of Corollary 23. □

**Proposition 13.** *Let $\underline{Q}$ be a lower transition rate operator. Then for any $t > 0$, any $x \in \mathcal{X}$ and any $A \subseteq \mathcal{X}$: $\underline{T}_t\mathbb{I}_A(x) > 0$ if and only if $A$ is lower reachable from $x$.*

*Proof.* *** still need to fix this *** If $\underline{T}_t(\mathbb{I}_A)(x) > 0$, then there is some $n$ such that

$$(I + \frac{t}{n}\underline{Q})^n(\mathbb{I}_A)(x) > 0,$$

which in turn implies that the property with the lower transition rates to sets holds for $A$ and $x$.

Next, I need to prove that, if the property with the lower transition rates holds for $A$ and $x$, then for $n$ large enough:

$$(I + \frac{t}{n}\underline{Q})^n(\mathbb{I}_A)(x) > 0,$$

□

## 8 Conclusions

*** ??? ***

As future work, we would like to develop *coefficients of ergodicity* that characterise whether $\underline{Q}$ is PF and that provide—tight—bounds on the rate of convergence. So far, we have found a coefficient of ergodicity whose positivity is sufficient—but not necessary—for $\underline{Q}$ to be PF and which, in that case, provides a conservative bound on the rate of convergence.

## Acknowledgements

Many thanks to Gert de Cooman, Matthias C. M. Troffaes, Stavros Lopatatzidis and Thomas Krak for stimulating discussions on the topic of continuous-time imprecise Markov chains.

## References

[1] Filip Hermans and Gert de Cooman. Characterisation of ergodic upper transition operators. *International Journal of Approximate Reasoning*, 53(4):573–583, June 2012.

[2] H L Royden and P M Fitzpatrick. *Real Analysis*. Prentice Hall, 4 edition, 2010.

[3] Damjan Škulj. Efficient computation of the bounds of continuous time imprecise Markov chains. *APPLIED MATHEMATICS AND COMPUTATION*, 250(C):165–180, January 2015.

[4] Matthias C. M. Troffaes and Gert de Cooman. *Lower previsions*. John Wiley & Sons, March 2014.

[5] Peter Walley. *Statistical reasoning with imprecise probabilities*. Chapman and Hall, London, 1991.

## A Proofs

### A.1 Proofs of results in Section 2

*** still need to include a proof for the norm properties ***

## A.2 Proofs of results in Section 3

*Proof of L7, L8 and L9.* L7 follows from Equation (1) because we know from L4 that $\|\underline{T}f\| \leq \|f\|$ for all $f \in \mathcal{L}(\mathcal{X})$. L8 follows from L7 and L4 (in that order). L9 follows from Equation (1) and L8. □

*Proof of Proposition 1.* The direct implication follows trivially from N1. For the converse implication, we provide a proof by contradiction. Assume that $\underline{T}_n f \to \underline{T} f$ for all $f \in \mathcal{L}(\mathcal{X})$. Assume *ex absurdo* that $\underline{T}_n \not\to \underline{T}$. Then since $\underline{T}_n \not\to \underline{T}$, it follows that $\limsup_{n\to\infty} \|\underline{T}_n - \underline{T}\| > 0$, which implies that there is some $\varepsilon > 0$ and an increasing sequence $n_k$, $k \in \mathbb{N}$, of natural numbers such that $\left\|\underline{T}_{n_k} - \underline{T}\right\| > \varepsilon$ for all $k \in \mathbb{N}$. Furthermore, for all $k \in \mathbb{N}$, it follows from $\left\|\underline{T}_{n_k} - \underline{T}\right\| > \varepsilon$ and Equation (1) that there is some $f_k \in \mathcal{L}(\mathcal{X})$ such that $\|f_k\| = 1$ and $\left\|\underline{T}_{n_k} f_k - \underline{T} f_k\right\| > \varepsilon$. Since the sequence $f_k$, $k \in \mathbb{N}$, is clearly bounded, it follows from the BolzanoWeierstrass theorem that it has a convergent subsequence, which implies that there is some $f \in \mathcal{L}(\mathcal{X})$ and an increasing sequence $k_i$, $i \in \mathbb{N}$, of natural numbers such that $\lim_{i\to\infty} \left\|f_{k_i} - f\right\| = 0$. Furthermore, since we have assumed that $\underline{T}_n f \to \underline{T} f$, it follows that

$$\lim_{i\to\infty} \left\|\underline{T}_{n_{k_i}} f - \underline{T} f\right\| = \lim_{n\to\infty} \|\underline{T}_n f - \underline{T} f\| = 0.$$

Hence, since it follows from L8 that

$$\begin{aligned} \left\|\underline{T}_{n_{k_i}} f_{k_i} - \underline{T} f_{k_i}\right\| &= \left\|\left(\underline{T}_{n_{k_i}} f_{k_i} - \underline{T}_{n_{k_i}} f\right) + \left(\underline{T} f - \underline{T} f_{k_i}\right) + \left(\underline{T}_{n_{k_i}} f - \underline{T} f\right)\right\| \\ &\leq \left\|\underline{T}_{n_{k_i}} f_{k_i} - \underline{T}_{n_{k_i}} f\right\| + \left\|\underline{T} f - \underline{T} f_{k_i}\right\| + \left\|\underline{T}_{n_{k_i}} f - \underline{T} f\right\| \\ &\leq \left\|f_{k_i} - f\right\| + \left\|f_{k_i} - f\right\| + \left\|\underline{T}_{n_{k_i}} f - \underline{T} f\right\|, \end{aligned}$$

we find that

$$\lim_{i\to\infty} \left\|\underline{T}_{n_{k_i}} f_{k_i} - \underline{T} f_{k_i}\right\| = 0.$$

Since $\left\|\underline{T}_{n_{k_i}} f_{k_i} - \underline{T} f_{k_i}\right\| > \varepsilon > 0$ for all $i \in \mathbb{N}$, this is a contradiction. □

## A.3 Proofs of results in Section 4

*Proof of Proposition 2.* Since we know from Proposition 3 that our definition of a regularly absorbing lower transition operator is equivalent to the definition in Reference [1], this result is identical to [1, Proposition 3]. □

*Proof of Proposition 3.* Consider any $x \in \mathcal{X}_{\mathrm{RA}}$, which implies that there is some $n \in \mathbb{N}$ such that $\min \overline{T}^n \mathbb{I}_x > 0$. It then follows from L4 that $\overline{T}^{n+1} \mathbb{I}_x = \overline{T}(\overline{T}^n \mathbb{I}_x) \geq \min \overline{T}^n \mathbb{I}_x > 0$, which implies that $\min \overline{T}^{n+1} \mathbb{I}_x > 0$. In the same way, we also find that $\min \overline{T}^{n+2} \mathbb{I}_x > 0$ and, by continuing in this way, that $\min \overline{T}^k \mathbb{I}_x > 0$ for all $k \geq n$. Since this holds for all $x \in \mathcal{X}_{\mathrm{RA}}$, it follows that $\mathcal{X}_{\mathrm{RA}} \subseteq \mathcal{X}'_{\mathrm{RA}}$. Since $\mathcal{X}'_{\mathrm{RA}}$ is clearly a subset of $\mathcal{X}_{\mathrm{RA}}$, this implies that $\mathcal{X}'_{\mathrm{RA}} = \mathcal{X}_{\mathrm{RA}}$. Hence, trivially, $\mathcal{X}_{\mathrm{RA}} \neq \emptyset$ if and only if $\mathcal{X}'_{\mathrm{RA}} \neq \emptyset$. The result now follows because it holds for all $x \in \mathcal{X} \setminus \mathcal{X}'_{\mathrm{RA}} = \mathcal{X} \setminus \mathcal{X}_{\mathrm{RA}}$ and all $n \in \mathbb{N}$ that

$$\overline{T}^n \mathbb{I}_{\mathcal{X}\setminus\mathcal{X}'_{\mathrm{RA}}}(x) = \overline{T}^n (1 - \mathbb{I}_{\mathcal{X}'_{\mathrm{RA}}})(x) = 1 - \underline{T}^n (\mathbb{I}_{\mathcal{X}'_{\mathrm{RA}}})(x) = 1 - \underline{T}^n (\mathbb{I}_{\mathcal{X}_{\mathrm{RA}}})(x),$$

where the second inequality follows from R1 and Equation (2). □

## A.4 Proofs of results in Section 5

*Proof of R5, R6, R7, R8 and R9.* R5 holds because it follows from Equation (3), R2 and R1 that

$$\underline{Q}(f)-\overline{Q}(f)=\underline{Q}(f)+\underline{Q}(-f)\leq\underline{Q}(f-f)=\underline{Q}(0)=0.$$

R6 holds because it follows from R1 and R2 that

$$\underline{Q}(f)=\underline{Q}(f)+\underline{Q}(\mu)\leq\underline{Q}(f+\mu)=\underline{Q}(f+\mu)+\underline{Q}(-\mu)\leq\underline{Q}(f).$$

R7 holds because it follows from Equation (3), R6, R2 and R4 (in that order) that

$$\begin{aligned}\overline{Q}(\mathbb{I}_x)(x)&=-\underline{Q}(-\mathbb{I}_x)(x)=-\underline{Q}(1-\mathbb{I}_x)(x)\\&=-\underline{Q}(\textstyle\sum_{y\in\mathcal{X}\setminus\{x\}}\mathbb{I}_y)(x)\leq-\textstyle\sum_{y\in\mathcal{X}\setminus\{x\}}\underline{Q}(\mathbb{I}_y)(x)\leq 0.\end{aligned}$$

R8 holds because it follows from R6, R2, R3, R4, R7 and R5 (in that order) that

$$\begin{aligned}\underline{Q}(f)(x)=\underline{Q}(f-\min f)(x)&\geq\textstyle\sum_{y\in\mathcal{X}}(f(y)-\min f)\underline{Q}(\mathbb{I}_y)(x)\\&\geq(f(x)-\min f)\underline{Q}(\mathbb{I}_x)(x)\\&\geq(\max f-\min f)\underline{Q}(\mathbb{I}_x)(x)\geq 2\left\|f\right\|\underline{Q}(\mathbb{I}_x)(x).\end{aligned}$$

We end by proving R9. Consider any $g\in\mathcal{L}(\mathcal{X})$ such that $\|g\|=1$. It then follows from R8 and R7 that

$$\underline{Q}(g)\geq 2\left\|g\right\|\min_{x\in\mathcal{X}}\underline{Q}(\mathbb{I}_x)(x)\geq-2\max_{x\in\mathcal{X}}\left|\underline{Q}(\mathbb{I}_x)(x)\right|.$$

Similarly, since $\|-g\|=\|g\|=1$, we also find that $\underline{Q}(-g)\geq-2\max_{x\in\mathcal{X}}\left|\underline{Q}(\mathbb{I}_x)(x)\right|$. By combining these two inequalities with Equation (3) and R5, it follows that

$$-2\max_{x\in\mathcal{X}}\left|\underline{Q}(\mathbb{I}_x)(x)\right|\leq\underline{Q}g\leq\overline{Q}g=-\underline{Q}(-g)\leq 2\max_{x\in\mathcal{X}}\left|\underline{Q}(\mathbb{I}_x)(x)\right|,$$

which implies that $\left|\underline{Q}(g)\right|\leq 2\max_{x\in\mathcal{X}}\left|\underline{Q}(\mathbb{I}_x)(x)\right|$. Since this is true for all $g\in\mathcal{L}(\mathcal{X})$ such that $\|g\|=1$, R9 now follows from Equation (1). □

*Proof of Proposition 4.* L2 and L3 follow trivially from R2 and R3. We only prove L1. Consider any $f\in\mathcal{L}(\mathcal{X})$. Then

$$\begin{aligned}(I+\Delta\underline{Q})f&=f+\Delta\underline{Q}f\\&=f+\Delta\sum_{x\in\mathcal{X}}\mathbb{I}_x\underline{Q}(f)(x)\\&\geq f+\Delta\sum_{x\in\mathcal{X}}(f(x)-\min f)\mathbb{I}_x\underline{Q}(\mathbb{I}_x)(x)\\&\geq f-\Delta\sum_{x\in\mathcal{X}}(f(x)-\min f)\mathbb{I}_x\left\|\underline{Q}\right\|\\&=f-\Delta\left\|\underline{Q}\right\|(f-\min f)=(f-\min f)(1-\Delta\left\|\underline{Q}\right\|)+\min f\geq\min f,\end{aligned}$$

where the first inequality follows from R8 and the second inequality follows from Equation (1) and the fact that $\|\mathbb{I}_x\|=1$. □

*Proof of Proposition 5.* Simply check each of the defining properties: R1 follows from L4, R2 follows from L2, R3 follows from L3 and R4 follows from L1. □

*Proof of R10, R11 and R12.* R10, R11 and R12 are trivial if $\underline{Q} = 0$. Therefore, we may assume that $\underline{Q} \neq 0$, which implies that $\|\underline{Q}\| > 0$. Now let $\underline{T} := I + {}^1\!/\!\|\underline{Q}\| \underline{Q}$. It then follows from Proposition 4 that $\underline{T}$ is a lower transition operator. We first prove R10. If $f_n \to f$, then $\underline{T} f_n \to \underline{T} f$ because of L8. Since $\underline{Q} = \|\underline{Q}\| (\underline{T} - I)$, this implies that $\underline{Q} f_n \to \underline{Q}$. R11 holds because

$$\begin{aligned}\|\underline{Q}f - \underline{Q}g\| &= \big\| \|\underline{Q}\| (\underline{T}f - f) - \|\underline{Q}\| (\underline{T}g - g) \big\| \\ &\leq \|\underline{Q}\| \|\underline{T}f - \underline{T}g\| + \|\underline{Q}\| \|f - g\| \leq 2 \|\underline{Q}\| \|f - g\|,\end{aligned}$$

where the last inequality follows from L8. Similarly, R12 holds because

$$\begin{aligned}\|\underline{Q}A - \underline{Q}B\| &= \big\| \|\underline{Q}\| (\underline{T}A - A) - \|\underline{Q}\| (\underline{T}B - B) \big\| \\ &\leq \|\underline{Q}\| \|\underline{T}A - \underline{T}B\| + \|\underline{Q}\| \|A - B\| \leq 2 \|\underline{Q}\| \|A - B\|,\end{aligned}$$

where the last inequality follows from L9. □

*Proof of Proposition 6.* The direct implication follows trivially from N1. We only prove the converse implication. Assume that $\underline{Q}_n f \to \underline{Q} f$ for all $f \in \mathcal{L}(\mathcal{X})$. For all $x \in \mathcal{X}$, this implies that $\underline{Q}_n(\mathbb{I}_x)(x) \to \underline{Q}(\mathbb{I}_x)(x)$, which in turn implies that there is some $c_x > 0$ such that $|\underline{Q}(\mathbb{I}_x)(x)| < c$ and $|\underline{Q}_n(\mathbb{I}_x)(x)| < c$ for all $n \in \mathbb{N}$. Let $c := \max_{x \in \mathcal{X}} c_x$. It then follows from R9 that $||\underline{Q}|| \leq 2c$ and $||\underline{Q}_n|| \leq 2c$ for all $n \in \mathbb{N}$. Choose any $0 < \Delta \leq {}^1\!/\!2c$. It then follows from Proposition 4 that $\underline{T} := I + \Delta \underline{Q}$ and $\underline{T}_n := I + \Delta \underline{Q}_n$, $n \in \mathbb{N}$, are lower transition operators. Furthermore, since $\underline{Q}_n f \to \underline{Q} f$ for all $f \in \mathcal{L}(\mathcal{X})$, it follows that $\underline{T}_n f \to \underline{T} f$ for all $f \in \mathcal{L}(\mathcal{X})$. By applying Proposition 1, we now find that $\underline{T}_n \to \underline{T}$, which implies that $\underline{Q}_n \to \underline{Q}$ because

$$\left\|\underline{Q}_n - \underline{Q}\right\| = \frac{1}{\Delta} \left\|\Delta\underline{Q}_n - \Delta\underline{Q}\right\| = \frac{1}{\Delta} \left\|(I + \Delta\underline{Q}_n) - (I + \Delta\underline{Q})\right\| = \frac{1}{\Delta} \|\underline{T}_n - \underline{T}\|.$$

□

## A.5 Proofs of results in Section 6

**Lemma 14.** *Let $\underline{Q}$ be a lower transition rate operator. Then for all $f \in \mathcal{L}(\mathcal{X})$, $\underline{T}_s f$ is continously differentiable on* $[0, \infty)$.

*Proof.* It follows from Equation (4) that $\underline{T}_s f$ is continuous on $[0, \infty)$. Therefore, since $\underline{Q}$ is a continuous operator [R10], $\underline{Q}\underline{T}_s f$ is also continuous on $[0, \infty)$. Because of Equation (4), this implies that $\underline{T}_s f$ is continuously differentiable on $[0, \infty)$. □

**Lemma 15.** *Let $\underline{Q}$ be a lower transition rate operator and let $\Gamma(s)$ be a continously differentiable map from $[0, t]$ to $\mathcal{L}(\mathcal{X})$ for which $\frac{d}{ds}\Gamma(s) \geq \underline{Q}\Gamma(s)$ for all $s \in [0, t]$. Then* $\min \Gamma(t) \geq \min \Gamma(0)$.

*Proof.* Since $\Gamma(s)$ is continously differentiable on $[0, t]$, it follows that for every $x \in \mathcal{X}$, $\Gamma(s)(x)$ is also continuously differentiable on $[0, t]$, which implies that it is absolutely continuous on $[0, t]$. Hence, since a minimum of a finite number of absolutely continuous functions is again absolutely continuous, we find that $\min \Gamma(s)$ is absolutely continuous on $[0, t]$, which implies—see Reference [2, Theorem 10, Section 6.5]—that $\min \Gamma(s)$ has a derivative $\frac{d}{ds} \min \Gamma(s)$ almost everwhere on $(0, t)$, that this derivative is Lebesgue integrable over $[0, t]$, and that

$$\min \Gamma(t) = \min \Gamma(0) + \int_0^t \Big(\frac{d}{ds} \min \Gamma(s)\Big) ds. \tag{8}$$

Consider now any $t^* \in (0,t)$ for which $\min\Gamma(s)$ has a derivative and consider any $x \in \mathcal{X}$ for which $\Gamma(t^*)(x) = \min\Gamma(t^*)$ [clearly, there is at least one such $x$]. Since $\Gamma(s)(x)$ is differentiable, $\frac{d}{ds}\Gamma(s)(x)$ exists in $t^*$. Assume *ex absurdo* that $\frac{d}{ds}\Gamma(s)(x)\big|_{s=t^*}$ is not equal to $\frac{d}{ds}\min\Gamma(s)\big|_{s=t^*}$ or, equivalently, that $\frac{d}{ds}(\Gamma(s)(x) - \min\Gamma(s))\big|_{s=t^*} \neq 0$. Then, because $\Gamma(s)(x) - \min\Gamma(s)$ is continous [since $\Gamma(s)(x)$ and $\min\Gamma(s)$ are (absolutely) continuous] and because $t^* \in (0,t)$ and $\Gamma(t^*)(x) - \min\Gamma(t^*) = 0$, it follows that there is some $t' \in (0,t)$ such that $\Gamma(t')(x) - \min\Gamma(t') < 0$ or, equivalently, such that $\Gamma(t')(x) < \min\Gamma(t')$. Since this is clearly a contradiction, it follows that

$$\frac{d}{ds}\Gamma(s)(x)\Big|_{s=t^*} = \frac{d}{ds}\min\Gamma(s)\Big|_{s=t^*}. \tag{9}$$

We also have that

$$\frac{d}{ds}\Gamma(s)(x)\Big|_{s=t^*} \geq \underline{Q}\big(\Gamma(t^*)\big)(x) \geq \big(\Gamma(t^*)(x) - \min\Gamma(t^*)\big)\underline{Q}(\mathbb{I}_x)(x) = 0,$$

where the second inequality follows from R8 and the last equality follows because $\Gamma(t^*)(x) = \min\Gamma(t^*)$. By combining this result with Equation (9), we find that, for all $t^* \in (0,t)$ for which $\min\Gamma(s)$ has a derivative, $\frac{d}{ds}\min\Gamma(s)\big|_{s=t^*} \geq 0$. It therefore follows from Equation (8) that $\min\Gamma(t) \geq \min\Gamma(0)$. □

*Proof of Proposition 7.* We first prove L1. Consider any $f \in \mathcal{L}(\mathcal{X})$. It follows from Lemma 14 that $\underline{T}_s f$ is continuously differentiable on $[0,t]$. Therefore, because of Equation (4), we infer from Lemma 15 that $\min\underline{T}_t f \geq \min\underline{T}_0 f$. Since $\underline{T}_0 f = f$, this implies that $\min\underline{T}_t f \geq \min f$, which in turn implies that $\underline{T}_t f \geq \min f$.

Let us now prove L2. Consider any $f, g \in \mathcal{L}(\mathcal{X})$. It follows from Lemma 14 that $\underline{T}_s f$, $\underline{T}_s g$ and $\underline{T}_s(f+g)$ are continuously differentiable on $[0,t]$, which implies that $\Gamma(s) := \underline{T}_s(f+g) - \underline{T}_s f - \underline{T}_s g$ is continuously differentiable on $[0,t]$. Furthermore, for all $s \in [0,t]$, it follows from Equation (4) and R2 that

$$\begin{aligned}
\frac{d}{ds}\Gamma(s) &= \frac{d}{ds}\underline{T}_s(f+g) - \frac{d}{ds}\underline{T}_s f - \frac{d}{ds}\underline{T}_s g \\
&= \underline{Q}\underline{T}_s(f+g) - \underline{Q}\underline{T}_s f - \underline{Q}\underline{T}_s g \\
&= \underline{Q}\big(\Gamma(s) + \underline{T}_s f + \underline{T}_s g\big) - \underline{Q}\underline{T}_s f - \underline{Q}\underline{T}_s g \geq \underline{Q}\Gamma(s).
\end{aligned}$$

Therefore, we infer from Lemma 15 that $\min\Gamma(t) \geq \min\Gamma(0)$. Since $\Gamma(0) = \underline{T}_0(f+g) - \underline{T}_0 f - \underline{T}_0 g = 0$, this implies that $\min\Gamma(t) \geq 0$, which in turn implies that $\Gamma(t) \geq 0$ or, equivalently, that $\underline{T}_t(f+g) \geq \underline{T}_t f + \underline{T}_t g$.

We end by proving L3. Consider any $f \in \mathcal{L}(\mathcal{X})$ and $\lambda \geq 0$. It then follows from Equation (4) and R3 that

$$\frac{d}{ds}(\lambda\underline{T}_s f) = \lambda\frac{d}{ds}\underline{T}_s f = \lambda\underline{Q}\underline{T}_s f = \underline{Q}(\lambda\underline{T}_s f) \text{ for all } s \geq 0.$$

Since we also have that $\lambda\underline{T}_0 f = \lambda f = \underline{T}_0(\lambda f)$, it follows that $\lambda\underline{T}_s f$ satisfies the same differential equation and boundary condition as $\underline{T}_s(\lambda f)$. Since we know that this differential equation and boundary condition lead to a unique solution on $[0,\infty)$, it follows that $\underline{T}_t(\lambda f) = \lambda\underline{T}_t(f)$. □

**Lemma 16.** *Let $\underline{Q}$ be a lower transition rate operator. Then*

$$\lim_{\Delta\to 0^+}\underline{T}_\Delta = I \text{ and } \lim_{\Delta\to 0^+} {}^1\!/\!{}_\Delta(\underline{T}_\Delta - I) = \underline{Q}.$$

*Proof.* For any $f \in \mathcal{L}(\mathcal{X})$, it follows from Equation (4) that $\underline{T}_t f$ is continuous on $[0,\infty)$, which implies that $\lim_{\Delta\to 0^+} \underline{T}_\Delta f = \underline{T}_0 f = f$. By applying Proposition 1, we find that $\lim_{\Delta\to 0^+} \underline{T}_\Delta = I$, thereby proving the first part of this Lemma. We end by proving the second part. For any $f \in \mathcal{L}(\mathcal{X})$, it follows from Equation (4) that

$$\lim_{\Delta\to 0^+} {}^1/_\Delta(\underline{T}_\Delta - I)(f) = \lim_{\Delta\to 0^+} {}^1/_\Delta(\underline{T}_\Delta f - f) = \lim_{\Delta\to 0^+} {}^1/_\Delta(\underline{T}_\Delta f - \underline{T}_0 f) = \underline{Q}\underline{T}_0 f = \underline{Q}f.$$

Therefore, and since, for all $\Delta > 0$, ${}^1/_\Delta(\underline{T}_\Delta - I)$ is a lower transition rate operator because of Proposition 5, it follows from Proposition 6 that $\lim_{\Delta\to 0^+} {}^1/_\Delta(\underline{T}_\Delta - I) = \underline{Q}$. □

*Proof of Proposition 8.* Since $\underline{T}_0 f := f$ for all $f \in \mathcal{L}(\mathcal{X})$, it follows trivially that $\underline{T}_0 = I$. Consider now any $t \geq 0$. In order to prove that $\frac{d}{dt}\underline{T}_t = \underline{Q}\underline{T}_t$, it suffices to show that for all $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\left\| \frac{\underline{T}_s - \underline{T}_t}{s-t} - \underline{Q}\underline{T}_t \right\| < \varepsilon \text{ for all } s \geq 0 \text{ such that } 0 < |t-s| < \delta. \tag{10}$$

So consider any $\varepsilon > 0$. If $\underline{Q} = 0$, Equation (10) is trivially true because, since $I$ clearly satisfies Equation (4), it follows from the unicity of the solution of Equation (4) that $\underline{T}_t = \underline{T}_s = I$. Therefore, in the remainder of this proof, we may assume that $\underline{Q} \neq 0$, which implies that $\|\underline{Q}\| \neq 0$. It then follows from Lemma 16 that there are $\delta_1 > 0$ and $\delta_2 > 0$ such that $\|\underline{T}_q - I\| < \varepsilon/4\|\underline{Q}\|$ for all $0 < q < \delta_1$ and $\|{}^1/_\Delta(\underline{T}_\Delta - I) - \underline{Q}\| < \varepsilon/2$ for all $0 < \Delta < \delta_2$. Now define $\delta := \min\{\delta_1, \delta_2\}$ and consider any $s \geq 0$ such that $0 < |t-s| < \delta$. Let $u := \min\{s,t\}$, $\Delta := |t-s|$ and $q := t-u$, which implies that $0 \leq q \leq \Delta < \delta \leq \delta_1$ and $0 < \Delta < \delta \leq \delta_2$. If $q = 0$, then $\underline{T}_q = \underline{T}_0 = I$ and therefore $\|\underline{Q}\underline{T}_q - \underline{Q}\| = \|\underline{Q} - \underline{Q}\| = 0$. If $q > 0$, it follows from R12 and Proposition 7 that $\|\underline{Q}\underline{T}_q - \underline{Q}\| \leq 2\|\underline{Q}\|\|\underline{T}_q - I\| < 2\|\underline{Q}\|\varepsilon/4\|\underline{Q}\| = \varepsilon/2$. Hence, in all cases, we find that $\|\underline{Q}\underline{T}_q - \underline{Q}\| < \varepsilon/2$. The result now holds because

$$\begin{aligned}
\left\| \frac{\underline{T}_s - \underline{T}_t}{s-t} - \underline{Q}\underline{T}_t \right\| &= \left\| \frac{\underline{T}_{\Delta+u} - \underline{T}_u}{\Delta} - \underline{Q}\underline{T}_{q+u} \right\| = \left\| \frac{\underline{T}_\Delta \underline{T}_u - \underline{T}_u}{\Delta} - \underline{Q}\underline{T}_q \underline{T}_u \right\| \\
&\leq \left\| \frac{\underline{T}_\Delta - I}{\Delta} - \underline{Q}\underline{T}_q \right\| \|\underline{T}_u\| \leq \left\| \frac{\underline{T}_\Delta - I}{\Delta} - \underline{Q}\underline{T}_q \right\| \\
&\leq \left\| \frac{\underline{T}_\Delta - I}{\Delta} - \underline{Q} \right\| + \|\underline{Q}\underline{T}_q - \underline{Q}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned}$$

where the second equality follows from Equation (5), the first inequality follows from N2 and the second inequality follows from Proposition 7 and L7. □

*Proof of Proposition 9.* The result is trivial if $t = 0$. In the remainder of this proof, we assume that $t > 0$. The result for $\underline{Q} = 0$ is also trivial because, since $I$ then clearly satisfies Equation (4), it follows from the unicity of the solution of Equation (4) that $\underline{T}_t = I$. Therefore, in the remainder of this proof, we assume that $\underline{Q} \neq 0$, which implies that $\|\underline{Q}\| \neq 0$. We will now prove that for every $\varepsilon > 0$, there is some $n \in \mathbb{N}$ such that

$$\left\| \underline{T}_t - (I + \frac{t}{k}\underline{Q})^k \right\| < \varepsilon \text{ for all } k \geq n.$$

So consider any $\varepsilon > 0$. It then follows from Lemma 16 that there is some $\delta > 0$ such that $\|{}^1/_\Delta(\underline{T}_\Delta - I) - \underline{Q}\| < \varepsilon/t$ for all $0 < \Delta < \delta$. Now choose $n \in \mathbb{N}$ such that

$n > \min\{t/\delta, t\,\|\underline{Q}\|\}$ and consider any $k \geq n$. Let $\Delta := t/k \leq t/n$, which implies that $0 < \Delta < \delta$ and $\Delta < 1/\|\underline{Q}\|$. Then

$$\begin{aligned}
&\left\|(\underline{T}_\Delta)^k - (I+\Delta\underline{Q})^k\right\| \\
&\quad= \left\|(\underline{T}_\Delta)^k - (\underline{T}_\Delta)^{k-1}(I+\Delta\underline{Q}) + (\underline{T}_\Delta)^{k-1}(I+\Delta\underline{Q}) - (I+\Delta\underline{Q})^k\right\| \\
&\quad\leq \left\|(\underline{T}_\Delta)^k - (\underline{T}_\Delta)^{k-1}(I+\Delta\underline{Q})\right\| + \left\|(\underline{T}_\Delta)^{k-1}(I+\Delta\underline{Q}) - (I+\Delta\underline{Q})^k\right\| \\
&\quad\leq \left\|\underline{T}_\Delta - (I+\Delta\underline{Q})\right\| + \left\|(\underline{T}_\Delta)^{k-1} - (I+\Delta\underline{Q})^{k-1}\right\| \left\|I+\Delta\underline{Q}\right\| \\
&\quad\leq \left\|\underline{T}_\Delta - (I+\Delta\underline{Q})\right\| + \left\|(\underline{T}_\Delta)^{k-1} - (I+\Delta\underline{Q})^{k-1}\right\|,
\end{aligned}$$

where the second inequality follows from Proposition 7 and L9 [by applying them repeatedly] and N2, and the third inequality follows from L7 and Proposition 4. By continuing in this way, we find that

$$\left\|(\underline{T}_\Delta)^k - (I+\Delta\underline{Q})^k\right\| \leq k\left\|\underline{T}_\Delta - (I+\Delta\underline{Q})\right\|.$$

Therefore, because

$$\left\|\underline{T}_\Delta - (I+\Delta\underline{Q})\right\| = \Delta\left\|\frac{\underline{T}_\Delta - I}{\Delta} - \underline{Q}\right\| < \Delta\frac{\varepsilon}{t} = \frac{t}{k}\frac{\varepsilon}{t} = \frac{\varepsilon}{k}$$

and because it follows from Equation (5) that $\underline{T}_t = (\underline{T}_{t/k})^k = (\underline{T}_\Delta)^k$, we find that

$$\left\|\underline{T}_t - (I+\frac{t}{k}\underline{Q})^k\right\| = \left\|(\underline{T}_\Delta)^k - (I+\Delta\underline{Q})^k\right\| \leq k\left\|\underline{T}_\Delta - (I+\Delta\underline{Q})\right\| < k\frac{\varepsilon}{k} = \varepsilon.$$

□

## A.6 Proofs of results in Section 7

*Proof of Proposition 10.* First assume that $\underline{Q}$ is ergodic. Then by definition, for all $f \in \mathcal{L}(\mathcal{X})$, $\lim_{s\to\infty} \underline{T}_s f$ exists and is a constant function. Therefore, for all $f \in \mathcal{L}(\mathcal{X})$, it follows from Equation (5) that

$$\lim_{n\to\infty} \underline{T}_t^n f = \lim_{n\to\infty} \underline{T}_{nt} f = \lim_{s\to\infty} \underline{T}_s f$$

exists and is a constant function, which implies that $\underline{T}_t$ is ergodic.

Next, assume that $\underline{T}_t$ is ergodic. This means that, for all $f \in \mathcal{L}(\mathcal{X})$, there is some $c_f \in \mathbb{R}$ such that

$$(\forall \varepsilon > 0)(\exists n \in \mathbb{N})(\forall k \geq n)\left\|\underline{T}_t^k f - c_f\right\| < \varepsilon. \tag{11}$$

Consider now any $f \in \mathcal{L}(\mathcal{X})$ and any $\varepsilon > 0$. It then follows from Equation (11) that there is some $n_\varepsilon \in \mathbb{N}$ such that $\left\|\underline{T}_t^{n_\varepsilon} f - c_f\right\| < \varepsilon$. Now let $s_\varepsilon := n_\varepsilon t$. Then for all $s \geq s_\varepsilon$, we have that

$$\left\|\underline{T}_s f - c_f\right\| = \left\|\underline{T}_s(f - c_f)\right\| = \left\|\underline{T}_{s-tn_\varepsilon}\underline{T}_t^{n_\varepsilon}(f - c_f)\right\| \leq \left\|\underline{T}_{s-tn_\varepsilon}\right\| \left\|\underline{T}_t^{n_\varepsilon}(f - c_f)\right\| < \varepsilon,$$

where the first equality follows from R1, the second equality follows from Equation (5), the first inequality follows from N1 and the last inequality follows from L7 and the fact that $\left\|\underline{T}_t^{n_\varepsilon} f - c_f\right\| < \varepsilon$. Hence, we have found that for all $\varepsilon > 0$, there is some $s_\varepsilon > 0$ such that $\left\|\underline{T}_s f - c_f\right\| < \varepsilon$ for all $s \geq s_\varepsilon$. In other words: $\lim_{s\to\infty} \underline{T}_s f = c_f$. Since this holds for all $f \in \mathcal{L}(\mathcal{X})$, $\underline{Q}$ is ergodic. □

**Lemma 17.** *Let $\underline{Q}$ be a lower transition rate operator. Consider any $f \in \mathcal{L}(\mathcal{X})$ and $x \in \mathcal{X}$ such that $f(x) > \min f$. Then for all $t \geq 0$: $\underline{T}_t f(x) > \min f$.*

*Proof.* Since we know from Lemma 14 that $\underline{T}_t f$ is continuously differentiable on $[0,\infty)$, we know that $r_t := \underline{T}_t f - \min f$ and therefore also $r_t(x)$ is continuously differentiable on $[0,\infty)$. For all $t \geq 0$, it follows from Proposition 7 and L1 that $r_t \geq 0$ and therefore also that

$$\frac{d}{dt} r_t(x) = \frac{d}{dt}\underline{T}_t f(x) = \underline{Q}(\underline{T}_t f)(x) = \underline{Q}(r_t)(x) \geq \sum_{y\in\mathcal{X}} r_t(y)\underline{Q}(\mathbb{I}_y)(x) \geq r_t(x)\underline{Q}(\mathbb{I}_x)(x),$$

where the second equality follows from Equation (4), the third equality follows from R6, the first inequality follows from R2 and R3 and the last inequality follows from R4. Since we also know that $r_0(x) = \underline{T}_0 f(x) - \min f = f(x) - \min f > 0$, this implies that

$$\underline{T}_t f(x) - \min f = r_t(x) \geq r_0(x) e^{\underline{Q}_x(\mathbb{I}_x)t} > 0 \text{ for all } t \geq 0.$$

□

**Lemma 18.** *Let $\underline{Q}$ be a lower transition rate operator. Consider any $f \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$ and $s \geq 0$ such that $\underline{T}_s f(x) > \min f$. Then for all $t \geq s$: $\underline{T}_t f(x) > \min f$.*

*Proof.* Because of Equation (5), it suffices to prove that $\underline{T}_{t-s}\underline{T}_s f(x) > \min f$. We consider two cases: $\min \underline{T}_s f > \min f$ and $\min \underline{T}_s f = \min f$; $\min \underline{T}_s f < \min f$ is not possible because of Proposition 7 and L1. If $\min \underline{T}_s f > \min f$, it follows from Proposition 7 and L1 that $\underline{T}_{t-s}\underline{T}_s f(x) \geq \min \underline{T}_s f > \min f$. If $\min \underline{T}_s f = \min f$, then $\underline{T}_s f(x) > \min \underline{T}_s f$ and therefore, because of Lemma 17, $\underline{T}_{t-s}\underline{T}_s f(x) > \min \underline{T}_s f = \min f$. □

**Lemma 19.** *Let $\underline{Q}$ be a lower transition rate operator. Consider any $f \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$ and $t, s > 0$. Then*

$$\underline{T}_t f(x) > \min f \Leftrightarrow \underline{T}_s f(x) > \min f.$$

*Proof.* For any $\tau \geq 0$, let

$$\mathcal{X}_\tau := \{y \in \mathcal{X} : \underline{T}_\tau f(y) > \min f\}. \tag{12}$$

It then follows from Lemma 18 that $\mathcal{X}_\tau$ is an increasing function of $\tau$:

$$\tau \leq \tau' \Rightarrow \mathcal{X}_\tau \subseteq \mathcal{X}_{\tau'}. \tag{13}$$

Assume *ex absurdo* that

$$(\forall \tau' > 0)\ (\forall \mathcal{X}' \subseteq \mathcal{X})\ (\exists \tau \in (0,\tau'])\ \mathcal{X}_\tau \neq \mathcal{X}'. \tag{14}$$

Choose any $\tau_1 > 0$. Then clearly, $\mathcal{X}_{\tau_1} \subseteq \mathcal{X}$. Due to Equation (14), we know that there is some $0 < \tau_2 < \tau_1$ such that $\mathcal{X}_{\tau_2} \neq \mathcal{X}_{\tau_1}$, which, because of Equation (13), implies that $\mathcal{X}_{\tau_2} \subset \mathcal{X}_{\tau_1}$. Similarly, we infer that there is some $0 < \tau_3 < \tau_2$ such that $\mathcal{X}_{\tau_2} \subset \mathcal{X}_{\tau_1}$. By continuing in this way, we obtain an infinite sequence of time points $\tau_1 > \tau_2 > \tau_3 > \cdots > \tau_i > \cdots > 0$ such that $\mathcal{X} \supseteq \mathcal{X}_{\tau_1} \supset \mathcal{X}_{\tau_2} \supset \mathcal{X}_{\tau_3} \supset \cdots \supset \mathcal{X}_{\tau_i} \supset \cdots$. Since $\mathcal{X}$ is a finite set, this is a contradiction, leading us to conclude that Equation (14) is false. This implies that there is some $\tau^* > 0$ and $\mathcal{X}^* \subseteq \mathcal{X}$ such that

$$(\forall \tau \in (0,\tau^*])\ \mathcal{X}_\tau = \mathcal{X}^*. \tag{15}$$

Fix any $\tau > \tau^*$ and choose $n \in \mathbb{N}$ high enough such that $2\tau/n \leq \tau^*$. It then follows from Equation (15) that $\mathcal{X}_{\tau/n} = \mathcal{X}_{2\tau/n} = \mathcal{X}^*$. Furthermore, because of Proposition 7, L1 and Equation (12), we know that $\underline{T}_{\tau/n} f(y) = \underline{T}_{2\tau/n} f(y) = \min f$ for all $y \in \mathcal{X} \setminus \mathcal{X}^*$. Therefore, we infer from Equation (12) that there is some $\lambda > 0$ such that

$$\underline{T}_{2\tau/n} f - \min f \leq \lambda(\underline{T}_{\tau/n} f - \min f),$$

which, because of Equation (5), Proposition 7 and L5 implies that

$$\underline{T}^2_{\tau/n}(f - \min f) = \underline{T}_{2\tau/n}(f - \min f) = \underline{T}_{2\tau/n} f - \min f \leq \lambda(\underline{T}_{\tau/n} f - \min f).$$

Hence, it follows from Proposition 7, L5, Equation (5) and L6 that

$$\underline{T}_\tau f - \min f = \underline{T}_\tau(f - \min f) = \underline{T}^n_{\tau/n}(f - \min f) \leq \lambda^{n-1}(\underline{T}_{\tau/n} f - \min f). \tag{16}$$

Consider now any $y \in \mathcal{X} \setminus \mathcal{X}^*$. Since $\underline{T}_{\tau/n} f(y) = \min f$, it follows from Equation (16) that $\underline{T}_\tau f(y) \leq \min f$, which in turn implies that $y \notin \mathcal{X}_\tau$. Since this holds for all $y \in \mathcal{X} \setminus \mathcal{X}^*$, we find that $\mathcal{X}_\tau \subseteq \mathcal{X}^* = \mathcal{X}_{\tau/n}$. Furthermore, since $\tau/n \leq \tau$, it follows from Equation (13) that $\mathcal{X}_{\tau/n} \subseteq \mathcal{X}_\tau$. Hence, we find that $\mathcal{X}_\tau = \mathcal{X}^*$. Since this is true for all $\tau > \tau^*$, it follows from Equation (15) that

$$\mathcal{X}_\tau = \mathcal{X}^* \text{ for all } \tau > 0.$$

Therefore, due to Equation (12), we find that

$$\underline{T}_t f(x) > \min f \Leftrightarrow x \in \mathcal{X}_t \Leftrightarrow x \in \mathcal{X}_s \Leftrightarrow \underline{T}_s f(x) > \min f.$$

□

**Lemma 20.** *Let $\underline{Q}$ be a lower transition rate operator. Consider any $f \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$ and $s \geq 0$ such that $\overline{T}_s f(x) > \min f$. Then for all $t \geq s$: $\overline{T}_t f(x) > \min f$.*

*Proof.* Because of Equations (2) and (5), it suffices to prove that $\overline{T}_{t-s}\overline{T}_s f(x) > \min f$. We consider two cases: $\min \overline{T}_s f > \min f$ and $\min \overline{T}_s f = \min f$; $\min \overline{T}_s f < \min f$ is not possible because of Proposition 7 and L4. If $\min \overline{T}_s f > \min f$, it follows from Proposition 7 and L4 that $\overline{T}_{t-s}\overline{T}_s f(x) \geq \min \overline{T}_s f > \min f$. If $\min \overline{T}_s f = \min f$, then $\overline{T}_s f(x) > \min \overline{T}_s f$ and therefore, it follows from Proposition 7, L4 and Lemma 17 that $\overline{T}_{t-s}\overline{T}_s f(x) \geq \underline{T}_{t-s}\overline{T}_s f(x) > \min \overline{T}_s f = \min f$. □

**Lemma 21.** *Let $\underline{Q}$ be a lower transition rate operator. Consider any $f \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$ and $t, s > 0$. Then*

$$\overline{T}_t f(x) > \min f \Leftrightarrow \overline{T}_s f(x) > \min f.$$

*Proof.* For any $\tau \geq 0$, let

$$\mathcal{X}_\tau := \{y \in \mathcal{X} : \overline{T}_\tau f(y) > \min f\}. \tag{17}$$

It then follows from Lemma 20 that $\mathcal{X}_\tau$ is an increasing function of $\tau$:

$$\tau \leq \tau' \;\Rightarrow\; \mathcal{X}_\tau \subseteq \mathcal{X}_{\tau'}. \tag{18}$$

Using an argument that is identical to that in Lemma 19, we find that this implies that there is some $\tau^* > 0$ and $\mathcal{X}^* \subseteq \mathcal{X}$ such that

$$(\forall \tau \in (0, \tau^*])\ \mathcal{X}_\tau = \mathcal{X}^*. \tag{19}$$

Fix any $\tau > \tau^*$ and choose $n \in \mathbb{N}$ high enough such that $2\tau/n \leq \tau^*$. It then follows from Equation (19) that $\mathcal{X}_{\tau/n} = \mathcal{X}_{2\tau/n} = \mathcal{X}^*$. Furthermore, because of Proposition 7, L4 and Equation (17), we know that $\overline{T}_{\tau/n} f(y) = \overline{T}_{2\tau/n} f(y) = \min f$ for all $y \in \mathcal{X} \setminus \mathcal{X}^*$. Therefore, we infer from Equation (17) that there is some $\lambda > 0$ such that

$$\overline{T}_{2\tau/n} f - \min f \leq \lambda(\overline{T}_{\tau/n} f - \min f),$$

which, because of Equations (2) and (5), Proposition 7 and L5 implies that

$$\overline{T}^2_{\tau/n}(f - \min f) = \overline{T}_{2\tau/n}(f - \min f) = \overline{T}_{2\tau/n} f - \min f \leq \lambda(\overline{T}_{\tau/n} f - \min f).$$

Hence, it follows from Proposition 7, L5, L6 and Equations (2) and (5) that

$$\overline{T}_{\tau} f - \min f = \overline{T}_{\tau}(f - \min f) = \overline{T}^n_{\tau/n}(f - \min f) \leq \lambda^{n-1}(\overline{T}_{\tau/n} f - \min f). \tag{20}$$

Consider now any $y \in \mathcal{X} \setminus \mathcal{X}^*$. Since $\overline{T}_{\tau/n} f(y) = \min f$, it follows from Equation (20) that $\overline{T}_{\tau} f(y) \leq \min f$, which in turn implies that $y \notin \mathcal{X}_\tau$. Since this holds for all $y \in \mathcal{X} \setminus \mathcal{X}^*$, we find that $\mathcal{X}_\tau \subseteq \mathcal{X}^* = \mathcal{X}_{\tau/n}$. Furthermore, since $\tau/n \leq \tau$, it follows from Equation (18) that $\mathcal{X}_{\tau/n} \subseteq \mathcal{X}_\tau$. Hence, we find that $\mathcal{X}_\tau = \mathcal{X}^*$. Since this is true for all $\tau > \tau^*$, it follows from Equation (19) that

$$\mathcal{X}_\tau = \mathcal{X}^* \text{ for all } \tau > 0.$$

Therefore, due to Equation (17), we find that

$$\overline{T}_t f(x) > \min f \Leftrightarrow x \in \mathcal{X}_t \Leftrightarrow x \in \mathcal{X}_s \Leftrightarrow \overline{T}_s f(x) > \min f.$$

□

**Proposition 22.** *Let $\underline{Q}$ be a lower transition rate operator. Then for all $f \in \mathcal{L}(\mathcal{X})$, $x \in \mathcal{X}$ and $t, s > 0$:*

$$\begin{aligned}
f(x) > \min f \;&\Rightarrow\; \underline{T}_t f(x) > \min f \;\Leftrightarrow\; \underline{T}_s f(x) > \min f;\\
f(x) < \max f \;&\Rightarrow\; \underline{T}_t f(x) < \max f \;\Leftrightarrow\; \underline{T}_s f(x) < \max f;\\
f(x) > \min f \;&\Rightarrow\; \overline{T}_t f(x) > \min f \;\Leftrightarrow\; \overline{T}_s f(x) > \min f;\\
f(x) < \max f \;&\Rightarrow\; \overline{T}_t f(x) < \max f \;\Leftrightarrow\; \overline{T}_s f(x) < \max f;
\end{aligned}$$

*Proof.* The first implication $[f(x) > \min f \Rightarrow \underline{T}_t f(x) > \min f]$ follows from Lemma 17 and the first equivalence $[\underline{T}_t f(x) > \min f \Leftrightarrow \underline{T}_s f(x) > \min f]$ follows from Lemma 19. Since $\overline{T}_0 f = f$, the third implication $[f(x) > \min f \Rightarrow \overline{T}_t f(x)]$ follows from Lemma 20. The third equivalence $[\overline{T}_t f(x) > \min f \Leftrightarrow \overline{T}_s f(x) > \min f]$ follows from Lemma 21. The rest of the result now follows directly because we know from Equation (2) that $\overline{T}_t f(x) = -\underline{T}_t(-f)(x)$, $\overline{T}_s f(x) = -\underline{T}_s(-f)(x)$ and $\max f = -\min(-f)$. □

**Corollary 23.** *Let $\underline{Q}$ be a lower transition rate operator. Then for all $A \subseteq \mathcal{X}$, all $x \in \mathcal{X}$ and all $t, s > 0$:*

$$\begin{aligned}
x \in A \;&\Rightarrow\; \underline{T}_t \mathbb{I}_A(x) > 0 \;\Leftrightarrow\; \underline{T}_s \mathbb{I}_A(x) > 0;\\
x \notin A \;&\Rightarrow\; \underline{T}_t \mathbb{I}_A(x) < 1 \;\Leftrightarrow\; \underline{T}_s \mathbb{I}_A(x) < 1;\\
x \in A \;&\Rightarrow\; \overline{T}_t \mathbb{I}_A(x) > 0 \;\Leftrightarrow\; \overline{T}_s \mathbb{I}_A(x) > 0;\\
x \notin A \;&\Rightarrow\; \overline{T}_t \mathbb{I}_A(x) < 1 \;\Leftrightarrow\; \overline{T}_s \mathbb{I}_A(x) < 1.
\end{aligned}$$

*Proof.* If $A = \emptyset$ or $A = \mathcal{X}$, the result follows trivially from Proposition 7 and L4. In all other cases, the result follows directly from Proposition 22, with $f = \mathbb{I}_A$. □

*Proof of Proposition 11.* If $t = 0$, we know from Proposition 8 that $\underline{T}_t = I$, which implies that $\underline{T}_t^n = I = \underline{T}_t$ and $\overline{T}_t^n = I = \overline{T}_t$ for all $n \in \mathbb{N}$. In that case, Definitions 3 and 4 are trivially equal. If $t > 0$, then since we know from Equations (5) and (2) that $\underline{T}_t^n = \underline{T}_{nt}$ and $\overline{T}_t^n = \overline{T}_{nt}$ for all $n \in \mathbb{N}$, the equivalence of Definitions 3 and 4 follows directly from Corollary 23. □