EE1080/AI1110/EE2102 Probability: HW 7

20th April, 2025

Bayes rule depending on X, Y:

$$
\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) f_X(x)}{\int f_{Y|X}(y|x) f_X(x) dx}, \quad X, Y \text{continuous} \\
f_{X|Y}(x|y) &= \frac{P_{Y|X}(y|x) f_X(x)}{\int P_{Y|X}(y|x) f_X(x) dx}, \quad X \text{ continuous }, Y \text{ discrete} \\
P_{X|Y}(x|y) &= \frac{P_{Y|X}(y|x) P_X(x)}{\sum\limits_x P_{Y|X}(y|x) P_X(x)}, \quad X, Y \text{discrete R.Vs} \\
P_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x) P_X(x)}{\sum\limits_x f_{Y|X}(y|x) P_X(x)}, \quad X \text{ discrete }, Y \text{continuous}
\end{aligned}
$$

Throughout these questions, X is the hidden parameter we are trying to estimate and Y is the observation you have. (Bertsekas has a slightly different notation. We'll go with the notation used in the class.)

1. Let X, Y be distributed as shown below:

$$
f_{Y|X}(y|x) = \begin{cases} xe^{-yx} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0 \\ 0 & \text{otherwise.} \end{cases}
$$

   (a) Find the MAP, ML, MMSE estimator for X on observing $Y = y$.

   (b) Now suppose you have n independent observations $Y_1 = y_1, \cdots, Y_n = y_n$ find the MAP, ML, MMSE estimators. (Independent observations imply that $f_{Y_1,\cdots,Y_n|X}(y_1, y_2, \cdots, y_n|x) = \prod\limits_{i=1}^{n} f_{Y_i|X}(y_i|x)$)

2. *Beta prior on Bias of a coin, Bernoulli observations:* Let X be Beta(α, β) distributed i.e., its density function is given by:

$$
f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1} \quad 0 \leq x \leq 1
$$

where $B(\alpha, \beta)$ is a normalization constant, $\alpha, \beta > 0$. For integer α, β, $B(\alpha, \beta) = \frac{(\alpha-1)!(\beta-1)!}{(\alpha+\beta-1)!}$. Let Y be defined as:

$$P_{Y|X}(y|x) = \begin{cases} x & y = 1 \\ 1-x & y = 0. \end{cases}$$

i.e., is Y conditioned over $X = x$ is distributed as Bernoulli(x).

(a) Find $E[X]$.

(b) What is the conditional density $f_{X|Y}(x|y)$? Can you identify the distribution ?

(c) What is the MMSE estimator of X on observing $Y = y$?

(d) Suppose you have n independent observations $Y_1 = y_1, \cdots, Y_n = y_n$ what is the MMSE estimator ?

3. *Beta prior on Bias of a coin, Binomial observations:* Similar to the earlier question, let $X \sim \text{Beta}(\alpha, \beta)$. Suppose $P_{Y|X}(y|x) = \binom{n}{y} x^y (1-x)^{n-y})$ i.e., Y conditioned over $X = x$ is a Binomial(n, x) random variable.

   (a) What is the MMSE estimator of X given observation $Y = y$?

   (b) What is the MAP estimator ? Does it have any similarity with the MAP estimator seen in the previous problem ?

4. Let $Y = XZ$ where X is Bernoulli(p) where $p = 1/4$ and Z is a normal random variable with mean 2 and variance 4.

   (a) What is the MMSE estimator of X on observing $Y = y$?

   (b) What are the MAP and ML estimators ?

   (c) Is the MMSE estimator useful in this case ?

5. Tintin and Prof. Calculus start work on top secret nuclear project, but Prof. Calculus will be late for any meeting by a random amount Y, uniformly distributed over interval $[0, x]$. The parameter x is unknown and is modeled as the value of a random variable X, uniformly distributed between zero and one hour. Assuming that Prof. Calculus was late by an amount y on their first meeting, how should Tintin use this information to update the distribution of X i.e., what is the conditional density $f_{X|Y}(x|y)$?

   (a) What is the MAP, ML estimate for X ?

   (b) What is the MMSE estimate for X ?

   (c) What is the linear MMSE estimate for X ?

   (d) Can you find the mean square error for the MMSE and linear MMSE estimators ?

6. *Gaussian Channel, MMSE estimation*

   (a) Let $Y_1, \cdots, Y_n$ be iid random variables and let $Y = Y_1 + Y_2 + \cdots + Y_n$. Show that:

   $$E[Y_i|Y] = \frac{Y}{n}.$$

   (b) Let $Y = X + W$ where X, W are independent zero mean Gaussian and of variance given by integers k, m respectively. Can you use the earlier part to find the MMSE estimator below:

   $$E[X|Y] = E[X|X + W]$$

   Hint: X is sum of m standard normal random variables.

   (c) Suppose X, W are independent Poisson random variables with integer means λ and μ respectively. What is the MMSE estimator ? Hint: Use the splitting property of Poisson random variables.

7. The joint PDF of random variables X and Θ is of the form:

   $$f_{Y,X}(y, x) = \begin{cases} c & (y, x) \in S \\ 0 & \text{otherwise,} \end{cases}$$

   where x is a constant and S is the set

   $$S = \{(y, x) \mid 0 \leq y \leq 2, 0 \leq x \leq 2, y - 1 \leq x \leq y\}.$$

   (a) Find MMSE estimate $\hat{x}(y)$ of X ?

   (b) Calculate $E[(X - \hat{x}(Y))^2 | Y = y]$, $E[\hat{x}(Y)]$ and $\mathrm{Var}(\hat{x}(Y))$.

   (c) Calculate the MSE $E[(X - \hat{x}(Y))^2$. It same as $E[\mathrm{Var}(X|Y)]$?

   (d) Calculate $\mathrm{Var}(X)$ using the total law of variance

   (e) Derive the linear MMSE estimator of X based on Y and calculate the mean squared error.