EE1080/AI1110/EE2120 Probability, EndTerm Exam

2nd May, 2025

Max. Marks: 50 **Time:** 3 hours.

Instructions

- You are welcome to answer all questions. The maximum marks will be capped at 50. That is, if your answers are evaluated to x marks then your score will be $\min\{50, x\}$.
- Please **write your roll number**, prominently in the first page of the answer sheet. Mark your **serial number** (used for attendance) at the top right corner of the answer sheet.
- No laptops, mobile devices, cheat sheets allowed. You may use calculators
- Please write supporting arguments for any of the statements you make. any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer.
- Highlight your final answer clearly for each part by drawing a box around it.

$\mathbb{R}$ is the set of real numbers

1. (12 marks) *Joint Density, Two functions of two random variables* X and Y have joint density function given by:

$$f_{X,Y}(x,y) = \frac{c}{x^2y^2} \quad x \geq 1,\ y \geq 2$$

   (a) (1) Find the value of c

   (b) (2) Find the marginals $f_X(x)$ and $f_Y(y)$

   (c) (1) Are X, Y independent ?

   (d) (2) Find the joint density function of U, V where $U = XY$ and $V = X/Y$. *(identify the ranges of values u, v.)*

   (e) (5) Find the marginals $f_U(u)$ and $f_V(v)$. *(identify the ranges of values u, v.)*

   (f) (1) Are U, V independent ?

2. (8 marks) *Moment Generating Functions (MGF) / Transforms*

   (a) (2) Let X be an Exp(λ) random variable with probability density function $f_X(x) = \lambda e^{-\lambda x}$ for $x \geq 0$. Find the MGF $M_X(s) = E[e^{sX}]$. Is it well-defined for every $s \in \mathbb{R}$?

   (b) (2) Find the m-th moment of X, $E[X^m]$ from the MGF $M_X(s)$

   (c) (3+1)Let $Y = \sum_{i=1}^{N} X_i$ where X_i are i.i.d Exp(λ) random variables, and N is Geometric (p) for some fixed p. Assume X_i and N are independent. Hint: MGF of N is given by:

$$M_N(s) = \frac{pe^s}{1-(1-p)e^s}$$

   i. Find the MGF of Y, $M_Y(s)$ (as a function of s, p, λ) ?

   ii. Find the pdf of Y (as a function of p, λ) ? *(Hint: Use MGF from part (i))*

3. (20 marks) A defective coin minting machine produces coins whose probability of heads is a random variable X with PDF

$$f_X(x) = \begin{cases} 2x & x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the number of heads seen on tossing n times with this coin. It is clear to see that conditioned over $X = x$, Y is a Binomial(n, x).

   (a) (2) Find $E[X]$ and Var(X).

   (b) (1+2+1) What is $P_{Y|X}(y|x)$? Find $P_Y(y)$ and $f_{X|Y}(x|y)$. *(Clearly identify the ranges of values x, y can take for each of them.)* Hints below:

   - Given two non-negative integers a, b:

$$\int_0^1 x^a(1-x)^b = \frac{a!\ b!}{(a+b+1)!}$$

   - $Z \sim$ Binomial(n, p) then

$$P_Z(k) = \binom{n}{k} p^k(1-p)^{n-k},$$
$$E[Z] = np, \text{Var}(Z) = np(1-p)$$

   (c) (1+2+2) Find $E[Y]$, Var(Y), Cov(X, Y) (Hint: try using total law of expectation, variance)

   (d) (2) Find the minimum mean square error (MMSE) estimate for the bias of the coin, X on observing the number of heads in n flips given by $Y = y$.

   (e) (2) Find the maximum likelihood (ML) estimate of X from $Y = y$.

   (f) (2) Find the maximum aposteriori probability (MAP) estimate of X from $Y = y$.

(g) (1) By staring at the MMSE estimator can you tell if the linear MMSE estimator is the same as MMSE estimator ?

(h) (2) Find the mean square error of the MMSE estimator (Hint: Use total law of variation for $\mathrm{Var}(X)$ or part (c).)

4. (8 marks) *Concentration Inequalities:* Let X be a Binomial $(n, p = 1/4)$ random variable. Bound $P(X > n/2)$ using (see hints of 3.b)

   (a) (1) Markov inequality,

   (b) (1) Chebyshev inequality,

   (c) (3) Chernoff bound. Hint: MGF of Binomial random variable is given by $M_X(s) = (pe^s + 1 - p)^n$

   (d) (1) What is the best upper bound among the three ?

   (e) (2) Use central limit theorem to approximate $P(X > n/2)$. Write this probability in terms of the CDF of standard normal random variable $\phi(.)$

5. (10 marks) *Convergence of Random Variables:*

   (a) (2) Indicate if the following statements are true or false:

      i. almost sure convergence implies convergence in mean square sense

      ii. almost sure convergence implies convergence in distribution

      iii. convergence in probability implies convergence in distribution

      iv. convergence in probability implies convergence in mean square sense

   (b) Let X_i's be i.i.d uniform$[0,1]$ random variables defined over sample space $\Omega = [0,1]$. Let $Y_n = \max(X_1, X_2, \cdots, X_n)$

      i. (4 marks) Find the CDF $F_{Y_n}(y)$, PDF $f_{Y_n}(y)$, $E[Y_n]$, $E[Y_n^2]$

      ii. (2) Does Y_n converge in distribution sense to 1 ?

      iii. (2) Does Y_n converge in probability sense to 1 ?

      iv. (2) Does Y_n converge in almost sure sense to 1 ?

      v. (2) Does Y_n converge in mean square sense to 1 ?

6. (6 marks) *Gaussian Random Vectors:* Let $X = (X_1, X_2, X_3)^T$ be a zero-mean 3-Gaussian vector with covariance matrix:

$$K_X = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 1 \end{bmatrix}$$

   (a) (2) Find the covariance matrix of

$$Y = AX, \text{ where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

   (b) (1) Find the joint density of $Y = (Y_1, Y_2)^T$

   (c) (3) Plot the points at which PDF evaluates to 80 percent of the maximum possible value.

Note that the pdf of a n-Gaussian random vector is given by:

$$f_X(x) = \frac{1}{(2\pi)^{\frac{n}{2}} (\det(K))^{\frac{1}{2}}} e^{-\frac{1}{2}x^T K^{-1} x}, \quad x \in \mathbb{R}^n$$