The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

Sinusoidal Input

$$r(t) = M\, sin\left(w_0 t\right)$$

$$R(s) = \frac{M\, w_0}{s^2 + w_0^2}$$

$$T(s) = \frac{1}{s + 1}$$

$$Y(s) = \frac{M\ w_0}{\left(s + 1\right)\left(s^2 + w_0^2\right)}$$

$$Y(s) = \frac{A_1}{s + 1} + \frac{A_2}{s + jw_0} + \frac{A_3}{s - jw_0}$$

$$A_1 = \frac{M\ w_0}{1 + w_0^2}$$

$$A_2 \;=\; \frac{M\;w_0}{\big(-jw_0\;+\;1\big)\big(-2jw_0\big)} \;=\; \frac{M}{\big(-jw_0\;+\;1\big)\big(-2j\big)}$$

$$A_3 \;=\; \frac{M\;w_0}{\big(jw_0\;+\;1\big)\big(2jw_0\big)} \;=\; \frac{M}{\big(jw_0\;+\;1\big)\big(2j\big)}$$

$$y(t) \;=\; A_1\;e^{-t} \;+\; \frac{M}{\big(-jw_0\;+\;1\big)\big(-2j\big)}\;e^{-jw_0 t} + \;\frac{M}{\big(jw_0\;+\;1\big)\big(2j\big)}\;e^{jw_0 t}$$

$$y(t) = A_1 e^{-t} + \frac{M}{(-jw_0 + 1)(-2j)} e^{-jw_0 t} + \frac{M}{(jw_0 + 1)(2j)} e^{jw_0 t}$$

$$T(jw_0) = \frac{1}{jw_0 + 1} = \left|T(jw_0)\right| \angle T(jw_0) = \left|T(jw_0)\right| e^{j\phi}$$

$$\frac{1}{-jw_0 + 1} = \left|T(jw_0)\right| e^{-j\phi}$$

$$y(t) \;=\; A_1\; e^{-t} \;+\; M\big|T(jw_0)\big|\left(\frac{e^{j\left(w_0 t \,+\, \phi\right)} \;-\; e^{-j\left(w_0 t \,+\, \phi\right)}}{2j}\right)$$

$$y(t) \;=\; A_1\; e^{-t} \;+\; M\;\big|T(jw_0)\big|\; sin\big(w_0 t \;+\; \phi\big)$$

As $t \to \infty$ $\qquad y(t) \;=\; M\;\big|T(jw_0)\big|\; sin\big(w_0 t \;+\; \phi\big)$

$$r(t) \;=\; M\; sin\big(w_0 t\big)$$

Frequency response

Transfer function $T(s)$

$$r(t) \;=\; M\,sin\big(w_0 t\big)$$

$$T(jw_0) \;=\; \big|T(jw_0)\big|\;\angle\phi$$

$$y(t) \;=\; M\,\big|T(jw_0)\big|\;sin\big(w_0 t \,+\, \phi\big)$$

Example

$$T(s) = \frac{1}{s+2}$$

$$\left|T(jw)\right| = \frac{1}{\sqrt{w^2+4}}$$

$$\angle T(jw) = -tan^{-1}\left(\frac{w}{2}\right)$$

$$T(s) = \frac{K\,(s+z_1)(s+z_2)\dots(s+z_k)}{s^m(s+p_1)(s+p_2)\dots(s+p_n)}$$

$$T(jw) = \frac{K\,(jw+z_1)(jw+z_2)\dots(jw+z_k)}{(jw)^m(jw+p_1)(jw+p_2)\dots(jw+p_n)}$$

$$\left|T(jw)\right| = \frac{K\,\left|jw+z_1\right|\left|jw+z_2\right|\dots\left|jw+z_k\right|}{\left|(jw)^m\right|\left|jw+p_1\right|\left|jw+p_2\right|\dots\left|jw+p_n\right|}$$

$$\begin{aligned}\log\left|T(jw)\right| = \log K + \log\left|jw+z_1\right| + \dots + \log\left|jw+z_k\right| \\ - \log\left|(jw)^m\right| - \log\left|jw+p_1\right| - \dots - \log\left|jw+p_n\right|\end{aligned}$$

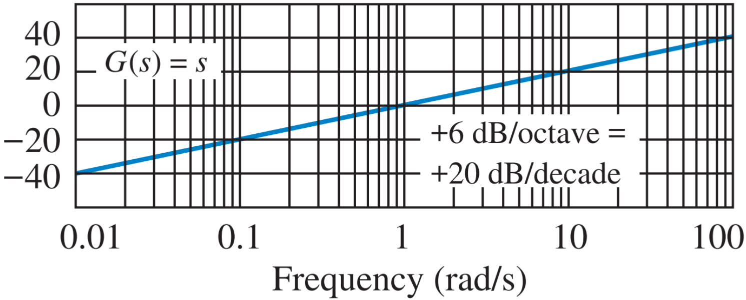
$$T(s) = s$$

$$T(jw) = jw$$

$$\left|T(jw)\right| = w$$

$$20 \log \left|T(jw)\right| = 20 \log w$$

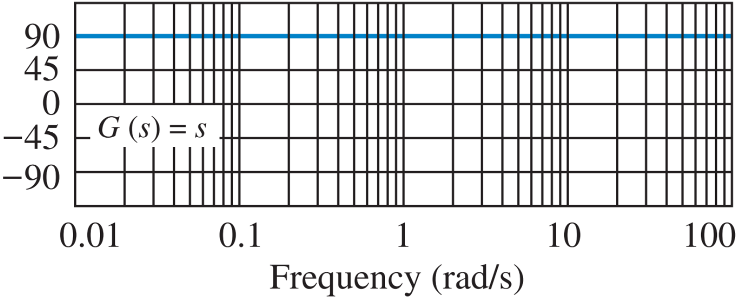
At $w = 1$ $\qquad 20 \log \left|T(jw)\right| = 20 \log 1 = 0$



40
20
0
−20
−40
G(s) = s
+6 dB/octave =
+20 dB/decade
0.01
0.1
1
10
100
Frequency (rad/s)

$$T(jw) = jw$$

$$\angle T(jw) = 90^\circ$$

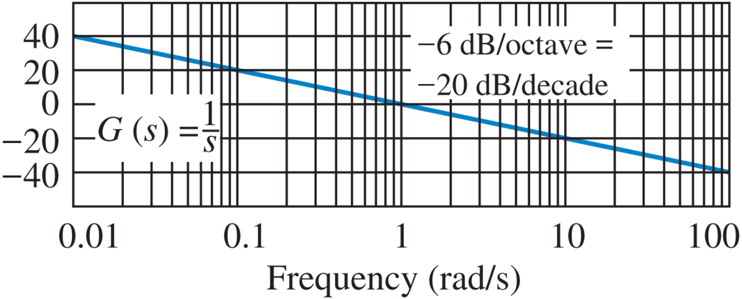


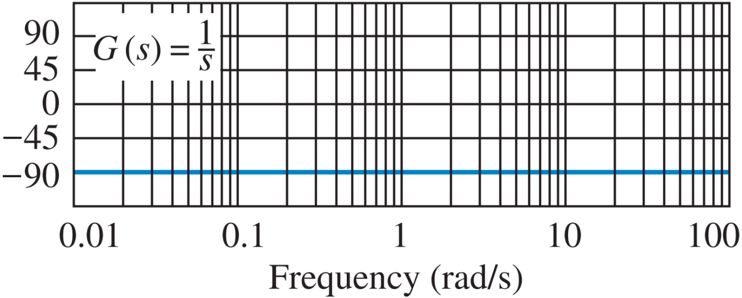
$$T(s) = \frac{1}{s}$$

$$T(jw) = \frac{1}{jw}$$

$$\left|T(jw)\right| = \frac{1}{w}$$

$$\angle T(jw) = -90^\circ$$


40
20
0
−20
−40
$G(s) = \frac{1}{s}$
−6 dB/octave =
−20 dB/decade
0.01
0.1
1
10
100
Frequency (rad/s)


90
45
0
−45
−90
$G(s) = \frac{1}{s}$
0.01
0.1
1
10
100
Frequency (rad/s)

Zero

$$T(s) = s + a$$

$$T(jw) = jw + a = a\left(\frac{jw}{a} + 1\right)$$

At $w \ll a$ $\qquad T(jw) = a$

$$20 \log \left|T(jw)\right| = 20 \log a$$

$$\text{At } w \gg a \qquad T(jw) = jw$$

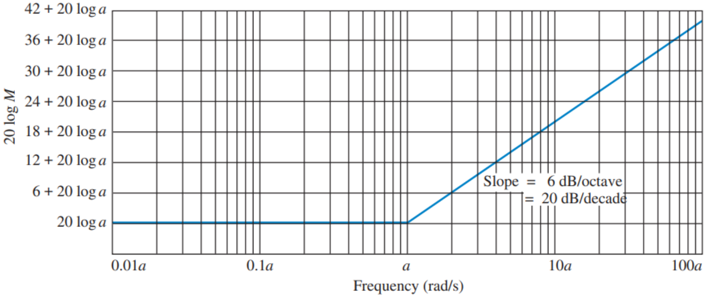
$$20 \log \left| T(jw) \right| = 20 \log w$$

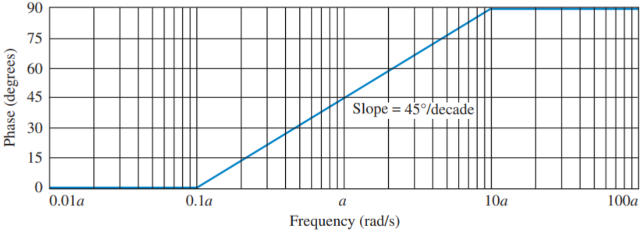
$$\text{At } w = a$$

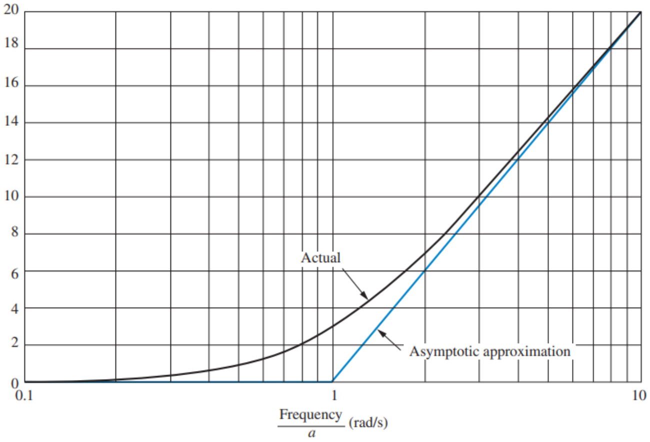
$$T(jw) = jw + a = a\left(\frac{jw}{a} + 1\right)$$

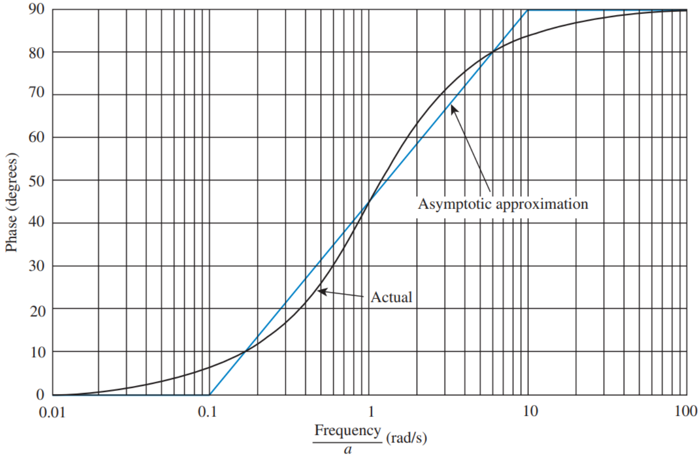
At $w \ll a$ $\qquad \angle T(jw) = 0^\circ$

At $w \gg a$ $\qquad \angle T(jw) = 90^\circ$


42 + 20 log a
36 + 20 log a
30 + 20 log a
24 + 20 log a
18 + 20 log a
12 + 20 log a
6 + 20 log a
20 log a
20 log M
Slope = 6 dB/octave
= 20 dB/decade
0.01a
0.1a
a
10a
100a
Frequency (rad/s)


90
75
60
45
30
15
0
Phase (degrees)
Slope = 45°/decade
0.01a
0.1a
a
10a
100a
Frequency (rad/s)

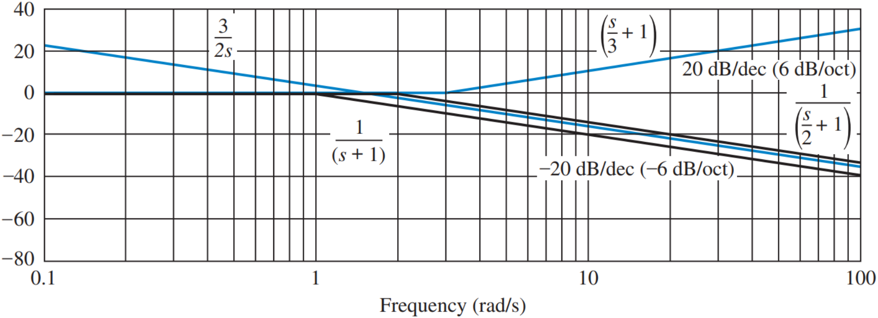

20
18
16
14
12
10
8
6
4
2
0
0.1
1
10
Actual
Asymptotic approximation
Frequency / a (rad/s)

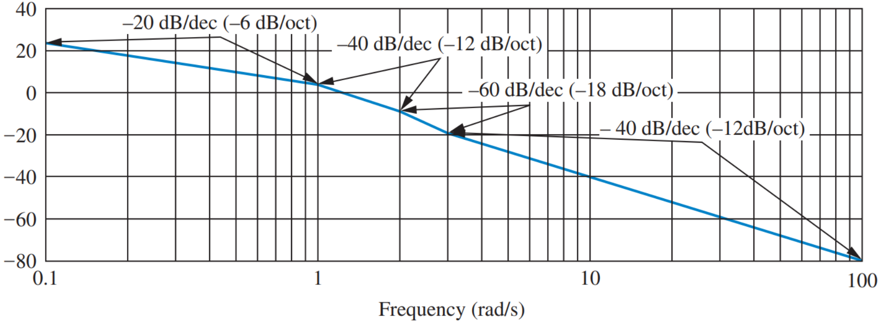


90
80
70
60
50
40
30
20
10
0
Phase (degrees)
0.01
0.1
1
10
100
$\frac{\text{Frequency}}{a}$ (rad/s)
Asymptotic approximation
Actual

$$G(s) = \frac{K(s+3)}{s(s+1)(s+2)}$$

$$G(s) = \frac{\frac{3K}{2}\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{2}+1\right)(s+1)}$$


40
20
0
−20
−40
−60
−80
0.1
1
10
100
Frequency (rad/s)
$\frac{3}{2s}$
$\left(\frac{s}{3}+1\right)$
20 dB/dec (6 dB/oct)
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{1}{(s+1)}$
−20 dB/dec (−6 dB/oct)



40
20
0
−20
−40
−60
−80
0.1
1
10
100
Frequency (rad/s)
–20 dB/dec (–6 dB/oct)
–40 dB/dec (–12 dB/oct)
–60 dB/dec (–18 dB/oct)
– 40 dB/dec (–12dB/oct)

	Uncompensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$
Dominant poles	$-1.794 \pm j3.501$
K	192.1
ζ	0.456
ω_n	3.934
$\%OS$	20
T_s	2.230
T_p	0.897
K_v	3.202
$e(\infty)$	0.312
Third pole	-12.41
Zero	None
Comments	Second-order approx. OK

$$\text{Real part of desired pole} = -3.588$$

$$\text{Imaginary part of desired pole} = 7.003$$

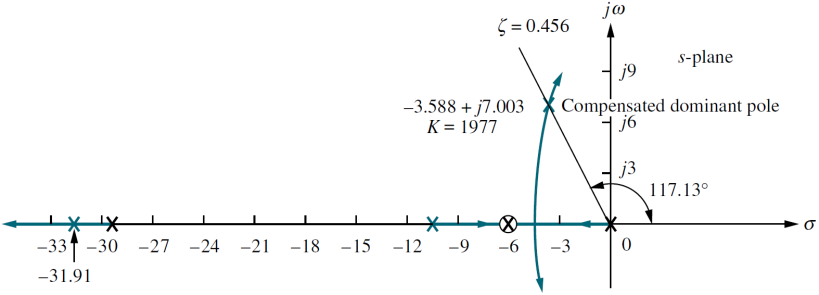
$$G_{Lead} = \frac{s + z_1}{s + p_1}$$

$$z_1 = 6$$

$$\text{Sum of angle to desired pole} = -164.65^\circ$$

$$\text{Required angle from compensator pole} = -15.35^\circ$$

$$G_{lead} = \frac{s + 6}{s + 29.1}$$



$j\omega$
$\zeta = 0.456$
s-plane
$j9$
$-3.588 + j7.003$
$K = 1977$
Compensated dominant pole
$j6$
$j3$
117.13°
σ
–33
–30
–27
–24
–21
–18
–15
–12
–9
–6
–3
0
–31.91

	Uncompensated	Lead-compensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$	$\dfrac{K}{s(s+10)(s+29.1)}$
Dominant poles	$-1.794 \pm j3.501$	$-3.588 \pm j7.003$
K	192.1	1977
ζ	0.456	0.456
ω_n	3.934	7.869
$\%OS$	20	20
T_s	2.230	1.115
T_p	0.897	0.449
K_v	3.202	6.794
$e(\infty)$	0.312	0.147
Third pole	−12.41	−31.92
Zero	None	None
Comments	Second-order approx. OK	Second-order approx. OK

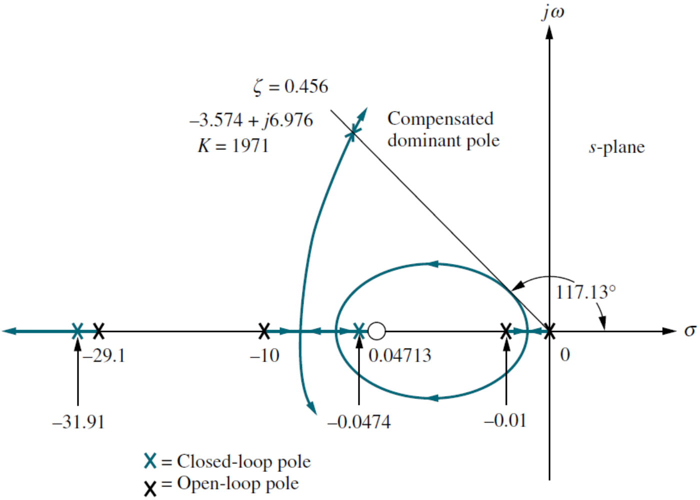
$$\text{Desired } K_v = 32.02$$

$$\text{Required change in } K_v = \frac{32.02}{6.794} = 4.713$$

$$G_{lag} = \frac{s + z_2}{s + p_2}$$

$$p_2 = 0.01$$

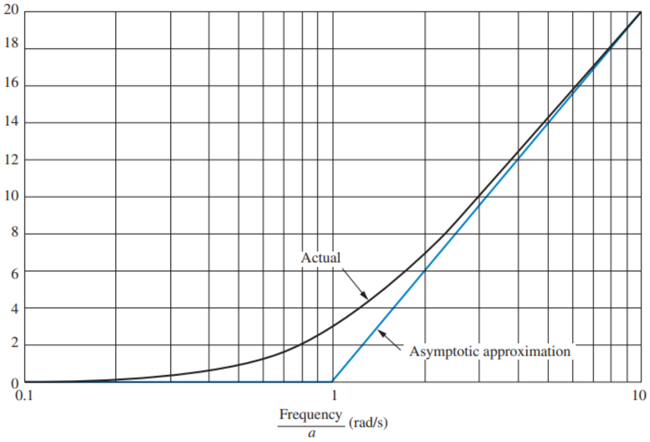
$$z_2 = 0.04713$$

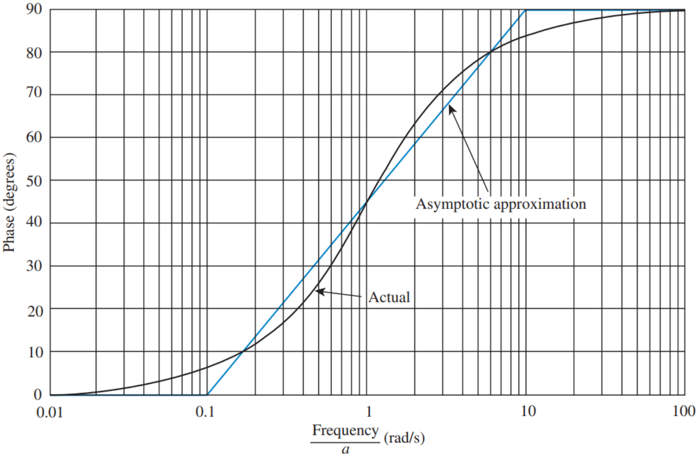


$j\omega$
$\zeta = 0.456$
$-3.574 + j6.976$
$K = 1971$
Compensated
dominant pole
s-plane
117.13°
σ
–29.1
–10
0.04713
0
–31.91
–0.0474
–0.01
= Closed-loop pole
= Open-loop pole

	Uncompensated	Lead-compensated	Lag-lead-compensated
Plant and compensator	$\dfrac{K}{s(s+6)(s+10)}$	$\dfrac{K}{s(s+10)(s+29.1)}$	$\dfrac{K(s+0.04713)}{s(s+10)(s+29.1)(s+0.01)}$
Dominant poles	$-1.794 \pm j3.501$	$-3.588 \pm j7.003$	$-3.574 \pm j6.976$
K	192.1	1977	1971
ζ	0.456	0.456	0.456
ω_n	3.934	7.869	7.838
$\%OS$	20	20	20
T_s	2.230	1.115	1.119
T_p	0.897	0.449	0.450
K_v	3.202	6.794	31.92
$e(\infty)$	0.312	0.147	0.0313
Third pole	-12.41	-31.92	$-31.91, -0.0474$
Zero	None	None	-0.04713
Comments	Second-order approx. OK	Second-order approx. OK	Second-order approx. OK

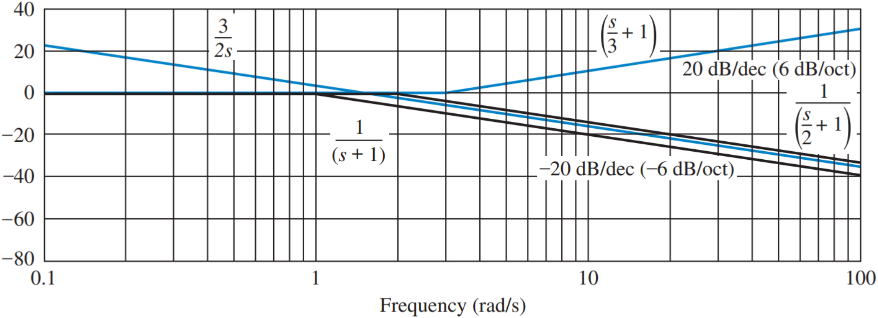
(rad/s)	Asymptotic	Actual	Asymptotic	Actual
0.01	0	0.00	0.00	0.57
0.02	0	0.00	0.00	1.15
0.04	0	0.01	0.00	2.29
0.06	0	0.02	0.00	3.43
0.08	0	0.03	0.00	4.57
0.1	0	0.04	0.00	5.71
0.2	0	0.17	13.55	11.31
0.4	0	0.64	27.09	21.80
0.6	0	1.34	35.02	30.96
0.8	0	2.15	40.64	38.66
1	0	3.01	45.00	45.00
2	6	6.99	58.55	63.43
4	12	12.30	72.09	75.96
6	15.56	15.68	80.02	80.54
8	18	18.13	85.64	82.87
10	20	20.04	90.00	84.29
20	26.02	26.03	90.00	87.14
40	32.04	32.04	90.00	88.57
60	35.56	35.56	90.00	89.05
80	38.06	38.06	90.00	89.28
100	40	40.00	90.00	89.43


20
18
16
14
12
10
8
6
4
2
0
0.1
1
10
Actual
Asymptotic approximation
Frequency / a (rad/s)

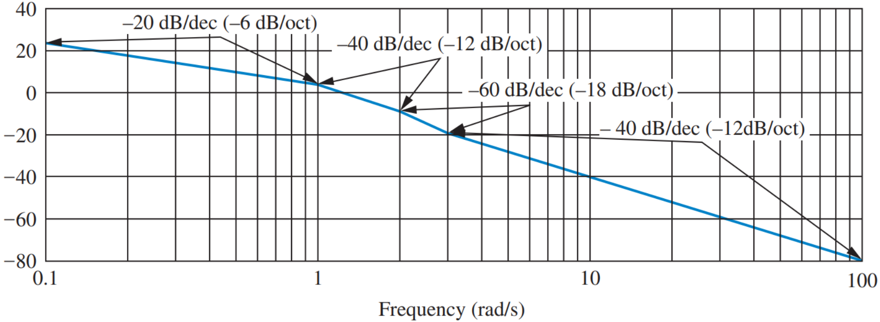

90
80
70
60
50
40
30
20
10
0
Phase (degrees)
0.01
0.1
1
10
100
Frequency/a (rad/s)
Asymptotic approximation
Actual

$$G(s) = \frac{(s+3)}{s(s+1)(s+2)}$$

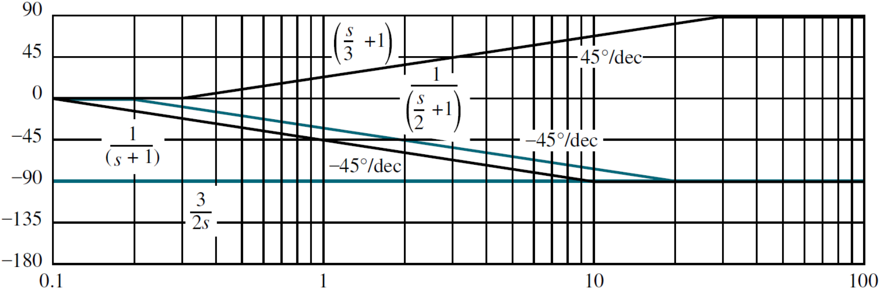
$$G(s) = \frac{\frac{3}{2}\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{2}+1\right)(s+1)}$$



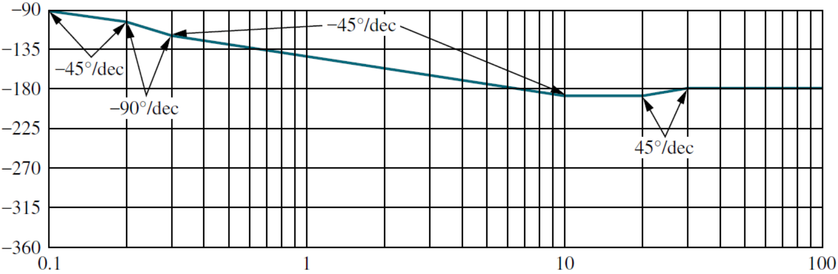
40
20
0
−20
−40
−60
−80
0.1
1
10
100
Frequency (rad/s)
$\frac{3}{2s}$
$\left(\frac{s}{3}+1\right)$
20 dB/dec (6 dB/oct)
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{1}{(s+1)}$
−20 dB/dec (−6 dB/oct)



–20 dB/dec (–6 dB/oct)
–40 dB/dec (–12 dB/oct)
–60 dB/dec (–18 dB/oct)
– 40 dB/dec (–12dB/oct)
40
20
0
−20
−40
−60
−80
0.1
1
10
100
Frequency (rad/s)



90
45
0
−45
−90
−135
−180
0.1
1
10
100
$\left(\frac{s}{3}+1\right)$
45°/dec
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{1}{(s+1)}$
−45°/dec
−45°/dec
$\frac{3}{2s}$

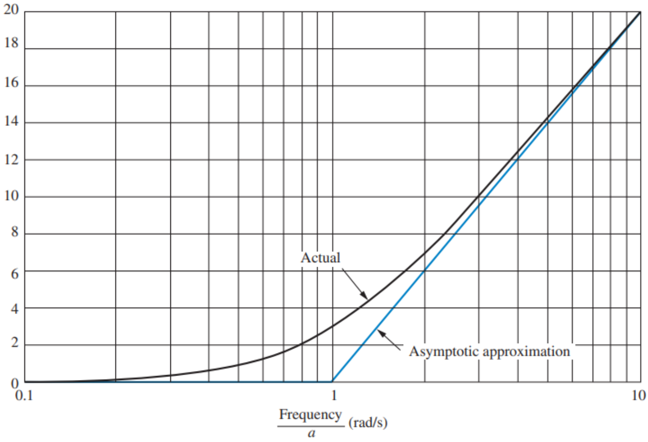


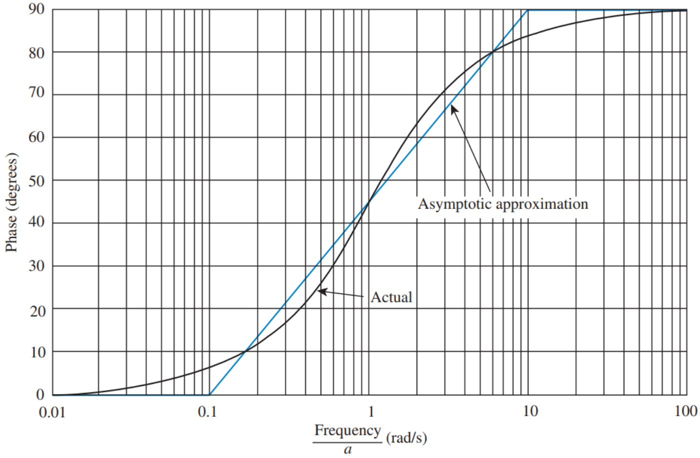
−90
−135
−180
−225
−270
−315
−360
0.1
1
10
100
−45°/dec
−45°/dec
−90°/dec
45°/dec

$$G(s) = \frac{s^2 + 2\zeta w_n s + w_n^2}{w_n^2}$$

$$\left|G(jw)\right| = \sqrt{\left(1 - \frac{w^2}{w_n^2}\right)^2 + \frac{4\zeta^2 w^2}{w_n^2}}$$

(rad/s)	Asymptotic	Actual	Asymptotic	Actual
0.01	0	0.00	0.00	0.57
0.02	0	0.00	0.00	1.15
0.04	0	0.01	0.00	2.29
0.06	0	0.02	0.00	3.43
0.08	0	0.03	0.00	4.57
0.1	0	0.04	0.00	5.71
0.2	0	0.17	13.55	11.31
0.4	0	0.64	27.09	21.80
0.6	0	1.34	35.02	30.96
0.8	0	2.15	40.64	38.66
1	0	3.01	45.00	45.00
2	6	6.99	58.55	63.43
4	12	12.30	72.09	75.96
6	15.56	15.68	80.02	80.54
8	18	18.13	85.64	82.87
10	20	20.04	90.00	84.29
20	26.02	26.03	90.00	87.14
40	32.04	32.04	90.00	88.57
60	35.56	35.56	90.00	89.05
80	38.06	38.06	90.00	89.28
100	40	40.00	90.00	89.43

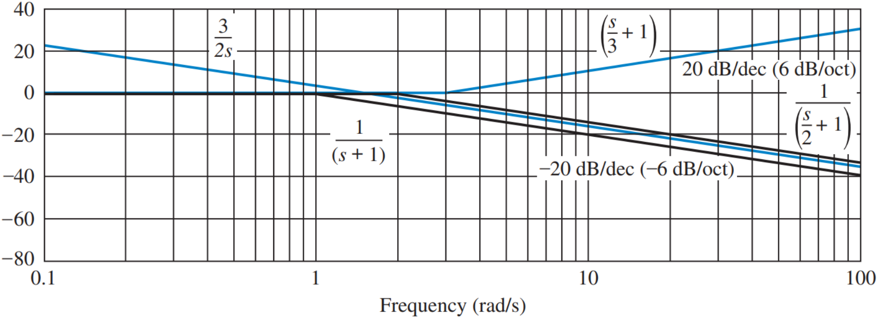

20
18
16
14
12
10
8
6
4
2
0
0.1
1
10
Actual
Asymptotic approximation
Frequency / a (rad/s)



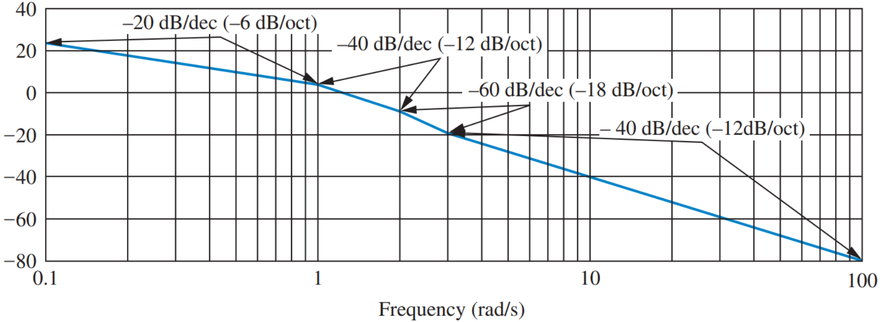
90
80
70
60
50
40
30
20
10
0
Phase (degrees)
0.01
0.1
1
10
100
Frequency/a (rad/s)
Asymptotic approximation
Actual

$$G(s) = \frac{(s+3)}{s(s+1)(s+2)}$$

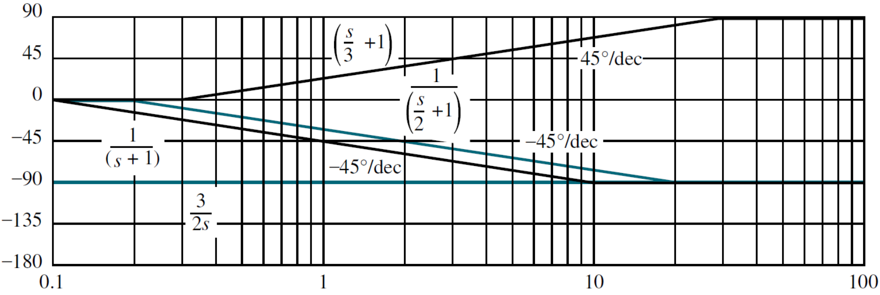
$$G(s) = \frac{\frac{3}{2}\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{2}+1\right)(s+1)}$$

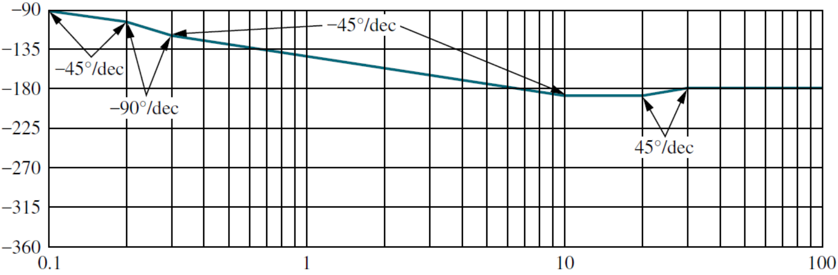


40
20
0
−20
−40
−60
−80
0.1
1
10
100
Frequency (rad/s)
$\frac{3}{2s}$
$\left(\frac{s}{3}+1\right)$
20 dB/dec (6 dB/oct)
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{1}{(s+1)}$
−20 dB/dec (−6 dB/oct)



40
20
0
−20
−40
−60
−80
–20 dB/dec (–6 dB/oct)
–40 dB/dec (–12 dB/oct)
–60 dB/dec (–18 dB/oct)
– 40 dB/dec (–12dB/oct)
0.1
1
10
100
Frequency (rad/s)


90
45
0
−45
−90
−135
−180
0.1
1
10
100
$\left(\frac{s}{3}+1\right)$
45°/dec
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{1}{(s+1)}$
−45°/dec
−45°/dec
$\frac{3}{2s}$



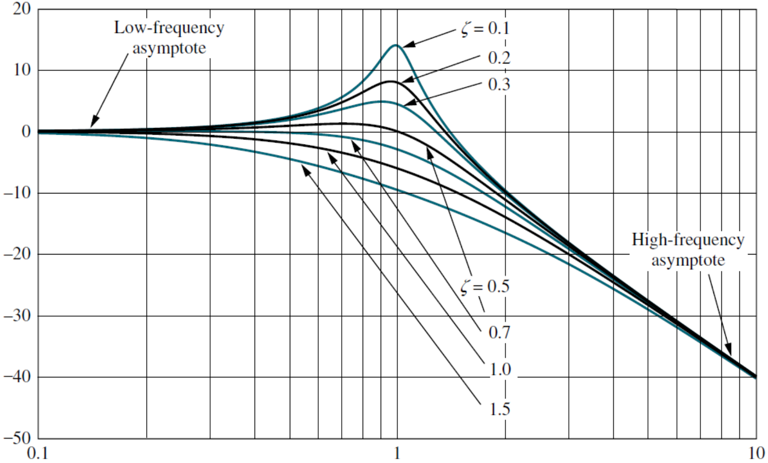
−90
−135
−180
−225
−270
−315
−360
0.1
1
10
100
−45°/dec
−45°/dec
−90°/dec
45°/dec

$$T(s) \;=\; \frac{w_n^2}{s^2 \;+\; 2\zeta w_n s \;+\; w_n^2}$$

$$\left|T(jw)\right| \;=\; \frac{1}{\sqrt{\left(1 - \dfrac{w^2}{w_n^2}\right)^2 \;+\; \dfrac{4\zeta^2 w^2}{w_n^2}}}$$

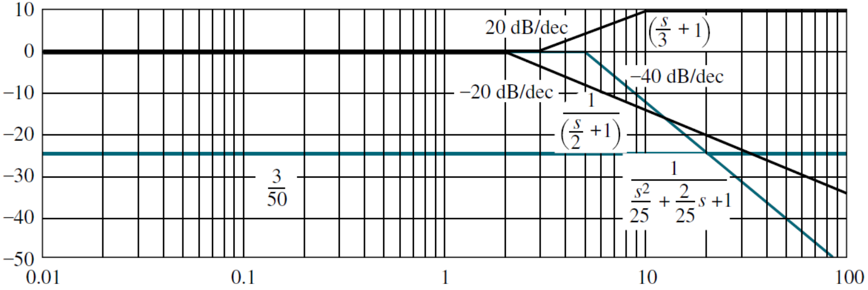
$$w_r = w_n\sqrt{1-2\zeta^2}$$

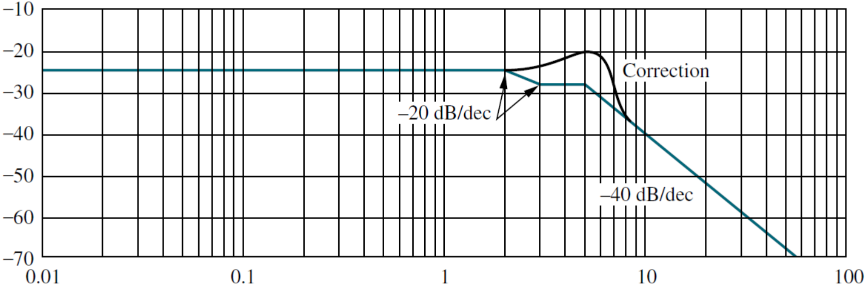
$$|T(jw)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$


20
10
0
−10
−20
−30
−40
−50
0.1
1
10
Low-frequency asymptote
High-frequency asymptote
ζ = 0.1
0.2
0.3
ζ = 0.5
0.7
1.0
1.5

$$T(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

$$T(s) = \frac{\frac{3}{50}\left(\frac{s}{3}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2s}{25}+1\right)}$$


10
0
−10
−20
−30
−40
−50
0.01
0.1
1
10
100
20 dB/dec
$\left(\frac{s}{3}+1\right)$
−40 dB/dec
−20 dB/dec
$\frac{1}{\left(\frac{s}{2}+1\right)}$
$\frac{3}{50}$
$\frac{1}{\frac{s^2}{25}+\frac{2}{25}s+1}$

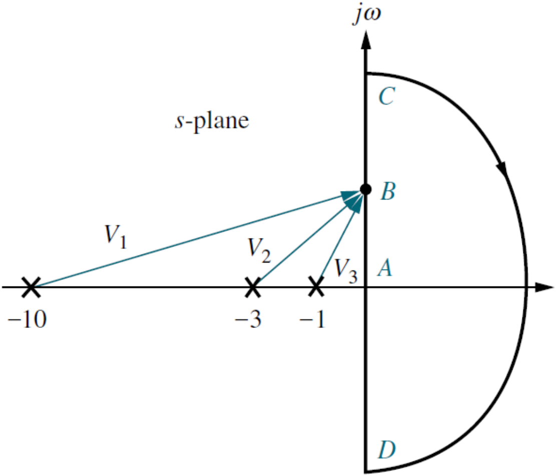

−10
−20
−30
−40
−50
−60
−70
0.01
0.1
1
10
100
Correction
–20 dB/dec
–40 dB/dec

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

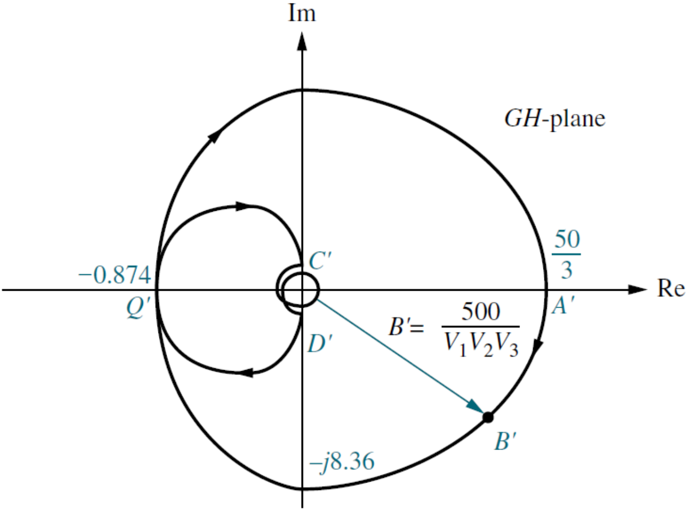
$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

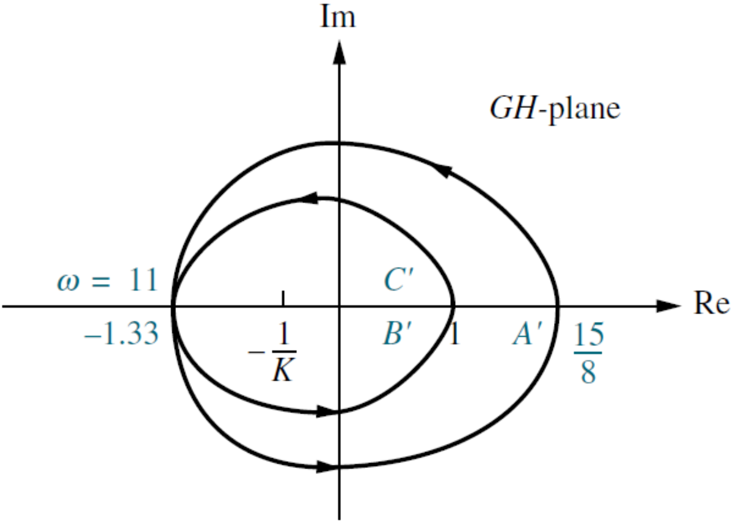
$$G(jw) = \frac{500}{(-14w^2+30)+j(43w-w^3)}$$



jω
s-plane
C
B
A
D
V1
V2
V3
−10
−3
−1


Im
GH-plane
50
3
Re
−0.874
C′
Q′
A′
B′= 500 / V1V2V3
D′
B′
–j8.36

$$G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}$$

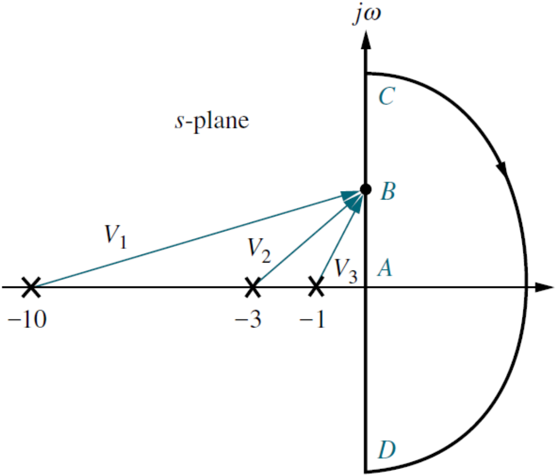

Im
GH-plane
ω = 11
C′
Re
–1.33
−1/K
B′
1
A′
15/8

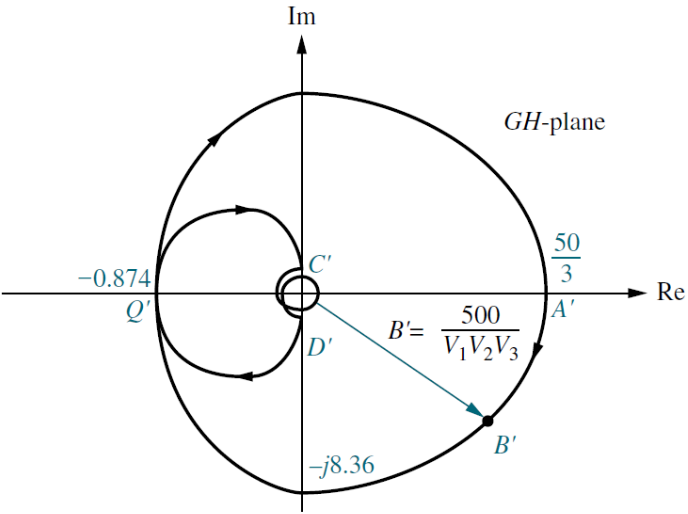
The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

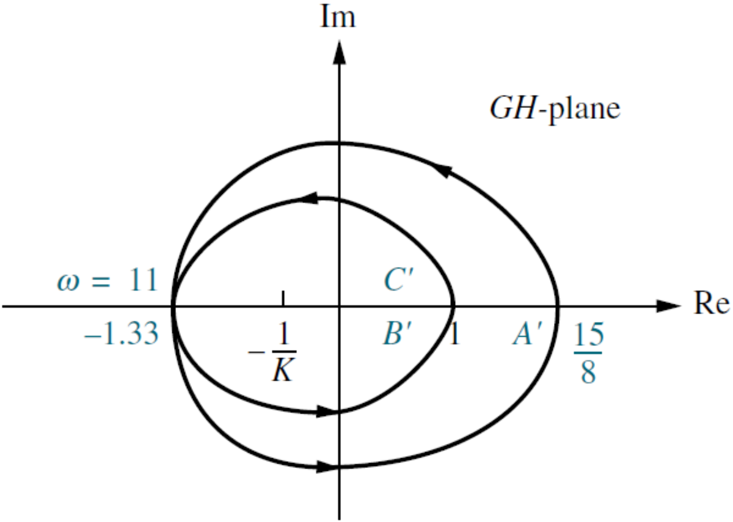
$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

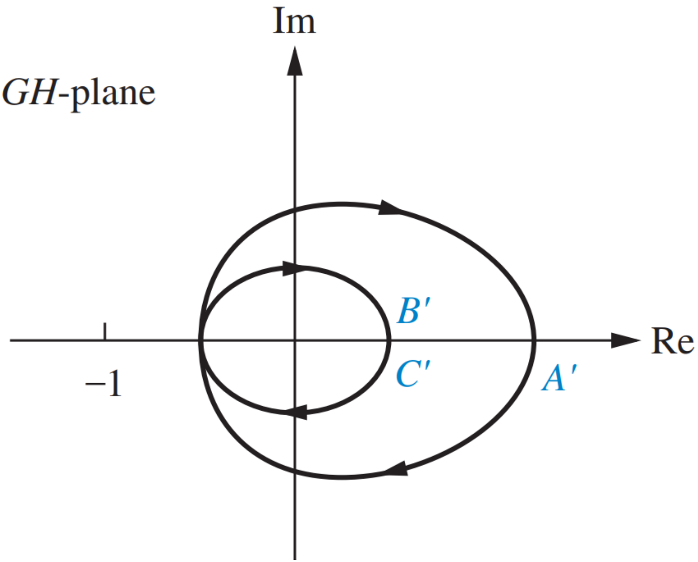
$$G(jw) = \frac{500}{(-14w^2+30)+j(43w-w^3)}$$


jω
s-plane
C
B
A
D
V1
V2
V3
−10
−3
−1


Im
GH-plane
C′
50/3
−0.874
Re
Q′
A′
B′= 500/(V1V2V3)
D′
B′
–j8.36

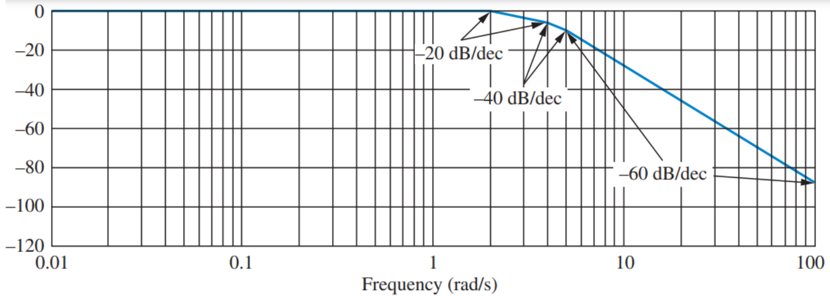
$$G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}$$

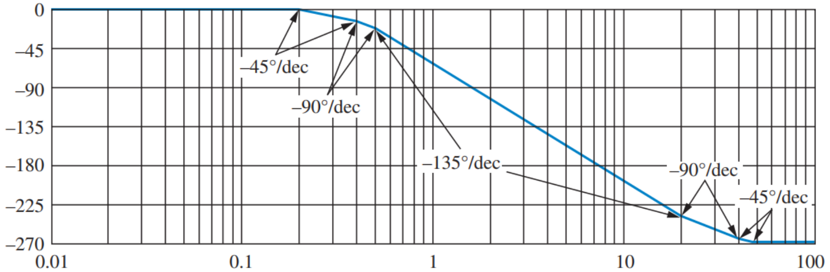

Im
GH-plane
C′
ω = 11
Re
–1.33
$-\frac{1}{K}$
B′
1
A′
$\frac{15}{8}$



Im
GH-plane
B'
Re
−1
C'
A'

$$G(s) = \frac{1}{(s+2)(s+4)(s+5)}$$


0
–20
–40
–60
–80
–100
–120
0.01
0.1
1
10
100
–20 dB/dec
–40 dB/dec
–60 dB/dec
Frequency (rad/s)

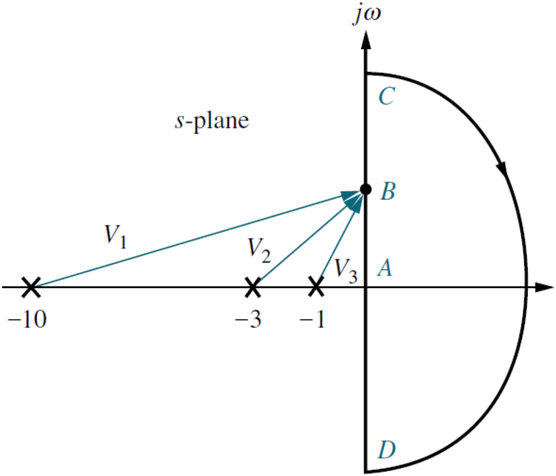

0
–45
–90
–135
–180
–225
–270
0.01
0.1
1
10
100
–45°/dec
–90°/dec
–135°/dec
–90°/dec
–45°/dec

The images are sourced from the Wikipedia website and the book Control Systems Engineering by Norman Nise. The author extends gratitude to these sources.

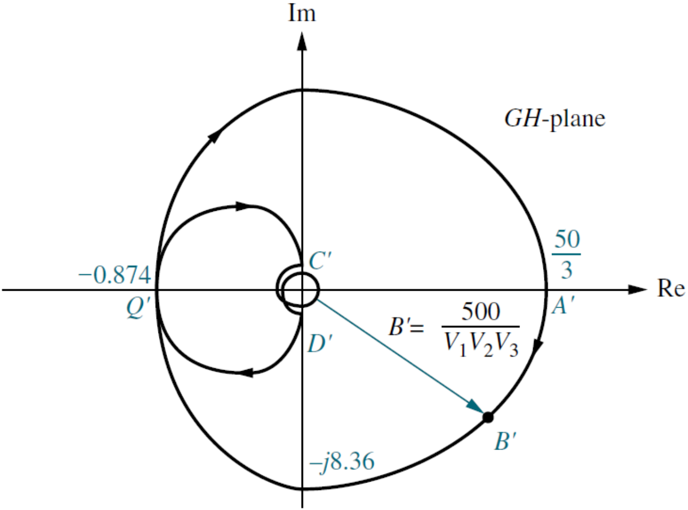
$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

$$G(s) = \frac{500}{(s+1)(s+3)(s+10)}$$

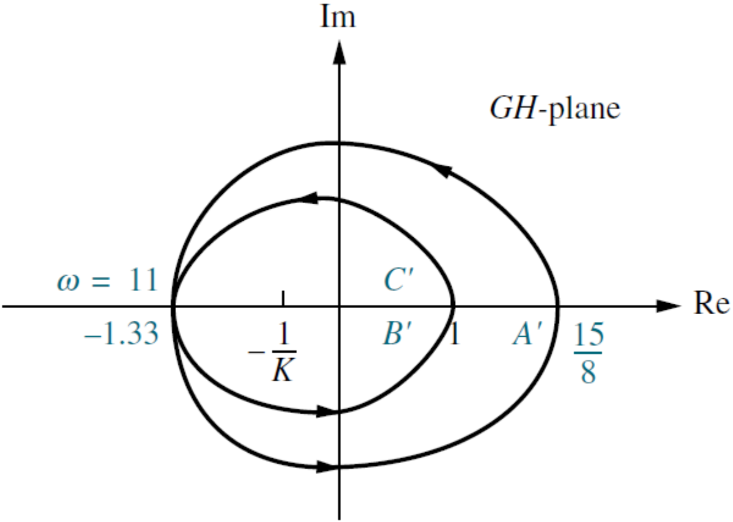
$$G(jw) = \frac{500}{(-14w^2+30)+j(43w-w^3)}$$

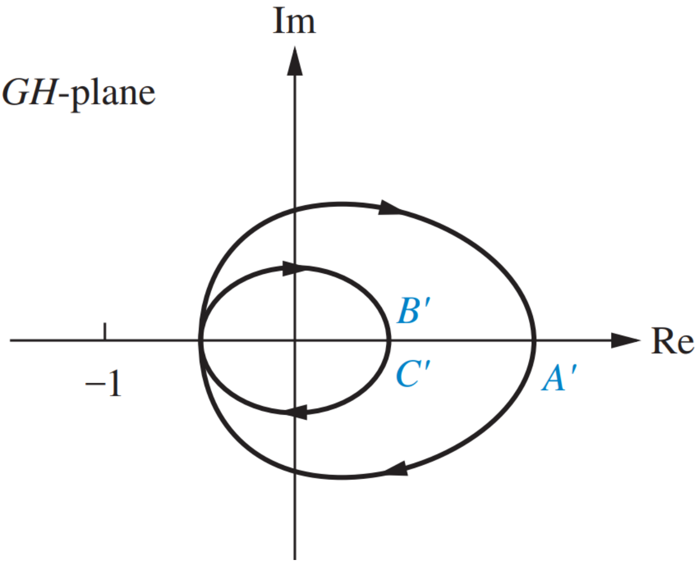


$j\omega$
s-plane
C
B
A
D
V_1
V_2
V_3
−10
−3
−1

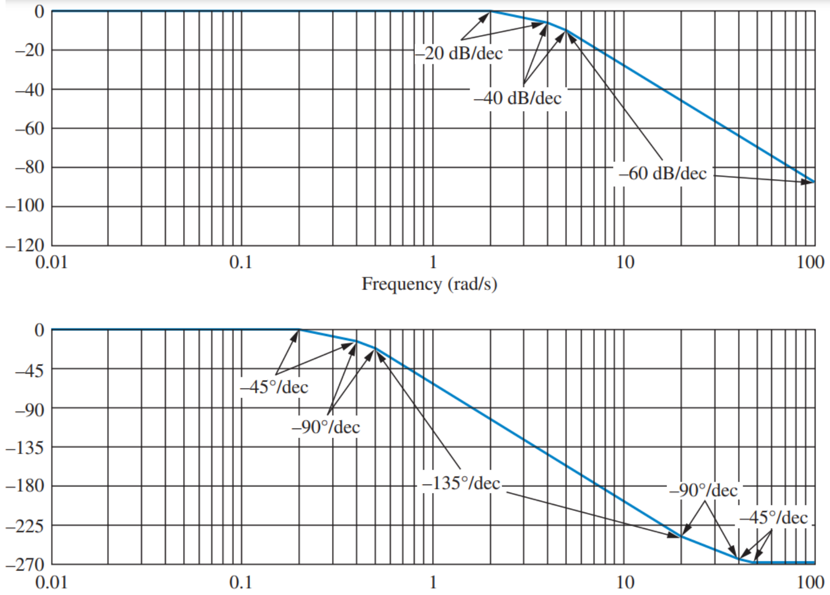

Im
GH-plane
50/3
Re
−0.874
Q'
C'
D'
A'
B' = 500/(V1V2V3)
B'
−j8.36

$$G(s) = \frac{K(s+3)(s+5)}{(s-2)(s-4)}$$

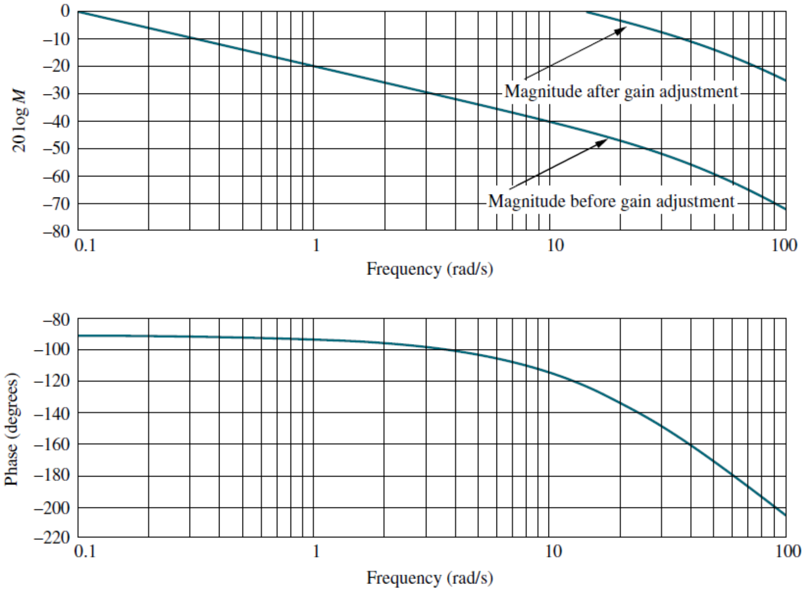

Im
GH-plane
ω = 11
C′
Re
−1.33
$-\frac{1}{K}$
B′
1
A′
$\frac{15}{8}$


Im
GH-plane
B′
C′
A′
Re
−1

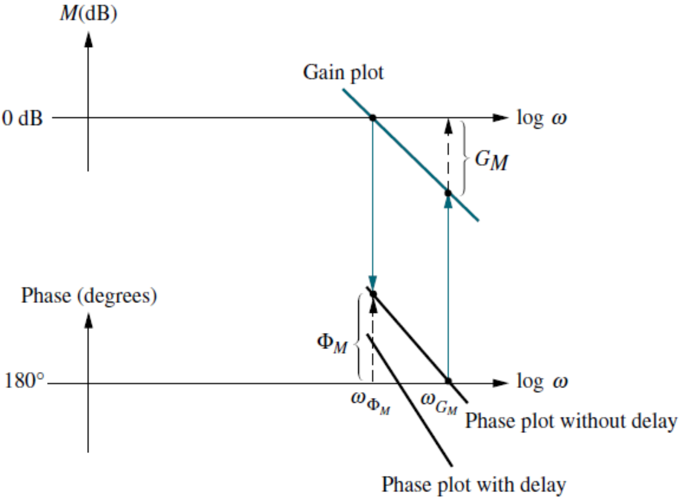
$$G(s) = \frac{1}{(s+2)(s+4)(s+5)}$$


0
–20
–40
–60
–80
–100
–120
0.01
0.1
1
10
100
Frequency (rad/s)
–20 dB/dec
–40 dB/dec
–60 dB/dec
0
–45
–90
–135
–180
–225
–270
0.01
0.1
1
10
100
–45°/dec
–90°/dec
–135°/dec
–90°/dec
–45°/dec

$$G(s) = \frac{100}{s(s+36)(s+100)}$$



0
−10
−20
−30
−40
−50
−60
−70
−80
20 log M
Magnitude after gain adjustment
Magnitude before gain adjustment
0.1
1
10
100
Frequency (rad/s)
−80
−100
−120
−140
−160
−180
−200
−220
Phase (degrees)
0.1
1
10
100
Frequency (rad/s)


M(dB)
Gain plot
0 dB
log ω
G_M
Phase (degrees)
Φ_M
180°
log ω
ω_{Φ_M}
ω_{G_M}
Phase plot without delay
Phase plot with delay