GATE: ST - 32.2023

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Question: Let $\{X_n\}_{n\geq 1}$ be a sequence of independent and identically distributed random variables each having a mean 4 and variance 9. If $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ for $n \geq 1$, then $\lim_{n\to\infty} \mathrm{E}\left[\left(\frac{Y_n-4}{\sqrt{n}}\right)^2\right]$ (in integer) equals __________. (GATE ST 2023)

Solution: For all X_i which as i.i.d's, mean $\mu = 4$ and variance $\sigma^2 = 9$,

$$Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i \tag{1}$$

The mean of a sum of i.i.d random variables is calculated as

$$\mathrm{E}[Y_n] = \mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] \tag{2}$$

$$= \frac{1}{n}\sum_{i=1}^{n} \mathrm{E}[X_i] \tag{3}$$

$$= \frac{1}{n}(n\mu) \tag{4}$$

$$= \mu \tag{5}$$

The variance of a sum of i.i.d random variables is calculated as

$$\mathrm{var}(Y_n) = \mathrm{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)^2\right] - \left(\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right]\right)^2 \tag{6}$$

$$= \frac{1}{n^2}\left\{\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] - \left(\mathrm{E}\left[\sum_{i=1}^{n} X_i\right]\right)^2\right\} \tag{7}$$

But

$$\mathrm{E}\left[\left(\sum_{i=1}^{n} X_i\right)^2\right] = \mathrm{E}\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_iX_j\right] \tag{8}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}\left[X_iX_j\right] \tag{9}$$

and

$$\left(\mathrm{E}\left[\sum_{i=1}^{n} X_i\right]\right)^2 = \left(\sum_{i=1}^{n} \mathrm{E}[X_i]\right)^2 \tag{10}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{E}[X_i]\,\mathrm{E}\left[X_j\right] \tag{11}$$

Putting (9) and (11) in (7), and using the definition of covariance,

$$\mathrm{var}(Y_n) = \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\mathrm{E}\left[X_iX_j\right] - \mathrm{E}[X_i]\,\mathrm{E}\left[X_j\right]\right)\right\} \tag{12}$$

$$= \frac{1}{n^2}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}\left(X_i, X_j\right)\right\} \tag{13}$$

As all the variables are i.i.d's and are thus uncorrelated,

$$\mathrm{cov}\left(X_i, X_j\right) = \begin{cases} 0 & \text{if } i \neq j \\ \mathrm{var}(X_i) & \text{if } i = j \end{cases} \tag{14}$$

Putting (14) in (13),

$$\mathrm{var}(Y_n) = \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{cov}(X_i, X_i)\right) \tag{15}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\mathrm{var}(X_i)\right) \tag{16}$$

$$= \frac{1}{n^2}\left(\sum_{i=1}^{n}\sigma^2\right) \tag{17}$$

$$= \frac{\sigma^2}{n} \tag{18}$$

Consider the term $\left(\frac{Y_n-\mu}{\sqrt{n}}\right)^2$. Calculating its expectation,

$$\mathrm{E}\left[\left(\frac{Y_n-\mu}{\sqrt{n}}\right)^2\right] = \frac{1}{n}\mathrm{E}\left[(Y_n-\mu)^2\right] \tag{19}$$

$$= \frac{1}{n}\mathrm{var}(Y_n) \tag{20}$$

$$= \frac{\sigma^2}{n^2} \tag{21}$$

Substituting $\sigma^2 = 9$ and $\mu = 4$, we get

$$\lim_{n\to\infty}\mathrm{E}\left[\left(\frac{Y_n-4}{\sqrt{n}}\right)^2\right] = \lim_{n\to\infty}\frac{9}{n^2} = 0 \tag{22}$$