EE1080/AI1110/EE2120 Probability, Quiz 2

3rd Feb, 2025

Max. Marks: 18. **Time:** 1 hour.

Instructions

- Please **write your roll number** and **course id** prominently in the first page of the answer sheet.
- No laptops, mobile devices etc. allowed.
- please write supporting arguments for any of the statements you make. any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer.

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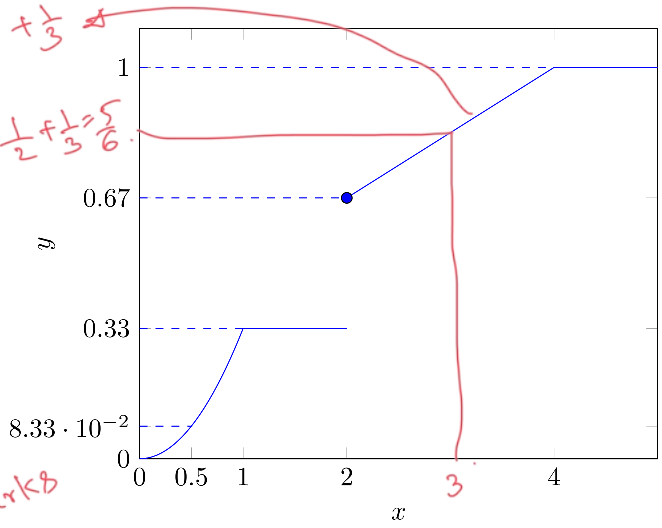
1. *(2)* Consider a sequence of five independent fair coin tosses. Define random variable X to be the number of times that heads is followed immediately by a tail. Find the pmf of X. (*Hint: Look for outcomes that result in* $X = 0$, $X = 2$).

2. *(2)* Given the following cumulative distribution function $F_X(.)$ of random variable X, please find the probabilities of the following events:



$y = \frac{1}{6}x + \frac{1}{3}$

$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

3

0.5 marks for each question.

   (a) $\{X = 2\}$ $\boxed{1/3}$

   (b) $\{X < 2\}$ $\boxed{1/3}$

   (c) $\{X = 2\} \cup \{0.5 \le X \le 1.5\}$ $\boxed{(1/3 - 1/12) + \frac{1}{3} = \frac{7}{12}}$

   (d) $\{X = 2\} \cup \{0.5 \le X \le 3\}$ $\boxed{\frac{5}{6} - \frac{1}{12} = \frac{9}{12} = \frac{3}{4}.}$

3. *(1+2)* BSC channel: Consider a binary symmetric channel with crossover (bit-flipping) probability α, shown below. Suppose the input X to the channel is 0 and 1 with probability p and $1-p$ respectively. Let Y be the output of the channel.

   (a) What is the probability that $Y \neq X$? $\boxed{\alpha}$

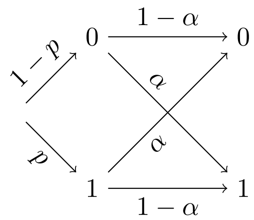


Figure 1: Binary Symmetric Channel (BSC)

   (b) Now consider the cascade of two such channels with crossover probabilities α and β respectively, as shown in figure below. Define the input X as before, and let Y be the output of the cascade. What is the probability that $Y \neq X$?

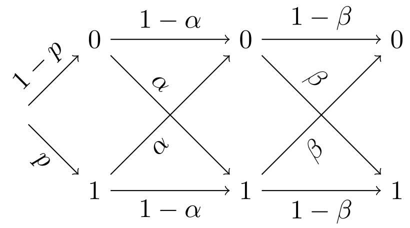


Figure 2: Cascading of two BSCs

4. *(2)* The number of times that a person contracts a cold in a given year is a Poisson random variable with parameter $\lambda = 5$. Suppose that a new wonder drug (based on large quantities of vitamin C) has just been marketed that reduces the Poisson parameter to $\lambda = 3$ for 75 percent of the population. For the other 25 percent of the population the drug has no appreciable effect on colds. If an individual tries the drug for a year and has 2 colds in that time, how likely is it that the drug is beneficial for him or her? (*Hint: Bayes rule*)

5. *(2+1)* On a given day, your golf score takes values in 101 to 110 with probability 0.1, independent of the other days. Determined to improve your score, you decider to play on three different days and declare as your score,

$$Y = \max(X_1, X_2, X_3)$$

i.e., the maximum across the three days.

   (a) Calculate the PMF of Y

   (b) By how much has your expected score improved as a result of playing on these days ?

$P(Y \neq X) = P(Y \neq X \mid X=1)\, P(X=1) + P(Y \neq X \mid X=0)\, P(X=0) = \alpha\,[P(X=1) + P(X=0)] = \alpha.$

① ⓐ $\Omega = \{H, T\}^5$. X : # times H is immediately followed by T

$\{X=0\} := \{$ HHHHH, THHHH, TTHHH, TTTHH, TTTTT, TTTTH $\}$

$\{X=2\} = \{$ HTHTH, HTHTT, HTHHT, HTTHT, HHTHT, THTHT $\}$

$|\Omega| = 2^5 = 32$

$P_X(0) = 6/32$, $P_X(2) = 6/32$

$P_X(1) = \frac{32-12}{32} = \frac{20}{32}$.

* Give partial marks (1) if they are able to identify some of the pmf's or events correctly.

3a $P(Y \neq X) = \alpha$.

3b $P(Y \neq X) = P(X=0)\left[\underbrace{\alpha}_{\text{flips first}}\underbrace{(1-\beta)}_{\text{no flip next}} + \underbrace{\beta}_{\text{flip}}\underbrace{(1-\alpha)}_{\text{no flip}}\right]$

$+ \left[\alpha(1-\beta) + \beta(1-\alpha)\right] P(X=1)$.

$= \alpha(1-\beta) + \beta(1-\alpha)$.

④ Let

$X_1 \sim \text{Poisson}(5)$

$X_2 \sim \text{Poisson}(3)$

$Z = \begin{cases} 1 & \text{w.p } \frac{3}{4} \\ 0 & \text{w.p } \frac{1}{4} \end{cases}$

$Z=1$ implies it is beneficial.

X: # times a person gets cold

$X \mid Z=1 \sim \text{Poisson}(3)$

$X \mid Z=0 \sim \text{Poisson}(5)$.

$$P(Z=1 \mid X=2) = \frac{P(X=2 \mid Z=1)\,P(Z=1)}{P(X=2 \mid Z=1)\,P(Z=1) + P(X=2 \mid Z=0)\,P(Z=0)}$$

$$= \frac{\left(e^{-3}\frac{3^2}{\cancel{2!}}\right)\frac{3}{\cancel{4}}}{\left[e^{-3}\frac{3^2}{\cancel{2!}}\left(\frac{3}{\cancel{4}}\right)\right] + \left(e^{-5}\frac{5^2}{\cancel{2!}}\left(\frac{1}{\cancel{4}}\right)\right)}$$

$$= \frac{e^{-3}\times 27}{e^{-3}\times 27 + e^{-5}\,25}$$

$$= \frac{27e^2}{e^2\,27 + 25}$$

5. $Y = \max(X_1, X_2, X_3)$.

$$P(Y = 100+i) = \left(\frac{1}{10}\right)^3 + \left(\frac{1}{10}\right)^2 \left({}^3C_1\right)\left(\frac{i-1}{10}\right) + \left({}^3C_2\right)\left(\frac{i-1}{10}\right)^2\left(\frac{1}{10}\right).$$

$$E[Y] = \sum_{i=1}^{10} (100+i)\, P(Y = 100+i)$$

$$= 100 + \sum_{i=1}^{10} i \left[\frac{1}{10^3} + \frac{1}{10^3} \times 3(i-1) + \frac{3(i-1)^2}{10^3}\right]$$

$$= 100 + \frac{1}{10^3} \underbrace{\sum_{i=1}^{10} \left(i + 3(i-1)\,i + 3(i-1)^2 i\right)}$$

$$= \sum_{i=1}^{10} 3i^2 - 2i + 3i(i^2 - 2i + 1)$$

$$= \sum_{i=1}^{10} 3i^2 - 2i + 3i^3 - 6i^2 + 3i$$

$$= \sum_{i=1}^{10} (3i^3 - 3i^2 + i)$$

$$= 3 \times \frac{(10)^2(11)^2}{4} - 3\frac{(10)(11)(21)}{6} + \frac{10 \times 9}{2}.$$

$$= 7965.$$

$$= \underline{\underline{107.965}}$$

$$E[X_i] = 100 + \sum_{i=1}^{10} \left(\frac{1}{10}\right) i = 100 + \frac{\cancel{10} \times 9}{2 \times \cancel{10}}$$

$$= 105.5$$

$\therefore$ Expectation increased by

$107.965 - 104.5 \simeq 2.5$

6. (*2+1+1*)Find the PMF of random variable X that represents the number of tosses need to see two heads. The coin is biased to see head with probability p and each toss is independent of the other. What is the expectation of X and how is X related to Geometric random variable ?

7. (*2*) There are 200 students in the 9th standard with 48 in section 1, 50 in section 2 and 52 in section three. Given section-wise average score in mathematics to be 60, 70, 65 and section wise variance to be 20, 30, 25, find the mean and variance of the mathematics score for the entire 9th standard.

6 X : # of tosses needed to see two heads.

$$P(X=k) = \binom{k-1}{1} p^2 (1-p)^{k-2}.$$

$$\_\ \_\ \_\ \_\ \underline{H}.$$

the last toss is a head. &
the first toss can be anywhere in first $(k-1)$ positions.

$$E[X] = E[X_1 + X_2] = E[X_1] + E[X_2].$$

where $X_1 \sim \text{Geometric}(p)$ &
$X_2 \sim \text{Geometric}(p)$

$$= \frac{2}{p}$$

150 students

48, 50, 52

Y : Section index

$$P_Y(y) = \begin{cases} \frac{48}{150} & y=1 \\ \frac{50}{150} & y=2 \\ \frac{52}{150} & y=3 \end{cases}$$

$$E[X|Y=y] = \begin{cases} 60 & y=1 \\ 70 & y=2 \\ 65 & y=3 \end{cases}$$

$$Var(X|Y=y) = \begin{cases} 20 & y=1 \\ 30 & y=2 \\ 25 & y=3 \end{cases}$$

$$E[X] = 60\times\frac{48}{150} + 70\times\frac{50}{150} + 65\times\frac{52}{150}$$

$$\approx 65$$

$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

$$E[\mathrm{Var}(X|Y)] = 20\times\frac{48}{150} + 30\times\frac{50}{150} + 25\times\frac{52}{150}$$

$$\approx 25$$

$$\mathrm{Var}[E[X|Y]] = 60^2\times\left(\frac{48}{150}\right) + 70^2\times\left(\frac{50}{150}\right) - (E[X])^2 + 65^2\times\left(\frac{52}{150}\right)$$

$$\simeq 16.$$

$$\Rightarrow \mathrm{Var}(X) = 16 + 25 = 41.$$