1. Introduction.
2. Vector addition/subtraction.
3. Dot product
4. Cross product.

EE1203: Vector Calculus

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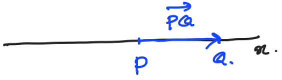
భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

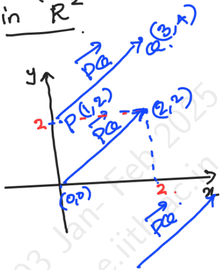
Vector $\vec{PQ}$:- A directed line segment from a point P (called initial point) to a point Q (terminal point), with P, Q being distinct points.

in 'R'.

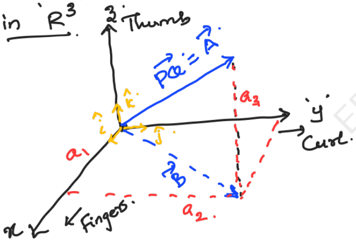


in 'R^2'.

$\vec{PQ} = (2,2)$



in 'R^3'.



"Right handed Configuration."

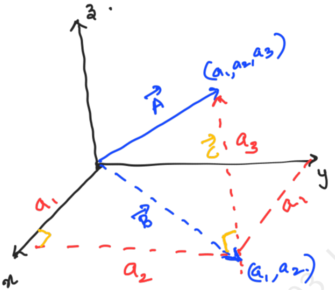
$$\vec{A} = (a_1, a_2, a_3)$$

$$= a_1\hat{i} + a_2\hat{j} + a_3\hat{k}.$$

Magnitude & direction of $\vec{A}$

Magnitude $\rightarrow$ $|\vec{A}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$\mathrm{dirc}(\vec{A}) = \dfrac{\vec{A}}{|\vec{A}|}$



$|\vec{B}|^2 = a_1^2 + a_2^2$

$|\vec{A}|^2 = |\vec{B}|^2 + a_3^2$

$|A| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$

If. $\vec{A} = (a_1, a_2, \ldots a_n) \in R^n.$

$\Rightarrow |\vec{A}| = \sqrt{a_1^2 + a_2^2 + \cdots a_n^2}$.

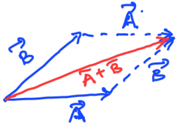
VECTOR ALGEBRA.

1. Vector Addition / subtraction.

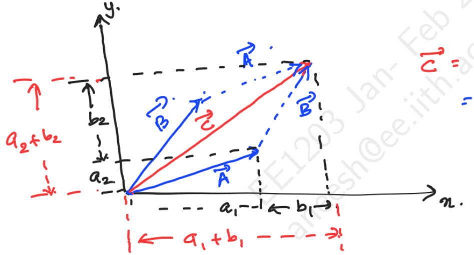
$\vec{A} = (a_1, a_2, a_3)$ in R^3 ; $\vec{B} = (b_1, b_2, b_3)$

$$\vec{A} + \vec{B} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$$

in $R^2 \Rightarrow \vec{A} + \vec{B} = (a_1 + b_1,\ a_2 + b_2)$



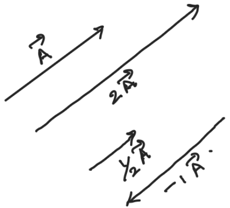
Geometry of vector addition:



$\vec{C} = \vec{A} + \vec{B}$

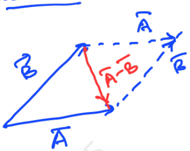
$= \vec{B} + \vec{A}$

2. Multiplying by a scalar $\alpha \in \mathbb{R}$.



$\rightarrow$ Subtraction: $\bar{A} - \bar{B}$

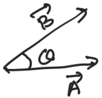
Try yourself:



3. Dot product.

Def: $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$ in $\mathbb{R}^3$

Geometric notation: $\vec{A} \cdot \vec{B} = |\vec{A}| \, |\vec{B}| \cos \theta$.

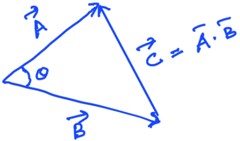


Meaning of this definition:

a) $\vec{A}\cdot\vec{A} = |\vec{A}|\,.\,|\vec{A}|\cos 0$; $\vec{A}\cdot\vec{A} = a_1a_1 + a_2a_2 + a_3a_3$.

$= |\vec{A}|^2$. $\qquad = a_1^2 + a_2^2 + a_3^2 = |\vec{A}|^2$.

b) $\underline{\vec{A}\cdot\vec{B}}$



By law of Cosine:

$\Rightarrow |\vec{C}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}||\vec{B}|\cos\theta$.

$$|\vec{C}|^2 = \vec{C}\cdot\vec{C} = (\vec{A}-\vec{B})\cdot(\vec{A}-\vec{B})$$

$$= \vec{A}\cdot\vec{A} - \vec{A}\cdot\vec{B} - \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B}$$

$$= |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A}\cdot\vec{B}$$

$\Rightarrow \vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$.

Observation:

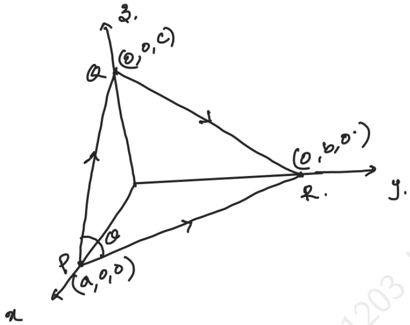
* The sign of $\vec{A}\cdot\vec{B}$ = +ve if $0 < \theta < 90^\circ$.

= 0 if $\theta = 0$.

= −ve if $90^\circ < \theta \leq 180^\circ$.

Applications of Dot product.

(i) To compute the lengths and angles



$\overline{PR} \cdot \overline{PQ} = |\overline{PQ}|\,|\overline{PR}| \cos\theta.$

$$\boxed{\cos\theta = \frac{\overline{PR} \cdot \overrightarrow{PQ}}{|\overline{PR}|\,|\overline{PQ}|}}$$

$\overline{PR} = (-a, b, 0)$ ✓

$\overline{PQ} = (-a, 0, c)$ ✓

$\overline{PR} \cdot \overline{PQ} = a^2.$ ✓

$|\overline{PR}| = \sqrt{a^2+b^2}$ ✓

$|\overline{PQ}| = \sqrt{a^2+c^2}.$ ✓

(ii). Detect orthogonality:

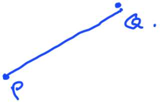
If $\bar{A} \cdot \bar{B} = 0 \Rightarrow$ (for non zero vectors)

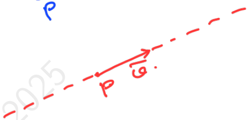
$\bar{A} \perp^{r} \bar{B} \Rightarrow \theta = \frac{\pi}{2}$

To define a line: (a) 2 distinct points.

OR.

(b) One point & direction. vector



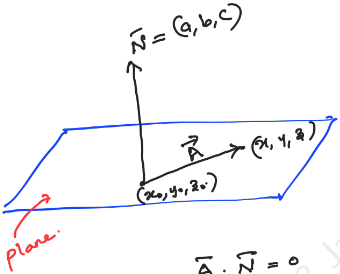


To define a plane: – (a) 3 distinct non collinear points uniquely define a plane.

(b) A direction & a point?

Ans: A point & a perpendicular (normal) direction uniquely define a plane.

Goal: (b) To write an equation of the plane containing a particular point in R^3 (x_0, y_0, z_0) with a normal vector (a, b, c)



Given $\bar{A} \perp^r \bar{N}$

$\Rightarrow \bar{A} \cdot \bar{N} = 0$

$$\vec{A} = (x - x_0)\hat{i} + (y - y_0)\hat{j} + (z - z_0)\hat{k}.$$

$$\bar{N} = a\hat{i} + b\hat{j} + c\hat{k}.$$

* idea: Every vector in plane is $\perp^r$ to the normal vector.

$\bar{A} \cdot \bar{N} = 0$

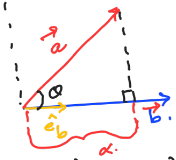
$\Rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$

$\Rightarrow ax + by + cz - (\underbrace{ax_0 + by_0 + cz_0}_{d}) = 0.$

$\Rightarrow \boxed{ax + by + cz = d.}$ ← Equation of the plane!

(iii) Projection of a vector onto another

* Component of a given vector along the direction of another.



$\hat{e}_b = \frac{\vec{b}}{|\vec{b}|}$.

Projection of $\vec{a}$ onto $\vec{b}$.
$\Rightarrow$ Component of $\vec{a}$ along unit vector $\hat{e}_b$.

$$\text{Scalar Proj}_{\vec{b}}\vec{a}\ (\alpha) = |\vec{a}|\cos\theta$$
$$= |\vec{a}|\,|\hat{e}_b|\cos\theta.$$
$$= \vec{a}\cdot\hat{e}_b$$

Projection of $\vec{a}$ onto $\vec{b}$ $= (\vec{a}\cdot\hat{e}_b)\,\hat{e}_b = \left(\vec{a}\cdot\frac{\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}$

Projection of $\vec{b}$ onto $\vec{a}$ $= (\vec{b}\cdot\hat{e}_a)\,\hat{e}_a$.

$\text{Proj}_{\vec{b}}\vec{a} \neq \text{proj}_{\vec{a}}\vec{b}$.

Component of any vector along $\hat{i} = \vec{A} \cdot \hat{i}$

along $\hat{j} = \vec{A} \cdot \hat{j}$.

along $\hat{k} = \bar{A} \cdot \hat{k}$

Cauchy-Schwarz Inequality.

$$\boxed{|\bar{v} \cdot \bar{\omega}| \leq |v||\omega|}$$

$$\bar{v} \cdot \bar{\omega} = |\bar{v}||\bar{\omega}| \cos\theta.$$

$$|\bar{v} \cdot \bar{\omega}| = |\cos\theta| \, |\vec{v}| \, |\bar{\omega}|$$

$$\Rightarrow \quad |\bar{v} \cdot \bar{\omega}| \leq |\bar{v}||\bar{\omega}| \qquad \because 0 \leq |\cos\theta| \leq 1$$

Goal : (a) If given three non-collinear points:

How to define the plane?