Lectures 9 and 10: Beyond diffusion and intro to SDEs

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(remember our swapped lecture with MA930)

Plan for this week

1. Beyond diffusion
2. Intro to stochastic differential equations

Beyond diffusion - Lévy processes

We can define processes other than diffusion processes using generators, e.g., processes that combine jumps and diffusion...

Definition (Lévy process)

A **Lévy process** $(X_t : t \geq 0)$ is a real-valued process with right-continuous paths and stationary, independent increments.

These processes have generators which have

- a part with **constant drift** $a \in \mathbb{R}$,
- constant **diffusion** $\sigma^2 \geq 0$,
- and a translation invariant **jump part** with density $q(z)$ that fulfills

$$\int_{|z|>1} q(z)dz < \infty \quad \text{and} \quad \int_{0<|z|<1} z^2 q(z)dz < \infty.$$

$$\mathcal{L}f(x) = af'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}} \left(f(x+z) - f(x) - zf'(x)\,\mathbb{1}_{(0,1)}(|z|)\right) q(z)dz,$$

Examples of Lévy processes

1. Diffusion processes are Lévy processes. In particular the **Brownian Motion** with $a = 0$, $\sigma^2 > 0$ and $q(z) \equiv 0$.
2. Jump processes are also Lévy processes. An example is the **Poisson process** with $a = \sigma = 0$ and $q(z) = \lambda\delta(z-1)$.
3. A new example: the process with $a = \sigma = 0$ and heavy-tailed jump distribution

$$q(z) = \frac{C}{|z|^{1+\alpha}} \quad \text{with} \quad C > 0 \text{ and } \alpha \in (0,2]$$

is called **α-stable symmetric Lévy process** or **Lévy flight**.

The Lévy flight is **self-similar**:

$$(X_{\lambda t} : t \geq 0) \sim \lambda^H (X_t : t \geq 0)\,, \quad \lambda > 0 \quad \text{with } H = 1/\alpha$$

and exhibits something we call **super-diffusive behaviour** with $\mathbb{E}[X_t^2] \propto t^{2/\alpha}$.

This is an example of a **Markov process which is not Gaussian**.

Beyond diffusion - anomalous diffusion

In general, we say that a process $(X_t : t \geq 0)$ exhibits **anomalous diffusion** if

$$\frac{\mathrm{Var}[X_t]}{t} \to \begin{cases} 0, & \text{(sub-diffusive)} \\ \infty, & \text{(super-diffusive)} \end{cases} \quad \text{as } t \to \infty \, .$$

This leads us to introduce a process in another extreme: one that is Gaussian but **not Markov**.

Definition (fractional Brownian motion)

A **fractional Brownian motion** (fBM) $(B_t^H : t \geq 0)$ with **Hurst index** $H \in (0,1)$ is a **mean-zero Gaussian process** with continuous paths, $B_0^H = 0$ and covariances given by

$= \min(t,s)$ $(H = \frac{1}{2})$

$$\mathbb{E}\left(B_t^H \, B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right) \quad \text{for all } s, t \geq 0 \, .$$

$s < t$

$H = \frac{1}{2}$. $\frac{1}{2}\left(t + s - |t-s|\right) = \frac{1}{2}(\cancel{t} + s - \cancel{t} + s)$

$= \frac{1}{2}(2s) = s$

Fractional Brownian Motion

Some properties of fBM:

- For $H = 1/2$, the fBM is the standard Brownian motion.
- The fBM has stationary Gaussian increments where for all $t > s \geq 0$

$$B_t^H - B_s^H \sim B_{t-s}^H \sim \mathcal{N}\left(0, (t-s)^{2H}\right).$$

  For $H \neq 1/2$, these increments are **not** independent and the process is **non-Markov**.

- The fBM is **self-similar**, i.e.

$$(B_{\lambda t}^H : t \geq 0) \sim \lambda^H (B_t^H : t \geq 0) \quad \text{for all} \quad \lambda > 0.$$

- The fBM exhibits **anomalous diffusion** with $\mathrm{Var}[B_t^H] = t^{2H}$. If
  - ⋆ $\mathbf{H > 1/2}$, it is super-diffusive with positively correlated increments.
  - ⋆ $\mathbf{H < 1/2}$ it is sub-diffusive with negatively correlated increments.

$$\mathbb{E}\left[B_1^H(B_{t+1}^H - B_t^H)\right] = \frac{(t+1)^{2H} - 2t^{2H} + (t-1)^{2H}}{2} \underset{t\to\infty}{\simeq} H(2H-1)t^{2(H-1)}$$

Spectral densities and noise

For a **stationary process** $(X_t : t \geq 0)$ we define **autocorrelation function**

$$c(t) := \mathrm{Cov}[X_s, X_{s+t}] \quad \text{for all } s, t \in \mathbb{R}\ .$$

The Fourier transform of this function is called the **spectral density**

$$S(\omega) := \int_{\mathbb{R}} c(t) e^{-i\omega t} dt.$$

We can use this to describe noise:

- **White noise** $(\xi_t : t \geq 0)$, is a stationary GP with mean zero and

$$c(t) = \delta(t) \quad \Rightarrow \quad S(\omega) \equiv 1.$$

- **fractional noise** $(\xi_t^H : t \geq 0)$, is a stationary GP formally defined as the "derivative" of the fractional BM. It has mean zero and

$$c(t) = \frac{H(2H-1)}{|t|^{2(1-H)}} \quad \Rightarrow \quad S(\omega) \propto= \frac{1}{|\omega|^{2H-1}}$$

- If $H \to 1$, $S(\omega) \propto \frac{1}{\omega}$ and we call this **1/f-noise** or **"pink noise"**. Similarly, if $H \to 0$ we have $S(\omega) \propto \omega$ and we have **"blue noise"**.

Check out this article on Wired and Wiki

Introduction to Stochastic Differential Equations

SDEs and some definitions

Let $(B_t : t \geq 0)$ be a standard BM. Then a diffusion process with drift $a(x,t)$ and diffusion $\sigma(x,t)$ solves the **Stochastic differential equation (SDE)**

$$dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t.$$

Here dB_t is white noise as described before, and we interpret it in its integrated form.

To understand why, use our intuition from ODEs, and "conclude" that the solution is given by

$$X_t - X_0 = \int_0^t a(X_s, s)ds + \int_0^t \sigma(X_s, s)dB_s,$$

where X_0 is the initial condition (which can be deterministic or random).

The **problem** here is that we have an integral that we don't know how to compute:

$$\mathcal{I} = \int_0^t \sigma(X_s, s)dB_s,$$

which is a **stochastic integral**!

The stochastic integral

The problem with the stochastic integral $\mathcal{I} = \int_0^t f(X_s, s)dB_s$ is that we are trying to integrate a stochastic process X_t or a function of a stochastic process $f(X_t)$ **with respect to another stochastic process**.

This means that the stochastic integral $\mathcal{I}$ **is a random variable**! So... How do we compute it?

Let's think about the definition of Riemann integral.

We discretise the interval:

and we define

$$\mathcal{I}(t) := \lim_{K\to\infty} \sum_{k=1}^{K} f(\tau_k) \left(B_{t_k} - B_{t_{k-1}}\right) .$$

The important part now is that the definition of the stochastic integral **depends on our choice of τ_k**!

Why is this a problem?

$t_k - t_{k-1} = \Delta t$

$K \Delta t = T$

Recall that B_t is a sBM, so we know that $B_{t_k} - B_{t_{k-1}}$ are increments of a BM and therefore they are independent and $B_{t_k} - B_{t_{k-1}} \sim \mathcal{N}(0, \Delta t)$.

This sort of makes sense to compute the integral. However, we would expect the limit to be **independent of the chosen** τ_k. This will **not** be the case for us.

Example: Let's try to compute the integral $\mathcal{I} = \int_0^T B_t \, dB_t$.

$\tau_k \in [t_{k-1}, t_k]$

$\mathcal{I} = \sum B_{\tau_k} (B_{t_k} - B_{t_{k-1}})$

Case 1: choose $\tau_k = t_{k-1}$ (left)

$\mathcal{I}^{(l)} = \sum B_{t_{k-1}} (B_{t_k} - B_{t_{k-1}}) \Rightarrow \mathbb{E}(\mathcal{I}^{(l)}) = \sum \mathbb{E}(B_{t_{k-1}} (B_{t_k} - B_{t_{k-1}})) = 0$

Case 2: choose $\tau_k = t_k$ (right)

$\mathcal{I}^{(r)} = \sum B_{t_k} (B_{t_k} - B_{t_{k-1}}) \Rightarrow \mathbb{E}(\mathcal{I}^{(r)}) = \sum \mathbb{E}(B_{t_k} (B_{t_k} - B_{t_{k-1}}))$

$= \sum \mathbb{E}((B_{t_k} - B_{t_{k-1}} + B_{t_{k-1}})(B_{t_k} - B_{t_{k-1}}))$

$= \sum \mathbb{E}((B_{t_k} - B_{t_{k-1}})^2 + \underbrace{B_{t_{k-1}}(B_{t_k} - B_{t_{k-1}})}_{=0}) = \sum \Delta t = T$

Various definitions of stochastic integral

X_t $\quad$ $f(x(t))$ $\quad$ $\frac{df}{dt} = f'(x(t))\,\frac{dx}{dt}$

This happens because the BM is a.s. non-differentiable; and this means it "varies too much" in the interval $[t_{k-1}, t_k]$.

Note that: in "normal" integrals $\int f(x)\ dg(x)$ it was required that $g(x)$ had bounded total variation $\rightarrow$ this is what fails here.

There is no way around this problem. So we always need to specify our choice of τ_k when computing stochastic integrals. The most popular choices are:

- $\tau_k = t_{k-1}$ $\quad\rightarrow$ **Itô stochastic integral**
  commonly used in finance and biology
- $\tau_k = \frac{t_k + t_{k-1}}{2}$ $\quad\rightarrow$ **Stratonovich stochastic integral**
  mostly used in physics and engineering
- $\tau_k = t_k$ $\quad\rightarrow$ **Klimontovich stochastic integral**
  commonly used in statistical physics

Itô's formula $\frac{df}{dt}(x(t)) = f'(x(t))\frac{dx}{dt}$ $\Rightarrow$ $df(X_t) = f'(X_t) \cdot dX_t$

$dX_t = \underline{a(X_t, t)\, dt} + \underline{\sigma(X_t, t)\, dB_t}$

In this module, we will only use the **Itô interpretation**. This is because it has a lot of nice properties that you would expect of an integral.

However, it doesn't have a very important property: **the chain rule does not hold**. To overcome this, we use one of the most important results in stochastic calculus...

Theorem (Itô's formula for diffusions)

Let $(X_t : t \geq 0)$ be a diffusion with generator $\mathcal{L}$ and $f : \mathbb{R} \to \mathbb{R}$ a smooth. Then

$$f(X_t) - f(X_0) = \int_0^t (\mathcal{L}f)(X_s)ds + \int_0^t \sigma(X_s, s)f'(X_s)dB_s.$$

or, equivalently in terms of SDEs

$$df(X_t) = \underline{a(X_t, t)}f'(X_t)dt + \frac{1}{2}\underline{\sigma^2(X_t, t)f''(X_t)dt} + \sigma\underline{(X_t, t)f'(X_t)}dB_t.$$

correction term!

Back to SDEs

$Y_t = f(X_t)$

$\hookrightarrow dY_t$

Recall we are looking into SDEs of the form

$$\boxed{dX_t = a(X_t, t)dt + \sigma(X_t, t)dB_t.}$$

$X_t = f^{-1}(Y_t)$

Suppose we want to change variables to some r.v. $Y_t = f(X_t)$ for some nice invertible function $f \in C^2$. Itô's formula for diffusions implies the following.

Proposition:

Let $(X_t : t \geq 0)$ be a diffusion process with drift $a(x,t)$ and diffusion $\sigma(x,t)$, and $f : \mathbb{R} \to \mathbb{R}$ a smooth invertible function. Then $(Y_t : t \geq 0)$ with $Y_t = f(X_t)$ is a diffusion process with $(x = f^{-1}(y))$

$$\textbf{drift} \quad a(x,t)f'(x) + \underline{\frac{1}{2}\sigma^2(x,t)f''(x)} \quad \text{and } \textbf{diffusion} \quad \sigma(x,t)f'(x),$$

i.e., it solves the SDE

$$\begin{aligned} dY_t &= \left(a(X_t,t)f'(X_t) + \tfrac{1}{2}\sigma^2(X_t,t)f''(X_t)\right) dt + \sigma(X_t,t)f'(X_t)\, dB_t \\ &= \underbrace{f'(X_t)\left(a(X_t,t)\, dt + \sigma(X_t,t)\, dB_t\right)} + \underbrace{f''(X_t)\tfrac{1}{2}\sigma^2(X_t,t)\, dt}_{\text{correction}} \end{aligned}$$

Example - geometric Brownian motion

We saw the geometric random walk in one of our problem sheets, and a similar process is the gBM: Let $Y_t := e^{\theta B_t}$, so that $Y_t = f(X_t)$ with $f(x) = e^{\theta x}$.

Since $f'(x) = \theta f(x)$ and $f''(x) = \theta^2 f(x)$ and the sBM is a diffusion with $a \equiv 0$, $\sigma^2 \equiv 1$, for $\theta \in \mathbb{R}$ we have that $(Y_t : t \geq 0)$ is a diffusion process with SDE

$$dY_t = \frac{\theta}{2} Y_t dt + \theta Y_t dB_t.$$

Alternatively, suppose X_t is a gBM and we start with its SDE

$$dX_t = \theta X_t \, dt + \sigma X_t \, dB_t \qquad \longrightarrow \qquad \frac{dX_t}{X_t} = \theta dt + \sigma dB_t.$$

So... Define $Y_t = f(X_t)$ with $f(x) = ln(x)$. We have

$$f'(x) = \frac{1}{x}, \qquad \text{and} \qquad f''(x) = -\frac{1}{2x^2},$$

and we can use Itô's formula.

gBM continued

Applying Itô's formula, we have

$$\begin{aligned} dY_t &= d(\ln X_t) = \left(\tfrac{1}{X_t}\theta X_t - \tfrac{1}{2X_t^2}\sigma^2 X_t^2\right)\, dt + \tfrac{1}{X_t}\sigma X_t\, dB_t \\ &= \left(\theta - \tfrac{\sigma^2}{2}\right)\, dt + \sigma\, dB_t. \end{aligned}$$

This is something we can integrate!

$$\ln\left(\frac{X_t}{X_0}\right) = \left(\theta - \frac{\sigma^2}{2}\right)\, t + \sigma\, B_t,$$

... and this gives

$$X_t = X_0 \exp\left(\left(\theta - \frac{\sigma^2}{2}\right)\, t + \sigma\, B_t\right).$$

A note on gBM

We can show that the law of the gBM is a log-normal (like with the gRW) with mean $\mathbb{E}(X_t) = X_0 e^{\theta t}$ and variance $\mathrm{Var}(X_t) = X_0^2 e^{2\theta t}(e^{\sigma^2 t} - 1)$.

The gBM is the most widely used model for stock price behaviour in mathematical finance.

Example - Ornstein-Uhlenbeck process

The OU process is a diffusion process which solves the SDE

$$dX_t = -\alpha X_t \, dt + \sigma \, dB_t, \quad X_0 = x, \quad \alpha, \sigma > 0.$$

We can solve this SDE analytically by considering the ODE analogue:

$$\frac{dx}{dt} = -\alpha x + f(t) \quad \Rightarrow x(t) = e^{-\alpha t} x(0) + \int_0^t e^{-\alpha(t-s)} f(s) \, ds.$$

So, for the OU process, we obtain

$$X_t = e^{-\alpha t} X_0 + \int_0^t e^{-\alpha(t-s)} \, dW_s,$$

which we can also check using Itô's formula.

More on the OU process

The OU process also has some nice properties. For example, if $X_0 = x$ is deterministic, then $X_t \sim \mathcal{N}\left(e^{-\alpha t}x, \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})\right)$, i.e.

$$\mathbb{E}(X_t) = e^{-\alpha t}x \quad \text{and} \quad \mathrm{Var}(X_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t})$$

Use Itô's isometry $\mathbb{E}\left(\int_0^T f\, dW_t\right)^2 = \mathbb{E}\left(\int_0^T f^2(t)\, dt\right)$.

- The OU process describes the movement of a Brownian particle moving within a fluid with random “kicks" due to friction with other particles.
- It is a mean-reverting process (the mean acts as an equilibrium state)
- In mathematical finance, it is used to model interest rates and currency exchange rates.

Other examples

Other examples of SDEs that appear in applications include

- Cox-Ingersoll-Ross model for interest rates

$$dX_t = \alpha(\beta - X_t)\,dt + \sigma\sqrt{X_t}\,dB_t,$$

- Stochastic Verhulst equation for population dynamics

$$dX_t = (\lambda X_t - X_t^2)\,dt + \sigma X_t\,dB_t$$

- Langevin equation (similar to OU with a particle also having potential energy)

$$\begin{cases} dQ_t &= P_t\,dt \\ dP_t &= (-\lambda P_t - V'(Q_t))\,dt + \sigma dB_t \end{cases}$$

... and many others.

A note on solving SDEs numerically

We can't always solve SDEs analytically, so we must sometimes revert to numerical techniques.

The most commonly used numerical integration technique for SDEs is the **Euler-Maruyama scheme**.

Using the Markov property of diffusions, we can assume that $a(X_t)$ and $\sigma(X_t)$ don't change too much in a small time interval, so we can write, for $[t_n, t_{n+1}]$ with $t_{n+1} - t_n = \Delta t$

$$\begin{aligned} X_{t_{n+1}} &= X_{t_n} + \int_{t_n}^{t_{n+1}} a(X_s)\, ds + \int_{t_n}^{t_{n+1}} \sigma(X_s)\, dB_s \\ &\approx X_{t_n} + a(X_{t_n}) \int_{t_n}^{t_{n+1}} ds + \sigma(X_{t_n}) \int_{t_n}^{t_{n+1}} dB_s \end{aligned}$$

If we let $X_n = X_{t_n}$, this gives

$$X_{n+1} = X_n + a(X_n)\, \Delta t + \sigma(X_n) \Delta B_n,$$

with $\Delta B_n = B_{t_{n+1}} - B_{t_n} \sim \mathcal{N}(0, \Delta t)$.

A little more on numerical solution of SDEs

The Euler-Maruyama scheme is often the best we can do, especially with constant σ. For example, it is possible to show it converges (in some sense) to the right process with an optimal rate.

However, an alternative (for non-constant σ) is the **Milstein scheme**, which improves on the approximation

$$\int_{t_n}^{t_{n+1}} \sigma(X_s)\, dB_s \approx \sigma(X_n) \int_{t_n}^{t_{n+1}} \, dB_s.$$

This scheme gives

$$X_{n+1} = X_n + a(X_n)\, \Delta t + \sigma(X_n)\Delta B_n + (\sigma'\sigma)(X_n) \left((\Delta B_n)^2 - \Delta t \right).$$

We will not discuss numerical solution of SDEs in this module, but if you are interested, see this **paper by Des Higham** or ask me to borrow his book :)