① $Y = X + Z$

$$f_X(x) = f_Z(x) = \frac{1}{\sqrt{2\pi \times 4}} e^{-\frac{1}{2}\left(\frac{x-2}{2}\right)^2}$$

ⓐ $f_{X,Y}(x, y) = f_{X,Z}(x, y-x)$

$(X, Y) \Longleftrightarrow (X, Z)$

$X = Z$

$Y = X + Z$

$$= f_X(x) f_Z(y-x).$$

$$= f_X(x) f_{Y|X}(y|x)$$

$$\Rightarrow \boxed{f_{Y|X}(y|x) = f_Z(y-x).}$$ 1 mark

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x) f_{Y|X}(y|x)\, dx.$$

$$= \int_{-\infty}^{\infty} f_X(x) f_Z(y-x)\, dx$$

$\mu = 2$

$\sigma = 2$

$$= \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} e^{-\frac{1}{2}\left(\frac{\overbrace{y-x-\mu}^{(y-2\mu)-(x-\mu).}}{\sigma}\right)^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{1}{2}\left[\left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{x-\mu}{\sigma}\right)^2 + \left(\frac{y-2\mu}{\sigma}\right)^2 - \frac{2(x-\mu)(y-2\mu)}{\sigma^2}\right]}$$

$$= \int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{1}{2}\left[2\left(\frac{x-\mu}{\sigma}\right)^2 - 2\sqrt{2}\left(\frac{x-\mu}{\sigma}\right)\frac{(y-2\mu)}{\sqrt{2}\sigma} + \left(\frac{y-2\mu}{\sqrt{2}\sigma}\right)^2 + \left(\frac{y-2\mu}{\sqrt{2}\sigma}\right)^2\right]}$$

$$= \frac{e^{-\left(\frac{y-2\mu}{\sqrt{2}\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma\sqrt{2}} \underbrace{\int_{-\infty}^{\infty} \frac{\sqrt{2}}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2(\sigma^2/2)}\left[(x-\mu) - \left(\frac{y-2\mu}{2}\right)\right]^2} dx}_{\text{density of } N(y/2,\ \sigma^2/2)}$$

$\therefore$ should sum to 1.

$$= \frac{e^{-\left(\frac{y-2\mu}{\sqrt{2}\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma\sqrt{2}}$$

Please give them marks if they assumed Sum of Gaussian is Gaussian.

$E[Y] = E[X] + E[Z] = 4,$

$Var(Y) = Var[X] + Var(Z) = 8$

$$\boxed{\Rightarrow Y \sim N(4, 8) \quad \Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi \times 8}} e^{-\frac{1}{2}\frac{(y-4)^2}{8}}}$$

① mark

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$f_Z(y-x)$

0.5

$$= \frac{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \cdot \cancel{\frac{1}{\sqrt{2\pi\sigma^2}}} e^{-\frac{1}{2}\left(\frac{y-x-\mu}{\sigma}\right)^2}}{\frac{1}{\cancel{\sqrt{2\pi\sigma^2}}\times 2} \cdot e^{-\frac{1}{2}\left(\frac{y-2\mu}{\sqrt{2}\sigma}\right)^2}}$$

0.5 $\leftarrow$ $$= \frac{1}{\sqrt{2\pi(\sigma^2/2)}} e^{-\frac{1}{2}\frac{(x-y/2)^2}{(\sigma^2/2)}}$$

(b) MMSE estimator. $\hat{x}_{MMSE}[y] = E[X|Y=y]$.

1.5

$E[X|Y] = y/2$ as

1.5

$X|Y=y$ is $N(y/2, \sigma^2/2)$.

(c) MAP estimator.

$$\hat{x}_{MAP}(y) = \arg\max_x f_{X|Y}(x|y) = \arg\max_x e^{-\frac{1}{2}\left(\frac{x-y/2}{\sigma^2/2}\right)^2}$$

0.5

1 $\leftarrow$ $= y/2$

$$\Rightarrow \hat{x}_{MAP}(y) = y/2$$

ML estimator

$$\hat{x}_{ML}(y) = \arg\max_x f_{Y|X}(y|x) = \arg\max_x f_Z(y-x)$$

0.5

1 $\leftarrow$ $$= \arg\max_x e^{-\frac{1}{2}\left(\frac{y-x-\mu}{\sigma}\right)^2}$$

$$= y - \mu.$$

$$\Rightarrow x^*_{ML}(y) = y - \mu = y - 2.$$

(d) $\mathrm{Var}(Y) = \mathrm{Var}(X) + \mathrm{Var}(Z) = 8 \rightarrow 1$

$E[Y] = E[X] + E[Z] = 4. \rightarrow 1$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = E[X(X+Z)]$$
$$= E[X^2 + XZ]$$
$$= E[X^2] + E[X]E[Z]$$
$$= (4+4) + 2\times 2.$$
$$= 12.$$

$$\mathrm{Cov}(X,Y) = 12 - 2\times 4 = 4 \rightarrow 2$$

(e) $$x^*_{LMMSE}(y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}(y - E[Y]) + E[X].$$

1

$$= \frac{4}{8}(y-4) + 2.$$

give marks even if there are calculation mistakes

1 $= y/2.$

(f) $E\left[\left(X - x^*_{LMMSE}(Y)\right)^2\right]$ $= E\left[\left(X - \frac{Y}{2}\right)^2\right]$

$= E\left[\left(X - \frac{(X+Z)}{2}\right)^2\right]$

$= \frac{1}{4}E\left[(X-Z)^2\right]$

$= \frac{1}{4}\left[8+8-2\times2\times2\right]$

$= 2.$

0.5 for writing this expression ✓

1.5.

$$= Var(X) - \frac{Cov(X,Y)^2}{Var(Y)}$$

$$= 4 - \frac{4^2}{8} = 2.$$

(g, h) Should have same error for LMMSE, MAP, MMSE estimator are all the same in this case: as these three estimators are the same.

3 marks.

ML error

$E\left[\left[X - \hat{x}_{ML}(Y)\right]^2\right]$

0.5 ✓

$= E\left[\left(X-(Y-2)\right)^2\right].$

$= E\left[X^2 + (Y-2)^2 - 2X(Y-2)\right]$

$= E[X^2] + E\left[(Y-2)^2\right] - 2E[X(Y-2)]$

$= Var(X) + (E[X])^2 + Var(Y-2) + (E[Y-2])^2 - 2E[XY] + 4E[X]$

2.5

$Var(Y-2) = Var(Y)$

$E[Y-2] = E[Y] - 2$

$= 4 + (2)^2 + 8 + (2)^2 - 2\times12 + 4\times2$

give partial marks for intermediate steps.

$= 4.$