1. Functions of multivariables.

2. Partial derivatives

(a) Tangent plane to the surface of a given function

(b) Maxima & minima.

EE1203: Vector Calculus

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16^{th} January, 2025

భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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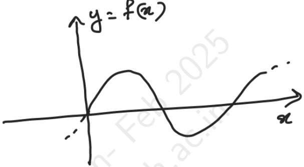
Functions of several variables and derivatives.

* Calculus: $\rightarrow$ Studying functions:

1. Functions of 1 variable: $f(x) = \sin(x)$

Graph of a function $f(x)$



2. Function of 2-variables:

Given (x, y) in space $\rightarrow$ get a number $f(x, y)$

Example: $f(x,y) = x^2 + y^2$ Domain $\in R^2$; Range $\in R$

$f(x,y) = \sqrt{y}$; only when $y \geq 0$ (y should not be negative)

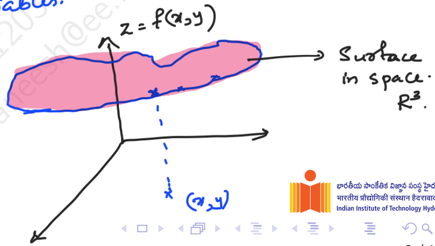
$f(x,y) = \dfrac{1}{x+y}$; only defined when $x+y \neq 0$.

Physical example : $f(x,y)$ = temperature at point (x,y)
or 3 or more variables.

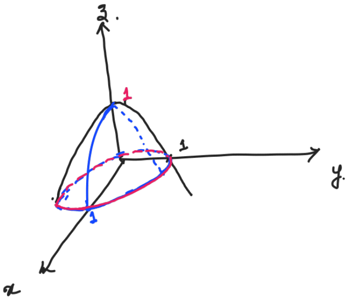
* Here we might again restrict in <u>2 or 3 variables</u> :
But tools we learn is common even if we have 'n' variables.

<u>Representation of function of 2 variables.</u>

→ Graph : $Z = f(x,y)$



Example: $f(x,y) = 1-x^2-y^2$.



in YZ plane $(x=0)$

$\Rightarrow z = 1-y^2$.

in XZ plane: $(y=0)$

$z = 1-x^2$.

$\Rightarrow$ where it hits the xy plane?

$\rightarrow$ at $z=0$

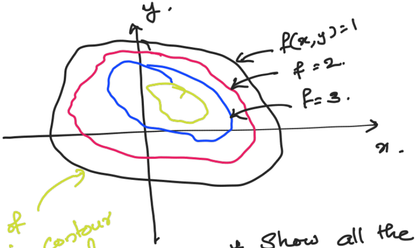
$\Rightarrow$ $1-x^2-y^2=0$

$x^2+y^2=1$ (Circle of $r=1$)

Example: Try to visualize (may be use a computer code)

$$\underline{z = y^2 - x^2}.$$

Contour Plot: Another elegant method to represent functions of 2-variables.



$\equiv$ Elevation of $z = f(x,y)$ onto xy plane.

$\Rightarrow$ simpler to read the map in contour plot.

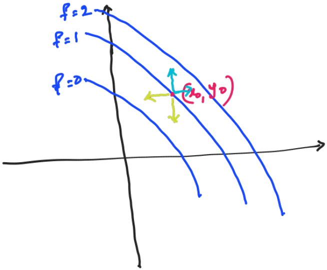
Each of these in contour plot is called level curves!

* Show all the points where $f(x,y) = c$ where c = Constant.

* Usually c is chosen at regular intervals.

* Equivallent of cutting the graph into horizontal planes

How is the contour plot useful?



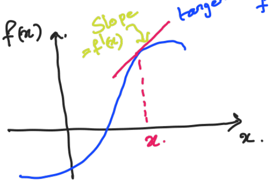
→ It tells us many insights!

* If x increases,
  → f increases (here)
* If y increases,
  * f increases.

$||^{ly}$. If x decreases, or y decreases } f decreases.

Partial derivatives:

Function of 1 variable : $f(x)$



Rate of change of $f(x)$ w.r.t change in x

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

* Please review your basics on continuity, differentiability concepts.

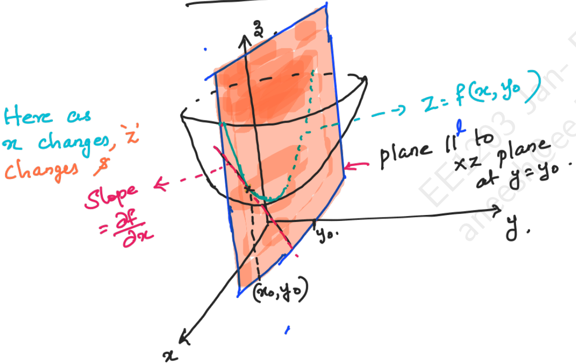
How to extend this idea for functions of 2 variables.

Partial derivatives :-

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}.$$

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0, y_0)}{\Delta y}.$$

How to visualize this concept?



How to compute?

To find $\frac{\partial f}{\partial x} = f_x$

* treat 'y' as constant & 'x' as variable & vice versa.

Usefullness of partial derivatives!

* minimization / maximization problems involving functions of multivariables.

Approximation formula:

If we change $x \rightsquigarrow x+\Delta x$

$y \rightsquigarrow y+\Delta y$.

$z=f(x,y)$: then $\boxed{\Delta z \approx f_x \Delta x + f_y \Delta y.}$ ⟵ one of the powerful formulae in partial derivatives

How to justify this formula?

⇓ neglecting higher orders

Justification to the above formula;

Approximate formula in 1 variable case

If at $x_0 \longrightarrow f(x)=f(x_0)$

Value at nearby point : $f(x) \approx f(x_0)+f'(x_0)(x-x_0)$

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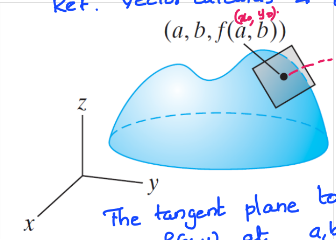
Ref: Vector calculus 4^{th} Ed; by "Susan Jane Colley".

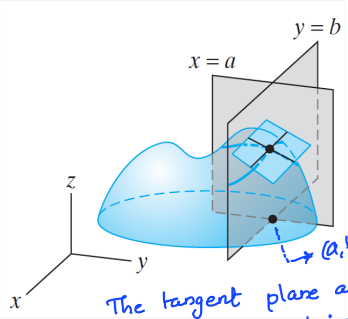


The tangent plane to $z = f(x,y)$ at $a, b, f(a,b)$

* f_x, f_y are slopes of 2 tangent lines.

If $\dfrac{\partial f}{\partial x}(a,b) = p$; $f(a,b) = z_0$.

$$\Rightarrow L_x = \begin{cases} z = z_0 + p(x-a) \\ y = b \end{cases}$$



If $\dfrac{\partial f}{\partial y}(a,b) = q$

$$\Rightarrow L_y = \begin{cases} z = z_0 + q(y-b) \\ x = a \end{cases}$$

The tangent plane at $(a, b, f(a,b))$ contains the Lines tangent to the curves formed between $z = f(x,y)$ & $x = a$; $z = f(x,y)$ & $y = b$.

L_1 & L_2 are both tangent to the surface $z = f(x,y)$

Together they determine a plane

$$\boxed{z = z_0 + p(x-a) + q(y-b)}$$

-- Eqn of tangent plane!

If we fix $y = b$ and vary 'x' we get first tangent line & fix $x = a$ and vary 'y' $\Rightarrow$ the second tangent line. ✓

* This tangent plane is near to the graph / surface $f(x,y)$ (as it is only an approximation eqn).

∴ Can you think of another way to define the above eqn using lines & planes in vector calculus.

Tangent plane Eqn. to the surface $z = f(x,y)$ at point $(a, b, f(a,b))$

$L_x \rightarrow$ tangent line to the traces of surface in plane $y = b$;

$L_y \rightarrow$ " " plane $x = a$.

If $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist at (a, b), and is continuous,

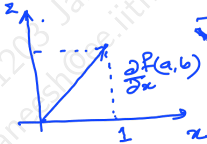
Equation of tangent plane is; $A(x-a) + B(y-b) + C(z - f(a,b)) = 0$.

if $\vec{n} = (A, B, C)$ (normal vector)

Can we find this normal vector ?

Tangent plane contains both L_x & L_y. $\Rightarrow$ If we know $\vec{V}_x, \vec{V}_y$ parallel to L_x, L_y respectively

$\Rightarrow \vec{n} = \vec{V}_x \times \vec{V}_y$.



$\vec{V}_x = \left(1, 0, \frac{\partial f(a,b)}{\partial x}\right)$

This is $\parallel^{l}$ to L_x.

$\parallel^{ly}$ $\vec{V}_y = \left(0, 1, \frac{\partial f(a,b)}{\partial y}\right)$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x(a,b) \\ 0 & 1 & f_y(a,b) \end{vmatrix}$$

$$= -\hat{i}\, f_x(a,b) - \hat{j}\, f_y(a,b) + \hat{k}$$

$\Rightarrow$ $\Rightarrow$ $f_x(a,b)(x-a) + f_y(y-b) - z + f(a,b) = 0$.

Maxima & minima.

* Optimization problems.

$\leadsto$ Find min/max of a function in 2 variables $f(x,y)$

* At a local min or max $f_x = 0$ and $f_y = 0$.

* From Approx Formula; $\Delta z = f_x \Delta x + f_y \Delta y$.

$\Rightarrow$ when $f_x = f_y = 0$ $\Delta z = 0$

$\Rightarrow$ The function does not change anymore

$\Rightarrow$ local max or local min.

$\Rightarrow$ Or it also means the tangent plane is horizontal to the graph of the function $z = f(x,y)$

Ex: