EE1080/AI1110/EE2102 Probability: HW2

28th Jan 2025

Assume discrete random variables throughout.

1. The sample space Ω of an experiment consists of all 8-dimensional binary vectors, e.g., every member of Ω has the form $\omega = (\omega_0, ..., \omega_7)$, where ω_i is 0 or 1. The probability law P assigns a probability of $1/2^8$ to each of the 2^8 elements in Ω.

   Find the pmfs describing the following random variables:

   (a) $W(\omega) =$ the number of 1's in the binary vector ω.

   (b) $Z(\omega) = \max(\omega_i)$

2. Gukesh and Pragya play a chess match in which first player to win a game wins the match. After 10 successive draws, the match is declared drawn. Each game is won by Pragya with probability 0.4, is won by Gukesh with probability 0.3 and declared a draw with probability 0.3, independent of the previous game.

   (a) What is the probability that Pragya wins the match ?

   (b) What is the PMF of the duration of the match ?

3. Consider a hierarchical storage system composed of n subsystems, each comprising k servers. Each subsystem can tolerate a single server failure, and the overall system can tolerate a single subsystem failure. Thus, in order for the overall system to fail, there has to be at least two subsystems that each have at least two server failures. Suppose servers fail independently with probability p.

   (a) Let X be the number of servers that failed in a particular subsystem. Find the pmf of X in terms of k and p. What is the probability p_0 that a particular subsystem fails?

   (b) Let Y be the number of subsystems that failed. Find the pmf of Y in terms of n and p_0. What is the probability p_1 that the overall system fails?

   (c) Give upper bounds on p_0 and p_1 using the union of events bound (Let A_{ij} be the event both server i and j fail for $1 \leq i < j \leq k$ and use the fact that $X \geq 2 = \cup_{1 \leq i < j \leq k} A_{ij}$.

1a : $\Omega = \{0,1\}^8$, $\{w\} = \frac{1}{2^8}$

$W(\omega) = \#$ of 1's in ω.

Binomial random variable with $n = 8$, $p = ½$.

$$P(W=k) = \binom{n}{k} p^k (1-p)^{n-k} = \binom{8}{k}\left(\frac{1}{2}\right)^8$$

$\forall\, k \in \{0, 1, \ldots 8\}$.

1b : $Z(\omega) = \max\limits_{i=1,2,..8}(\omega_i) \in \{0,1\}$. $\omega = (\omega_1, \omega_2, \ldots \omega_8) \in \{0,1\}^8$.

$$Z(\omega) \begin{cases} 0 & \text{only if } \omega_1 = \omega_2 = \ldots \omega_8 = 0 \Rightarrow \omega = (0 \ldots 0) \\ 1 & \text{if } \omega \neq (0 \ldots 0). \end{cases}$$

$$P(Z=0) = \frac{1}{2^8}, \quad P(Z=1) = 1 - \frac{1}{2^8}.$$

Z is a Bernoulli$\left(1 - \frac{1}{2^8}\right)$ R.V.

2 . $\_\ \_\ \_\ \_\ \ldots\ \_$

10

Prob that Pragya wins is 0.4 } H (0.7)
" Gukesh " 0.3
" draw is 0.3. } T (0.3)

(a) Prob that pragya wins the match

A_i : event that Pragya wins at the i-th game. $\Rightarrow$ all the prev $(i-1)$ games are a draw.

$$P(A_i) = (0.3)^{i-1} \times (0.4)$$

$$\text{Prob that pragya wins} = \sum_{i=1}^{10} P(A_i)$$
$$= \sum_{i=1}^{10} (0.3)^{i-1} (0.4).$$

(b) PMF of the duration of the match :

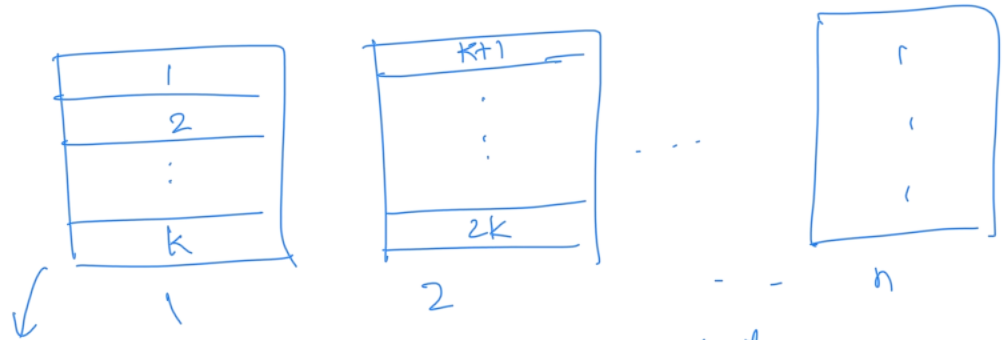
Let X be a r.v representing duration of a match.

$$P(X=k) = (0.3)^{k-1}(0.7). \qquad k=1,2,3,\dots 9.$$

$$P(X=10) = (0.3)^{9} \times 1$$

check that $\sum_{k=1}^{10} P(X=k) = 1.$

(3).



Each subsystem can tolerate one server failure.

$\Rightarrow$ For a subsystem to fail you need atleast 2 server failures.

System can tolerate one sub-system failure

$\Rightarrow$ for system to fail you need atleast 2 sub-system failure.

Servers fail with probability p independently.

(a) X : # servers that failed within subsystem.

$\cancel{x} \in \{0, 1, 2, \dots k\}.$

PMF of X

$$P_X(x) = P(X=x) = \binom{k}{x} p^x (1-p)^{k-x}.$$

X is a Binomial (k, p) R.V.

What is the probability p_0 that the particular subsystem fails?

$$P_0 = P(X \geq 2) = \sum_{x=2}^{k} P_X(x)$$

$$= 1 - \sum_{x=0}^{1} P_X(x)$$

$$\boxed{p_0 = 1 - \binom{k}{0}(1-p)^k - \binom{k}{1} p(1-p)^{k-1}}$$

(b) Y : # of subsystem that failed.

Each subsystem fails w.p P_0.

$$P_Y(y) = P(Y=y) = \binom{n}{y}(P_0)^y(1-P_0)^{n-y}$$

Y is a Binomial (n, P_0) R.V.

What is the probability that overall system fails?

$$P_1 = P(Y \geq 2) = 1 - P(Y \leq 1)$$

$$= 1 - \{P(Y=0) + P[Y=1]\}$$

$$= 1 - \underbrace{\binom{n}{0}}_{1}(1-P_0)^n - \underbrace{\binom{n}{1}}_{n}(1-P_0)^{n-1}P_0.$$

(c) $P_0 = P(X \geq 2) = P\left(\bigcup_{1 \leq i < j \leq k} A_{ij}\right)$

i, j are not equal

A_{ij} is the event that servers i, j fail

$$P_0 \leq \sum_{1 \leq i < j \leq k} \underbrace{P(A_{ij})}_{p^2} = \binom{k}{2} p \cdot p = \binom{k}{2} p^2$$

union bound.

$$P_0 \leq \binom{k}{2} p^2.$$

$$\boxed{\sum_{x=2}^{k} \binom{k}{x} p^x (1-p)^{k-x}}$$

B_{ij} event that subsystems i, j fail.

$$P_1 \leq \binom{n}{2} P_0^2 \leq \binom{n}{2}\left(\binom{k}{2}p^2\right)^2$$

$$= \binom{n}{2}\binom{k}{2}^2 p^4.$$

4. Binary Symmetric Channel: Repetition Coding: A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability 0.2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?

5. Binomial and Poisson PMFs: Show that the PMF $P_X(x)$ of Binomial (n, p) is monotonically non-decreasing as long as $x \leq (n+1)p$ and is monotonically decreasing for $x \geq (n+1)p$. Similarly Poisson PMF $P_Y(y)$ is non-decreasing for $y \leq \lambda$ and decreasing for $y > \lambda$

6. A total of 4 buses carrying 148 students from the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.

   (a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?

   (b) Compute $E[X]$ and $E[Y]$

   (c) Find $Var(X)$ and $Var(Y)$

7. If $E[X] = 1$ and $Var(X) = 5$, find $E[(2+X)^2]$ and $Var(4+3X)$.

8. Let X be a non-negative random variable ie., $P(X \geq 0) = 1$. Show that

$$E[X] = \sum_{k=0}^{\infty} P(X > k)$$

   Similarly show that for any random variable X:

$$E[X] = \sum_{k=0}^{\infty} P(X > k) - \sum_{k=0}^{\infty} P(X < -k)$$

9. A product that is sold seasonally yields a net profit of b dollars for each unit sold and a net loss of l dollars for each unit left unsold when the season ends. The number of units of the product that are ordered at a specific department store during any season is a random variable having probability mass function $p(i)$, $i \geq 0$. If the store must stock this product in advance, determine the number of units the store should stock so as to maximize its expected profit.

10. Quiz Problem: In an examination a student can choose the order in which two questions(Ques A and Ques B) must be attempted.

B.S.C channel :

$$X \xrightarrow{\text{input}} \boxed{\text{Channel}} \longrightarrow Y.$$

$$P(Y=0 \mid X=0) = P(Y=1 \mid X=1) = 0.8$$
$$P(Y=1 \mid X=0) = P(Y=0 \mid X=1) = 0.2.$$

Prob of error incurred in this channel is 0.2. if there is no coding.

$$P(Y \neq X) = 0.2.$$

Repetition coding

$$X \longrightarrow \boxed{\text{Encoding}} \xrightarrow{XXXXX} \boxed{\text{Channel}} \xrightarrow{Y_1 Y_2 Y_3 Y_4 Y_5} \boxed{\text{Decoding}} \longrightarrow \hat{X}.$$

majority logic decoding.

This is looked at in coding theory.

$$P(\hat{X} \neq X)$$

$$= P(\text{maj}\, Y_1 \cdots Y_5 \neq X).$$

$$= P(Z_1 + \cdots Z_5 > 2) \text{ bit flips}$$

$Y = X \oplus Z.$ XOR (addition in binary)

$Z \sim \text{Ber}(p=0.2)$ R.V

$$0 \xrightarrow{\text{Encoding}} 00000 \xrightarrow{\text{channel}} 00111 \xrightarrow{\text{decoder}} \text{"1" error}$$

more than two bits flipped

$Z = Z_1 + \cdots Z_5$

$Z = \text{Binomial}(5, 0.2)$

$$= P(Z > 2)$$

$$= \sum_{z=3}^{5} P_Z(z) \rightarrow \binom{5}{z} (0.2)^z (0.8)^{5-z}.$$

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(5) Please try this (from Bertsekas) Solutions available online.

$\boxed{k \quad k+1}$ $\qquad k < \lambda$

$$P_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1}.$$

$$\frac{P_X(k)}{P_X(k+1)} = \frac{\cancel{\frac{n!}{(n-k)\,(n-k)!\,k!}} \, \cancel{p^k} \, \cancel{(1-p)^{n-k}}}{\cancel{\frac{n!}{(n-k-1)!\,(k+1)!}} \, p^{k+1} \, \cancel{(1-p)^{n-k-1}}}$$

$k+1$

$$= \left(\frac{k+1}{n-k}\right)\left(\frac{1-p}{p}\right).$$

updated after discussion.

i.e., P_X is increasing.

$\downarrow$

If $k+1 \le np$

then $$\frac{P_X(k)}{P_X(k+1)} = \left(\frac{1-p}{p}\right)\left(\frac{k+1}{n-k}\right) = \left(\frac{1-p}{p}\right)\left(\frac{\frac{k}{n}+\frac{1}{n}}{1-\frac{k}{n}}\right) < \left(\frac{1-p}{p}\right)\left(\frac{p}{1-p}\right) = 1.$$

assume $k+1 \le np$.

$k < np$.

$-k > -np$

$\frac{1}{-k} < \frac{1}{-np}$.

$k > np$

$$\frac{P_X(k)}{P_X(k+1)} > \left(\frac{1-p}{p}\right)\frac{np}{(1-p)n} > 1.$$

$\downarrow$

$P_X(k)$ is decreasing.

$k > np$.

$k+1 > np$.

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⑥ 4 buses carrying 148 students. $\Omega = \{1, 2, \ldots 148\}$.

40, 33, 25, 50. $P(\{\omega\}) = 1/148$.

Both X & Y take values in 40, 33, 25, 50.

$$P(X=x) = \begin{cases} \frac{40}{148} & x = 40 \\ 33/148 & x = 33 \\ 25/148 & x = 25 \\ 50/148. & x = 50 \end{cases}$$

$$X(\omega) = \begin{cases} 40 & \omega \in \{1, \ldots 40\} \\ 33 & \omega \in \{41, \ldots 73\} \\ 25 & \omega \in \{74, \ldots\}. \\ 50 & \end{cases}$$

$$P(X=40) = P(\{\omega \in \{1, \ldots 40\}\}) = \frac{40}{148}.$$

(a) Intuitively

$E[X] \geq E[Y]$

as R.V X has more probability for larger values.

$$\sum_{i=1}^{4} x_i P_X(x_i).$$

$$\sum_{i=1} x_i P_Y(x_i).$$

$x_1 > x_2 > x_3 > x_4$

$P_X(x_1) \geq P_X(x_2) \geq \cdots$

$$P(Y=y) = \begin{cases} \frac{1}{4} & y = 40 \\ 1/4 & y = 33 \\ 1/4 & y = 25 \\ 1/4 & y = 50 \end{cases}$$

$$E[X] - E[Y]$$
$$= \sum_{i=1}^{4} x_i \underbrace{\left[P_X(x_i) - P_Y(x_i)\right]}_{\text{not necessarily positive all the time}}$$

Try this

Consider X, Y that have same support

x_1, x_2, x_3, x_4.

$$P_X(x_i) = \frac{x_i}{\sum_{i=1}^{4} x_i}$$

$$P_Y(x_i) = 1/4.$$

Come up with an x_1, x_2, x_3, x_4 such that $E[X] < E[Y]$.

* updated after discussion

This is not possible

$$E[X] = \frac{x_1^2 + x_2^2 + x_3^2 + x_4^2}{\Sigma x_i}$$

$$E[Y] = \frac{\Sigma x_i}{4}.$$

From Cauchy-Schwartz inequality.

$$(\Sigma x_i)^2 \le (x_1^2 + x_2^2 + \cdots x_4^2)\, 4.$$

In general, for any two vectors $\langle x \cdot y \rangle \le \|x\| \|y\|$.

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(7). $E[X] = 1$, $Var(X) = 5$, $E[X^2] = Var(X) + (E[X])^2 = 6.$

$$E\left[(2+X)^2\right] = E\left[4 + 4X + X^2\right]$$
$$= 4 + 4E[X] + E[X^2].$$

$$Var(\cancel{4} + 3X) = 9\, Var(X).$$

(8)

$$E[X] = \sum_{k=0}^{\infty} P(X > k)$$

X is a non-negative R.V.

$$\sum_{k=0}^{\infty} P(X > k) = \sum_{k=0}^{\infty} \sum_{x=k+1}^{\infty} P(X = x)$$

$$= 0 \times P(X=0)$$
$$+ \; 1 \times P(X=1)$$
$$+ \; 2 \times P(X=2)$$
$$+ \; \cdots$$
$$= \sum_{x=0}^{\infty} x\, P(X=x) = E[X].$$

(k, x)

$(0, 1)$

$(0, 2)$

$(1, 2)$

$(1, 3)$

$\vdots$

9 (Sheldon M. Ross textbook example)

profit : b dollars
loss : l dollars.

s : # of product unit stocked

$$R_s(d) = \begin{cases} bd - (s-d)l & d < s \\ bs & d \geq s \end{cases}$$

R_s → profit, d → demand

Find an s that

Maximizes $E[R_s]$.

$$\max_s E[R_s].$$

$$E[R_s] = \sum_{d \geq 0} R_s(d)\, P_D(d)$$

D → demand r.v.

$$\boxed{E[g(X)] = \sum_x g(x) P_X(x)}$$

$$= \sum_{d \leq s} \big(bd - (s-d)l\big) P(d) + \underbrace{\sum_{d > s} bs \cdot P(d)}_{bs\left[1 - \sum_{d \leq s} P(d)\right]}$$

$$E[R_s] = sb - \sum_{d=1}^{s} (s-d) P(d)(b+l)$$

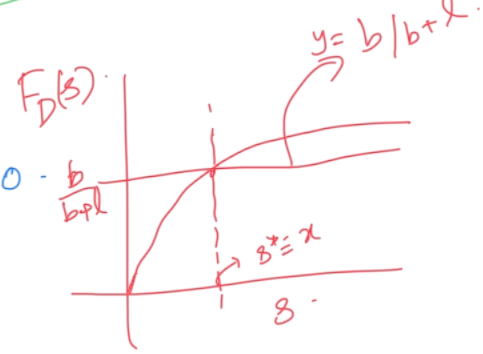
$$E[R_{s+1}] = (s+1)b - \sum_{d=1}^{s+1} (s+1-d) P(d)(b+l)$$

For what s is

$$E[R_{s+1}] > E[R_s]$$

$$b - \sum_{d=1}^{s} P(d)(b+l) > 0$$

$$\sum_{d=1}^{s} P(d) < \frac{b}{b+l}$$



$$\boxed{\text{If } F_D(s) < \frac{b}{b+l} \text{ then } E[R_{s+1}] > E[R_s]}$$

Set the stock prize as s^* st. $F_D(s^*) \geq \frac{b}{b+l}$.

1 2 3.
3 1 2

Swapped 1 and 2 first followed by swap of 2 and 3.

(12) $Y = \min(X_1, X_2, X_3)$, X_1, X_2, X_3 iid.

$$F_Y(y) = 1 - (1 - F_{X_1}(y))^3$$

$$(1 - F_{X_1}(y))^3 = 1 - F_Y(y)$$

$P(Y \leq y)$

$$= P(Y > y).$$

$$P(\{X_1 > y\}\{X_2 > y\}, \{X_3 > y\}) = P(\{X_1 > y\})\, P(\{X_2 > y\})\, P(\{X_3 > y\})$$

$$= (1 - F_{X_1}(y))^3$$

If X_1, X_2 are independent

$$P_{X_1}(x_1)\, P_{X_2}(x_2) = P_{X_1, X_2}(x_1, x_2)$$

for all x_1, x_2

$$P(X_1 \in A,\ X_2 \in B) = P(X_1 \in A)\, P(X_2 \in B)$$

To show

$$\{X_1 > y,\ X_2 > y,\ X_3 > y\} = \{Y > y\}.$$

$\quad A'' \qquad\qquad ''B$

If $\omega \in A$, $\quad X_i(\omega) > y \quad i = 1, 2, 3.$

$$\Rightarrow \min(X_1(\omega), X_2(\omega), X_3(\omega)) > y$$

$$\Rightarrow Y(\omega) > y$$

$$\Rightarrow A \subseteq B.$$

$\omega \in B$, $\quad Y(\omega) > y.$

$$X_i(\omega) \geq \min(X_1(\omega), X_2(\omega), X_3(\omega)) > y. \quad \forall\ i = 1, 2, 3$$

$$\Rightarrow \omega \in A.$$

Recap of $E[X] > E[Y]$.

X is a R.V taking values in $\{x_1, x_2, x_3, x_4\}$

Y is " " "

$$P_X(x_i) = \frac{x_i}{\sum_{i=1}^{4} x_i} \qquad i \in \{1, 2, 3, 4\}$$

$$P_Y(x_i) = \frac{1}{4}.$$

$$E[X] = \sum_{i=1}^{4} x_i P_X(x_i) = \frac{\sum_{i=1}^{4} x_i^2}{\sum_{i=1}^{4} x_i}$$

$$E[Y] = \sum_{i=1}^{4} \frac{x_i}{4}$$

$$v = (x_1, x_2, x_3, x_4).$$

$$u = (1, 1, 1, 1)$$

$$\langle u \cdot v \rangle \le \|v\|_2 \|u\|_2 \quad \sqrt{4}.$$

$$\sum_{i=1}^{4} x_i \qquad \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$\left(\sum x_i\right)^2 \le \left(\sum x_i^2\right) 4.$$

$$E[Y] = \left(\frac{\sum x_i}{4}\right) \le \frac{\left(\sum x_i^2\right)}{\sum x_i} = E[X]$$

X takes values in $\{x_1, x_2, x_3, x_4\}$

What is the maximum $E[X]$ possible?

(b) $Y = \min(X_1, X_2, X_3)$.

$X_i \in [101 \dots 110]$ uniformly.

$$P(Y = 100 + i) \qquad i = 1, 2, \dots 10$$

$$= \left(\frac{1}{10}\right)^3 + \left({}^3C_2\right)\left(\frac{1}{10}\right)^2 \cdot \left(\frac{10-i}{10}\right).$$

$$+ \left({}^3C_1\right)\left(\frac{1}{10}\right)\left(\frac{10-i}{10}\right)^2$$

$$E[Y] = \sum_{i=1}^{10} (100+i) \, P(Y = 100+i).$$

$$E[X_i] = \sum_{i=1}^{10} (100+i)\frac{1}{10}$$

$$= 100 + \frac{\cancel{10}\times 11}{2\times\cancel{10}} = 105.5$$

(13)

$$X \sim \text{Poisson}(\lambda_1)$$

$$Y \sim \text{Poisson}(\lambda_2).$$

$$Z = X + Y.$$

$$P_Z(z) = \sum_{\substack{(x,y):\\ x+y=z}} P_{X,Y}(x,y)$$

independence

$$= \sum_{x=0}^{z} \sum_{y=z-x} P_X(x)\, P_Y(y)$$

$$= \sum_{x=0}^{z} P_X(x)\, P_Y(z-x).$$

$$= \sum_{x=0}^{z} \left[e^{-\lambda_1}\frac{\lambda_1^x}{x!}\right]\left[e^{-\lambda_2}\frac{(\lambda_2)^{z-x}}{(z-x)!}\right]$$

$$e^{-\lambda_1+\lambda_2}\frac{(\lambda_1+\lambda_2)^z}{z!}$$

$$= \frac{(\lambda_1+\lambda_2)^z e^{-(\lambda_1+\lambda_2)}}{z!} \underbrace{\sum_{x=0}^{z} \frac{\lambda_1^x}{(\lambda_1+\lambda_2)^x}\frac{(\lambda_2)^{z-x}}{(\lambda_1+\lambda_2)^{z-x}}\frac{z!}{x!\,(z-x)!}}_{1}$$

$$p = \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right) \qquad n = z$$

(b) Splitting

$$N = \text{Poisson}(\lambda) \text{ R.V.}$$

$X :=$ # 1's transmitted in a slot.
R.Vs

$$X \mid N = n = X_1 + X_2 + \cdots X_n$$

$X \mid N = n$ is a Binomial(n, p) R.V.

$$P_X(x) = \sum_{n=0}^{\infty} P_{X\mid\{N=n\}}(x) \, P(\{N=n\}).$$

$$P_{X\mid\{N=n\}}(x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

$$P(\{N=n\}) = \frac{e^{-\lambda} \lambda^n}{n!}$$

$x \leq n$

$$P_X(x) = \sum_{n=0}^{\infty} \binom{n}{x} p^x (1-p)^{n-x} \frac{e^{-\lambda} \lambda^n}{n!}$$

$n \geq x$.

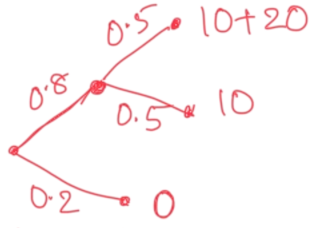
$\lambda^n \to \lambda^x \lambda^{n-x}$

$e^{-\lambda} \to e^{-\lambda p} e^{-\lambda(1-p)}$

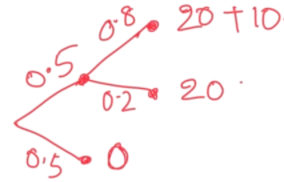
$$= \sum_{n=x}^{\infty} \frac{\cancel{n!}}{x!(n-x)!} \frac{(\lambda p)^x ((1-p)\lambda)^{n-x} e^{-\lambda}}{\cancel{n!}}$$

$$= \frac{(\lambda p)^x}{x!} e^{-\lambda p} \underbrace{\sum_{n=x}^{\infty} \frac{e^{-\lambda(1-p)}}{(n-x)!} (\lambda(1-p))^{n-x}}_{1}$$

$(0.8 \times 0.5)\, 30$
$+ (0.8)(0.5) 10.$
$= (0.4)(40).$
$= 16.$



A first



B first.

$(0.5 \times 0.8)(30)$
$+(0.5 \times 0.2)(20)$
$12 + 2 = 14.$

(a) If first question is answered wrong, the student gets zero marks.

(b) If the first question is answered correctly and the second question is not answered correctly, the student gets the marks only for the first question.

(c) If both questions are answered correctly, the student gets the sum of the marks of the two questions.

The probability of answering Question A and B correctly are 0.8 and 0.5 respectively. And the marks for each question A and B are 10 and 20 respectively. Assuming that the student always wants to maximize her expected marks in the examination, in which order should she attempt the questions and what is the expected marks for that order.

G 1 2 3.
3 2 1
swap of 1and3.

11. Extension of Quiz problem: Consider there are n questions to be answered. Question i will be answered correctly with probability p_i and the person will receive a reward v_i on correctly answering the question. The Quiz terminates on a wrong answer. The problem here is to choose the best ordering for the answers. Show that it is optimal to answer questions in non-decreasing order of $p_i v_i/(1 - p_i)$

Solns in Bertsekas excercise.

$L(\; i_1, i_2, i_3 \ldots, i_n)$

12. Minimum of random variables:

(a) Consider you have three identical, independent random variables X_1, X_2, X_3 with cumulative distribution function $F_{X_1}(.)$ and $Y = \min(X_1, X_2, X_3)$. Show that $F_Y(y) = 1 - (1 - F_{X_1}(y))^3$.

(b) On a given day, your golf score takes values in 101 to 110 with probability 0.1, independent of the other days. Determined to improve your score, you decider to play on three different days and declare as your score the score the minimum Y, of scores X_2, X_2, X_3 on the different days.

i. Calculate the PMF of Y

ii. By how much has your expected score improved as a result of playing on these days ?

13. Poisson R.V:

(a) (Combining) Let X, Y be Poisson random variables with means λ_1, λ_2 respectively. Find the PMF of random variable $Z = X + Y$.

(b) (Splitting) A transmitter sends out either a 1 with probability p or 0 with probability $1 - p$ independent of earlier transmissions. The number of packets transmitted in a slot has a Poission PMF with parameter λ. Show that the number of 1's transmitted in a slot has a Poission PMF with parameter $n\lambda$

14. Tossing a coin till a pattern appears: What is the PMF of number of tosses, expected number of tosses, variance for the following scenarios ?

T T T T H . T - - - H

$2/p$

Bertsekas solutions

(a) Toss a coin until you see two heads.

(b) Toss a coin until you see two successive heads.

(c) Toss a coin until you see two successive heads or tails.

PMF is very hard.

in the textbook.

Simular to Geometric R.V Expectation

H_1 : event that first toss is head

H_2 : event that second toss is head.

H_1 & T_1.

$$E[X] = P(H_1)E[X|H_1] + P(T_1)E[X|T_1]$$

$$= p\,E[X|H_1] + (1-p)\left[1+E[X]\right]$$

H_2 & T_2.

$$E[X|H_1] = \underbrace{P(H_2)}_{=P(H_2|H_1)} \underbrace{E[X|H_1\cap H_2]}_{2} + \underbrace{E[X|H_1\cap T_2]}_{(2+E[X])} \underbrace{P(T_2)}_{=P(T_2|H_1)}$$

updated after discussion in class. (Proof below)

$$E[X|B] = \sum_{x\in\mathcal{X}} P_{X|B}(x)\,x.$$

$$P_{X|B}(x) = P(\{X=x\}|B) = \frac{P(\{X=x\}\cap B)}{P(B)}$$

$$= \frac{\sum_{i=1}^{m} P(\{X=x\}\cap A_i\cap B)}{P(B)} = \sum_{i=1}^{m} \frac{P(A_i\cap B)\,P_{X|A_i\cap B}(x)}{P(B)}$$

$$= \sum_{i=1}^{m} P(A_i|B)\, P_{X|A_i \cap B}(x)$$

$$E[X|B] = \sum_{x \in \mathcal{X}} x \sum_{i=1}^{m} P(A_i|B)\, P_{X|A_i \cap B}(x)$$

$$= \sum_{i=1}^{m} P(A_i|B) \sum_{x \in \mathcal{X}} x\, P_{X|A_i \cap B}(x)$$

$$\boxed{E[X|B] = \sum_{i=1}^{m} P(A_i|B)\, E[X|A_i \cap B].}$$

$$\boxed{E[X] = \sum_{i=1}^{m} P(A_i)\, E[X|A_i].}$$