EE2100: Matrix Analysis

Review Notes - 28

Topics covered :

1. Eigen Values and Eigen Vectors

---

1. **Motivation for Eigen Values and Eigen Vectors**: Consider a system of equations of the form

$$\mathbf{x}_k = \mathbf{A}^k \mathbf{x}_0 \tag{1}$$

where $\mathbf{x}_0 \in \mathcal{R}^n$ and $\mathbf{A} \in \mathcal{R}^{n \times n}$. In terms of linear transformations, computing $\mathbf{x}_k$ (often referred to as computing the output in this notes) can be thought of as applying k sequential linear transformations to $\mathbf{x}_0$. Equations of the form (1) appear quite frequently in the several domains. The aim of the analysis to follow is to provide some insight into the concept of Eigen values and Eigen Vectors.

(a) Let $\mathbf{x}_0$ be such that $\mathbf{A}\mathbf{x}_0 = \lambda \mathbf{x}_0$ (i.e., the output of the linear transformation applied to $\mathbf{x}_0$ results in an output that is in the same direction as $\mathbf{x}_0$). In such a scenario, it can be shown that the output $\mathbf{x}_k$ can be computed as

$$\mathbf{x}_k = \lambda^k \mathbf{x}_0 \tag{2}$$

Equation 2 indicates that, if the input $\mathbf{x}_0$ is such that $\mathbf{A}\mathbf{x}_0 = \lambda \mathbf{x}_0$, then, the output $\mathbf{x}_k$ can be computed directly using the value of λ (thereby avoiding the need to raise the matrix to a power). Further, based on the values of λ, it is possible to predict the behavior of $\mathbf{x}_k$ as $k \to \infty$. For example, if $|\lambda| < 1$, then $\mathbf{x}_k \to \mathbf{0}$ as $k \to \infty$.

Although, the forgoing analysis indicates that computing the solution and understanding its behavior is much easier if the input follows the condition $\mathbf{A}\mathbf{x}_0 = \lambda \mathbf{x}_0$, it should be kept in mind that not all inputs satisfy the condition. The analysis to follow considers this scenario.

(b) Let $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ denote a collection of linearly independent vectors that satisfy the condition $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ (at a later point, it will be shown that there can be a maximum of n vectors that satisfy this condition). Since $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ satisfy the $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$, it is known that

$$\mathbf{A}^k \mathbf{v}_i = \lambda_i^k \mathbf{v}_i \tag{3}$$

Further, since $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$ are linearly independent, it is possible to express any $\mathbf{x}_0$ as

$$\mathbf{x}_0 = \sum_{i=1}^{n} \alpha_i \mathbf{v}_i \tag{4}$$

Accordingly, the output (i.e., $\mathbf{x}_k$) can be computed as

$$\mathbf{x}_k = \mathbf{A}^k \left(\sum_{i=1}^{n} \alpha_i \mathbf{v}_i \right) = \sum_{i=1}^{n} \alpha_i \mathbf{A}^k \mathbf{v}_i = \sum_{i=1}^{n} \alpha_i \lambda_i^k \mathbf{v}_i \tag{5}$$

Similar to the previous scenario, it is possible to compute the output in a simpler manner, if there are n linearly independent vectors that satisfy the equation $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Further, it is possible to predict the nature of $\mathbf{x}_k$ as $k \to \infty$ using the values of λ. It is very interesting to note that, if we define a matrix $\mathbf{V}$ whose column vectors are $(\mathbf{v}_1, \cdots, \mathbf{v}_n)$, then

$$\mathbf{AV} = \mathbf{A}[\mathbf{v}_1, \cdots, \mathbf{v}_n] = [\mathbf{A}\mathbf{v}_1, \cdots, \mathbf{A}\mathbf{v}_n] = [\lambda_1 \mathbf{v}_1, \cdots, \lambda_n \mathbf{v}_n] = \mathbf{VD} \tag{6}$$

where $\mathbf{D}$ is a diagonal matrix whose diagonal entries are given by $D_{ii} = \lambda_i$. Using 6, it can be concluded that any square matrix can be written as

$$\mathbf{A} = \mathbf{VDV}^{-1} \tag{7}$$

if there are n linearly independent vectors that satisfy the equation $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$. Alternatively, it is possible to diagonalize a square matrix using

$$\mathbf{V}^{-1}\mathbf{AV} = \mathbf{D} \tag{8}$$

if there are n linearly independent vectors that satisfy the equation $\mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i$.

2. Eigen Value/s and Eigen Vector/s: Let $\mathbf{A} \in \mathcal{R}^{n \times n}$. The value λ is called an Eigen value of $\mathbf{A}$, if there is some non-zero $\mathbf{x} \in \mathcal{R}^n (\neq \mathbf{0})$ such that $\mathbf{Ax} = \lambda \mathbf{x}$. The vector $\mathbf{x}$ that satisfies $\mathbf{Ax} = \lambda \mathbf{x}$ is called the Eigen vector associated with the Eigen value λ. Eigen vectors are those vectors that do not change their direction even when the linear transformation associated with $\mathbf{A}$ is applied to them.

3. If $\mathbf{v}$ is the Eigen vector of $\mathbf{A}$, then $\alpha \mathbf{v}$, where $\alpha \mathcal{R}$ is also an Eigen value (with the same Eigen value) of $\mathbf{A}$. For the purpose of analysis, out of all possible Eigen vectors associated with a given Eigen value i.e., $\alpha \mathbf{v}$, we often consider the vector whose norm is 1 as the Eigen vector associated with the Eigen value.

4. Eigen Space: If λ is an Eigen value of $\mathbf{A}$, then, the collection of Eigen vectors associated with the λ is called as the Eigen space associated with Eigen value λ. Eigen space can be be represented as the **Span** of linearly independent vectors associated with a given Eigen value. If consider $\mathbf{0}$ in the span, it can be shown that the Eigen space is a Subspace.

5. **Examples**: The Eigen value associated with a zero matrix $\mathbf{0} \in \mathcal{R}^{n \times n}$ is 0 and all $\mathbf{x} \in \mathcal{R}^n$ are the Eigen Vectors. Similarly, the Eigen value associated with an Identity matrix $\mathbf{I} \in \mathcal{R}^{n \times n}$ is 1 and all $\mathbf{x} \in \mathcal{R}^n$ are the Eigen Vectors.