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EE1101: Circuits and Network Analysis

Lecture 29: Second-Order Circuits

October 13, 2025

Topics :

1. Method of Undetermined Coefficients
2. Sinusoidal Forcing Functions

Method of Undetermined Coefficients - Second-Order Circuits

DE : $\frac{d^2x}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = f(t) \rightarrow A\cos(\omega t + \phi)$

Instead solve $\frac{d^2\hat{x}}{dt^2} + 2\xi\omega_n \frac{d\hat{x}}{dt} + \omega_n^2 \hat{x} = Ae^{j(\omega t+\phi)} \longrightarrow (2)$

and $x = \mathrm{Re}\{\hat{x}\}$

general sol: $\hat{x}(t) = \hat{x}_c(t) + \hat{x}_p(t) \rightarrow$ guess.

$\hookrightarrow$ Complementary or homogeneous sol

guess for $\hat{x}_p(t)$: $\hat{x}_p(t) = \underbrace{r_p e^{j(\omega t + \phi - \theta_p)}}$

$\downarrow$ That need to be Computed

How to Compute r_p and θ_p :

$\frac{d\hat{x}_p}{dt} = j\omega \hat{x}_p(t)$

$\frac{d^2\hat{x}_p}{dt^2} = -\omega^2 \hat{x}_p(t)$

Plug in $\hat{x}_p$, $d\hat{x}_p/dt$, $\frac{d^2\hat{x}_p}{dt^2}$ into (2)

$(-\omega^2 + j2\xi\omega_n\omega + \omega_n^2)\hat{x}_p = Ae^{j(\omega t+\phi)}$

$[\underbrace{(\omega_n^2 - \omega^2) + j(2\xi\omega_n\omega)}_{Be^{j\phi_B}}]\hat{x}_p = Ae^{j(\omega t+\phi)}$

$B = \sqrt{(\omega_n^2 - \omega^2)^2 + 4\xi^2\omega_n^2\omega^2}$ $\phi_B = \tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right)$

DE $\frac{d^2x}{dt^2} + 2\xi\omega_n\frac{dx}{dt} + \omega_n^2 x = K$

$\downarrow$ $f(t)$

gen : $x = x_c(t) + x_p(t)$

guess for x_p : $x_p = C$

$\uparrow$ need to be Computed

Sinusoidal Forcing Functions

How to compute r_p and θ_p:

$$\frac{d\hat{x}_p}{dt} = j\omega \hat{x}_p(t)$$
$$\frac{d^2\hat{x}_p}{dt^2} = -\omega^2 \hat{x}_p(t)$$

Plug in $\hat{x}_p$, $d\hat{x}_p/dt$, $\frac{d^2\hat{x}_p}{dt^2}$ into (2)

$$(-\omega^2 + j2\xi\omega_n\omega + \omega_n^2)\hat{x}_p = Ae^{j(\omega t+\phi)}$$

$$[\underbrace{(\omega_n^2-\omega^2) + j(2\xi\omega_n\omega)}_{Be^{j\phi_B}}]\hat{x}_p = Ae^{j(\omega t+\phi)}$$

$$B = \sqrt{(\omega_n^2-\omega^2)^2 + 4\xi^2\omega_n^2\omega^2} \qquad \phi_B = \tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

$$\hat{x}_p = \frac{A}{B}e^{j(\omega t+\phi-\phi_B)} \longrightarrow (I)$$

when (I) & $\hat{x}_p = r e^{j(\omega t+\phi-\theta)}$ are compared

$$r = \frac{A}{B} = \frac{A}{\sqrt{(\omega_n^2-\omega^2)^2 + 4\xi^2\omega_n^2\omega^2}}$$

$$\theta = \phi_B = \tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

Complete sol for sinusoidal $f(t)$: (1) Compute $\hat{x}_c(t)$ (don't use init cond).

(2) Compute $\hat{x}_p(t)$

(3) Real part of sol $\hat{x}(t) = \hat{x}_c(t) + \hat{x}_p(t)$

(4) apply Init Cond to get specific sol.

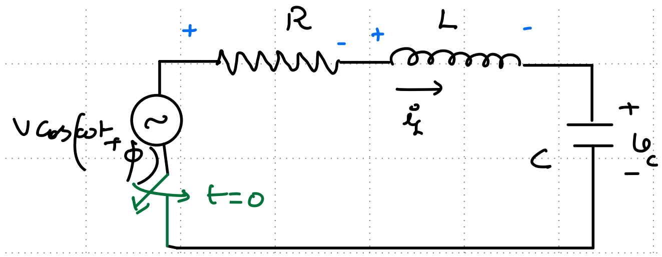
Example

$R = 30\,\Omega$, $L = 10\,H$, $C = 0.1\,F$

$\omega = 100\pi$

$\phi = \pi/4.$

$V = 10$



$i_L(0) = 10A$

$V_c(0) = 0V$

Step ①: Setup the DE and Init Cond

$$R i_L + L\frac{di_L}{dt} + v_C = v_S \longrightarrow ①$$

$i_L(0) = 10$

diff ① w.r.t t, $$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{1}{L}\frac{dv_s}{dt}$$

at $t = 0+$, $$v_s(0) = R i_L(0) + L\left.\frac{di_L}{dt}\right|_{t=0} + \underbrace{v_C(0)}_{} = R i_L(0) + L\left.\frac{di_L}{dt}\right|_{t=0} + 0$$

$$\left.\frac{di_L}{dt}\right|_{t=0} = \frac{v_s(0) - R i_L(0)}{L}$$

$$= -29.293$$

Step 2: Compute Complementary sol

$$i(t) = c_1 e^{-0.4t} + c_2 e^{-2.6t}.$$

Example

$$\frac{d^2 i_L}{dt^2} + \frac{R}{L}\frac{di_L}{dt} + \frac{1}{LC} i_L = \frac{1}{L}\frac{dv_s}{dt} = \frac{1}{L}\frac{d}{dt}\left(V\cos(\omega t + \phi)\right)$$

$$= -\frac{V\omega}{L}\sin(\omega t + \phi)$$

$$= \underbrace{\frac{V\omega}{L}}_{A}\cos(\omega t + \underbrace{\phi + \pi/2}_{\phi'})$$

Step 3: Compute $\hat{x}_p(t)$: response to $Ae^{j(\omega t + \phi')}$

$$= r\, e^{j(\omega t + \phi' - \theta)}$$

$$r = \frac{A}{B}, \quad \theta = \phi_B$$

where $B = \sqrt{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}$, $\phi_B = \tan^{-1}\left(\frac{2\xi\omega_n\omega}{\omega_n^2 - \omega^2}\right)$

Step 4: Real part of sol : $x(t) = Re\{\hat{x}_c(t) + \hat{x}_p(t)\}$

Step 5: Determine the constants/parameters in the solution using initial conditions

Method of Undetermined Coefficients - ~~Constant Forcing~~ Sinusoidal Function

$$x = Re\{\hat{x}_c(t) + \hat{x}_p(t)\}$$

$$= \underbrace{Re\{\hat{x}_c(t)\}}_{\to 0} + Re\{\hat{x}_p(t)\}$$

as $t \to \infty$ $\hat{x}_c(t) \to 0$ except when $\xi = 0$

as $t \to \infty$; $x(t) = Re\{\hat{x}_p(t)\} = Re\{ r_p e^{j(\omega t + \phi - \theta_p)}\}$

$$= r_p \cos(\omega t + \phi - \theta_p)$$

In steady state

$$v_s(t) = V\cos(\omega t + \phi) \longrightarrow \frac{V}{\sqrt{2}}\angle\phi$$

$$I_L(t) = r_p\cos(\omega t + \phi - \theta_p) \longrightarrow \frac{r_p}{\sqrt{2}}\angle\phi - \theta_p$$

} by constructing the phasor domain eq of the ckt.