Lecture-15

1. Concept of Flux - normal form of Green's theorem.

2. Surface integral & Flux.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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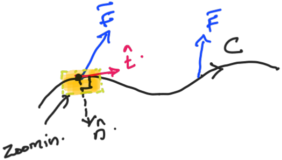
Flux: – Another line integral

$C \to$ plane curve ; $\bar{F}$ – a vector field then

Flux of $\bar{F}$ across C is .

$$\boxed{\text{Flux} = \int_C (\underbrace{\bar{F}\cdot\hat{n}}_{\text{normal component of } \bar{F}})\, ds}$$

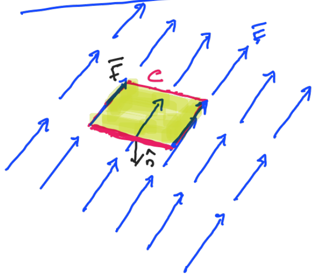
; $\hat{n}$ normal unit vector $\hat{}$. Clockwise to $\hat{t}$. (tangent unit vector) on the curve .



<u>Interpretation!</u> For $\bar{F}$ a velocity field

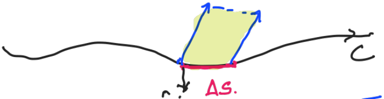
→ Flux measures how much fluid passes through 'C' per unit time.

Zoom in version of curve & $\vec{F}$.



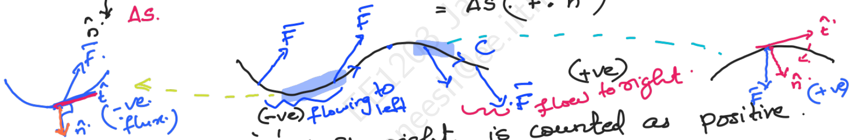
* Vector field $\bar{F}$ is uniform in the small portion of the curve we considered.

* Shaded portion: the total amount of field passes through small portion of 'c' considered in unit time. (form a parallelogram)



Area = Base (height)

$= \Delta S (\bar{F} \cdot \hat{n})$



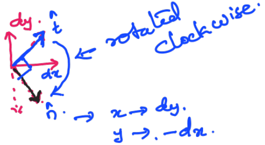
What flows across 'c' left-right is counted as positive & what flows right-to-left as negative.

* In otherwords, if $\bar{F}$ in the direction of $\hat{n}$ +ve.
$\bar{F}$ opposite to $\hat{n}$ $\rightarrow$ -ve.

Calculation using components:

For work: $\hat{t}\, ds = \langle dx, dy \rangle$

$\hat{n}\, ds = \langle dy, -dx \rangle$



So if $\vec{F} = (P(x,y), Q(x,y))$

$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_C (P, Q)\cdot(dy, -dx)$$

$$= \int_C -Q\, dx + P\, dy.$$

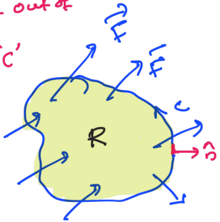
Now apply Green's theorem for flux.

If 'c' encloses a region 'R' counter clockwise and if $\vec{F}$ is defined in 'R' and 'c'; then $\vec{F} = \langle P, Q \rangle$

$$\oint_C \vec{F}\cdot\hat{n}\, ds = \iint_R (\underbrace{\operatorname{div} \vec{F}}_{\text{divergence of } \vec{F}\ (\nabla\cdot F)})\, dA$$

$\operatorname{div} \vec{F} = P_x + Q_y = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$.

Flux out of 'R' through 'C'



Apply: Green's theorem:

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA.$$

$N = P$, $M = -Q$.

$$= \iint_R \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)}_{=\nabla\cdot \vec{F}} dA$$

$\therefore$ Basically Flux can be calculated by normal form of Green's theorem!

Surface integrals:-

Vector field in space: $\langle P, Q, R \rangle$

$\downarrow$ $f(x, y, z)$

Flux:

We have seen flux of $\bar{F}$ in 2-D $\Rightarrow \int_c \bar{F} \cdot \hat{n} \, ds$.

In 3D $\rightarrow$ Flux is measured through a surface.

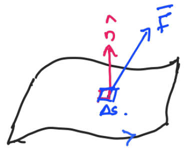
$\Rightarrow$ Surface integral.

Definition of Flux in 3D

If we have a vector field $\bar{F}$; surface in space 's'.

$\hat{n} \rightarrow$ unit normal vector to the surface.

2 choices for $\hat{n}$: Choose side of the surface. (orientation) : Normal convention $\hat{n}$ out of the surface.



$$\boxed{\text{Flux} = \iint_S \bar{F} \cdot \hat{n} \; dS}$$

(surface area element)

$$\boxed{\hat{n}\, dS = \vec{dS}}$$

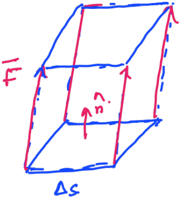
$dS = dx\,dy$ (Cartesian.

$dS = a\, d\phi\, dz$ (Cylindrical)

$dS = r^2 \sin\theta\, d\theta\, d\phi$ (Spherical)

Geometric Interpretation:

If $\bar{F}$ is a velocity field, Flux = amount of matter through 'S' per unit time.

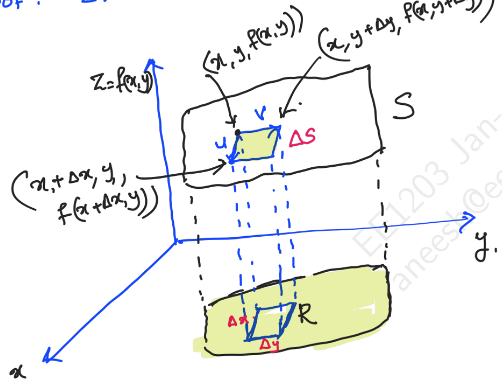


Total flux: through the surface!

Volume = Base area · height

$$= \iint_S dS \, (\bar F \cdot \hat n)$$

Proof: If 'S' is the graph of $z = f(x,y)$



$\hat n \Delta S = \bar u \times \bar v$

By linear approx.

$f(x+\Delta x, y) \approx f(x,y) + \Delta x f_x.$

$f(x, y+\Delta y) \approx f(x,y) + \Delta y f_y.$

$\bar{u} = \langle \Delta x, 0, \Delta x f_x \rangle$; $\bar{v} = \langle 0, \Delta y, \Delta y f_y \rangle$

$= \langle 1, 0, f_x \rangle \Delta x$ $\qquad = \langle 0, 1, f_y \rangle \Delta y.$

$$\bar{u} \times \bar{v} = \begin{vmatrix} i & j & k \\ \Delta x & 0 & \Delta x f_x \\ 0 & \Delta y & \Delta y f_y \end{vmatrix} = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} \Delta x \Delta y$$

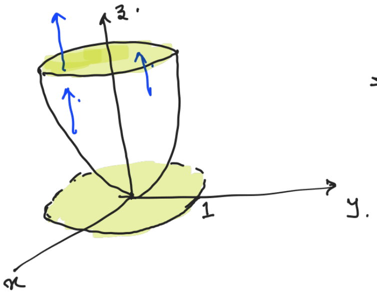
$$\hat{n} \Delta S = \bar{u} \times \bar{v} = \langle -f_x, -f_y, 1 \rangle \Delta x \Delta y.$$

As $\Delta x \to 0$ $\Delta y \to 0$.

$$\boxed{\hat{n}\, dS = \langle -f_x, -f_y, 1 \rangle\, dx\, dy.}$$

Example: $\bar{F} = z\hat{k}$ through portion of paraboloid $z = x^2+y^2$ above unit disk. Find the flux of $\bar{F}$.

Solution:



$$\iint_S \vec{F}\cdot\hat{n}\,dS \;;\; \vec{F} = z\hat{k}.$$

$$\Rightarrow \hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\, dx\,dy$$

$$f(x,y) = x^2+y^2.$$

$$f_x = 2x$$

$$f_y = 2y.$$

$$\Rightarrow \hat{n}\,dS = \langle -2x, -2y, 1\rangle\, dx\,dy.$$

$$\iint_S \vec{F}\cdot\hat{n}\,dS = \iint_S \langle 0,0,z\rangle\cdot\langle -2x,-2y,1\rangle\, dx\,dy$$

$$= \iint_S z\,dx\,dy = \iint_S (x^2+y^2)\,dx\,dy$$

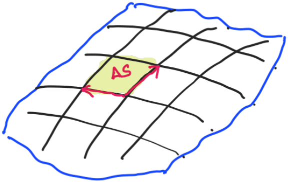
More general case: Suppose when the surface is too complicated so that you cannot express z in terms of x & y.

* · Given parametric description of surface 'S'

$$S = \begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$$

ie; $\langle x, y, z \rangle = \bar{r} = \bar{r}(u,v)$

Now how to express $\hat{n}\, dS$ in terms of $du\, dv$?



Sides: Small change in 'u'

$$\Rightarrow \frac{\partial \bar{r}}{\partial u} \Delta u = \left\langle \frac{\partial x}{\partial u} \Delta u, \frac{\partial y}{\partial u} \Delta u, \frac{\partial z}{\partial u} \Delta u \right\rangle$$

$$\frac{\partial \bar{r}}{\partial v} \Delta v = \left\langle \frac{\partial x}{\partial v} \Delta v, \frac{\partial y}{\partial v} \Delta v, \frac{\partial z}{\partial v} \Delta v \right\rangle$$

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$$\hat{n}\,\Delta S = \left(\frac{\partial \bar{r}}{\partial u}\Delta u\right)\times\left(\frac{\partial \bar{r}}{\partial v}\Delta v\right)$$

$$= \left(\frac{\partial \bar{r}}{\partial u}\times\frac{\partial \bar{r}}{\partial v}\right)\Delta u\,\Delta v$$

$\Delta s \to 0.$

$$\boxed{\hat{n}\,dS = \left(\frac{\partial \bar{r}}{\partial u}\times\frac{\partial \bar{r}}{\partial v}\right)du\,dv.}$$

Verify this with.

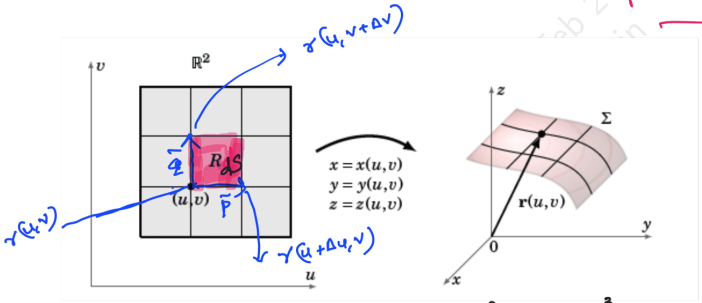
$x = x$

$y = y$

$z = f(x,y)$

You should get

$$\hat{n}\,dS = \langle -f_x, -f_y, 1\rangle\,dx\,dy.$$



Parametrization of a surface in $\mathbb{R}^3$.

Ref: Vector Calculus by Michael Corral.

$$r(u,v) = x(u,v)\,i + y(u,v)\,j + z(u,v)\,\hat{k}$$

$$\frac{\partial r}{\partial u}(u,v) = \frac{\partial x}{\partial u}\,i + \frac{\partial y}{\partial u}\,j + \frac{\partial z}{\partial u}\,\hat{k}$$

$$\frac{\partial r}{\partial v}(u,v) = \frac{\partial x}{\partial v}\,i + \frac{\partial y}{\partial v}\,j + \frac{\partial z}{\partial v}\,\hat{k}.$$

Linear approx

$$\frac{\partial \bar{r}}{\partial u} \approx \frac{\bar{r}(u+\Delta u, v) - \bar{r}(u,v)}{\Delta u};$$

$$\frac{\partial \bar{r}}{\partial v} \approx \frac{\bar{r}(u, v+\Delta v) - \bar{r}(u,v)}{\Delta v}.$$

$$dS\,\hat{n} = \bar{p} \times \bar{q}$$

$$\bar{p} = \bar{r}(u+\Delta u, v) - \bar{r}(u,v) = \frac{\partial \bar{r}}{\partial u}\Delta u.$$

$$\bar{q} = \bar{r}(u, v+\Delta v) - \bar{r}(u,v) = \frac{\partial \bar{r}}{\partial u}\Delta v.$$

$$\boxed{\hat{n}\,dS = \bar{p} \times \bar{q} = \left(\frac{\partial \bar{r}}{\partial u} \times \frac{\partial \bar{r}}{\partial v}\right) du\,dv}$$