EE1080/AI1110/EE2102 Probability: HW4

26th Feb 2025

1 Joint Density

1. *Buffon's needle problem:* Consider a needle of length ℓ dropped across a surface with parallel lines spaced at distance d. What is the probability that the needle intersects with any of the lines ? Assume $d > \ell$ so that the needle can intersect at most one line. Assume that the smallest distance from the center of the needle to the nearest line is random variable X and the angle made the needle with the line's axis is Θ. Assume that $X \sim \text{Uniform}(0, \frac{d}{2})$, $\Theta \sim \text{Uniform}(0, \pi/2)$ and are independent of each other.

2. Puchku and Boltu decide to meet at the river front. If each of them independently arrive at a time uniformly distributed between 12 noon and 1 p.m., find the probability that the first to arrive has to wait longer than 10 minutes.

2 Conditional Density, Expectation

3. Suppose V is a uniform random variable,

$$f_V(v) = \begin{cases} 1/2 & 0 \leq v < 2, \\ 0 & \text{otherwise.} \end{cases}$$

   (a) Find the conditional PDF, $f_{V|\{V>1\}}$

   (b) Find the conditional PDF, $f_{V|\{1/2<V<3/2\}}$.

   (c) Find the conditional CDF, $F_{V|\{1/2<V<3/2\}}$

4. The random variable X has the PDF

$$f_X(x) = \begin{cases} cx^{-2} & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Determine the value of c

(b) Let A be the event that $X > 1.5$. Calculate $P(A)$ and the conditional PDF of X given that A has occurred

5. We start with a stick of length ℓ. We break it at a point which is chosen according to a uniform distribution and keep the piece of length Y, that contains the left end of the stick. We then repeat the same process on the piece we are left with, and let X be the length of the remaining piece after breaking for the second time.

   (a) What is the joint density of Y and X

   (b) Find the marginal PDF of X

   (c) Use the PDF of X to evaluate $E[X]$

   (d) Evaluate $E[X]$ by exploiting the relation that $X = Y.(X/Y)$.

6. Let X be a Gaussian random variable $N(0, 1)$. Suppose an event A is the event that X is positive. What is $E(X|A)$?

3 Total Law of Expectation, Variance

7. If $X_1, X_2, ..., X_n$ are i.i.d., find $E[X_1|X_1 + X_2 + ... + X_n = 1]$. *(Hint: $E[X \mid X = x] = x$.)* Suppose in addition that $X_1, X_2, \cdots, X_n$ are zero mean, and let

$$Y_n = \sum_{i=1}^{n} X_i$$

   Find $E(Y_{10}|Y_5)$. What about $E(Y_{10}|Y_2, Y_5)$?

8. (Sum of random number of random variables) Let $X_1, X_2, X_3, \cdots$ be i.i.d random variables and N be a random variable independent of them. Given $E[N], Var(N)$ and $E[X_1], Var(X_1)$, find $E[X_1 + X_2 + \cdots + X_N]$ and $Var(X_1 + \cdots + X_N)$. If N is a Poisson(λ) random variable and $X_1, X_2, X_3, \cdots$, be i.i.d Gaussian random variables with mean μ and variance σ^2. Find $E[X_1 + X_2 + \cdots + X_N]$ and $Var(X_1 + \cdots + X_N)$.

9. *Wald's equation:* Let $X_1, X_2, X_3, \cdots$ be i.i.d random variables. Let $N = \min\{n : X_n > x\}$. Show that $E[X_1 + \cdots + X_N] = E[X]E[N]$.

10. Prof. Calculus and Tintin are collaborating on a top-secret mission. There meeting is scheduled to start at 10am. Tintin promptly arrives at 10am. Prof. Calculus being absent minded arrives at a time that is uniform in 9am to 11am. Let X be the time in hours between 9am and when Prof. Calculus arrives. If Prof arrives before 10am then the meeting will last for 3 hours. If he arrives after 10am, then the meeting time is uniformly distributed between 0 and $3 - X$ hours. The meeting starts when Prof and Tintin meet. Tintin gets irritated when Prof is late and will end the

collaboration after the second meeting for which Prof is late by more than 45 mins. All meetings are independent of others.

(a) What is the expected number of hours Tintin waits for Prof. Calculus to arrive?

(b) What is the expected duration of any particular meeting

(c) What is the expected number of meeting they will have before ending their collaboration ?

4 Independence, Correlation

11. Let X and Y be two random variables.

    (a) If $Var(X + 3Y) = Var(X - 3Y)$, must X and Y be uncorrelated?

    (b) If $Var(X) = Var(Y)$, must X and Y be uncorrelated?

    (c) Find the sense of the following inequality: $\geq, =,$ or $\leq$. $Var(X_1 + X_2)$ vs. $Var(X_1) + Var(X_2)$ if $\rho(X_1, X_2) > 0$.

    (d) Let X and Y have correlation coefficient $\rho(X, Y)$. What is the correlation coefficient between X and $aY + b$, where a and b are constants with $a > 0$?

12. If $\rho(X, Y) = 1$ show that $\hat{Y} = Y - E[Y]$ is a positive multiple of $\hat{X} = X - E[X]$ with probability 1. *(Hint: Look at $E[(\hat{X} - c\hat{Y})^2]$ for $c = E[\hat{X}\hat{Y}]/E[\hat{Y}^2]$)*

13. Let $X_1, ..., X_n$ be random variables with fixed variances $Var(X_i)$, $i = 1, ..., n$ and let $Y = \sum_{i=1}^{n} X_i$ be their sum. What is the maximum value that $Var(Y)$ can take if we are allowed to vary the covariances?

14. X, Y are said to be joint normal if the joint density can be expressed as:

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1 - r^2}} e^{-\frac{1}{2(1-r^2)}\left((\frac{x-\mu_x}{\sigma_x})^2 + (\frac{y-\mu_y}{\sigma_y})^2 - 2r(\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y})\right)}$$

    where $\mu_x, \mu_y, \sigma_x, \sigma_y, r$ are the parameters of the distribution. Find

    (a) Find $f_X(x), f_Y(y)$. Is X a normal distribution ? If so, what are the mean, variance

    (b) Find $f_{X|Y}(x|y)$. Is $X|Y$ a normal distribution ? If so, what are the mean, variance

    (c) Find $Cov(X, Y)$

    (d) If X, Y are uncorrelated does it imply that X, Y are independent?

15. Let X have a normal distribution with zero mean and unit variance. Let Z be independent of X, with $P(Z = 1) = P(Z = -1) = 1/2$. Let $Y = ZX$,

(a) What is the pdf of Y ?

(b) Are X and Y uncorrelated ?

(c) Are X and Y independent ?

(d) Are X and Y jointly Gaussian ?

16. If Z is a standard normal random variable, what is $Cov(Z, Z^2)$?

17. Suppose that Y is a normal random variable with mean 0 and variance 1, and suppose also that the conditional distribution of X, given that $Y = y$, is normal with mean y and variance 1.

    (a) Argue that the joint distribution of X, Y is the same as that of $Y + Z, Y$ when Z is a standard normal random variable that is independent of Y.

    (b) Use the result of part (a) to argue that X, Y has a bivariate normal distribution.

    (c) Find $E[X]$, $Var(X)$, and $Cov(X, Y)$.

    (d) Find $E[Y|X = x]$.

    (e) What is the conditional distribution of Y given that $X = x$?

5 Bayes rule, MAP

18. A defective coin minting machine produces coins whose probability of heads is a random variable P with PDF

$$f_P(p) = \begin{cases} pe^p & p \in [0, 1] \\ 0 & \text{otherwise} \end{cases}$$

A coin produced by this machine is selected and tossed repeatedly with successive tosses assumed independent.

(a) Find the probability that the coin toss results in heads.

(b) Given that a coin toss results in heads, find the conditional PDF of P

(c) Given that the first coin toss resulted in heads, find the conditional probability of heads on the next toss ?

6 Function of Random Variables

19. If X is uniformly distributed over (a, b), what random variable, having a linear relation with X, is uniformly distributed over $(0, 1)$?

20. If X is uniformly distributed over $(-1, 1)$, find

(a) $P(|X| > \frac{1}{2})$;

(b) the density function of the random variable $|X|$.

21. If X, Y are independent $exp(\lambda_1), exp(\lambda_2)$, random variables, what is the pdf of $X + Y$, $\min(X, Y)$?

22. If X_1, X_2 are independent uniform$[0, 1]$ random variables, what is the pdf of $X_1 + X_2$. What is the pdf of $X_1 + X_2 + \cdots + X_n$ given $X_1, \cdots, X_n$ are i.i.d uniform $[0, 1]$ random variables.

23. Show that sum of two independent normal random variables is a normal random variable. Assume the mean and variances to be μ_1, μ_2 and σ_1^2, σ_2^2 respectively for the two random variables.

24. X and Y have joint density function:

$$f(x, y) = \frac{1}{x^2 y^2}, \quad x \geq 1, y \geq 1$$

(a) Compute the joint density function of U = XY, V = X/Y

(b) What are the marginal densities?