The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

Specifications of Control Systems

- Transient response
- Stability
- Steady state response

Steady State Error

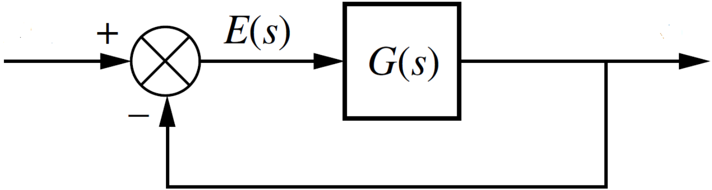
$$e(t) = R(t) - Y(t)$$

$$\text{Steady State Error (SSE)} \quad = \quad \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\big(R(s) - Y(s)\big)$$

SSE is meaningful only when system is BIBO stable

SSE depends on both system and input signal

Derivation of $E(s)$



$$E(s) = R(s) - Y(s)$$

$$Y(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Steady State Error

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

$$r(t) = u(t) \qquad\qquad R(s) = \frac{1}{s}$$

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

$$K_p = \lim_{s \to 0} G(s) = \text{Position Error Constant}$$

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + K_p}$$

K_p

$$K_p = \lim_{s \to 0} G(s) = \text{Position Error Constant}$$

$$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_k)}{s^l(s+p_1)(s+p_2)\dots(s+p_k)}$$

$$\text{Type of } G(s) := l$$

K_p for Type 0 $G(s)$

Type 0 $$G(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$K_p = \lim_{s\to 0} G(s) = \frac{z_1 z_2 \ldots z_m}{p_1 p_2 \ldots p_n} = \text{finite}$$

$$\lim_{t\to\infty} e(t) = \frac{1}{1+K_p} = \text{finite}$$

K_p for Type 1 $G(s)$

Type 1 $$G(s) = \frac{(s + z_1)(s + z_2)\dots(s + z_m)}{s(s + p_1)(s + p_2)\dots(s + p_n)}$$

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + K_p} = 0$$

K_p for Type 2 $G(s)$

Type 2 $$G(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{s^2(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$K_p = \lim_{s \to 0} G(s) = \infty$$

$$\lim_{t \to \infty} e(t) = \frac{1}{1+K_p} = 0$$

Summary of Step Input

Type 0	$K_p = \text{Finite}$	$\text{SSE} = \dfrac{1}{1 + K_p}$
Type 1	$K_p = \infty$	$\text{SSE} = 0$
Type 2	$K_p = \infty$	$\text{SSE} = 0$

$$G(s) = K$$

$$K_p = K$$

$$\lim_{t \to \infty} e(t) = \frac{1}{1 + K}$$

K increases $\quad\Longrightarrow\quad$ steady state error decreses

Steady State Error: Ramp Input

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

$$r(t) = tu(t) \qquad\qquad R(s) = \frac{1}{s^2}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{s + sG(s)} = \frac{1}{\lim_{s\to 0} sG(s)}$$

$$K_v = \lim_{s\to 0} sG(s) = \text{ Velocity Error Constant}$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_v}$$

K_v for Type 0 $G(s)$

Type 0 $$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_v = \lim_{s\to 0} sG(s) = 0$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_v} = \infty$$

K_v for Type 1 $G(s)$

Type 1 $$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{s(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_v = \lim_{s\to 0} sG(s) = \frac{z_1 z_2 \dots z_m}{p_1 p_2 \dots p_n} = \text{Finite}$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_v} = \text{Finite}$$

K_v for Type 2 $G(s)$

Type 2 $$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{s^2(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_v = \lim_{s\to 0} sG(s) = \infty$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_v} = 0$$

Summary of Ramp Input

Type 0	$K_v = 0$	$\text{SSE} = \infty$
Type 1	$K_v = \text{Finite}$	$\text{SSE} = \text{Finite}$
Type 2	$K_v = \infty$	$\text{SSE} = 0$

Steady State Error: Parabola Input

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} E(s) = \lim_{s\to 0} \frac{sR(s)}{1 + G(s)}$$

$$r(t) = \frac{1}{2}t^2 u(t) \qquad\qquad R(s) = \frac{1}{s^3}$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

$$K_a = \lim_{s\to 0} s^2 G(s) = \text{ Acceleration Error Constant}$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_a}$$

K_a for Type 0 $G(s)$

Type 0 $$G(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_a} = \infty$$

K_a for Type 1 $G(s)$

Type 1
$$G(s) = \frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{s(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$K_a = \lim_{s\to 0} s^2 G(s) = 0$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_a} = \infty$$

K_a for Type 2 $G(s)$

Type 2 $$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{s^2(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$K_a = \lim_{s\to 0} s^2 G(s) = \frac{z_1 z_2 \dots z_m}{p_1 p_2 \dots p_n} = \text{Finite}$$

$$\lim_{t\to\infty} e(t) = \frac{1}{K_a} = \text{Finite}$$

Summary of Parabola Input

Type 0	$K_a = 0$	SSE $= \infty$
Type 1	$K_a = 0$	SSE $= \infty$
Type 2	$K_a =$ Finite	SSE $=$ Finite

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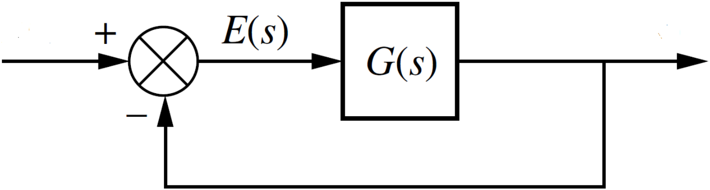
Steady State Error

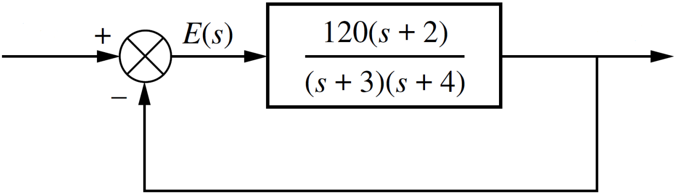


$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Summary: Steady State Error

		Type 0		Type 1		Type 2	
Input	**Steady-state error formula**	**Static error constant**	**Error**	**Static error constant**	**Error**	**Static error constant**	**Error**
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

Example 1



$$T(s) = \frac{120(s+2)}{s^2 + 127s + 252}$$

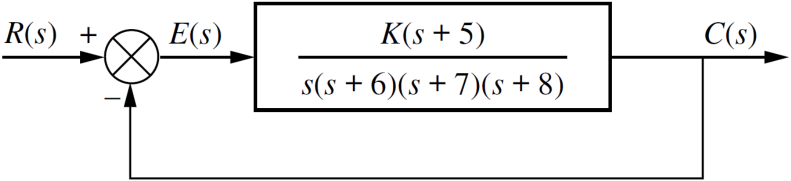
Closed loop poles: $-124.98, \quad -2.016$

$$K_p = 20 \qquad e(\infty) = \frac{1}{21}$$

$$K_v = 0 \qquad e(\infty) = \infty$$

$$K_a = 0 \qquad e(\infty) = \infty$$

Example 2

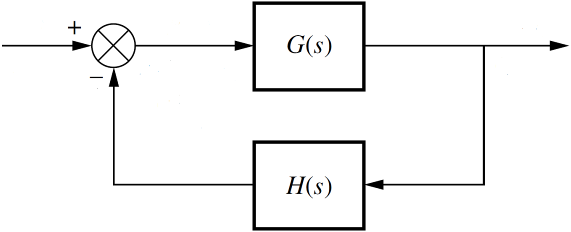


Find K such that there is 10 % steady state error.

$$K_v = \frac{K \times 5}{6 \times 7 \times 8}$$

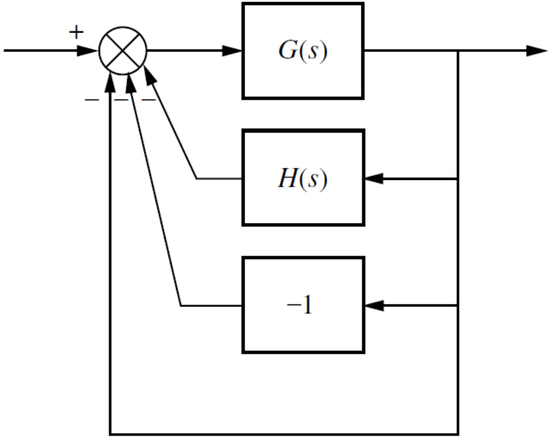
$$K_v = \frac{1}{0.1} \qquad K = 672$$

Non-unity Feedback

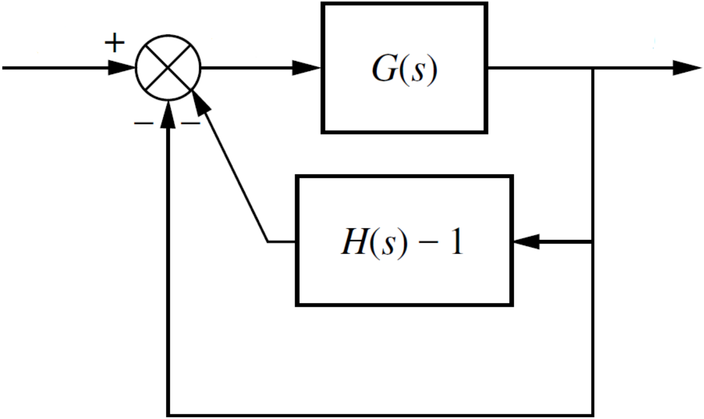


$$T(s) = \frac{G(s)}{1 + G(s)H(s)}$$

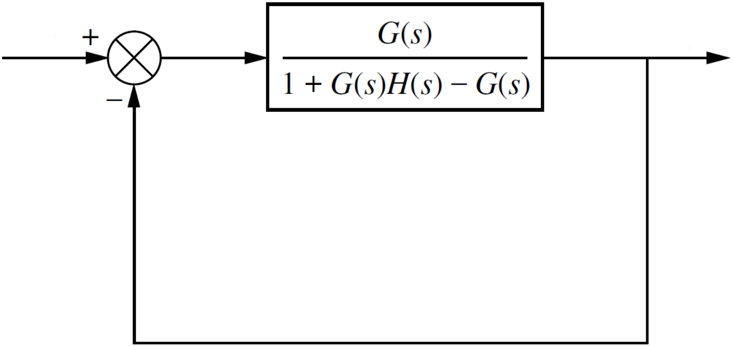
Non-unity Feedback



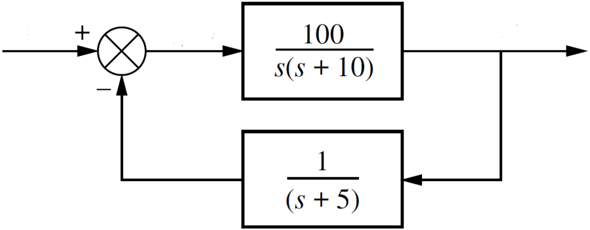
Non-unity Feedback



Non-unity Feedback



Non-unity Feedback

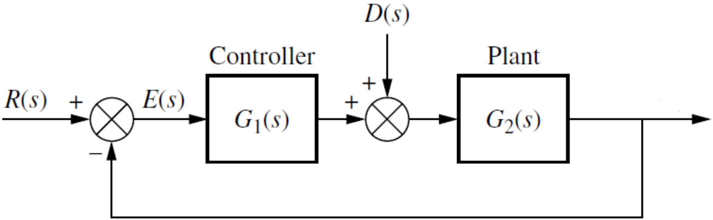


$$G_{eq} = \frac{100(s+5)}{s^3+15s^2-50s-400}$$

$$K_p = -1.25$$

$$e(\infty) = -4$$

Steady State Error with Disturbance



$$Y(s) = G_1(s)G_2(s)E(s) + G_2(s)D(s)$$

$$E(s) = R(s) - Y(s)$$

$$E(s) = \frac{1}{1 + G_1(s)G_2(s)} R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$e(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s) - \lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$e(\infty) = e_R(\infty) \quad + \quad e_D(\infty)$$

$$e(\infty) = -\lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

$$R(s) = \frac{1}{s}$$

$$e_D(\infty) = -\frac{1}{\lim_{s \to 0} \frac{1}{G_2(s)} + \lim_{s \to 0} G_1(s)}$$

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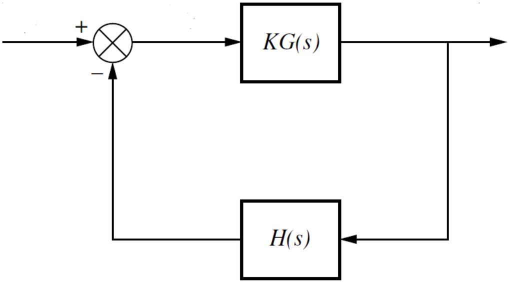
Complex Function

$$F(s) = s + 3$$

$$s_1 = -4 + j0 \qquad F(s_1) = -1 + j0 = 1\angle 180^\circ$$

$$s_2 = 0 + j5 \qquad F(s_2) = 3 + j5 = \sqrt{3^2 + 5^2}\angle tan^{-1}\left(\frac{5}{3}\right)$$

Control System



$$\text{OLTF} = K\,G(s)\,H(s)$$

$$\text{CLTF} = \frac{KG(s)}{1 + KG(s)H(s)}$$

Example

$$G(s) = \frac{s+1}{s(s+2)} \qquad\qquad H(s) = \frac{s+3}{s+4}$$

$$\text{OLTF} = \frac{K(s+1)(s+3)}{s(s+2)(s+4)}$$

$$\text{CLTF} = \frac{K(s+1)(s+4)}{s(s+2)(s+4) + K(s+1)(s+3)}$$

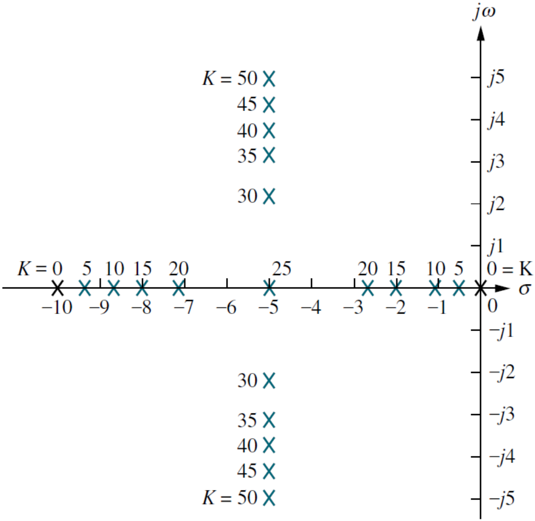
Example

$$G(s) = \frac{1}{s(s+10)} \qquad\qquad H(s) = 1$$

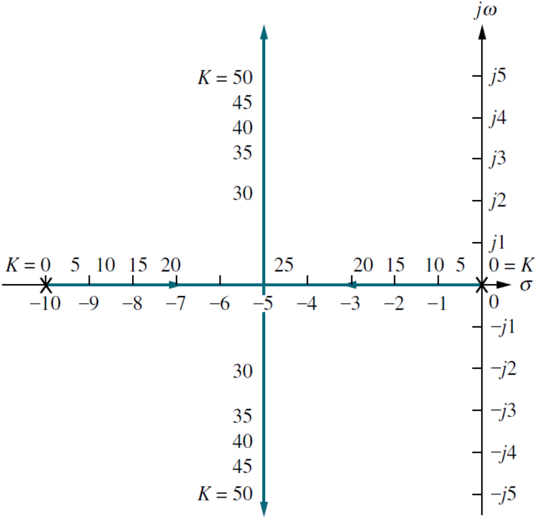
$$T(s) = \frac{K}{s^2 + 10s + K}$$

$$\text{Closed-loop poles} = \frac{-10 \pm \sqrt{100 - 4K}}{2}$$

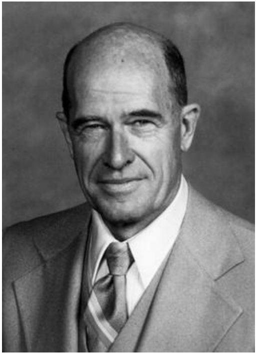
K	Closed-loop poles	
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
25	-5	-5
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$



jω
j5
j4
j3
j2
j1
0 = K
σ
0
−j1
−j2
−j3
−j4
−j5
K = 50
45
40
35
30
K = 0
5
10
15
20
25
20
15
10
5
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
30
35
40
45
K = 50

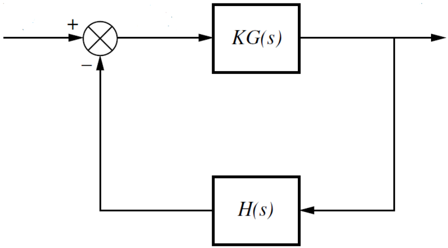

jω
K = 50
45
40
35
30
j5
j4
j3
j2
j1
K = 0
5
10
15
20
25
20
15
10
5
0 = K
σ
−10
−9
−8
−7
−6
−5
−4
−3
−2
−1
0
−j1
−j2
−j3
−j4
−j5
30
35
40
45
K = 50

Walter Evans (1920-1999)



- Root locus
- Year: 1948

Properties



$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

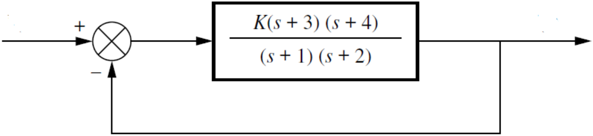
Conditions

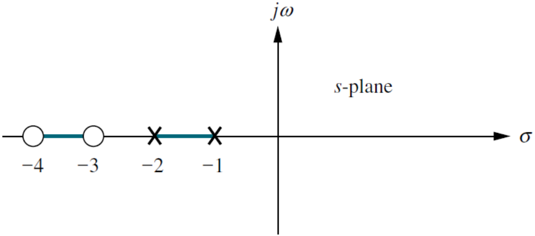
$$1 + KG(s_1)H(s_1) = 0$$

$$KG(s_1)H(s_1) = -1 = 1\angle(2k+1)180^\circ \qquad k = 0, \pm 1, \pm 2, \ldots$$

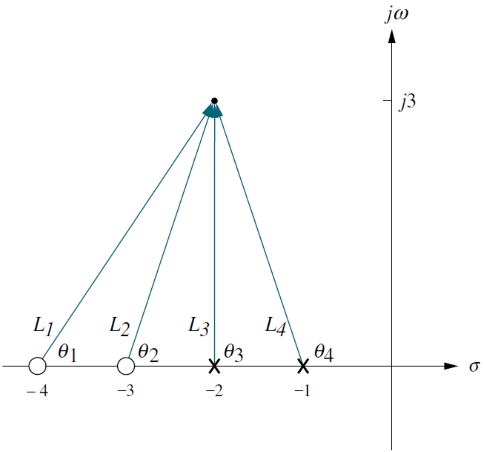
$$\angle G(s_1)H(s_1) = (2k+1)180^\circ \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$\left|KG(s_1)H(s_1)\right| = 1$$





$$s_1 = -2 + j3$$



$$
\begin{aligned}
\angle G(s_1)H(s_1) &= \theta_1 + \theta_2 - \theta_3 - \theta_4 \\
&= 56.31^\circ + 71.57^\circ - 90^\circ - 108.43^\circ \\
&= -70.55^\circ
\end{aligned}
$$

$$s_2 = -2 + j\frac{1}{\sqrt{2}}$$

$$\left|KG(s_2)H(s_2)\right| = 1$$

$$K = \frac{1}{\left|G(s_2)H(s_2)\right|} = \frac{L_3 L_4}{L_1 L_2}$$

$$K = 0.33$$

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$$y(t) = \left(\frac{1}{16} - \frac{1}{9}e^{-t} + \frac{7}{144}e^{-4t} + \frac{1}{12}te^{-t}\right)u(t)$$

$$y_1(t) = \left(\frac{1}{9}e^{-t} - \frac{1}{9}e^{-4t} - \frac{1}{3}te^{-4t}\right)$$

$$\frac{\pi - \theta}{w_d} = 1.3 \qquad \frac{\pi}{w_d} = 2.2$$

$$\theta = 0.409\pi \qquad \zeta = cos\theta = 0.282$$

$$w_n = 1.488$$

$$T(s) = \frac{4.431}{s^2 + 0.840s + 2.215}$$

$$\zeta w_n = 0.840$$

$$\zeta = 0.282 \qquad\qquad w_n = 2.979$$

$$T(s) = \frac{17.748}{s^2 + 1.680s + 8.874}$$

$$P(s) = s^6 - 2s^5 + 3s^4 - 12s^3 - 13s^2 - 10s - 15$$

$$P(s) = \left(s^2 - 2s - 2\right)\left(s^4 + 6s^2 + 5\right)$$

$$P(s) = (s - 3)(s + 1)(s^2 + 5)(s^2 + 1)$$

Conditions

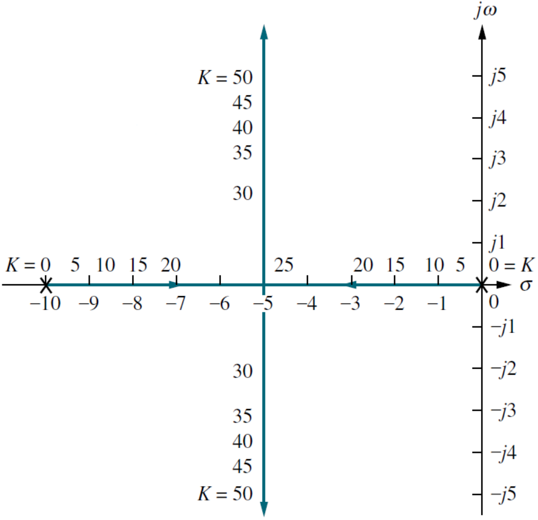
$$1 + KG(s_1)H(s_1) = 0$$

$$KG(s_1)H(s_1) = -1 = 1\angle(2k+1)180^\circ \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$\angle G(s_1)H(s_1) = (2k+1)180^\circ \qquad k = 0, \pm 1, \pm 2, \ldots$$

$$\left|KG(s_1)H(s_1)\right| = 1$$

$$G(s) = \frac{1}{s(s+10)}$$



Rules

1. Number of branches in root locus $=$ Number of closed-loop poles

2. Root locus is symmetric around real axis

3. If the sum of number of open-loop zeros and poles on the real axis and to the right hand side of point s_1 on the real axis is odd, then point s_1 is on the root locus

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

$$G(s) = \frac{N_G(s)}{D_G(s)} \qquad\qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

$$T(s) = \frac{KN_G(s)N_H(s)}{D_G(s)D_H(s) + K\,N_G(s)N_H(s)}$$

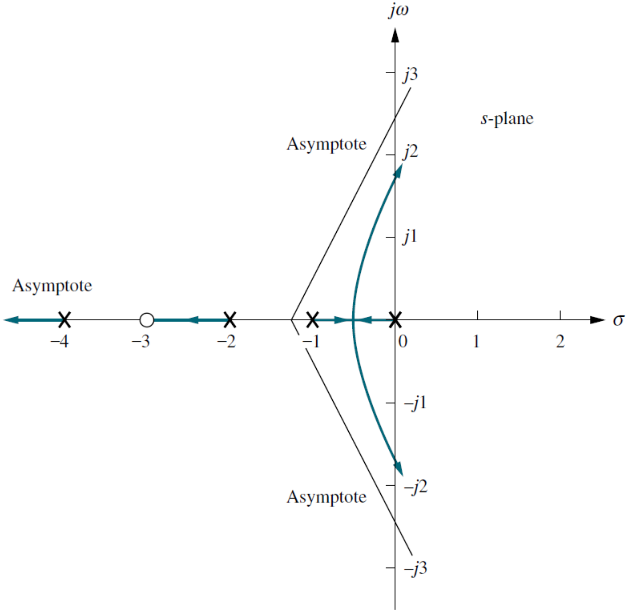
$$T(s) = \frac{N_G(s)N_H(s)}{\frac{1}{K}D_G(s)D_H(s) + N_G(s)N_H(s)}$$

1. Root locus starts at open-loop poles and ends at open-loop zeros

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)(s+4)}$$

$$\text{Asymptote intersection} = \frac{-1-2-4+3}{3} = -\frac{4}{3}$$

$$\text{Asymptote angle} = \frac{(2k+1)180}{3} = 60^\circ,\ 180^\circ,\ 300^\circ$$


jω
j3
s-plane
Asymptote
j2
j1
Asymptote
σ
−4
−3
−2
−1
0
1
2
−j1
−j2
Asymptote
−j3

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Rules

1. Number of branches in root locus $=$ Number of closed-loop poles

2. Root locus is symmetric around real axis

3. If the sum of number of open-loop zeros and poles on the real axis and to the right hand side of point s_1 on the real axis is odd, then point s_1 is on the root locus

4. Root locus begins at open-loop poles and ends at open-loop zeros

Asymptotes

5. The number of finite poles $>$ The number of finite zeros

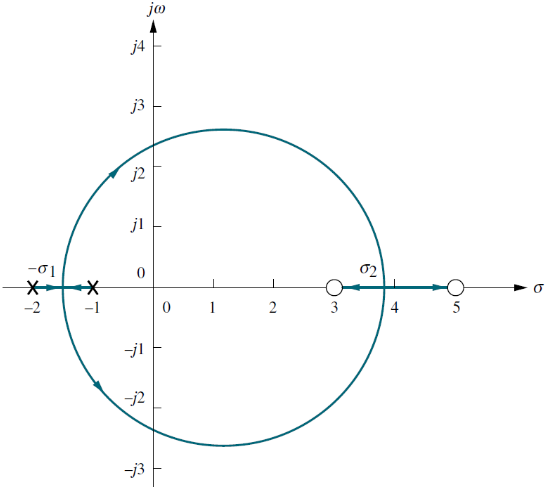
$$\Longrightarrow \quad \text{Number of asymptotes} = \big(\text{Number of finite poles}\big) - \big(\text{Number of finite zeros}\big)$$

σ: Intersection of asymptotes on real axis

$$\sigma = \frac{\text{Sum of real part of poles} - \text{Sum of real part of zeros}}{\text{Number of finite poles} - \text{Number of finite zeros}}$$

$$\text{Angle of asymptotes} = \frac{(2q+1)180^\circ}{\text{Number of finite poles} - \text{Number of finite zeros}}$$

$$G(s)H(s) = \frac{(s-3)(s-5)}{(s+1)(s+2)}$$



$$K = -\frac{1}{G(\sigma)H(\sigma)}$$

$$\frac{dK}{d\sigma} = 0$$

$$K = -\frac{(s+1)(s+2)}{(s-3)(s-5)} = -\frac{\sigma^2 + 3\sigma + 2}{\sigma^2 - 8\sigma + 15}$$

$$\frac{dK}{d\sigma} = \frac{11\sigma^2 - 26\sigma - 61}{\left(\sigma^2 - 8\sigma + 15\right)^2}$$

$$\sigma = -1.45 \qquad \sigma = 3.82$$

$$T(s) = \frac{*}{(1 + K)s^2 + (3 - 8K) + (2 + 15K)}$$

$$K = \frac{3}{8}$$

$$s_1 = 2.355j$$

Angle of Departure

$$G(s)H(s) = \frac{(s+3)(s+4)}{s^2+2s+17)}$$

$$tan^{-1}\left(\frac{4}{3}\right) + tan^{-1}\left(\frac{4}{2}\right) - 90^\circ - \phi = 180^\circ$$

$$\phi = 206.565^\circ$$

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

Angle of Departure

$$G(s)H(s) = \frac{(s+3)(s+4)}{(s^2+2s+17)}$$

$$tan^{-1}\left(\frac{4}{3}\right) + tan^{-1}\left(\frac{4}{2}\right) - 90^\circ - \theta_4 = 180^\circ$$

$$\theta_4 = 206.565^\circ$$

$$G(s)H(s) = \frac{s^2 - 4s + 20}{(s+2)(s+4)}$$

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

$$G(s)H(s) = \frac{s^2 - 4s + 20}{(s+2)(s+4)}$$

$$s_1 = -0.5 + 0.866j \qquad G(s_1)H(s_1) = 2 - 2.887j$$

$$\text{Angle of } G(s_1)H(s_1) = -55.284^\circ$$

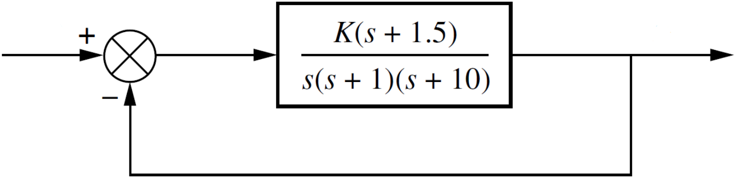
$$s_2 = -0.75 + 1.299j \qquad G(s_2)H(s_2) = 0.256 - 3.638j$$

$$\text{Angle of } G(s_2)H(s_2) = -85.977^\circ$$

$$s_3 = -1.6725 + 2.897j \qquad G(s_3)H(s_3) = -2.766 + 0.001j$$

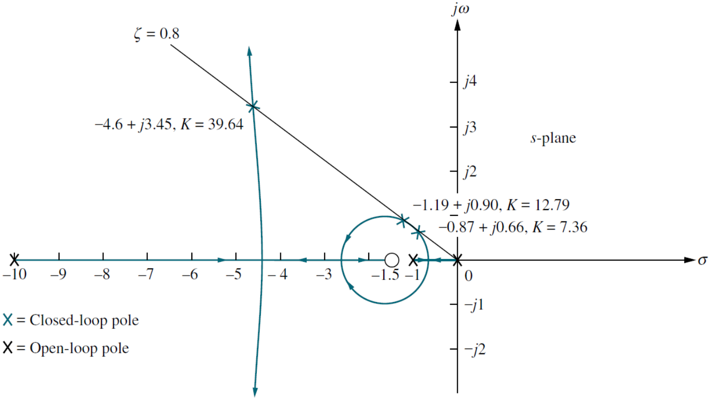
$$\text{Angle of } G(s_3)H(s_3) = 180^\circ$$

$$K = 0.362$$



Required percent overshoot: 1.52 %

$\zeta = 0.8$



jω
ζ = 0.8
−4.6 + j3.45, K = 39.64
j4
j3
j2
s-plane
−1.19 + j0.90, K = 12.79
−0.87 + j0.66, K = 7.36
σ
−10
−9
−8
−7
−6
−5
– 4
−3
−1.5
−1
0
−j1
−j2
X = Closed-loop pole
X = Open-loop pole

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

$$G(s)H(s) = \frac{s^2 - 4s + 20}{(s+2)(s+4)}$$

Second Order Approximation

$$T(s) = \frac{w_n^2}{(s+\alpha)(s^2+2\zeta w_n + w_n^2)}$$

$$T(s) = \frac{s+\beta}{(s^2+2\zeta w_n + w_n^2)}$$

$$T(s) = \frac{s+\beta}{(s+\alpha)(s^2+2\zeta w_n + w_n^2)}$$

Example

$$G(s) = \frac{1}{s(s+1)(s+3)} \qquad\qquad H(s) = 1$$

$$T(s) = \frac{1}{s(s+1)(s+3)+K}$$

Example

$$G(s) = \frac{s + 2.5}{s(s + 1)(s + 3)} \qquad H(s) = 1$$

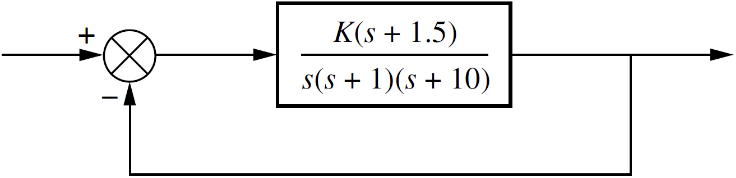
$$T(s) = \frac{s + 2.5}{s(s + 1)(s + 3) + K(s + 2.5)}$$

Procedure for higher order systems

1. Sketch the root locus of $G(s)H(s)$
2. Assume second order system without zeros. Find K at which 2 poles give require transient specification
3. Find remaining closed-loop poles at that K
4. Find closed-loop zeros
5. Check whether second-order approximation is valid

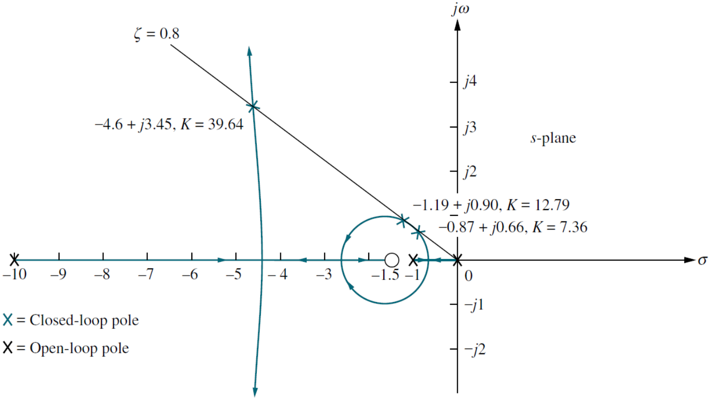
$$G(s) = \frac{s+1}{s+3} \qquad H(s) = \frac{s+2}{s+5}$$

$$T(s) = \frac{(s+1)(s+5)}{(s+3)(s+5) + K(s+1)(s+2)}$$



Required percent overshoot: 1.52 %

$\zeta = 0.8$


jω
ζ = 0.8
−4.6 + j3.45, K = 39.64
j4
j3
j2
s-plane
−1.19 + j0.90, K = 12.79
−0.87 + j0.66, K = 7.36
σ
−10
−9
−8
−7
−6
−5
− 4
−3
−1.5
−1
0
−j1
−j2
X = Closed-loop pole
X = Open-loop pole

Case	Closed-loop poles	Closed-loop zero	Gain	Third closed-loop pole	Settling time	Peak time
1	$-0.87 \pm j0.66$	$-1.5 + j0$	7.36	-9.25	4.60	4.76
2	$-1.19 \pm j0.90$	$-1.5 + j0$	12.79	-8.61	3.36	3.49
3	$-4.60 \pm j3.45$	$-1.5 + j0$	39.64	-1.80	0.87	0.91

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

$$T(s) = \frac{KG(s)}{1 - KG(s)H(s)}$$

$$KG(s)H(s) = 1$$

Angle of $KG(s)H(s) = k\ 360^\circ$

1. Number of branches: same as negative feedback

2. Symmetry: same as negative feedback

3. Real axis: Point s_1 is on the root locus if the number of open-loop zeros and open-loop poles on the real axis to the right of s_1 is even

4. Start and end point: same as negative feedback

5. Asymptotes:

   Intersection with real axis: same

$$\text{Angle} \;=\; \frac{k\,360^\circ}{P-Z}$$

5. Intersection with *jw* axis: Angle of $G(s)H(s) == \ k\ 360^{\circ}$

6. Break in and break away point: same

$$G(s) = \frac{s+3}{s(s+1)(s+2)(s+4)}$$

$$\text{Intersection of asymptotes} = -\frac{4}{3}$$

$$\text{Angle} = 0^\circ,\ 120^\circ,\ 240^\circ$$

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

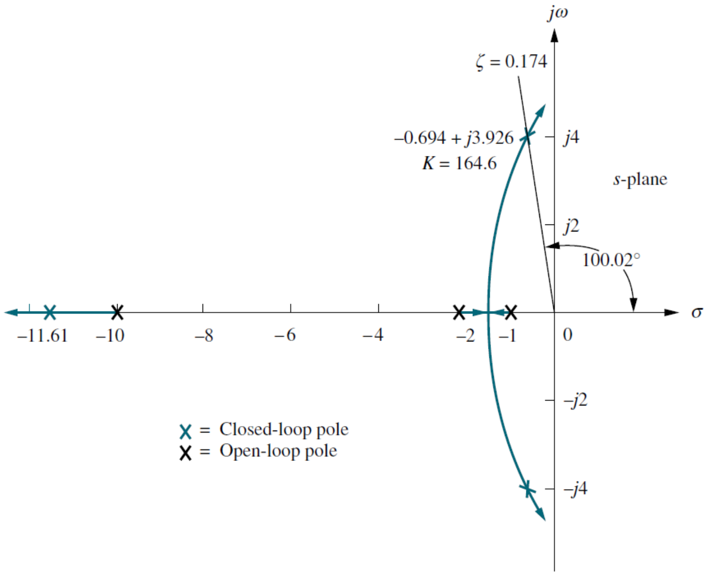
$$G(s) = \frac{s}{s + 1}$$

$$T(s) = \frac{Ks}{s + 1 + Ks} = \frac{Ks}{(1 + K)s + 1}$$

$$T(s) = \frac{Ks}{s + 1 - Ks} = \frac{Ks}{(1 - K)s + 1}$$

$$G(s) = \frac{1}{(s+1)(s+2)(s+10)}$$

$$\zeta = 0.174 \qquad\qquad \theta = 100.02^\circ$$

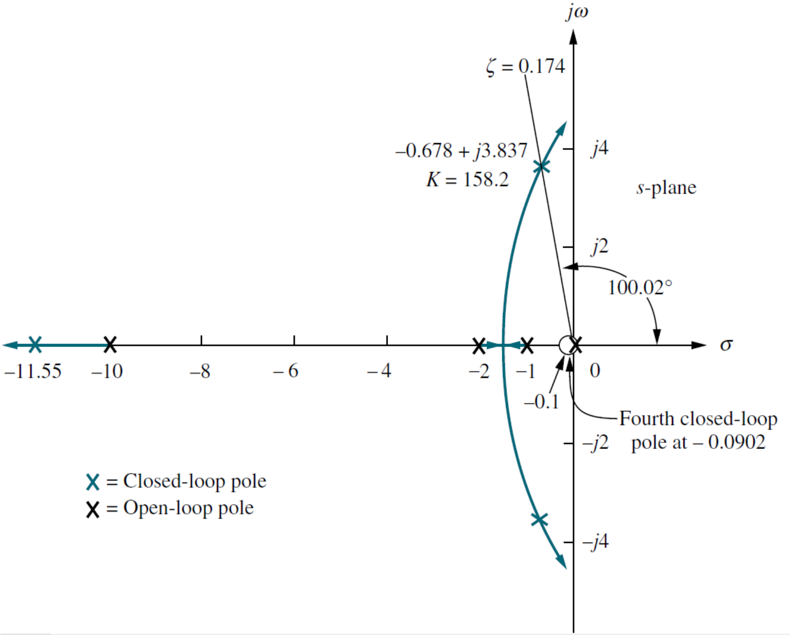


$j\omega$
$\zeta = 0.174$
$-0.694 + j3.926$
$K = 164.6$
$j4$
s-plane
$j2$
100.02°
σ
−11.61
−10
−8
−6
−4
−2
−1
0
$-j2$
$-j4$
= Closed-loop pole
= Open-loop pole

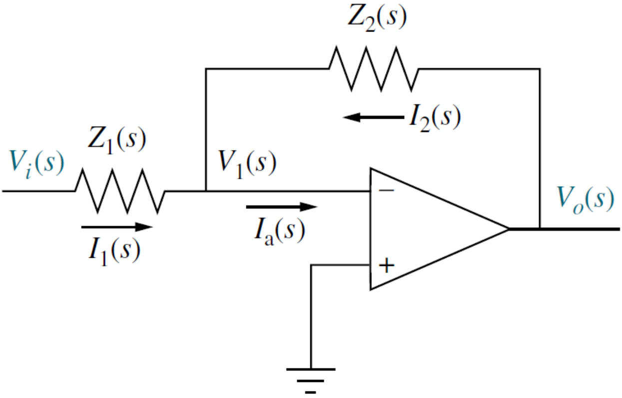
$$K_p = 8.231$$

$$SSE = 0.108$$

$$G_c(s) = \frac{s + 0.1}{s}$$


jω
ζ = 0.174
–0.678 + j3.837
K = 158.2
j4
s-plane
j2
100.02°
σ
–11.55
–10
–8
– 6
– 4
–2
–1
0
–0.1
Fourth closed-loop
pole at – 0.0902
–j2
X = Closed-loop pole
X = Open-loop pole
–j4

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.



$$\frac{V_O}{V_i} = \frac{Z_2(s)}{Z_1(s)}$$

$$Z_1(s) = R_1 \qquad Z_2(s) = R_2 \qquad \frac{Z_2(s)}{Z_1(s)} = \frac{R_2}{R_1}$$

$$Z_1(s) = R_1 \qquad Z_2(s) = \frac{1}{Cs} \qquad \frac{Z_2(s)}{Z_1(s)} = \frac{1}{RCs}$$

$$Z_1(s) = R_1 \qquad Z_2(s) = R_2 + \frac{1}{Cs} \qquad \frac{Z_2(s)}{Z_1(s)} = \frac{R_2}{R_1} \frac{\left(s + \frac{1}{R_2 C}\right)}{s}$$

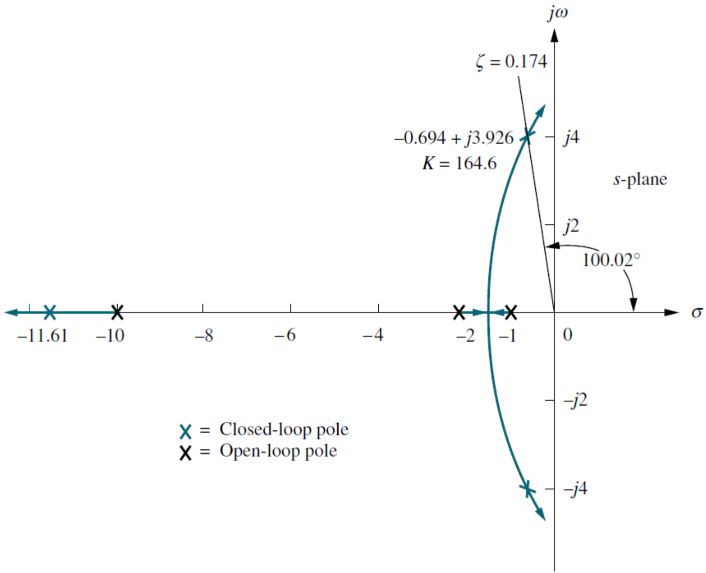
$$G(s) = \frac{s}{s + 1}$$

$$T(s) = \frac{Ks}{s + 1 + Ks} = \frac{Ks}{(1 + K)s + 1}$$

$$T(s) = \frac{Ks}{s + 1 - Ks} = \frac{Ks}{(1 - K)s + 1}$$

$$G(s) = \frac{1}{(s+1)(s+2)(s+10)}$$

$$\zeta = 0.174 \qquad\qquad \theta = 100.02^{\circ}$$



jω
ζ = 0.174
–0.694 + j3.926
K = 164.6
j4
s-plane
j2
100.02°
σ
–11.61
–10
–8
–6
–4
–2
–1
0
–j2
–j4
= Closed-loop pole
= Open-loop pole

$$K_p = 8.231$$

$$SSE = 0.108$$

$$SSE_{new} = \frac{SSE}{10}$$

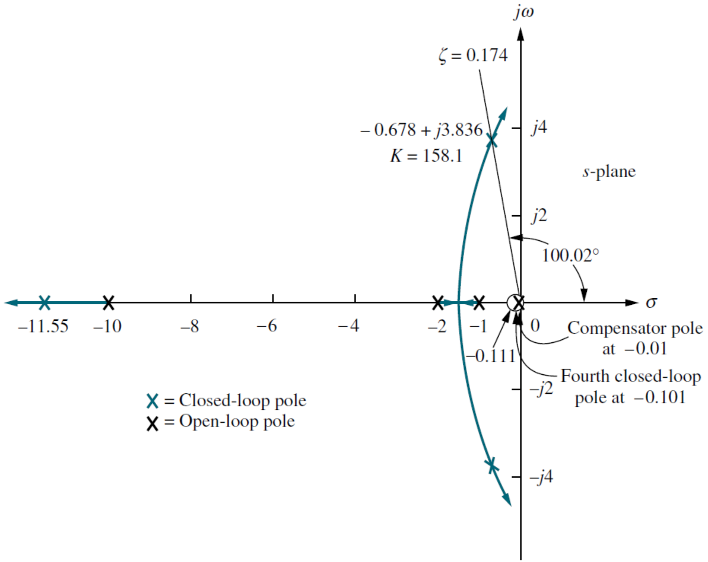
$$\frac{1}{1 + K_p^{new}} = \frac{1}{10(1 + K_p)}$$

$$K_P^{new} = 11.13 K_p$$

$$G_c(s) = \frac{s + z_1}{s + p_1}$$

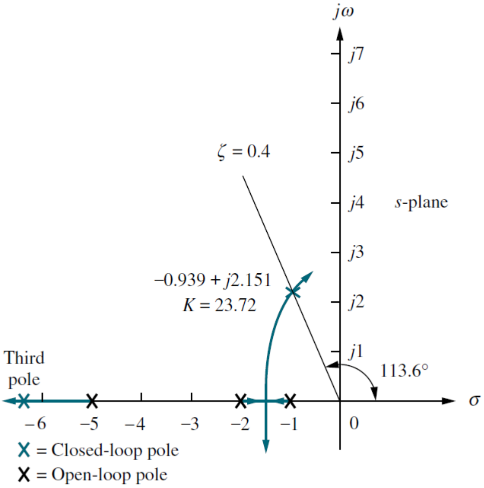
$$\frac{z_1}{p_1} = 11.13$$

$$p_1 = 0.01 \qquad\qquad z_1 = 0.111$$

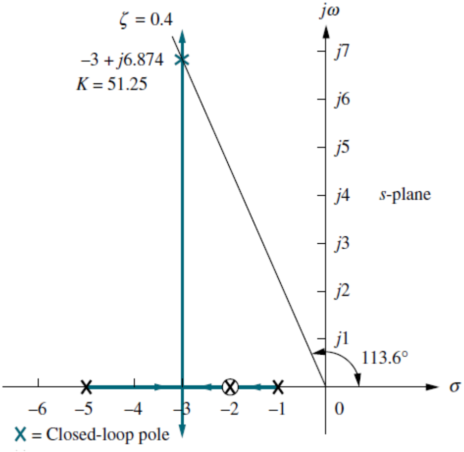


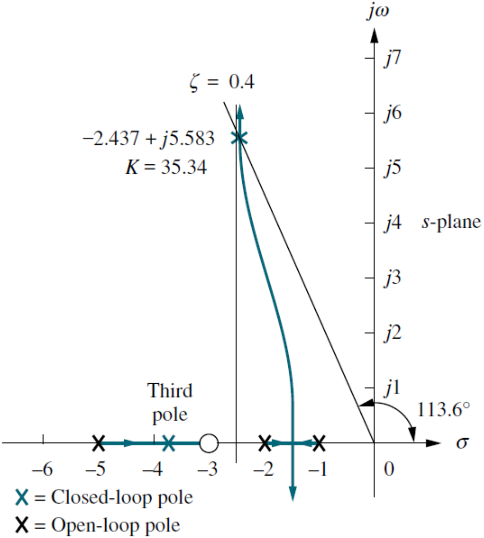
jω
ζ = 0.174
– 0.678 + j3.836
K = 158.1
j4
s-plane
j2
100.02°
σ
–11.55
–10
–8
–6
–4
–2
–1
0
–0.111
Compensator pole at –0.01
Fourth closed-loop pole at –0.101
–j2
–j4
X = Closed-loop pole
X = Open-loop pole

Parameter	Uncompensated	Lag-compensated
Plant and compensator	$\dfrac{K}{(s+1)(s+2)(s+10)}$	$\dfrac{K(s+0.111)}{(s+1)(s+2)(s+10)(s+0.01)}$
K	164.6	158.1
K_p	8.23	87.75
$e(\infty)$	0.108	0.011
Dominant second-order poles	$-0.694 \pm j3.926$	$-0.678 \pm j3.836$
Third pole	-11.61	-11.55
Fourth pole	None	-0.101
Zero	None	-0.111



$j\omega$
$j7$
$j6$
$j5$
$j4$
s-plane
$j3$
$j2$
$j1$
$\zeta = 0.4$
$-0.939 + j2.151$
$K = 23.72$
Third
pole
113.6°
σ
−6
−5
−4
−3
−2
−1
0
= Closed-loop pole
= Open-loop pole


ζ = 0.4
jω
−3 + j6.874
K = 51.25
j7
j6
j5
j4
s-plane
j3
j2
j1
113.6°
σ
−6
−5
−4
−3
−2
−1
0
X = Closed-loop pole



$j\omega$
$j7$
$\zeta = 0.4$
$j6$
$-2.437 + j5.583$
$K = 35.34$
$j5$
$j4$
s-plane
$j3$
$j2$
$j1$
Third
pole
113.6°
σ
–6
–5
–4
–3
–2
–1
0
X = Closed-loop pole
X = Open-loop pole

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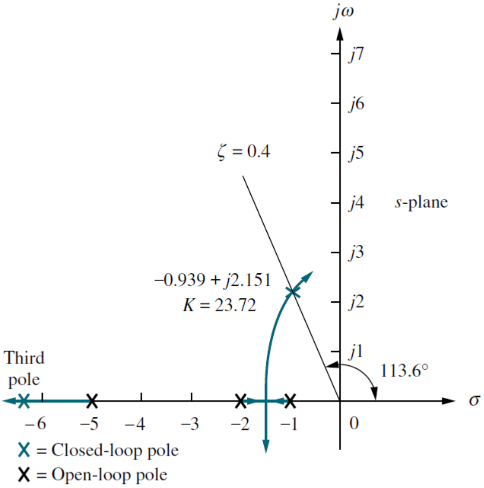
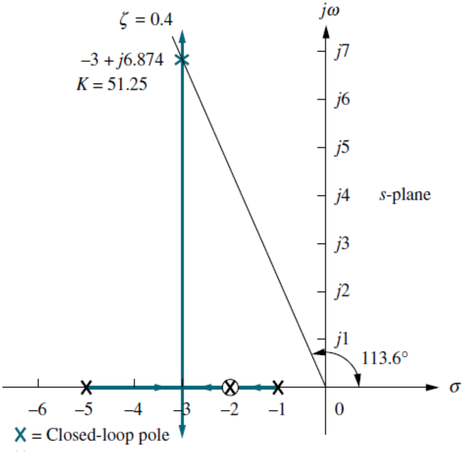
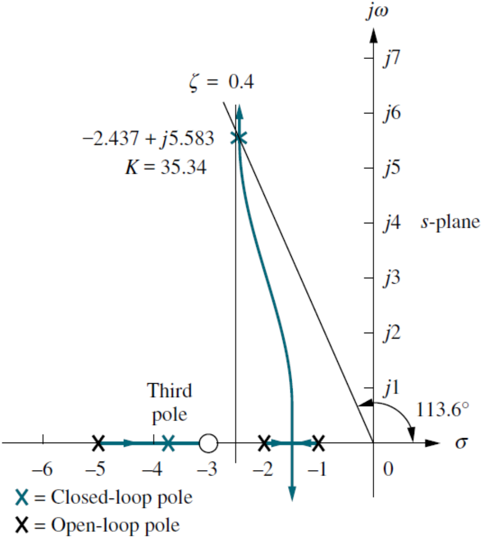


jω
j7
j6
j5
j4
s-plane
j3
j2
j1
ζ = 0.4
−0.939 + j2.151
K = 23.72
113.6°
Third
pole
σ
−6
−5
−4
−3
−2
−1
0
= Closed-loop pole
= Open-loop pole



ζ = 0.4
–3 + j6.874
K = 51.25
jω
j7
j6
j5
j4
s-plane
j3
j2
j1
113.6°
σ
–6
–5
–4
–3
–2
–1
0
= Closed-loop pole

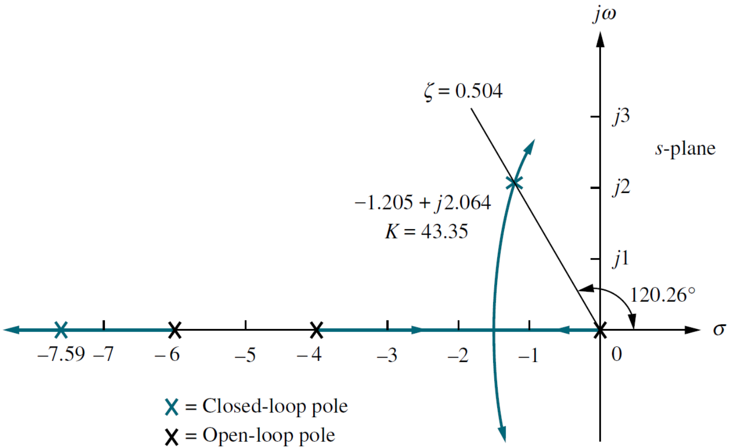


jω
j7
j6
j5
j4
j3
j2
j1
s-plane
ζ = 0.4
−2.437 + j5.583
K = 35.34
Third
pole
113.6°
σ
–6
–5
–4
–3
–2
–1
0
X = Closed-loop pole
X = Open-loop pole

$$G(s) = \frac{1}{s(s+4)(s+6)}$$

$$OS = 16\% \qquad \zeta = 0.504$$

$$\theta = 59.735^\circ$$



jω
ζ = 0.504
j3
s-plane
j2
−1.205 + j2.064
K = 43.35
j1
120.26°
σ
–7.59 –7
– 6
–5
– 4
–3
–2
–1
0
X = Closed-loop pole
X = Open-loop pole

$$T_s = 3.320$$

$$\text{compensated settling time} = 1.107$$

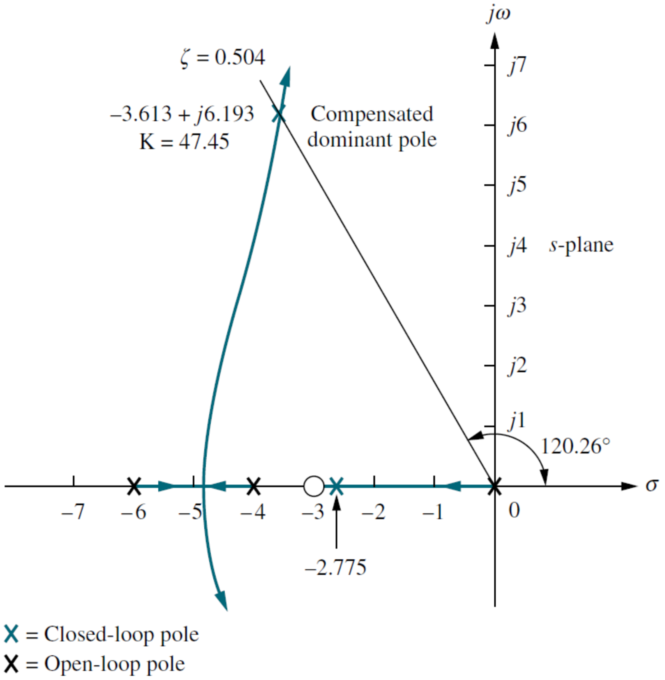
$$\text{Real part of pole} = \frac{4}{1.107} = 3.613$$

$$\text{Imaginary part of pole} = 6.193$$

$$\text{Sum of angle to pole} = -275.6^\circ$$

$$\text{Angle contribution from compensator zero to pole to} = 95.6^\circ$$

$$z_1 = 3.006$$


jω
ζ = 0.504
–3.613 + j6.193
K = 47.45
Compensated
dominant pole
j7
j6
j5
j4
s-plane
j3
j2
j1
120.26°
σ
–7
–6
–5
–4
–3
–2
–1
0
–2.775
X = Closed-loop pole
X = Open-loop pole

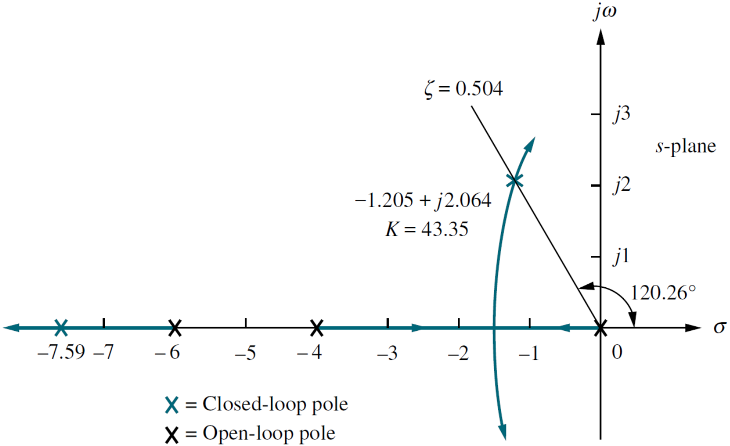
	Uncompensated	Simulation	Compensated	Simulation
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$		$\frac{K(s+3.006)}{s(s+4)(s+6)}$	
Dominant poles	$-1.205 \pm j2.064$		$-3.613 \pm j6.193$	
K	43.35		47.45	
ζ	0.504		0.504	
ω_n	2.39		7.17	
$\%OS$	16	14.8	16	11.8
T_s	3.320	3.6	1.107	1.2
T_p	1.522	1.7	0.507	0.5
K_v	1.806		5.94	
$e(\infty)$	0.554		0.168	
Third pole	−7.591		−2.775	
Zero	None		−3.006	
Comments	Second-order approx. OK		Pole-zero not canceling	

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

$$G(s) = \frac{1}{s(s+4)(s+6)}$$

$$OS = 16\% \qquad \zeta = 0.504$$

$$\theta = 59.735^\circ$$


jω
ζ = 0.504
j3
s-plane
j2
−1.205 + j2.064
K = 43.35
j1
120.26°
σ
–7.59
–7
–6
–5
–4
–3
–2
–1
0
= Closed-loop pole
= Open-loop pole

Angle contribution from compensator zero to pole $= 95.6^\circ$

$$G_c(s) = s + 3.006$$

	Uncompensated	Simulation	Compensated	Simulation
Plant and compensator	$\dfrac{K}{s(s+4)(s+6)}$		$\dfrac{K(s+3.006)}{s(s+4)(s+6)}$	
Dominant poles	$-1.205 \pm j2.064$		$-3.613 \pm j6.193$	
K	43.35		47.45	
ζ	0.504		0.504	
ω_n	2.39		7.17	
$\%OS$	16	14.8	16	11.8
T_s	3.320	3.6	1.107	1.2
T_p	1.522	1.7	0.507	0.5
K_v	1.806		5.94	
$e(\infty)$	0.554		0.168	
Third pole	-7.591		-2.775	
Zero	None		-3.006	
Comments	Second-order approx. OK		Pole-zero not canceling	

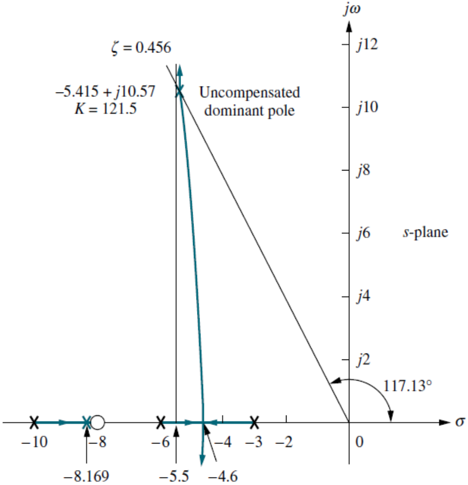
Plant and compensator	$\frac{K}{s(s+4)(s+6)}$	$\frac{K(s+5)}{s(s+4)(s+6)(s+42.96)}$	$\frac{K(s+4)}{s(s+4)(s+6)(s+20.09)}$	$\frac{K(s+2)}{s(s+4)(s+6)(s+8.971)}$
Dominant poles	$-1.007 \pm j2.627$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$	$-2.014 \pm j5.252$
K	63.21	1423	698.1	345.6
ζ	0.358	0.358	0.358	0.358
ω_n	2.813	5.625	5.625	5.625
$\%OS^*$	30 (28)	30 (30.7)	30 (28.2)	30 (14.5)
T_s^*	3.972 (4)	1.986 (2)	1.986 (2)	1.986 (1.7)
T_p^*	1.196 (1.3)	0.598 (0.6)	0.598 (0.6)	0.598 (0.7)
K_v	2.634	6.9	5.791	3.21
$e(\infty)$	0.380	0.145	0.173	0.312
Other poles	-7.986	$-43.8, -5.134$	-22.06	$-13.3, -1.642$
Zero	None	-5	None	-2
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK	No pole-zero cancellation

$$G(s) = \frac{s + 8}{(s + 3)(s + 6)(s + 10)}$$

20% overshoot

$$\text{Desired peak time} = \frac{2}{3} \text{ of uncompensated peak time}$$

$$\text{Desired steady state error under step input} = 0$$



jω
j12
j10
j8
j6
j4
j2
ζ = 0.456
−5.415 + j10.57
K = 121.5
Uncompensated
dominant pole
s-plane
117.13°
σ
−10
−8
−6
−4
−3
−2
0
−8.169
−5.5
−4.6

Plant and compensator	$\dfrac{K(s+8)}{(s+3)(s+6)(s+10)}$
Dominant poles	$-5.415 \pm j10.57$
K	121.5
ζ	0.456
ω_n	11.88
$\%OS$	20
T_s	0.739
T_p	0.297
K_p	5.4
$e(\infty)$	0.156
Other poles	-8.169
Zeros	-8
Comments	Second-order approx. OK

$$\text{Peak time of uncompensated system} \ = \ 0.297$$

$$\text{New peak time} \ = \ \frac{2}{3}(0.297)$$

$$\text{Imaginary part of desired pole} \ = \ 15.87$$

$$\text{Real part of desired pole} \ = \ 8.13$$

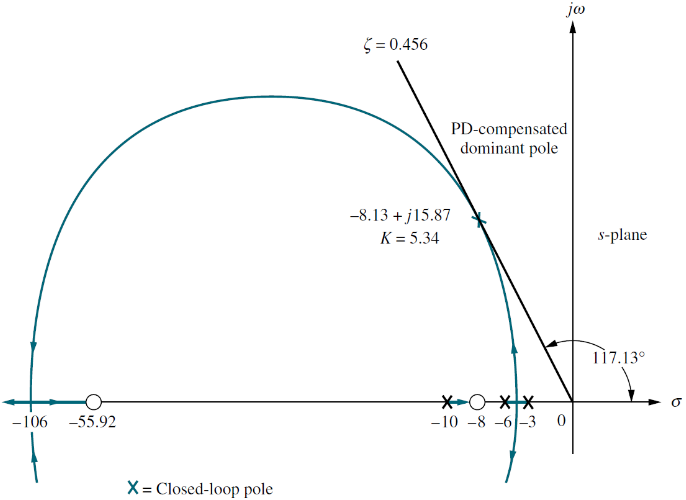
$$\text{Total angle from all zeros and poles to desired pole} = -198.37^\circ$$

$$\text{Angle required from zero} = 18.37^\circ$$

$$z_c = 55.92$$

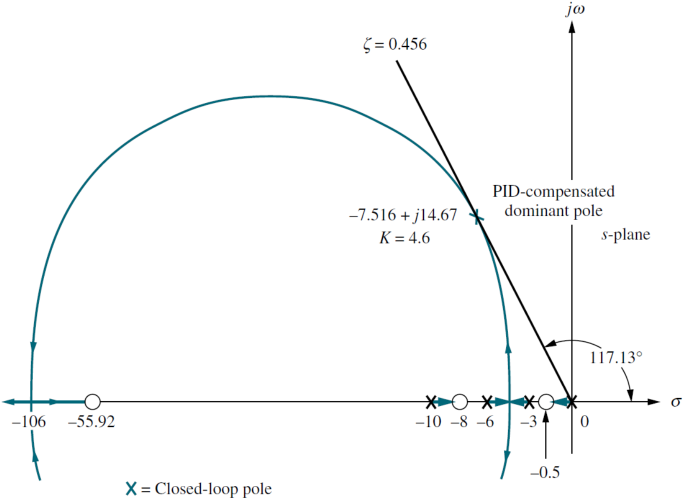
$$G_{PD}(s) = (s + 55.92)$$

Plant and compensator	$\dfrac{K(s+8)}{(s+3)(s+6)(s+10)}$	$\dfrac{K(s+8)(s+55.92)}{(s+3)(s+6)(s+10)}$
Dominant poles	$-5.415 \pm j10.57$	$-8.13 \pm j15.87$
K	121.5	5.34
ζ	0.456	0.456
ω_n	11.88	17.83
$\%OS$	20	20
T_s	0.739	0.492
T_p	0.297	0.198
K_p	5.4	13.27
$e(\infty)$	0.156	0.070
Other poles	-8.169	-8.079
Zeros	-8	$-8, -55.92$
Comments	Second-order approx. OK	Second-order approx. OK


jω
ζ = 0.456
PD-compensated
dominant pole
−8.13 + j15.87
K = 5.34
s-plane
117.13°
σ
−106
−55.92
−10
−8
−6
−3
0
X = Closed-loop pole

$$G_{PI}(s) = \frac{s + 0.5}{s}$$

$$G_{PI}(s)G_{PD}(s)G(s) = \left(s + 55.92\right)\left(\frac{s + 0.5}{s}\right)\left(\frac{s + 8}{(s + 3)(s + 6)(s + 10)}\right)$$



jω
ζ = 0.456
PID-compensated
dominant pole
−7.516 + j14.67
K = 4.6
s-plane
117.13°
σ
−106
−55.92
−10
−8
−6
−3
0
−0.5
X = Closed-loop pole

Plant and compensator	$\dfrac{K(s+8)}{(s+3)(s+6)(s+10)}$	$\dfrac{K(s+8)(s+55.92)}{(s+3)(s+6)(s+10)}$	$\dfrac{K(s+8)(s+55.92)(s+0.5)}{(s+3)(s+6)(s+10)s}$
Dominant poles	$-5.415 \pm j10.57$	$-8.13 \pm j15.87$	$-7.516 \pm j14.67$
K	121.5	5.34	4.6
ζ	0.456	0.456	0.456
ω_n	11.88	17.83	16.49
$\%OS$	20	20	20
T_s	0.739	0.492	0.532
T_p	0.297	0.198	0.214
K_p	5.4	13.27	∞
$e(\infty)$	0.156	0.070	0
Other poles	-8.169	-8.079	-8.099, -0.468
Zeros	-8	-8, -55.92	-8, -55.92, -0.5
Comments	Second-order approx. OK	Second-order approx. OK	Zeros at -55.92 and -0.5 not canceled

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

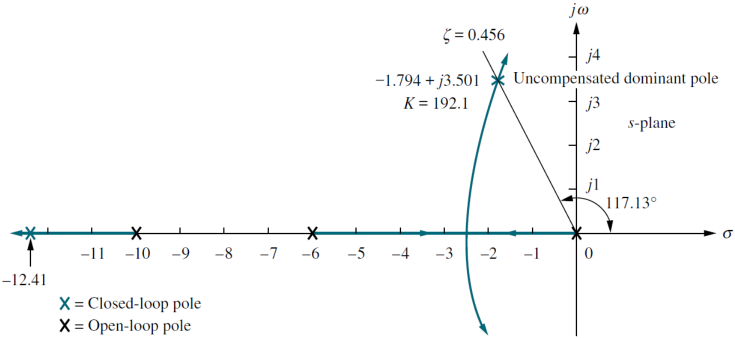
$$G(s) = \frac{1}{s(s+6)(s+10)}$$

$$OS = 20\% \qquad\qquad \zeta = 0.456$$

$$\theta = 117.13^{\circ}$$

Desired settling time $=$ 0.5 (Original settling time)

Desired settling time $=$ 0.1 (Original steady-state error)



$j\omega$
$\zeta = 0.456$
$-1.794 + j3.501$
$K = 192.1$
Uncompensated dominant pole
$j4$
$j3$
$j2$
$j1$
s-plane
117.13°
σ
–11
–10
–9
–8
–7
–6
–5
–4
–3
–2
–1
0
–12.41
X = Closed-loop pole
X = Open-loop pole

	Uncompensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$
Dominant poles	$-1.794 \pm j3.501$
K	192.1
ζ	0.456
ω_n	3.934
$\%OS$	20
T_s	2.230
T_p	0.897
K_v	3.202
$e(\infty)$	0.312
Third pole	-12.41
Zero	None
Comments	Second-order approx. OK

$$\text{Real part of desired pole} = -3.588$$

$$\text{Imaginary part of desired pole} = 7.003$$

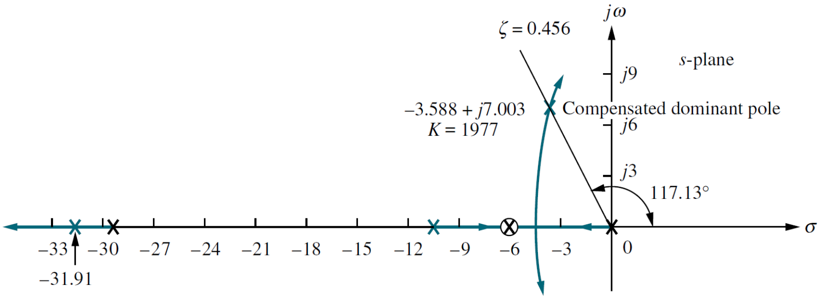
$$G_{Lead} = \frac{s + z_1}{s + p_1}$$

$$z_1 = 6$$

$$\text{Sum of angle to desired pole} = -164.65^\circ$$

$$\text{Required angle from compensator pole} = -15.35^\circ$$

$$G_{lead} = \frac{s + 6}{s + 29.1}$$



$j\omega$
$\zeta = 0.456$
s-plane
$j9$
$-3.588 + j7.003$
$K = 1977$
Compensated dominant pole
$j6$
$j3$
117.13°
σ
−33
−30
−27
−24
−21
−18
−15
−12
−9
−6
−3
0
−31.91

	Uncompensated	Lead-compensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$	$\dfrac{K}{s(s+10)(s+29.1)}$
Dominant poles	$-1.794 \pm j3.501$	$-3.588 \pm j7.003$
K	192.1	1977
ζ	0.456	0.456
ω_n	3.934	7.869
$\%OS$	20	20
T_s	2.230	1.115
T_p	0.897	0.449
K_v	3.202	6.794
$e(\infty)$	0.312	0.147
Third pole	−12.41	−31.92
Zero	None	None
Comments	Second-order approx. OK	Second-order approx. OK

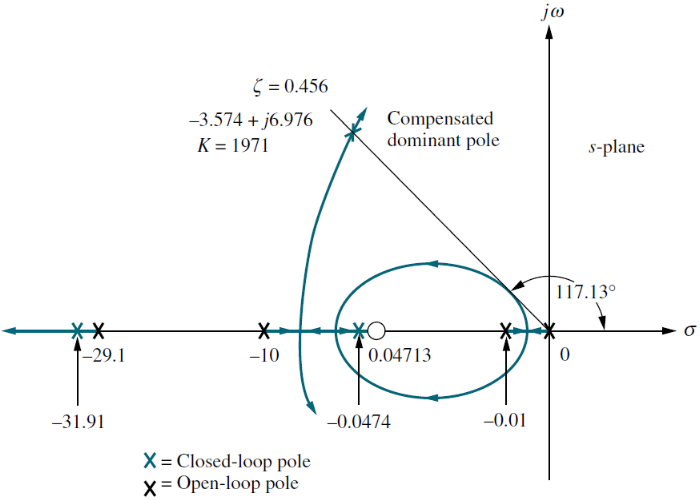
$$\text{Desired } K_v = 32.02$$

$$\text{Required change in } K_v = \frac{32.02}{6.794} = 4.713$$

$$G_{lag} = \frac{s + z_2}{s + p_2}$$

$$p_2 = 0.01$$

$$z_2 = 0.04713$$


$j\omega$
$\zeta = 0.456$
$-3.574 + j6.976$
$K = 1971$
Compensated
dominant pole
s-plane
117.13°
σ
–29.1
–10
0.04713
0
–31.91
–0.0474
–0.01
= Closed-loop pole
= Open-loop pole

	Uncompensated	Lead-compensated	Lag-lead-compensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$	$\dfrac{K}{s(s+10)(s+29.1)}$	$\dfrac{K(s+0.04713)}{s(s+10)(s+29.1)(s+0.01)}$
Dominant poles	$-1.794 \pm j3.501$	$-3.588 \pm j7.003$	$-3.574 \pm j6.976$
K	192.1	1977	1971
ζ	0.456	0.456	0.456
ω_n	3.934	7.869	7.838
$\%OS$	20	20	20
T_s	2.230	1.115	1.119
T_p	0.897	0.449	0.450
K_v	3.202	6.794	31.92
$e(\infty)$	0.312	0.147	0.0313
Third pole	-12.41	-31.92	-31.91, -0.0474
Zero	None	None	-0.04713
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK