EE2100: Matrix Analysis

Review Notes - 27

Topics covered :

1. Computation of Determinant of a Matrix.

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1. Axiomatic definition of determinant: The determinant of a matrix (typically denoted by $\mathbf{det}(\mathbf{A})$), defined for square matrices, is a function that satisfies the following conditions

   (a) Multilinearity: Let $\mathbf{A} \in \mathcal{R}^{n\times n}$ and let $f(\mathbf{A})$ denote the determinant of $\mathbf{A}$. To understand multilinearity, we analyze determinant as a function that operates on column vectors of $\mathbf{A}$ i.e. as $f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$. We say that the function is multilinear if $f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \alpha\mathbf{u}_i + \beta\mathbf{v}_i, \cdots, \mathbf{a}_n) = \alpha f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{u}_i, \cdots, \mathbf{a}_n) + \beta f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{v}_i, \cdots, \mathbf{a}_n)$.

   (b) Alternating property: Let $f(\mathbf{A}) = f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ denote the determinant of $\mathbf{A}$, then $f(\mathbf{a}_2, \mathbf{a}_1, \cdots, \mathbf{a}_n) = -f(\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$ i.e., the determinant should change in sign if two columns are interchanged.

   (c) Unity for Identity matrix: The determinant of an identity matrix is unity.

2. Determinant of Permutation matrices: A matrix $\mathbf{P} \in \mathcal{R}^{n\times n}$ is called a permutation matrix (in the context of this course) if its column vectors are linearly independent and have the standard basis vectors i.e., $\mathbf{e}_i$ where $(i \leq n)$ as the column vectors. For example, the matrix $\mathbf{P} = (\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2)$ and $\mathbf{P} = (\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2)$ are permutation matrices. Using the alternating property of the determinant it can be shown that $\mathbf{det}(\mathbf{P}) = \pm 1$.

3. Computing the determinant of 2×2 matrix using Axiomatic definition: Let $\mathbf{A}$ denote a matrix whose entries are

$$\mathbf{A} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \tag{1}$$

   In terms of column vectors, $\mathbf{det}(\mathbf{A}) = \mathbf{det}(\mathbf{a}_1, \mathbf{a}_2)$, where $\mathbf{a}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$. Using multilinearity property of the determinant and the value of determinant of permutation matrices, $\mathbf{det}(\mathbf{A})$ can be computed as

$$\begin{aligned}
\mathbf{det}(\mathbf{a}_1, \mathbf{a}_2) &= \mathbf{det}(a\ \mathbf{e}_1 + b\ \mathbf{e}_2, \mathbf{a}_2) \\
&= a\ \mathbf{det}(\mathbf{e}_1, \mathbf{a}_2) + b\ \mathbf{det}(\mathbf{e}_2, \mathbf{a}_2) \\
&= a\ \mathbf{det}(\mathbf{e}_1, c\ \mathbf{e}_1 + d\ \mathbf{e}_2) + b\ \mathbf{det}(\mathbf{e}_2, c\ \mathbf{e}_1 + d\ \mathbf{e}_2) \\
&= ac\ \cancelto{0}{\mathbf{det}(\mathbf{e}_1, \mathbf{e}_1)} + ad\ \cancelto{1}{\mathbf{det}(\mathbf{e}_1, \mathbf{e}_2)} + bc\ \cancelto{-1}{\mathbf{det}(\mathbf{e}_2, \mathbf{e}_1)} + bd\ \cancelto{0}{\mathbf{det}(\mathbf{e}_2, \mathbf{e}_2)} \\
&= ad - bc
\end{aligned} \tag{2}$$

4. Computing the determinant of 3×3 matrix using Axiomatic definition: Let $\mathbf{A}$ denote a matrix whose entries are

$$\mathbf{A} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \tag{3}$$

In terms of column vectors, $\mathbf{det}\,(\mathbf{A}) = \mathbf{det}\,(\mathbf{a}_1, \mathbf{a}_2)$, where $\mathbf{a}_1 = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\mathbf{a}_2 = \begin{bmatrix} c \\ d \end{bmatrix}$. Using multilinearity property of the determinant and the value of determinant of permutation matrices, $\mathbf{det}\,(\mathbf{A})$ can be computed as (writing only the non-zero terms)

$$\begin{aligned} \mathbf{det}\,(\mathbf{a}_1, \mathbf{a}_2) &= aei\ \mathbf{det}\,(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) + afh\ \mathbf{det}\,(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) + bdi\ \mathbf{det}\,(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) + bfg\ \mathbf{det}\,(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) + \\ &\quad ceg\ \mathbf{det}\,(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) + cdh\ \mathbf{det}\,(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) \\ &= aei - afh - bdi + bfg - ceg + cdh \end{aligned} \tag{4}$$

5. In general for an $n \times n$ matrix, expanding the determinant using the property of multilinearity results in n^n terms out of which only $n!$ terms would be non-zero.

6. Let $\mathbf{A}$ and $\mathbf{B}$ be two matrices such that $\mathbf{AB}$ and $\mathbf{BA}$ are defined. Then, it can be shown that $\mathbf{det}(\mathbf{AB}) = \mathbf{det}(\mathbf{BA})$.

7. It can be shown using the axiomatic definition that the determinant of the upper triangular matrix (or a lower triangular matrix) is the product of the diagonal terms.

8. In general, one way to compute the determinant involves two steps. First, the given matrix is decomposed into $\mathbf{LU}$ and subsequently, the determinant is computed as the product of the determinants of $\mathbf{L}$ and $\mathbf{U}$ i.e.,

$$\mathbf{det}(\mathbf{A}) = \mathbf{det}(\mathbf{LU}) = \left(\prod_{i=1}^{n} U_{ii}\right)\left(\prod_{i=1}^{n} L_{ii}\right) \tag{5}$$

9. The determinant of any matrix $\mathbf{A}$ containing a zero column vector is 0 and the determinant of any matrix that has linearly dependent columns (i.e., not a full-rank) is also 0 (can be shown using the axiomatic definition of determinant).