Lec 7 : 20th Jan

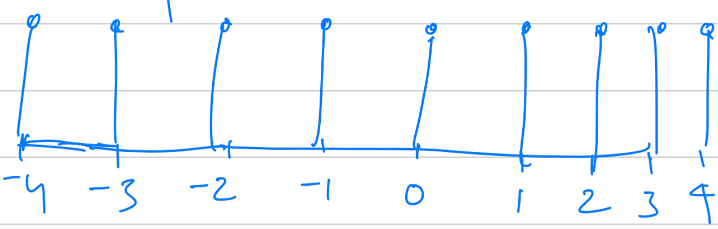
Function of a R.V.

$$Y = g(X).$$

Example: X be a R.V taking values in $\mathcal{X} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$

$$P_X(x) = P(X = x) = \frac{1}{9} \quad \forall x \in \mathcal{X}$$

$$Y = |X|.$$



What is the PMF of Y.

$$\mathcal{Y} = \{0, 1, 2, 3, 4\}$$

$$P_Y(y) = P(Y = y)$$

$$= \begin{cases} P_X(0) = 1/9 & y = 0 \\ P_X(1) + P_X(-1) = 2/9 & y = 1 \\ 2/9 & y = 2 \\ \vdots & \vdots \end{cases}$$

Expectation of a Random Variable.

Consider that you are running N number of times and let's say you observe that the random variable is taking value x, n_x number of times where $x \in \mathcal{X}$

Made with Goodnotes

$$\sum_{x \in \mathcal{X}_0} n_x = N.$$

Suppose you are interested in average value of these outcomes

$$\frac{\sum_{x \in \mathcal{X}_0} x \cdot n_x}{N} = \sum_{x \in \mathcal{X}_0} x \cdot \underbrace{\left(\frac{n_x}{N}\right)}_{\text{relative frequency.}}$$

$E[X]$ is defined as

$$E[X] = \sum_{x \in \mathcal{X}_0} x \, P_X(x)$$

We consider that $E[X]$ is well defined if absolute value converges i.e.,

$$\boxed{E[|X|] < \infty \quad \text{ie,} \quad \sum_{x \in \mathcal{X}_0} |x| \, P_X(x) < \infty}$$

Examples:

① $\mathcal{X}_0$ is finite

$E[|X|]$ is also bounded in finite.

$$E[X] = \sum_{x \in \mathcal{X}_0} x \, P_X(x)$$

$$\leq \sum_{x \in \mathcal{X}_0} x_{max} \, P_X(x) \qquad \max \mathcal{X}_0 = x_{max}$$

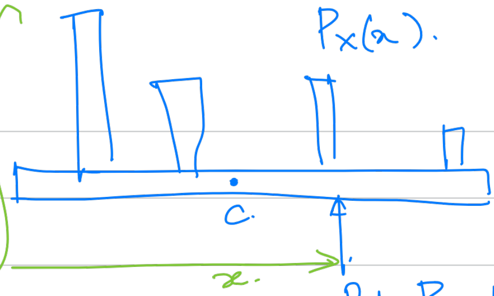
$$\leq x_{max}.$$

Can similarly show $E[X] \geq x_{min}$.

② $\mathcal{X}$ is infinite.

Consider a R.V taking values $\{2, 4, 8, 16, \dots\} = \mathcal{X}$

$$P(X = 2^n) = \frac{1}{2^n}.$$



Ref: Bertsekas

Can be thought centre of gravity of ab.

$$\sum_{x \in \mathcal{X}} (x - c) P_X(x) = 0.$$

Right & left torques balance.

i.e., $\sum_{x : \mathcal{X}, x > c} (x - c) P_X(x) = \sum_{x \in \mathcal{X}, x < c} (c - x) P_X(x)$.

$$\sum_{n=1}^{\infty} P(X = 2^n) = \sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2} + \frac{1}{2^2} + \dots$$

$$= 1.$$

$$E[X] = \sum_{n=1}^{\infty} 2^n P(X = 2^n).$$

$$= \sum_{n=1}^{\infty} 2^n \frac{1}{2^n} \quad \text{not finite.}$$

$$S_m = \sum_{n=1}^{m} 1$$

1 2 3 4 5 ....

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③ R.V X takes values $\{(-2)^n \mid n = 1, 2, \dots\}$

$$= \{-2, 4, -8, 16, \dots\}$$

with probability

$$P(X = (-2)^n) = \frac{1}{2^n}.$$

$$E[X] = \sum_{n=1}^{\infty} \frac{1}{2^n} (-2)^n. = \sum_{n=1}^{\infty} (-1)^n$$

$$S_m = \sum_{n=1}^{m} (-1)^n. = \begin{cases} 0 & m \text{ is even} \\ -1 & m \text{ is odd.} \end{cases}$$

$S_1 \quad S_2 \quad S_3 \quad S_4 \quad S_5 \quad \ldots$

$0 \quad -1 \quad 0 \quad -1 \quad 0 \quad -1$

Properties of Expectation

① Let $Y = g(X)$ then

$$E[Y] = \sum_{y \in \mathcal{Y}} P_Y(y)\, y.$$

$$= \sum_{y \in \mathcal{Y}} y \sum_{x : g(x) = y} P_X(x).$$

$$= \sum_{y \in \mathcal{Y}} \sum_{x : g(x) = y} y\, P_X(x).$$

$$= \sum_{y \in \mathcal{Y}} \sum_{x : g(x) = y} g(x)\, P_X(x).$$

$$= \sum_{x \in \mathcal{X}} g(x)\, P_X(x)$$

$y=0 \qquad \overbrace{x : g(x) = 0}^{A_0}$

$y=1 \qquad \underbrace{x : g(x) = 1}_{A_1}$

$\vdots$

$$\bigcup_{y \in \mathcal{Y}} A_y = \mathcal{X}$$

$$\boxed{E[Y] = E[g(X)] = \sum_{x \in \mathcal{X}} g(x)\, P_X(x)}$$

Moments of a Random Variable

m-th moment of R.V X is defined as

$$E[X^m] = \sum_{x \in \mathcal{X}} x^m P_X(x).$$

Variance of random variable.

indicates how far R.V is from its mean.

$$Var(X) = E\left[(X - E[X])^2\right].$$

$$= \sum_{x \in \mathcal{X}} (x - E[x])^2 P_X(x)$$

② $$Var(X) = E[X^2] - (E[X])^2.$$

$$Var(X) = \sum_{x \in \mathcal{X}} \left(x^2 + (E[X])^2 - 2x E[X]\right) P_X(x)$$

$$= \underbrace{\sum_{x \in \mathcal{X}} x^2 P_X(x)}_{E[X^2]} + \underbrace{\sum_{x \in \mathcal{X}} (E[X])^2 P_X(x)}_{=(E[X])^2} - \underbrace{\sum_{x \in \mathcal{X}} 2x E[X] P_X(x)}_{2E[X] \sum_{x \in \mathcal{X}} x P_X(x) = 2(E[X])^2}$$

$$= E[X^2] + (E[X])^2 - 2(E[X])^2$$

$$\boxed{Var(X) = E[X^2] - (E[X])^2.}$$

③ Let $Y = aX + b$.

To show $E[Y] = a\,E[X] + b$

$$= E[aX+b]$$

$$= \sum_{x \in \mathcal{X}} (ax+b)\, P_X(x)$$

$$= a\,E[X] + b.$$

④ $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$.

$\mathrm{Var}(Y)$

$$= E[(Y - E[Y])^2] = E[(aX + \cancel{b} - (aE[X] + \cancel{b}))^2]$$

$$= E[a^2 (X - E[X])^2].$$

$$= a^2 \sum_{x \in \mathcal{X}} (x - E[X])^2 P_X(x)$$

$$= a^2\,\mathrm{Var}(X).$$

Example random variables

① **Bernoulli (p).** $\quad \mathcal{X} = \{0, 1\}$

$$P(X = x) = \begin{cases} p & x = 1 \\ 1-p & x = 0. \end{cases}$$

$$E[X] = p \times 1 + (1-p) \times 0.$$
$$= p.$$

$$\mathrm{Var}[x] = E[x^2] - (E[x])^2$$

$$E[x^2] = p \cdot 1^2 + (1-p) \cdot 0^2$$
$$= p.$$

$$\mathrm{Var}(X) = p - p^2 = p(1-p).$$

$p = \frac{1}{2}$ it is maximum.