① $Y \sim N(0,1)$

$X \mid Y=y \sim N(y, 1)$.

$$f_{X,Y}(x,y) = f_Y(y)\, f_{X|Y}(x|y).$$

$$= \frac{e^{-y^2/2}}{\sqrt{2\pi}} \; \frac{e^{-\frac{(x-y)^2}{2}}}{\sqrt{2\pi}}$$

$$f_{Y,Z}(y,z) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \; \frac{e^{-z^2/2}}{\sqrt{2\pi}}$$

Z, Y independent

$$(W = Y+Z,\; Z) \Longleftrightarrow (Y, Z)$$

$$W = g_1(Y, Z) = Y+Z$$
$$g_2(Z) = Z.$$

$$f_{W,Z}(w,z) = \frac{f_{Y,Z}(y,z)}{J(y,z)}.$$

st $w = y+z$.

$$J(y,z) = \left| \begin{matrix} \frac{\partial g_1(y,z)}{\partial y} & \frac{\partial g_1(y,z)}{\partial z} \\ \frac{\partial g_2(y,z)}{\partial y} & \frac{\partial g_2(y,z)}{\partial z} \end{matrix} \right| = \left| \begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix} \right|$$

$= 1$

$$\Rightarrow f_{W,Z}(w,z) = f_{Y,Z}(w-z, z).$$

$$= \frac{e^{-z^2/2}}{\sqrt{2\pi}} \; \frac{e^{-(w-z)^2/2}}{\sqrt{2\pi}}$$

$$= f_{X,Y}(w, z).$$

Made with Goodnotes

(b) $$\begin{pmatrix}X\\Y\end{pmatrix} = \underbrace{\begin{bmatrix}1&1\\0&1\end{bmatrix}}_{A}\begin{bmatrix}Z\\Y\end{bmatrix}.$$

as the joint distribution of X, Y is same as $Y+Z, Y$

$\Rightarrow$ $\begin{pmatrix}X\\Y\end{pmatrix}$ is Gaussian random vector / X, Y are jointly Gaussian

as we can write it as a matrix A multiplying standard normal iid Gaussian vector.

(c) $E[X] = E[E[X|Y]]$.

$E[X|Y=y] = y$ by defn. as $X|Y=y \sim N(y,1)$

$\Rightarrow$ $E[X] = E[E[X|Y]] = E[Y] = 0$.

$Var(X) = E[Var(X|Y)] + Var(E[X|Y])$

$= E[1] + Var(Y)$.

as $Var(X|Y=y) = 1$ by defn as $X|Y=y \sim N(y,1)$
$E[X|Y=y] = y$ " "

$\Rightarrow$ $Var(X) = 2$.

$Cov(X,Y) = E[XY] - \underbrace{E[X]E[Y]}_{0}$

$= E[E[XY|Y]]$.

$= E[Y\,E[X|Y]]$.

$= E[Y^2] = Var(Y) + (E[Y])^2 = 1$.

Made with Goodnotes

(e) $$f_{Y|X}(y|x) = \frac{f_{X|Y}(x|y)\, f_Y(y)}{f_X(x)}.$$

$X \sim N(0, 2)$.

(1) Since we have shown $\begin{pmatrix} X \\ Y \end{pmatrix}$ is jointly Gaussian we know X should be Gaussian too.

as $X = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$.

(2) The parameters of distribution are $E[X]$, $Var(X)$ determined earlier.

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-y)^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2}}{\frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{x^2}{2\cdot 2}}}$$

$$= \frac{1}{\sqrt{2\pi}\left(\frac{1}{\sqrt{2}}\right)} e^{-\left(y-\frac{x}{2}\right)^2}.$$

(f) $E[Y|X=x] = x/2$. from $f_{Y|X}(y|x)$ above.

Could've used the linear MMSE expression to compute this directly.

note that this notation is slightly different from notes as usually we find $E[X|Y=y]$ to estimate X.

$$E[Y|X=x] = \frac{Cov(X,Y)}{Var(X)}(x - E[X]) + E[Y].$$

$$= \frac{Cov(X,X)}{Var(X)}\, x = \frac{x}{2}.$$

(2). Will change notation of qn for ease to match with class notes.

Rephrased Qn:

Let
$$Y_1 = X + N_1 + \alpha N_2$$
$$Y_2 = X + 3N_1 + \alpha N_2.$$

(a) Find estimate for X from Y_1, Y_2.

Made with Goodnotes

Two ways: ① Using the formula we derived.

$$\hat{x}_{LMMSE}(y) = R_{XY} K_Y^{-1} y.$$

$$K_Y = \begin{bmatrix} Var(Y_1) & Cov(Y_1, Y_2) \\ Cov(Y_1, Y_2) & Var(Y_2) \end{bmatrix}.$$

$$R_{X,Y} = \begin{bmatrix} Cov(X, Y_1) & Cov(X, Y_2) \end{bmatrix}$$

$$Cov(X, Y_1) = E[XY_1] = E[X(X + N_1 + \alpha N_2)]$$

both have mean 0

$$= E[X^2 + N_1 X + \alpha N_2 X]$$

$$= E[X^2] + E[N_1] E[X] + \alpha E[N_2] E[X]$$

X, N_1, N_2 are independent

$$= E[X^2] = 1.$$

Similarly

$$Cov(X, Y_2) = E[X Y_2] = 1.$$

$$Var(Y_1) = E[Y_1^2] = E[(X + N_1 + \alpha N_2)^2]$$

rest of the terms vanish.

$$= E[X^2] + E[N_1^2] + \alpha^2 E[N_2^2].$$

$$= 2 + \alpha^2.$$

$$Var(Y_2) = E[(\cancel{X} + 3N_1 + \alpha N_2)^2].$$

$$= E[X^2] + 9E[N_1^2] + \alpha^2 E[N_2^2]$$

$$= 10 + \alpha^2.$$

Made with Goodnotes

$$\hat{x}_{LMMSE}(y_1, y_2) = R_{X,Y} K_Y^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2+\alpha^2 & 4+\alpha^2 \\ 4+\alpha^2 & 10+\alpha^2 \end{bmatrix}^{-1} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$Cov(Y_1, Y_2) = E[(X+3N_1+\alpha N_2)(X+N_1+\alpha N_2)]$$

all the other terms are 0

$$= E[X^2] + 3E[N_1^2] + \alpha^2 E[N_2^2].$$

$$= 4+\alpha^2.$$

$$\Rightarrow \hat{x}_{LMMSE}(y_1, y_2) = \frac{1}{4+4\alpha^2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 10+\alpha^2 & -(4+\alpha^2) \\ -(4+\alpha^2) & (2+\alpha^2) \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\det(K_y) = (2+\alpha^2)(10+\alpha^2) - (4+\alpha^2)^2.$$

$$20 + \cancel{\alpha^4} + 12\alpha^2 - 16 - \cancel{\alpha^4} - 8\alpha^2.$$

$$4 + 4\alpha^2.$$

$$= \frac{1}{4(1+\alpha^2)} \begin{bmatrix} 6 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \frac{(6y_1 - 2y_2)}{4(1+\alpha^2)}.$$

<u>MSE for LMMSE estimator</u>

$$E\left[X - \left(\frac{6Y_1 - 2Y_2}{4(1+\alpha^2)}\right)\right]^2.$$

$$6Y_1 - 2Y_2 = 6(X + N_1 + \alpha N_2).$$

$$- 2(X + 3N_1 + \alpha N_2)$$

$$= 4(X + \alpha N_2)$$

Made with Goodnotes

$$= E\left[\left(X - \frac{(X+\alpha N_2)}{(1+\alpha^2)}\right)^2\right].$$

$$= E\left[\left(X(1+\alpha^2) - X - \alpha N_2\right)^2\right] \cdot \frac{1}{(1+\alpha^2)^2}.$$

$$= E\left[\left(\alpha^2 X - \alpha N_2\right)^2\right] \frac{1}{(1+\alpha^2)^2}.$$

$$= \frac{\alpha^2}{(1+\alpha^2)^2} E\left[\left(\alpha X - N_2\right)^2\right].$$

$$= \frac{\alpha^2}{(1+\alpha^2)^{\cancel{2}}} \cancel{(\alpha^2+1)} = \frac{\alpha^2}{1+\alpha^2}.$$

(c) At $\alpha = 0$.

3. (a) $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

$X_1 = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{M_1} X.$

$E[X_1] = M_1 \mu = 1$ or The first element of μ.

$Var(X_1) = M_1 K_X M_1^T = 1.$

first element of K_X.

$$f_{X_1}(x_1) = \frac{e^{-\frac{(x_1-1)^2}{2}}}{\sqrt{2\pi}}$$

(b) $f_{X_2, X_3 | X_1}(x_2, x_3 | x_1) = \dfrac{f_{X_1, X_2, X_3}(x_1, x_2, x_3)}{f_{X_1}(x_1)}$

Made with Goodnotes

$$K_X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \qquad = \frac{\dfrac{e^{-\frac{(x_1\, x_2\, x_3)\, K^{-1} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}{2}}}{(2\pi)^{3/2} (\det(K))^{1/2}}}{\dfrac{e^{-x_1^2/2}}{\sqrt{2\pi}}}$$

$$\bar{X} = \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}$$

Refer to Appendix of Lec 37.

$$K_{\bar{X}, X_1} = \begin{pmatrix} \mathrm{Cov}(X_2, X_1) \\ \mathrm{Cov}(X_3, X_1) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

$$K_X = 1.$$

Can show

$$X_2\, X_3 \mid X_1 = x_1 \sim N(\hat{\mu}, R_{X_2, X_3}).$$

$$\hat{\mu} = R_{\bar{X}, X_1} K_{X_1}^{-1} x_1.$$

$$\bar{X} = \begin{pmatrix} X_2 \\ X_3 \end{pmatrix}.$$

$$\Rightarrow \hat{\mu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} x_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}.$$

$$R_{X_2, X_3} = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}.$$

$$\Rightarrow f_{X_2 X_3 \mid X_1}(x_2, x_3 \mid x_1) = \frac{e^{-\frac{1}{2}\left[\frac{(x_2 - x_1)^2}{4} + \frac{x_3^2}{9}\right]}}{(2\pi) \times 6}$$

(c) $Y = 2X_1 + X_2 + X_3$

$$= \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}.$$

$$\Rightarrow E[Y] = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} E[X] = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$$

$$= 2 + 5 + 2 = 9.$$

$$\Rightarrow \mathrm{Var}(Y) = E[YY^T] = E\left[\begin{pmatrix} 2 & 1 & 1 \end{pmatrix} X\, X^T \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]$$

Made with Goodnotes

$$= [2\ 1\ 1]\, K_X \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= [2\ 1\ 1] \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 6+6+9$$

$$= 21.$$

$$f_Y(y) = \frac{e^{-\frac{(y-9)^2}{2 \times 21}}}{\sqrt{2\pi \times 21}}$$

(4).

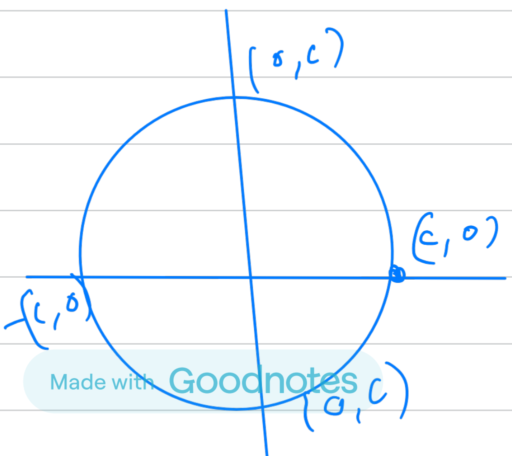
Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$$\frac{e^{-\frac{x^T K^{-1} x}{2}}}{2\pi (\det(K))^{1/2}} = \frac{0.8}{2\pi (\det(K))^{1/2}}$$

$$\Rightarrow \frac{x^T K^{-1} x}{2} = \log\left(\frac{1}{0.8}\right).$$

(a) $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = K^{-1}$

$$\Rightarrow x_1^2 + x_2^2 = 2\log\left(\frac{1}{0.8}\right) = c$$



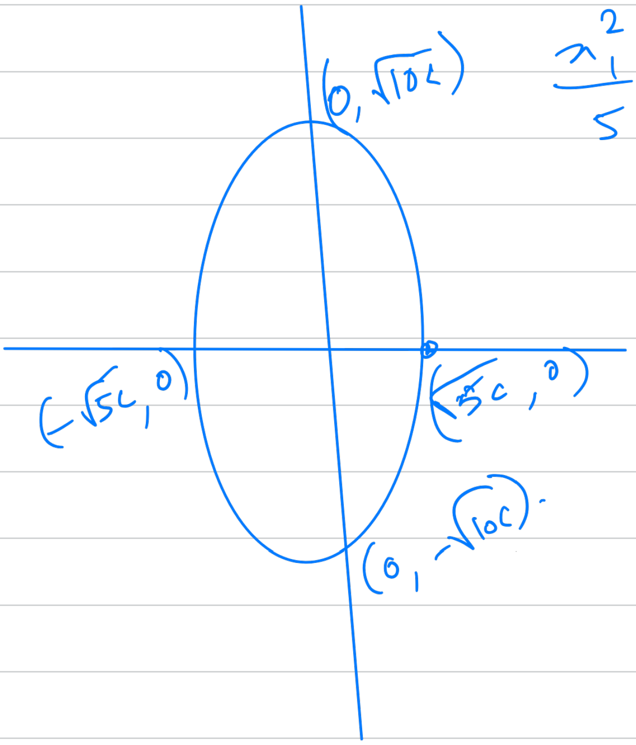
Made with Goodnotes

(b) $K = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$.

$$x^T K^{-1} x = c = 2\log\left(\frac{1}{0.8}\right).$$

$$[x_1 \ x_2]\begin{bmatrix} 1/5 & \\ & \frac{1}{10} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c$$

$$\frac{x_1^2}{5} + \frac{x_2^2}{10} = c.$$



(c) $K = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$,

To find eigen values

$$\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)^2 - 4 = 0$$

$$16 - 8\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 8\lambda + 12 = 0.$$

$D = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$.

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 6\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

$$4x_1 + 2x_2 = 6x_1.$$

$$\underline{x_2 = x_1}$$

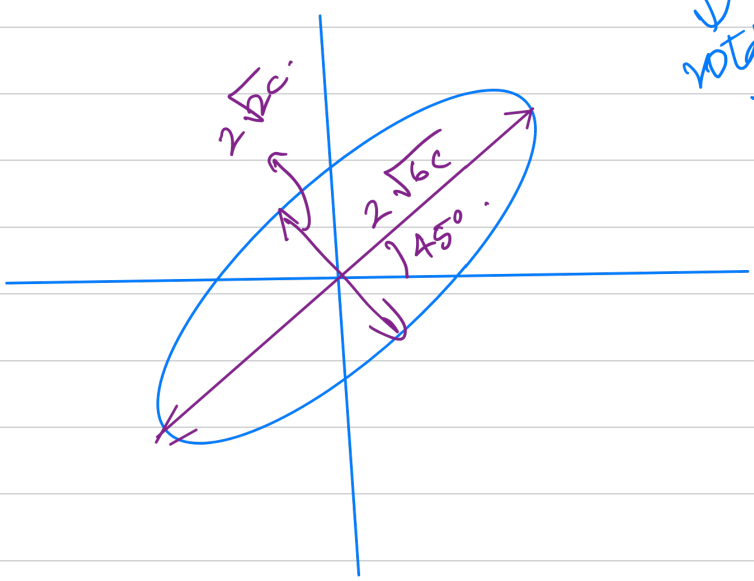
$$\underline{q_1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Made with Goodnotes

$$\Rightarrow \quad q_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ since it is orthogonal to q_1

$$K = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{Q} \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

$$\underbrace{x^T Q}\, D^{-1}\, \underbrace{Q^T x}_{\text{rotated vector}} = c.$$



(5) $K = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$ Nope. $\det(K) < 0$.

Can't be since K is p.s.d its $\det(K) \geq 0$

(6) $$\|z - y\|_2^2 = (z-y)^T (z-y).$$

$$\|Qz - Qy\|_2^2 = [Q(z-y)]^T Q(z-y)$$

$$= (z-y)^T \underbrace{Q^T Q}(z-y)$$

$$= (z-y)^T I (z-y)$$

$$= \|z-y\|_2^2$$

Made with Goodnotes

7

$$K = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}.$$

(b) Eigen vectors of $K' = \alpha K$ and K will be same

Say $Kx = \lambda x$.

$$\alpha K x = \alpha \lambda x.$$

$$\Rightarrow K' x = (\alpha \lambda) x.$$

Eigen values get scaled by α.

(a). Can find eigen values of $K' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$

and scale them by $1/4$.

$$(3-\lambda)^2 - 1 = 0.$$

$$8 - 6\lambda + \lambda^2 = 0.$$

$$\lambda = 4, 2.$$

$\Rightarrow$ eigen values of K are

$4/4, 2/4 = 1, 1/2$

(c) K^m since K is p.s.d $\exists\, Q, D$ diagonal st

Q orthonormal

$$K = QDQ^T$$

$$K^2 = (QDQ^T)(QDQ^T)$$

$$= QD^2Q^T.$$

Made with Goodnotes

Similarly $K^m = Q D^m Q^T$.

$\Rightarrow$ eigen vectors of K^m are same as that of K

eigen values of K^m are m-th power of eigen values of K.

Made with Goodnotes

$$K = \begin{bmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{bmatrix}.$$

We can pick a vector to have 0's at 3rd position

$$(a_1 \;\; a_2 \;\; 0)\, K \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \geq 0.$$

$$(a_1 \;\; a_2) \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \geq 0.$$

$$\Rightarrow \det \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \geq 0$$

[otherwise it would imply that there is a -ve eigen vector for matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$]

$$\Rightarrow 1 - \rho^2 \geq 0.$$

(X, Y) are jointly Gaussian

$$K = \begin{bmatrix} \sigma_x^2 & \rho \sigma_x \sigma_y \\ \rho \sigma_x \sigma_y & \sigma_y^2 \end{bmatrix}.$$

(a) Find $f_{X|V}(x|v)$ where $V = Y^3$.

(b) Find $f_{X|U}(x|u)$ where $U = Y^2$.

$$(X, Y) \Longleftrightarrow (X, V) \qquad V = g(Y)$$

invertible function

$$J(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & g'(y) \end{bmatrix}.$$

$$f_{X,V}(x, v) = \frac{f_{X,Y}(x, g^{-1}(v))}{g'(g^{-1}(v))}$$

Made with Goodnotes

$$f_V(v) = \frac{f_Y(g^{-1}(v))}{g'(g^{-1}(v))}.$$

$$f_{X|V}(x|v) = \frac{f_{X,V}(x,v)}{f_V(v)}.$$

$$= \frac{f_{X,Y}(x, v^{1/3})}{f_Y(v^{1/3})} = f_{X|Y}(x|v^{1/3}).$$

$$X|V=v \sim N\left(\rho \, \sigma_x/\sigma_y \, v^{1/3}, \; (1-\rho^2)\sigma_x^2\right)$$

$$\text{''} \quad \frac{Cov(X,Y)}{Var(Y)} v^{1/3}$$

Check HW4 solns for detailed proof for $f_{X|Y}$

$U = Y^2$.

$$f_U(u) = \frac{f_Y(\sqrt{u})}{2\sqrt{u}} + \frac{f_Y(-\sqrt{u})}{|2\sqrt{u}|}.$$

Similarly. $$= \frac{f_Y(\sqrt{u}) + f_Y(-\sqrt{u})}{|2\sqrt{u}|}$$

$$f_{X,U}(x,u) = \frac{f_{X,Y}(x,\sqrt{u}) + f_{X,Y}(x,-\sqrt{u})}{2|(\sqrt{u})|}.$$

$$\Rightarrow f_{X|U}(x|u) = \frac{f_{X,Y}(x,\sqrt{u}) + f_{X,Y}(x,-\sqrt{u})}{}$$

Made with Goodnotes

$f_Y(\sqrt{u}) + f_Y(-\sqrt{u})$.

$$f_Y(y) = \frac{e^{-y^2/2\sigma_y^2}}{\sqrt{2\pi}}$$

$$f_Y(\sqrt{u}) = f_Y(-\sqrt{u}) = \frac{e^{-u/2\sigma_y^2}}{\sqrt{2\pi}}.$$

$$f_U(u) = 2 f_Y(\sqrt{u}).$$

$$f_{X|U}(x|u) = \frac{f_{X,Y}(x,\sqrt{u})}{2 f_Y(\sqrt{u})} + \frac{f_{X,Y}(x,-\sqrt{u})}{2 f_Y(\sqrt{u})}.$$

$$= \frac{1}{2} f_{X|Y}(x|\sqrt{u}) + \frac{1}{2} f_{X|Y}(x|-\sqrt{u})$$

$\therefore$ $X|U=u$ is a mixture of two Gaussian distributions.

$$N\left(\rho \frac{\sigma_x}{\sigma_y}\sqrt{u}, (1-\rho^2)\sigma_x^2\right), \quad N\left(-\rho \frac{\sigma_x}{\sigma_y}\sqrt{u}, (1-\rho^2)\sigma_x^2\right)$$

$X|U=u$ is not Gaussian but a mixture of Gaussian distributions.

Made with Goodnotes