Network Theorems

- **Substitution Theorem**: Consider a network with an element connected between two nodes. We wish to analyze if it is possible to replace this element with a different one without effecting the response of the circuit. To understand this further, let a and b denote the two nodes where the element is connected. From an analysis point of view, the two terminals constitute a port and the rest of the circuit can be represented as a black box (as shown in Fig. 1). The properties of the rest of the network (or black box) can be represented by the open-circuit voltage V_{oc}, the short-circuit current I_{sc} at the port and the equivalent resistance R_{eq} [1].

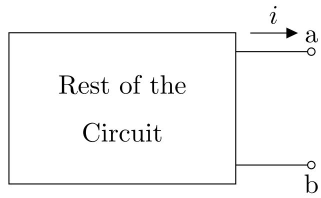

Figure 1: Representation of a circuit with port.

[1] It can be the input resistance looking into the port or the output resistance looking out of the port

Recall, that a graphical way to solve the circuit parameters at the port when a circuit element is connected at the port is to draw the terminal characteristics of the element and that of the port (also referred to as the load line analysis).

Although not commonly noted, when the terminal characteristics of the port and the element connected across it are plotted, the current reference chosen for the port is opposite to that of the element. Typically, when plotting the terminal characteristics of the port, the current reference is chosen to be out of the black box (or rest of the circuit) and into the element. This is opposite to the current reference chosen for the element, which is typically chosen to be out of the element and into the rest of the circuit [2]. For a linear circuit, the terminal characteristics of the rest of the circuit is a straight line with a negative slope, intersecting the voltage axis at V_{oc} and the current axis at I_{sc}.

[2] Recall the discussions on load line analysis

On the other hand, for the element connected across the port, the terminal characteristics can be drawn based on the type of element with current reference chosen to be from the rest of the circuit into the element. For example, if a resistor R is connected across the port, the terminal characteristics is a straight line with a positive slope, intersecting the origin. The intersection of the two terminal characteristics gives the voltage across and current through the element connected across the port.

The **key idea** behind the substitution theorem is to note that any modifications to the element connected across the port will not change the response of the circuit as along as the terminal characteristics of the new element intersects the terminal characteristics of the port at the same point as that of the original element. This is because, the voltage across and current through the element connected across the port will remain unchanged and hence, the rest of the circuit will not be affected.

To illustrate this further, consider a scenario wherein the circuit is resistive in nature (i.e., V_{oc} and I_{sc} are real values) and the element connected across the port is a resistor R. The typical terminal characteristics of the port and the resistor are shown in Fig. 2. The intersection of the two terminal characteristics

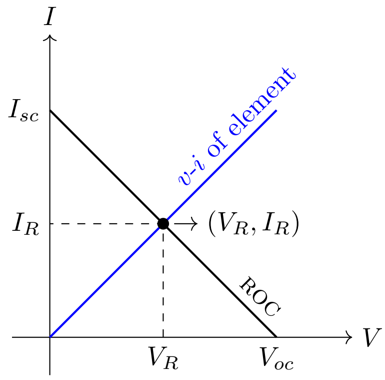

Figure 2: Terminal characteristics of the port and a resistor connected across it.

gives the voltage across and current through the resistor R. An observation into Fig. 2 indicates that the point of intersection (denoted by (V_R, I_R)) will remain unchanged if the new element has either of the following terminal characteristics:

- A straight line parallel to x-axis or voltage-axis and passing through (V_R, I_R). This corresponds to a voltage source of value V_R.
- A straight line parallel to y-axis or current-axis and passing through (V_R, I_R). This corresponds to a current source of value I_R.

Hence, replacing the resistor R with either a voltage source of value V_R or a current source of value I_R will not change the response of the circuit.

Although the concept is illustrated using a purely resistive circuit and a resistor connected across the port, the key idea is more general. It applies to any circuit configuration and any type of element connected at the port. The essential principle is to ensure that the new element's terminal characteristics intersect with those of the port at the same point as the original element. This guarantees that the overall behavior of the circuit remains unchanged.

More formally, the Substitution Theorem can be stated as follows:

Theorem 1. *Consider a linear circuit with an element connected between two nodes a and b. The response of the circuit is unaltered if the element is replaced by a voltage source of value equal to the voltage across the element or a current source of value equal to the current through the element.*

Although, the usefulness of the substitution theorem may not be immediately apparent, it finds significant application in deriving Thevenin's and Norton's Theorems.

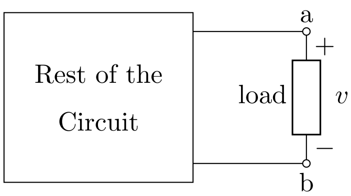

Figure 3: Linear circuit with load connected between nodes a and b.

- **Thevenin's Theorem**: Consider a linear circuit wherein the goal is to compute the response associated with a particular element (referred to as the load) connected between two nodes a and b (as shown in Fig. 3). In the first sight, this looks like a standard circuit analysis problem and the response can be computed using any of the circuit analysis techniques discussed so far. Lets make the problem a little more interesting by assuming that the load is variable in nature. That is, the value of the load can be changed and for each value of the load, we wish to compute the response associated with it. In such scenarios, it may not be efficient to compute the analysis from the scratch for each value of the load. Instead, it is desirable to have a method that allows for an efficient computation of the response for different load values. Thevenin's and Norton's Theorems provide such a method.

We will focus primarily on Thevenin's and Norton's theorems in the context of steady-state sinusoidal analysis[3]. However, it is important to note that these

[3] In the class, we have derived these theorems in the context of DC circuits

theorems are more general and can be applied to other types of circuit analysis as well, including transient analysis and DC analysis.

To understand the idea behind Thevenin's Theorem, we again consider the two nodes a and b where the load is connected as a port and represent the rest of the circuit as a black box (as shown in Fig. 3). From the subsitution theorem, it is known that the response of the circuit will remain unchanged if the load is replaced by either a voltage source of value equal to the voltage across the load or a current source of value equal to the current through the load. Let us replace the load (whose impedance is known) with a current source of value $\mathbf{I}$ (in Phasor-form) as shown in Fig. 4. The value of $\mathbf{I}$ is equal to the current through the load before replacement.

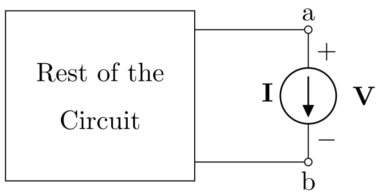

Figure 4: Load replaced by a current source of value i.

Now this might look a little strange, since it appears that we have complicated the problem by replacing a known load with an unknown current source. The only advantage is that, if we can somehow determine the voltage across the current source (denoted by $\mathbf{V}$ in Fig. 4), we can easily compute the response associated with any load connected across nodes a and b. This is because, once $\mathbf{V}$ is known, the current through any load connected across nodes a and b can be computed as $\mathbf{I} = \frac{\mathbf{V}}{\mathbf{z}_l}$, where $\mathbf{z}_l$ is the impedance of the load.

Since the rest of the circuit is assumed to be linear [4], the voltage $\mathbf{V}$ across the current source (or between terminals a and b) can be expressed as the sum of two components (based on superposition principle):

[4] This is a key assumption for the validity of Thevenin's and Norton's theorems

- The first component is the voltage between terminals a and b when the current source is set to 0 (i.e., open circuited). This voltage is referred to as the **open-circuit voltage** and is denoted by $\mathbf{V_{oc}}$.
- The second component is the voltage between terminals a and b when all independent sources in the rest of the circuit are set to 0 (i.e., voltage sources are short circuited and current sources are open circuited) and the current source of value $\mathbf{I}$ is active. Recall from port analysis that this voltage can be expressed as $-\mathbf{z}_{eq}\mathbf{I}$, where $\mathbf{z}_{eq}$ is the equivalent impedance looking into the port (i.e., terminals a and b) with all independent sources set to 0.

Hence, the voltage $\mathbf{V}$ across terminals a and b can be expressed as:

$$\mathbf{V} = \mathbf{V_{oc}} - \mathbf{z}_{eq}\mathbf{I} \tag{1}$$

Note that in (1), the current $\mathbf{I}$ represents the current through the load and $\mathbf{V}$ represents the voltage across the load. The voltage across the load and the current through are related as

$$\mathbf{V} = \mathbf{z}_l\mathbf{I} \tag{2}$$

From (1) and (2), we have

$$\mathbf{z}_l\mathbf{I} = \mathbf{V_{oc}} - \mathbf{z}_{eq}\mathbf{I} \tag{3}$$

Rearranging the above equation, we have

$$\mathbf{I} = \frac{\mathbf{V_{oc}}}{\mathbf{z}_{eq} + \mathbf{z}_l} \tag{4}$$

This indicates that the current through the load can be computed using only three parameters: the open-circuit voltage $\mathbf{V_{oc}}$, the equivalent impedance $\mathbf{z}_{eq}$ and the load impedance $\mathbf{z}_l$. Note that both $\mathbf{V_{oc}}$ and $\mathbf{z}_{eq}$ are independent of the load impedance $\mathbf{z}_l$. Hence, once these two parameters are computed for a given circuit, the current through any load connected across terminals a and b can be easily computed using (4).

Further, the voltage across the load is given by

$$\mathbf{V} = \mathbf{z}_l\left(\frac{\mathbf{V_{oc}}}{\mathbf{z}_{eq} + \mathbf{z}_l}\right) \tag{5}$$

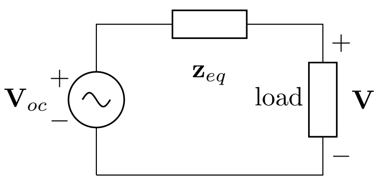

Figure 5: Thevenin equivalent circuit.

Equations (4) and (5) correspond to that of a voltage source of value $\mathbf{V_{oc}}$ in series with an impedance $\mathbf{z}_{eq}$ connected across the load (as shown in Fig. 5). This leads to the formal statement of Thevenin's Theorem:

Theorem 2. *Any linear circuit with two terminals can be replaced by an equivalent circuit consisting of a voltage source in series with an impedance connected across the terminals. The voltage source is equal to the open-circuit voltage at the terminals and the impedance is equal to the equivalent impedance looking into the terminals with all independent sources set to zero.*

The open-circuit voltage $\mathbf{V_{oc}}$ and the equivalent impedance $\mathbf{z}_{eq}$ are referred to as the **Thevenin voltage** and **Thevenin impedance** respectively.

- **Norton's Theorem**: Norton's theorem is closely related to Thevenin's theorem and provides an alternative way to represent a circuit as seen from a pair of terminals. The way the Norton's equivalent circuit is derived is similar to that of Thevenin's circuit, but instead of replacing the load with a current source, we replace it with a voltage source of value equal to the voltage across the load before replacement (as shown in Fig. 6).

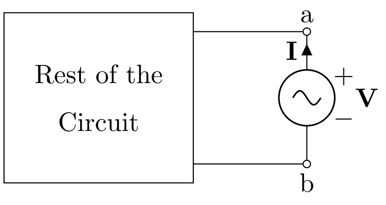

Figure 6: Load replaced by a voltage source of value v.

  Using superposition principle, the current $\mathbf{I}$ associated with the voltage source (or between terminals a and b) can be expressed as the sum of two components:

  - The first component is the current between terminals a and b when the voltage source is set to 0 (i.e., short circuited). This current is referred to as the **short-circuit current** and is denoted by $\mathbf{I_{sc}}$.
  - The second component is the current between terminals a and b when all independent sources in the rest of the circuit are set to 0 (i.e., voltage sources

are short circuited and current sources are open circuited) and the voltage source of value $\mathbf{V}$ is active. From port analysis, this current can be expressed as $-\frac{\mathbf{V}}{\mathbf{z}_{eq}}$, where $\mathbf{z}_{eq}$ is the equivalent impedance looking into the port (i.e., terminals a and b) with all independent sources set to 0.

Hence, the current $\mathbf{I}$ through terminals a and b can be expressed as:

$$\mathbf{I} = \mathbf{I_{sc}} - \frac{\mathbf{V}}{\mathbf{z}_{eq}} \tag{6}$$

Note that in (6), the voltage $\mathbf{V}$ represents the voltage across the load and the current $\mathbf{I}$ represents the current through the load. The voltage across the load and the current through are related as

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{z}_l} \tag{7}$$

From (6) and (7), we have

$$\frac{\mathbf{V}}{\mathbf{z}_l} = \mathbf{I_{sc}} - \frac{\mathbf{V}}{\mathbf{z}_{eq}} \tag{8}$$

Rearranging the above equation, we have

$$\mathbf{V} = \frac{\mathbf{I_{sc}}\mathbf{z}_{eq}\mathbf{z}_l}{\mathbf{z}_{eq} + \mathbf{z}_l} \tag{9}$$

This indicates that the voltage across the load can be computed using only three parameters: the short-circuit current $\mathbf{I_{sc}}$, the equivalent impedance $\mathbf{z}_{eq}$ and the load impedance $\mathbf{z}_l$. Note that both $\mathbf{I_{sc}}$ and $\mathbf{z}_{eq}$ are independent of the load impedance $\mathbf{z}_l$. Hence, once these two parameters are computed for a given circuit, the voltage across any load connected across nodes a and b can be easily computed using (9).

Further, the current through the load is given by

$$\mathbf{I} = \frac{\mathbf{I_{sc}}\mathbf{z}_{eq}}{\mathbf{z}_{eq} + \mathbf{z}_l} \tag{10}$$

Equations (9) and (10) correspond to that of a current source of value $\mathbf{I_{sc}}$ in parallel with an impedance $\mathbf{z}_{eq}$ connected across the load (as shown in Fig. 7). This leads to the formal statement of Norton's Theorem:

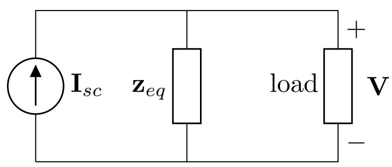

Figure 7: Norton equivalent circuit.

Theorem 3. *Any linear circuit with two terminals can be replaced by an equivalent circuit consisting of a current source in parallel with an impedance connected across the terminals. The current source is equal to the short-circuit current at the terminals and the impedance is equal to the equivalent impedance looking into the terminals with all independent sources set to zero.*

The short-circuit current $\mathbf{I_{sc}}$ and the equivalent impedance $\mathbf{z}_{eq}$ are referred to as the **Norton current** and **Norton impedance** respectively. Note that the Norton impedance is identical to the Thevenin impedance.

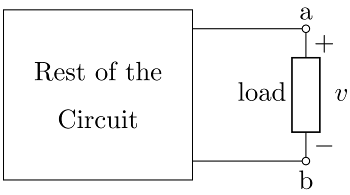

Figure 8: Linear circuit with load connected between nodes a and b.

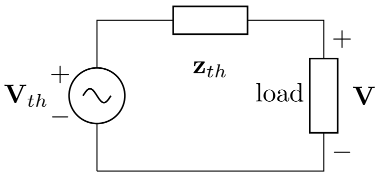

Figure 9: Rest of the circuit replaced by its Thevenin equivalent circuit.

- **Condition for Maximum Power Transfer**: Consider a linear circuit with a load connected between two nodes a and b (see Fig. 8). The goal is to determine the value of the load impedance $\mathbf{z}_l$ that maximizes the power delivered to it.

  Using Thevenin's theorem, the rest of the circuit (i.e., excluding the load) can be replaced by its Thevenin equivalent circuit (as shown in Fig. 9). We will consider three scenarios and determine the condition for maximum power transfer in each case:

  – **Case A: Resistive Circuit with Resistive Load**: In this case, both the Thevenin impedance $\mathbf{z}_{th}$ and the load impedance $\mathbf{z}_l$ are purely resistive in nature. Let $\mathbf{z}_{th} = R_{th}$ and $\mathbf{z}_l = R_l$. The power delivered to the load is given by

  $$P = |\mathbf{I}|^2 R_l = \left(\frac{V_{th}}{R_{th} + R_l}\right)^2 R_l \tag{11}$$

  The value of R_l that maximizes [5] the power delivered to the load (say R_l^*) is given by

  $$R_l^* = \left\{ R_l : \frac{dP}{dR_l} = 0 \right\} \tag{12}$$

  Taking the derivative of P with respect to R_l and setting it to zero results in

  $$R_l^* = R_{th} \tag{13}$$

  Thus, maximum power is transferred to the load when the load resistance is equal to the Thevenin resistance of the rest of the circuit. It is interesting to note that under the condition of maximum power transfer, the voltage across the load is half the open circuit voltage and this may not be desirable in practical scenarios.

  – **Case B: Complex Circuit with Resistive Load**: In this case, the rest of the circuit comprises of complex impedances while the load is purely resistive in nature. Let $\mathbf{z}_{th} = R_{th} + jX_{th}$ and $\mathbf{z}_l = R_l$. The power delivered to the load is given by

  $$P = |\mathbf{I}|^2 R_l = \left(\frac{V_{th}}{\sqrt{(R_{th} + R_l)^2 + X_{th}^2}}\right)^2 R_l \tag{14}$$

  The value of R_l that maximizes[5] the power delivered to the load (say R_l^*) is given by

  $$R_l^* = \left\{ R_l : \frac{dP}{dR_l} = 0 \right\} \tag{15}$$

  Taking the derivative of P with respect to R_l and setting it to zero results in

  $$R_l^* = |\mathbf{z}_{th}| = \sqrt{R_{th}^2 + X_{th}^2} \tag{16}$$

  Thus, maximum power is transferred to the load when the load resistance is

[5] The first-order condition is true for minima as well. Verify with the second order condition at optimum value to confirm that it corresponds to a maximum.

[5]

equal to the magnitude of the Thevenin impedance of the rest of the circuit[6].

– **Case C: Complex Circuit with Complex Load**: In this case, both the rest of the circuit and the load comprise of complex impedances. Let $\mathbf{z}_{th} = R_{th} + jX_{th}$ and $\mathbf{z}_l = R_l + jX_l$. The power delivered to the load is given by

$$P = |\mathbf{I}|^2 R_l = \left(\frac{V_{th}}{\sqrt{(R_{th} + R_l)^2 + (X_{th} + X_l)^2}} \right)^2 R_l \tag{17}$$

The values of R_l and X_l that maximizes[7] the power delivered to the load (say R_l^* and X_l^*) are given by

$$(R_l^*, X_l^*) = \left\{ (R_l, X_l) : \frac{dP}{dR_l} = 0, \frac{dP}{dX_l} = 0 \right\} \tag{18}$$

Taking the derivatives of P with respect to R_l and X_l and setting them to zero results in

$$R_l^* = R_{th}, \quad X_l^* = -X_{th} \tag{19}$$

Thus, maximum power is transferred to the load when the load impedance is equal to the complex conjugate of the Thevenin impedance of the rest of the circuit[8].

[6] What will be the voltage across the load under this condition?

[7] The first-order condition i.e., $\nabla P = 0$ is true for minima as well. To verify that it corresponds to a maximum, the second order condition must be verified. The second order condition is: The Hessian matrix must be negative definite at the optimum point.

[8] What will be the voltage across the load under this condition?