Mid Term Solutions :

① 52 cards equally distributed among 4 people.

$\#$ of possible ways to do so $= \binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \dfrac{52!}{(13!)^4}$.

ⓐ Prob that each one gets an ace $= \dfrac{\#\text{ possible ways to distribute such that each person gets an ace}}{\#\text{ possible ways to distribute}}$

$\#$ ways to distribute ace's equally among 4 people

$$= \frac{4!\binom{48}{12}\binom{36}{12}\binom{24}{12}\binom{12}{12}}{\frac{52!}{(13!)^4}}$$

$$= \frac{4!\,\frac{48!}{(12!)^4}}{\frac{52!}{(13!)^4}} = \frac{4!\,(13)^4}{52\times 51\times 50\times 49}$$

ⓑ Prob that one user gets all 13 spades.

$$= \frac{\binom{4}{1}\binom{13}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}{\binom{52}{13}\binom{39}{13}\binom{26}{13}\binom{13}{13}}$$

choose a user that gets spades.

$$= \frac{\binom{4}{1}}{\binom{52}{13}}$$

② ⓐ $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i).$$

$$\Rightarrow \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \leq \lim_{n\to\infty} \sum_{i=1}^{n} P(A_i)$$

$$= \sum_{i=1}^{\infty} P(A_i)$$

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i).$$

ⓑ $P\left(\bigcap_{i=1}^{\infty} B_i\right) = P\left(\left(\bigcup_{i=1}^{\infty} B_i^c\right)^c\right)$

$$= 1 - P\left(\bigcup_{i=1}^{\infty} B_i^c\right).$$

$$P\left(\bigcup_{i=1}^{\infty} B_i^c\right) \leq \sum_{i=1}^{\infty} P(B_i^c)$$

Since $P(B_i)=1$, $P(B_i^c)=0$.

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} B_i^c\right) \leq 0.$$

$\Rightarrow P\left(\bigcup_{i=1}^{\infty} B_i^c\right) = 0$ as probability can't be negative.

$$\Rightarrow P\left(\bigcap_{i=1}^{\infty} B_i\right) = 1 - P\left(\bigcup_{i=1}^{\infty} B_i^c\right) = 1.$$

ⓒ E_i = event that trial is successful

(i) $P\left(\bigcap_{i=1}^{n} E_i\right) = \prod_{i=1}^{n} P(E_i)$ as all trial are independent

$$= p^n,$$

(2) $P\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i\right) \rightarrow$ Follows from Continuity property.

$P\left(\bigcup_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} E_i^c\right)$

$\Rightarrow 1 - P\left(\bigcup_{i=1}^{\infty} E_i^c\right) = \lim_{n\to\infty} 1 - P\left(\bigcup_{i=1}^{n} E_i^c\right)$

$\Rightarrow P\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} P\left(\bigcap_{i=1}^{n} E_i\right)$

$\therefore P\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} p^n$.

prob that all trials are successful

$$= \begin{cases} 1 & p = 1 \\ 0 & p < 1. \end{cases}$$

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(3) (a) $P(A\cap B) > 0$

$$P(A\cup B \mid A\cap B) = \frac{P((A\cup B)\cap(A\cap B))}{P(A\cap B)} = \frac{P(A\cap B)}{P(A\cap B)} = 1.$$

$$P(A\cap B \mid A\cup B) = \frac{P(A\cap B)}{P(A\cup B)} \leq 1.$$

$\therefore$ True.

(b)

$$P(A^c \mid B) = 1 - P(A\mid B)$$
$$\leq 1 - P(C\mid B) \quad \text{as } P(A\mid B) \geq P(C\mid B).$$
$$= P(C^c \mid B).$$

$\therefore$ False

(c) Need not be true. (False)

For example: $\Omega = \{H,T\}^2$. $P(\omega) = \frac{1}{4}$ then

outcomes of two coin tosses

A : event that you see two heads

B : event " " two tails.

$A\cap B = \phi$. $P(A\cap B) = 0$

$\neq P(A)P(B)$.

Made with Goodnotes

(d)

$$P(A\cup B) = P(A)+P(B) \quad \text{as } A, B \text{ are disjoint}$$
$$= 1.2 \quad \text{contradicts normalization}$$

$\therefore$ False.

(c)

$$P(A \mid A\cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{P(A)}{P(A)+P(B\setminus A)}$$

$$P(A \mid B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A\cap B)}{P(A\cap B)+P(B\setminus A)}.$$

$$P(A) \geq P(A\cap B).$$

$$\frac{1}{P(A)} \leq \frac{1}{P(A\cap B)}.$$

$$1+\frac{P(B\setminus A)}{P(A)} \leq \frac{P(B\setminus A)}{P(A\cap B)}+1.$$

$$\frac{1}{1+\frac{P(B\setminus A)}{P(A)}} \geq \frac{1}{1+\frac{P(B\setminus A)}{P(A\cap B)}}.$$

$\therefore$ The statement is true.

4 (a) Let p_i be the probability that Alice is successful in ith round.

$$P_1 = p^2.$$

$$P_2 = \underbrace{(1-p^2)}_{\text{Prob that Alice not successful in round 1}} \cdot \underbrace{(1-q^2)}_{\text{prob that Bob not successful in round 1}} \cdot p^2 \Big\} \text{ Prob that Alice is successful in round 2.}$$

$$P_3 = \left[(1-p^2)(1-q^2)\right]^2 p^2.$$

$\vdots$

$$\text{Prob that Alice wins} = \sum_{i=1}^{\infty} P_i = \sum_{i=1}^{\infty}\left[(1-p^2)(1-q^2)\right]^{i-1} p^2$$

$$= \frac{p^2}{1-(1-p^2)(1-q^2)} = \frac{p^2}{p^2+q^2-p^2q^2}$$

(b) **Soln 1:** $a_{i,j}$ = Prob that Alice wins given that Alice has accumulated i heads so far & Bob j heads so far.

$$a_{0,0} = p^2 + (1-p)(1-q)\cdot a_{0,0} + (1-p)q(1-q)a_{0,1} + p(1-p)(1-q)a_{1,0} + p(1-p)q(1-q)a_{1,1} \rightarrow ⓪$$

Alice sees T in 1st round

$$\boxed{a_{1,1} = p + (1-p)(1-q)a_{1,1} \qquad a_{1,1} = \frac{p}{p+q-pq}} \rightarrow ①$$

$$a_{1,0} = p + (1-p)(1-q)a_{1,0} + (1-p)q(1-q)a_{1,1}$$

$$a_{1,0}[p+q-pq] = p + \frac{(1-q)(1-p)qp}{(p+q-pq)}$$

$$\boxed{a_{1,0} = \frac{p}{p+q-pq} + \frac{qp(1-q)(1-q)}{(p+q-pq)^2}} \rightarrow ②$$

$$a_{0,1} = p^2 + (1-p)(1-q)a_{0,1} + p(1-p)(1-q)a_{1,1}$$

$$a_{0,1}(p+q-pq) = p^2 + \frac{p^2(1-p)(1-q)}{p+q-pq} = \frac{p^2}{p+q-pq}$$

$$\Rightarrow \boxed{a_{0,1} = \frac{p^2}{(p+q-pq)^2}} \rightarrow ③$$

Substitute ①, ②, ③ in ⓪ to find $a_{0,0}$.

$$(p+q-pq)\,a_{0,0} = p^2 + (1-p)\,q(1-q)\frac{p^2}{(p+q-pq)^2} + p(1-p)(1-q)\left[\frac{p}{p+q-pq} + \frac{qp(1-p)(1-q)}{(p+q-pq)^2}\right] + pq(1-p)(1-q)\frac{p}{p+q-pq}$$

$$= p^2\left[1 + \frac{(1-p)(1-q)}{(p+q-pq)}\left[\frac{q}{p+q-pq} + 1 + \underbrace{\frac{q(1-p)(1-q)}{(p+q-pq)} + q}_{\frac{q}{p+q-pq}}\right]\right]$$

$$\frac{1}{p+q-pq}$$

$$a_{0,0} = \frac{p^2}{p+q-pq}\left[\frac{1}{p+q-pq} + \frac{(1-p)(1-q)}{p+q-pq}\left[\frac{2q}{p+q-pq}\right]\right]$$

Solution 2 :

P_i := Probability that Alice wins in round i.

$$P_1 = p^2$$

$$P_2 = \underbrace{p(1-p)}_{\substack{HT \\ \text{Alice}}}\ \underbrace{(1-q^2)}_{\substack{T \text{ or } HT \\ \text{for Bob}}}\ \underbrace{p}_{\substack{H \\ \text{Alice}}}$$

$$+ \underbrace{(1-p)}_{T \text{ Alice}}\ \underbrace{[1-q^2]}_{\substack{T \text{ or } HT \\ \text{for Bob}}}\ \underbrace{p^2}_{\substack{HH \\ \text{Alice}}}$$

$$P_3 = \underbrace{\left[\binom{2}{1}[p(1-p)]^1\,[(1-p)]^1\right]}_{\text{One H in previous two rounds by Alice}}\ \underbrace{\left[\binom{2}{1}q(1-q)^2 + (1-q)^3\right]}_{\text{atmost one H in previous two rounds}}\ \underbrace{p}_{\text{H in third round}}$$

T HT
HT T

T T
HT T
T HT

$$+\underbrace{\left[(1-p)^2\right]}_{\substack{\text{no heads in}\\ \text{previous two rounds by}\\ \text{Alice}}}\underbrace{\left[\binom{2}{1}q(1-q)^2+(1-q)^3\right]}_{\substack{\text{atmost one H}\\ \text{in previous}\\ \text{two}\\ \text{rounds.}}}p^2 .$$

T T

T. T

H T T

T H T

$$P_i = \underbrace{\left[\binom{i-1}{1}p(1-p)(1-p)^{i-2}\right]}_{\substack{\text{exactly one head}\\ \text{in prev }(i-1)\text{ rounds}\\ \text{by Alice}}}\underbrace{\left[\binom{i-1}{1}q(1-q)(1-q)^{i-2}+(1-q)^{i-1}\right]}_{\substack{\text{atmost 1 head in prev}(i-1)\\ \text{rounds, by Bob}}}p.$$

$$+\underbrace{\left[(1-p)^{i-1}\right]}_{\substack{0\text{ heads in}\\ \text{prev }(i-1)\\ \text{rounds}\\ \text{by Alice.}}}\left[\binom{i-1}{1}q(1-q)^{i-1}+(1-q)^{i-1}\right]p^2$$

$$= p(1-q)^{i-1}\left[(i-1)q+1\right]\left[p(1-p)^{i-1}+(i-1)p(1-p)^{i-1}\right]$$

$$= p^2(1-q)^{i-1}(1-p)^{i-1}\left[(i-1)q+1\right]i.$$

Made with Goodnotes

Probability of Alice winning,

$$W = \sum p^2 \, i \left[(1-q)(1-p)\right]^{i-1}$$

$$+ \sum q p^2 \, i(i-1) \left[(1-q)(1-p)\right]^{i-1}$$

$$\sum_{i=1}^{\infty} x^i = \frac{1}{1-x} \quad \text{If } x < 1.$$

$$\sum_{i=1}^{\infty} i \, x^{i-1} = \frac{+1}{(1-x)^2}.$$

$$\sum_{i=1}^{\infty} i(i-1) \, x^{i-2} = \frac{2}{(1-x)^3}$$

$$\therefore \quad W = p^2 \left[\frac{1}{\left[1-(1-q)(1-p)\right]^2} + \frac{(1-q)(1-p)\, 2q}{\left[1-(1-q)(1-p)\right]^3} \right]$$

5 (a) $Y \mid X = +1 = Z \mid X = +1.$

Note that Z is independent of X.

$$\therefore f_{Y|X}(y|+1) = f_{Z|X}(y|+1).$$

$$= f_Z(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2}$$

$$f_{Y|X}(y|-1) = f_{Z|X}(-y|-1).$$

$$= f_Z(-y) = \frac{e^{-\frac{1}{2}\left(\frac{y+\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\sigma}.$$

(b) soln1: $E[Y] = E\{E[Y|X]\}$

soln2. $E[Y] = E[XZ] = E[X]E[Z] = \left(\frac{-1}{2}\right)(2) = -1.$

$= P(X=+1)E[Y|X=+1] + P(X=-1)E[Y|X=-1]$

$= \frac{1}{4}E[Z|X=+1] + \frac{3}{4}E[-Z|X=-1]$

Z is independent of X. $= \frac{1}{4}E[Z] + \frac{3}{4}E[-Z]$

$= \left(\frac{1}{4} - \frac{3}{4}\right)E[Z] = \frac{-1}{2} \times \mu = -1.$

Soln1: $\mathrm{Var}(Y) = \mathrm{Var}(E[Y|X]) + E[\mathrm{Var}(Y|X)].$

$\mathrm{Var}(Y|X=+1) = \mathrm{Var}(Z|X=+1) = \mathrm{Var}(Z) = \sigma^2 = 4.$

$\mathrm{Var}(Y|X=-1) = \mathrm{Var}(-Z|X=-1) = \mathrm{Var}(-Z) = \sigma^2 = 4.$

$E[\mathrm{Var}(Y|X)] = \frac{1}{4}\times 4 + \frac{3}{4}\times 4 = 4.$

$\mathrm{Var}(E[Y|X]) = P(X=+1)\left[E[Y|X=+1] - E[Y]\right]^2 + P(X=-1)\left\{E[Y|X=-1] - E[Y]\right\}^2$

$= \frac{1}{4}\left[(2+1)^2\right] + \frac{3}{4}\left[-2+1\right]^2.$

$= \frac{1}{4}\times 9 + \frac{3}{4}\times 1 = 3.$

$\Rightarrow \mathrm{Var}(X) = 7.$

Soln 2: $\frac{1}{4}E[Y^2|X=1] + E\{Y^2|X=-1\}\,3/4 = E[Y^2].$

$E[Y^2] = (\sigma^2 + \mu^2)\frac{1}{4} + [\sigma^2 + (-\mu)^2]\,3/4 = \sigma^2 + \mu^2$

$= 4 + 4 = 8.$

$E[Y] = -1.$

$\therefore \mathrm{Var}(Y) = E[Y^2] - (E[Y])^2 = 8 - 1 = 7.$

(c) $\mathrm{Cov}(X, Y) = E[XY] - E[X]E[Y]$

$= E[X^2 Z] - E[X]E[Y]$

$= E[X^2]E[Z] - \left(\frac{-1}{2}\right)(-1)$

$= 1 \cdot 2 - \frac{1}{2} = 3/2 \neq 0$.

$$\rho(X, Y) = \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}$$

$\mathrm{Var}(X) = 1 - \left(\frac{-1}{2}\right)^2 = 3/4$.

$$\Rightarrow \rho(X, Y) = \frac{3/2}{\left(\sqrt{\frac{3}{4}}\right)\sqrt{7}} = \sqrt{\frac{3}{7}}.$$

(d) $f_Y(y) = f_{Y|\{X=+1\}}(y)\,\frac{1}{4} + \frac{3}{4} f_{Y|\{X=-1\}}(y)$ Total probability theorem

$= f_{Z|\{X=+1\}}(y)\,\frac{1}{4} + \frac{3}{4} f_{Z|\{X=-1\}}(-z)$ as $Y = XZ$

$= \frac{1}{4} f_Z(y) + \frac{3}{4} f_Z(-y)$ as Z is independent of X.

$$= \frac{1}{4}\frac{e^{-\left(\frac{y-2}{2}\right)^2\frac{1}{2}}}{\sqrt{2\pi}\,2} + \frac{3}{4}\frac{e^{-\left(\frac{y+2}{2}\right)^2}}{\sqrt{2\pi}\,2}.$$

Not a normal distribution

(e) $$P(X = +1 \mid Y = y) = \frac{f_{Y|\{X=+1\}}(y)\, P(X=+1)}{f_Y(y)}$$

$$= \frac{\frac{1}{4}\frac{e^{-\left(\frac{y-2}{2}\right)^2\frac{1}{2}}}{\sqrt{2\pi}\,2}}{\frac{1}{4}\frac{e^{-\left(\frac{y-2}{2}\right)^2\frac{1}{2}}}{\sqrt{2\pi}\,2} + \frac{3}{4}\frac{e^{-\left(\frac{y+2}{2}\right)^2\frac{1}{2}}}{\sqrt{2\pi}\,2}}$$

$$= \frac{1}{(1 + 3e^{-y})}$$

$$P(X=-1 \mid Y=y) = \frac{f_{Y|\{X=-1\}}(y)\; P(X=-1)}{f_Y(y)}$$

$$= \frac{\frac{3}{4}\,\frac{e^{-\left(\frac{y+2}{2}\right)^2 \frac{1}{2}}}{\sqrt{2\pi}\,2}}{\frac{1}{4}\,\frac{e^{-\left(\frac{y-2}{2}\right)^2 \frac{1}{2}}}{\sqrt{2\pi}\,2} + \frac{3}{4}\,e^{-\left(\frac{y+2}{2}\right)^2 \frac{1}{2}}}$$

$$= \frac{3e^{-y}}{1+3e^{-y}}$$

6 a $$\hat{x}_{MAP}(y) = \begin{cases} 1 & \text{if } 1 \geq 3e^{-y} \text{ i.e., } y \geq \ln 3 \\ -1 & \text{if } y \leq \ln 3 \end{cases}$$

b $$\hat{x}_{ML}(y) = \begin{cases} 1 & \text{if } f_{Y|X}(y|1) \geq f_{Y|X}(y|-1) \\ -1 & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|1) \geq f_{Y|X}(y|-1)$$

$$\Updownarrow$$

$$f_Z(y) \geq f_Z(-y)$$

$$\Updownarrow$$

$$\frac{e^{-\left(\frac{y-\mu}{\sigma}\right)^2 \frac{1}{2}}}{\cancel{\sqrt{2\pi}\sigma}} \geq \frac{e^{-\left(\frac{y+\mu}{\sigma}\right)^2 \frac{1}{2}}}{\cancel{\sqrt{2\pi}\sigma}}$$

$$\Updownarrow$$

$$e^{\frac{4y\mu}{2\sigma}} \geq 1 \iff y \geq 0$$

$$\hat{x}_{ML}(y) = \begin{cases} +1 & y \geq 2 \\ -1 & \text{otherwise} \end{cases}$$

(c)

$$P(\hat{x}_{MAP}(Y) \neq X) = P(X=1)\, P(\hat{x}_{MAP}(Y) \neq 1 \mid X=+1)$$
$$+ P(X=-1)\, P(\hat{x}_{MAP}(Y) \neq -1 \mid X=-1)$$

$$= \frac{1}{4} P(Y < \ln 3 \mid X=+1)$$
$$+ \frac{3}{4} P(Y > \ln 3 \mid X=-1).$$

$$= \frac{1}{4} P(Z < \ln 3) + \frac{3}{4} P(-Z > \ln 3).$$

$\frac{Z-2}{2}$ is normal R.V.

$$= \frac{1}{4} P\left(\frac{Z-2}{2} < \frac{\ln 3 - 2}{2}\right) + \frac{3}{4} P\left(\frac{Z-2}{2} < \frac{-\ln 3 - 2}{2}\right)$$

$$= \frac{1}{4} \Phi\left(\frac{\ln 3 - 2}{2}\right) + \frac{3}{4} \Phi\left(\frac{-\ln 3 - 2}{2}\right).$$

$$= \frac{1}{4} \Phi(-0.45) + \frac{3}{4} \Phi(-1.549).$$

$$= \frac{1}{4}(1 - \Phi(0.45)) + \frac{3}{4}(1 - \Phi(1.549)).$$

$$= \frac{1}{4}(1 - 0.67364) + \frac{3}{4}(1 - 0.93943)$$

$$= 1 - 0.872 \approx 0.13$$

$$P(\hat{x}_{ML}(Y) \neq X) = \frac{1}{4} P(\hat{x}_{ML}(Y) \neq 1 \mid X=1)$$
$$+ \frac{3}{4} P(\hat{x}_{ML}(Y) = -1 \mid X=-1)$$

$$= \frac{1}{4} P(Y \leq 0 \mid X=+1) + \frac{3}{4} P(Y > 0 \mid X=-1)$$

$$= \frac{1}{4} P(Z \leq 0) + \frac{3}{4} P(-Z > 0).$$

$$= \frac{1}{4} P(Z \le 0) + \frac{3}{4} P(Z < 0)$$

$$= P(Z \le 0) = P\left(\frac{Z-2}{2} \le -1\right)$$

$$= 1 - \Phi(1) = (1 - 0.84) \approx 0.16$$

MAP is better as it has smaller error

(d) One can't do better than MAP error of 0.13 due to the result shown in class

(7) $Z = |X - Y|$

$$F_Z(z) = P(Z \le z) = P(|X - Y| \le z)$$

$$= P(-z \le X - Y \le z)$$

$$= \int_0^1 \int_{\max\{y-z,0\}}^{\min\{y+z,1\}} dx\, dy$$

$$= \int_0^1 \min\{y+z, 1\} - \max\{y-z, 0\}\, dy$$

$$= \int_0^{1-z} (y+z)\, dy + \int_{1-z}^1 dy - \int_z^1 (y-z)\, dy$$

$$= \frac{y^2}{2} + zy \Big|_0^{1-z} + z - \left(\frac{y^2}{2} - zy\right)\Big|_z^1$$

$$= \frac{(1-z)^2}{2} + z(1-z) + z + \left(\frac{z^2}{2} - z^2\right) - \left(\frac{1}{2} - z\right)$$

$$= \frac{\cancel{1} + \cancel{z^2} - \cancel{2z}}{2} + z(1-z) + \cancel{z} - \cancel{\frac{z^2}{2}} - \cancel{\frac{1}{2}} + z$$

$$= z(2 - z) \qquad 0 \le z \le 1$$

(b) $f_Z(z) = 2 - 2z \qquad 0 \le z \le 1$

Sanity check
$\int (2-2z)\, dz$
$2z - z^2 \Big|_0^1$
$= 1$

(c) $E[|X - Y|] = \int z(2 - 2z)$

$$= \int 2z - 2z^2 \, dz$$

$$= \frac{\cancel{2} z^2}{\cancel{2}} - \frac{2}{3} z^3 \Big|_0^1 = \frac{1}{3}.$$

8) N: # of bits

X_i : 1 if ith bit is 1 otherwise 0.

$$Y = X_1 + X_2 + \cdots X_N$$

$E[X_1 + \cdots X_N] = E[X_1] E[N]$ (shown in class)

$$E[X_1 + \cdots X_N] = \sum_{n=0}^{\infty} E[X_1 + \cdots X_N \| N = n] \, P(N=n)$$

$$= \sum_{n=0}^{\infty} E[X_1 + \cdots X_n | N = n] \, P(N=n)$$

$$= \sum_{n=0}^{\infty} n \, E[X_1] \, P(N=n)$$

iid and independent of N

$$= E[X_1] E[N].$$

$$= p \cdot \lambda.$$

b) $$P(Y=y) = \sum_{n=y}^{\infty} P(Y=y | N=n) \, P(N=n)$$

$$P(Y=y | N=n) = P(\underbrace{X_1 + \cdots X_n}_{\text{Bernoulli}(n,p)} = y).$$

$$= \binom{n}{y} p^y (1-p)^{n-y}.$$

$$P(Y=y) = \sum_{n=y}^{\infty} \binom{n}{y} p^y (1-p)^{n-y} \frac{e^{-\lambda} \lambda^n}{n!}$$

$$= \sum_{n=y}^{\infty} \frac{\cancel{n!}}{y! \, (n-y)!} \frac{(\lambda p)^y (\lambda(1-p))^{n-y} e^{-\lambda p} e^{-\lambda(1-p)}}{\cancel{n!}}$$

Made with Goodnotes

$$= \frac{(\lambda p)^y}{y!} e^{-\lambda p} \underbrace{\boxed{\sum_{n=y}^{\infty} \frac{(\lambda(1-p))^{n-y}}{(n-y)!} e^{-\lambda(1-p)}}}_{e^{\lambda(1-p)}}$$

$$= \frac{(\lambda p)^y}{y!} e^{-\lambda p}.$$

$\Rightarrow Y$ is Poisson(λp).