GATE: ST - 32.2023

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Question: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having a mean 4 and variance 9. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $n \geq 1$, then $\lim_{n\to\infty} E\left[\left(\frac{Y_n-4}{\sqrt{n}}\right)^2\right]$ (in integer) equals __________.

Solution: For all X_i which as i.i.d's, mean $\mu = 4$ and variance $\sigma^2 = 9$,

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

$$\implies Y_n - 4 = \frac{1}{n}\sum_{i=1}^{n} (X_i - 4) \tag{2}$$

$$\implies \frac{Y_n - 4}{\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n} (X_i - 4) \tag{3}$$

$$\tag{4}$$

As all X_i are i.i.d's, and thus all $X_i - 4$ are i.i.d's

$$\frac{Y_n - 4}{\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n} (X_i - 4) \tag{5}$$

$$\implies \left(\frac{Y_n - 4}{\sqrt{n}}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - 4)\left(X_j - 4\right) \tag{6}$$

$$\implies E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = E\left[\frac{1}{n^3}\sum_{i=1}^{n}\sum_{j=1}^{n} (X_i - 4)\left(X_j - 4\right)\right] \tag{7}$$

$$= \frac{1}{n^3}E\left[2\sum_{i=1}^{n}\sum_{j=i+1}^{n} (X_i - 4)\left(X_j - 4\right) + \sum_{i=1}^{n}(X_i - 4)^2\right] \tag{8}$$

$$= \frac{1}{n^3}\left(2\sum_{i=1}^{n} (E[X_i] - 4)\left(E\left[X_j\right] - 4\right) + \sum_{i=1}^{n} E\left[(X_i - 4)^2\right]\right) \tag{9}$$

$$= \frac{1}{n^3}\left(2\sum_{i=1}^{n} (\mu - 4)(\mu - 4) + \sum_{i=1}^{n} E\left[(X_i - \mu)^2\right]\right) \tag{10}$$

$$= \frac{1}{n^3}\left(2\sum_{i=1}^{n} 0 + \sum_{i=1}^{n} \sigma^2\right) \tag{11}$$

$$= \frac{n\sigma^2}{n^2} = \frac{9}{n^2} \tag{12}$$

Taking limits,

$$\lim_{n\to\infty} E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = \lim_{n\to\infty} \frac{9}{n^2} = 0 \tag{13}$$