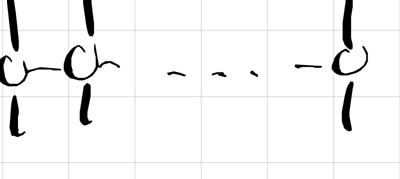
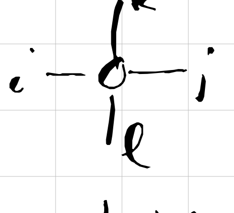
We aim to represent the Rydberg Hamiltonian

$$H=\sum_i \omega_i \sigma_i^x - \delta_i n_i + \sum_{i,j} C_{ij} n_i n_j$$

where $n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ as an MPO



To facilitate calculations let us write a rank 4 tensor



as a matrix valued matrix

$$\begin{pmatrix} A_{00} & A_{01} & \\ A_{10} & A_{11} & \cdots \\ & & \ddots \end{pmatrix}$$

Where the matrix A_{ij} has indices k, l

Furthermore let us take the multiplication of the inner matrices to be the kronecker product $\otimes$

Then, an MPO will be a series of matrices, and performing the multiplication is equivalent to contracting the bonds and reshaping all the physical in and out indices into one fat index for in and out respectively.

For example let

$$H = (X \;\; \mathbb{1}) \begin{pmatrix} \mathbb{1} & 0 \\ X & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ X \end{pmatrix}$$

$$= \begin{pmatrix} X\otimes\mathbb{1} + \mathbb{1}\otimes X & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ X \end{pmatrix}$$

$$= X\otimes\mathbb{1}\otimes\mathbb{1} + \mathbb{1}\otimes X\otimes\mathbb{1} + \mathbb{1}\otimes\mathbb{1}\otimes X$$

$$= X_1 + X_2 + X_3.$$

This immediately shows that we can implement the single qubit part of H with a bond dimension of 2. Namely let the action on qubit i be given by the 2x2 matrix

$A_i = \omega_i \sigma_x - \delta_i n$, then

$$H_1 = (A_1 \;\; \mathbb{1}) \begin{pmatrix} \mathbb{1} & 0 \\ A_2 & \mathbb{1} \end{pmatrix} \cdots \begin{pmatrix} \mathbb{1} & 0 \\ A_{n-1} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbb{1} \\ A_N \end{pmatrix}$$

With a bit of creativity, we can add the Rydberg interaction.

We write $H = S L_1 \dots L_a M R_1 \dots R_b E$

such that $a+b+3=N$, and $a = \lfloor \frac{N-2}{2} \rfloor$

$$S = (A_1 \;\; \mathbb{1} \;\; n) \qquad E = \begin{pmatrix} \mathbb{1} \\ A_N \\ n \end{pmatrix}$$

unless $N=2$ then $E = \begin{pmatrix} \mathbb{1} \\ A_2 \\ C_{12} n \end{pmatrix}$

$$L_i = \begin{array}{c} \\ 2+i \\ \downarrow \end{array} \overset{3+i \longrightarrow}{\begin{pmatrix} \mathbb{1} & 0 & \cdots & & 0 \\ A_{i+1} & \mathbb{1} & 0 & \cdots 0 & n \\ C_{1,i+1} n & & \ddots & & 0 \\ \vdots & & & & \vdots \\ C_{i,i+1} n & & & \mathbb{1} & 0 \end{pmatrix}}$$

$$R_i = \begin{array}{c} \\ 3+i \\ \downarrow \end{array} \overset{2+i \longrightarrow}{\begin{pmatrix} \mathbb{1} & 0 & \cdots & & 0 \\ A_k & \mathbb{1} & C_{k,k+1} n & \cdots & C_{k,N} n \\ n & 0 & \cdots & & 0 \\ 0 & 0 & \mathbb{1} & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & & \mathbb{1} \end{pmatrix}} \qquad k = a+i+1$$

$$M = \begin{pmatrix} \mathbb{1} & 0 & 0 & \cdots & 0 \\ A_{a+2} & \mathbb{1} & C_{a+2,a+3} n & \cdots & C_{a+2,N} n \\ C_{1,a+2} n & 0 & & & \\ \vdots & & & C_{ij} \mathbb{1} & \\ & & & i < a+2 & \\ & & & j > a+2 & \\ C_{a+1,a+2} n & 0 & & & \end{pmatrix}$$