Gaussian - 9.3.19

EE22BTECH11039 - Pandrangi Aditya Sriram*

Question: Suppose X is a binomial distribution $B\left(6, \frac{1}{2}\right)$. Show that $X = 3$ is the most likely outcome. (Hint : $P(X = 3)$ is the maximum among all $P(x_i), x_i = 0, 1, 2, 3, 4, 5, 6$)
Solution:

$$X \sim Bin\left(6, \frac{1}{2}\right) \tag{1}$$

Thus,

$$\implies \begin{cases} n = 6 \\ p = \frac{1}{2} \\ q = 1 - p = \frac{1}{2} \end{cases} \tag{2}$$

The probability of getting exactly k successes in n trials is given by

$$p_X(k) = {}^nC_k p^k q^{n-k} \tag{3}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} \tag{4}$$

$$= {}^nC_k \left(\frac{1}{2}\right)^6 \tag{5}$$

We know that ${}^nC_k$ can be written as,

$${}^nC_k = \frac{n!}{(n-k)!k!} \tag{6}$$

If pmf is the greatest, then ${}^nC_k$ is the maximum for $k \in [0, n]$, Therefore It can be said that,

$${}^nC_k \geq {}^nC_{k-1} \quad \text{and} \tag{7}$$

$${}^nC_k \geq {}^nC_{k+1} \tag{8}$$

From (6) and (7), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k+1)!(k-1)!} \tag{9}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!} \frac{k}{n-k+1} \tag{10}$$

$$\implies 1 \geq \frac{k}{n-k+1} \tag{11}$$

$$\therefore k \leq \frac{n+1}{2} \tag{12}$$

From (6) and (8), we can state that

$$\frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k-1)!(k+1)!} \tag{13}$$

$$\implies \frac{n!}{(n-k)!k!} \geq \frac{n!}{(n-k)!k!} \frac{n-k}{k+1} \tag{14}$$

$$\implies 1 \geq \frac{n-k}{k+1} \tag{15}$$

$$\therefore k \geq \frac{n-1}{2} \tag{16}$$

From (12) and (16), we can state that

$$\frac{n-1}{2} \leq k \leq \frac{n+1}{2} \tag{17}$$

We know that, $k \in \mathbb{W}$ and $k \in [0, n]$ and from (17),

$$k = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2} \quad \text{or} \quad \frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases} \tag{18}$$

As,

$$n = 6 \tag{19}$$

$$\implies k = \frac{n}{2} = 3 \tag{20}$$

$\therefore X = 3$ is the most likely outcome.

$$p_X(3) = {}^6C_3 \left(\frac{1}{2}\right)^6 \tag{21}$$

$$= \frac{5}{16} \tag{22}$$

The binomial distribution $X \sim Bin\left(6, \frac{1}{2}\right)$ can be approximated as a Gaussian distribution using the Mean μ and Standard Deviation σ parameters.

$$\mu = np = 6 \times \frac{1}{2} = 3 \tag{23}$$

$$\sigma = \sqrt{npq} = \sqrt{6 \times \frac{1}{2} \times \frac{1}{2}} = \sqrt{\frac{3}{2}} \tag{24}$$

Thus, the Gaussian (normal) approximation is:

$$X \sim \mathcal{N}\left(3, \frac{3}{2}\right) \tag{25}$$

$$p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{26}$$

$$= \frac{1}{\sqrt{3\pi}} e^{-\frac{(x-3)^2}{3}} \tag{27}$$

The plots are given as follows:

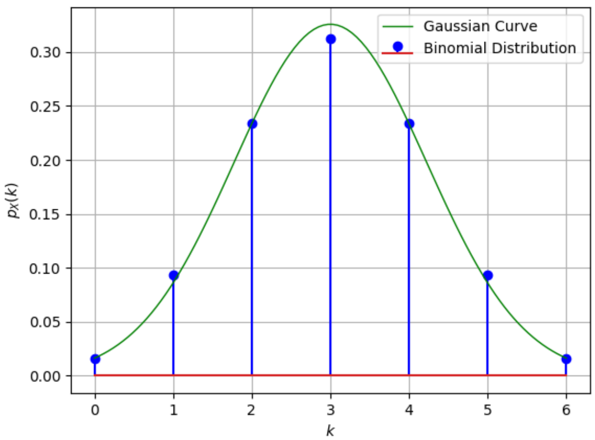

Fig. 0. Binomial Distribution and Gaussian Approximation