.

EE1101: Circuits and Network Analysis

Lecture 25: Sinusoidal Response of First-Order Circuits

September 23, 2025

Topics :

1. Sinusoidal Response using complex exponentials

2. Role of Phasors and Impedances

Sinusoidal Response using Complex Exponentials

The DE that describes a first-order ckt excited by a sinusoidal source is

$$\frac{dx}{dt} + P(t)\,x(t) = \varphi(t) \quad \text{where} \quad \varphi(t) = A\cos(\omega t + \phi) \longrightarrow (1)$$

In addition $P(t) = P$

Complex exponential Route:- $\hat{x}$ to a Complex exponent $e^{j(\omega t + \phi)}$

x is $Re\{\hat{x}\} \leftarrow$

DE with Complex exponential input

$$\frac{d\hat{x}}{dt} + P\hat{x}(t) = A e^{j(\omega t + \phi)} \longrightarrow (2)$$

$$\text{Integrating factor} = e^{\int P dt} = e^{Pt} \quad (\because P(t) = P) \longrightarrow (3)$$

Mul (2) with I F & upon simplification

$$\frac{d}{dt}\left(e^{Pt}\hat{x}(t)\right) = A e^{Pt} e^{j(\omega t + \phi)} = A e^{(P + j\omega)t} e^{j\phi} \longrightarrow (4)$$

Integrating on both sides of (4)

$$e^{Pt}\hat{x}(t) = \frac{A e^{j\phi}}{(P + j\omega)} e^{(P + j\omega)t} + C \longrightarrow (5)$$

Sinusoidal Response using Complex Exponentials

$$\hat{x}(t) = \underbrace{\frac{Ae^{j\phi}}{(p+j\omega)} e^{j\omega t}}_{\text{I}} + \underbrace{c e^{-pt}}_{\text{II}}$$

Term (II) $\longrightarrow 0$ as $t \to \infty$ if $p>0$ (true for ckts)

Term (I) : as $t \to \infty$: sinusoidal in nature

for first order ckts : sinusoidal excitation

$\Downarrow$

sinusoidal response in steady state.

Steady state response:

$$\hat{x}_{ss}(t) = \frac{\overbrace{A e^{j\phi}}}{p+j\omega} e^{j\omega t} = \underbrace{\frac{\sqrt{2}\vec{A}}{\vec{Z}} e^{j\omega t}}. \quad \text{where } \vec{Z} = \underline{p+j\omega}$$

Steady state response x in Phasor form $\Big\}$ $\vec{X}_{ss} = \dfrac{\sqrt{2}\,\vec{A}}{\vec{Z}}$ (in phasor form)

$$= \frac{Ae^{j\phi}}{Ze^{j\theta}} = \frac{A}{Z} e^{j(\phi-\theta)}$$

Steady state response in time-domain $\Big\}$ $x_{ss}(t) = \dfrac{A}{Z} \cos(\omega t + \phi - \theta)$

where $Z = \sqrt{p^2+\omega^2}$ and $\theta = \tan^{-1}\left(\omega/p\right)$

Sinusoidal Response using Complex Exponentials

$$\hat{x}(t) = \frac{A e^{j\phi}}{(p+j\omega)} e^{j\omega t} + c e^{-pt}$$

$$= \frac{(p-j\omega)}{p^2+\omega^2} A\left[\cos(\omega t+\phi) + j\sin(\omega t+\phi)\right] + c e^{-pt}$$

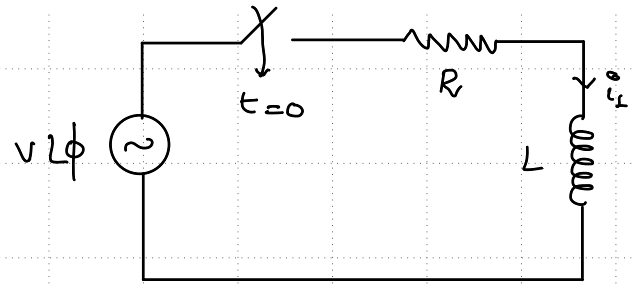
$$= \frac{A}{p^2+\omega^2}\left\{\left[p\cos(\omega t+\phi) + \omega\sin(\omega t+\phi)\right] + j\left[p\sin(\omega t+\phi) - \omega\cos(\omega t+\phi)\right]\right\} + c e^{-pt}$$

$$= \frac{A}{\sqrt{p^2+\omega^2}}\left[\cos(\omega t+\phi-\theta) + j\sin(\omega t+\phi-\theta)\right] + \underbrace{c e^{-pt}}_{\in \mathbb{R}} \quad \text{where } \theta = \tan^{-1}(\omega/p)$$

$$\mathrm{Real}\{\hat{x}(t)\} = x(t) = \underbrace{\frac{A}{\sqrt{p^2+\omega^2}}\cos(\omega t+\phi-\theta)}_{\text{Steady state resp}} + \underbrace{c e^{-pt}}_{\text{Transient response.}}$$

Example

Example 1:-



$$\frac{di}{dt} + \underbrace{\frac{R}{L}}\, i = \frac{V_m}{L}\cos(\omega t+\phi) \quad \text{where } V_m = \sqrt{2}\,V$$

(for $t>0$)

$$P = R/L$$

$$i(t) = \left(\frac{V_m}{L}\right)\frac{1}{\sqrt{R^2/L^2+\omega^2}}\cos(\omega t+\phi-\theta) + c^{-R/L\,t} \quad \text{where } \theta = \tan^{-1}\left(\omega/(R/L)\right) = \tan^{-1}(\omega L/R)$$

$$i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t+\phi-\theta) + ce^{-R/L\,t} \leftarrow$$

given $i_L(0)=0$ $\Rightarrow$ $c = -\dfrac{V_m}{\sqrt{R^2+\omega^2L^2}}\cos(\phi-\theta)$.

$$i(t) = \frac{V_m}{\sqrt{R^2+\omega^2L^2}}\left[\cos(\omega t+\phi-\theta) - \cos(\phi-\theta)e^{-R/L\,t}\right]$$

Steady state sol in phasor form:

$$\vec{I} = \frac{V_m}{\sqrt{2}\sqrt{R^2+\omega^2L^2}}e^{j(\phi-\theta)} = \frac{Ve^{j\phi}\,e^{-j\theta}}{\sqrt{R^2+\omega^2L^2}} = \frac{Ve^{j\phi}}{\sqrt{R^2+\omega^2L^2}\,e^{j\theta}} = \frac{\vec{V}}{\underset{\sim}{\vec{Z}}}$$

Example use of Phasors for Computing S.S. response.

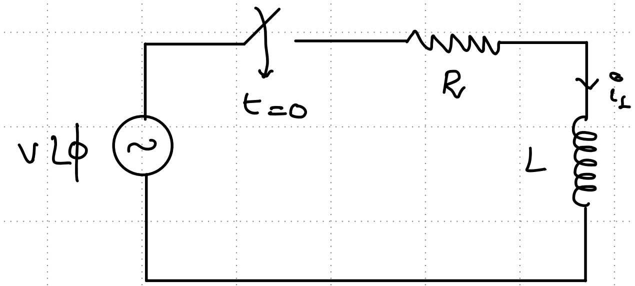
① replace Sources by Phasor representation

② Replace R, L & C by Impedances.

③ Apply Node|mesh analysis ⟹ Circuit resp in phasor form

⇓

Convert it to time domain.



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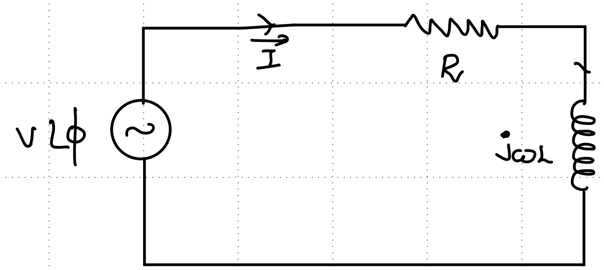
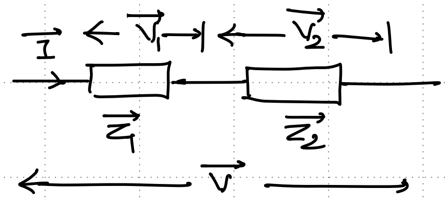


by KVL : $(R+j\omega L)\vec{I} = \vec{V}$

$$\Rightarrow \vec{I} = \frac{\vec{V}}{(R+j\omega L)}$$

Series :



$$\vec{V} = \vec{I}\vec{Z_1} + I\vec{Z_2}$$

$$= \vec{I}(\vec{Z_1}+\vec{Z_2}) \Rightarrow \frac{\vec{V}}{\vec{I}} = \vec{Z_1}+\vec{Z_2}$$

Parallel : $\vec{I} = \vec{I_1} + \vec{I_2}$

$$= \frac{\vec{V}}{\vec{Z_1}} + \frac{\vec{V}}{\vec{Z_2}}$$

$$\frac{\vec{V}}{\vec{I}} = \frac{\vec{Z_1}\vec{Z_2}}{\vec{Z_1}+\vec{Z_2}}$$