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Lecture 6:
Reversibility, and countably infinite state spaces

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Reminder about class swap in a couple of weeks!

Plan for today

1. Reversibility
2. Communicating classes
3. Explosions
4. Some time to catch up (next week we'll start the continuous state space case)

Reversibility

I mentioned before that we would look again at reversibility and the detailed balance condition. The main nice thing that comes out of reversibility is that we can “reverse time". Before we see how, let us first look at the following:

Proposition (Time reversal)

Let $(X_t : t \in [0, T])$ be a finite state, irreducible CTMC with generator G^X on a compact time interval which is **stationary**, i.e. $X_t \sim \pi$ for $t \in [0, T]$. Then the **time reversed chain**

$$(Y_t : t \in [0, T]) \quad \text{with} \quad Y_t := X_{T-t}$$

is a stationary CTMC with generator

$$g^Y(x, y) = \frac{\pi(y)}{\pi(x)} g^X(y, x)$$

and stat. prob. π.

Note that: This means that the definition of stationary chains can be extended to negative times: $(X_t : t \in \mathbb{R})$, with the time reversed chain given by $Y_t := X_{-t}$.

Reversibility

Proof.

$$\mathbb{P}(Y_t = y \mid Y_s = x) = \mathbb{P}(X_{-t} = y \mid X_{-s} = x)$$

$$= \frac{\mathbb{P}(X_{-t} = y \cap X_{-s} = x)}{\mathbb{P}(X_{-s} = x)}$$

$$= \frac{\overbrace{\mathbb{P}(X_{-s} = x \mid X_{-t} = y)}^{g(x,y)} \cdot \overbrace{\mathbb{P}(X_{-t} = y)}^{\pi(y)}}{\underbrace{\mathbb{P}(X_{-s} = x)}_{\pi(x)}} \qquad \square$$

Reversibility

Proof.

- An analogous statement (with similar proof!) holds for stationary, finite state, irreducible DTMCs, with
$$p^Y(x,y) = \frac{\pi(y)}{\pi(x)} p^X(y,x).$$
- Stationary chains with reversible π are **time-reversible**, i.e.,
$$g^Y(x,y) = g^X(x,y), \qquad (\text{ or } p^Y(x,y) = p^X(x,y)).$$
- The time reversal of non-stationary MCs is in general **not a homogeneous MC**, and using Bayes' Theorem (for DTMCs) we get
$$p^Y(x,y;n) = \frac{\pi_{N-n-1}(y)}{\pi_{N-n}(x)} p^X(y,x).$$

Countably infinite state space

The last topic we will look at for CTMCs is what happens when we are in a **countably infinite state space**. This is the case, e.g., of a simple random walk in $\mathbb{Z}$.

Note that: The SRW is a DTMC!, and the results in this section are valid for both CTMCs and DTMCs.

The reason we care about this, is that for infinite state space, Markov chains can get 'lost at infinity' and have no stationary distribution.
Again, think of the SRW. It is irreducible, but it is not ergodic and does **not** have a stationary distribution!

So... What can we do? We need to do a bit more to be able to characterise these Markov chains.
Yesterday we defined holding times. Today we will work with a "similar" time:

Return times

We define the first **return time** to a state x by $\quad T_x := \inf\{t > J_1 : X_t = x\}$

For DTMCs return times are defined as $\quad T_x := \inf\{n \geq 1 : X_n = x\}$

Classification of states and communication classes

We will use the return times to say something about the states of our MCs:

Definition (classification of states)

A state $x \in S$ is called

- **transient**, if $\mathbb{P}[T_x = \infty | X_0 = x] > 0$
- **null recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] = \infty$
- **positive recurrent**, if $\mathbb{P}[T_x < \infty | X_0 = x] = 1$ and $\mathbb{E}[T_x | X_0 = x] < \infty$

and these properties partition S into **communicating classes**.

- For an irreducible MC all states are either transient, null or positive recurrent.
- A MC has a unique stationary distribution if and only if it is positive recurrent, and in this case

$$\pi(x) = \frac{1}{\mathbb{E}[T_x | X_0 = x]} \mathbb{E}\left[\int_0^{T_x} \mathbb{1}_x(X_s) ds \Big| X_0 = x \right].$$

Explosions

A CTMC with an infinite transient component in S can exhibit **explosion**.

Definition

For a CTMC with jumping times J_n, we define the **explosion time** by

$$J_\infty := \lim_{n \to \infty} J_n \in (0, \infty].$$

The chain is called **non-explosive** if $\mathbb{P}[J_\infty = \infty] = 1$, otherwise it is **explosive**.

Note that: if the chain is explosive, what this means is that the chain does *infinite jumps in a finite amount of time*.

Explosions

It is easy to see that if the exit rates are uniformly bounded, i.e.

$$\sup_{x \in S} |g(x,x)| < \infty,$$

then the chain is non-explosive, and this is always the case if S is finite.

However, an example of an explosive CTMC is a **pure birth chain**. Consider $X_0 = 1$ and the generator of this CTMC is

$$g(x,y) = \alpha_x \delta_{y,x+1} - \alpha_x \delta_{y,x}\ , \quad x, y \in S = \mathbb{N}_0.$$

If $\alpha_x \to \infty$ fast enough (e.g. $\alpha_x = x^2$) we get

$$\mathbb{E}[J_\infty] = \sum_{x=1}^{\infty} \mathbb{E}[W_x] = \sum_{x=1}^{\infty} \frac{1}{\alpha_x} < \infty$$

since holding times $W_x \sim \mathrm{Exp}(\alpha_x)$. This implies $\mathbb{P}[J_\infty = \infty] = 0 < 1$.