EE2100: Matrix Analysis

Review Notes - 37

Topics covered :

1. Condition Number of a Matrix

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1. Let $\mathbf{A} \in \mathcal{R}^{n \times n}$ be a non-singular matrix. Accordingly, it is possible to express $\mathbf{A}$ (using singular value decomposition) as

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T \tag{1}$$

where $\mathbf{\Sigma} \in \mathcal{R}^{n \times n} = \mathbf{Diag}(\sigma_1, \cdots, \sigma_n)$, $\mathbf{U}, \mathbf{V} \in \mathcal{R}^{n \times n}$ are orthonormal matrices (i.e., $\mathbf{U}^{-1} = \mathbf{U}^T$ and $\mathbf{V}^{-1} = \mathbf{V}^T$). The inverse of $\mathbf{A}$ (since, it is presumed to be non-singular) can be computed as

$$\mathbf{A} = \mathbf{V}\mathbf{\Sigma}^{-1}\mathbf{U}^T \tag{2}$$

where $\mathbf{\Sigma}^{-1} = \mathbf{Diag}\left(\frac{1}{\sigma_n}, \cdots, \frac{1}{\sigma_1}\right)$. Thus, the singular values of $\mathbf{A}^{-1}$ are $\left(\frac{1}{\sigma_n}, \cdots, \frac{1}{\sigma_1}\right)$.

2. Condition number of a Matrix: Consider the scenario of solving system of linear equations $\mathbf{Ax} = \mathbf{b}$. While solving system of linear equations (using any programming language) one can potentially encounter many errors (most noticeably, errors due to round off). Condition number of a matrix quantifies the maximum possible error (relative) in solution $\mathbf{x}$ caused due to errors (relative) in $\mathbf{b}$. Further, condition number of a matrix can also be thought of as an indicator for the proximity of a matrix to be singular (recollect the arguments covered in the class).

3. Let $\mathbf{\Delta x}$ denote the error in the solution (for the system of linear equations $\mathbf{Ax} = \mathbf{b}$) caused due to error in b (denoted by $\Delta\mathbf{b}$). Accordingly,

$$\mathbf{A}\left(\mathbf{x} + \Delta x\right) = \mathbf{b} + \Delta\mathbf{b} \implies \mathbf{Ax} = \mathbf{b} \text{ and } \mathbf{A}\Delta\mathbf{x} = \Delta\mathbf{b} \tag{3}$$

The condition number of the matrix (denoted by $\kappa(\mathbf{A})$) is the maximum possible value (or the upper bound) of the ratio $\frac{\|\Delta\mathbf{x}\|/\|\mathbf{x}\|}{\|\Delta\mathbf{b}\|/\|\mathbf{b}\|}$.

$$\frac{\|\Delta\mathbf{x}\|}{\|\Delta\mathbf{b}\|}\frac{\|\mathbf{b}\|}{\|\mathbf{x}\|} \ \leq \ \kappa(\mathbf{A}) \tag{4}$$

4. The norm of a matrix $\mathbf{A}$ (square, in the current context) is the largest singular value (σ_1).

$$\|\mathbf{A}\| \ = \ \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} = \sigma_1 \tag{5}$$

According to the definition of norm (given by (5)), the upper bound on the ration $\frac{\|\mathbf{Ax}\|}{\|\mathbf{x}\|} = \frac{\|\mathbf{b}\|}{\|\mathbf{x}\|}$ (using (3)) is given by σ_1 (or $\|\mathbf{A}\|$) i.e.,

$$\frac{\|\mathbf{b}\|}{\|\mathbf{x}\|} \ \leq \ \sigma_1 \text{ (positive, by convention).} \tag{6}$$

Further, if $(\sigma_1, \cdots \sigma_n)$ denote the singular values of $\mathbf{A}$ (note that, if $\mathbf{A}$ is singular, the singular values will be 0), the singular values of $\mathbf{A}^{-1}$ are $\left(\frac{1}{\sigma_n}, \cdots, \frac{1}{\sigma_1}\right)$, with $\frac{1}{\sigma_n}$ being the largest singular value.

5. The upper bound on the ratio of $\frac{\|\Delta\mathbf{x}\|}{\|\Delta\mathbf{b}\|}$ can be computed using the norm of $\mathbf{A}^{-1}$. Accordingly,

$$\frac{\|\Delta\mathbf{x}\|}{\|\Delta\mathbf{b}\|} \leq \underbrace{\|\mathbf{A}^{-1}\|}_{\frac{1}{\sigma_n}} \tag{7}$$

Using (4) and (6)-(7), the condition number of a matrix is given by

$$\kappa(\mathbf{A}) = \frac{\sigma_1}{\sigma_n} = \|\mathbf{A}\| \, \|\mathbf{A}^{-1}\| \tag{8}$$

Note, that if $\mathbf{A}$ is singular, its condition number is infinite.