EE2100: Matrix Analysis

Review Notes - 32

Topics covered :

1. QR Decomposition

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1. Consider a matrix $\mathbf{A} \in \mathcal{R}^{m\times n}$. The column vectors of $\mathbf{A}$ are denoted by $\mathbf{a}_i \in \mathcal{R}^m$ where $i \in (1, \cdots, n)$. Using the Grahm-Schmidt Algorithm, it is possible to generate orthogonal vectors from $[\mathbf{a}_1, \cdots, \mathbf{a}_n]$. If $\mathbf{u}_1, \cdots, \mathbf{u}_n$ and $\mathbf{q}_1, \cdots, \mathbf{q}_n$ denote the orthogonal vectors and orthonormal vectors generated by Grahm-Schmidt respectively, then

$$
\begin{aligned}
\mathbf{u}_1 &= \mathbf{a}_1 \text{ and } \mathbf{q}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|} \\
\mathbf{u}_2 &= \mathbf{a}_2 - \mathbf{Proj}_{\mathbf{q}_1}\mathbf{a}_2 \text{ i.e., } \mathbf{u}_2 = \mathbf{a}_2 - (\mathbf{a}_2 \cdot \mathbf{q}_1)\,\mathbf{q}_1 \text{ and } \mathbf{q}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} \\
&\vdots \qquad \vdots \\
\mathbf{u}_k &= \mathbf{a}_k - \sum_{i=1}^{k-1}\mathbf{Proj}_{\mathbf{q}_i}\mathbf{a}_k \text{ i.e., } \mathbf{u}_k = \mathbf{a}_k - \sum_{i=1}^{k-1}(\mathbf{a}_k \cdot \mathbf{q}_i)\,\mathbf{q}_i \text{ and } \mathbf{q}_k = \frac{\mathbf{u}_k}{\|\mathbf{u}_k\|}
\end{aligned} \tag{1}
$$

It is interesting to note that the column vectors $\mathbf{a}_1, \cdots, \mathbf{a}_n$ can be represented in terms of the orthonormal vectors $\mathbf{q}_1, \cdots, \mathbf{q}_n$ as

$$
\begin{aligned}
\mathbf{a}_1 &= \|\mathbf{u}_1\|\mathbf{q}_1 \\
\mathbf{a}_2 &= (\mathbf{a}_2 \cdot \mathbf{q}_1)\,\mathbf{q}_1 + \|\mathbf{u}_2\|\mathbf{q}_2 \\
\vdots &= \vdots \\
\mathbf{a}_k &= \sum_{i=1}^{k-1}(\mathbf{a}_k \cdot \mathbf{q}_i)\,\mathbf{q}_i + \|\mathbf{u}_k\|\mathbf{q}_k
\end{aligned} \tag{2}
$$

Equation (2) indicates that $\mathbf{A}$ (which is a collection of column vectors $[\mathbf{a}_1 \cdots \mathbf{a}_n]$) can be represented as

$$\mathbf{A} = \mathbf{QR} \tag{3}$$

where $\mathbf{Q} = [\mathbf{q}_1 \cdots \mathbf{q}_n]$ is an orthonormal matrix and the matrix $\mathbf{R}$ is an upper triangular matrix given by

$$
\mathbf{R} = \begin{bmatrix}
\|\mathbf{u}_1\| & \mathbf{a}_2\cdot\mathbf{q}_1 & \mathbf{a}_3\cdot\mathbf{q}_1 & \cdots & \mathbf{a}_k\cdot\mathbf{q}_1 & \cdots & \mathbf{a}_n\cdot\mathbf{q}_1 \\
0 & \|\mathbf{u}_2\| & \mathbf{a}_2\cdot\mathbf{q}_2 & \cdots & \mathbf{a}_k\cdot\mathbf{q}_2 & \cdots & \mathbf{a}_n\cdot\mathbf{q}_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \|\mathbf{u}_k\| & \cdots & \mathbf{a}_n\cdot\mathbf{q}_k \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \cdots & 0 & \|\mathbf{u}_n\|
\end{bmatrix} \tag{4}
$$

The entries of $\mathbf{R}$ are given by

$$R_{ij} = \begin{cases} \|\mathbf{u}_i\| & \text{when } i = j \\ \mathbf{a}_j \cdot \mathbf{q}_i & \text{when } j > i \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Thus, a rectangular matrix can be represented as $\mathbf{A} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthonormal matrix (and hence $\mathbf{Q}^{-1} = \mathbf{Q}^T$) and $\mathbf{R}$ is an upper triangular matrix. This is often referred to as QR Decomposition of a matrix (Note: There are other numerically stable algorithms for QR factorization. In this course, the idea of QR factorization is introduced through the Grahm-Schmidt approach).

2. Consider the scenario of solving $\mathbf{Ax} = \mathbf{b}$ where $\mathbf{A} \in \mathcal{R}^{m \times n}$, $\mathbf{x} \in \mathcal{R}^n$ and $\mathbf{b} \in \mathcal{R}^m$. The least-squares solution to such a system of linear equations is obtained by solving

$$\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b} \tag{6}$$

If $\mathbf{A}$ is represented as $\mathbf{QR}$, the solution can be obtained by solving

$$\mathbf{R}^T \underbrace{\mathbf{Q}^T\mathbf{Q}}_{\mathbf{I}} \mathbf{Rx} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b} \implies \mathbf{R}^T\mathbf{Rx} = \mathbf{R}^T\mathbf{Q}^T\mathbf{b} \tag{7}$$

Computing $\mathbf{x}$ using (7) is a preferred approach from the point of view of numerical stability (an aspect covered at a later point in the course).