Lec 1 : Basics of Probability

Papoulis & Pillai

Probability is used to model uncertainity we see in nature. Roll a die, you can't be certain about the outcome.

A random experiment has uncertain outcomes and we want to know how likely it is to see (observe) a particular event.

Probability theory provides you a mathematical framework to analyze these random experiments.

$\boxed{\text{Kolomogorov}}$ ↓ Axioms based

Approaches to Probability

ⓐ Classic Definition.

A is an event and probability of event A,

$$P(A) = \frac{\#\ \text{outcomes that favour } A}{\text{Total number of outcomes that are possible.}}$$

Example: Roll two dice, find the probability that sum of the numbers is 9.

ⓐ Total number of outcomes $2, 3, \ldots 12$ = 11.

(1,1) (6,6)

outcomes that favour sum being 9 = 1

Probability $1/11$.

ⓑ Outcomes is actual two tuple that indicates the die outcome

(1, 1) (1, 2) . . . (1, 6)

Suggested reading

Bertrand's Paradox

Probability that a randomly chosen chord has size $> \sqrt{3}r$

$(2,1) \quad \cdot \quad \cdot \quad (2,6)$

$\vdots$

$(6,1) \quad - \quad - \quad \cdot \quad (6,6)$

\# of possible outcomes $= 36$

favourable outcomes $= 4$.

$(3,6)$
$(6,3)$
$(4,5)$
$(5,4)$

The probability $= 1/9 = \frac{4}{36}$

(b) Relative frequency approach.

Repeat an experiment n number of times and define n_A to be the \# times you observed event A.

$$P(A) = \lim_{n \to \infty} \frac{n_A}{n}.$$

Drawbacks to this approach

1. have to run this experiment infinite \# of times.

(c) Axiomatic approach to probability.

Probability model is defined through a probability space $(\Omega, \mathcal{F}, P)$.

Ω: Sample Space

$\mathcal{F}$: Event Space

P: Probability metric or function.

<u>Quick basics on sets</u>

A Set is a collection of objects that are referred to as elements.

$x \in S \qquad x$ is an element of S.

Made with Goodnotes

$$S = \{x_1, x_2, \dots x_n\}$$

$|S| = n$.

cardinality of a set i.e., total number of elements in it.

If cardinality is finite then S is a finite set

S is a countably infinite if it can be mapped using a one-to-one function to the set of natural numbers $\mathbb{N} = \{1, 2, 3, \dots\}$

Examples :

① $\mathbb{Z} = \{\underset{x_1}{0}, \underset{x_2}{1}, \underset{x_3}{-1}, \underset{x_4}{2}, -2, \dots\}$

$\mathbb{N} \subset \mathbb{Z}$.

$\phi : S \longrightarrow \mathbb{N}$

$0 \longrightarrow 1$

$1 \longrightarrow 2$

$-1 \longrightarrow 3$

$2 \longrightarrow 4$

$-2 \longrightarrow 5$

$\vdots$

$\phi(i) = 2i$ i is +ve

$= 2|i|+1$ $i \leq 0$

② Set of even numbers

$S = \{\underset{x_1}{2}, \underset{x_2}{4}, \underset{x_3}{6}, \dots\}$

③ Set of rational numbers $\mathbb{Q}$.

any number that can be represented in p/q form where $p, q \in \mathbb{Z}$, $q \neq 0$ are rational numbers

$$\begin{array}{llll}
\overset{1}{(1,1)} & \overset{2}{(1,2)}\;5 & \overset{6}{(1,3)}, & 7 \ldots \\
3\,(2,1) & (2,2) & (2,3)\;8, & \ldots \\
4\,(3,1) & (3,2)\;9 & \ldots & \\
\vdots & & & \\
10\,(4,1) & & &
\end{array}$$

If an infinite set is not countable then it is referred to as uncountable.

Example:

① $S = \{0,1\}^{\infty} \longrightarrow$ Set of infinite length binary sequences

Suppose S is countable then

$$S = \{x_1, x_2, x_3, \ldots\} = \{0,1\}^{\infty}.$$

Let $y \in \{0,1\}^{\infty}$ such that

$$y = (\bar{x}_{11}\ \bar{x}_{22}\ \bar{x}_{33}\ \bar{x}_{44} \ldots)$$

$$\bar{x}_{11} = x_{11} \oplus 1$$

cantor's diagonalization

$y \neq x_n$ for any $n \in \mathbb{N}$.

Example

diagonals picked

$x_1 = 0\,0011 \ldots$

$x_2 = 0\,1\,100 \ldots$

$x_3 = 01\,0\,00 \ldots$

$x_4 = 011\,0\,1 \ldots$

$\ddots$

$y = (00\,10 \ldots)$ diagonals flipped.

② $S = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

$x_1 \quad 0.\,1\,92 \ldots$

$x_2 \quad 0.\,2\,0\,3 \ldots$

$\vdots \qquad 0 \quad \ddots \quad 0$

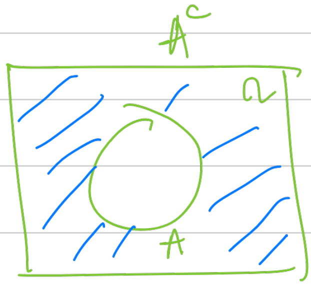
Can use similar proof to show $[0,1]$ is uncountable

Suggested Reading for uncountable/countable sets:
Principles of Mathematical Analysis by W. Rudin

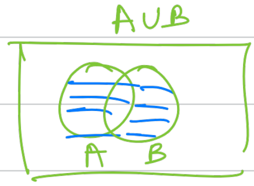
Set operations : Let us assume universe to be Ω.

$$A^c = \{x \in \Omega \mid x \notin A\}$$

$$\bigcup_{i=1}^{\infty} A_i = \{x \in \Omega \mid \exists\, i \in \mathbb{N} \text{ st } x \in A_i\}$$

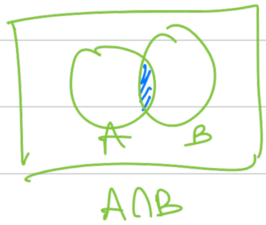


$$A \cup B = \{x \in A \text{ or } x \in B\}$$



$$A \cap B = \{x \in A \text{ and } x \in B\}$$

$$\bigcap_{i=1}^{\infty} A_i = \{x \in \Omega \mid \forall i \in \mathbb{N},\ x \in A_i\}$$



Commutative : $A \cup B = B \cup A$, $A \cap B = B \cap A$.

Distributive : $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Associative : $(A \cup B) \cup C = A \cup (B \cup C)$.

$(A \cap B) \cap C = A \cap (B \cap C)$.

Demorgan's laws

$$\left(\bigcap_{i=1}^{\infty} A_i\right)^c = \bigcup_{i=1}^{\infty} A_i^c$$

Excercise

$$\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

$(A^c)^c = A$, $A \cap A^c = \phi$, $A \cup A^c = \Omega$.

Probability Space $(\Omega, \mathcal{F}, P)$

Ω : Sample space and it is collection of all possible outcomes of your experiment

doesn't include all outcomes

✗ $\Omega = \{1, 2, 3, 5, 6\}$

Should include all possible outcomes of an experiment

Every outcome is distinct and mutually exclusive

✗ $\Omega = \{1, 2, 2 \text{ or } 3, 3, 4, 5, 6\}$

not mutually exclusive.

finite

(a). $\Omega = \{1, 2, 3, 4, 5, 6\}$

(b). $\Omega = \{(1,1)\ (1,2)\ \dots\ (1,6)\ \dots\ (6,6)\}$

(c). Roll a coin until you see a head.

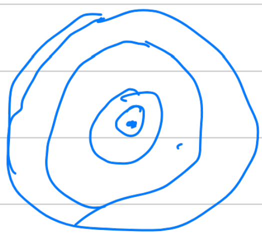
Countably infinite

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

uncountable

(d) $\Omega = [0, 1]$

(e) $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ $[0, 1]$



Bi-weekly exam options

Wed 9–10 ? x 5-6 x.

Thu 9–10 x

BT till 9 4 pm
free