Lecture 8: Generators as operators and diffusion processes

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Plan for today

1. Generators as operators
2. Scaling limits and diffusion processes

Generators as operators

Generators as operators

Recall from week 2 that, for a CTMC $(X_t \ : \ t \geq 0)$ with discrete state space S, we could write an ODE for the distribution at time t:

$$\frac{d}{dt}\langle\pi_t| = \langle\pi_t|G.$$

Furthermore, we also know that, given a function $f \ : \ S \to \mathbb{R}$, we can compute its expectation:

$$\mathbb{E}(f(X_t)) = \sum_{x \in S} \pi_t(x) f(x) = \langle\pi_t|f\rangle.$$

Therefore, we can use this to write an ODE to $\mathbb{E}(f(X_t))$:

$$\frac{d}{dt}\mathbb{E}\left[f(X_t)\right] = \frac{d}{dt}\langle\pi_t|f\rangle = \langle\pi_t|G|f\rangle = \mathbb{E}\left[(Gf)(X_t)\right] \ .$$

Generators as operators

We can do the same when $S = \mathbb{R}$, and this motivates the definition of the generator as an (differential) operator acting on functions $f : S \to \mathbb{R}$:

$$G|f\rangle(x) = (G\,f)(x) = \sum_{y \neq x} g(x,y)\big[f(y) - f(x)\big].$$

Note that: When we do this, we usually write $\mathcal{L}$ instead of G but I will try to be clear when doing that :)

Example: For **jump processes** with $S = \mathbb{R}$ and rate density $r(x,y)$, the generator is

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}} r(x,y)\big[f(y) - f(x)\big]\,dy.$$

Example: Brownian motion

Let us see what the generator is for the **Brownian motion**.

Recall that we mentioned yesterday that the transition density solves the **heat equation**:

$$p_t(x,y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2}{2\sigma^2 t}\right), \quad \text{solves} \quad \frac{\partial}{\partial t} p_t(x,y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p_t(x,y)$$

Using this, we obtain, for $f \in C^2(\mathbb{R})$,

$$\frac{d}{dt}\mathbb{E}_x\big(f(X_t)\big) = \int_{\mathbb{R}} \partial_t p_t(x,y) f(y) dy = \frac{\sigma^2}{2} \int_{\mathbb{R}} \partial_y^2 p_t(x,y) f(y) dy.$$

Now we integrate by parts:

$$\frac{\sigma^2}{2} \int_{\mathbb{R}} \partial_y^2 p_t(x,y) f(y) dy = \frac{\sigma^2}{2} \int_{\mathbb{R}} p_t(x,y) \partial_y^2 f(y) dy = \mathbb{E}_x\big((\mathcal{L}f)(X_t)\big).$$

This means that the **generator of BM** is

$$(\mathcal{L}f)(x) = \frac{\sigma^2}{2} \Delta f(x) \qquad \left(\text{or } \overbrace{\frac{\sigma^2}{2} f''(x)} \right).$$

Brownian motion as scaling limit

An interesting consequence of this is that we can see the Brownian motion as the scaling limit of a jump process:

Proposition:

Let $(X_t : t \geq 0)$ be a jump process on $\mathbb{R}$ with **translation invariant rates** $r(x, y) = q(y - x)$ which have

- **mean zero** $\int_{\mathbb{R}} q(z)\, z\, dz = 0$
- **finite second moment** $\sigma^2 := \int_{\mathbb{R}} q(z)\, z^2\, dz < \infty$.

Then, for all $T > 0$ the rescaled process $\big(\epsilon X_{t/\epsilon^2} : t \in [0, T]\big)$ converges in distribution to a BM with generator $\mathcal{L} = \frac{1}{2}\sigma^2\Delta$ for all $T > 0$ as $\epsilon \to 0$, i.e.

$$\big(\epsilon X_{t/\epsilon^2} : t \in [0, T]\big) \quad \longrightarrow \quad \big(B_t : t \in [0, T]\big) \quad \text{as } \epsilon \to 0.$$

Proof. Taylor expansion of the generator for test functions $f \in C^3(\mathbb{R})$, and tightness argument for continuity of paths (requires fixed interval $[0, T]$).

Diffusion processes

Diffusion processes

We can now define a general class of Markov processes.

Definition

A **diffusion process** with **drift** $a(x,t) \in \mathbb{R}$ and **diffusion** $\sigma(x,t) > 0$ is a real-valued process with continuous paths and generator

$$(\mathcal{L}f)(x) = a(x,t)\, f'(x) + \frac{1}{2}\sigma^2(x,t)\, f''(x).$$

Examples.

- The **Ornstein-Uhlenbeck process** is a diffusion process with generator

$$(\mathcal{L}f)(x) = -\alpha\, x\, f'(x) + \frac{1}{2}\sigma^2\, f''(x)\ , \quad \alpha, \sigma^2 > 0.$$

  It has a Gaussian stationary distribution $\mathcal{N}(0, \sigma^2/(2\alpha))$.
  If the initial distribution π_0 is Gaussian, this is a **Gaussian process**.

- **Brownian bridge** is a Gaussian diffusion with $X_0 = 0$ and generator

$$(\mathcal{L}f)(x) = -\frac{x}{1-t}\, f'(x) + \frac{1}{2}\, f''(x).$$

Time evolution of diffusion processes

Generators are defined on functions f of the state space. However, they are very useful, as they tell us a lot about the evolution of the underlying probability distributions.

Recall that the generator is given by

$$(\mathcal{L}f)(x) = a(x,t)\, f'(x) + \frac{1}{2}\sigma^2(x,t)\, f''(x).$$

Time evolution of the mean:

Use $\frac{d}{dt}\mathbb{E}[f(X_t)] = \mathbb{E}[(\mathcal{L}f)(x_t)]$ with $f(x) = x$ to obtain

$$\frac{d}{dt}\mathbb{E}[X_t] = \mathbb{E}[a(X_t, t)]$$

$f'(x) = 1$, $f''(x) = 0$

Time evolution of diffusion processes

Time evolution of the transition density:

With $X_0 = x$ we have for $p_t(x, y)$

$$\int_{\mathbb{R}} \frac{\partial}{\partial t} p_t(x, y) f(y) dy = \frac{d}{dt} \mathbb{E}[f(X_t)] = \int_{\mathbb{R}} p_t(x, y) \mathcal{L} f(y) dy \quad \text{for any } f.$$

As before, we can use integration by parts to get the **Fokker-Planck equation**:

$$\boxed{\frac{\partial}{\partial t} p_t(x, y) = -\frac{\partial}{\partial y} \left(a(y, t) p_t(x, y) \right) + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left(\sigma^2(y, t) p_t(x, y) \right).}$$

$$\int p_t(x,y) \left(a(y,t) f'(y) + \frac{1}{2} \sigma^2(y,t) f''(y) \right) dy$$

$$= \int \underbrace{p_t(x,y) a(y,t)}\, f'(y)\, dy + \frac{1}{2} \int p_t(x,y) \sigma^2(y,t) f''(y)\, dy$$

$$= - \int \partial_y \left(p_t(x,y) a(y,t) \right) f\, dy + \frac{1}{2} \int \partial_y^2 \left(p_t(x,y) \sigma^2(y,t) \right) f(y)\, dy$$

Time evolution of diffusion processes

Finally, we can also look at **stationary distributions** for time-independent $a(y) \in \mathbb{R}$ and $\sigma^2(y) > 0$.

A stationary distribution p^* satisfies $\frac{\partial p^*}{\partial t} = 0$ and so we have

$$\frac{d}{dy}\big(a(y)p^*(y)\big) = \frac{1}{2}\frac{d^2}{dy^2}\big(\sigma^2(y)p^*(y)\big).$$

With this, we can solve for a stationary density (modulo normalisation fixing $p^*(0)$)

$$p^*(x) = p^*(0)\exp\Big(\int_0^x \frac{2a(y) - (\sigma^2)'(y)}{\sigma^2(y)}dy\Big).$$

Note the need for computing a normalisation constant here - connection to MCMC