GATE: ST - 32.2023

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Question: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having a mean 4 and variance 9. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $n \geq 1$, then $\lim_{n\to\infty} E\left[\left(\frac{Y_n-4}{\sqrt{n}}\right)^2\right]$ (in integer) equals ___________.

Solution: For all X_i which as i.i.d's, mean $\mu = 4$ and variance $\sigma^2 = 9$,

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

$$\implies E\left[Y_n\right] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \tag{2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} E\left[X_i\right] \tag{3}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mu \tag{4}$$

$$= \frac{n\mu}{n} \tag{5}$$

$$= \mu \tag{6}$$

As they are independent and identically distributed:

$$\implies Var\left(Y_n\right) = E\left[Y_n^2\right] - E^2\left[Y_n\right] \tag{7}$$

$$= E\left[\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i^2 + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} X_i X_j\right)\right] - \mu^2 \tag{8}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n} E\left[X_i^2\right] + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} E\left[X_i\right]E\left[X_j\right]\right) - \mu^2 \tag{9}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\left(\sigma^2 + \mu^2\right) + 2\sum_{i=1}^{n}\sum_{j=i+1}^{n} \mu^2\right) - \mu^2 \tag{10}$$

$$= \frac{1}{n^2}\left(n\left(\sigma^2 + \mu^2\right) + 2\frac{n(n-1)}{2}\mu^2\right) - \mu^2 \tag{11}$$

$$= \frac{1}{n^2}\left(n\sigma^2 + n\mu^2 + n^2\mu^2 - n\mu^2\right) - \mu^2 \tag{12}$$

$$= \frac{\sigma^2}{n} \tag{13}$$

Here, $E[Y_n] = 4$ and $Var[Y_n] = \frac{9}{n}$. As n is large, we can approximate it as a normal distribution, $Y_n \sim \mathcal{N}\left(4, \frac{9}{n}\right)$. Taking a standard normal variable $Z \sim \mathcal{N}(0,1)$

$$\frac{Y_n - 4}{\sqrt{\frac{9}{n}}} = Z \tag{14}$$

$$\implies \left(\frac{Y_n - 4}{\sqrt{n}}\right)^2 = \frac{9}{n} Z^2 \tag{15}$$

$$\implies E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = \frac{9}{n} E\left(Z^2\right) \tag{16}$$

$$= \frac{9}{n} \tag{17}$$

Taking limits,

$$\lim_{n\to\infty} E\left[\left(\frac{Y_n - 4}{\sqrt{n}}\right)^2\right] = \lim_{n\to\infty} \frac{9}{n} = 0 \tag{18}$$