① Lagrange multipliers: Constrained optimization.

② Divergence of a vector field

EE1203: Vector Calculus

Aneesh Sobhanan
Department of Electrical Engineering
IIT Hyderabad, India

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Constrained optimization: Lagrange multipliers.

* We saw how to find max, min of functions without any constraints on the variables.

* How about having constraints on the variables.

Example 1: For a rectangle whose perimeter is 20 m, find the dimension that will maximize the area?

width $= x$
height $= y$.
Area: $A = xy$.
perimeter $P = 2x + 2y$.

Solution:

maximize $\rightarrow F(x, y) = xy$

Given $2x + 2y = 20$.

Using single variable calculus;

$2x+2y = 20$ - - - -> interval $y \Rightarrow [0, 10]$

$x = 10 - y.$

$F(y) = y(10-y) = 10y - y^2.$

$F'(y) = 10 - 2y = 0$; $x = 10 - 5 = 5$

$\Rightarrow y = 5$

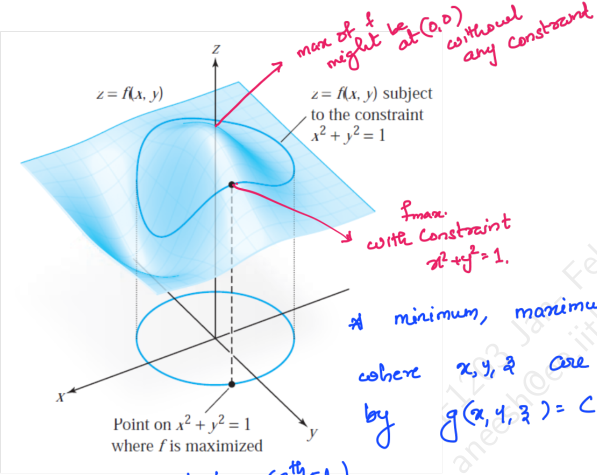
At $y=5$; $F''(y) = -2 < 0 \Rightarrow$ $y=5$ is local maxima.

$\Rightarrow$ it is global maximum on the interval $[0, 10]$

$\therefore$ The maximum area occurs for rectangle with

$x = 5m$; $y = 5m$.

Example 2: The geometric meaning of maximizing $f(x,y)$ subject to the constraint $x^2+y^2=1$.

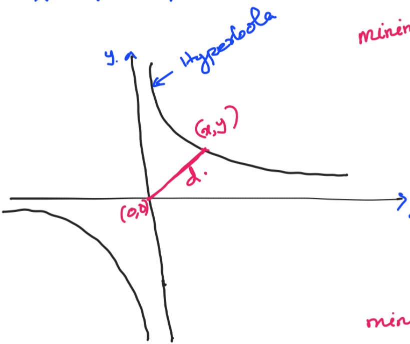


* minimum, maximum of a function $f(x,y,z)$ where x, y, z are not-independent, but related by $g(x,y,z) = c$

Ref: Vector Calculus (6^{th} Ed)
Jerrold E Marsden
Anthony Tromba.

* First observation: We cannot use our usual method of critical point, as critical point of 'f' may not satisfy the given constrained condition. $g = c$

Example: point closest to origin on hyperbola $xy = 3$.



minimize distance of point (x,y) to the origin given $xy=3$.

ie; point (x,y) on $xy=3$, which is nearest to origin.

$$d = \sqrt{x^2+y^2}.$$

minimizing 'd' is equivallent to minimizing $d^2 = x^2+y^2$.

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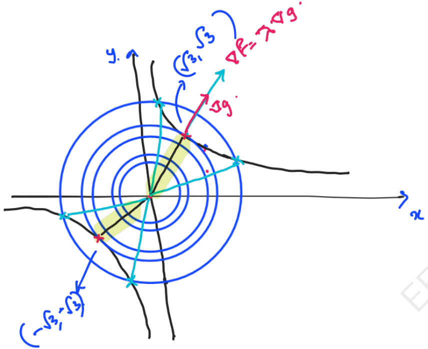
Problem here is;

$\Rightarrow$ minimize $f(x,y) = x^2+y^2$.

Given $g(x,y) = c$

here "$xy = 3$".



$\Rightarrow$ Observation:

* – The smallest circle that hit the hyperbola is the tangent circle to hyperbola!

* This is the minimum of $f = x^2+y^2$; which has solution on Hyperbola.

* below this value of 'f' no solution satisfying $xy = 3$.

$\therefore$ It is the minimum of 'f', given hyperbola as boundary or Constraints.

* When the function is minimum, the level curve of 'f' is tangent to the hyperbola. (In general, at minimum of function f the level curve of 'f' will be tangent to the given constraints $g=c$)

How to find (x,y) where level curves of $f(x,y)$ & $g(x,y)$ are tangent to each other?

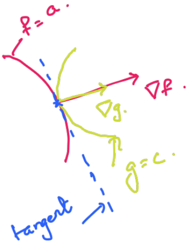
* two level curves are tangent $\Rightarrow$ they have same tangent line.

$\Rightarrow$ Their normal vectors should be parallel.

$\Rightarrow$

* Gradients of both the function should be parallel to each other.

$$\nabla f \parallel^{l} \nabla g$$



$\Rightarrow \quad \nabla f = \lambda \nabla g.$

$\hookrightarrow$ Lagrange multiplier.

$$\Rightarrow \left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y. \end{aligned}\right\} \text{ unknowns } x, y \text{ \& } \lambda.$$

Constraint: $xy = 3$

$\Rightarrow g = 3.$

$f = x^2 + y^2$

$g = xy.$

$$\left\{\begin{aligned} & f_x = \lambda g_x. \\ & \Rightarrow 2x = \lambda y \quad \text{- - - (1)} \\ & 2y = \lambda x \quad \text{- - - (2)} \\ & g = c \\ & \Rightarrow xy = 3 \quad \text{- - - (3)} \end{aligned}\right.$$

Three s/m of equations in 3 variables.

$$\left.\begin{aligned} 2x - \lambda y &= 0 \\ \lambda x - 2y &= 0 \end{aligned}\right\} \Rightarrow \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$xy = 3$

Trivial solution $(0,0)$ not satisfied

as $xy = 3$

$\Rightarrow 0 \neq 3.$

Other solution exist when $\begin{vmatrix} 2 & -\lambda \\ \lambda & -2 \end{vmatrix} = 0$

$$\Rightarrow -4 + \lambda^2 = 0 \Rightarrow \lambda = \pm 2.$$

$\lambda = +2 \Rightarrow \Rightarrow x = 4$

$y = x.$

$\Rightarrow xy = 3$

$\Rightarrow x^2 = 3 \Rightarrow x = \pm\sqrt{3}$

$(\sqrt{3}, \sqrt{3})$ or $(-\sqrt{3}, -\sqrt{3})$

$\underline{\lambda = -2.}$

$\Rightarrow x = -y.$

$xy = 3$

$\Rightarrow -x^2 = 3$

$\rightarrow$ no solution.

Why this Lagrange multiplier method

$\nabla f = \lambda \nabla g$ ← valid for optimization with constraints?

* At constrained min/max in any direction along level $g = c$; the rate of change of 'f' should be zero.

Let $\hat{u}$ be any tangent vector to $g=c$ (Constraint)

$\Rightarrow$ By directional derivative concept

rate of change of function 'f' along $\hat{u}$

$\Rightarrow \nabla f \cdot \hat{u}$

But here rate of change of f should be zero.

$\Rightarrow \nabla f \perp^r$ to $\hat{u}$ (tangent vector in $g=c$)

$\Rightarrow \nabla f \perp^r$ to level set '$g=c$'

∇g also $\perp^r$ to level set '$g=c$'

$\Rightarrow \boxed{\nabla f = \lambda \nabla g.}$

* Drawback: It does not tell whether a solution is maxima / minima.

* Check all solutions of Lagrange multiplier equations & find min/max from the corresponding values of functions at each of the solutions!

Divergence operator. ($\vec{\nabla}\cdot$)

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}. \quad \text{(Del operator)}$$

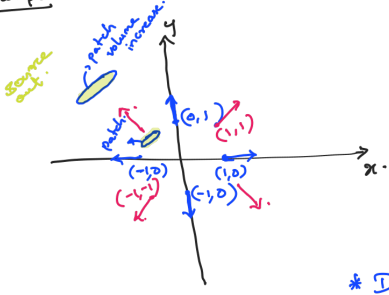
$$\vec{f} = \left(f_1(x,y,z),\ f_2(x,y,z),\ f_3(x,y,z) \right) \quad \text{(vector function.)}$$

$$\mathrm{div}(\vec{f}) = \vec{\nabla}\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \dashrightarrow \text{Scalar function.}$$

Physical meaning: $\text{div}(\vec{f})$ tells how much this vector field $\vec{f}$ locally sourcing out or sinking in.

if $\text{div}(\vec{f}) = 0 \Rightarrow$ net in/out flow is zero.

Example 1: $\vec{f}(x,y) = x\hat{i} + y\hat{j}$.



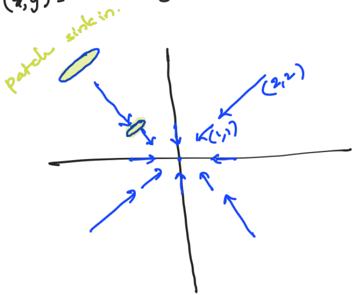
* Intuition is the divergence of $\vec{f}$ should be +ve; as the field values sourcing out from origin.

$$\nabla \cdot \vec{f} = \frac{\partial}{\partial x} x + \frac{\partial}{\partial y} y = +2 \quad (+\text{ve})$$

* Divergence +ve $\Rightarrow$ Volume grows/expands.

Example 2.

$$\bar{F}(x,y) = -x\hat{i} - y\hat{j}$$



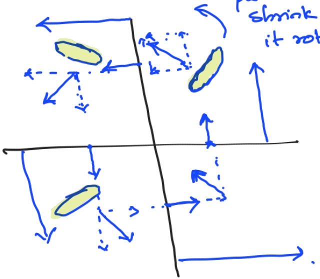
$$\nabla\cdot\bar{F} = -1-1 = -2 < 0.$$

$\text{div}(\vec{F}) \Rightarrow$ -ve.

→ Here things are sinking in or flow is converging.

Divergence -ve $\Rightarrow$ Volume shrinks.

Example 3 : $f(x,y) = -y\,\hat{i} + x\,\hat{j}$



patch neither shrink / expand
it rotated!

x	y	$\vec{f}$
1	0	$\hat{j}$
0	1	$-\hat{i}$
-1	0	$-\hat{j}$
0	-1	$\hat{i}$
1	1	$-\hat{i}+\hat{j}$
-1	1	$-\hat{i}-\hat{j}$
-1	-1	$\hat{i}-\hat{j}$
1	-1	$\hat{i}+\hat{j}$

$\vec{\nabla}\cdot\vec{f} = \dfrac{\partial(-y)}{\partial x} + \dfrac{\partial}{\partial y}(x) = 0 \Rightarrow$ Divergence free vector field $\Rightarrow$ $\boxed{\text{incompressible field.}}$

* Characterized by rotation! (here it is rotational)

$\boxed{\begin{array}{l}\vec{\nabla}\cdot\vec{f}=0; \\ \text{* it is not sufficient condition for rotation nature of vector fields or rotational field.}\end{array}}$