Instructions:

- Marks will be awarded only if the problems are solved strictly using the vector calculus approach.
- Marks will be awarded only if all assumptions (if any) are explicitly stated and clearly justified in the answer sheets.
- Vectors are denoted by bold letters.

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1. (a) **(5 Marks)** Sketch the region R labeling all the relevant points, of the following:

$$\iint_R f dA = \int_0^2 \int_{x^2}^{2\sqrt{2x}} f(x,y)dydx$$

   (b) **(5 Marks)** Rewrite the double integral as an iterated integral with the order interchanged.

2. Consider the function:

$$F(x,y,z) = z\sqrt{x^2+y} + 2\frac{y}{z}$$

   (a) **(5 Marks)** The point $P_0 : (1,3,2)$ lies on the surface $F(x,y,z) = 7$. Find the equation of the tangent plane to the surface $F = 7$ at P_0.

   (b) **(5 Marks)** If starting at P_0 a small change were to be made in only one of the variables, which one would produce the largest change (in absolute value) in F? If the change in this variable was of size 0.1, approximately how large would be the change in F be?

   (c) **(5 Marks)** What distance from P_0 in the direction $\pm(2,2,-1)$ will produce an approximate change in F of size 0.1 units, according to the (already computed) linearization of F?.

3. **(5 Marks)** Let the surface S be the elliptical paraboloid $z = x^2 + 4y^2$ lying beneath the plane $z = 1$. We define the orientation of S by taking the inner normal vector $\mathbf{n}$ to the surface, which is the normal having a positive $\mathbf{k}$- component. Find the flux of $\nabla \times \mathbf{F}$ across S in the direction $\mathbf{n}$ for the vector field $\mathbf{F} = y\mathbf{i} - xz\mathbf{j} + xz^2\mathbf{k}$. (**2 Marks** explicitly for drawing the correct curve of intersection along with surface inner normal orientation $\mathbf{n}$)

4. **(5 Marks)** Calculate the outward flux of the vector field

$$\mathbf{F}(x,y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$$

   across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$

5. (a) **(2.5 Marks)** Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

   (b) **(2.5 Marks)** Sketch and explain the geometry of the solution with all the relevant points labeled.

(PTO)

6. **(5 Marks)** Let S be the entire closed surface of the region inside the cylinder: $x^2 + y^2 = 4$, bounded above by the plane $z = 5$ and below by the paraboloid :$z = 4 - (x^2 + y^2)$. Evaluate the total flux of the vector Field: $\mathbf{F} = (x + y, y + z, z + x)$. Comment on the result obtained: Does the flux indicate that more is entering the surface or leaving the surface?

7. (a) **(2.5 Marks)** Show that the best linear approximation L of $f(x, y)$ at (a, b) is given by:

$$L(x, y) = f(x, y) + \nabla f(a, b) \cdot (x - a, y - b)$$

   (b) **(2.5 Marks)** Explain the geometric meaning of the direction vector $\nabla f(a, b)$

8. $\mathbf{F}(x, y, z) = (yz)\mathbf{i} + (-xz)\mathbf{j} + \mathbf{k}$. Let S be the portion of surface of the paraboloid $z = 4 - x^2 - y^2$ which lies above the first octant; and let C be the closed curve $C = C_1 + C_2 + C_3$, where the curves C_1, C_2, C_3 are the three curves formed by intersecting S with the xy, yz, xz planes respectively (so that C is the boundary of S). Orient C so that it is traversed CCW when seen from above in the first octant.

   (a) **(5 Marks)** Use Stokes' Theorem to compute the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ by using the surface integral over the capping surface S.

   (b) **(5 Marks)** Set up and evaluate the loop integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ directly by parametrizing each piece of the curve C and then adding up the three line integrals.

9. **(10 Marks)** A symmetrical coffee percolator holds 24 cups when full. The interior has a circular cross-section which tapers from a radius of 3 cm at the center to 2 cm at the base and top, which are 12 cm apart. The bounding surface is parabolic. At what height h from the base should the mark for the 6-cup level be placed? (Write the equation that determines h as an integral, without solving for h). Marks will be awarded only for the final, most simplified form of the integral equation.