

EE1080/AI1110/EE2102 Probability: HW3

10th Feb 2025

1. Consider non-negative R.V. Show that $E[X^n] = \int nx^{n-1}P(X > x)dx$.
Hint: $Y = X^n, E[Y] = \int P(Y > y)dy$ and do a variable change $y = x^n$.
2. Let X be uniformly distributed in the unit interval $[0, 1]$. Consider the random variable $Y = g(X)$, where

$$g(x) = \begin{cases} 1 & x < 1/3 \\ 2 & x > 1/3 \end{cases}$$

Find the expected value of Y by first deriving the PMF. Verify the result using the expected value rule

3. Laplace random variable: Let X have the PDF

$$f_X(x) = \frac{\lambda}{2}e^{-\lambda|x|}$$

Verify that f_X satisfies the normalization condition, evaluate the mean and variance of X and the cumulative distribution function.

4. Let X be a random variable with p.d.f

$$f_X(x) = \begin{cases} c(x - \frac{3}{x^2}) & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases}$$

What is the value of c and CDF of X .

5. Find the CDF of cauchy random variable with p.d.f shown below:

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad x \in \mathbb{R}.$$

6. Group of construction workers takes time X (in hours) to finish a task.
 The density function of time X is:

$$f_X(x) = \begin{cases} cxe^{-\sqrt{x}} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

What is the probability that workers take more than 10 hours to finish the task ?

7. Simulating a continuous R.V: Let X be a random variable with F as its CDF.
- Show that $U = F(X)$ is a uniform random variable in $[0, 1]$.
 - Use this fact to generate an exponential random variable X , with parameter λ from the uniform random variable U
 - Generalize this to generate discrete random variable X with CDF $F_X(k)$.
8. Acchu goes to the bank to make a withdrawal, and is equally likely to find 0 or 1 customers ahead of him. The service time of the customers ahead, if present is exponentially distributed with parameter λ . What is the CDF of Acchu's waiting time.
9. A point is chosen at random on a line segment of length L . Interpret this statement and find probability that the ratio of the shorter to the longer segment is less than $1/4$.
10. Consider two continuous R.Vs Y and Z and a random variable $X = Y$ w.p p and $X = Z$ w.p $(1 - p)$.
- Show that the PDF is as given below. (*Hint: Find CDF of x using total probability theorem*)
- $$f_X(x) = pf_Y(x) + (1 - p)f_Z(x)$$
- Calculate the CDF of two-sided exponential random variable that has PDF given by
- $$f_X(x) = \begin{cases} p\lambda e^{\lambda x} & x \leq 0 \\ (1 - p)\lambda e^{-\lambda x} & x \leq 0 \end{cases}$$
11. Mixed random variables: The taxi stand and the bus stop near Puchku's home are in the same location. Puchku goes there at a given time and if a taxi is waiting (this happens with probability $2/3$) she boards it. Otherwise she waits for bus or taxi to come, whichever comes first. The next taxi will arrive in time that is uniformly distributed between 0 and 10 minutes while the next bus will arrive exactly in 5 minutes. Find the CDF and the expected value of Puchku's waiting time.
12. An absent minded professor schedules two student appointments for the same time. The appointment durations are independent of each other and exponentially distributed with mean 30 minutes. What is the expected value of time between arrival of first student and departure of the second student.
13. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 a.m., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 a.m.

- (a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 a.m. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?
- (b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 a.m?
14. If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain
- (a) at least 50 who are in favor of the proposition;
 - (b) between 60 and 70 inclusive who are in favor;
 - (c) fewer than 75 in favor.
15. Let X be a random variable that takes values between 0 and c . Show that $Var(X) \leq c^2/4$.
16. For exponential random variable show that $E[X^k] = \frac{k!}{\lambda^k}$.
17. Median of a continuous R.V having distribution F is the value m such that $F(m) = 1/2$. Find the median of X if X is:
- (a) uniform in $[a, b]$
 - (b) normal with parameters μ, σ^2
 - (c) exponential with rate λ