Lecture 2: Discrete time Markov chains

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Plan for today

1. Definition of stochastic processes and Markov chains
2. Some examples
3. Some properties of Markov chains
4. (if we have time) Stationary distributions

Discrete Time Markov Chains

Discrete-time stochastic processes, and the Markov property

We want to an efficient description for processes such as the simple random walk from yesterday. We will start with some definitions and then will look at some useful examples and properties.

A **discrete-time stochastic process** with **state space** S is a sequence

$$Y_0, Y_1, \ldots = (Y_n : n \in \mathbb{N}_0)$$

of random variables taking values in S.

The process is called **Markov**, if for all $A \subset S$, $n \in \mathbb{N}_0$ and $s_0, \ldots, s_n \in S$ we have

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s_n, \ldots, Y_0 = s_0) = \mathbb{P}(Y_{n+1} \in A | Y_n = s_n).$$

A Markov process (MP) is called **homogeneous** if for all $A \subset S$, $n \in \mathbb{N}_0$ and $s \in S$

$$\mathbb{P}(Y_{n+1} \in A | Y_n = s) = \mathbb{P}(Y_1 \in A | Y_0 = s).$$

If S is discrete, the MP is called a **Markov chain (MC)**.

Some notation

We will be working with the generic probability space Ω, which is the **path space**

$$\Omega = D(\mathbb{N}_0, S) := S^{\mathbb{N}_0} = S \times S \times \dots$$

Note that Ω is uncountable, even when S is finite.

For a given $\omega \in \Omega$ the function $n \mapsto Y_n(\omega)$ is called a **sample path**.

Up to finite time N and with finite S, $\Omega_N = S^{N+1}$ is finite.

Example 1

Consider the simple random walk from yesterday, with a random walker starting at $X_0 = 0$. We have

- $Y_0 = X_0 = 0$.
- The state space $S = \mathbb{Z}$.
- Up to time N, $\mathbb{P}$ is a distribution on the finite path space Ω_N with

$$\mathbb{P}(\omega) = \begin{cases} p^{\text{\# of right-steps}} q^{\text{\# of left-steps}} & \text{, path } \omega \text{ possible} \\ 0 & \text{, path } \omega \text{ not possible} \end{cases}$$

- There are only 2^N paths in Ω_N with non-zero probability.
- If $p = q = 1/2$ all paths have the same probability $(1/2)^N$.

$N = 3$ $p^2 q$

More examples

- A **generalised random walk** is a random walk with $Y_0 = 0$ and increments $X_{n+1} = Y_{n+1} - Y_n \in \mathbb{R}$. This is a Markov process with $S = \mathbb{R}$ and $\Omega_N = \mathbb{R}^N$. It has an uncountable number of possible paths.

- A sequence $Y_0, Y_1, \ldots \in S$ of iid rv's is also a Markov process with state space S.

- Let $S = \{1, \ldots, 52\}$ be a deck of cards, and $Y_1, \ldots, Y_{52}$ be the cards drawn at random without replacement. Is this a Markov process?

Homogeneous MCs and the Chapman-Kolmogorov equations

Let $(X_n : n \in \mathbb{N}_0)$ be a homogeneous DTMC with **discrete** state space S.

Then we can define the **transition function**

$$p_n(x, y) := \mathbb{P}[X_n = y | X_0 = x] = \mathbb{P}[X_{k+n} = y | X_k = x] \quad \text{for all } k \geq 0.$$

Proposition: Chapman-Kolmogorov equations

If $(X_n : n \in \mathbb{N}_0)$ is a homogeneous DTMC, $p_n(x, y)$ is well defined and fulfills the **Chapman Kolmogorov equations**:

$$p_{k+n}(x, y) = \sum_{z \in S} p_k(x, z)\, p_n(z, y) \quad \text{for all } k, n \geq 0,\ x, y \in S.$$

Proof.

We use the law of total probability, the Markov property and homogeneity

$$p_{n+k}(x,y) = \mathbb{P}(X_{n+k} = y \mid X_0 = x)$$

$$\overset{\text{LTP}}{=} \sum_{z \in S} \mathbb{P}(X_{n+k} = y \mid X_k = z, X_0 = x)\, \mathbb{P}(X_k = z \mid X_0 = x)$$

$$\overset{\text{Markov}}{=} \sum_{z \in S} \mathbb{P}(X_{n+k} = y \mid X_k = z)\, \mathbb{P}(X_k = z \mid X_0 = x)$$

$$\overset{\text{homog}}{=} \sum_{z \in S} \underbrace{\mathbb{P}(X_n = y \mid X_0 = z)}_{p_n(z,y)}\, \mathbb{P}(X_k = z \mid X_0 = x)$$

$$p_n(z,y)\; p_k(x,z)$$

Transition matrices

It is convenient to write all of this in matrix form. For this, we define the matrix $P_n = \big(p_n(x,y) : x, y \in S\big)$, and can rewrite the Chapman-Kolmogotov equations as:

$$P_{n+k} = P_n \, P_k$$

and, in particular,

$$P_{n+1} = P_n \, P_1 .$$

which gives us a recursion relation to the matrices P_n.

With $P_0 = \mathbb{I}$, the obvious solution to this recursion is $P_n = P^n$, where we define the **transition matrix**

$$P = P_1 = \big(p(x,y) : x, y \in S\big).$$

The transition matrix P and the initial condition $X_0 \in S$ completely determine a homogeneous DTMC, since for all $k \geq 1$ and all events $A_1, \ldots, A_k \subset S$

$$\mathbb{P}[X_1 \in A_1, \ldots, X_k \in A_k] = \sum_{s_1 \in A_1} \cdots \sum_{s_k \in A_k} p(X_0, s_1) p(s_1, s_2) \cdots p(s_{k-1}, s_k).$$

Properties of transition matrices and some notation

Note that there is no reason to have a fixed X_0 and instead we can work with an **initial distribution**

$$\pi_0(x) := \mathbb{P}[X_0 = x].$$

The distribution at time n is then

$$\pi_n(x) = \sum_{y \in S} \sum_{s_1 \in S} \cdots \sum_{s_{n-1} \in S} \pi_0(y) p(y, s_1) \cdots p(s_{n-1}, x)$$

In this case, we can also write

$$\langle \pi_n | = \langle \pi_0 | P^n,$$

where $\langle \cdot |$ denotes a row vector.

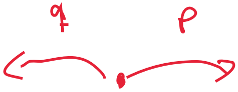
Finally, the transition matrix P is **stochastic**, i.e.

$$p(x, y) \in [0, 1] \quad \text{and} \quad \sum_{y \in S} p(x, y) = 1 \; ,$$

or equivalently, the column vector $|1\rangle = (1, \ldots, 1)^T$ is an **eigenvector** of P with **eigenvalue** 1: $\quad P|1\rangle = |1\rangle$

Quick example

Back to the simple random walk...



$X_0 = 0 \Rightarrow \pi_0 = \delta_0$

$$\mathbb{P}(X_1 = y \mid X_0 = 0) = p\,\delta_{1,x_1} + q\,\delta_{-1,x_1}$$

$$P \begin{pmatrix} \cdots & 0 & q & 0 & p & 0 & \cdots \end{pmatrix}$$

$$\pi_1 = \pi_0 P = \begin{pmatrix} 0 \\ \vdots \\ q \\ 0 \\ p \\ 0 \\ \vdots \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow -1 \\ \leftarrow 0 \\ \leftarrow 1 \\ \\ \\ \end{array}$$

$$\pi_2 = (0 \cdots \; q^2 \;\; 0 \;\; 2pq \;\; 0 \;\; p^2 \;\; 0 \cdots)$$

Example: Random walk with boundaries

To make our lives simpler, we can look at MCs with finite state space. A good example of this are random walks with boundaries.

Let $(X_n : n \in \mathbb{N}_0)$ be a simple random walk on $S = \{1, \dots, L\}$ with

$$p(x, y) = p\delta_{y,x+1} + q\delta_{y,x-1}.$$

In this case, we need to tell it what happens once we reach the boundary (1 or L). We can have the following boundary conditions:

- **periodic** if $\quad p(L, 1) = p\ , \quad p(1, L) = q,$
- **absorbing** if $\quad p(L, L) = 1\ , \quad p(1, 1) = 1,$
- **closed** if $\quad p(1, 1) = q\ , \quad p(L, L) = p,$
- **reflecting** if $\quad p(1, 2) = 1\ , \quad p(L, L-1) = 1.$

P periodic, $L = 4$

$$\begin{pmatrix} 0 & p & 0 & q \\ q & 0 & p & 0 \\ 0 & q & 0 & p \\ p & 0 & q & 0 \end{pmatrix}$$

$\pi_0 = (1, 0, 0, 0)$

$\Rightarrow \pi_1 = \pi_0 P$

$= (0, p, 0, q)$