Lecture 7:
Processes with continuous state space

Susana Gomes

October 28^{th}, 2021

Plan for today

1. Summary of what we have done so far
2. Generalisation to continuous state space and a couple of examples
3. Stationary independent increments and Brownian motion

Processes with continuous state space

Markov processes with $S = \mathbb{R}$

Recall the definition of continuous time Markov process from week 3...

- A **continuous-time stochastic process** with state space S is a family $(X_t : t \geq 0)$ of random variables taking values in S.

- The process is called **Markov** if, for all $A \subset S$, $n \in \mathbb{N}$, $t_1 < \ldots < t_{n+1} \in [0, \infty)$ and $s_1, \ldots, s_n \in S$, we have

$$\mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n, \ldots, X_{t_1} = s_1) = \mathbb{P}(X_{t_{n+1}} \in A | X_{t_n} = s_n).$$

- A Markov process (MP) is called **homogeneous** if for all $A \subset S$, $t, u > 0$ and $s \in S$

$$\mathbb{P}(X_{t+u} \in A | X_u = s) = \mathbb{P}(X_t \in A | X_0 = s).$$

We spent the last 3 weeks talking about what happens when the state space S is finite. The next couple of weeks will be focused on when $S = \mathbb{R}$.

Kernels, densities, and the Chapman-Kolmogorov equations

Let $(X_t : t \geq 0)$ be a homogeneous MP as in previous slide, with state space $S = \mathbb{R}$.

For all $t \geq 0$ and (measurable) $A \subset \mathbb{R}$, the **transition kernel** for all $x \in \mathbb{R}$

$$P_t(x, A) := \mathbb{P}(X_t \in A | X_0 = x) = \mathbb{P}(X_{t+u} \in A | X_u = x) \quad \forall u \geq 0$$

is well defined.

A

Proposition (Chapman-Kolmogorov equations)

If $P_t(x, A)$ is absolutely continuous, we can define the **transition density** p_t

$$P_t(x, A) = \int_A p_t(x, y)\, dy$$

replace $\sum_{s \in A}$ by $\int_A dy$

and it fulfills the **Chapman Kolmogorov equations**

$$p_{t+u}(x, y) = \int_{\mathbb{R}} p_t(x, z)\, p_u(z, y)\, dz \quad \text{for all } t, u \geq 0,\ x, y \in \mathbb{R}\,.$$

A note on finite dimensional distributions

Similarly to what happened with CTMCs, we need to say something about time and finite dimensional distributions.

As before, the transition densities and the initial distribution $p_0(x)$ describe all **finite dimensional distributions (fdds)**.

This means that for all $n \in \mathbb{N}$, $0 < t_1 < \ldots < t_n$ and $x_1, \ldots x_n \in \mathbb{R}$, we can write.

$$\begin{aligned} &\mathbb{P}(X_{t_1} \leq x_1, \ldots, X_{t_n} \leq x_n) = \\ &\qquad = \int_{\mathbb{R}} p_0(z_0)\, dz_0 \int_{-\infty}^{x_1} p_{t_1}(z_0, z_1) dz_1 \cdots \int_{-\infty}^{x_n} p_{t_n - t_{n-1}}(z_{n-1}, z_n) dz_n \end{aligned}$$

However, there is no general solution formula for the CK equations and we have to consider several types of processes separately.

Example 1: Jump processes

We can think of two extremes for continuous state space MPs. One of them is Gaussian processes, which have **continuous movement**. The opposite extreme is when our MP has **discrete movements**.

Our first example considers the discrete movements case: jump processes.

Jump processes are similar to CTMCs, except that now the state space is continuous.

Jump processes

A **jump process** is a Markov process $(X_t : t \geq 0)$ with state space $S = \mathbb{R}$ characterised by

- a **jump rate density** $r(x, y) \geq 0$, and
- a uniformly bounded **total exit rate** $R(x) = \int_{\mathbb{R}} r(x, y)\, dy < \bar{R} < \infty$ for all $x \in \mathbb{R}$.

In this case, we can simplify the Chapman-Kolmogorov equations...

Jump processes - Kolmogorov-Feller equation

To try and obtain a better expression for the transition rates, we can try to solve the Chapman-Kolmogorov equations.

- Recall that for DTMCs we obtained a recurrence relation that told us what P was.
- Similarly, for CTMCs we used it to obtain the forward-backward equations, which gave us G.

To do this, we make an **ansatz** for the transition function as $\Delta t \to 0$:

$$p_{\Delta t}(z, y) = r(z, y)\Delta t + \big(1 - R(z)\Delta t\big)\delta(y - z)$$

and plug this into the Chapman Kolmogorov equations.

Jump processes - Kolmogorov-Feller equation

The Chapman-Kolmogorov equations say:

$$p_{t+u}(x,y) = \int_{\mathbb{R}} p_t(x,z)\, p_u(z,y)\, dz \quad \text{for all } t,u \geq 0,\ x,y \in \mathbb{R}\ .$$

Plugging the previous ansatz, we obtain

$$p_{t+\Delta t}(x,y) - p_t(x,y) = \int_{\mathbb{R}} p_t(x,z)\, p_{\Delta t}(z,y)\, dz - p_t(x,y) =$$

$$= \int_{\mathbb{R}} p_t(x,z) r(z,y) \Delta t\, dz + \int_{\mathbb{R}} \big(1 - R(z)\Delta t - 1\big) p_t(x,z) \delta(y-z) dz.$$

This allows us to get the **Kolmogorov-Feller equation** (x is a fixed initial condition)

$$\frac{\partial}{\partial t} p_t(x,y) = \int_{\mathbb{R}} \Big(p_t(x,z) r(z,y) - p_t(x,y) r(y,z) \Big)\, dz.$$

As for CTMC sample paths $t \mapsto X_t(\omega)$ are piecewise constant and right-continuous.

Example 2: Gaussian processes

The other extreme example (with continuous movement) is that of Gaussian processes, which we define now.

We say that the random variable $\mathbf{X} = (X_1, \dots, X_n) \sim \mathcal{N}(\mu, \Sigma)$ is a **multivariate Gaussian** in $\mathbb{R}^n$ if it has the following Probability Density Function (PDF):

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det \Sigma}} \exp\Big(-\frac{1}{2} \langle \mathbf{x} - \mu | \, \Sigma^{-1} \, | \mathbf{x} - \mu \rangle \Big),$$

with **mean** $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$ and **covariance matrix**

$$\Sigma = (\sigma_{ij} : i, j = 1, \dots, n) \,, \quad \sigma_{ij} = \mathrm{Cov}[X_i, X_j] = \mathbb{E}\big[(X_i - \mu_i)(X_j - \mu_j)\big].$$

Σ is symmetric and invertible (unless in degenerate cases with vanishing variance, which we won't look at that much).

Definition: Gaussian process

A stochastic process $(X_t : t \geq 0)$ with state space $S = \mathbb{R}$ is a **Gaussian process** if for all $n \in \mathbb{N}$, $0 \leq t_1 < \dots < t_n$ the vector $(X_{t_1}, \dots, X_{t_n})$ is a multivariate Gaussian.

Some quick notes on Gaussian processes

We will spend some time over the next couple of weeks discussing Gaussian processes (Brownian motion, other diffusion processes), but for now we will just see one of its key features:

Proposition

All the finite dimensional distributions of a Gaussian process $(X_t : t \geq 0)$ are fully characterized by their **mean** and **covariance function**

$$m(t) := \mathbb{E}[X_t] \quad \text{and} \quad \sigma(s,t) := \mathrm{Cov}[X_s, X_t].$$

In fact, it is also possible to prove the following:

Proposition

For any function $\mu : [0,\infty) \to \mathbb{R}$ and any non-negative definite function
C $\cancel{\sigma} : [0,\infty) \times [0,\infty) \to \mathbb{R}$, there exists a Gaussian process X_t such that

$$\mathbb{E}[X_t] = \mu(t) \quad \text{and} \quad \mathrm{Cov}[X_s, X_t] = C(s,t).$$

Stationary independent increments and Brownian motion

Stationary independent increments

The best way to look at continuous-time, continuous state space processes is by considering their increments.

Definition

A stochastic process $(X_t : t \geq 0)$ has **stationary increments** if

$$X_t - X_s \sim X_{t-s} - X_0 \quad \text{for all } 0 \leq s \leq t.$$

It has **independent increments** if for all $n \geq 1$ and $0 \leq t_1 < \cdots < t_n$

$$\{X_{t_{k+1}} - X_{t_k} : 1 \leq k < n\} \quad \text{are independent.}$$

Example. The Poisson process we defined last week, $(N_t : t \geq 0) \sim PP(\lambda)$ has stationary independent increments with $N_t - N_s \sim \text{Poi}\big(\lambda(t-s)\big)$.

Stationary indep. increments and Gaussian Processes

The following is a very useful property:

Proposition:

The following two statements are equivalent for a stochastic process $(X_t : t \geq 0)$:

- X_t has stationary independent increments and $X_t \sim \mathcal{N}(0, t)$ for all $t \geq 0$.
- X_t is a Gaussian process with $m(t) = 0$ and $\sigma(s, t) = \min\{s, t\}$.

Note that stationary independent increments have **stable distributions** such as Gaussian or Poisson.

Brownian motion

One of the most famous (Gaussian) stochastic processes which you will probably have heard of before is the Brownian motion.
There are many ways that people choose to define it (for an example of an alternative, check 2019 lecture notes in the module resources), and we will use this one:

Definition

We define a **Standard Brownian motion** (SBM) $(B_t : t \geq 0)$ to be a real-valued stochastic process such that

(i) $B_0 = 0$

(ii) B_t is continuous almost surely, i.e.,

$$\mathbb{P}\big[\{\omega : t \mapsto B_t(\omega) \text{ is continuous in } t \geq 0\}\big] = 1.$$

(iii) $B_t - B_s \sim \mathcal{N}(0, t - s)$ for all $0 \leq s \leq t$

(iv) B_t has independent increments, i.e., $\forall n \in \mathbb{N},\ \forall 0 \leq t_1 < t_2 < \cdots < t_n$ we have that $B_{t_1}, \cancel{B_{t_2 - t_1}}, \dots, \cancel{B_{t_n - t_{n-1}}}$ are independent random variables.

$B_{t_2} - B_{t_1}$ $B_{t_n} - B_{t_{n-1}}$

Brownian motion and Wiener

Wiener proved in 1923 that “the Brownian motion exists":

Theorem (Wiener, 1923):

There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which standard Brownian motion exists.

For this reason, the SBM is often also called a Wiener process, and a lot of books (or me, if I am distracted!) will write W_t instead of B_t!

His proof is beyond the scope of this module, but the idea is to construct the process on $\Omega = \mathbb{R}^{[0,\infty)}$ using **Kolmogorov's extension theorem**.

This shows that for every ‘consistent’ description of finite dimensional distributions (fdds) there exists a 'canonical' process $X_t[\omega] = \omega(t)$ characterised by a law $\mathbb{P}$ on Ω.

The main problem is to show that there exists a 'version' of the process that has continuous paths, i.e. $\mathbb{P}$ can be chosen to concentrate on continuous paths ω.

Properties of Brownian motion (1)

The Brownian motion has several useful properties (which is why it is so widely used!)
First of all, from the definition (and previous proposition), it follows:

- The SBM is a time-homogeneous Gaussian process.
- We have $m(t) = \mathbb{E}(B_t) = 0$ and

$$\mathrm{Cov}(B_t, B_s) = \mathbb{E}(B_t B_s) = \min(t, s)$$

- For all $a \leq b$, we have

$$\mathbb{P}(B_t \in (a, b)) = \frac{1}{2\pi t} \int_a^b \exp\left(-\frac{x^2}{2t}\right) \, dx$$

Note that: the SBM can be seen as the limit of a random walk, and this can be seen from the "functional central limit theorem" (Donsker's Theorem), which we will not cover.

Properties of Brownian motion (2)

Some more “advanced" properties are the following:

- For $\sigma > 0$ and a given number μ, $\sigma B_t + \mu$ is a (general) BM with $B_t \sim \mathcal{N}(\mu, \sigma^2 t)$.

  Its transition density is given by a Gaussian PDF

$$p_t(x, y) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(y-x)^2}{2\sigma^2 t}\right)$$

- This transition density is also called the **heat kernel**, since it solves the **heat/diffusion equation**

$$\frac{\partial}{\partial t} p_t(x, y) = \frac{\sigma^2}{2} \frac{\partial^2}{\partial y^2} p_t(x, y) \quad \text{with} \quad p_0(x, y) = \delta(y - x).$$

  We will see more about this later!

- The BM has the following scaling properties: If B_t is a SBM, so is

$$\begin{aligned} X_t &:= B_{t+s} - B_s \quad \text{with fixed } s > 0, \text{ and} \\ Y_t &:= B_{ct}/\sqrt{c} \quad \text{with fixed } c > 0\ . \end{aligned}$$

Properties of Brownian motion (3)

Finally, some very useful properties:

- The SBM is **self-similar** with **Hurst exponent** $H = 1/2$, i.e.

$$(B_{\lambda t} : t \geq 0) \sim \lambda^H (B_t : t \geq 0) \quad \text{for all } \lambda > 0.$$

- It is also **Hölder continuous**: For all $T > 0$ and $0 < \alpha < \frac{1}{2}$ there exists a random variable C such that

$$|B_t - B_s| \leq C|t - s|^\alpha, \quad \forall 0 \leq s, t \leq T.$$

- Most importantly, if we see the SBM as a function, $t \mapsto B_t$, it is $\mathbb{P} - a.s.$ **not differentiable** at t for all $t \geq 0$!
  If it was, then for fixed $h > 0$ define $\xi_t^h := (B_{t+h} - B_t)/h \sim \mathcal{N}(0, 1/h)$, a mean-0 Gaussian process with covariance

$$\sigma(s,t) = \begin{cases} 0 & , |t-s| > h \\ (h - |t-s|)/h^2 & , |t-s| < h \end{cases}.$$

The (non-existent) derivative $\xi_t := \lim_{h \to 0} \xi_t^h$ is called **white noise** and is formally a mean-0 Gaussian process with covariance $\sigma(s,t) = \delta(t-s)$.