Power in AC Circuits

- **Power in AC Circuits**: Consider a generic two-terminal AC circuit element with voltage $v(t)$ and current $i(t)$. The references for voltage and current are chosen according to passive sign convention (as shown in Fig. 1). Let the voltage and current be given by

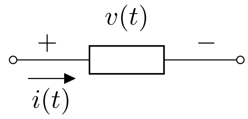


Figure 1: Generic AC Circuit Element

$$v(t) = V_m \cos(\omega t + \phi_v), \quad i(t) = I_m \cos(\omega t + \phi_i) \tag{1}$$

where V_m and I_m are the peak values of voltage and current, ω is the angular frequency, and ϕ_v and ϕ_i are the phase angles of voltage and current, respectively. In the phasor domain, the voltage and current can be represented as $\mathbf{V} = \frac{V_m}{\sqrt{2}}\angle\phi_v$ and $\mathbf{I} = \frac{I_m}{\sqrt{2}}\angle\phi_i$, respectively.

The instantaneous power $s(t)$ associated with the element is defined as

$$s(t) = v(t) \cdot i(t) \tag{2}$$

The unit of instantaneous power is Volt-Ampere (VA). Lets assume that the time reference for analysis is chosen such that $\phi_v = 0$. Thus, the voltage and current can be rewritten as

$$v(t) = V_m \cos(\omega t), \quad i(t) = I_m \cos(\omega t - \theta) \tag{3}$$

where $\theta = \phi_v - \phi_i$ is the phase difference between voltage and current. Under this scenario, the instantaneous power can be expressed as

$$\begin{aligned} s(t) &= V_m I_m \cos(\omega t)\cos(\omega t - \theta) \\ &= \frac{V_m I_m}{2}\left[\cos(2\omega t - \theta) + \cos(\theta)\right] \end{aligned} \tag{4}$$

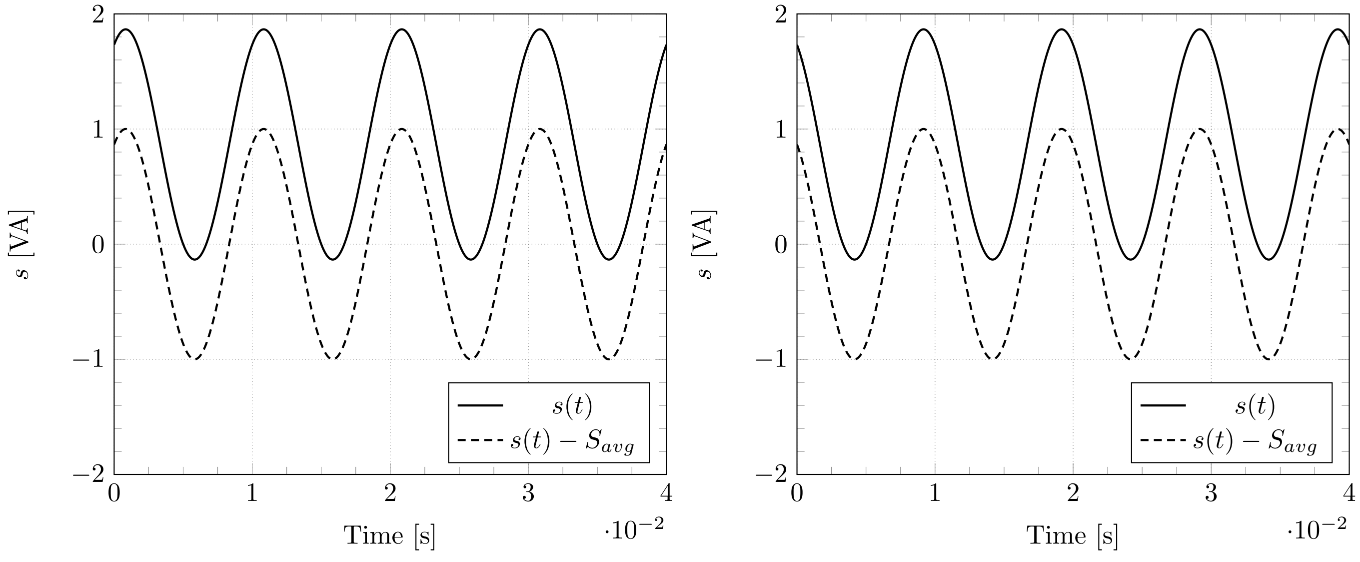
Note that the instantaneous power consists of two components namely (a) a time-varying component with frequency 2ω and (b) a DC component. The average value of the instantaneous power over one complete cycle (of either the voltage or instantaneous power) is given by

$$S_{avg} = \frac{1}{T}\int_0^T s(t)\,dt = \frac{V_m I_m}{2}\cos(\theta) \tag{5}$$

Example 1: Let $v(t) = \sqrt{2}\cos(100\pi t)$ V and $i(t) = \sqrt{2}\cos\left(100\pi t - \frac{\pi}{6}\right)$ A. The instantaneous power associated with the element is given by

$$s(t) = \left[\cos\left(200\pi t - \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}\right]$$

The waveform of instantaneous power is shown in Fig. 2(a). The average value of instantaneous power over one complete cycle is given by $S_{avg} = \frac{\sqrt{3}}{2}$ W. Now, if



(a) when $\theta = \frac{\pi}{6}$ (b) when $\theta = -\frac{\pi}{6}$

Figure 2: Instantaneous Power Waveform for Example 1

the current is given by $i(t) = \sqrt{2}\cos\left(100\pi t + \frac{\pi}{6}\right)$ A, then the instantaneous power associated with the element is given by

$$s(t) = \left[\cos\left(200\pi t + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}\right]$$

The waveform of instantaneous power is shown in Fig. 2(b). The average value of instantaneous power over one complete cycle is again given by $S_{avg} = \frac{\sqrt{3}}{2}$ W. Note that the average power is the same in both cases even though the current waveforms are different. This is because the phase difference between voltage and current is the same in both cases.

More importantly, note that it is very hard to infer the phase difference between voltage and current just by looking at the instantaneous power waveform. In this example, the phase difference between voltage and current is $\frac{\pi}{6}$ in the first case and $-\frac{\pi}{6}$ in the second case. However, the instantaneous power waveforms look very similar in both cases. Infact, the amplitude of oscillation $(s(t) - S_{avg})$ is identical in both cases and the only difference that can be observed in the phase of the oscillating comopnent, which may be very hard to compute/infer in practice. This implies that the instantaneous power waveform does not provide any information about whether the current is leading or lagging the voltage. In view of this, it is important to define other power quantities that can provide more insights about the power associated with an AC circuit element.

- **Active, Reactive, Apparent, and Complex Power**: In order to gain a better understading of power in AC circuits, lets rewrite the instantaneous power given in Equation (4) as

$$s(t) = \underbrace{\frac{V_m I_m}{2}\cos\theta\left[1 + \cos(2\omega t)\right]}_{p(t)} + \underbrace{\frac{V_m I_m}{2}\sin\theta\sin(2\omega t)}_{q(t)} \tag{6}$$

where $p(t)$ is the instantaneous active power and $q(t)$ is the instantaneous reactive power. The average value of instantaneous active power over one complete cycle is defined as Active power (denoted by P) and is given by

$$P = \frac{1}{T}\int_0^T p(t)\,dt = \frac{V_m I_m}{2}\cos\theta = VI\cos\theta \tag{7}$$

where $V = \frac{V_m}{\sqrt{2}}$ and $I = \frac{I_m}{\sqrt{2}}$ are the RMS values of voltage and current, respectively. On the other hand, the peak value of instantaneous reactive power is defined as Reactive power (denoted by Q) and is given by

$$Q = \frac{V_m I_m}{2}\sin\theta = VI\sin\theta \tag{8}$$

In terms of active and reactive power, the instantaneous power can be expressed as

$$s(t) = P\,[1 + \cos(2\omega t)] + Q\sin(2\omega t) \tag{9}$$

It is interesting to note that the average value of instantaneous power over one complete cycle is equal to the active power, i.e., $S_{avg} = P$. Further, the reactive power does not contribute to the average power consumed by the element.

In terms of phasors, the active and reactive power can be expressed as

$$P = \mathrm{Re}\{\mathbf{VI}^*\}, \quad Q = \mathrm{Im}\{\mathbf{VI}^*\} \tag{10}$$

where $\mathbf{I}^*$ is the complex conjugate of the current phasor $\mathbf{I}$. The complex power (denoted by $\mathbf{S}$) is defined as

$$\mathbf{S} = P + jQ = VI\,(\cos\theta + j\sin\theta) \;=\; \mathbf{VI}^* \tag{11}$$

The magnitude of complex power is called apparent power (denoted by $|S|$ or just S) and is given by

$$|S| = \sqrt{P^2 + Q^2} \tag{12}$$

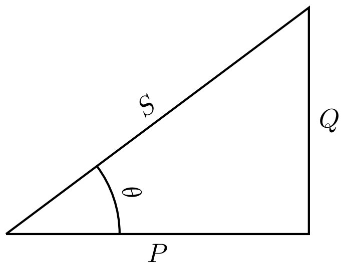


Figure 3: Power Triangle

The active, reactive, and apparent power are often represented using a power triangle as shown in Fig. 3. The phase angle difference between the voltage and current phasors i.e., θ is defined as the power factor angle and can be computed from power as $\theta = \tan^{-1}(Q/P)$ or $\theta = \cos^{-1}(P/S)$. The power factor (PF) of the element is defined as

$$\mathrm{PF} = \cos\theta = \frac{P}{S} \tag{13}$$

The unit of active power is Watt (W), reactive power is Volt-Ampere-Reactive (VAR), and apparent power is Volt-Ampere (VA).

Example 2: Consider the element in Example 1. The RMS values of voltage and current are $V = 1$ V and $I = 1$ A, respectively. The phase difference between voltage and current is $\theta = \frac{\pi}{6}$ in the first case and $\theta = -\frac{\pi}{6}$ in the second case. The active, reactive, apparent, and complex power associated with the element in both cases can be computed as follows:

Case A: When $\theta = \frac{\pi}{6}$

- Active Power, $P = VI\cos\theta = \frac{\sqrt{3}}{2}$ W
- Reactive Power, $Q = VI\sin\theta = \frac{1}{2}$ VAR
- Apparent Power, $S = \sqrt{P^2 + Q^2} = 1$ VA
- Complex Power, $\mathbf{S} = P + jQ = \frac{\sqrt{3}}{2} + j\frac{1}{2}$ VA
- Power Factor angle $\theta = \frac{\pi}{6}$.
- Power Factor, $\mathrm{PF} = \cos\theta = \frac{\sqrt{3}}{2} = 0.866$ (lagging)

Case B: When $\theta = -\frac{\pi}{6}$

- Active Power, $P = VI\cos\theta = \frac{\sqrt{3}}{2}$ W
- Reactive Power, $Q = VI\sin\theta = -\frac{1}{2}$ VAR
- Apparent Power, $S = \sqrt{P^2 + Q^2} = 1$ VA
- Complex Power, $\mathbf{S} = P + jQ = \frac{\sqrt{3}}{2} - j\frac{1}{2}$ VA
- Power Factor angle $\theta = -\frac{\pi}{6}$.
- Power Factor, $\mathrm{PF} = \cos\theta = \frac{\sqrt{3}}{2} = 0.866$ (leading)

Note that the active, apparent, and power factor are the same in both cases. However, the reactive and complex power are different in both cases. This is because the reactive power is positive when current lags voltage [1] and negative when current leads voltage [2]. Thus, the reactive and complex power can provide information about whether the current is leading or lagging the voltage.

[1] characteristic of an inductive element
[2] characteristic of a capacitive element