

Convergence of Random Variables

Ref: Principles of Math. Analysis by Rudin

① Convergence of sequence of real numbers.

For a sequence of real numbers a_1, a_2, \dots

$= (a_n)_{n \geq 1}$, $(a_n)_{n \geq 1}$ is said to converge to a if the following holds:

$\forall \epsilon > 0, \exists N$ st.

$$|a_n - a| \leq \epsilon \quad \forall n \geq N.$$

Example: ① $a_n = \frac{1}{n}$, $\lim_{n \rightarrow \infty} a_n = 0$.

let $a = 0$,

for $\epsilon > 0$.

$$|a_n| \leq \epsilon \quad \forall n \geq \lceil \frac{1}{\epsilon} \rceil$$

$$n \geq \lceil \frac{1}{\epsilon} \rceil \Rightarrow n \geq \frac{1}{\epsilon}$$

$$\Rightarrow \frac{1}{n} \leq \epsilon$$

② $a_n = (-1)^n$ doesn't converge.

1 -1 1 -1 ...

Let $\epsilon = \frac{1}{100}$, N be some number

Suppose $|(-1)^n - a| < \frac{1}{100}$
then $|-1 - a| > \frac{1}{100}$

We'll show that $\exists \epsilon$ st $\forall N \in \mathbb{N}$, $\exists n \geq N$ st

$$|a_n - a| > \epsilon.$$

Made with Goodnotes there are infinitely many n st $|a_n - a| > \epsilon$ for some ϵ .

$$(U a_n)^\complement = \cap A_n^\complement$$

② Some properties

① $a_n \rightarrow a, b_n \rightarrow b$ then $a_n + b_n \rightarrow a+b$
and $a_n - b_n \rightarrow a-b$.

② $b_n \leq a_n \leq u_n$ then $\lim b_n \leq \lim a_n \leq \lim u_n$

③ Any monotonic sequence that is bounded converges.

Example: $a_n = \frac{1}{n} \geq 0$ ② let $a_1 \geq a_2 \geq \dots \geq 0$. then $\lim a_n = a^*$
exists

$a_n = 1 - \frac{1}{n} \leq 1$. ③ Let $a_1 \leq a_2 \leq \dots \leq u$ then $\lim a_n = a^*$
exists.

④ convergence of series (sum sequence).

$$S_n = \sum_{i=1}^n a_i, \quad \sum_{i=1}^{\infty} a_i \text{ is finite or not.}$$

$$S_1, S_2, S_3, \dots$$

$$\text{If } S_n \rightarrow s = \sum_{i=1}^{\infty} a_i < \infty.$$

Lemma 1: Suppose S_n converges to s , then

$$\hat{S}_n = \sum_{i=n}^{\infty} a_i \rightarrow 0.$$

Proof:

$$\hat{S}_n = s - S_{n-1}$$

$$\lim_{n \rightarrow \infty} \hat{S}_n = \lim_{n \rightarrow \infty} (s - S_{n-1}) = s - \lim_{n \rightarrow \infty} S_{n-1} \\ = s - s = 0.$$

Lemma 2: Suppose S_n converges to s , then $a_n \rightarrow 0$.

Proof:

$$a_n = s_n - s_{n-1}$$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} s_n - \lim_{n \rightarrow \infty} s_{n-1} = s - s = 0.$$

Suppose $S_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$ S_n converges for $p > 1$
and diverges for $p \leq 1$.

We'll use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

$$\sum \frac{1}{n^2} \leq \sum \frac{1}{n(n-1)}$$

$$= 1 + \frac{1}{2} \left(\frac{1}{1} - \frac{1}{2} \right) \leq 2.$$

② Pointwise convergence of random variables

② Random Processes for Engineers, Bruce Hayek.

Let x_1, x_2, \dots be random variables defined over the same sample space (Ω, \mathcal{F}, P) . $x_i : \Omega \rightarrow \mathbb{R}$.

For every $\omega \in \Omega$ if $x_n(\omega) \rightarrow x(\omega)$. i.e.,

$$\lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)$$

then x_1, x_2, \dots converge to R.V x pointwise.

Example: $x_n(\omega) = \omega^n \quad \forall \omega \in [0, 1] = \Omega$.

$$\lim_{n \rightarrow \infty} x_n(\omega) = \lim_{n \rightarrow \infty} \omega^n = \begin{cases} 0 & \omega \in [0, 1), \\ 1 & \text{if } \omega = 1 \end{cases}$$

Say $x(\omega) = \begin{cases} 0 & \omega \in [0, 1) \\ 1 & \text{otherwise} \end{cases}$

then $(x_n)_{n \geq 1}$ converges to x pointwise.

③ Almost-Sure Convergence:

x_1, x_2, \dots, x are R.Vs defined over same (Ω, \mathcal{F}, P) then

$x_n \rightarrow x$ almost surely iff

$$P\left(\left\{\omega : \lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)\right\}\right) = 1.$$

Example a $x_n(\omega) = \omega^n, x(\omega) = 0. \quad \forall \omega \in [0, 1]$.

$$\left\{\omega : \lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)\right\} = [0, 1].$$

$$P\left(\left\{\omega : \lim_{n \rightarrow \infty} x_n(\omega) = x(\omega)\right\}\right) = P(\{w \in [0, 1]\}) = 1.$$

b) min of uniform R.Vs. let x_1, x_2, \dots be iid

(Bertsekas). uniform(0,1) R.Vs and $T_n = \min\{x_1, x_2, \dots, x_n\}$

$$T_n(\omega) = \min\{x_1(\omega), \dots, x_n(\omega)\}$$

Can define R.V Υ as

$$\Upsilon(\omega) = \lim_{n \rightarrow \infty} \Upsilon_n(\omega).$$

$$= \inf_n \Upsilon_n(\omega)$$

$$\{X_i > \epsilon\} \cap \dots \cap \{X_i > \epsilon\}$$

The limit exist because

$$\Upsilon_1(\omega) \geq \Upsilon_2(\omega) \geq \dots$$

$$\geq 0$$

$$\Upsilon(\omega) \leq \Upsilon_n(\omega) \quad \forall n$$

i.e., $(\Upsilon_n(\omega))$ is a monotonically decreasing sequence lower bounded by 0,

If $\Upsilon > \epsilon \Rightarrow \Upsilon_n > \epsilon$

$$P(\Upsilon > \epsilon) \leq P(\Upsilon_n > \epsilon) = P\left(\bigcap_{i=1}^n \{X_i > \epsilon\}\right)$$

$$= \prod_{i=1}^n P(X_i > \epsilon) = (1 - \epsilon)^n$$

$$P(\Upsilon > \epsilon) \leq \lim_{n \rightarrow \infty} (1 - \epsilon)^n = 0.$$

$$\Rightarrow P(\Upsilon > \epsilon) = 0 \text{ for } \epsilon > 0.$$

$$\Rightarrow P(\Upsilon = 0) = 1 \text{ as } \Upsilon \text{ is non-negative R.V.}$$

$$P\left(\{\omega : \lim_{n \rightarrow \infty} \Upsilon_n(\omega) = 0\}\right) = P(\{\omega : \Upsilon(\omega) = 0\})$$

as we defined $\Upsilon(\omega)$ to be 1.

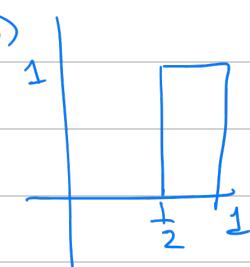
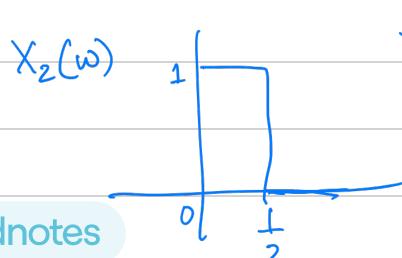
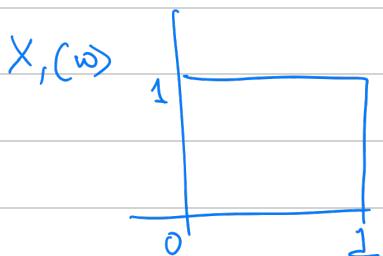
$$\Rightarrow \Upsilon_n \rightarrow 0 \text{ a.s.}$$

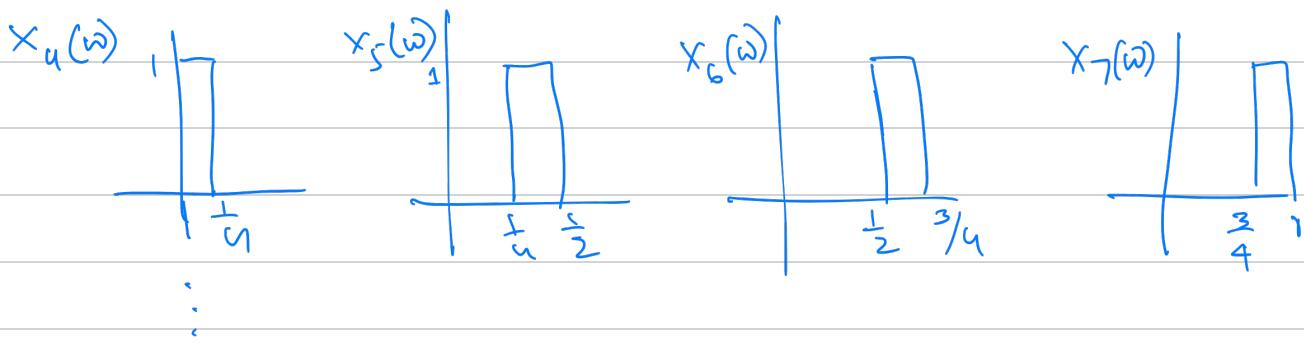
(almost surely). $\lim_{n \rightarrow \infty} \Upsilon_n(\omega)$

Lecture 31

c) Moving, shrinking rectangles

$$\Omega = [0, 1].$$

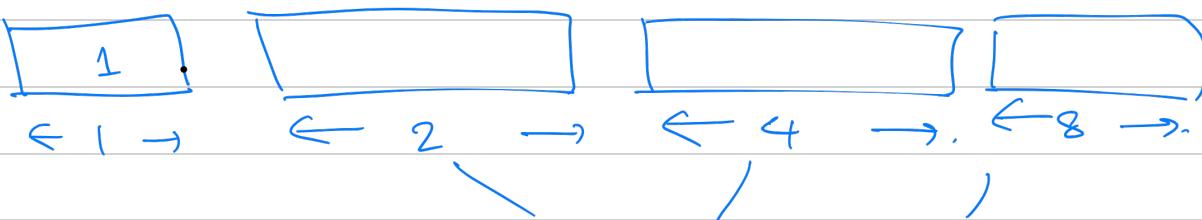




$$(x_n(\omega))_{n \geq 1} = (1 \ 1 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \dots)$$

This sequence doesn't converge as 1 appears "infinitely often"

$$(x_n(\omega))_{n \geq 1} \quad x_1(\omega) \quad x_2(\omega) \quad x_3(\omega) \quad x_4(\omega) \dots x_7(\omega) \dots$$



exactly one element is 1 and rest are zeroes.

This sequence doesn't converge as 1 appears infinitely often.

$\Rightarrow x_n$ doesn't converge to 0 a.s
 " " " any R.V a.s.

④ Convergence in probability: Let X_1, X_2, \dots, X be R.V.s defined over sample space Ω , then $X_n \rightarrow X$ in probability if $\forall \epsilon > 0$

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0.$$

/ Examples @ Moving, Shrinking Rectangles.

$$\text{Let } k = \lfloor \log_2 n \rfloor, \quad i = n - 2^k + 1. \Rightarrow n = 2^k + i$$

$$X_n(\omega) = \begin{cases} 1 & \omega \in \left[\frac{(i-1)}{2^k}; \frac{i}{2^k}\right] \\ 0 & \text{Otherwise} \end{cases}$$

$$\text{If } n=1, k=0, i=1 \quad X_1(\omega) = 1 \quad \omega \in [0, 1]$$

$$n=2, k=1, i=1$$

$$X_2(\omega) = 1 \quad \omega \in [0; \frac{1}{2}]$$

$$n=3, k=1, i=2$$

$$X_3(\omega) = 1 \quad \omega \in [\frac{1}{2}, 1].$$

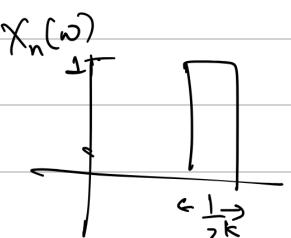
⋮

$$X(\omega) = 0 \quad \omega \in [0, 1].$$

For some $\epsilon > 0$, let $k = \lfloor \log_2 n \rfloor$

$$P(|X_n - X| > \epsilon) = \frac{1}{2^k}.$$

$$\leq \frac{2}{n}.$$



$$\log_2 n - 1 \leq k \leq \lfloor \log_2 n \rfloor$$

$$\frac{n}{2} \leq 2^k \leq n. \therefore$$

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) \leq \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

$$\Rightarrow \lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0$$

$X_n \rightarrow X$ in probability.

(b) Weak Law of Large numbers.: X_i 's are iid with mean μ & variance σ^2

$$S_n = \sum_{i=1}^n X_i, \quad E[S_n] = \mu, \quad \text{Var}(S_n) = \frac{\sigma^2}{n}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \frac{\text{Var}\left(\frac{S_n}{n}\right)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2}$$

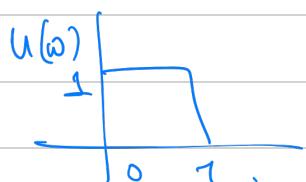
$$\lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \lim_{n \rightarrow \infty} \frac{\sigma^2}{n\epsilon^2} = 0.$$

$$\Rightarrow \lim_{n \rightarrow \infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0.$$

$\Rightarrow \frac{S_n}{n} \rightarrow \mu$ in probability sense.

(c) $X_n(\omega) = U(\omega)$ $\omega \in [0, 1]$

$$X(\omega) = 1 - U(\omega).$$



does X_n converge to X in probability sense?

$$P(|X_n - X| > \epsilon) = P(|2U - 1| > \epsilon).$$

$$= P(2U - 1 > \epsilon) + P(2U - 1 < -\epsilon)$$

$$= P(U > \frac{1+\epsilon}{2}) + P(U < \frac{1-\epsilon}{2}).$$

$$= \left[1 - \left(\frac{1+\epsilon}{2} \right) \right] + \cdot \left(\frac{1-\epsilon}{2} \right)$$

$$= (1-\epsilon) \cdot$$

$\Rightarrow X_n$ doesn't converge to X in probability.

"We'll define convergence in distribution which is further weaker form of convergence".



Strong law of large numbers (SLLN)

$$\frac{S_n}{n} \xrightarrow{\text{a.s.}} \mu \quad \text{where } S_n = \sum_{i=1}^n X_i$$

and X_i 's are iid

with mean μ &

We'll also assume $E[S_n^4] < \infty$.

Variance σ^2 .

We want to show

(Bertsekas)

$$\lim_{n \rightarrow \infty} F_{Z_n}(z) = F_Z(z)$$

Moment Generating function

$$\lim_{n \rightarrow \infty} M_{Z_n}(s) = M_Z(s)$$



Follows from inversion property.

$$M_{Z_n}(s) = E[e^{s Z_n}]$$

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sqrt{n}\sigma}$$

$$\bar{X}_i = X_i - \mu$$

$$= E\left[e^{\frac{s}{\sqrt{n}\sigma}(\sum_{i=1}^n (X_i - \mu))}\right]$$

$$= E\left[\prod_{i=1}^n e^{\frac{s}{\sqrt{n}\sigma} \bar{X}_i}\right]$$

Independence \hookrightarrow

$$= \prod_{i=1}^n E\left[e^{\frac{s}{\sqrt{n}\sigma} \bar{X}_i}\right]$$

$$= \frac{\sum_{i=1}^n (X_i - \mu)}{\sqrt{n}\sigma}$$

MGF defn \hookleftarrow

$$= \prod_{i=1}^n M_{\bar{X}_i}\left(\frac{s}{\sqrt{n}\sigma}\right).$$

identical \hookleftarrow

$$\bar{X}_i \text{ s. } = M_{\bar{X}_i}\left(\frac{s}{\sqrt{n}\sigma}\right)^n$$

$$\log M_{Z_n}(s) = n \log M_{\bar{X}_i}\left(\frac{s}{\sqrt{n}\sigma}\right).$$

$$\lim_{n \rightarrow \infty} \log M_{Z_n}(s) = \lim_{n \rightarrow \infty} \frac{\log M_{\bar{X}_i}\left(\frac{s}{\sqrt{n}\sigma}\right)}{\frac{1}{\sqrt{n}}}.$$

Let $L(s) = \log M_{\bar{X}_i}(s)$. $= \lim_{n \rightarrow \infty} \frac{L\left(\frac{s}{\sqrt{n}\sigma}\right)}{\frac{1}{\sqrt{n}}}$

$$L'(s) = \frac{1}{M_{\bar{X}_i}(s)} M'_{\bar{X}_i}(s)$$

$$= \lim_{n \rightarrow \infty} \frac{(-2)L'\left(\frac{s}{\sqrt{n}\sigma}\right) \frac{s}{\sigma}}{(-1)n^{-\frac{3}{2}}}$$

$$L''(s) = -\frac{1}{(M_{\bar{X}}(s))^2} \left(M'_{\bar{X}}(s) \right)^2 + \frac{1}{M_{\bar{X}}(s)} M''_{\bar{X}}(s).$$

$$= \lim_{n \rightarrow \infty} \frac{L''\left(\frac{s}{\sqrt{n}\sigma}\right) \frac{s}{\sigma}}{\frac{1}{2} (\sqrt{n})^{-3/2}}.$$

$$L''(0) = -\frac{(M'_{\bar{X}}(0))^2}{(M_{\bar{X}}(0))^2} \rightarrow E[\bar{x}] = 0.$$

$$= \lim_{n \rightarrow \infty} \frac{\left(\frac{s}{\sigma}\right)^2 L''\left(\frac{s}{\sqrt{n}\sigma}\right) \frac{1}{n}^{-3/2}}{2 \times \left(\frac{-1}{2}\right) n^{-3/2}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \left(\frac{s}{\sigma}\right)^2 L''\left(\frac{s}{\sqrt{n}\sigma}\right)$$

$$= \sigma^2$$

$$\lim_{n \rightarrow \infty} \log M_{Z_n}(s) = \frac{s^2}{2},$$

$$\Rightarrow \lim_{n \rightarrow \infty} M_{Z_n}(s) = e^{s^2/2}.$$

$= M_Z(s)$ i.e., MGf of



Standard normal R.V

$$\lim_{n \rightarrow \infty} F_{Z_n}(z) = F_Z(z).$$

Example: "Exit Poll"

let p be the fraction of voters that support Trump for office. n "randomly selected" voters are interviewed & M_n fraction of them support him.

"randomly selected" \rightarrow chosen independently & uniformly from the pool.

Each voter's response is $X_i \sim \text{Ber}(\phi)$.

$$M_n = \frac{X_1 + X_2 + \dots + X_n}{n}$$

X_i 's are iid.

How accurately does M_n predict " ϕ "?

We want to say with "high confidence" that
 $M_n \in [\phi - \epsilon, \phi + \epsilon]$.

High confidence
⇒

$$P(|M_n - \phi| \leq \epsilon) \geq 1 - \delta.$$

$$P(|M_n - \phi| \geq \epsilon) \leq \frac{\text{Var}(M_n)}{\epsilon^2}$$

$$E[M_n] = E\left[\frac{X_1 + \dots + X_n}{n}\right]$$

$$= \frac{\phi}{n} \leq \frac{1}{4n\epsilon^2}$$

$$= n \frac{E[X_i]}{n}$$

$$= \phi$$

$$\text{Var}[M_n] = \frac{1}{n^2} \text{Var}(X_1 + \dots + X_n)$$

$$\text{If } \frac{1}{4n\epsilon^2} < \delta.$$

$$= \frac{1}{n} \text{Var}(X_i)$$

$$= \frac{1}{n} \phi(1-\phi)$$

$$P(|M_n - \phi| \leq \epsilon)$$

$$= 1 - P(|M_n - \phi| > \epsilon)$$

$$\geq 1 - \delta.$$

Suppose we want to declare that exit poll is 95% confident that the estimate M_n is accurate upto 10^{-2} error. What should the sample size of poll be?

$$\epsilon = 10^{-2}, \quad 1 - \alpha = \frac{95}{100}, \quad \alpha = \frac{5}{100}$$

Pick n st

$$\frac{1}{4n\epsilon^2} \leq \frac{5}{100}$$

$$\Rightarrow n \geq \frac{100}{4 \times 5 \times \epsilon^2} = 5 \times 10^4.$$

You'll need 50K sample size to be 95%.
Confident within 10^{-2} error.

How does n value look like using CLT based approximation.

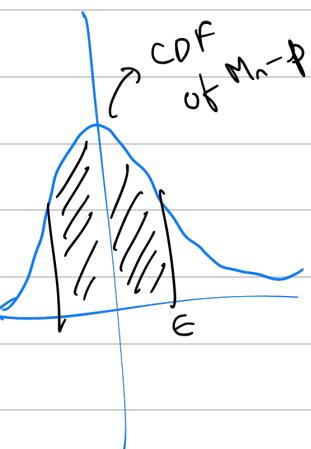
$$M_n = \frac{S_n}{n}$$

$$\mu = p$$

$$Z_n = \frac{S_n - np}{\sqrt{n}\sigma}$$

$$\sigma^2 = p(1-p)$$

$$= \frac{n M_n - n \mu}{\sqrt{n}\sigma} = \left(\frac{M_n - \mu}{\sigma/\sqrt{n}} \right)$$



$$\text{If } Z_n \approx N(0, 1)$$

$$M_n \approx N(\mu, \frac{\sigma^2}{n}), \quad M_n - p \approx N(0, \frac{\sigma^2}{n})$$

$$P(|M_n - p| > \epsilon) = 2 P(M_n - p > \epsilon).$$

Normal
RV with
mean 0.
Symmetric

$$= 2 P\left(\frac{M_n - p}{\sigma/\sqrt{n}} > \frac{\epsilon}{\sigma/\sqrt{n}}\right)$$

$$= 2 \left[1 - \Phi\left(\frac{\epsilon}{\sigma/\sqrt{n}}\right) \right]$$

$$\sigma^2 = p(1-p).$$

$$\sigma^2 \leq \frac{1}{4}$$

$$\sigma \leq \frac{1}{2}, \frac{1}{\sigma} \geq 2.$$

CDF is monotonically increasing

$$\Phi\left(\frac{\epsilon}{\sigma}\right) > \Phi\left(\frac{2\epsilon}{\sigma}\right)$$

$$\leq 2 \left[1 - \Phi \left(2 \sqrt{n} \right) \right]$$

$$G = 10^2, \quad S = 5/100 \quad \leq 8.$$

$$2 \left(1 - \Phi \left(2 \times 10 \sqrt{n} \right) \right) \leq \frac{5}{100}. \\ \Rightarrow n \geq 9604$$

Berry Esseen Theorem : Let X_1, X_2, \dots are iid

$$Z_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sigma \sqrt{n}}$$

$$|F_{Z_n}(z) - F_z(z)| \leq \frac{C}{\sigma^3 \sqrt{n}} \text{ where}$$

$$\int = E[|X|^3] < \infty \quad \text{and} \quad \sigma^2 = E[X^2] < \infty \\ \text{and} \quad E[X] = 0.$$

$$C \geq \frac{1}{\sqrt{2\pi}} + 0.011.$$

Why look at "infinitely-often" sets?

To show $X_n \xrightarrow{a.s.} X$, it is enough to show $P(A) = 0$

$$\text{where } A = \left\{ \omega : \lim_{n \rightarrow \infty} X_n(\omega) \neq X(\omega) \right\}$$

$$= \left\{ \omega : \exists \epsilon > 0, \forall N; \exists n > N \text{ st}, |X_n(\omega) - X(\omega)| > \epsilon \right\}$$

$$\text{Let } A_n = \left\{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \right\}.$$

$$A_\epsilon = \bigcap_{n=1}^{\infty} \bigcup_{n \geq N} A_n = \left\{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \text{ i.o.} \right\}.$$

Also referred
as limsup An
 $\limsup_{n \rightarrow \infty} A_n$

$$\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) \leq 0$$

Lemma 2 : If A_1, A_2, \dots are independent events such that

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

then $P(\limsup_{n \rightarrow \infty} A_n) = 1$.

Proof: $\limsup_{n \rightarrow \infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n.$

$$B_1 \supset B_2 \supset B_3 \dots$$

$$\begin{aligned} P(\limsup_{n \rightarrow \infty} A_n) &= P\left(\bigcap_{N=1}^{\infty} B_N\right). \\ &= \lim_{N \rightarrow \infty} P(B_N). \end{aligned}$$

$$P(B_N) = P\left(\bigcup_{n=N}^{\infty} A_n\right).$$

$$\text{Let us define } B_{N,M} = \bigcup_{n=N}^{M} A_n$$

$$B_{N,N} = A_N$$

$$B_{N,N} \subseteq B_{N,N+1} \subset B_{N,N+2} \dots \quad B_{N,N+1} = A_N \cup A_{N+1}$$

$$P(B_N) = P\left(\bigcup_{M=N}^{\infty} B_{N,M}\right) = \lim_{M \rightarrow \infty} P(B_{N,M})$$

Made with Goodnotes

$$P(B_{N,M}) = P\left(\bigcup_{n=N}^{M} A_n\right) = 1 - P\left(\bigcap_{n=N}^{M} A_n^c\right)$$

$$\text{as } A_n \text{ s are independent.} = 1 - \prod_{n=N}^M (1 - P(A_n))$$

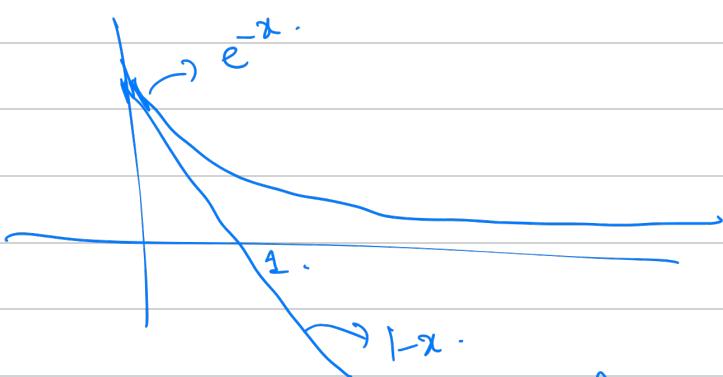
$$P(\limsup_{n \rightarrow \infty} A_n) = \lim_{N \rightarrow \infty} P(B_N).$$

$$= \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} P(B_{N,M}).$$

$$= \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} 1 - \prod_{n=N}^M (1 - P(A_n))$$

- $P(A_n)$.

$\Rightarrow -e^{-\infty} \leq -(1 - P(A_n))$



$$P(\limsup_{n \rightarrow \infty} A_n) \geq \lim_{N \rightarrow \infty} \lim_{M \rightarrow \infty} 1 - e^{-\sum_{n=N}^M P(A_n)}$$

$$\text{as } \sum_{n=1}^{\infty} P(A_n) = \infty = \lim_{N \rightarrow \infty} 1 - e^{-\sum_{n=N}^{\infty} P(A_n)}$$

$$\Rightarrow \sum_{n=N}^{\infty} P(A_n) = \infty = \lim_{N \rightarrow \infty} 1 - e^{-\infty}$$

as $\sum_{n=1}^N P(A_n)$ is finite

$= 1.$

$$P(\limsup_{n \rightarrow \infty} A_n) \geq 1.$$

$$\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) = 1.$$

(d) m.s convergence $\not\Rightarrow$ a.s convergence x_i 's are independent

R.Vs defined as

$$P(X_n = x) = \begin{cases} 1 - \frac{1}{n} & x = 0 \\ \frac{1}{n} & x = 1. \end{cases}$$

$$E[X_n^2] = \frac{1}{n} \cdot 1^2 + \left(1 - \frac{1}{n}\right) \cdot 0^2 = \frac{1}{n}.$$

$\Rightarrow X_n \rightarrow 0$ in m.s sense

$$\text{let } A_n = \{ \omega : |X_n(\omega)| > \epsilon \}, \quad P(A_n) = \frac{1}{n}.$$

$$\begin{aligned} P(A_1 \cap A_2) &= P(X_1 > \epsilon, X_2 > \epsilon) \\ &= P(X_1 > \epsilon) P(X_2 > \epsilon) \quad \text{as } X_1 \text{ & } X_2 \text{ are independent} \\ &= P(A_1) P(A_2) \end{aligned}$$

A_1, A_2, \dots are independent events.

$$\sum_n P(A_n) = \sum_n \frac{1}{n} = \infty$$

Borel Cantelli lemma 2 $\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) = 1 \Rightarrow X_n \not\rightarrow 0$ in a.s sense.

(e)

a.s convergence $\not\Rightarrow$ m.s convergence.

$X_n \rightarrow 0$?

$$P(X_n = x) = \begin{cases} 1 - \frac{1}{n^2} & x = 0 \\ \frac{1}{n^2} & x = n^2. \end{cases}$$

$$\begin{aligned} E[X_n^2] &= \left(1 - \frac{1}{n^2}\right) 0^2 + (n^2)^2 \frac{1}{n^2} \\ &= n^2. \end{aligned}$$

$\Rightarrow X_n \not\rightarrow 0$ in m.s sense.

$$A_n = \{ \omega : |X_n(\omega)| > \epsilon \}, \quad P(A_n) = \frac{1}{n^2}.$$

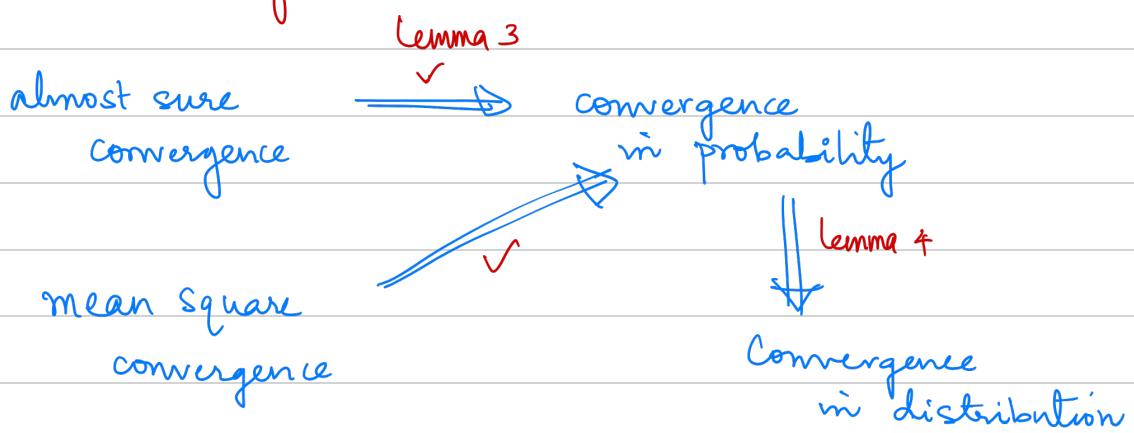
$$\Rightarrow \sum_n P(A_n) = \sum_n \frac{1}{n^2} < \infty$$

Borel Cantelli lemma 1. $\Rightarrow P(\limsup_{n \rightarrow \infty} A_n) = 0$

$\Rightarrow X_n \rightarrow 0$ in a.s sense.

(8)

Hierarchies of convergence notions



Lemma 3: If $X_n \rightarrow X$ in almost sure sense then $X_n \rightarrow X$ in probability.

Proof: let $A_n = \{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \}$.

To show $X_n \rightarrow X$ in probability
we need to prove

$$\lim_{n \rightarrow \infty} P\left(\bigcup \{ |X_n - X| > \epsilon \}\right) = 0.$$

To show $\lim_{n \rightarrow \infty} P(A_n) = 0$

We have that $P\left(\limsup_{n \rightarrow \infty} A_n\right) = 0$

as $X_n \rightarrow X$ in almost sure sense.

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n. \\ \underbrace{\qquad\qquad\qquad}_{B_N}.$$

$$B_1 \supset B_2 \supset \dots$$

$$A_N \subseteq B_N.$$

$$P(A_N) \leq P(B_N).$$

$$\Rightarrow \lim_{N \rightarrow \infty} P(A_N) \leq \lim_{N \rightarrow \infty} P(B_N)$$

$$B_1 \supset B_2 \supset \dots$$

Continuity
property of
probability.

$$\begin{aligned}
 &= P\left(\bigcap_{N=1}^{\infty} B_N\right) \\
 &= P\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n\right) \\
 &= P\left(\limsup_{n \rightarrow \infty} A_n\right) \\
 &= 0 \\
 \Rightarrow \lim_{N \rightarrow \infty} P(A_N) &\leq 0 \\
 \Rightarrow \lim_{N \rightarrow \infty} P(A_N) &= 0 \\
 \Rightarrow X_n \rightarrow X \text{ in probability.}
 \end{aligned}$$

Lemma 4: If $X_n \rightarrow X$ in probability then $X_n \rightarrow X$ in distribution.

To show $\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x)$.
given that

$$\lim_{n \rightarrow \infty} P(|X_n - X| > \epsilon) = 0.$$

$$F_{X_n}(x) = P(X_n \leq x).$$



$$\begin{aligned}
 P(A) &= P(A \cap \Omega) \\
 &= P(A \cap B) \\
 &\quad + P(A \cap B^c)
 \end{aligned}$$

$= P\left(\{X_n \leq x\} \cap \{X > x + \epsilon\}\right)$
 $+ P\left(\{X_n \leq x\} \cap \{X \leq x + \epsilon\}\right)$
 $\leq P(\{X \leq x + \epsilon\})$
 $= F_X(x + \epsilon)$

If $w \in \{X_n \leq x\} \cap \{X > x + \epsilon\}$

$$\Rightarrow w \in \{X - X_n > \epsilon\}$$

$$F_{X_n}(x) \leq P(X - X_n > \epsilon) + F_x(x + \epsilon).$$

$$\leq P(|X - X_n| > \epsilon) + F_x(x + \epsilon).$$

$$\lim_{n \rightarrow \infty} F_{X_n}(x) \leq \lim_{n \rightarrow \infty} P(|X - X_n| > \epsilon) + F_x(x + \epsilon)$$

$$= F_x(x + \epsilon).$$

$$1 - F_{X_n}(x) = P(X_n > x).$$

$$= P(X_n > x, X > x - \epsilon)$$

$$\leq P(X \geq x - \epsilon) + P(X_n > x, X \leq x - \epsilon)$$

$$\leq P(X_n - X \geq \epsilon).$$

$$\leq P(X \geq x - \epsilon) + P(X_n - X \geq \epsilon).$$

$$\leq 1 - F_x(x - \epsilon) + P(|X_n - X| \geq \epsilon).$$

$$F_x(x - \epsilon) - P(|X_n - X| \geq \epsilon) \leq F_{X_n}(x).$$

$$F_x(x - \epsilon) - \lim_{n \rightarrow \infty} P(|X_n - X| \geq \epsilon) \leq \lim_{n \rightarrow \infty} F_{X_n}(x)$$

$$\Rightarrow F_x(x - \epsilon) \leq \lim_{n \rightarrow \infty} F_{X_n}(x)$$

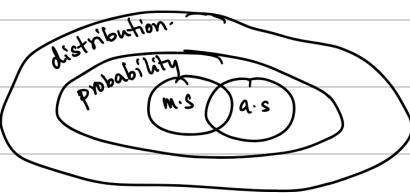
$$\Rightarrow F_x(x - \epsilon) \leq \lim_{n \rightarrow \infty} F_{X_n}(x) \leq f_x(x + \epsilon) \quad \forall \epsilon > 0.$$

$$\Rightarrow \lim_{n \rightarrow \infty} F_{X_n}(x) = \lim_{\epsilon \rightarrow 0} f_x(x + \epsilon)$$

$$= F_x(x)$$

for $x \in \mathbb{R}$

Continuous
points of F_x .



⑨ Strong Law of Large numbers.: Let $x_1, x_2 \dots$ are iid R.Vs

$$S_n = \sum_{i=1}^n x_i$$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu \quad \text{given } E[x^4] < \infty.$$

For simplicity we'll show this proof for x_i 's with mean 0.

It is equivalent to showing

$$\frac{S_n - \mu n}{n} \xrightarrow{a.s} 0.$$

$$\frac{\sum_{i=1}^n (x_i - \mu)}{n}$$

We'll show $\frac{S_n}{n} \xrightarrow{a.s} 0$ and assume x_i 's to be of 0 mean.

$$A_n = \left\{ \omega : \left| \frac{S_n}{n} \right| > \epsilon \right\}.$$

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P\left(\left| \frac{S_n}{n} \right| > \epsilon\right).$$

$$P\left(\left| \frac{S_n}{n} \right| > \epsilon\right) = P\left(\left(\frac{S_n}{n} \right)^4 > \epsilon^4\right).$$

$$\leq \frac{E[S_n^4]}{n^4 \epsilon^4}.$$