

GATE: ST - 14.2023

EE22BTECH11039 - Pandrangi Aditya Sriram*

Question: Consider the probability space (Ω, \mathcal{G}, P) where $\Omega = [0, 2]$ and $\mathcal{G} = \{\emptyset, \Omega, [0, 1], (1, 2]\}$. Let X and Y be two functions on Ω defined as

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in [0, 1] \\ 2 & \text{if } \omega \in (1, 2] \end{cases}$$

and

$$Y(\omega) = \begin{cases} 2 & \text{if } \omega \in [0, 1.5] \\ 3 & \text{if } \omega \in (1.5, 2]. \end{cases}$$

Then which one of the following statements is true?

- (A) X is a random variable with respect to \mathcal{G} , but Y is not a random variable with respect to \mathcal{G} .
- (B) Y is a random variable with respect to \mathcal{G} , but X is not a random variable with respect to \mathcal{G} .
- (C) Neither X nor Y is a random variable with respect to \mathcal{G} .
- (C) Both X and Y are random variables with respect to \mathcal{G} .

(GATE ST 2023)

Solution:

- 1) For X to be a random variable with respect to \mathcal{G} :

$$X^{-1}(X(\omega)) \in \mathcal{G} \quad \forall X(\omega) \in \mathcal{T}_1 \quad (1)$$

where \mathcal{T}_1 is the range of $X(\omega)$.

If $X(\omega) = 1$:

$$X^{-1}(X(\omega)) = X^{-1}(1) \quad (2)$$

$$= [0, 1] \quad (3)$$

$$\in \mathcal{G} \quad (4)$$

If $X(\omega) = 2$:

$$X^{-1}(X(\omega)) = X^{-1}(2) \quad (5)$$

$$= (1, 2] \quad (6)$$

$$\in \mathcal{G} \quad (7)$$

$\therefore X$ is a random variable with respect to \mathcal{G} .

- 2) For Y to be a random variable with respect to \mathcal{G} :

$$Y^{-1}(Y(\omega)) \in \mathcal{G} \quad \forall Y(\omega) \in \mathcal{T}_2 \quad (8)$$

where \mathcal{T}_2 is the range of $Y(\omega)$.

If $Y(\omega) = 2$:

$$Y^{-1}(Y(\omega)) = Y^{-1}(2) \quad (9)$$

$$= [0, 1.5] \quad (10)$$

$$\notin \mathcal{G} \quad (11)$$

$\therefore Y$ is not a random variable with respect to \mathcal{G} .