1. Curl of a vector field.

2. Laplace operator
   & Laplace's equation.

3. Double integral (review)

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Curl of a vector field $\vec{F}$:

$$\vec{F} = F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}.$$

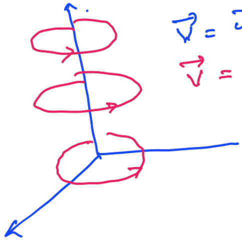
$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \rightarrow \text{Gives vector.}$$

Geometric interpretation:

Curl measures rotation component in a vector field.

Example :



Linear velocity

$\vec{v} = \bar{\omega} \times \vec{r}$.

If angular velocity

$\bar{\omega} = \omega \hat{k}$.

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
(position vector)

$\vec{v} = \langle -\omega y, \omega x, 0 \rangle$

$$\nabla \times \bar{V} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= i\left(-\frac{\partial}{\partial z}(\omega x)\right) - j\left(0 - \frac{\partial}{\partial z}\omega y\right) + \hat{k}\left(\frac{\partial}{\partial x}\omega x + \frac{\partial}{\partial y}\omega y\right)$$

$$= 2\omega \hat{k}.$$

$\Rightarrow$ Vector field $\vec{V}$ has rotation component $|2\omega|$ in the direction or axis along $\hat{k}$.

* It shows the vorticity (a measure of local rotational motion) is twice the angular velocity.

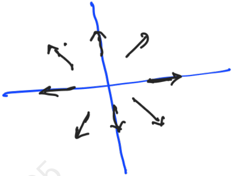
Ex 1: $\vec{F}(x,y) = x\hat{i} + y\hat{j}$

$\nabla \cdot \vec{F} \neq 0$; $|\nabla \times \vec{F}| = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix}$$

$$= \hat{k}(0) = 0.$$

$\Rightarrow$ No rotational component in $\vec{F}$.

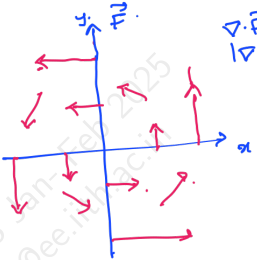


Curl free or irrotational vector field.

Ex 2:

$$\vec{F}(x,y) = -y\,\hat{i} + x\,\hat{j}.$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = 2\hat{k}. \Rightarrow \text{rotational vector field.}$$



$\nabla \cdot \vec{F} = 0$

$|\nabla \times \vec{F}| \neq 0$

* : "More intuition about curl: when we do stokes theorem"

* By right hand rule configuration.

Curl +ve $\Rightarrow$ Anticlockwise rotation. $(x \times y \rightarrow +z)$

-ve $\Rightarrow$ Clockwise. $(y \times x \rightarrow -z)$

$$\boxed{\begin{array}{l} * \ \boxed{\text{Curl of grad} = 0 \\ \vec{\nabla} \times (\nabla f) = 0 \quad \text{for all } f} \\ * \ \text{Div of Curl} = 0 \\ \quad \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 \quad \text{for all } \vec{F} \end{array}}$$

$\rightarrow$ This idea is used to check the vector field is a gradient field or not!

$\leftarrow$ verify yourself!

scalar field.

$\nabla f \longrightarrow$ vector field

$\nabla \cdot \vec{F} \longrightarrow$ Scalar field

vector field

$\nabla \times \vec{F} \longrightarrow$ Vector field

vector field

$\nabla \cdot (\nabla f) \rightarrow \nabla^2 f \longrightarrow$ scalar. { Laplacian operator }.

scalar

Laplace operator: ∇^2.

$$\nabla \cdot \nabla = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

$$\nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Divergence of the gradient of a scalar field.

Divergence → the spread/flow behaviour

gradient → directional rate of change.

⇒ Used to describe, a) wave propagation.
b) Heat conduction
c) Diffusion.

} ∇^2 used where Local variation in field influence its behaviour.

Laplace's Equation:

Steady, incompressible, irrotational flow of $\vec{V} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$

Steady : $\frac{\partial \vec{V}}{\partial t} = 0$;

incompressible : $\nabla \cdot \vec{V} = 0$. $\quad \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} = 0$.

irrotational : $\nabla \times \vec{V} = 0$.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ V_1 & V_2 & 0 \end{vmatrix} = \left(\frac{\partial}{\partial x} V_2 - \frac{\partial}{\partial y} V_1 \right) \hat{k} = 0.$$

These are satisfied automatically if $\vec{V} = \nabla \phi$ ← scalar field. (scalar potential field)

for a real-valued potential ϕ

that satisfies $\boxed{\nabla^2 \phi = 0}$ → Laplace's Eqn.

$$\bar{\nabla}\cdot\bar{V} = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j\right)\cdot\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right)$$

$$= \underbrace{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}}_{\text{since } \nabla^2\phi = 0} = 0.$$

✓ incompressible.

$\boxed{\text{This is true when '}\phi\text{' satisfies laplace's Eqn}}$

$\bar{V} = \nabla\phi$ (Gradient field) automatically satisfy curl free condition $\Rightarrow$ irrotational.

$$\nabla\times\bar{V} = \nabla\times(\nabla\phi) = 0$$

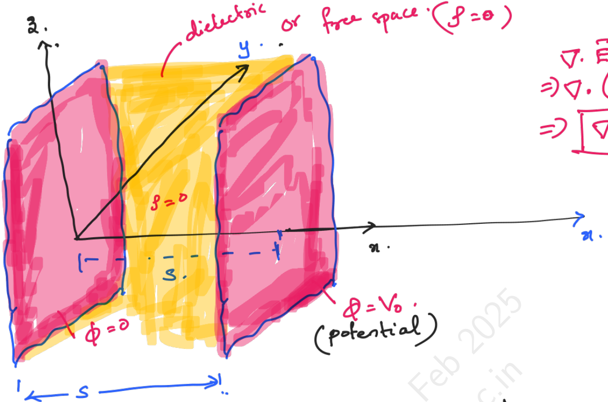
Application of Laplace's Eqn:

Problem: Two parallel plates (very large infinite in size) separated by 's'. Find Electric field between the plates?

"with applied scalar potential"

* Her laplace equation can be used since between parallel plate it is charge free. $\nabla\cdot\bar{E} = \rho/\epsilon_0$. here $\rho = 0$.

Here $\nabla\cdot\bar{E} = 0$



$\nabla \cdot \bar{E} = 0$

$\Rightarrow \nabla \cdot (-\nabla \phi) = 0$

$\Rightarrow \boxed{\nabla^2 \phi = 0}$

Since plates are infinitely large in yz plane

$(x, y, z) \approx (x_0, y, z.)$ $\Rightarrow$ only $\frac{d^2\phi}{dx^2}$ is there in $\nabla^2\phi$ equation.

Laplace Equation & boundary conditions are,

$$\nabla^2 \phi = 0 \Rightarrow \frac{d^2\phi}{dx^2} = 0$$

and $\phi = \begin{cases} 0 \; ; & \text{at } x = 0. \\ V_0 \; ; & \text{at } x = s. \end{cases}$

$$\frac{d^2\phi}{dx^2} = 0 \Rightarrow \qquad \phi(x) = c_1 x.$$

$$V_0 = c_1 s$$

$$c_1 = \frac{V_0}{s}.$$

$$\Rightarrow \phi(x) = \frac{V_0}{s} x.$$

The electric field is $\bar{E} = -\nabla\phi$

$$= -\frac{\partial}{\partial x} \frac{V_0}{s} x \cdot \hat{i}.$$

$\Rightarrow \; E_x = -\frac{V_0}{s} \; ; \quad E_y = E_z = 0.$

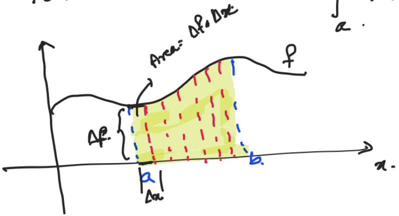
$\Rightarrow \; \vec{E}$ field is constant vector normal to plates.

This is an excellent approx to the potential & field between two plates (but far from edges), (no 'fringe' field)

(Since here 's' $<<$ (y dim & z dim))

Double integrals: (Review)

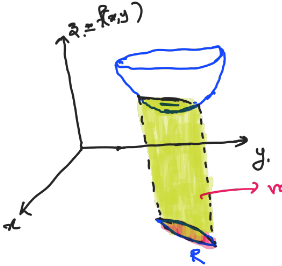
Function of 1 variable: $\int_a^b f(x)\,dx$ = area below the graph of $f(x)$ over $[a,b]$



$$\sum_{i=1}^{n} \Delta f \cdot \Delta x_i$$

$$\Delta x_i \to 0 \Rightarrow \int_a^b f(x)\,dx. \checkmark$$

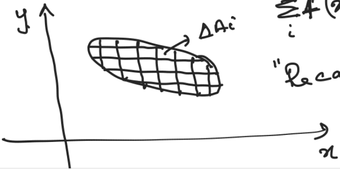
In double integral: Volume below the graph $z = f(x,y)$ over a region 'R' in xy plane.



volume of this solid

$$\iint_R z \underbrace{dx\,dy}_{dA}.$$

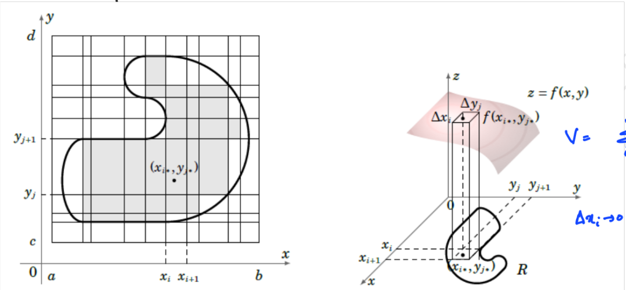
Definition: Cut 'R' into small area ΔA_i



$\sum_i f(x_i, y_i)\, dA_i$ Take limit $\Delta A_i \to 0$

$\Rightarrow \iint$

"Recall Reimann integral in single variable calculus"



(a) Subrectangles inside the region R

(b) Parallelepiped over a subrectangle, with volume $f(x_{i*}, y_{j*})\Delta x_i \Delta y_j$

$$V = \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{m}}_{R} f(x_i, y_j)\, \underbrace{\Delta x_i \Delta y_j}_{dA_i}$$

$$\Delta x_i \to 0 \,;\ \Delta y_j \to 0 \Rightarrow \iint_R f(x,y)\, \underbrace{dx\, dy}_{dA}$$

Ref: Vector Calculus by - Michael Corral