EE1080/AI1110/EE2102 Probability: HW1

13 January 2025

1. A six-sided die is loaded in a way that each even face is twice as likely as each odd face. All even faces are equally likely as are all the odd faces. Construct a probability model (define sample, event spaces and probability function) for a single roll of this die and find the probability that the outcome is less than 4.

2. Use union-bound for union of finite number of sets ($P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i)$) to show:

   (a) following lower bound on probability of intersection of events

   $$P(A_1 \cap A_2 \cdots \cap A_n) \geq P(A_1) + P(A_2) + \cdots + P(A_n) - (n-1)$$

   (b) union-bound for union of countably-infinite number of sets: *(Hint: Use continuity property)*

   $$P(\cap_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

3. Suppose A and B are events with very high probability: say $P(A) = 0.9$ and $P(B) = 0.9$. Then which of the following are true (check all that apply)

   (a) Either A or B occurs with probability at least 0.9

   (b) If A occurs, then B does not occur with probability at most 0.12

   (c) Both A and B occur with probability 0.81

   (d) Both A and B occur with probability at most 0.9

   (e) If A does not occur, then B also does not occur with probability at least 0.5

   (f) Both A and B occur with probability at least 0.8

4. (a) Mr. Jones has two children. The older child is a girl. What is the probability that both children are girls?
   (b) Mr. Smith has two children. At least one of them is a boy. What is the probability that both children are boys? Be sure to carefully justify your answers.

5. Given events A and B that are independent of each other. Show that
   (a) A and B^c are independent.
   (b) A^c and B^c are independent.
6. Use total probability theorem to show that:
   $$P(A) = P(A \cap B) + P(A \cap C) + P(A \cap B^c \cap C^c) - P(A \cap B \cap C).$$
7. Suppose an experiment involves picking a pair of whole natural numbers. The probability of picking (k, m) is $p(k, m) = p^c(1-p)^{k+m}$.
   (a) Find the value of c for which this is a valid probability assignment.
   (b) Assume p = 0.25 for the rest of the questions. Find the probability that k is equal to m
   (c) Find the probability that k is at least as large as m.
   (d) The probability that k is odd is . . ..
   (e) Are the events: A : k is even, B : m is even independent?
8. A coin is tossed twice. Alice claims that the event of two heads is at least as likely if we know that the first toss is head than if we know that at least one of the tosses is a head. What should be bias (probability of head) of the coin for Alice to be right ?
9. Suppose $A\Delta B$ denotes the symmetric difference of A and B i.e., $A\Delta B = (A \cup B) \setminus (A \cap B)$. Then show that $P(A\Delta B) \leq P(A\Delta C) + P(C\Delta B)$.
10. A medical test for a certain disease has a sensitivity of 90% and a specificity of 95%. That is, if a person has the disease, the test will correctly identify the person as positive 90% of the time, and if a person does not have the disease, the test will correctly identify the person as negative 95% of the time. Suppose that 1% of the population has the disease. If a person tests positive for the disease, what is the probability that the person actually has the disease?
11. Alice searches for her term paper in her filing cabinet, which has several drawers. She knows that she left her term paper in drawer j with probability $p_j > 0$. The drawers are so messy that even if she correctly guesses that the term paper is in drawer i, the probability of her finding it is only d_i. Alice searches in a particular drawer say i but the search is unsuccessful. Conditioned on this event what is the probability that her paper is in drawer j, where $j \neq i$?
12. Each of k jars contains m white and n black balls. A ball is randomly chosen from jar 1 and transferred to jar 2, then a ball is randomly chosen from jar 2 and transferred to jar 3, etc. Finally, a ball is randomly chosen from jar k. Show that the probability that the last ball is white is the same as the probability that the first ball is white, i.e., it is $m/(m+n)$.

13. Let A and B be events with $P(A) > 0$ and $P(B) > 0$. We say that an event B suggests an event A if $P(A|B) > P(A)$, and does not suggest event A if $P(A|B) < P(A)$.

    (a) Show that B suggests A if and only if A suggests B.

    (b) Assume that $P(B^c) > 0$. Show that B suggests A if and only if B^c does not suggest A.

    (c) We know that a treasure is located in one of two places, with probabilities β and $1 - \beta$, respectively, where $0 < \beta < 1$. We search the first place and if the treasure is there, we find it with probability $p > 0$. Show that the event of not finding the treasure in the first place suggests that the treasure is in the second place.

14. Consider a coint that comes up with heads with probability p and tails with probability $1-p$. Let q_n be the probability that after n independent tosses, there have been an even number of heads. Derive a recursion that relates q_n to q_{n-1} and solve it to establish the formula:

$$q_n = (1 + (1 - 2p)^n)/2$$

15. Eight rooks are placed in distinct square of an 8×8 chessboard, with all possible placements being equally likely. Find the probability that all the rooks are safe from one another i.e, there is no row of column with more than one rook.

16. Birthday problem:

    (a) Find the probability that a collection of 10 people have birthdays on distinct days of an year.

    (b) How many people do you need approximately to have atleast two people with the same birthday ? (use the fact that $1 - x \leq e^{-x}$)