EE1205: Signals and Systems - TA

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1) The Z-transform of $p(n)$ is defined as

$$P(z) = \sum_{n=-\infty}^{\infty} p(n)z^{-n} \tag{1.1}$$

2) If

$$p(n) = p_1(n) * p_2(n), \tag{2.1}$$
$$P(z) = P_1(z)P_2(z) \tag{2.2}$$

3) For a Geometric progression

$$x(n) = x(0)\, r^n u(n), \tag{3.1}$$
$$\implies X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} = \sum_{n=0}^{\infty} x(0)\, r^n z^{-n} \tag{3.2}$$
$$= \sum_{n=0}^{\infty} x(0) \left(rz^{-1}\right)^n \tag{3.3}$$
$$= \frac{x(0)}{1 - rz^{-1}}, \quad |z| > |r| \tag{3.4}$$

4) Substituting $r = 1$ in (3.4),

$$u(n) \overset{Z}{\longleftrightarrow} ZU(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.1}$$

5) From (1.1) and (4.1),

$$U(z) = \sum_{n=-\infty}^{\infty} u(n)z^{-n} \tag{5.1}$$
$$\implies \frac{dU(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} nu(n)z^{-n} \tag{5.2}$$
$$\therefore nu(n) \overset{Z}{\longleftrightarrow} Z\frac{z^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{5.3}$$

6) For an AP,

$$x(n) = [x(0) + nd]\, u(n) = x(0)u(n) + dnu(n) \tag{6.1}$$
$$\implies X(z) = \frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{6.2}$$

upon substituting from (4.1) and (5.3).

7) From (**??**), the sum to n terms of a GP can be expressed as

$$y(n) = x(n) * u(n) \tag{7.1}$$

where $x(n)$ is defined in (3.1). From (2.2), (3.4) and (4.1),

$$Y(z) = X(z)\,U(z) \tag{7.2}$$

$$= \left(\frac{x(0)}{1 - rz^{-1}}\right)\left(\frac{1}{1 - z^{-1}}\right) \quad |z| > |r| \cap |z| > |1| \tag{7.3}$$

$$= \frac{x(0)}{(1 - rz^{-1})(1 - z^{-1})} \quad |z| > |r| \tag{7.4}$$

which can be expressed as

$$Y(z) = \frac{x(0)}{r - 1}\left(\frac{r}{1 - rz^{-1}} - \frac{1}{1 - z^{-1}}\right) \tag{7.5}$$

using partial fractions. Again, from (3.4) and (4.1), the inverse of the above can be expressed as

$$y(n) = x(0)\left(\frac{r^{n+1} - 1}{r - 1}\right)u(n) \tag{7.6}$$

8) For the AP $x(n)$, the sum of first $n + 1$ terms can be expressed as

$$y(n) = \sum_{k=0}^{n} x(k) \tag{8.1}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) \tag{8.2}$$

$$= x(n) * u(n) \tag{8.3}$$

Taking the Z-transform on both sides, and substituting (6.2) and (4.1) (making use of the convolution property of Z-transform),

$$Y(z) = X(z)U(z) \tag{8.4}$$

$$\implies Y(z) = \left(\frac{x(0)}{1 - z^{-1}} + \frac{dz^{-1}}{(1 - z^{-1})^2}\right)\frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{8.5}$$

$$= \frac{x(0)}{(1 - z^{-1})^2} + \frac{dz^{-1}}{(1 - z^{-1})^3}, \quad |z| > 1 \tag{8.6}$$

But

$$\frac{1}{(1 - z^{-1})^2} = z\frac{z^{-1}}{(1 - z^{-1})^2} \tag{8.7}$$

$$\text{and w.k.t. } x(n+1) \overset{\mathcal{Z}}{\longleftrightarrow} zX(z), \quad |z| > 1 \tag{8.8}$$

From (5.3) and (8.8),

$$(n+1)u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{(1 - z^{-1})^2}, \quad |z| > 1 \tag{8.9}$$

Also, w.k.t

$$nx(n) \overset{\mathcal{Z}}{\longleftrightarrow} -z\frac{d}{dz}X(z) \tag{8.10}$$

$$\therefore \frac{z^{-1}}{(1 - z^{-1})^3} = -\frac{z}{2}\frac{d}{dz}\left[\frac{1}{(1 - z^{-1})^2}\right] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2}n\left[(n+1)u(n)\right] \tag{8.11}$$

Therefore, taking inverse Z-transform of (8.6)

$$y(n) = x(0)\left[(n+1)u(n)\right] + \frac{d}{2}\left[n(n+1)u(n)\right] \tag{8.12}$$

$$= \frac{n+1}{2}\left(2x(0) + nd\right)u(n) \tag{8.13}$$