7th Feb.

If x_1 and x_2 are independent

$$Var(X_1+X_2) = Var(X_1) + Var(X_2)$$

$$X = (X_1 + X_2 + \dots X_n).$$

$X \sim$ Binomial(n, p)

X_i's are iid Bernoulli(p) R.V.s.

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right] \underset{\text{Linearity of Expectation}}{=} \sum_{i=1}^{n} E[X_i]$$

$$Var(X) = Var\left(\sum X_i\right)$$

(independent) $\leftarrow$ $$= \sum_{i=1}^{n} Var(X_i) = n\,Var(X_1)$$

$$Var(X_1) = E[X_1^2] - (E[X_1])^2 = 1^2 \cdot p - p^2$$

$$= np(1-p)$$

Sample mean

$$S = \frac{X_1 + X_2 + \dots X_n}{n}.$$

$$E[S] = p.$$

$$Var[S] = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$$

as $n \to \infty$ $Var(S) \to 0$.

(b) If $Var(S) = 0$ then $S = E[S]$ w.p 1.

(a) If $E[X^2] = 0$ then $X = 0$ w.p 1.

If $x \neq 0$ w.p > 0.

then $E[X^2] = \sum_{x} x^2 P_X(x)$

$$= \sum_{x \neq 0} x^2 P_X(x) > 0.$$

$$Var(S) = E[\underbrace{(X - E[X])^2}].$$

$$Y = X - E[X].$$

If $E[Y^2] = 0$ then $Y = 0$ w.p 1.

$$\Rightarrow \quad X = E[X] \quad \text{w.p } 1.$$

$$\hat{S}_n = \frac{X_1 + X_2 + \cdots X_n}{\sqrt{n}}.$$

$$Var(\hat{S}_n) = \frac{np(1-p)}{n} = \underline{\underline{p(1-p)}}.$$

$$E[\hat{S}_n] = \frac{np}{\sqrt{n}}. = \underline{\underline{\sqrt{n}\,p}}.$$

CDF of

$\hat{S}_n - E[\hat{S}_n]$ converges to that of Normal distribution

Made with Goodnotes

Lec 15: 10th Feb, 2025

1 Expectation:

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx.$$

Expectation is finite and well-defined if

$$E[|X|] = \int_{-\infty}^{\infty} |x| \, f_X(x) \, dx < \infty$$

1.1 Properties:

① X is a non-negative Random Variable.

$$E[X] = \int_{x=0}^{\infty} P(X > x) \, dx = \int_{x=0}^{\infty} (1 - F_X(x)) \, dx$$

$$= \int_{x=0}^{\infty} \left(\int_{t=x}^{\infty} f_X(t) \, dt \right) dx$$

(x, t) s.t. $t \geq x$

$$= \int_{t=0}^{\infty} \left(\int_{x=0}^{t} dx \right) f_X(t) \, dt.$$

$$= \int_{t=0}^{\infty} t \, f_X(t) \, dt = E[X].$$

② Assume $g(X)$ to be a non-negative function $g : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$.

$$E[g(X)] = \int_{x=-\infty}^{\infty} g(x) \, f_X(x) \, dx.$$

$$Y = g(x)$$

$$E[Y] = \int_{y=0}^{\infty} P(Y > y)\, dy.$$

$$= \int_{y=0}^{\infty} P(g(x) > y)\, dy.$$

$$= \int_{y=0}^{\infty} \left[\int_{x : g(x) > y} f_X(x)\, dx \right] dy.$$

$$= \int_{x=-\infty}^{\infty} \underbrace{\left[\int_{y=0}^{g(x)} dy. \right]} f_X(x)\, dx.$$

$$= \int_{x=-\infty}^{\infty} g(x) f_X(x)\, dx.$$

③ $E[X] = \int_{x=0}^{\infty} P(X > x)\, dx - \int_{x=0}^{\infty} P(X < -x)\, dx.$

④ $E[g(X)] = \int_{x=-\infty}^{\infty} g(x) f_X(x)\, dx.$ for any g.

⑤ Scaling and Shifting

$$E[aX + b] = a\,E[X] + b.$$

Using ④

$$E[aX+b] = \int_{x=-\infty}^{\infty} (ax + b) f_X(x)\, dx$$

$$= a \int x f_X(x)\, dx + b \int f_X(x)\, dx$$

$$= a\,E[X] + b$$

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$$Var(X) = E[(X - E[X])^2]$$

$$Y = aX + b$$

$$Var(aX+b) = E[(Y - E[Y])^2]$$

$$Y - E[Y] = (aX + \cancel{b}) - (aE[X] + \cancel{b})$$

$$= a(X - E[X]).$$

$$Var(Y) = E[a^2 (X - E[X])^2]$$

$$= a^2 E[(X - E[X])^2] = a^2 Var(X).$$

⑥ Show That $Var(X) = E[X^2] - (E[X])^2$.

Examples

① Uniform $([a,b])$.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_a^b \frac{x}{b-a}\, dx$$

$$= \frac{x^2}{2(b-a)}\Big|_a^b = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{a+b}{2}.$$

$$E[X^2] = \int_a^b \frac{x^2}{(b-a)}\, dx.$$

$$= \frac{x^3}{3(b-a)}\Big|_a^b = \frac{b^3 - a^3}{3(b-a)}.$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{b^3 - a^3}{3(b-a)} - \left(\frac{a+b}{2}\right)^2.$$

$$\left[\underset{a}{} \overset{\frac{a+b}{2}}{\downarrow} \underset{b}{}\right]$$

$$\left[\overset{0}{\downarrow} \right]$$

$$\mathrm{Var}(X) = E\left[(X - E[X])^2\right]$$

$$= \int_a^b \left(x - \left(\frac{a+b}{2}\right)\right)^2 \left(\frac{1}{b-a}\right) dx$$

$$y = x - \left(\frac{a+b}{2}\right).$$

$$dy = dx.$$

$$= \int_{-\left(\frac{b-a}{2}\right)}^{\left(\frac{b-a}{2}\right)} \frac{y^2}{(b-a)} dy$$

$$= \left.\frac{y^3}{3(b-a)}\right|_{-\left(\frac{b-a}{2}\right)}^{\left(\frac{b-a}{2}\right)}$$

$$= \frac{2(b-a)^3}{2^3 \times 3(b-a)} = \frac{(b-a)^2}{12}.$$

② Exponential R.V (λ)

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$E[X] = \int_0^\infty x\, f_X(x)\, dx$$

$$= \int_0^\infty \underbrace{x}_{u(x)} \underbrace{\lambda e^{-\lambda x}}_{v'(x)} dx.$$

$v(x) = -e^{-\lambda x}$

$v'(x) = \lambda . e^{-\lambda x}.$

Integration by parts:

$$\int u(t)\, v'(t)\, dt = u(t)\, v(t) - \int u'(t)\, v(t)\, dt$$

$\frac{\partial}{\partial t}$ of RHS is

$$\cancel{u'(t)\, v(t)} + u(t)\, v'(t) - \cancel{u'(t)\, v(t)}$$

$\frac{\partial}{\partial t}$ of LHS $u(t)\, v'(t).$

$$= -x e^{-\lambda x}\Big|_0^\infty + \int 1 \, e^{-\lambda x} \, dx$$

$$= 0 + \frac{e^{-\lambda x}}{(-\lambda)}\Big|_0^\infty$$

$$= \frac{1}{\lambda}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$E[X^2] = \int_0^\infty x^2 e^{-\lambda x} \, dx$$ (Excercise)

$$= \frac{2}{\lambda^2}$$

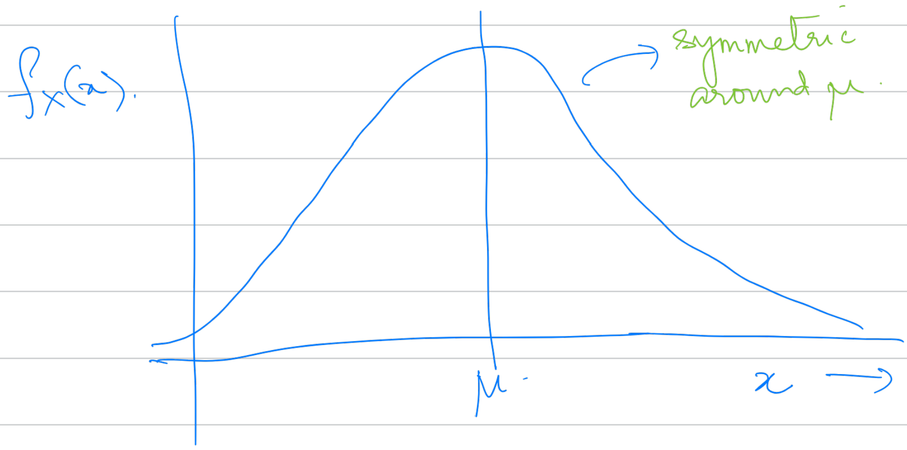
$$Var(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

③ Gaussian R.V or Normal R.V

$$X \sim N(\mu, \sigma^2)$$

Normal R.V described through parameters μ and σ^2.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



$$S_n = \frac{\Sigma X_i}{\sqrt{n}}$$

$$Var(S_n) = \frac{n \, Var(X_i)}{n}$$

X_i's are independent

$$= Var(X_i)$$

$$E[S_n] = \frac{n \, E[X_i]}{\sqrt{n}} = \sqrt{n} \, E[X_i]$$

$$\hat{S}_n = \frac{\sum_{i=1}^{n} (X_i - E[X_i])}{\sqrt{n}}$$

$$E[\hat{S}_n] = 0$$

$$Var(\hat{S}_n) = Var(S_n) = Var(X_1)$$

$$F_{\hat{S}_n}(x) \xrightarrow[n\to\infty]{} F_N(x)$$

where N is normal distribution with 0 mean & Var(X) variance

Made with Goodnotes

Is this a valid p.d.f ?

$$\int_{-\infty}^{\infty} f_X(x)\, dx. = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Let $y = \frac{x-\mu}{\sigma}$. , $dy = \frac{\partial x}{\sigma}$.

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\cancel{\sigma}} e^{-\frac{1}{2}y^2} (dy)\,\cancel{\sigma}$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = I$$

$$I^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\left(\frac{x^2+y^2}{2}\right)} dx\, dy.$$

$x = r\cos\theta$, $y = r\sin\theta$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$.

$$dx\, dy = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} \partial r\, \partial\theta.$$

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$I^2 = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} \frac{1}{2\pi} e^{-r^2/2}\, r\, dr\, d\theta.$$

$$= \int_{r=0}^{\infty} e^{-r^2/2}\, r\, dr. = 1$$

$I = 1.$ It's a valid p.d.f.

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}\, dx.$$

$\frac{x-\mu}{\sigma} = y.$

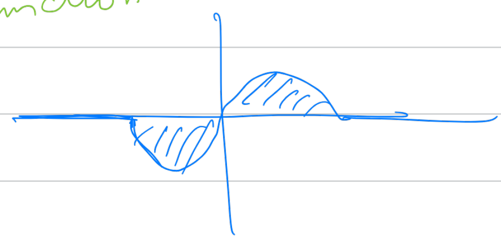
$x = \sigma y + \mu$

$$= \int_{-\infty}^{\infty} (\sigma y + \mu) \frac{e^{-y^2/2}\, \cancel{\sigma}\, dy}{\sqrt{2\pi}\, \cancel{\sigma}}$$

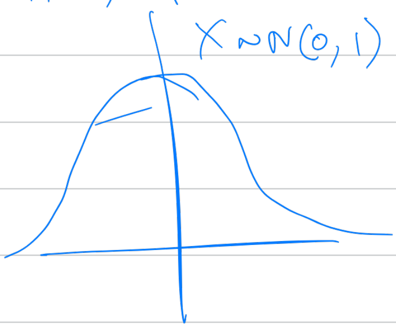
$$= \mu + \underbrace{\int_{-\infty}^{\infty} \sigma \frac{y\, e^{-y^2/2}}{\sqrt{2\pi}}\, dy}_{0 \text{ since it is an odd function}}$$

odd function

$f(x) = -f(-x)$



$f_X(x)$ for $X \sim N(0,1)$



$f_X(x) = f_X(-x)$

even function

$$Var(X) = E\left[\left(X - E[X]\right)^2\right].$$

$y = \frac{x-\mu}{\sigma}$

$$= \int_{-\infty}^{\infty} \frac{(x-\mu)^2\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}\, dx$$

$$= \int_{-\infty}^{\infty} \frac{\sigma^2 y^2\, e^{-y^2/2}}{\sqrt{2\pi}\, \cancel{\sigma}}\, \cancel{\sigma}\, dy$$

$$= \sigma^2 \int_{-\infty}^{\infty} \underbrace{y}_{u(y)} \underbrace{\frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{v'(y)} \, dy .$$

$$v(y) = -e^{-y^2/2}$$
$$v'(y) = y e^{-y^2/2} .$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \left[\underbrace{-y e^{-y^2/2}}_{u(y)v(y)} \Big|_{-\infty}^{\infty} + \underbrace{\int_{-\infty}^{\infty} e^{-y^2/2} \, dy}_{\sqrt{2\pi}} \right] .$$

$$\int u'(y) v(y) .$$

$$= \underline{\underline{\sigma^2}} .$$