Sinusoidal Response of First-Order Circuits

- To understand the sinusoidal response of a first-order circuit to sinusoidal inputs, we will solve a first-order constant coefficient differential equation of the form

$$\frac{dx}{dt} + px = A\cos\left(\omega t + \phi\right) \tag{1}$$

where p, A, ω, and ϕ are constants. Note that p is assumed to be constant (since we are dealing with circuits where the parameters are constant)[1].

[1] Infact, p is typically positive, since the circuit parameters are non-negative

Although the response can be obtained using the intergrating factor approach in a straightforward manner, it is often more convienient to solve the response of the circuit to a complex exponential input and then take the real part of the solution. Thus, we will first solve the differential equation

$$\frac{d\hat{x}}{dt} + p\hat{x} = Ae^{j(\omega t+\phi)} \tag{2}$$

where $\hat{x}$ is the complex-valued function that denotes the response of the circuit to the complex exponential input. The solution to the original problem can then be obtained as $x\left(t\right) = \Re\left\{\hat{x}\left(t\right)\right\}$.

It is to be noted that the response to a complex exponential input can also be obtained using the integrating factor appraoch. The integrating factor for the the differential equation (2) is

$$\text{integrating factor} = u\left(t\right) = e^{pt} \tag{3}$$

Multiplying both sides of the differential equation (2) by the integrating factor (3) and rearranging results in

$$\frac{d}{dt}\left[\hat{x}e^{pt}\right] = Ae^{pt}e^{j(\omega t+\phi)} \tag{4}$$

Integrating on both sides of the equation (4) gives

$$\hat{x}e^{pt} = \int Ae^{pt}e^{j(\omega t+\phi)}dt + c \tag{5}$$

where c is the constant of integration. The integral on the right-hand side of the equation (5) can be evaluated as

$$\hat{x}e^{pt} = \frac{A}{p+j\omega}e^{pt}e^{j(\omega t+\phi)} + c \implies \hat{x} = \frac{A}{p+j\omega}e^{j(\omega t+\phi)} + ce^{-pt} \tag{6}$$

The term ce^{-pt} denotes the transient response of the circuit and since $p > 0$, this term decays to zero as $t \to \infty$. The term $\frac{A}{p+j\omega}e^{j(\omega t+\phi)}$ denotes the steady-state response of the circuit. Thus, the steady-state response of the circuit to a complex exponential input is

$$\hat{x}_{ss} = \frac{A}{p + j\omega} e^{j(\omega t + \phi)} \tag{7}$$

Note that the steady state response to the complex exponential input is also a complex exponential with the same frequency as that of the input. The amplitude and phase of the steady-state response can be obtained from $\frac{A}{p+j\omega}$.

Returning to the original problem, the response to the sinusoidal input can be obtained as

$$\begin{aligned} x(t) =& \Re\left\{\frac{A}{p + j\omega} e^{j(\omega t + \phi)} + ce^{-pt}\right\} \\ =& \frac{A}{p^2 + \omega^2}\left[p\cos\left(\omega t + \phi\right) + \omega \sin\left(\omega t + \phi\right)\right] + ce^{-pt} \end{aligned} \tag{8}$$

Equation (8) can be rewritten in a more compact form as

$$x(t) = \frac{A}{\sqrt{p^2 + \omega^2}} \cos\left(\omega t + \phi - \theta\right) + ce^{-pt} \quad \text{where} \quad \theta = \tan^{-1}\left(\frac{\omega}{p}\right) \tag{9}$$

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$$x_{ss}(t) = \frac{A}{\sqrt{p^2 + \omega^2}} \cos\left(\omega t + \phi - \theta\right) \quad \text{where} \quad \theta = \tan^{-1}\left(\frac{\omega}{p}\right) \tag{10}$$

It is interesting to note that the steady-state response to the sinusoidal input is also a sinusoid with the same frequency as that of the input. The amplitude scaling and the phase shift of the steady-state response is determined by the complex number $p + j\omega$. The fact that the steady-state response has the same frequency as that of the input is a general property of linear time-invariant (LTI) systems. Further, since the response is determined by the complex number $p+j\omega$, it is often convenient to represent the **steady-state response using phasors**. The phasor representation of the input and the steady-state response are given by

$$\text{Input: } \mathbf{F} = \frac{A}{\sqrt{2}} e^{j\phi} \quad \text{and} \quad \mathbf{X}_{ss} = \frac{A}{\sqrt{2}\sqrt{p^2 + \omega^2}} e^{j\phi - \theta} = \frac{\mathbf{F}}{p + j\omega} \tag{11}$$

The sinusoidal steady-state response in the phasor form is the ratio of the input phasor to the complex number $p+j\omega$. When the goal is to compute the sinusoidal-state response of a circuit, it is often easier to work with phasors.

Key observations:

- The complete response (also known as the time-domain response) is the sum of the steady-state response and the transient response.
- The steady-state response to a sinusoidal input is a sinusoid with the same

frequency as that of the input.

- The transient response decays to zero as $t \to \infty$ if $p > 0$ (which is typically the case for physical systems).
- It is often convenient to adopt phasors when the objective is to compute the sinusoidal steady-state response.

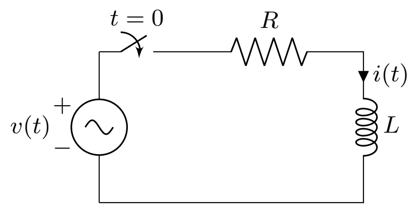

Figure 1: RL circuit for Example 1.

- **Example 1:** Compute the time-domain response of the RL circuit shown in Fig. 1 to the sinusoidal input $V_m \cos(\omega t + \phi)$ assuming that the initial current through the inductor is 0. The differential equation characterizing the response of the circuit is

$$L\frac{di}{dt} + Ri = V_m \cos(\omega t + \phi) \tag{12}$$

In the standard form, (12) can be written as

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V_m}{L}\cos(\omega t + \phi) \tag{13}$$

Comparing (13) with the standard form of the first-order differential equation, we identify $p = \frac{R}{L}$ and $A = \frac{V_m}{L}$. Thus, the complete response of the circuit can be obtained using (9) as

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}\cos(\omega t + \phi - \theta) + ce^{-\frac{R}{L}t} \quad \text{where} \quad \theta = \tan^{-1}\left(\frac{\omega L}{R}\right) \tag{14}$$

The constant c can be determined using the initial condition $i(0) = 0$ as

$$c = -\frac{V_m}{\sqrt{R^2 + (\omega L)^2}}\cos(\phi - \theta) \tag{15}$$

Thus, the complete response of the circuit is

$$i(t) = \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}\cos(\omega t + \phi - \theta) - \frac{V_m}{\sqrt{R^2 + (\omega L)^2}}\cos(\phi - \theta)\, e^{-\frac{R}{L}t} \tag{16}$$

The phasor representation of the input and the steady-state response are given by

$$\text{Input: } \mathbf{V} = \frac{V_m}{\sqrt{2}}e^{j\phi} \quad \text{and} \quad \mathbf{I}_{ss} = \frac{V_m}{\sqrt{2}\sqrt{R^2 + (\omega L)^2}}e^{j\phi - \theta} = \frac{\mathbf{V}}{R + j\omega L} \tag{17}$$

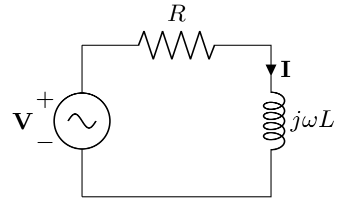

Figure 2: Phasor domain representation of the RL circuit for Example 1.

It is to be noted that the complex number $R + j\omega L$ respresents the sum of impedances of the resistor and the inductor at the frequency ω. The phasor domain representation of the circuit is shown in Fig. 2. It is important to note that the phasor domain representation of the circuit is obtained by replacing the inductor with its impedance $j\omega L$ and can only be used to compute the sinusoidal steady-state response of the circuit.

- **Equivalent Impedance and Divider Circuits:** The analysis carried out for a first-order differential equation for sinusoidal inputs indicate that the concepts of

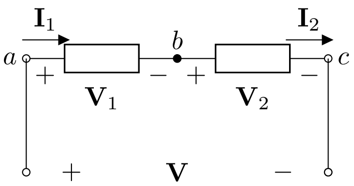

Figure 3: Two elements connected in series.

phasors and impedance can be used to efficiently compute the sinusoidal steady-state response of a circuit. It is perhaps an appropriate time to look at elements connected in series and parallel and derive the equivalent impedance of such connections. For two elements connected in series (as shown in Fig. 3), application of KVL requires

$$\mathbf{V} = \mathbf{V}_1 + \mathbf{V}_2 \tag{18}$$

Since the same current flows through both the elements, we have $\mathbf{I} = \mathbf{I}_1 = \mathbf{I}_2$. Using the definition of impedance, we have

$$\mathbf{V} = \mathbf{I}\mathbf{Z}_{eq} = \mathbf{I}\mathbf{Z}_1 + \mathbf{I}\mathbf{Z}_2 \tag{19}$$

Thus, the equivalent impedance of two elements connected in series is

$$\mathbf{Z}_{eq} = \mathbf{Z}_1 + \mathbf{Z}_2 \tag{20}$$

The voltage across each element can be computed to be

$$\mathbf{V}_1 = \mathbf{I}\mathbf{Z_1} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V} \quad \text{and} \quad \mathbf{V}_2 = \mathbf{I}\mathbf{Z_2} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{V} \tag{21}$$

Thus, the voltage divider rule can be applied to compute the voltage across each element in a series connection.

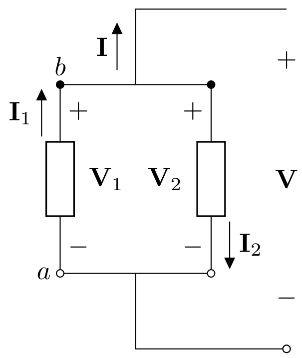

Figure 4: Two elements connected in parallel.

For two elements connected in parallel (as shown in Fig. 4), application of KCL requires

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2 \tag{22}$$

Since the same voltage is applied across both the elements, we have $\mathbf{V} = \mathbf{V}_1 = \mathbf{V}_2$. Using the definition of impedance, we have

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}_{eq}} = \frac{\mathbf{V}}{\mathbf{Z}_1} + \frac{\mathbf{V}}{\mathbf{Z}_2} \tag{23}$$

Thus, the equivalent impedance of two elements connected in parallel is

$$\frac{1}{\mathbf{Z}_{eq}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} \tag{24}$$

The current through each element can be computed to be

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z_1}} = \frac{\mathbf{Z}_2}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \quad \text{and} \quad \mathbf{I}_2 = \frac{\mathbf{V}}{\mathbf{Z_2}} = \frac{\mathbf{Z}_1}{\mathbf{Z}_1 + \mathbf{Z}_2}\mathbf{I} \tag{25}$$

Thus, the current divider rule can be applied to compute the current through each element in a parallel connection.

- **Example 2:** Compute the time-domain response of the RC circuit shown in Fig. 5 to the sinusoidal input $V_m \cos(\omega t + \phi)$ assuming that the initial voltage across the capacitor is 0. The differential equation characterizing the response of the circuit is

$$RC\frac{dv_C}{dt} + v_C = V_m \cos(\omega t + \phi) \tag{26}$$

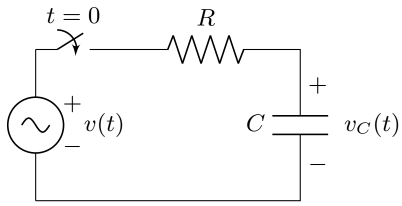

Figure 5: RC Circuit for Example 2.

In the standard form, (26) can be written as

$$\frac{dv_C}{dt} + \frac{1}{RC}v_C = \frac{V_m}{RC}\cos(\omega t + \phi) \tag{27}$$

Comparing (27) with the standard form of the first-order differential equation, we identify $p = \frac{1}{RC}$ and $A = \frac{V_m}{RC}$. Thus, the complete response of the circuit can be obtained using (9) as

$$v_C(t) = \frac{V_m}{\sqrt{1 + (\omega RC)^2}}\cos(\omega t + \phi - \theta) + ce^{-\frac{t}{RC}} \quad \text{where} \quad \theta = \tan^{-1}(\omega RC) \tag{28}$$

The constant c can be determined using the initial condition $v_C(0) = 0$ as

$$c = -\frac{V_m}{\sqrt{1 + (\omega RC)^2}}\cos(\phi - \theta) \tag{29}$$

Thus, the complete response of the circuit is

$$v_C(t) = \frac{V_m}{\sqrt{1 + (\omega RC)^2}}\cos(\omega t + \phi - \theta) - \frac{V_m}{\sqrt{1 + (\omega RC)^2}}\cos(\phi - \theta)\, e^{-\frac{t}{RC}} \tag{30}$$

The phasor representation of the input and the steady-state response are given by

$$\text{Input: } \mathbf{V} = \frac{V_m}{\sqrt{2}}e^{j\phi} \quad \text{and} \quad \mathbf{V}_{C,ss} = \frac{V_m}{\sqrt{2}\sqrt{1 + (\omega RC)^2}}e^{j\phi - \theta} = \frac{\mathbf{V}}{1 + j\omega RC} \tag{31}$$

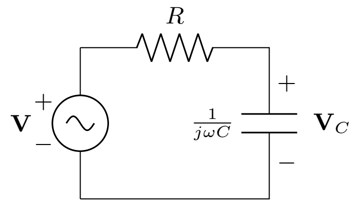

Figure 6: Phasor domain representation of the RC circuit for Example 2.

The phasor domain representation of the circuit is shown in Fig. 6. The impedance of the capacitor at the frequency ω is $\frac{1}{j\omega C}$. Note that, the voltage across the capacitor in phasor form can be computed using the voltage divider rule as

$$\mathbf{V}_C = \mathbf{V}\frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{\mathbf{V}}{1 + j\omega RC} \tag{32}$$

Equation 32 matches with the expression for the steady-state phasor response obtained in (31).

- **Using Phasors to Compute Steady-state Response of Circuits with Sinusoidal Sources:** The analysis of first-order circuits with sinusoidal sources indicates that the concepts of phasors and impedance can be used to efficiently compute the sinusoidal steady-state response of a circuit [2]. The procedure to analyze a circuit with sinusoidal sources using phasors is as follows

Step 1: Represent all sources in the phasor form.

Step 2: Replace all circuit elements with their equivalent impedances at the frequency

[2] Although, a formal proof for this statement is beyond the scope of this course, we will take this to be true for higher order circuits.

of the sources.

Step 3: Apply circuit laws or analysis techniques (such as nodal or mesh analysis) to compute the circuit response (in phasor form).

Step 4: Convert the phasor domain response to the time-domain response (if required).