Node Analysis

- The goal of any circuit analysis technique is to determine the voltages and currents in the circuit. The most common techniques for circuit analysis are **node analysis** and **mesh analysis**. To begin with, we will focus on node analysis, which is one of the most powerful and widely used techniques for solving electric circuits (especially in the context of computer-aided circuit simulation).

- In node analysis, the goal is to determine the voltages at all the nodes in the circuit with respect to a reference node (or ground). The potential difference between a node and the reference is called the **node voltage**[1].

  If the voltages at all nodes are known, then any circuit quantities of interest (such as branch currents, power, etc.) can be determined.

  Proof: Since the potentials at all nodes are known,

  - the potential difference [2] across a circuit element can be computed as the difference of potentials at its terminals (or nodes) - say v_{ab} for the purpose of discussion,
  - the current through the circuit element can be computed (except for sources) as[3]

  $$i = \frac{v_a - v_b}{R} \tag{1}$$

    In the case of a voltage source, the current through it can be computed by applying KCL at the connected nodes.
  - the power associated with a circuit element can be computed as the product of the voltage across it and the current through it[4].

- **Overview of the Node Analysis Method:** Node analysis involves a systemic way of applying KCL at various nodes (typically nodes of unknown potential) of a network to derive a set of equations that can be solved to determine the node voltages (unknown). From a mathematical point of view, the set of equations must be linearly independent so that the solution is unique[5].

  A note on convention: The basis of node analysis is the application of KCL at vaious nodes in the circuit. For the resulting set of equations to be consistent, it is essential to choose a unique and consistent convention for applying KCL. For this course, we will use the following convention: The currents in the branches connected to a node are first expressed as the currents leaving the node. Subsequently, KCL is applied in the form that the sum of the currents leaving the node is equal to zero. This is mathematically expressed as

$$\sum_{q \in p}^{n} i_{pq} = 0 \tag{2}$$

[1] Nodes are typically labeled with lowercase letters, and the node voltage is written as v_a instead of v_{ao}, with a as the node and o as the reference.

[2] Pay attention to the chosen reference and polarity when indicating v and i.

[3] For steady-state analysis of AC circuits, the resistance is replaced by impedance.

[4] Whether the element is acting as a source or load depends on the value and the chosen sign convention.

[5] We assume the circuit equations are linearly independent, although this may not always hold in practice. Use your knowledge of linear algebra to identify situations where this assumption may fail.

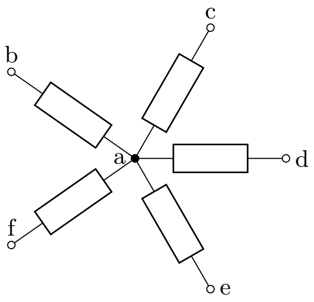

Figure 1: Node with connected circuit elements

where $q \in p$ indicates the branches connected to node p (with q representing the other node of each branch) and i_{pq} is the current in the associated branch leaving node p.

To understand the adopted convention, consider a generic scenario as shown in Fig. 1.

In this case, node a is connected to five branches, each with a different node at the other end. Application of KCL at node a using the chosen convention would yield

$$i_{ab} + i_{ac} + i_{ad} + i_{ae} + i_{af} = 0 \tag{3}$$

Note that the reference for the currents in the individual elements do not matter, when the equation is expressed in the form 3.

To solve for the unknown node potentials, the branch currents should be expressed in terms of node voltages wherever possible. For example, if the connected element is a resistor (say between nodes a and b), the corresponding branch current can be written as [6]

$$i_{ab} = \frac{v_a - v_b}{R} \tag{4}$$

[6] In steady state analysis of AC circuits, resistance is replaced by impedance and voltages and currents are phasors.

If the connected element is a current source, the branch current is already known and no further substitution is needed.

However, if the element is a voltage source, the branch current is both unknown and cannot be directly expressed in terms of node voltages. In such cases, it becomes necessary to either (a) introduce a new unknown variable to represent the branch current, or (b) apply an alternative technique such as *supernode analysis* to relate the node voltages across the source.

Given this, we will first focus on circuits comprising current sources and resistors to develop a clear understanding of the node analysis technique. Once we are familiar with the method, we will extend it to include circuits with voltage sources using supernode analysis.

- **Node analysis for circuits with current sources and resistors**: For circuits comprising current sources and resistors, the steps involved in finding the unknown node potentials are as follows:

  1. Identify the reference node (ground) and label all other nodes with unique identifiers.
  2. Apply KCL at each node (except the reference node) using the chosen convention. Note that in such circuits, the potentials of all nodes except the reference node are considered unknown.
  3. Express branch currents in terms of node potentials (in the case of a resistor), or use the known current value (in the case of a current source).

4. Combine the resulting expressions to form a system of linear equations. The number of equations should match the number of unknown node potentials.

5. Solve the system of equations to determine the unknown node potentials.

With sufficient practice, one can often write down the node equations directly without explicitly writing out KCL at each node. Furthermore, the use of techniques from linear algebra (such as matrix representation of equations and its solution) provides an alternative and efficient formulation—particularly useful when solving large circuits[7]. Typically, the equations obtained using node analysis, when represented in matrix form, are written as

$$\mathbf{G}\mathbf{v} = \mathbf{i} \tag{5}$$

[7] Developing a steady-state circuit solver requires formulating node analysis equations in matrix form and solving them via Gaussian elimination or LU decomposition.

where $\mathbf{G}$ is the conductance matrix, $\mathbf{v}$ is the vector of unknown node potentials, and $\mathbf{i}$ is the vector of equivalent currents[8].

[8] Note the notation: v denotes instantaneous voltage, while phasors are denoted using $\mathbf{V}$ or $\vec{V}$. Here, $\mathbf{v}$ is the vector of unknown node potentials. In handwritten notes, we write this vector as $[v]$.

It is important to note that the resulting set of equations and the corresponding representation using matrices is dependent on the convention chosen for applying KCL. Furthermore, in matrix representation, the order in which the vector of unknown node potentials is defined directly affects the position of entries in $\mathbf{G}$ and $\mathbf{i}$.

- Example: Analyze the circuit shown in Fig. 2 using node analysis.

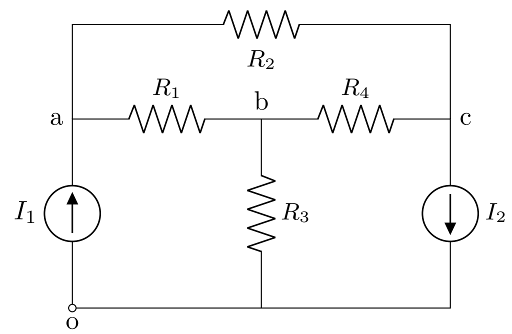

Figure 2: Circuit with current sources and resistors

Solution: Application of KCL at node a gives

$$\underbrace{I_{ao}}_{-I_1} + I_{ab} + I_{ac} = 0 \implies -I_1 + \left(\frac{V_a - V_b}{R_1}\right) + \left(\frac{V_a - V_c}{R_2}\right) = 0$$

$$\text{i.e.,} \quad \left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_a - \frac{1}{R_1} V_b - \frac{1}{R_2} V_c = I_1$$

Application of KCL at node b gives

$$I_{ba} + I_{bo} + I_{bc} = 0 \implies \left(\frac{V_b - V_a}{R_1}\right) + \frac{V_b}{R_3} + \left(\frac{V_b - V_c}{R_4}\right) = 0$$

$$\text{i.e.,} \quad -\frac{1}{R_1} V_a + \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) V_b - \frac{1}{R_4} V_c = 0$$

Application of KCL at node c gives

$$\underbrace{I_{co}}_{I_2} + I_{cb} + I_{ca} = 0 \implies I_2 + \left(\frac{V_c - V_b}{R_4}\right) + \left(\frac{V_c - V_a}{R_2}\right) = 0$$

$$\text{i.e.,} \quad -\frac{1}{R_2}V_a - \frac{1}{R_4}V_b + \left(\frac{1}{R_2} + \frac{1}{R_4}\right)V_c = -I_2$$

The unknown node potentials can be computed by solving the resulting set of equations[9]. A matrix-based representation of the resulting system of linear equations is

$$\begin{bmatrix} \left(\frac{1}{R_1} + \frac{1}{R_2}\right) & -\frac{1}{R_1} & -\frac{1}{R_2} \\ -\frac{1}{R_1} & \left(\frac{1}{R_1} + \frac{1}{R_3} + \frac{1}{R_4}\right) & -\frac{1}{R_4} \\ -\frac{1}{R_2} & -\frac{1}{R_4} & \left(\frac{1}{R_2} + \frac{1}{R_4}\right) \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} I_1 \\ 0 \\ -I_2 \end{bmatrix}$$

Important Observations[10]:

1. Under the chosen convention, the diagonal entries of the $\mathbf{G}$ matrix are the sum of conductances of all elements connected to a given node. The off-diagonal entries are the negatives of the conductance values of elements connected between the corresponding pair of nodes.
2. The entries in the current vector represent the net current injected into each corresponding node.

- **Questions for Further Thought**[11]

  1. Analyze the validity of the following statements. Provide a formal proof or a counterexample in each case:
     (a) The conductance matrix is always symmetric.
     (b) The conductance matrix, as derived using the adopted convention, is always of full rank.
     (c) If the conductance matrix is not of full rank (determine whether this is possible), can selecting a different reference node and reformulating the system result in a full-rank conductance matrix?
  2. Determine whether the conductance matrix, as defined using the adopted convention, is positive semi-definite or negative semi-definite.
  3. What does the expression $\mathbf{v}^T\mathbf{G}\mathbf{v}$ represent?
  4. Let $s = \mathbf{v}^T\mathbf{G}\mathbf{v}$. Is $s = 0$ possible? Justify your answer.

- **Node analysis for circuits with voltage sources**: For circuits containing voltage sources, the methodology depends on the nodes between which the voltage source is connected. If one end of the voltage source is connected to a reference node, the potential at the other end can be computed using the known value of the

[9] In this course, you are free to choose matrix-based methods or non-matrix approaches to solve system of linear equations.

[10] These observations are not limited to the specific example under study.

[11] Concepts from other courses, particularly linear algebra, will offer valuable insights for addressing these questions.

voltage source. The branch current through the voltage source can be computed by applying KCL at the connected nodes. The voltages of all the other unknwn nodes can be obtained by applying node analysis as described earlier.

On the other hand, if the voltage source is connected between two non-reference nodes, the node analysis technique must be adjusted to account for the voltage source. In such cases, we can adopt one of the following two approaches:

1. Introduce a new variable: One simple (though not necessarily the most efficient) approach is to define a new variable representing the branch current through the voltage source. This requires an addition equations to make the set of equations consistent. The additional equation can be obtained from the known potential difference across the voltage source. The resulting set of equations can then be solved to determine the unknown node potentials and the branch current through the voltage source.

   For illustration, consider the circuit shown in Fig. 3, where there are n nodes with unknown voltages and a voltage source between nodes a and d. Application of KCL at all nodes (as mentioned earlier) except a and d gives $n-2$ equations.

   However, at nodes a and d, the branch current through the voltage source is unknown. To resolve this, introduce a new variable i_x to represent the branch current through the voltage source. Using this variable, the KCL equations at nodes a and b can be written as

$$\text{at Node a}: \underbrace{i_{ad}}_{-i_x} + i_{ab} + i_{ac} + i_{ae} = 0$$
$$\text{at Node d}: \underbrace{i_{da}}_{i_x} + i_{dg} + i_{df} = 0$$

   The above equations together with the equations obtained from the other nodes (except a and d) form a set of n equations with $n+1$ unknowns (the node potentials and the branch current through the voltage source). An additional equation is required to complete the system. This equation comes from the known potential difference across the voltage source

$$v_a - v_d = v_s$$

   Now, with $n+1$ equations, we can solve for the unknown node potentials and the branch current through the voltage source.

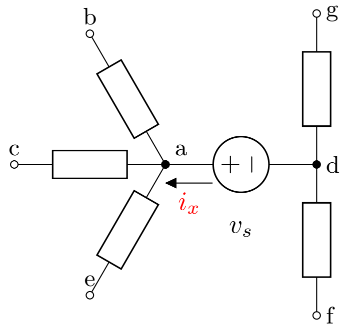

Figure 3: Circuit with voltage source

2. Supernode analysis: An alternative approach is to use the concept of a **supernode**, a fictitious node that treats the two nodes connected by a voltage source as a single entity. In this framework, the unknown branch current through the voltage source can be cleverly eliminated by adding the expressions obtained from applying Kirchhoff's Current Law (KCL) at either end of

the voltage source. This results in an equation containing the currents leaving all branches connected to either end of the voltage source, except the current through the voltage source itself. Thus, the two ends can be combined into a fictitious node, and an equation can be written to represent the currents associated with the supernode.

The second equation is derived from the potential difference (PD) across the voltage source, which relates the voltages at the two ends of the supernode. Together, these two equations i.e., the one relating currents at the supernode and the one from the potential difference, complete the system of equations needed to solve for the unknown node voltages in the circuit.

The basis of supernode analysis can be illustrated using the circuit shown in Fig. 3. By applying KCL at the two ends of the voltage source (nodes a and d), the resulting equations can be combined to eliminate the unknown current through the voltage source. This leads to the following equation

$$\underbrace{i_{ad}}_{-i_x} + i_{ab} + i_{ac} + i_{ae} + i_{dg} + i_{df} + \underbrace{i_{da}}_{i_x} + i_{dg} + i_{df} = 0$$

$$\implies i_{ab} + i_{ac} + i_{ae} + i_{df} + i_{dg} = 0$$

The above equation is referred to as current equation of the supernode formed by nodes a and d. The current equation for the supernode, along with the equation for the potential difference (i.e., $v_a - v_d = v_s$) provides the two equations necessary to formulate the system of equations.

- Example: Analyze the circuit shown in Fig. 4 using node analysis.

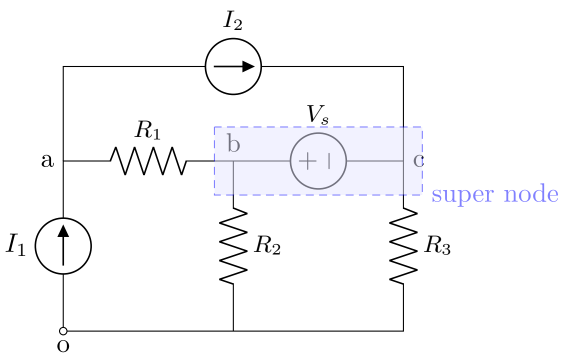

Figure 4: Circuit with voltage source between non-reference nodes

Solution: Application of KCL at node a gives

$$\underbrace{I_{ao}}_{-I_1} + I_{ab} + \underbrace{I_{ac}}_{I_2} = 0 \implies -I_1 + \left(\frac{V_a - V_b}{R_1}\right) + I_2 = 0$$

$$\text{i.e., } \frac{1}{R_1}V_a - \frac{1}{R_1}V_b = I_1 - I_2$$

The nodes b and c constitute the supernode. The current equation for the supernode (to recall, obtained by summing the currents leaving all branches connected to nodes b and c except the current through the voltage souce) is given by

$$I_{ba} + I_{bo} + I_{co} + I_{ca} = 0 \implies \left(\frac{V_b - V_a}{R_1}\right) + \frac{V_b}{R_2} + \frac{V_c}{R_3} - I_2 = 0$$
$$\text{i.e.,}\ -\frac{1}{R_1}V_a + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)V_b + \frac{1}{R_3}V_c = I_2$$

The equation that relates the potentials at either ends of the supernode is given by

$$V_b - V_c = V_s$$

The unknown node potentials can be computed by solving the resulting set of equations. A matrix-based representation of the resulting system of linear equations is

$$\begin{bmatrix} \frac{1}{R_1} & -\frac{1}{R_1} & 0 \\ -\frac{1}{R_1} & \left(\frac{1}{R_1} + \frac{1}{R_2}\right) & \frac{1}{R_3} \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} I_1 - I_2 \\ I_2 \\ V_s \end{bmatrix}$$

- **Questions for Further Thought**

  1. Let $\mathbf{G}$ denote the matrix associated with the system of linear equations obtained from supernode analysis.

     (a) Can $\mathbf{G}$ become singular? If so, under what conditions?.

     (b) Is $\mathbf{G}$ positive semi-definite (PSD)? Justify your answer.