Gaussian - 9.3.19

EE22BTECH11039 - Pandrangi Aditya Sriram*

Question: Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)

Solution:

RV	Values	Description
X	$\{0, 1, 2, 3, 4, 5, 6\}$	Outcomes of the binomial distribution
Y	$[-\infty, \infty]$	Outcomes of the Gaussian distribution

TABLE 0
Random Variables

1) **Binomial:**

$$X \sim Bin\left(6, \frac{1}{2}\right) \tag{1}$$

Thus,

$$\implies \begin{cases} n = 6 \\ p = \frac{1}{2} \\ q = 1 - p = \frac{1}{2} \end{cases} \tag{2}$$

The probability of getting exactly k successes in n trials is given by

$$p_X(k) = {}^nC_k p^k q^{n-k} \tag{3}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \tag{4}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^6 \tag{5}$$

We know that, for $k \in \mathbb{W}$ and $k \in [0, n]$, the maximum of ${}^nC_k$ occurs at

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \tag{6}$$

As,

$$n = 6 \tag{7}$$

$$\implies k = \frac{n}{2} = 3 \tag{8}$$

$\therefore X = 3$ is the most likely outcome.

$$p_X(3) = {}^6C_3 \left(\frac{1}{2}\right)^6 \tag{9}$$

$$= \frac{5}{16} \tag{10}$$

2) **Gaussian:** The binomial distribution $X \sim Bin\left(6, \frac{1}{2}\right)$ can be approximated as a Gaussian distribution $Y \sim \mathcal{N}\left(\mu, \sigma^2\right)$ using the Mean μ and Standard Deviation σ parameters.

$$\mu = np = 6 \times \frac{1}{2} = 3 \tag{11}$$

$$\sigma^2 = npq = 6 \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{2} \tag{12}$$

Thus, the Gaussian (normal) approximation is:

$$Y \sim \mathcal{N}\left(3, \frac{3}{2}\right) \tag{13}$$

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{14}$$

$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}} \tag{15}$$

At the most likely outcome, a maximum occurs. Thus,

$$\frac{d}{dx}\left(p_Y(x)\right) = 0 \tag{16}$$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}}\right) = 0 \tag{17}$$

$$\frac{-2}{3\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}} (x - 3) = 0 \tag{18}$$

$$\implies x = 3 \tag{19}$$

$\therefore$ Y = 3 is the most likely outcome.

The plots are given as follows:

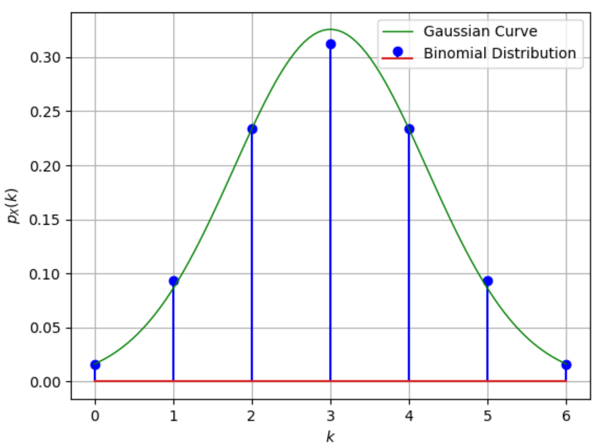

Fig. 2. Binomial Distribution and Gaussian Approximation