# EE1080/AI1110/EE2102 Probability: HW5

18th March 2025

## 1 Joint Density of Two Functions of Two Random Variables

1. Find the joint density of $R = \sqrt{X^2 + Y^2}$, $\Theta = \tan^{-1}(Y/X)$ given:
   (a) $X$ and $Y$ are independent random variables both uniformly distributed over $(0, 1)$.
   (b) $f_{x,y} = 1/\pi$ for $x^2 + y^2 \leq 1$.
2. Let $X, Y$ be independent standard normal random variables, determine the joint density function of
$$U = X, \quad V = X/Y.$$
   Use this result to show that $V$ has a Cauchy distribution.
3. If $U$ is uniform on $[0, 2\pi]$ and $Z$, independent of $U$, is exponential with rate 1, find the joint density of $X, Y$ defined by:
$$X = \sqrt{2Z}\cos(U), \quad Y = \sqrt{2Z}\sin(U).$$
4. Let $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ be the order statistics of a set of $n$ independent uniform $(0, 1)$ random variables $X_1, X_2, \cdots, X_n$.
   (a) Find the density of $X_{(i)}$ for $i \in [n]$.
   (b) Find the conditional distribution of $X_{(n)}$ given that $X_{(1)} = s_1, X_{(1)} = s_2, \cdots, X_{(n-1)} = s_{n-1}$
   (c) Prove that for $1 \leq k \leq n + 1$:
$$P(X_{(k)} - X_{(k-1)} > t) = (1 - t)^n$$
   where $X_{(0)} = 0, X_{(n+1)} = 1$ and $0 < t < 1$.
   (d) Find joint density of $R, M$ where $R = X_{(n)} - X_{(1)}$ and $M = (X_{(n)} + X_{(1)})/2$.

## 2 Moment Generating Functions

5. Find third and fourth moments of standard normal random variable $X$.

6. Find third, fourth and fifth moments of an exponential random variable with parameter $\lambda$.

7. A nonnegative integer-valued random variable $X$ has one of the following two expressions as its transform:

   (a) $M(s) = e^{2(e^{e^s - 1} - 1)}$

   (b) $M(s) = e^{2(e^{e^s} - 1)}$

   Explain why one of the two cannot possibly be the transform ? Use the true transform to find $P(X = 0)$.

8. Find the PDF of the continuous random variable $X$ associated with the transform

$$M(s) = \frac{1}{3}\frac{2}{2-s} + \frac{2}{3}\frac{3}{3-s}$$

9. Let $X$ be a random variable that takes non-negative integer values and is associated with transform of the form:

$$M_X(s) = c\frac{3 + 4e^{2s} + 2e^{3s}}{3 - e^s}$$

10. Let $X$, $Y$ and $Z$ be independent random variables, where $X$ is Bernoulli with parameter $1/3$, $Y$ is exponential with parameter 2, and $Z$ is Poisson with parameter 3.

    (a) Consider a new random variable $U = XY + (1 - X)Z$. Find the transform associated with $U$.

    (b) Find the transform associated with $2Z + 3$

    (c) Find the transform associated with $Y + Z$