Problem Set 1 Discussion : Jan 20

(1) $P(\{i\}) = \begin{cases} x. & i \text{ is odd} \\ 2x & i \text{ is even} \end{cases}$

$\Omega = \{1, 2, 3, 4, 5, 6\}.$

$\sum_{i=1}^{6} P(\{i\}) = 1$ to solve for x.

$A :=$ Prob that experiment outcome is ≤ 4.
$= \{1, 2, 3, 4\}$

$P(A) = \sum_{i \in A} P(\{i\})$

(2) (a) $P(A_1 \cap \cdots A_n).$

$= P\left(\left(\bigcup A_i^c\right)^c\right)$

$= 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right)$

$\geq 1 - \sum_{i=1}^{n} \underbrace{P(A_i^c)}_{(1 - P(A_i))}$

$= 1 - (n) + \sum_{i=1}^{n} P(A_i)$

Union bound
$P\left(\bigcup_{i=1}^{n} A_i\right)$
$\leq \sum_{i=1}^{n} P(A_i)$

(b). $P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right)$

$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

$\lim_{n \to \infty} P\left(\bigcup_{i=1}^{n} A_i\right) \leq \lim_{n \to \infty} \sum_{i=1}^{n} P(A_i)$

x_n, y_n be two sequences such that

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty} P(A_i).$$ $x_n \le y_n$ then $\lim x_n \le \lim y_n$

③.

$P(A) = P(B) = 0.9.$

ⓐ. $P(A \cup B) \ge 0.9$? ✓

ⓑ $P(B^c | A) \le 0.12$?

ⓒ $P(A \cap B) = 0.81$? ✓

ⓓ $P(A \cap B) \le 0.9$?. ✓

ⓔ $P(B^c | A) \ge 0.5$?

ⓕ $P(A \cap B) \ge 0.8$. ? ✓

$\boxed{A \cup B \supseteq A \quad A \cup B \supseteq B.}$

$\boxed{A \cap B \subseteq A. \quad A \cap B \subseteq B.}$

$$P(A \cap B) \ge P(A) + P(B) - (2-1)$$
$$= 0.9 + 0.9 - 1$$
$$= 0.8.$$

If A and B are highly likely (Prob close to 1)

then $P(A \cap B)$ is also very likely.

$P(A_i) \ge 1 - \delta. \quad \forall i = 1, 2, \dots n$

$$P(A_1 \cap A_2 \dots \cap A_n) \ge n(1-\delta) - (n-1)$$
$$= n - n\delta - (n-1)$$
$$= (1 - n\delta).$$

$$P(B^c | A) = \frac{P(A \cap B^c)}{P(A)} \le \frac{0.1}{0.9} = \frac{1}{9} = 0.11..$$

We know $P(A)$ and we have lower bound on $P(A \cap B)$.

$P(A) = P(A \cap B) + P(A \cap B^c)$

$0.9 - P(A \cap B) = P(A \cap B^c)$

$P(A \cap B) \ge 0.8.$

$P(A \cap B^c) \le 0.9 - 0.8 = 0.1$

Made with Goodnotes

$$P(B^c \mid A^c) = \frac{P(B^c \cap A^c)}{P(A^c)}$$

x

	B	B^c
A	$(1-y)\times 0.9$	$y \times 0.9$
A^c	$(1-x)\times 0.1$	$x \times 0.1$

x can be any value in $\{0,1\}$

$$= \frac{P(A^c) - P(B \cap A^c)}{P(A^c)}.$$

⑥.

$A \cap (B \cup C) \to A \cap B$, $A \cap (C \cap B^c)$

$A \cap (B \cup C)^c \to A \cap B^c \cap C^c$.

$A_1, A_2, \ldots A_n$ are partitioning Ω.

$$P(A) = \sum_{i=1}^{n} P(A \cap A_i)$$

$$P(A) = P(A \cap B) + P(A \cap C \cap B^c) + P(A \cap B^c \cap C^c)$$

$$P(A \cap C) = P(A \cap C \cap B^c) + P(A \cap C \cap B).$$

7. ⓐ.

$$\Omega = \{ (k,m) : k, m \in W \}$$

$$P(\{(k,m)\}) = p^c (1-p)^{k+m}.$$

$$\sum_{m \in W} \sum_{k \in W} P(\{(k,m)\}) = 1.$$

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} p^c (1-p)^{k+m} = 1.$$

$$\left(\sum_{k=0}^{\infty} (1-p)^k \right) \left(\sum_{m=0}^{\infty} (1-p)^m \right) = \cancel{p}^{-c}$$

$$1 + (1-p) + (1-p)^2 + \ldots = S.$$

$$1 + (1-p) + (1-p)^2 + \ldots = S(1-p) + 1$$

$$\boxed{= \frac{1}{1-(1-p)} = \frac{1}{p}}$$

$S(1-p)+1=S$

$1 = S(p)$.

LHS $= p^{-2}$

and RHS $= p^{-c}$

$\Rightarrow c=2$.

(b) $$\sum_{k=0}^{\infty} P(\{k,k\}) = \sum_{k=0}^{\infty} p^2(1-p)^{2k}$$

(c) $$\sum_{m=0}^{\infty}\sum_{k=m}^{\infty} p^2 (1-p)^{k+m}$$

$$p^2 \sum_{m=0}^{\infty} (1-p)^m \underbrace{\left[\sum_{k=m}^{\infty} (1-p)^k\right]}$$

$$(1-p)^m \underbrace{\sum_{k=0}^{\infty} (1-p)^k}_{\frac{1}{p}} = (1-p)^m + (1-p)^{m+1} + \dots$$

$$= p^{\cancel{2}} \sum_{m=0}^{\infty} (1-p)^m \frac{(1-p)^m}{\cancel{p}}$$

$$= p \sum_{m=0}^{\infty} (1-p)^{2m}.$$

$$= p \frac{1}{1-(1-p)^2}.$$

(d) $P(k \text{ is odd})$.

(8). A: two heads $= \{HH\}$ $\quad \Omega = \{HH, HT, TH, TT\}$

B : first toss is head. $= \{HT, HH\}$

C : atleast one of the tosses is head. $= \{HT, HH, TH\}$.

Prob of seeing head $= p$.

$$P(A|B) \geq P(A|C) \qquad P(\{HH\}) = p^2$$

$$P(\{HT\}) = P(\{TH\}) = p(1-p)$$

$$P(\{TT\}) = (1-p)^2.$$

$$\frac{P(A\cap B)}{P(B)} = \frac{p^2}{p^2+p(1-p)}$$

$$P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{p^2}{p^2+2p(1-p)}.$$

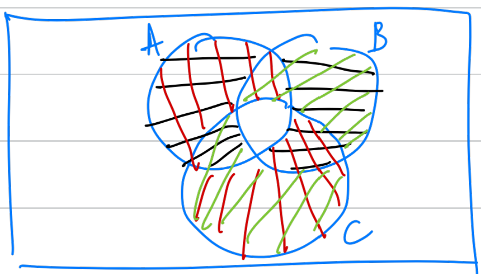
$$\frac{p^2}{p^2+p(1-p)} - \frac{p^2}{p^2+2p(1-p)}$$

$$= p^2\left[\frac{(\cancel{p^2}+2p(1-p)) - (\cancel{p^2}+p(1-p))}{P(B)\,P(C)}\right]$$

$$= \frac{p^2\; p(1-p)}{P(B)P(C)} \geq 0.$$

(9). $(A\Delta B) \subseteq (A\Delta C) \cup (C\Delta B)$.

Using union bound $\quad P(A\Delta B) \leq P(A\Delta C) + P(C\Delta B)$.



$\equiv$ $A\Delta B$

$\equiv$ $A\Delta C$.

$\equiv$ $B\Delta C$.

$A\Delta B \subseteq (A\Delta C) \cup (B\Delta C)$

$x \in A \Delta B$

$\Rightarrow x \in A$ or $x \in B$
but noth both.

① $x \in A$, $x \notin B$.

ⓐ Suppose $x \notin C$.

$\Rightarrow x \in A \Delta C$.

ⓑ Suppose $x \in C$

$\Rightarrow x \in B \Delta C$

② $x \in B$, $x \notin A$.

ⓐ Suppose $x \notin C$

$x \in B \Delta C$.

ⓑ .

If $x \in A \Delta B$.

$\Rightarrow x \in (A \Delta C) \cup (B \Delta C)$

$\Rightarrow A \Delta B \subseteq (A \Delta C) \cup (B \Delta C)$

$P(A \Delta C) \leq P((A \Delta C) \cup (B \Delta C))$

$\cancel{A} \subseteq B$
$P(A) \leq P(B)$

$\leq P(A \Delta C) + P(B \Delta C)$

union bound

11. A_j : event that term paper is in drawer j.

$P(A_j) = p_j$

$\sum_{j=1}^{n} P(A_j) = 1$.

B_i = prob of finding term paper from drawer i.

$\sum_{j=1}^{n} P_j = 1$.

$$P(B_i \mid A_j) = \begin{cases} d_i & j = i \\ 0. & j \neq i \end{cases}$$

$P(A_j \mid B_i^c) = P(A_j \cap B_i^c)$ $j \neq i$

$$\overline{\quad\quad} \\ P(B_i^c).$$

$$P(A_j \cap B_i^c) = P(B_i^c \mid A_j)\, P(A_j)$$

$$= [1 - P(B_i \mid A_j)]\, P(A_j)$$

$$= P(A_j) = p_j.$$

$$P(B_i^c) = \sum_{j=1}^{n} P(B_i^c \cap A_j)$$

$$= P(B_i^c \cap A_i) + \sum_{j \neq i} P(B_i^c \cap A_j)$$

$$= \underbrace{P(B_i^c \mid A_i)}_{(1-d_i)\,p_i}\, \underbrace{P(A_i)}\, + \sum_{j \neq i} \underbrace{P(B_i^c \mid A_j)}_{1 \cdot p_j}\, P(A_j)$$

$$(1-d_i)\,p_i + \boxed{\sum_{j \neq i} p_j} = (1-p_i).$$

$$(1-d_i)\,p_i + (1-p_i).$$

$$\frac{P(A_j \cap B_i^c)}{P(B_i^c)}$$

$$\frac{p_j}{(1-d_i)\,p_i + (1-p_i)}$$