Capacitors

- **Capacitors**: From the point of view of circuits, a capacitor can be defined as a two-terminal element whose voltage-current relation is given by

$$i(t) = \frac{dq(t)}{dt} \tag{1}$$

where $q(t)$ is the charge stored in the capacitor at time t and $i(t)$ is the current flowing through the capacitor. The charge $q(t)$ stored in the capacitor is given by

$$q(t) = Cv(t) \tag{2}$$

The capacitance C of a capacitor, measured in farads (F), is generally a positive quantity determined by its physical construction, specifically, the surface area of the plates, the distance between them, and the properties of the dielectric material. In the types of circuits we'll be analyzing, C is considered constant. Under this assumption, the voltage across the capacitor (also known as the induced voltage) $v_c(t)$ can be expressed as

$$v_c(t) = \frac{1}{C}\int i(t)dt \text{ or } i(t) = C\frac{dv_c(t)}{dt} \tag{3}$$

The circuit symbol for a capacitor is shown in Fig. 1. Note that the voltage-current relation of a capacitor (given by 3) is true when a passive sign convention is used.

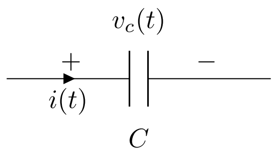

Figure 1: Circuit symbol for a capacitor.

- **(Optional) Brief overview of operation of a capacitor**: To understand the operation of a capacitor, let us consider the simple geometry of a parallel plate capacitor, as shown in Fig. 2. The capacitor is connected to a current source $i(t)$ and this results in a change $q(t)$ deposited in the top plate and $-q(t)$ in the bottom plate[1]. Since the plates are conductive, the charges are uniformly distributed and the resulting charge density on the plates is given by $\sigma = \frac{q(t)}{A}$, where A is the area of the plates.

[1] Can you explain why the charges are equal and opposite?

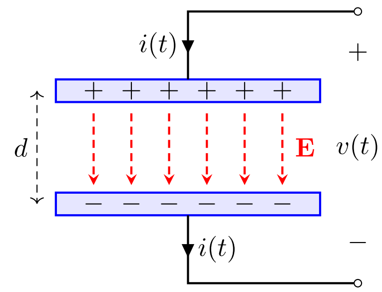

Figure 2: Geometry of a parallel plate capacitor. (2D view)

Coming to the physical construction of a capacitor, it consists of two conductive plates separated by a small distance d. Let the area of the parallel plates be such that $d << \sqrt{A}$. This ensures that the electric field between the plates is uniform and fringing effects at the edges of the plates can be neglected. Further, let us assume that the two parallel plates are lying on $z = d$ plane and $z = 0$ plane, respectively. The space between the plates is filled with a dielectric material of permittivity ϵ. The electric flux density $\mathbf{D}$ in the region between the plates can be computed using Gauss law as [2]

[2] The Gaussian surface is a rectangular box that encloses the top plate.

$$\mathbf{D} = \underbrace{\frac{\sigma}{2}(-\hat{z})}_{\text{due to } \sigma} + \underbrace{\frac{-\sigma}{2}(\hat{z})}_{\text{Due to } -\sigma} = -\sigma\hat{z} \tag{4}$$

The electric field intensity between the plates is given by

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = -\frac{\sigma}{\epsilon}\hat{z} = -\frac{q(t)}{\epsilon A}\hat{z} \tag{5}$$

The voltage across the plates is given by

$$v(t) = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = \frac{q(t)d}{\epsilon A} \tag{6}$$

Equation 6 indicates the voltage across the capacitor is directly proportional to the charge stored in it. The constant of proportionality is determined by the physical construction of the capacitor. The ratio of charge to voltage is defined as the capacitance C of the capacitor. For the case of a parallel plate capacitor, the capacitance is given by

$$C = \frac{\epsilon A}{d} \tag{7}$$

The time-varying current causes a time-varying charge $q(t)$ on the plates, which in turn results in a time-varying electric flux density $\mathbf{D}(t)$. According to Maxwell's equations, a time-varying electric flux density produces a time-varying current density $\mathbf{J}_d(t)$, known as the displacement current density, given by

$$\mathbf{J}_d(t) = \frac{\partial \mathbf{D}(t)}{\partial t} = -\frac{1}{A}\frac{dq(t)}{dt}\hat{z} = -\frac{i(t)}{A}\hat{z} \tag{8}$$

The displacement current (though not a real current) plays a crucial role in ensuring the continuity of current in circuits involving capacitors. The displacement current has the same magnitude as the conduction current $i(t)$ in the wires connected to the capacitor, but it flows in the opposite direction (from the bottom plate to the top plate). This is essential for maintaining the continuity of current in the circuit, as there is no physical conduction current flowing through the dielectric between the plates. Since the displacement current and the actual current in the wires have the same magnitude, we can use the same symbol $i(t)$ to represent both. This is a common practice in circuit analysis and helps simplify the analysis of circuits involving capacitors. The current through a capacitor is related to the voltage as

$$i(t) = i_d(t) = \frac{dq}{dt} = C\frac{dv(t)}{dt} \quad \text{(from Eq. 6)} \tag{9}$$

- **Computing capacitor voltage when the current is known**: The voltage across a capacitor can be determined if the current through the capacitor is known. Rearranging Eq. 3, we have

$$dv_c(t) = \frac{1}{C}i(t)dt \tag{10}$$

Integrating both sides with respect to time (note the limits on either side)

$$\int_{v_c(t_0)}^{v_c(t)} dv_c(\tau) = \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau \tag{11}$$

where t_0 is some initial time and τ is a dummy variable of integration. Evaluating the integrals, we get

$$v_c(t) = v_c(t_0) + \frac{1}{C}\int_{t_0}^{t} i(\tau)d\tau \tag{12}$$

- Example 1: Consider the circuit shown in Fig. 3. The switch is closed at $t = 0$ and the current source $v(t) = I$ (a constant) is applied to the circuit. If the capacitor voltage is 0 at $t = 0$, determine the expression for the capacitor voltage $v(t)$ for $t > 0$.

  When the switch is closed at $t = 0$, the current through the capacitor can be expressed as

  $$i_c(t) = Iu(t) = C\frac{dv_c(t)}{dt} \tag{13}$$

  where $u(t)$ is the unit step function. The voltage across the capacitor for $t > 0$ is thus given by

  $$\begin{aligned} v_c(t) &= v_c(0) + \frac{1}{C}\int_{0}^{t} i_c(\tau)d\tau \\ &= \frac{I}{C}t, \quad t > 0 \end{aligned} \tag{14}$$

  Note that the capacitor voltage increases linearly with time when a constant current is applied to the capacitor. Specifying the capacitor voltage at $t = 0$ ensures that the complete solution is obtained. Interestingly, note that the specified initial condition is $v_c(0) = 0$, which is same as the voltage across the capacitor just before the switch is closed (i.e., for $t < 0$). This is not a coincidence, and is a direct consequence of the continuity property of capacitor voltage, which we will discuss next.

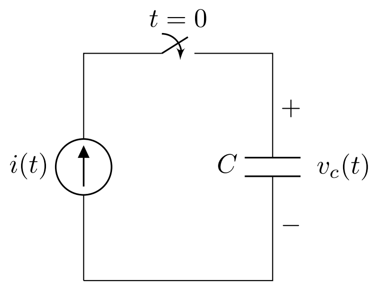

Figure 3: Circuit for Example 1.

- **Continuity of charge stored in a capacitor/capacitor voltage**: From the voltage-current relation of a capacitor (Eq. 3), we can see that an instantaneous change (like a step jump) in the capacitor voltage $v_c(t)$ would require an impulse current through the capacitor. However, in practical circuits, it is not possible to have an impulse signal, as this would require an infinite amount of energy. Therefore, the capacitor voltage $v_c(t)$ cannot change instantaneously in a real circuit. This implies that the charge associated $q(t)$ with a capacitor cannot change instantaneously, since $q(t) = Cv(t)$.

  In other words, the voltage across a capacitor is continuous, and any sudden change in voltage would require an infinite current through the capacitor, which is not physically realizable. This property of capacitors is often referred to as the capacitor voltage continuity property. It is an important consideration in circuit analysis and design, as it affects the transient response of circuits containing capacitors.

- Example 2: Consider the circuit shown in Fig. 3. The switch is closed at $t = 0$ and the current source $i(t) = I_m \cos(\omega t + \phi_i)$ is applied to the circuit.

When the switch is closed at $t = 0$, the current through the capacitor can be expressed as

$$i_c(t) = I_m \cos(\omega t + \phi_i)\, u(t) \tag{15}$$

The voltage across the capacitor for $t > 0$ is thus given by [3]

$$v_c(t) = \frac{I_m}{C} \int \cos(\omega t + \phi_i)\, u(t) d\tau = \frac{I_m}{\omega C} \sin(\omega t + \phi_i) + c, \quad t > 0 \tag{16}$$

where c is a constant. To determine the value of c, we use the voltage continuity principle. Since the capacitor voltage cannot change instantaneously, we have

$$v(0) = 0 \implies c = -\frac{I_m}{\omega C} \sin(\phi_i) \tag{17}$$

Substituting this value of c back into the expression for $v_c(t)$, we get

$$v_c(t) = \frac{I_m}{\omega C} \left[\sin(\omega t + \phi_i) + \frac{I_m}{\omega C} \sin(\phi_i) \right], \quad t > 0 \tag{18}$$

Unlike Example 1, where the initial current through the inductor was explicitly given, we were still able to determine the complete expression for the capacitor voltage in this case by using the continuity property of capacitor voltage.

[3] Note that in Example 2, we are using indirect integration, where we treat the constant c as an unknown and evaluate it later using the initial condition. This approach involves integrating the current over time and adding the constant to the solution. In contrast, in Example 1, we used direct integration, where the limits of integration were chosen to account for the initial condition automatically, eliminating the need for an explicit constant. Both methods are equivalent, but the handling of the constant differs between the two.

- **Sinusoidal (steady-state) response of an capacitor**: The steady-state sinusoidal response of an capacitor can be determined from the voltage-current relation of the capacitor. Let the current through the capacitor in sinusoidal stead-state be given by

$$i(t) = I_m \cos(\omega t + \phi_i) \tag{19}$$

The voltage across the capacitor is given by

$$v_c(t) = \frac{1}{C} \int i(t) dt = \frac{I_m}{\omega C} \sin(\omega t + \phi_i) \tag{20}$$

Note that in this case, the constant of integration is not explicitly included. This is because our focus is on the sinusoidal steady-state response of the capacitor, which describes the behavior after all transient effects have decayed.[4]

[4] To be clearly defined at a later point.

When considering only the steady-state response, we can assume the capacitor voltage is purely sinusoidal. Since both voltage and current are sinusoidal in this case, they can be conveniently represented using phasor notation as

$$\mathbf{I} = \frac{I_m}{\sqrt{2}} \angle \phi_i \quad \text{and} \quad \mathbf{V} = \frac{V_m}{\sqrt{2}} \angle \phi_v = \frac{I_m}{\omega C \sqrt{2}} \angle \left(\phi_i - \frac{\pi}{2} \right) = \frac{I}{\omega C} \angle \left(\phi_i - \frac{\pi}{2} \right) \tag{21}$$

This shows that in the sinusoidal steady-state, the current through a capacitor leads the voltage across it by $\frac{\pi}{2}$.

In sinusoidal steady-state circuit analysis, it is usually easier to work with phasors and complex impedances instead of time-domain waveforms. The impedance of a circuit element is defined as the ratio of its phasor voltage to its phasor current. For a capacitor, this impedance is expressed as

$$\mathbf{z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V\angle\phi_i}{I\angle\phi_i} = \frac{I\angle\left(\phi_i - \frac{\pi}{2}\right)}{\omega C}\frac{1}{I\angle\phi_i} = \frac{1}{j\omega C} \tag{22}$$

The impedance of a capacitor is purely imaginary and negative. It decreases as the frequency increases, while its phase angle remains fixed at $-\frac{\pi}{2}$. This means that for the same voltage, the current magnitude increases at higher frequencies. This characteristic makes capacitors useful in applications like filters and tuning circuits. The current through a capacitor in sinusoidal-steady state is given by

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{z}} = (j\omega C)\,\mathbf{V} \tag{23}$$

- **Power associated with a Capacitor**: The complex power associated with a capacitor (in sinusoidal steady-state) is given by

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (V\angle\phi_v)\left((j\omega C)\,\mathbf{V}\right)^* = -j\omega C\; V^2 \tag{24}$$

The active power associated with a capacitor is zero ($P = 0$), while the reactive power is negative ($Q = -\omega C V^2 < 0$). This indicates that a capacitor does not dissipate any real power; instead, it stores energy in its electric field during one half of the AC cycle and releases it back to the circuit during the other half. The instantaneous power associated with a capacitor is given by

$$s(t) = p(t) + q(t) = v(t)i(t) = -\omega C V^2 \sin(2\omega t) \tag{25}$$

The power factor of the capacitor ($\cos\theta = \frac{P}{|\mathbf{S}|}$) is zero. It is often a common practice to write the power factor of a capacitor as 0 leading, since the current leads the voltage by $\frac{\pi}{2}$.

- **Energy associated with a Capacitor**: To compute the energy associated with a capacitor, consider a simplified scenario, where in $i(t) = I_m \cos\omega t$. The voltage in steady-state is given by $v(t) = \frac{I_m}{\omega C}\sin\omega t$. The energy stored (since v and i are referred using passive sign convention) in the capacitor at any time t is given by

$$\begin{aligned} E(t) &= \int_0^t v(\tau)i(\tau)d\tau = \frac{I_m^2}{\omega C}\int_0^t \sin(\omega\tau)\cos(\omega\tau)d\tau \\ &= \frac{I_m^2}{2\omega C}\int_0^t \sin(2\omega\tau)d\tau = \frac{I_m^2}{4\omega^2 C}\left[1 - \cos(2\omega t)\right] \\ &= \frac{1}{2}CV_m^2\sin^2(\omega t) = \frac{1}{2}Cv^2(t) \end{aligned} \tag{26}$$

Note that the energy associated with the capacitor is always non-negative. This should not be misunderstood as energy continuously accumulating in the capacitor's electric field. The stored energy rises as the voltage across the capacitor increases, and it decreases when the voltage drops, meaning energy is returned to the circuit. The energy reaches its maximum when the voltage is at its peak value V_m, and it becomes zero when the voltage is zero. The maximum energy stored in the capacitor is expressed as $E_{max} = \frac{1}{2}CV_m^2$.