

$$\textcircled{1} \quad f_{x,y}(x,y) = 1 \text{ for } 0 \leq x \leq 1, 0 \leq y \leq 1$$

$$\textcircled{a} \quad u = x, \quad v = x + y.$$

$$f_{u,v}(u,v) = \frac{f_{x,y}(u, v-u)}{|J(x,y)|} \quad \begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq v \leq 2 \end{array}$$

$$|J(x,y)| = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1$$

$$\Rightarrow f_{u,v}(u,v) = f_{x,y}(u, v-u) \quad \begin{array}{l} 0 \leq u \leq 1 \\ 0 \leq v \leq 2 \end{array}$$

$$= \begin{cases} 1 & \begin{array}{l} 0 \leq u \leq 1, 0 \leq v \leq 2 \\ \text{and } u \leq v \leq 1+u. \end{array} \\ 0 & \text{otherwise} \end{cases}$$

$$\textcircled{b} \quad f_v(v) = \int_{-\infty}^{\infty} f_{u,v}(u,v) du.$$

$$= \int_{\max\{0, v-1\}}^{\min\{v, 1\}} 1 du. \quad \text{for } 0 \leq v \leq 2$$

$$= \min\{v, 1\} - \max\{0, v-1\}.$$

$$= \begin{cases} v & 0 \leq v \leq 1 \\ 2-v & 1 \leq v \leq 2 \end{cases}$$

$$\textcircled{2} \quad S_j = \sum_{i=1}^j Z_i, \quad S_j \sim N(0, j). \quad \forall j = 1, 2, \dots, n.$$

$$\text{Let } \hat{S}_j = S_n - S_j.$$

$$= \sum_{i=j+1}^n Z_i. \quad \text{Then } \hat{S}_j \sim N(0, n-j).$$

Clear to see that \hat{S}_j is independent of S_j .

$$f_{S_j, \hat{S}_j}(s, \hat{s}) = f_{\hat{S}_j}(\hat{s}) f_{S_j}(s).$$

$$(S_j, \hat{S}_j) \iff (S_j, s_n).$$

$s_n = S_j + \hat{S}_j$ (exactly like $q_n^m - 1$).

$$f_{S_j, s_n}(s, s') = f_{S_j, \hat{S}_j}(s, s' - s).$$

$$f_{S_n | S_k}(s|y) = \frac{f_{S_k, s_n}(y, s)}{f_{S_k}(y)}.$$

$$= \frac{f_{S_k, \hat{S}_k}(y, s-y)}{f_{S_k}(y)}$$

due to
independence. \leftarrow

$$= \frac{f_{\hat{S}_k}(s-y) f_{S_k}(y)}{f_{S_k}(y)}$$

$$= \frac{1}{\sqrt{2\pi(n-k)}} e^{-\frac{(s-y)^2}{2(n-k)}}$$

$$\textcircled{b} \quad f_{S_k | S_n}(s|x) = \frac{f_{S_k, s_n}(s, x)}{f_{S_n}(x)}$$

$$= \frac{f_{S_k, \hat{S}_k}(s, x-s)}{f_{S_n}(x)}$$

$$= \frac{\frac{1}{\sqrt{2\pi k}} e^{-\frac{s^2}{k}} \frac{1}{\sqrt{2\pi(n-k)}} e^{-\frac{(x-s)^2}{n-k}}}{\frac{1}{\sqrt{2\pi n}} e^{-\frac{x^2}{n}}} \quad \text{for } k < n$$

$$\text{for } k=n \quad f_{S_n|S_n}(s|x) = \begin{cases} 1 & s=x \\ 0 & \text{otherwise} \end{cases}$$

③ $X_{(1)}, X_{(2)}, X_{(3)}$ ordered statistics of X_1, X_2, X_3 .

$$\begin{aligned} \text{Let } 0 \leq x_1 < x_2 \leq 1 \quad f_{X_{(1)}, X_{(2)}}(x_1, x_2) &= \int_{-\infty}^{\infty} f_{X_1, X_2, X_3}(x_1, x_2, x_3) dx_3 \\ &= \int_{x_2}^1 3! \cdot \prod_{i=1}^3 f(x_i) dx_3 \\ &= \int_{x_2}^1 6 dx_3 \\ &= 6(1-x_2). \end{aligned}$$

$$\Rightarrow f_{X_{(1)}, X_{(2)}}(x_1, x_2) = \begin{cases} 6(1-x_2) & 0 \leq x_1 < x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

⑥ $P(X_{(2)} \geq X_{(1)} + d, X_{(3)} \geq X_{(2)} + d).$

$$\begin{aligned} &= \int_0^{1-d} \int_{x_1+d}^{1-d} \int_{x_2+d}^1 f_{X_1, X_2, X_3}(x_1, x_2, x_3) dx_3 dx_2 dx_1 \\ &= \int_0^{1-d} \int_{x_1+d}^{1-d} \int_{x_2+d}^1 3! dx_3 dx_2 dx_1 \\ &= \int_0^{1-d} \int_{x_1+d}^{1-d} 3! (1-x_2-d) dx_2 dx_1 \\ &= \int_0^{1-d} 3! \cdot \frac{-(1-x_2-d)^2}{2} \Big|_{x_1+d}^{1-d} dx_1 \\ &= \frac{-\cancel{(1-x_1-2d)^3}}{\cancel{3!}} \Big|_0^{1-2d} = (1-2d)^3. \end{aligned}$$

④

The PMF defined by

$$P_X(i) = \frac{1}{2^i} \quad \text{exactly results in MGIF seen in the qn}$$

It is also unique due to the inversion property.

X is a Geometric ($\frac{1}{2}$) R.V.

⑤

$$M_X(s) = E[e^{Xs}]$$

$$= \int_0^1 f_p(p) E[e^{Xs} | P=p] dp.$$

$$= \int_0^1 E[e^{Xs} | P=p] dp.$$

$X | P=p \sim \text{Binomial}(n, p)$

$$\Rightarrow E[e^{Xs} | P=p] = (1-p+pe^s)^n.$$

$$\Rightarrow M_X(s) = \int_0^1 (1-p+pe^s)^n dp.$$

$$\text{let } x = (1-p+pe^s).$$

$$dx = dp (e^s - 1).$$

$$\Rightarrow M_X(s) = \int_1^{e^s} \frac{x^n}{e^s - 1} dx$$

$$= \left[\frac{x^{n+1}}{(n+1)(e^s - 1)} \right]_1^{e^s}$$

$$= \frac{1}{n+1} \frac{(e^s)^{n+1} - 1}{e^s - 1}$$

$$= \frac{1}{n+1} \left[e^{ns} + e^{(n-1)s} + \dots + 1 \right].$$

⑥

$$P_X(x) = \frac{1}{n+1} \quad \forall x \in \{0, 1, \dots, n\}$$

is the PMF that corresponds to the MGF
seen above.