Conditional Independence.

Events A, B are said to be independent conditioned over C if

$$P(A \cap B \mid C) = P(A \mid C)\, P(B \mid C).$$

Conditional independence doesn't imply pairwise independence & vice-versa is also true i.e.,

pairwise independence $\not\Rightarrow$ conditional independence.

Example:

A : 1st toss is a head
B : 2nd toss is a head
C : Two tosses have different outcomes.

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

$$P(B \mid C) = \frac{1}{2}.$$

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = 0.$$

A & B are not conditionally independent.

$$P(A \cap B) = P(B)\, P(A).$$

$$\frac{1}{4} \quad\quad = \frac{1}{2} \times \frac{1}{2}$$

A & B are independent.

We are give two coins blue one and a red one.

$$P(B) = \frac{1}{2} = P(R) = P(B^c).$$

$$\Omega = \{ (B, H, H)\ (B, H, T)\ (B, T, T), (B, T, H)\ (R, H, H)\ (R, H, T)\ (R, T, T), (R, T, H) \}$$

8 possibi in total.

H_1 : the event that first toss is head

H_2 : event that second toss is head

We'll show H_1 & H_2 are conditionally independent (given B).

but not independent:

Let

$$P(H_2|B) = P(H_1|B) = 0.9$$
$$P(H_2|B^c) = P(H_1|B^c) = 0.1$$

Then

Remark

$P(H_1|B)$ and $P(H_1|B^c)$ need not sum to 1 as in this example.

By definition H_1 and H_2 are conditionally independent given B.

$$P(H_1 \cap H_2 | B) = P(H_1|B)\ P(H_2|B) = (0.9)^2.$$

$$P(H_1 \cap H_2) = P(H_1 \cap H_2 \cap B) + P(H_1 \cap H_2 \cap B^c)$$

$$= P(H_1 \cap H_2 | B)\ P(B) + P(H_1 \cap H_2 | B^c)\ P(B^c)$$

$$= (0.9)^2 \frac{1}{2} + (0.1)^2 \frac{1}{2}.$$

$$0.41 = \frac{1}{2}\left((0.9)^2 + (0.1)^2\right)$$

0.81

$\neq$

$P(H_1)\,P(H_2) = \frac{1}{4}$.

$$P(H_1) = P(H_1|B)\ P(B) + P(H_1|B^c)\ P(B^c)$$
$$= (0.9)\frac{1}{2} + (0.1)\frac{1}{2} = \frac{1}{2}$$

$$P(H_2) = 1/2.$$

(cbde)

abcde.
bacde.

Counting principle:

\# of 5 letter words where no letter is repeating

$26 \times \boxed{25 \times 24 \times 23 \times 22}$

Can divide the counting procedure into r stages.

First stage has n_1 options
For every option in first stage you have n_2 options in second stage.
For every option in first $(i-1)$ stages there are n_i options in ith stage.

Total number of options $= n_1 \times n_2 \times n_3 \cdots n_r$.

Example: Find the number of possible subsets of $S = \{1, 2, \ldots n\}$.

$$\overbrace{2 \times 2 \times \cdots 2}^{n \text{ times}}$$
$$= 2^n$$

2 options: "to pick" or "not pick" an element.

Permutations: There are n distinct objects, we would like to count # of ways to arrange k distinct objects, picked from n in a sequence.

$$^nP_k = n \times (n-1) \times \cdots (n-(k-1))$$

$$= \frac{n!}{(n-k)!} \qquad n! \longrightarrow n \times (n-1) \cdots \times 1.$$

Combinations: Count the # ways to pick k distinct objects from n distinct objects.

abcde.
bcade.

We'll count the number of 'k' length sequences that can be formed by picking k distinct elements from n objects in a slightly "different way."

} We know that this count should be ^nP_k from earlier.

Let # ways to choose k distinct objects $= A$.

ways to arrage k objects in a sequence

$$= \frac{k!}{(k-k)!} = {}^kP_k = k!$$

Choosing k and arranging them can also be counted as

$$\underbrace{A}\cdot k! = {}^nP_k.$$

$$\binom{n}{k} = {}^nC_k := \Rightarrow A = \frac{^nP_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Also gives count on number of ways to partition any set of size n into sets of sizes k and n-k.

Partitioning a set of size n into r subsets of sizes $n_1, n_2, n_3, \dots n_r$ respectively

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{\overbrace{n-n_1-n_2-\cdots-n_{r-1}}^{=n_r}}{n_r}$$

$$\frac{n!}{\cancel{(n-n_1)!}\,n_1!}\;\frac{\cancel{(n-n_1)!}}{n_2!\,\cancel{(n-n_1-n_2)!}}\;\frac{\cancel{(n-n_1-n_2)!}}{(n_3)!\ldots}\cdots$$

$$= \frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Combinatorial identities

$$2^n = \sum_{k=0}^{n} \binom{n}{k}$$

# of k-size subsets that are possible.

$S = \{1, 2, \ldots, n\}$

$X \subset S.$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

that has n and K element subsets of n.

that doesn't have n.

$$n\, 2^{n-1} = \sum_{k=1}^{n} k \binom{n-1}{k-1}.$$

Bertsekas

Random variables.

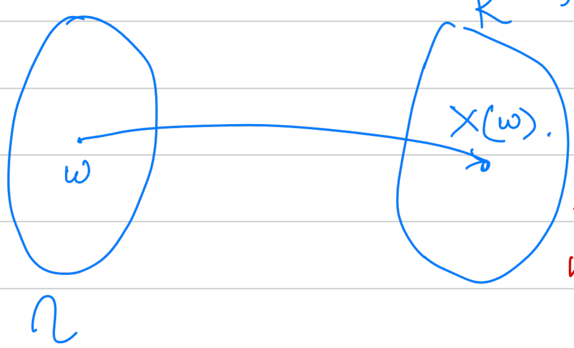
A random variable X defined over $(\Omega, \mathcal{F}, P)$ is a function defined over Ω taking real values.

$$X : \Omega \longrightarrow \mathbb{R}.$$

satisfying the condition that, $x \in \mathbb{R}$:

$$\underbrace{\{\omega \in \Omega \mid X(\omega) \le x\}}_{\in \mathcal{F}}$$

$\subseteq \Omega$.

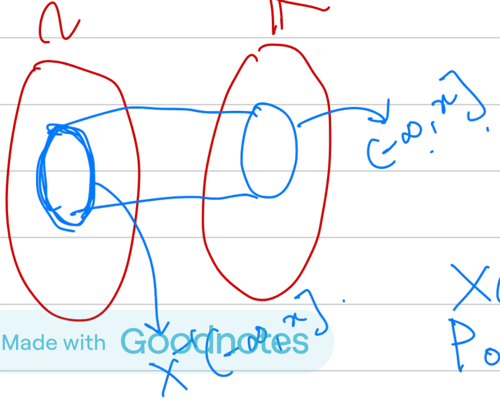


$X(\omega) \le x.$

$X(\omega) \in (-\infty, x]$

$\omega \in X^{-1}((-\infty, x])$

$X^{-1}((-\infty, x]) \in \mathcal{F}$



Consider $\Omega = \{HH, TH, HT, TT\}.$

$\mathcal{F} = \{\phi, \Omega, \{HT, TH\}, \{TT, HH\}\}$

$X(\omega)$: # of heads in ω.

Possible values that $X(\omega)$ can take are 0, 1, 2.

$$\{X \leq 0\} = \{\omega \in \Omega \mid X(\omega) \leq 0\}$$
$$= \{TT\}. \notin \mathcal{F}.$$

If we choose $\mathcal{F}$ to be collection of all possible subsets of Ω Then X is a random variable

2^{Ω}. power set

Theorem: Given a sample space Ω and event space $\mathcal{F}$. let $X : \Omega \longrightarrow \mathbb{R}$ be a random variable. Then following hold

By defn $\forall x \in \mathbb{R}$

$$X^{-1}((-\infty, x]) \in \mathcal{F}$$

$A \in \mathcal{F}$

$A^c \in \mathcal{F}$.

$$X^{-1}((x, \infty)) \in \mathcal{F}$$

(1) $X^{-1}((-\infty, x)) \in \mathcal{F}$ unbounded

(2) $X^{-1}(\{x\}) \in \mathcal{F}$

(3) $X^{-1}([x_1, x_2]) \in \mathcal{F}$.

(4) $X^{-1}((x_1, x_2)) \in \mathcal{F}$.

bounded

Idea to show (1)

$$A_n = X^{-1}\left(\left(-\infty, x - \tfrac{1}{n}\right]\right)$$

$$\bigcup_{n=1}^{\infty} A_n = X^{-1}((-\infty, x))$$

$x \notin A_n \ \forall n$.

Any $x-\epsilon$ can find n st $x-\epsilon \in A_n$

Distribution Function also refered to as cumilative distribution function (CDF) of random variable X is defined as:

$$F_X(x) = P(X \leq x)$$

random variable

$$= P(\{\omega \in \Omega \mid X(\omega) \leq x\}).$$

$$F_X : \mathbb{R} \longrightarrow [0, 1].$$

Examples: $\Omega = \{HH, HT, TH, TT\}$, $\mathcal{F} = 2^{\Omega}$

$$P(\{\omega\}) = \frac{1}{4} \qquad \omega \in \Omega.$$

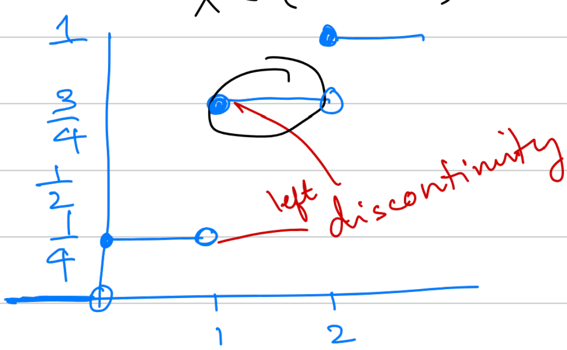
$$X(\omega) := \#\text{ of heads in } \omega.$$

$$\{X \le x\} = \begin{cases} \phi. & \boxed{x < 0} \\ \{TT\} & 1 > x \ge 0. \\ \{TT, TH, HT\} & 2 > x \ge 1 \\ \Omega. & x \ge 2 \end{cases}$$

$$\{\omega \in \Omega \mid X(\omega) \le x\}$$

$$F_X(x) = P(X \le x).$$

$X \in \{0, 1, 2\}$



$$= \left\{ \begin{array}{ll} 0. & x < 0 \\ P(\{TT\}) = \frac{1}{4} ✓ & 0 \le x < 1 \\ P(\{TT, TH, HT\}) = \frac{3}{4} & 1 \le x < 2 \\ 1 & x \ge 2. \end{array} \right\}.$$

$$P(X \le 1) = P(\{\omega : X(\omega) \le 1\}).$$
$$= \sum_{x = 0,1} P(\{\omega : X(\omega) = x\})$$
$$= P(\{TT\}) + P(\{HT, TH\})$$
$$= \frac{1}{4} + 2 \times \frac{1}{4} = 3/4.$$

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