Lecture-16

1. Divergence theorem.

2. Stokes' theorem:

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Indian Institute of Technology Hyderabad

Divergence Theorem: or "Gauss-Green theorem"

Green's for flux.

→ 3D analogue of

If 'S' is a closed surface enclosing a region 'D' oriented with $\hat{n}$ outward; and $\vec{F}$ defined everywhere in D, then

Green's for flux in 2D.

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_S \nabla\cdot\vec{F}\,dA.}$$

3D analogy.

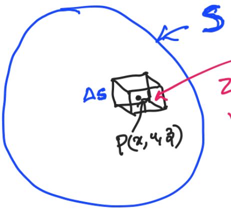
Divergence theorem:

$$\boxed{\underset{S}{\oiint} \vec{F}\cdot \underbrace{d\vec{S}}_{\hat{n}\,dS} = \iiint_D \operatorname{div}\vec{F}\,dV}$$

$$\boxed{\begin{aligned}&\operatorname{div}(P\hat{i} + Q\hat{j} + R\hat{k}) \\ &= \underbrace{P_x}_{\frac{\partial P}{\partial x}} + Q_y + R_z.\end{aligned}}$$

$\boxed{* \text{ If } \nabla\cdot\vec{F} = 0 \rightarrow \text{we say } \vec{F} \text{ is solenoidal.}}$

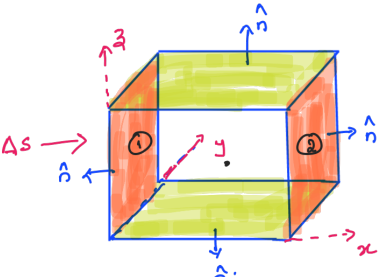
An intuitive proof for divergence theorem:



Zoom version shown next.

$$\oiint_{\Delta S} \vec{F}\cdot\vec{ds} \ ?$$

$$\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$



$$\oiint_{\Delta S} \vec{F}\cdot\vec{dS} = -F_1(x,y,z)\,\Delta y\,\Delta z$$

$$+ F_1(x+\Delta x, y, z)\,\Delta y\,\Delta z.$$

$+$ 4 other terms.

$$\Rightarrow \Delta x \Delta y \Delta z \left[\frac{F_1(x+\Delta x, y, z) - F_1(x,y,z)}{\Delta x} \right] = \frac{\partial F_1}{\partial x}$$

$$+ \Delta x \Delta y \Delta z \left[\frac{F_2(x, y+\Delta y, z) - F_2(x,y,z)}{\Delta y} \right] = \frac{\partial F_2}{\partial y}$$

$$+ \Delta x \Delta y \Delta z \left[\frac{F_3(x, y, z+\Delta z) - F_3(x,y,z)}{\Delta z} \right] = \partial F_3/\partial z.$$

$\Delta x \Delta y \Delta z = \Delta V.$

As $\Delta S \to 0 \Rightarrow \oint_{\Delta S} F \cdot \overline{dS} = \frac{\partial F_1}{\partial x} \Delta V + \frac{\partial F_2}{\partial y} \Delta V + \frac{\partial F_3}{\partial z} \Delta V.$

$$= (\nabla \cdot F) \Delta V.$$

Now for entire Surface.

$$\boxed{\oiint_S \vec{F}\cdot d\vec{s} = \iiint_V (\nabla\cdot\vec{F})\, dV.}$$

It intuitively explain the total flux entering and leaving the volume enclosed by the surface.

* Gauss's law in Engineering electromagnetics.
* Continuity equation of Hydrodynamics

$\Big\}$ Examples using divergence theorem.

Stoke's Theorem:

Review : Curl operator

$\bar{F} = \langle P, Q, R \rangle$; $\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

– measures rotation part of vector field

* direction : axis of rotation.

Line integral → Double integral : Green's theorem. in 2D

Line integral → Double integral : Stokes' theorem. in 3D space!

$$\boxed{\oint_C \bar{F} \cdot \hat{t}\, ds = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, dS.}$$

CCW circulation.

← If C closed curve in space;

S → any surface bounded by C.

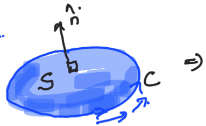
F defined everywhere in 'S'.

* Orientation:

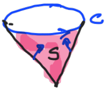
Need orientation of C & 'S' to be compatible.

1. If I walk along C in the positive direction, →(CCW) with 'S' to my left, then $\hat{n}$ is pointing up.

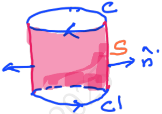
Ex. 1.



Ex 2:

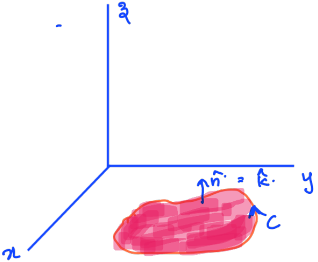


Ex 3:



Comparing Stokes' with Green's theorem:

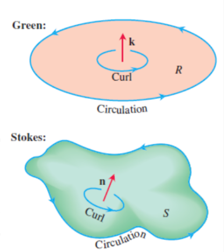
S = portion of xy-plane bounded by curve 'C', which goes counter clockwise.



$\vec{F} = \langle P, Q, R \rangle$

$\oint \vec{F} \cdot \underbrace{\hat{t}\, ds}_{d\vec{r}}$

$\Rightarrow \oint \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy + \underbrace{R\,dz}$ $z=0,\ dz=0$ at xy plane.



Ref: Thomas' Calculus.

Stokes

$$\iint_S (\nabla \times \vec{F}) \cdot \underset{= \hat{k}}{\hat{n}}\, \underbrace{dS}_{dx\,dy}.$$

$(\nabla \times \vec{F}) \cdot \hat{k} = z$-component of Curl F.

$= Q_x - P_y.$

$\Rightarrow$

$$\oint \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \underset{= \hat{k}}{\hat{n}}\, \underbrace{dS}_{dx\,dy} = \iint_S (Q_x - P_y)\, dx\,dy$$

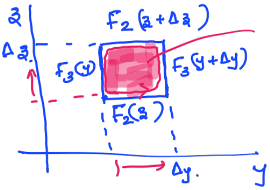
$\Rightarrow$ Green's theorem in 2D.!

* Green's theorem is a special case of Stoke's theorem in xy-plane.

Intuitive understanding of Stokes' theorem.

Say, $\bar{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$.

$$(\nabla \times \bar{F})_i = \hat{i}\underbrace{\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)}_{x\text{-component of Curl } F.}$$



If we integrate (loop integral) around this square.

$$\oint \bar{F} \cdot \underbrace{\hat{E}\, ds}_{d\bar{l}} = \oint \bar{F} \cdot d\bar{l}$$

$$= F_2(z)\,\Delta y - F_2(z+\Delta z)\,\Delta y + F_3(y+\Delta y)\,\Delta z - F_3(y)\,\Delta z$$

$$= -\left\{ \frac{F_2(z+\Delta z) - F_2(z)}{\Delta z} \Delta z \Delta y . \right\}$$

$$+ \frac{F_3(y+\Delta y) - F_3(y)}{\Delta y} \Delta y \Delta z ,$$

$$\oint \bar{F} \cdot \bar{dl} = -\frac{\partial F_2}{\partial z} \Delta y \Delta z + \frac{\partial F_3}{\partial y} \Delta y \Delta z .$$

$$= (dA) \underbrace{\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right]}_{(\nabla \times F) \cdot \hat{i}}$$

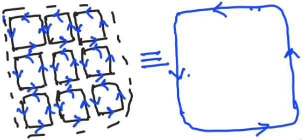
$\hat{n}$ to yz plane.

$$= \hat{i} (dA) . \nabla \times \bar{F}$$

$$= \hat{n}\, dS \cdot (\nabla \times \bar{F}) .$$

ie; $\boxed{\oint \bar{F} \cdot \bar{dl} = \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, dS}$ ← stokes theorem.

It works for any surface. ! extend this tiny loop idea to a large arbitrary surface.



So stokes theorem holds good for any arbitrary surface 'S' enclosed by curve 'C'.

* Intuitively, it states total circulation of a vector field F around the boundary of a surface 'S' (it is 'C' here) is equal to sum of infinitesimal rotations (curl) of F over the entire surface.

Interesting points on Stokes' theorem:

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times F) \cdot \hat{n} \, dS.$$

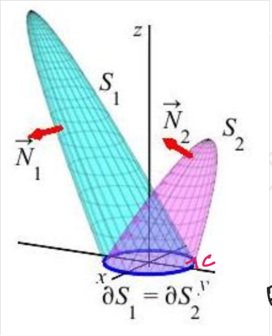
If two different oriented surfaces S_1 & S_2 have the same boundary 'C' their curl integral should be equal (by Stoke's theorem)

$$\Rightarrow \boxed{\iint_{S_1} \nabla \times \bar{F} \cdot \hat{n}_1 \, dS = \iint_{S_2} (\nabla \times \bar{F}) \cdot \hat{n}_2 \, dS.}$$

These two surface integral gives counterclockwise circulation integral ; as long as unit normal vectors n_1 & n_2 correctly orient the surface!

$\Rightarrow$ The curl integral is independent of the surface and depends only only the circulation around the boundary curve!

This is similar to the path independence nature of line integral: $\rightarrow$ True when $\vec{F} = \nabla f$ $\uparrow$ scalar field.



$\longrightarrow$ Surface independent surface integrals! (Stoke's theorem)

Ref: MyMath Apps.