Lecture 12:
Networks - basic definitions

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Plan for today

1. Graphs and some initial examples
2. Chat about assignment 2.

Networks - basic definitions and characteristics

Graphs - definition

Everything we will learn about networks is based on a concept which most of you have probably seen before - a graph. Today we will review basic definitions and concepts of graph theory which will be useful in the next few weeks.

Definition:

A **graph** (or **network**) $G = (V, E)$ consists of a finite set $V = \{1, \ldots, N\}$ of **vertices** (or **nodes**, **points**), and a set $E \subset V \times V$ of **edges** (or **links**, **lines**).

The graph G is called **undirected** if $(i, j) \in E$ implies $(j, i) \in E$, otherwise we say it is **directed**.

Two nodes $i, j \in V$ are **adjacent** or **neighbouring** if $(i, j) \in E$.

The structure of the graph is encoded in the **adjacency** (or **connectivity**) **matrix** which is defined as

$$A = (a_{ij} : i, j \in V) \quad \text{where} \quad a_{ij} = \begin{cases} 1 & , (i,j) \in E \\ 0 & , (i,j) \notin E \end{cases} .$$

We denote the number of edges by $K = |E|$ for directed, or $K = |E|/2$ for undirected graphs.

Some space for examples

Some things to note about graphs

The definition of a graph is quite general, and with the way it is written in the previous slide we could allow for *self-edges*, i.e., edges like (i, i), or *multiple edges* (multiple instances of (i, j). We do not allow for this in this module.

One can also consider **weighted graphs**, which are graphs with edge weights $w_{ij} \in \mathbb{R}$. These can be used to represent continuous- or discrete-time Markov chains.

In general graphs can also be infinite, but we will focus on finite graphs. Many of the following graph characteristics only make sense in the finite case.

Graphs - paths and shortest paths

A **path** γ_{ij} of length $l = |\gamma_{ij}|$ from vertex i to j is sequence of vertices

$$\gamma_{ij} = (v_1 = i, v_2, \ldots, v_{l+1} = j) \quad \text{with} \quad (v_k, v_{k+1}) \in E \text{ for all } k = 1, \ldots, l,$$

and $v_k \neq v_{k'}$ for all $k \neq k' \in \{1, \ldots, l\}$ (i.e. each vertex is visited only once).

If such a path exists, we say that vertex i is **connected** to j (write $i \to j$).

A **cycle** is a closed path γ_{ii} of length $|\gamma_{ii}| > 2$.

Shortest paths between vertices i, j are called **geodesics**. They are not necessarily unique, and their length d_{ij} is called the **distance** from i to j.

If $i \not\to j$ we set $d_{ij} = \infty$.

Graphs - connectivity

We say that a graph is **connected** if $d_{ij} < \infty$ for all $i, j \in V$.

We can define the **diameter** of the graph G by

$$\mathrm{diam}(G) := \max\{d_{ij} : i, j \in V\} \in \mathbb{N}_0 \cup \{\infty\},$$

and the **characteristic path length** of the graph G by

$$L = L(G) := \frac{1}{N(N-1)} \sum_{i,j \in V} d_{ij} \in [0, \infty].$$

Undirected graphs must have $d_{ij} = d_{ji}$ (which is finite if $i \leftrightarrow j$), and they can be decomposed into **connected components**, where we write

$$C_i = \{j \in V : j \leftrightarrow i\} \quad \text{for the component containing vertex } i.$$

Graphs - degrees

An important characteristic of any graph is the degree.

Definition:

The **in-** and **out-degree** of a node $i \in V$ is defined as

$$k_i^{\text{in}} = \sum_{j \in V} a_{ji} \quad \text{and} \quad k_i^{\text{out}} = \sum_{j \in V} a_{ij}.$$

$k_1^{\text{in}}, \dots k_N^{\text{in}}$ is called the **in-degree sequence**. With it, we can define the **in-degree distribution**:

$$\left(p^{\text{in}}(k) : k \in \{0, \dots, K\}\right) \quad \text{with} \quad p^{\text{in}}(k) = \frac{1}{N} \sum_{i \in V} \delta_{k, k_i^{\text{in}}},$$

which gives the fraction of vertices with in-degree k. The same holds for out-degrees.

In undirected networks, we simply write $k_i = k_i^{\text{in}} = k_i^{\text{out}}$ and $p(k)$.

Some properties of the degree

Here are some things to note about the degree of a graph:

- We have that, for an undirected graph, $\sum_{i \in V} k_i = \sum_{i,j \in V} a_{ij} = |E|$ (but this is also true for directed graphs)
- It is common to compute the average degree and the variance, which are given by

$$\langle k \rangle = \frac{1}{N} \sum_{i \in V} k_i = \frac{|E|}{N} = \sum_k k p(k), \qquad \text{and} \qquad \sigma^2 = \langle k^2 \rangle - \langle k \rangle^2.$$

- If a graph is such that each vertex has the same degree $k_i \equiv k$, we call it a **regular graph** (and it is usually undirected).
- Graphs where the degree distribution $p(k)$ shows a power law decay, i.e. $p(k) \propto k^{-\alpha}$ for large k, are often called **scale-free**.
- Real-world networks are often scale-free with exponent around $\alpha \approx 3$.

Graphs - first examples

Here are some examples of graphs:

- The **complete graph** K_N with N vertices is an undirected graph where all $N(N-1)/2$ vertices $E = ((i,j) : i \neq j \in V)$ are present.

- **Regular lattices** $\mathbb{Z}^d$ with edges between nearest neighbours are examples of regular graphs with degree $k = 2d$.

A special example

A particularly useful example of a graph is a tree:

Definition:

A **tree** is an undirected graph where any two vertices are connected by exactly one path.

In a tree, vertices with degree 1 are called **leaves**.

A **rooted tree** is a tree in which one vertex $i \in V$ is the designated **root**.

Then the graph can be directed, where all vertices point towards or away from the root.

Trees and cycles

Recall that a **cycle** is a closed path γ_{ii} of length $|\gamma_{ii}| > 2$. Using this, we can see that G is a tree if and only if

1. it is connected and has no cycles;
2. it is connected but is not connected if a single edge is removed;
3. it has no cycles but a cycle is formed if any edge is added.