1. A mosquito flies at a constant speed according to the equation;

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t \\ 1 \\ 1-t \end{bmatrix}$$

A spiderweb, with a patient spider, hangs in the position of the plane $2x + 3y + 5z = 15$. Will the mosquito get caught in the web, and if so, when and where? **10 Marks**

**Solution:** given mosquito flies at a constant speed according to the equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4+t \\ 1 \\ 1-t \end{bmatrix}$$

which is also written as

$$x = 4 + t, y = 1, z = 1 - t \tag{1}$$

A spiderweb lies on the plane $2x + 3y + 5z = 15$.
the mosquito get caught in web, means the trajectory of mosquito intersects the plane where spider was located
To find at what time intersection happens, substitute $x = 4 + t, y = 1, z = 1 - t$ , into the plane equation

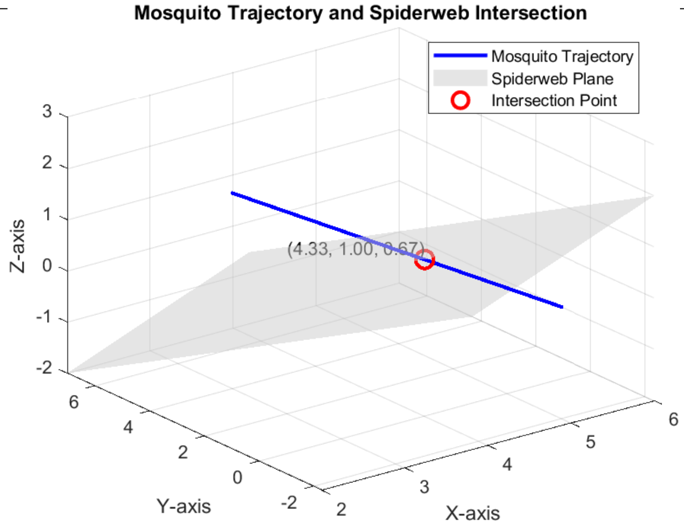
$$\begin{aligned} 2(4+t) + 3(1) + 5(1-t) &= 15 \\ 8 + 2t + 3 + 5 - 5t &= 15. \\ 16 - 3t &= 15. \\ -3t &= -1. \\ t &= \frac{1}{3}. \end{aligned}$$

now we have to find the position of mosquito at time $t = \frac{1}{3}$.
substitute $t = \frac{1}{3}$. in (1).

$$\begin{aligned} x = 4 + t = 4 + \frac{1}{3} &= \frac{12}{3} + \frac{1}{3} = \frac{13}{3} \\ y &= 1 \\ z = 1 - t = 1 - \frac{1}{3} &= \frac{2}{3}. \end{aligned}$$

Thus, the position of the mosquito when it intersects the plane is:

$$\mathbf{r} = \begin{bmatrix} \frac{13}{3} \\ 1 \\ \frac{2}{3} \end{bmatrix}.$$




2. Describe the geometric meaning of the following mappings in cylindrical coordinates (Preferably accompanied by a graphical diagram for demonstration).
**Solution:**
   (a) $(r, \theta, z) \mapsto (r, \theta, -z)$ **$2\frac{1}{2}$ Marks**
   (b) $(r, \theta, z) \mapsto (r, \theta + \pi, -z)$ **$2\frac{1}{2}$ Marks**
   (c) $(r, \theta, z) \mapsto (-r, \theta - \pi/4, z)$ **5 Marks**

**Solution**

(a) This mapping reflects a point across the $xy$ plane. In cylindrical coordinates, the $z-$ coordinate represents the height above or below the $xy-$ plane. Changing $z$ to $-z$ effectively mirrors the point across to $xy-$ plane.

(b) This mapping changes both the angular position and the height of the point:
1.The radial distance $r$ and the height $z$ are inverted as in part (a).
2.The angular coordinate $\theta$ is increased by $\pi$. Geometrically, this corresponds to a rotation by 180 around the $z$-axis. In other words, the point is reflected across the origin in the $xy-$ plane.

(c) Negating $r$ reflects the point across the origin, which is equivalent to adding $\pi$ to $\theta$. The transformation then subtracts an additional $\pi/4$ from $\theta$ leading to an overall shift of $3\pi/4$ in $\theta$

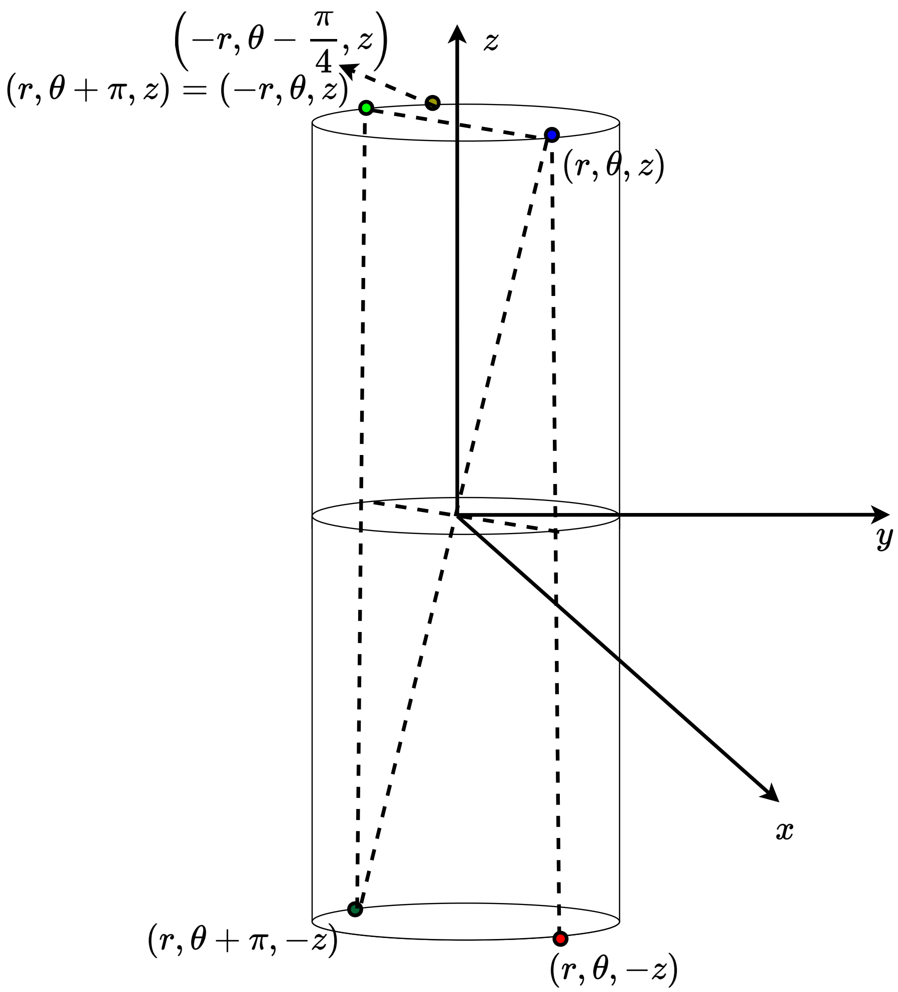

$\left(-r, \theta - \frac{\pi}{4}, z\right)$
$z$
$(r, \theta + \pi, z) = (-r, \theta, z)$
$(r, \theta, z)$
$y$
$x$
$(r, \theta + \pi, -z)$
$(r, \theta, -z)$

Figure 1

3. Using spherical coordinates $(\rho, \phi, \theta)$, show that

$$\phi = \cos^{-1}\frac{\vec{u}.\hat{k}}{|\vec{u}|};$$

where $\vec{u} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$. Interpret geometrically. **5 Marks**
**Solution:**

Given :

$$\vec{u} = x\hat{\imath} + y\hat{\jmath} + z\hat{k},$$

then we get

$$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$$

The polar angle $\phi$ is defined as the angle between the vector $\vec{u}$ and the $z$-axis $(\hat{k})$. The dot product between $\vec{u}$ and $\hat{k}$ is:

$$\vec{u} \cdot \hat{k} = z$$

The magnitude of $\vec{u}$ is:

$$|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$$

Using the formula for the cosine of the angle between two vectors:

$$\cos\phi = \frac{\vec{u} \cdot \hat{k}}{|\vec{u}|}$$

Taking the inverse cosine:

$$\phi = \cos^{-1}\left(\frac{\vec{u} \cdot \hat{k}}{|\vec{u}|}\right)$$

### Geometric Interpretation

The angle $\phi$ is the polar angle in spherical coordinates, which represents the angle between the position vector $\vec{u}$ and the $z$-axis. It ranges from 0 to $\pi$.

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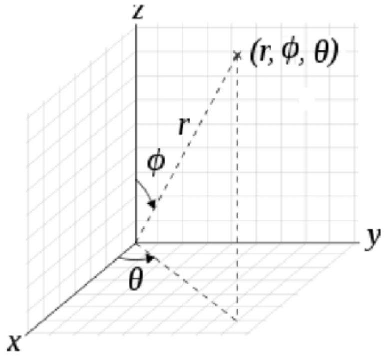
4. Find the unit vector which is at an angle $30^0$ to $\hat{\imath}$ and making equal angles with $\hat{\jmath}$ and $\hat{k}$? **5 Marks**
**Solution:**
Let the unit vector be $\vec{u}$ that makes an angle of $30°$ with $\hat{\imath}$ and makes equal angles with $\hat{\jmath}$ and $\hat{k}$ Let the unit

$$\vec{u} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}.$$




Since $\vec{u}$ is a unit vector, its magnitude must be 1:

$$\|\vec{u}\| = \sqrt{a^2 + b^2 + c^2} = 1.$$

Equal Angles with $\hat{j}$ and $\hat{k}$ The condition that $\vec{u}$ makes equal angles with $\hat{j}$ and $\hat{k}$ implies:

$$b = c.$$

Thus, the unit vector becomes:

$$\vec{u} = a\hat{i} + b\hat{j} + b\hat{k}.$$

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Angle with $\hat{i}$ The angle between $\vec{u}$ and $\hat{i}$ is given by:

$$\cos(30^\circ) = \frac{\vec{u} \cdot \hat{i}}{\|\vec{u}\|}.$$

The dot product $\vec{u} \cdot \hat{i} = a$, and $\|\vec{u}\| = 1$. Thus:

$$\cos(30^\circ) = a.$$

From trigonometry, $\cos(30^\circ) = \frac{\sqrt{3}}{2}$. Therefore:

$$a = \frac{\sqrt{3}}{2}.$$

—

Magnitude Condition Using the magnitude condition $\|\vec{u}\| = 1$, we have:

$$\sqrt{a^2 + b^2 + c^2} = 1.$$

Substitute $a = \frac{\sqrt{3}}{2}$ and $b = c$:

$$\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + b^2 + b^2} = 1.$$

Simplify:

$$\sqrt{\frac{3}{4} + 2b^2} = 1.$$

Square both sides:

$$\frac{3}{4} + 2b^2 = 1.$$

$$2b^2 = \frac{1}{4}.$$

$$b^2 = \frac{1}{8}.$$

$$b = \pm\sqrt{\frac{1}{8}} = \pm\frac{1}{2\sqrt{2}} = \pm\frac{\sqrt{2}}{4}.$$

—

Final Unit Vector The unit vector is:

$$\vec{u} = \frac{\sqrt{3}}{2}\hat{i} \pm \frac{\sqrt{2}}{4}\hat{j} \pm \frac{\sqrt{2}}{4}\hat{k}.$$

Thus, the two possible unit vectors are:

$$\boxed{\vec{u} = \frac{\sqrt{3}}{2}\hat{i} + \frac{\sqrt{2}}{4}\hat{j} + \frac{\sqrt{2}}{4}\hat{k}}$$

or

$$\boxed{\vec{u} = \frac{\sqrt{3}}{2}\hat{i} - \frac{\sqrt{2}}{4}\hat{j} - \frac{\sqrt{2}}{4}\hat{k}}.$$

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5. The temperature at a point $(x, y)$ is $T(x, y)$, measured in degrees Celcius. A bug crawls so that its position after $t$ seconds is given by $x = \sqrt{1+t}, y = 2 + \frac{1}{3}t$, where $x, y$ are measured in centimeters. The temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$. How fast is the temperature rising on the bug's path after 3 seconds? **10 Marks**

**Solution:**

To know how fast is the temperature rising on the bug's path after 3 seconds we have to find the rate of change of temperature with respect to time. According to total derivative formula:

$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt}$$

Given,

$$x(t) = \sqrt{1+t}, \quad y(t) = 2 + \frac{1}{3}t$$

$$\frac{dx}{dt} = \frac{1}{2\sqrt{1+t}}, \quad \frac{dy}{dt} = \frac{1}{3}.$$

Let us find the values of $x(t)$ and $y(t)$ at t=3,

$$x(3) = \sqrt{1+3} = \sqrt{4} = 2$$

$$y(3) = 2 + \frac{1}{3}(3) = 2 + 1 = 3$$

Given,

$$T_x(2,3) = 4, \quad T_y(2,3) = 3$$

Substitute values in the total derivative formula

$$\frac{dT}{dt} = T_x(2,3) \cdot \frac{dx}{dt} + T_y(2,3) \cdot \frac{dy}{dt}$$
$$\frac{dT}{dt} = 4 \cdot \frac{1}{2\sqrt{1+3}} + 3 \cdot \frac{1}{3}$$

Simplifying we get

$$\frac{dT}{dt} = 2$$

The temperature on the bug's path after 3 seconds is rising at a rate of **2°C/s** (Celcius/sec).

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6. You are walking on the graph of $f(x,y) = y\cos(\pi x) - x\cos(\pi y) + 10$, standing at the point $(2,1,13)$. Find $x, y-$ direction you should walk in to stay at the same level. **10 Marks**

**Solution:**

To stay at the same level, your movement must not cause $f(x,y)$ to change. This means the movement direction $(dx, dy)$ should be perpendicular to the gradient of $f(x,y)$,as the gradient points in the direction of the steepest increase.

### Step 1: Compute the gradient

The gradient $\nabla f(x,y)$ is given by:

$$\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right).$$

The function is:

$$f(x,y) = y\cos(\pi x) - x\cos(\pi y) + 10.$$

**Partial derivatives:**

$$\frac{\partial f}{\partial x} = \frac{d}{dx}\left(y\cos(\pi x) - x\cos(\pi y) + 10\right) = -y\pi\sin(\pi x) - \cos(\pi y),$$
$$\frac{\partial f}{\partial u} = \frac{d}{dy}\left(y\cos(\pi x) - x\cos(\pi y) + 10\right) = \cos(\pi x) + x\pi\sin(\pi y).$$

### Step 2: Evaluate the gradient at $(x, y) = (2, 1)$

At $(x, y) = (2, 1)$:

$$\sin(2\pi) = 0, \quad \cos(2\pi) = 1,$$
$$\sin(\pi) = 0, \quad \cos(\pi) = -1.$$

Substitute $x = 2$ and $y = 1$ into the partial derivatives:

$$\frac{\partial f}{\partial x} = -(1)\pi \sin(2\pi) - \cos(\pi) = 0 - (-1) = 1,$$
$$\frac{\partial f}{\partial y} = \cos(2\pi) + (2)\pi \sin(\pi) = 1 + 0 = 1.$$

Thus, the gradient at $(2, 1)$ is:

$$\nabla f(2, 1) = (1, 1).$$

### Step 3: Condition for staying at the same level

If you walk in a direction $(dx, dy)$, the rate of change of $f(x, y)$ in that direction is given by the dot product:

$$\nabla f(x, y) \cdot (dx, dy) = 0.$$

At $(2, 1)$, $\nabla f(2, 1) = (1, 1)$, so:

$$1 \cdot dx + 1 \cdot dy = 0.$$

Simplify:

$$dx + dy = 0.$$

### Step 4: Solution

The condition $dx + dy = 0$ describes all possible directions $(dx, dy)$ where you stay at the same level. For example:

- Choose $dx = 1$, then $dy = -1$, giving the direction $(1, -1)$.
- Alternatively, choose $dx = -1$, then $dy = 1$, giving $(-1, 1)$.

### Final Answer

The direction $(dx, dy)$ should satisfy:

$$\boxed{dx + dy = 0 \quad \text{or equivalently, any multiple of } (1, -1)}.$$