Time-domain Response of Second-order Circuits with Sinusoidal and Constant Forcing Functions

- **Sinusoidal Response of Second-order Circuits:** To analyze the sinusoidal response of a second-order circuit, we first introduce a method to solve a standard second-order differential equation with sinusoidal forcing function. The form of differential equation that we will consider is

$$\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2} + 2\zeta\omega_n\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \omega_n^2x(t) = f(t) \tag{1}$$

where $f(t) = A\left(\cos\omega t + \phi\right)$. As mentioned while solving sinusoidal response of first-order circuits, an efficient way to solve this differential equation is to use the complex forcing function $F(t) = Ae^{j(\omega t+\phi)}$. The solution of the differential equation with complex forcing function is a complex function $\hat{x}(t)$, and the solution of the original differential equation is the real part of $\hat{x}(t)$. Thus, we will first solve the differential equation

$$\frac{\mathrm{d}^2\hat{x}(t)}{\mathrm{d}t^2} + 2\zeta\omega_n\frac{\mathrm{d}\hat{x}(t)}{\mathrm{d}t} + \omega_n^2\hat{x}(t) = F(t) \tag{2}$$

and then take the real part of $\hat{x}(t)$ to obtain $x(t)$. The general solution of the differential equation with comlex forcing function is the sum of the complementary solution $\hat{x}_c(t)$ and a particular solution $\hat{x}_p(t)$, i.e.,

$$\hat{x}(t) = \hat{x}_c(t) + \hat{x}_p(t) \tag{3}$$

where $\hat{x}_c(t)$ is the solution of the corresponding homogeneous differential equation and $\hat{x}_p(t)$ is a specific solution of the non-homogeneous differential equation. The complementary solution is computed in the same manner as before, while the particular solution can be obtained either using the method of undetermined coefficients or the method of variation of parameters. Here, we will use the method of undetermined coefficients. Since the forcing function is a complex exponential function, we assume that the particular solution is also a complex exponential function of the same frequency, i.e.,

$$\hat{x}_p(t) = re^{j(\omega t+\phi-\theta)} \tag{4}$$

where r and θ are constants to be determined. Substituting $\hat{x}_p(t)$ into (2), we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(re^{j(\omega t+\phi-\theta)}\right) + 2\zeta\omega_n\frac{\mathrm{d}}{\mathrm{d}t}\left(re^{j(\omega t+\phi-\theta)}\right) + \omega_n^2\left(re^{j(\omega t+\phi-\theta)}\right) = Ae^{j(\omega t+\phi)} \tag{5}$$

which leads to

$$\left(-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2\right)re^{j(\omega t+\phi-\theta)} = Ae^{j(\omega t+\phi)} \tag{6}$$

Equating the magnitudes and angles on both sides, we obtain

$$r = \frac{A}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \text{and} \quad \theta = \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right) \tag{7}$$

Thus, the particular solution is

$$\hat{x}_p(t) = \frac{A}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} e^{j\left[\omega t + \phi - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)\right]} \tag{8}$$

The general solution is then

$$\hat{x}(t) = \hat{x}_c(t) + \hat{x}_p(t) \tag{9}$$

and the solution of the original differential equation is

$$x(t) = \text{Re}\{\hat{x}(t)\} = \text{Re}\{\hat{x}_c(t)\} + \text{Re}\{\hat{x}_p(t)\} \tag{10}$$

The real part of the particular solution can be expressed as

$$\text{Re}\{\hat{x}_p(t)\} = \frac{A}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos\left[\omega t + \phi - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)\right] \tag{11}$$

Typically, the complementary solution $\hat{x}_c(t)$ (and hence its real part) decays to zero as $t \to \infty$ for $\zeta > 0$ (except for the case when $R = 0$). Therefore, the steady-state solution of the original differential equation is

$$x_{ss}(t) = \frac{A}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \cos\left[\omega t + \phi - \tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right)\right] \tag{12}$$

It is **important** to note that, when solving for the response, the initial conditions should be applied to the general solution $\hat{x}(t)$ to determine the constants in the complementary solution $\hat{x}_c(t)$.

- Another impotant point to note is that the chosen form of particular solution $\hat{x}_p(t)$ is valid only when $\omega \neq \omega_n$. When $\omega = \omega_n$, the chosen form of particular solution becomes a solution of the corresponding homogeneous differential equation, and hence cannot be used to find a particular solution. In this case, we need to assume a different form of particular solution, which is

$$\hat{x}_p(t) = t\left(re^{j(\omega t + \phi - \theta)}\right) \tag{13}$$

where r and θ are constants to be determined. We will study this case in detail when we discuss resonance in second-order circuits.

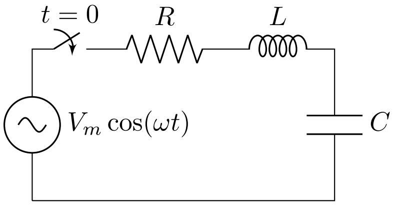

Figure 1: Circuit for Example 1.

- Example 1: Compute the sinusoidal steady-state response of the circuit shown in Fig.1. Assume that the circuit is initially at rest, i.e., $i(0) = 0$ and $v_C(0) = 0$. The switch closes at $t = 0$.

The second-order differential equation that governs the current response ($i(t)$) of the circuit (in standard form) is

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + \frac{R}{L}\frac{\mathrm{d}i(t)}{\mathrm{d}t} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{\mathrm{d}v_s(t)}{\mathrm{d}t}$$

where $v_s(t) = V_m\cos(\omega t)$. Comparing the above differential equation with the standard form, we identify

$$2\zeta\omega_n = \frac{R}{L}, \quad \omega_n^2 = \frac{1}{LC}, \quad \text{and} \quad f(t) = -\frac{\omega V_m}{L}\sin(\omega t)$$

The corresponding complex forcing function is

$$F(t) = -\frac{\omega V_m}{L}e^{j\left(\omega t - \frac{\pi}{2}\right)}$$

Since the circuit has some resistance $R > 0$, the complementary solution will decay to zero as $t \to \infty$. Therefore, the steady-state response can be obtained using (12), with $A = \frac{\omega V_m}{L}$ and $\phi = -\frac{\pi}{2}$. Thus, the steady-state response of the current is

$$i_{ss}(t) = re^{j(\omega t+\phi+\theta)} \quad \text{where} \quad r = \frac{\frac{\omega V_m}{L}}{\sqrt{\left(\frac{1}{LC}-\omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}} \quad \text{and} \quad \theta = -\tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC}-\omega^2}\right)$$

Having obtained the steady-state response, let us validate if the result is consistant with that obtained using phasor analysis. The phasor of the source voltage is

$$\mathbf{V}_s = \frac{V_m}{\sqrt{2}}\angle\phi^\circ$$

The impedances of the circuit elements are

$$\mathbf{Z}_R = R, \quad \mathbf{Z}_L = j\omega L, \quad \mathbf{Z}_C = \frac{1}{j\omega C}$$

The phasor of the current is

$$\mathbf{I} = \frac{\mathbf{V}_s}{\mathbf{Z}_R + \mathbf{Z}_L + \mathbf{Z}_C} = \frac{\frac{V_m}{\sqrt{2}}\angle\phi^\circ}{R + j\left(\omega L - \frac{1}{\omega C}\right)} = \frac{\frac{V_m}{\sqrt{2}}\angle\phi^\circ}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\angle\tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)}$$

The time-domain steady-state response of the current is

$$i_{ss}(t) = \sqrt{2}|\mathbf{I}|\cos\left(\omega t + \angle\mathbf{I}\right) = \frac{V_m}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos\left[\omega t + \phi - \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)\right]$$

which is the same as that obtained using the differential equation approach.

- Example 2: Compute the complete response of the circuit shown in Fig.1. Assume that the circuit is initially at rest, i.e., $i(0) = 0$ and $v_C(0) = 0$. The switch closes at $t = 0$. Assume that $R = 2$, $L = 1$, $C = 0.5$, $V_m = 1$, and $\omega = 1$.

For the given values of R, L, and C, the differential equation governing the current response is

$$\frac{\mathrm{d}^2 i(t)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}i(t)}{\mathrm{d}t} + 2i(t) = -\sin(t)$$

The corresponding complex forcing function is

$$F(t) = -e^{j\left(t-\frac{\pi}{2}\right)}$$

The characteristic equation of the corresponding homogeneous differential equation is

$$s^2 + 2s + 2 = 0$$

The roots of the characteristic equation are

$$s_{1,2} = -1 \pm j$$

Thus, the complementary solution is

$$i_c(t) = 2r_1 e^{-t}\cos(t+\theta_1)$$

where r_1 and θ_1 are constants to be determined using the initial conditions. The particular solution is obtained using (8), with $A = 1$ and $\phi = -\frac{\pi}{2}$, which gives

$$i_p(t) = \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)$$

The general solution is

$$i(t) = i_c(t) + i_p(t) = 2r_1 e^{-t}\cos(t+\theta_1) + \frac{1}{\sqrt{2}}\cos\left(t - \frac{\pi}{4}\right)$$

To determine the constants r_1 and θ_1, we use the initial conditions. At $t = 0$, we have

$$i(0) = 2r_1\cos(\theta_1) + \frac{1}{\sqrt{2}}\cos\left(-\frac{\pi}{4}\right) = 0$$

which leads to

$$r_1\cos(\theta_1) = -\frac{1}{2\sqrt{2}} \tag{14}$$

To apply the second initial condition, we need to compute $\frac{\mathrm{d}i(t)}{\mathrm{d}t}$, which is

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = 2r_1 e^{-t}\left(-\cos(t+\theta_1) - \sin(t+\theta_1)\right) - \frac{1}{\sqrt{2}}\sin\left(t - \frac{\pi}{4}\right)$$

At $t = 0$, we have

$$\frac{\mathrm{d}i(0)}{\mathrm{d}t} = 2r_1\left(-\cos(\theta_1) - \sin(\theta_1)\right) - \frac{1}{\sqrt{2}}\sin\left(-\frac{\pi}{4}\right) = 0$$

which leads to

$$r_1\left(\cos(\theta_1)+\sin(\theta_1)\right) = -\frac{1}{2\sqrt{2}} \tag{15}$$

From (14) and (15), we have

$$\tan(\theta_1) = 1 \quad \Rightarrow \quad \theta_1 = \frac{3\pi}{4}$$

and

$$r_1 = -\frac{1}{4}$$

Thus, the complete response of the current is

$$i(t) = -\frac{1}{2}e^{-t}\cos\left(t+\frac{3\pi}{4}\right) + \frac{1}{\sqrt{2}}\cos\left(t-\frac{\pi}{4}\right)$$

- **Response to a Constant Forcing Function:** When the forcing function is a constant, i.e., $f(t) = K$, we can use the same approach to solve the differential equation. The Solution of the differential equation has two parts, namely, the complementary solution and the particular solution. The complementary solution is obtained by solving the corresponding homogeneous differential equation, while the particular solution can be assumed to be a constant, i.e., $x_p(t) = C$. Substituting $x_p(t)$ into the differential equation, we have

$$\omega_n^2 C = K \quad \Rightarrow \quad C = \frac{K}{\omega_n^2}$$

Thus, the particular solution is

$$x_p(t) = \frac{K}{\omega_n^2}$$

The general solution is then

$$x(t) = x_c(t) + x_p(t)$$

where $x_c(t)$ is the complementary solution. The constants in the complementary solution can be determined using the initial conditions.