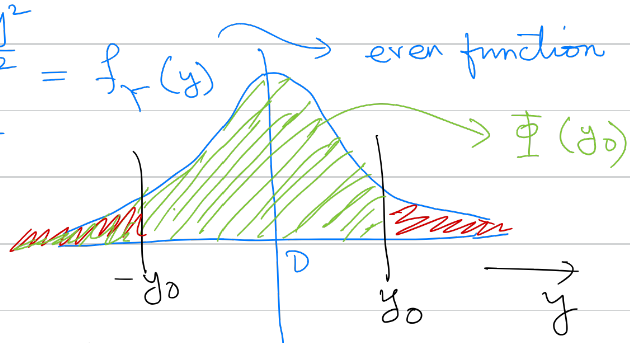
Lec 16 — 11th Feb

Normal R.V $X \sim N(\mu, \sigma^2)$

$$f_X(x) = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sqrt{2\pi}\,\sigma}$$

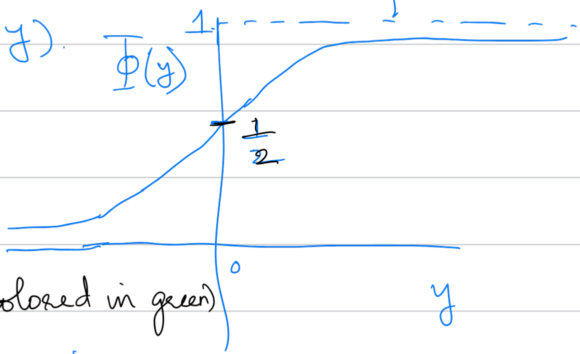
Standard Normal R.V $Y \sim N(0, 1)$

$$\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} = f_Y(y)$$



CDF of Y is

$$F_Y(y) = P(Y \le y) = \Phi(y).$$



Lemma 1: $\Phi(y) = 1 - \Phi(-y) \quad \forall y$

Proof: $\Phi(y_0) = P(Y \le y_0)$. (colored in green)

y_0 is +ve

$$= 1 - P(Y \ge y_0)$$
$$= 1 - P(Y \le -y_0) \quad \text{(Symmetry, colored in red).}$$
$$= 1 - \Phi(-y_0).$$

$$\boxed{\Phi(y_0) + \Phi(-y_0) = 1}$$

$$\Rightarrow 2\Phi(0) = 1 \Rightarrow \Phi(0) = 1/2.$$

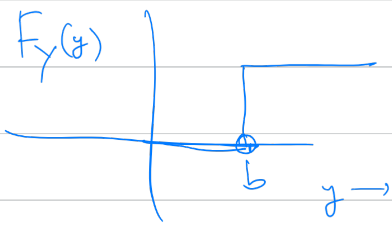
Lemma 2: p.d.f of scaled & shifted R.V.

Let $Y = aX + b$. Then $f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$.

Proof:

$$F_Y(y) = P(Y \le y) = P(aX + b \le y)$$
$$= P(aX \le y - b).$$

$Y = b$ (if $a = 0$).



$$= \begin{cases} P\left(X \le \frac{y-b}{a}\right) & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$f_Y(y) = \frac{\partial F_Y(y)}{\partial y} = \begin{cases} \frac{\partial}{\partial y} F_X\left(\frac{y-b}{a}\right) = f_X\left(\frac{y-b}{a}\right)\frac{1}{a} & a > 0 \\ -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases}$$

$$= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right).$$

Lemma 3:

Suppose X is a normal random variable with mean μ and variance σ^2. Then $Y = \frac{X-\mu}{\sigma}$ is a standard normal random variable.

$$E[Y] = \frac{E[X] - \mu}{\sigma} = 0.$$

$$Var[Y] = \frac{1}{\sigma^2} Var(X) = 1.$$

$$Y = aX + b, \quad a = \frac{1}{\sigma}, \quad b = -\frac{\mu}{\sigma}.$$

From previous Lemma:

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$= \sigma f_X\left(\frac{y + \frac{\mu}{\sigma}}{\frac{1}{\sigma}}\right) = \sigma f_X(\sigma y + \mu)$$

$$= \cancel{\sigma} \frac{e^{-\frac{1}{2}\left(\frac{\sigma y + \cancel{\mu} - \cancel{\mu}}{\sigma}\right)^2}}{\sqrt{2\pi}\,\cancel{\sigma}} = \frac{e^{-\frac{1}{2}y^2}}{\sqrt{2\pi}}$$

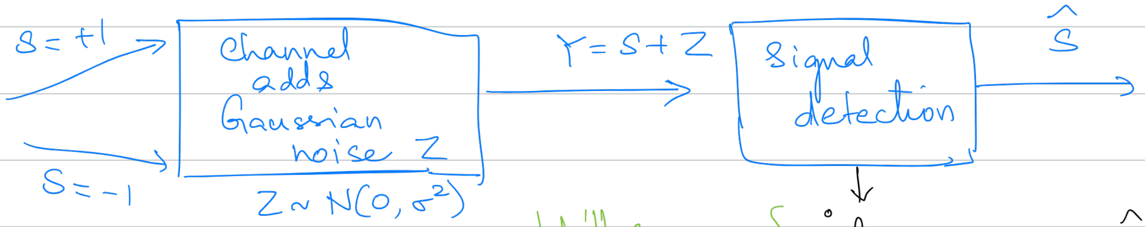
Y is a standard normal R.V.

Excercise

Lemma 4: If $X \sim N(\mu, \sigma^2)$ is a normal R.V then $Y = aX + b$ is also a normal R.V. with mean $a\mu + b$ and variance $a^2\sigma^2$

Example:

Signal detection



We'll see in later lectures why this is a good scheme $\left\{ \begin{array}{l} \text{if } Y \geq 0, \quad \hat{S} = 1 \\ \text{Otherwise } \quad \hat{S} = -1 \end{array} \right.$

Assume that $S=-1$ is transmitted, what is the probability of error i.e., what is the probability that $\hat{S}$ is detected as 1.

$$Y \mid S=-1 \sim N(-1, \sigma^2).$$
$$Y \mid S=+1 \sim N(+1, \sigma^2)$$

$$P(\hat{S} \neq S \mid S=-1) = P(\hat{S}=1 \mid S=-1) = P(Y \geq 0 \mid S=-1).$$
$$= P(-1 + Z \geq 0).$$
$$= P(Z \geq 1).$$
$$= P\left(\frac{Z}{\sigma} \geq \frac{1}{\sigma}\right).$$
$$= 1 - P\left(\frac{Z}{\sigma} \leq \frac{1}{\sigma}\right).$$

$\frac{Z}{\sigma}$ is a standard normal R.V. $\Big\}$ $= 1 - \Phi\left(\frac{1}{\sigma}\right).$

Excercise:

$$P(\hat{S} = -1 \mid S = 1) = 1 - \Phi\left(\frac{1}{\sigma}\right).$$
$$= P(Z \leq -1).$$
$$= P\left(\frac{Z}{\sigma} \leq -\frac{1}{\sigma}\right)$$
$$= \Phi\left(-\frac{1}{\sigma}\right). = 1 - \Phi\left(\frac{1}{\sigma}\right).$$

$$P(\hat{S} \neq S)$$
$$= P(\hat{S} \neq S \mid S=1)\, P(S=1)$$
$$+ P(\hat{S} \neq S \mid S=-1)\, P(S=-1)$$

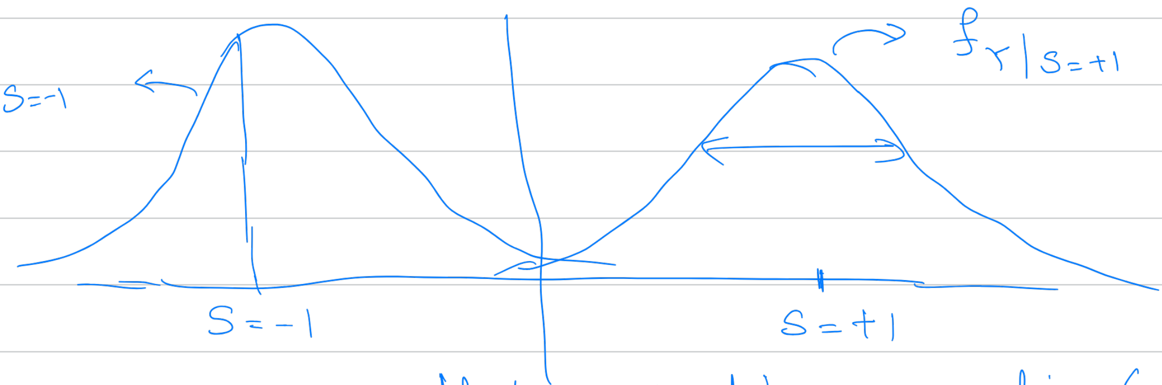
$$= P(S=1)\left(\Phi\left(\frac{-1}{\sigma}\right)\right) + P(S=-1)\,\Phi\left(\frac{-1}{\sigma}\right)$$

$$= \underline{\Phi\left(\frac{-1}{\sigma}\right)} = 1 - \Phi\left(\frac{1}{\sigma}\right).$$

$$\sigma \uparrow \qquad \frac{1}{\sigma} \downarrow \qquad \Phi\left(\frac{1}{\sigma}\right) \downarrow \qquad -\Phi\left(\frac{1}{\sigma}\right) \uparrow$$

If noise has larger variance then the signal detection error is high.

We have $\{ f_{Y|S=-1}$ not formally defined this will do in next few lectures. (conditional p.d.f).



Higher signal to noise ratio $\left(\frac{1}{\sigma^2}\uparrow\right)$

$\Rightarrow$ lower noise variance $(\sigma \downarrow)$

$\Rightarrow$ lower bit error rate $\left(1-\Phi\left(\frac{1}{\sigma}\right)\downarrow\right)$

Joint CDF of X and Y.

$$F_{X,Y}(x,y) = P(X \le x,\ Y \le y).$$

$$F_{X,Y} : \mathbb{R}^2 \longrightarrow [0,1].$$

Properties of joint CDF.

① $$\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y)$$

$$\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$$

$$\lim_{x\to\infty}\lim_{y\to\infty} F_{X,Y}(x,y) = \lim_{x\to\infty} F_X(x) = 1.$$

(2) $$\lim_{x \to -\infty} F_{X,Y}(x, y) = 0$$

$$\lim_{y \to -\infty} F_{X,Y}(x, y) = 0.$$

(3) Monotonic property. If $x_1 \leq x_2$, $y_1 \leq y_2$

$$F_{X,Y}(x_1, y_1) \leq F_{X,Y}(x_2, y_2).$$

(4) Right continuity:

$$\lim_{\substack{\Delta x \to 0^+ \\ \Delta y \to 0^+}} F_{X,Y}(x + \Delta x, y + \Delta y) = F_{X,Y}(x, y).$$

(5). $P(x_1 < X \leq x_2, \; y_1 < Y \leq y_2).$

$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2)$$
$$+ F_{X,Y}(x_1, y_1).$$