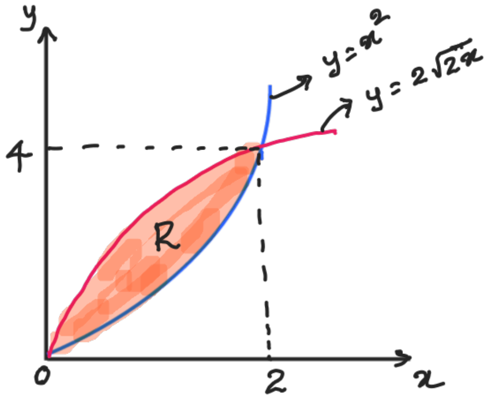
Solutions

Q1. (a)



$$R = \begin{cases} x^2 \le y \le 2\sqrt{x} \\ 0 \le x \le 2 \end{cases}$$

(b) $$R = \begin{cases} y^2/8 \le x \le \sqrt{y} \\ 0 \le y \le 4 \end{cases} \Rightarrow \iint_R f\, dA = \int_0^4 \int_{y^2/8}^{\sqrt{y}} f(x,y)\, dx\, dy.$$

Q2.

(a) $$F_x = \frac{\partial F}{\partial x} = \frac{xz}{(x^2+y)^{1/2}} \Rightarrow F_x(1,3,2) = 1$$

$$F_y = \frac{\partial F}{\partial y} = \frac{z}{2(x^2+y)^{1/2}} + \frac{2}{z} \Rightarrow F_y(1,3,2) = \frac{3}{2}$$

$$F_z = \frac{\partial F}{\partial z} = (x^2+y)^{1/2} - \frac{2y}{z^2} \Rightarrow F_z(1,3,2) = \frac{1}{2}$$

Normal to the tangent plane:

$$\hat{n} = \nabla F_{(1,3,2)} = \left(1, \frac{3}{2}, \frac{1}{2}\right)$$

$$P_0 = (1, 3, 2)$$

Solutions

Tangent plane Equation :

$$1(x-1) + \frac{3}{2}(y-3) + \frac{1}{2}(z-2) \quad \text{OR}$$

$$2x + 3y + z = 13$$

(b) At $P_0 = (1,3,2)$ we have $|F_y| = \frac{3}{2} > |F_x|, |F_z|$

So, a change in y produces the largest change in F.

Change in 'F' with $\Delta y = 0.1$ is;

$$\Delta F = F_y \Delta y = \frac{3}{2}(0.1) = 0.15$$

(c) Change in F in the direction $\pm(2,2,-1)$ is;

$$\left.\frac{dF}{ds}\right|_{P_0,\hat{u}} = \underbrace{\hat{u} \cdot \nabla F(P_0)}_{\text{Directional derivative}} \qquad \text{where, } \hat{u} = \pm\frac{1}{3}(2,2,-1)$$

$$= \pm\frac{1}{3}(2,2,-1)\cdot\left(1, \frac{3}{2}, \frac{1}{2}\right) = \pm\frac{3}{2}$$

$$\frac{\Delta F}{\Delta s} \approx \left.\frac{dF}{ds}\right|_{P_0,\hat{u}} \Rightarrow \Delta s = \frac{\Delta F}{\left.\frac{dF}{ds}\right|_{P_0,\hat{u}}}$$

$$= \frac{0.1}{\left(\frac{3}{2}\right)} = 0.06$$

$$\Rightarrow \Delta s = 0.06$$

Solutions

Q3.

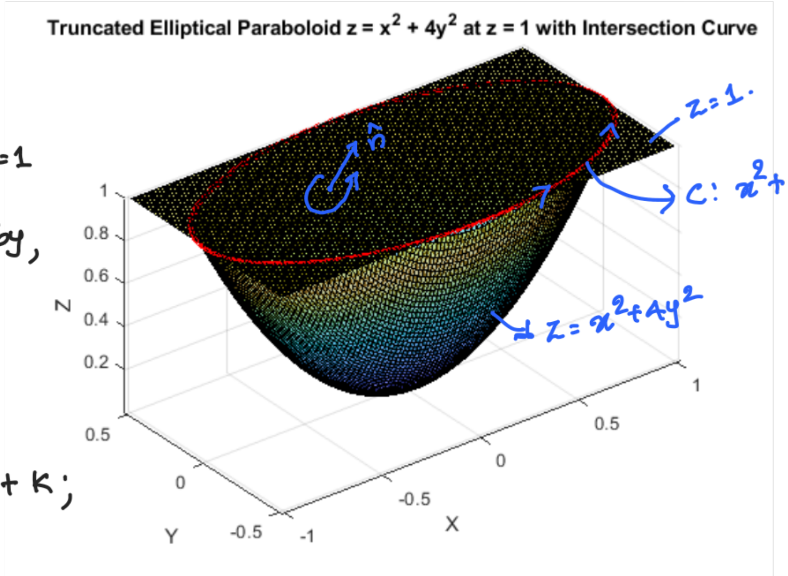
The curve C is;

$x^2+4y^2=1$ in the plane $z=1$

We can parametrize ellipse by,

$x=\cos t,\quad y=\frac{1}{2}\sin t,\quad z=1$

for $0\leq t\leq 2\pi$



$C:\quad r(t)=(\cos t)\,i+\frac{1}{2}(\sin t)\,j+k;$

$0\leq t\leq 2\pi$.

By stokes's theorem;

$$\iint_S (\nabla\times\bar{F})\cdot\hat{n}\,dS=\oint_C \bar{F}\cdot d\bar{r}$$

So we need to just calculate $\oint_C \bar{F}\cdot d\bar{r}$ here;

$\bar{F}=y\hat{i}-xz\hat{j}+xz^2\hat{k}$

$\bar{F}(r(t))=\frac{1}{2}\sin t\,\hat{i}-\cos t\,\hat{j}+\cos t\,\hat{k}$

and $\frac{d\bar{r}}{dt}=-\sin t\,\hat{i}+\frac{1}{2}\cos t\,j$

$$\oint \bar{F}\cdot d\bar{r}=\int_0^{2\pi}F(r(t))\cdot\frac{d\bar{r}}{dt}\,dt=\int_0^{2\pi}\left(-\frac{1}{2}\sin^2 t-\frac{1}{2}\cos^2 t\right)dt=-\pi$$

Thus by Stokes' theorem the flux of the curl across 'S' in the normal direction 'n' is;

$$\iint_S \nabla\times\bar{F}\cdot\hat{n}\,dS=-\pi$$

Solutions

Q4. By Green's theorem; (normal form)

$\bar{F} = \langle P, Q \rangle$

$$\oint_C \bar{F}\cdot\hat{n}\,ds = \oint_C P\,dy - Q\,dx$$

$$= \iint_R (\nabla\cdot F)\,dx\,dy.$$

$F = 2e^{xy}\hat{i} + y^3\hat{j}$

$\nabla\cdot\bar{F} = 2ye^{xy} + 3y^2$

$$\Rightarrow \text{Flux} = \int_{-1}^{+1}\int_{-1}^{+1}(2ye^{xy}+3y^2)\,dx\,dy = \int_{-1}^{1} 2e^{xy}+3xy^2\Big|_{x=-1}^{+1} dy.$$

$$= \int_{-1}^{+1}(2e^{y}+6y^2-2e^{-y})\,dy = \left[2e^{y}+2y^3+2e^{-y}\right]_{-1}^{+1} = 4$$

Flux = 4.

Q5.

(a) By Lagrange multiplier method;

$f(x,y) = 3x+4y$; $\quad g(x,y) = x^2+y^2-1$

$\nabla f = \lambda\nabla g$ ----- (1)

$3\hat{i}+4\hat{j} = 2x\lambda\hat{i} + 2y\lambda\hat{j}$ --- (2)

$$\left.\begin{array}{l} 2x\lambda = 3 \text{ ----(3)} \Rightarrow x = \frac{3}{2\lambda} \\ 2y\lambda = 4 \text{ ---(4)} \Rightarrow y = \frac{2}{\lambda} \end{array}\right\} \Rightarrow x, y \text{ have same sign.}$$

Also $x^2+y^2=1$ ---(5)

Solutions

Solving (3), (4) & (5)

$$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 - 1 = 0 \quad \Rightarrow \quad 9 + 16 = 4\lambda^2$$

$$\Rightarrow \lambda = \pm\frac{5}{2}$$

$$x = \frac{3}{2\lambda} = \pm\frac{3}{5}, \quad y = \frac{2}{\lambda} = \pm\frac{4}{5}$$

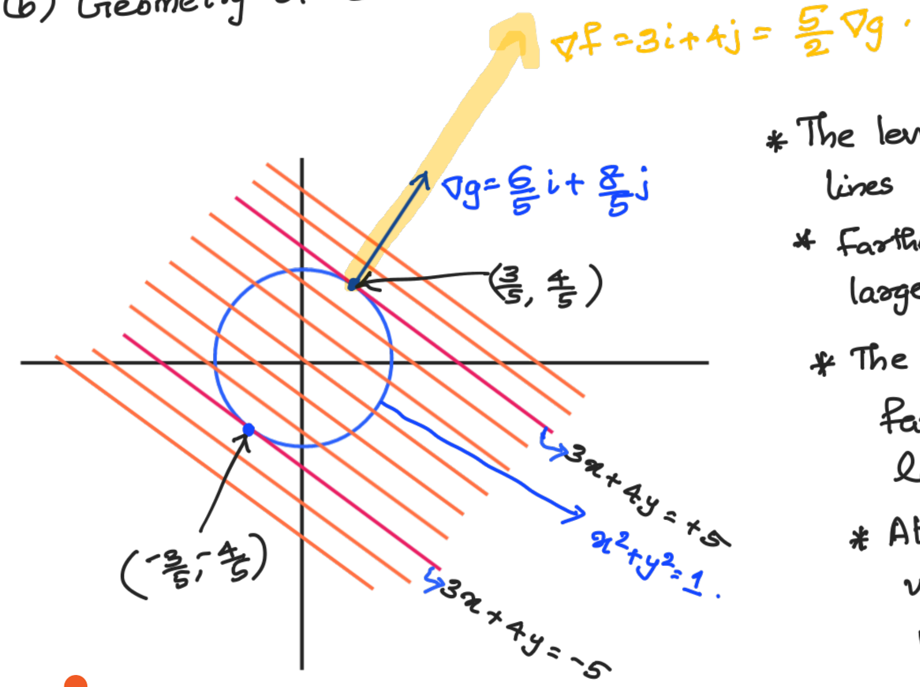
$\therefore f(x,y) = 3x+4y$ has extreme values at $(x,y) = \pm\left(\frac{3}{5}, \frac{4}{5}\right)$

On substituting we get

$f(x,y)_{max}$ on $g(x,y) = +5$ at $\left(+\frac{3}{5}, \frac{4}{5}\right)$

$f(x,y)_{min}$ on $g(x,y) = -5$ at $\left(-\frac{3}{5}, -\frac{4}{5}\right)$

(b) Geometry of solution:



* The level curves of $f(x,y)$ are the lines $3x+4y=c$
* Farther the line from the origin larger the absolute value of 'f'.
* The lines tangent to circle are farthest from origin which also lies on the circle.
* At the point of tangency any vector normal to the line is normal to circle also

$\therefore \nabla f = \lambda \nabla g$; Here $\nabla f = \frac{5}{2}\nabla g$.

Solutions

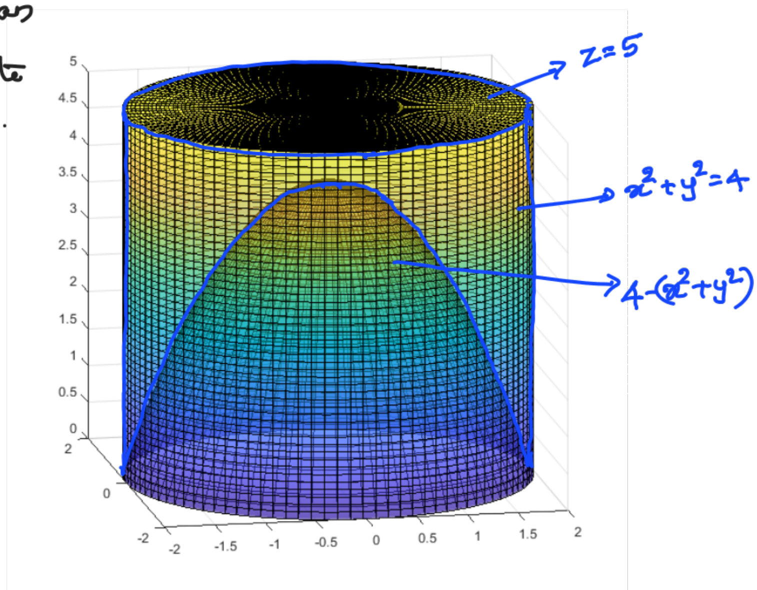
Q6. Since 'S' encloses a volume, we can use divergence theorem to calculate flux.

$$\oiint_S \bar{F}\cdot\hat{n}\,dS = \iiint_V (\nabla\cdot\bar{F})\,dV.$$

$$\bar{F} = (x+y)\hat{i} + (y+z)\hat{j} + (z+x)\hat{k}.$$

$$\nabla\cdot\bar{F} = \frac{\partial \bar{F}}{\partial x} + \frac{\partial \bar{F}}{\partial y} + \frac{\partial \bar{F}}{\partial z} = +3$$

$$\text{Flux} = \iiint_V 3\,dV.$$



The Blue outline shows the closed surface 'S'.

Applying Cylindrical Co-ordinate;

$x = r\cos\theta$; $y = r\sin\theta$; $z = z$.

From $x^2+y^2=4 \Rightarrow 0 \le r \le 2.$

$0 \le \theta \le 2\pi$

$4-r^2 \le z \le 5 \quad (\because x^2+y^2=r^2.)$

$dV = r\,dz\,dr\,d\theta.$

$$\iiint_V 3\,dV = 3\int_0^{2\pi}\int_0^{2}\int_{4-r^2}^{5} r\,dz\,dr\,d\theta. = \boxed{+36\pi}$$

Flux is +ve ($+36\pi$) $\Rightarrow$ More flux leaving the surface.

Solutions

it was given as $f(x,y)$ in Question.

Q7. (a) $L(x,y) = \boxed{f(a,b)} + \nabla f(a,b) \cdot (x-a, y-b)$

By definition of linear approximation (Taylor' expansion)

$$L(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b) \quad \text{---- (1)}$$

Now Gradient of $f(x,y)$ at (a,b);

$$\nabla f(a,b) = \left(\frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b)\right)$$

$$\nabla f(a,b) \cdot (x-a, y-b) = \frac{\partial f}{\partial x}(x-a) + \frac{\partial f}{\partial y}(y-b)$$

Substituting in Eqn (1)

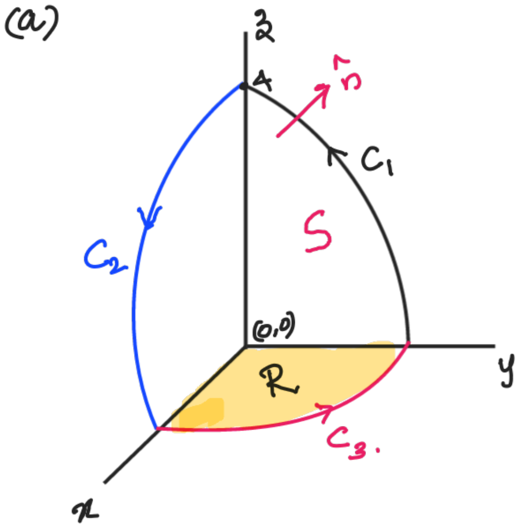
$$\Rightarrow L(x,y) = f(a,b) + \nabla f(a,b) \cdot (x-a, y-b) \quad \text{(QED)}$$

(b) * $\nabla f(a,b)$ points in the direction of maximum increase of $f(x,y)$
* It is perpendicular to the level curves of $f(x,y)=c$
* The magnitude $|\nabla f(a,b)|$ represents rate of increase in that direction.

Solutions

Q8.

(a)



$f(x,y) = 4 - x^2 - y^2$; $\bar{F} = yz\,\hat{i} - xy\,\hat{j} + \hat{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & -xy & 1 \end{vmatrix} = (x, y, -2z)$$

$$\hat{n}\,dS = (-f_x, -f_y, 1)\,dA = (2x, 2y, 1)\,dA$$

$$(\nabla \times \bar{F}) \cdot \hat{n}\,dS = (2x^2 + 2y^2 - 2z)\,dA$$

$$= \left(2x^2 + 2y^2 - 2(4 - x^2 - y^2)\right) dA$$

$$= 4(x^2 + y^2 - 2)\,dA$$

Limits of integration on R are; $0 \le r \le 2$; $0 \le \theta \le \frac{\pi}{2}$

By stoke's theorem

$$\oint \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\,dS$$

$$= 4 \iint_R (\underbrace{x^2 + y^2}_{r^2} - 2)\,dA = 4 \int_0^{\pi/2} \int_0^2 (r^2 - 2)\, r\,dr\,d\theta.$$

$$= 4 \cdot \frac{\pi}{2} \cdot \left[\frac{r^4}{4} - r^2 \Big|_0^2 \right]$$

$$= 2\pi (4 - 4) = 0$$

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\,dS = 0$$

Solutions

(b) $\bar{F} = (yz, -xz, 1) \Rightarrow \int_C \bar{F}\cdot d\bar{r} = \int_C (yz)dx - (xz)dy + 1\,dz$.

C_1 in yz-plane $\Rightarrow x=0$; $dx=0$

$y=t$, $z=4-t^2$

$dz=(-2t)dt$; t goes from 2 to 0.

$\downarrow$ $z=0$ $\quad$ $\downarrow$ $z=4$

$$\Rightarrow \int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} 1\,dz = \int_2^0 (-2t)\,dt = 4$$

C_2 in xz-plane ; $y=0$; $dy=0$; $x=t$, $z=4-t^2$

$dz=(-2t)dt$ $\quad$ t goes from $0 \rightarrow 2$

$(x=0)$ $\quad$ $(x=2)$

$$\Rightarrow \int_{C_2} \bar{F}\cdot d\bar{r} = \int_{C_2} 1\cdot dz = \int_0^2 (-2t)\,dt = -4$$

C_3 in xy-plane: $z=0$; $dz=0$

$$\int_{C_3} \bar{F}\cdot d\bar{r} = 0.$$

Now; $\oint_C \bar{F}\cdot d\bar{r} = \int_{C_1+C_2+C_3} \bar{F}\cdot d\bar{r} = 4+(-4)+0 = 0$.

Stoke's theorem is verified hence through this example.

Solutions

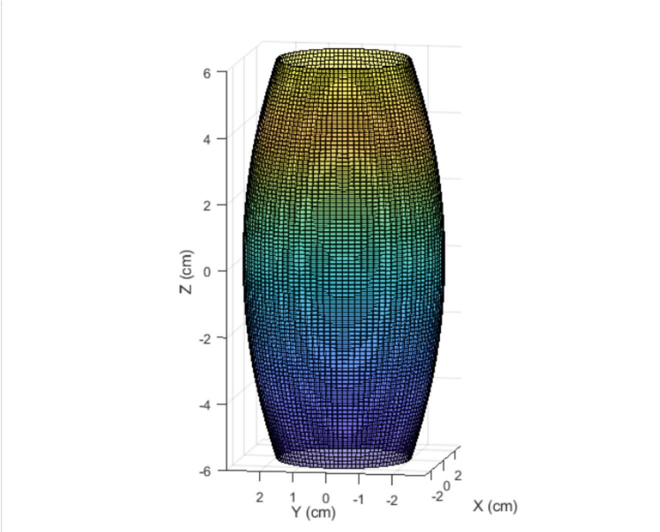
Q9.

$$r(z) = 3 - \frac{z^2}{36}$$

$$-6 \leq z \leq +6.$$

Using Cylindrical Co-ordinates.
Volume element is

$$dV = r\,dr\,d\theta\,dz$$



Total volume is;

$$V = \int_{\theta=0}^{2\pi} \int_{z=-6}^{+6} \int_{r=0}^{r(z)} r\,dr\,dz\,d\theta.$$

$$= \int_{\theta=0}^{2\pi} d\theta \int_{z=-6}^{+6} dz \int_{r=0}^{3-\frac{z^2}{36}} r\,dr = \int_0^{2\pi} d\theta \int_{-6}^{+6} \frac{\left(3-\frac{z^2}{36}\right)^2}{2}\,dz$$

$$\int_{-6}^{+6} \frac{\left(3-\frac{z^2}{36}\right)^2}{2}\,dz = \int_{-6}^{+6} \frac{9-\frac{z^2}{6}+\frac{z^4}{1296}}{2}\,dz$$

$$= \frac{1}{2}\int_{-6}^{+6} 9\,dz - \frac{1}{12}\int_{-6}^{6} z^2\,dz + \frac{1}{2592}\int_{-6}^{+6} z^4\,dz$$

$$V = 2\pi\left[\frac{1}{2}\int_{-6}^{+6} 9\,dz - \frac{1}{12}\int_{-6}^{+6} z^2\,dz + \frac{1}{2592}\int_{-6}^{+6} z^4\,dz\right]$$

$$V = 86.4\pi$$

Solutions

Since total volume = 24 cups, we need to solve for height 'h' where volume is 6 cups.

$$V_h = \frac{6}{24} V = \frac{1}{4} V = \frac{1}{4}(86.4\pi) = 21.6\pi.$$

$$21.6\pi = 2\pi \int_{-6}^{h} \frac{\left(3 - \frac{z^2}{36}\right)^2}{2} \, dz.$$

$$\Rightarrow \boxed{10.8 = \int_{-6}^{h} \frac{\left(3 - \frac{z^2}{36}\right)^2}{2} \, dz.}$$

This step (the final integral form) will be getting full marks.

Solutions

Solutions

Solutions

Solutions