Kirchoff's Laws for AC Circuits

- The circuit principles that govern the behavir of DC circuits are the (a) Kirchoff Current Law (KCL) and (b) Kirchoff Voltage Law (KVL). The Kirchoff's current law is derived from the continuity equation and the Kirchoff's voltage law is derived from the electrostatic field property that states that the line integral of the electric field around any closed path is zero. However, when dealing with time-varying scenarios (as is the case with AC circuits), the Kirchoff's voltage law is not strictly valid. This is because, in the presence of time-varying magnetic fields, the line integral of the electric field around a closed path is not zero. We will first discuss the validity of KVL in the context of AC circuits and then move on to KCL.

- **Kirchoff's Voltage Law (KVL) for AC Circuits:** In orde to understand the validity of KVL for AC circuits, we will consider a simple series circuit comprising of series connected resistors, as shown in Fig. 1. The circuit is excited by a time-varying voltage source $v_s(t)$ and for the purpose of this discussion, we will assume that the source is a sinusoidal source, i.e., $v_s(t) = V_m \cos(\omega t)$.

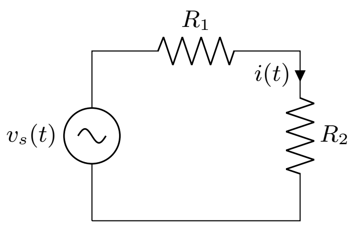

Figure 1: A simple series circuit with time-varying voltage source.

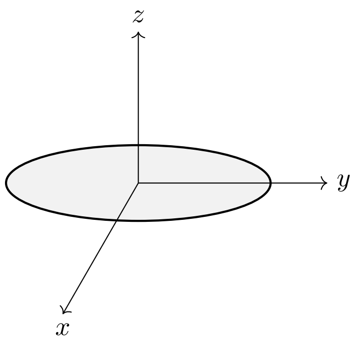

Figure 2: Idealized physical layout of the circuit.

To make the analysis tractable, we consider a simplified physical layout that the circuit could represent. We'll assume the circuit lies entirely in the xy-plane, with its elements connected in a circular loop and the wires treated as ideal conductors. The circuit is assumed to be placed in free space. This idealized setup allows us to focus on the fundamental factors that determine whether KVL remains valid under time-varying conditions.

When a time-varying current $i(t)$ flows around the loop in the counter-clockwise direction, it produces a time-varying magnetic field $\mathbf{B}(t)$ along the z-axis. This holds primarily along the axis of the loop. Off-axis, the magnetic field becomes more complex, with components that vary in both magnitude and direction. However, since our goal is to explore the conditions under which KVL can be applied, we will proceed with this simplified, on-axis approximation in the discussion that follows. The magnetic field intensity at the center of the loop can be computed using the Ampere's law and is given by

$$\mathbf{B}(t) = \frac{N\mu i(t)}{2R}\hat{z} \tag{1}$$

where R is the radius of the loop, μ is the permeability and N is the number of turns in this loop (if it is a single turn loop, then $N = 1$). The magnetic flux $\phi(t)$ linking the loop is given by

$$\phi(t) = \mathbf{B}(t) \cdot \mathbf{ds} = \frac{N\mu A}{2R} i(t) = \frac{N\mu\pi R^2}{2R} i(t) = \frac{N\mu\pi R}{2} i(t) \tag{2}$$

where A is the cross-sectional area of the loop. If the loop has multiple turns, the total flux linkage $\lambda(t)$ is given by

$$\lambda(t) = N\phi(t) = \frac{N^2 \mu \pi R}{2} i(t) \tag{3}$$

Note that the flux linkage $\lambda(t)$ is proportional to the current $i(t)$ flowing in the loop. The constant of proportionality is determined by the geometric and material properties of the loop. According to Faraday's law of electromagnetic induction, the non-conservative induced voltage $e(t)$ in the loop is given by

$$e(t) = \frac{d\lambda(t)}{dt} = \frac{N^2 \mu \pi R}{2} \frac{di(t)}{dt} \tag{4}$$

Assuming a sinusoidal current $i(t) = I_m \cos \omega t$, the induced voltage can be expressed as

$$e(t) = -\frac{N^2 \mu \pi R}{2} \omega I_m \sin(\omega t) \tag{5}$$

The induced voltage $e(t)$ is thus dependent on the frequency ω of the current $i(t)$, the amplitude I_m of the current, and the physical properties of the loop (i.e., N, μ, and R). For circuits with small dimensions (i.e., small R) and low frequencies (i.e., small ω), the induced voltage $e(t)$ can be negligible compared to the voltages across the circuit elements and can be assumed to be equal to zero. Under these conditions, the Kirchoff's voltage law can be applied to AC circuits, and the algebraic sum of the voltages around any closed loop is zero:

$$\sum v_k(t) = 0 \tag{6}$$

where $v_k(t)$ represents the voltage across the k^{th} element in the loop. It is interesting to note that there are scenraios where the circuit dimensions are relatively small and the frequency is relatively low, yet the induced voltage $e(t)$ is not negligible. This can occur in circuits with a large number of turns (i.e., large N) or in circuits made of materials with high permeability (i.e., large μ), such as inductors. In such scenarios, the induced voltage can no longer be ignored and must be accounted for as an additional circuit element (an induced electromotive force), which modifies the application of KVL accordingly.

In summary, Kirchhoff's Voltage Law (KVL) can be reliably applied to AC circuits under the assumption that the induced voltage $e(t)$, resulting from time-varying magnetic fields, is negligible. This is typically valid for circuits with small physical dimensions and low operating frequencies. However, when dealing with circuits of large dimensions and/or high frequencies, the induced voltage $e(t)$ may become significant, causing KVL to no longer hold strictly. In such cases, circuit analysis must explicitly account for the effects of time-varying magnetic fields and the resulting induced electromotive forces to accurately describe the behavior of the system.

- **Kirchoff's Current Law (KCL) for AC Circuits:** The Kirchoff's current law states that the algebraic sum of currents entering a node is equal to the algebraic sum of currents leaving the node. This law is derived from the continuity equation, which is based on the principle of conservation of charge. The continuity equation is given by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{7}$$

where $\mathbf{J}$ is the current density and ρ is the charge density. Integrating the continuity equation over a volume V enclosing a node and applying the divergence theorem, we get

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{d}{dt} \int_V \rho dV \tag{8}$$

where S is the surface enclosing the volume V. The left-hand side of the equation represents the net current flowing out of the volume V, while the right-hand side represents the rate of change of charge within the volume V. For KCL to hold, the right-hand side of the equation must be zero, which implies that the charge within the volume V is constant over time. This is a valid assumption for most practical circuits, where the charge accumulation at a node is negligible. Therefore, under this assumption, we can write

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = 0 \tag{9}$$

which leads to the Kirchoff's current law

$$\sum i_k(t) = 0 \tag{10}$$

where $i_k(t)$ represents the current flowing into or out of the node. It is important to note that KCL holds true for both DC and AC circuits, as it is based on the fundamental principle of conservation of charge, which is valid regardless of whether the currents are time-varying or not.