GATE: ST - 32.2023

EE22BTECH11039 - Pandrangi Aditya Sriram*

Question: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having a mean 4 and variance 9. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $n \geq 1$, then $\lim\limits_{n\to\infty} E\left[\left(\frac{Y_n-4}{\sqrt{n}}\right)^2\right]$ (in integer) equals ______.

Solution: For X_i, mean $\mu = 4$ and variance $\sigma^2 = 9$. Since all X_i are i.i.d's, by Central Limit Theorem, as $n \to \infty$

$$Y_n \sim \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right) \tag{1}$$

$$\implies Y_n \sim \mathcal{N}\left(4, \frac{9}{n}\right) \tag{2}$$

Taking a standard normal variable $Z \sim \mathcal{N}(0, 1)$

$$\frac{Y_n - 4}{\sqrt{\frac{9}{n}}} = Z \tag{3}$$

$$\implies \left(\frac{Y_n - 4}{\sqrt{n}}\right)^2 = \frac{9}{n}Z^2 \tag{4}$$

$$\implies E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = \frac{9}{n}E\left(Z^2\right) \tag{5}$$

$$= \frac{9}{n} \tag{6}$$

Taking limits,

$$\lim_{n\to\infty} E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = \lim_{n\to\infty} \frac{9}{n} = 0 \tag{7}$$