① ⓐ $\mathrm{Var}(M_n) = \dfrac{\mathrm{Var}(X_1)}{n}$

$= \dfrac{1\,m}{n} \leq 10^{-2}\,m$

$\Rightarrow n \geq 100.$

ⓑ $P(|M_n - h| < 0.05) \geq 0.99.$

i.e., $P(|M_n - h| \geq 0.05) \leq 0.01.$

From Chebyshev inequality

$$P(|M_n - h| \geq 0.05) \leq \frac{\mathrm{Var}(M_n)}{(0.05)^2} = \frac{400}{n}.$$

enough to have $\dfrac{400}{n} \leq 0.01 \Rightarrow n \geq 40{,}000.$

ⓒ $$P(|M_n - h| \geq 0.05) \leq \frac{\mathrm{Var}(M_n)}{(0.05)^2} = \frac{\mathrm{Var}(X_1)}{n(0.05)^2}$$

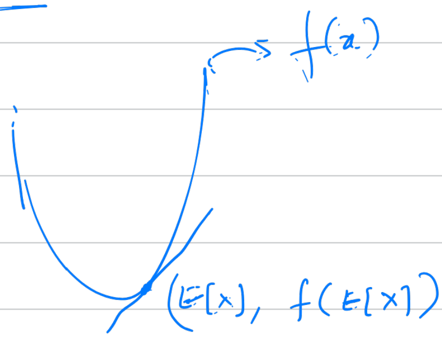
Using $\mathrm{Var}(X) \leq \dfrac{(b-a)^2}{4}$ if $X \in [a,b]$ w.p. 1 $\quad \leq \dfrac{(2-1.4)^2}{n\,4\,(0.05)^2} = \dfrac{(0.6)^2}{(0.05)^2 \times 4n}.$

Enough to have $\dfrac{0.36 \times \cancel{400}^{\,100}}{\cancel{4}\,n} \leq 0.01.$

$n \geq 3{,}600.$

② $l(x)$ is the tangent of $f(x)$ at $x = E[x]$.

$l(x) \leq f(x) \quad \forall x.$ (see the end)



$\Rightarrow E[l(X)] \leq E[f(X)]$

Let $l(x) = ax + b$. $\{$ can check that $a = f'(E[X])$,

$b = f(E[X]) - f'(E[X])\,E[X]$

$l(x) = f'(E[X])\,(x - E[X]) + f(E[X]).$

Made with Goodnotes

$$\underbrace{a\,E[X] + b}_{\text{by defn.}} \leq E[f(X)].$$

$$l(E[X]) = f(E[X]) \quad \text{as } l(x) \text{ is defined as tangent at } E[X]$$

$$\Rightarrow \quad f(E[X]) \leq E[f(X)].$$

Let $x_2 = E[X]$, be the point where tangent passes through $f(x)$ and let x_1 be a closest point to x_2 st $l(x_1) > f(x_1)$.

then $l(\lambda x_1 + (1-\lambda)x_2) = \lambda\, l(x_1) + (1-\lambda)\, l(x_2)$ for any $\lambda \in [0,1]$

$> \lambda f(x_1) + (1-\lambda)\, l(x_2)$

$= \lambda f(x_1) + (1-\lambda) f(x_2)$

$\geq f(\lambda x_1 + (1-\lambda) x_2).$

Let $x^* = \lambda x_1 + (1-\lambda)x_2$. then

x^* is closer to x_2 than x_1 but

$l(x^*) > f(x^*)$ contradicts our assumption

(3) $E[(f(X) - f(Y))(g(X) - g(Y))] \geq 0$

as f and g are both monotonic

either $f(X) - f(Y) \geq 0$ & $g(X) - g(Y) \geq 0$

or $f(X) - f(Y) < 0$ & $g(X) - g(Y) < 0$.

$$E[f(X)g(X) + f(Y)g(Y) - f(Y)g(X) - g(Y)f(X)]$$

$$= E[f(X)g(X)] + E[f(Y)g(Y)] - E[f(Y)g(X)] - E[g(Y)f(X)].$$

$$= \underbrace{2E[f(X)g(X)]}_{\text{as } X, Y \text{ are identical}} - \underbrace{2E[g(Y)]E[f(X)]}_{X \text{ and } Y \text{ are independent \& identical.}}.$$

$$= 2\left[E[f(X)g(X)] - E[g(X)]E[f(X)]\right].$$

$\geq 0.$

$$\Rightarrow \quad E[f(x)g(x)] \geq E[f(x)]\,E[g(x)]$$

④ $$P(X > a) = P(X + b > a + b).$$

$$\leq P(|X+b| > a+b).$$

$$\leq \frac{E[(X+b)^2]}{(a+b)^2}$$

$$= \frac{E[X^2 + 2Xb + b^2]}{(a+b)^2} = \frac{\sigma^2 + b^2}{(a+b)^2}.$$

$$P(X > a) \leq \min_b \underbrace{\frac{\sigma^2 + b^2}{(a+b)^2}}_{f(b)}$$

$$f'(b) = \frac{2b}{(a+b)^2} - \frac{2(\sigma^2 + b^2)}{(a+b)^3}.$$

$$= 0.$$

$$\Rightarrow \quad 2b(a+\cancel{b}) - 2(\sigma^2 + \cancel{b^2}) = 0.$$

$$b = \sigma^2/a$$

$$\Rightarrow \quad P(X > a) \leq \frac{\sigma^2(1 + \sigma^2/a^2)}{\left(a + \frac{\sigma^2}{a}\right)^2} = \frac{\sigma^2}{\sigma^2 + a^2}.$$

Using it for any X. (ones without zero mean).

$$P(X - \mu > a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

$$P(X - \mu < -a) \leq \frac{\sigma^2}{\sigma^2 + a^2}.$$

(5)

$$P(X \geq i) = P(e^{Xt} \geq e^{ti})$$

$$\leq \frac{E[e^{Xt}]}{e^{ti}} = M_X(t)\, e^{-ti}$$

$$P(X \geq i) \leq \inf_{t>0} M_X(t)\, e^{-ti}$$

$$M_X(t) = E[e^{Xt}] = \sum_{x=0}^{\infty} \frac{e^{-\lambda} \lambda^x}{x!} e^{xt}$$

$$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}$$

$$P(X \geq i) \leq \inf_{t>0} e^{\lambda(e^t-1)} e^{-ti}$$

$$= e^{-\sup_{t>0} \{ \underbrace{ti - \lambda(e^t - 1)}_{f_i(t)} \}}$$

$$\frac{\partial f_i(t)}{\partial t} = i - \lambda e^t$$

$$= 0$$

$$\Rightarrow i = \lambda e^t.$$

$$P(X \geq i) \leq \inf_{t>0} e^{-ti} e^{\lambda(e^t - 1)}.$$

$$= \left(\frac{\lambda}{i}\right)^i e^{(i-\lambda)}.$$

Made with Goodnotes

(6)

$$X_i = \begin{cases} 1 & \text{if the } i\text{-th women is paired with a man.} \\ 0 & \text{otherwise} \end{cases}$$

$$X = \sum_{i=1}^{100} X_i.$$

We want to find $P(X \geq 30)$.

$$E[X] = 100\, E[X_1].$$

$$P(X_1 = 1) = \frac{100}{199} = 1 - P(X_1 = 0).$$

$$\Rightarrow E[X] = \frac{100^2}{99} \;;\; E[X_i^2] = \frac{100}{199}.$$

$$E[X^2] = E\left[\left(\sum_{i=1}^{100} X_i\right)^2\right]$$

$$= E\left[\sum_{i=1}^{100} X_i^2 + 2 \sum_{1 \leq i < j \leq 100} X_i X_j\right]$$

$$= 100\, E[X_1^2] + 2 \times \binom{100}{2} E[X_1 X_2]$$

$$P(X_1 = 1, X_2 = 1) = P(X_1 = 1)\, P(X_2 = 1 \mid X_1 = 1).$$

$\Rightarrow$ women 1 is paired with a man so X_2 can't be paired with women 1 or the man that she's paired with

$$\Rightarrow P(X_2 = 1 \mid X_1 = 1) = \frac{100 - 1}{200 - 3} = \frac{99}{197}$$

$$P(X_1 = 1) = \frac{100}{199}.$$

$$P(X_2 = 1, X_1 = 1) = \frac{100 \times 99}{199 \times 197}.$$

$$E[X^2] = 100 \times \frac{100}{199} + \frac{(100 \times 99)^2}{199 \times 197}.$$

$$Var(X) = E[X^2] - (E[X])^2.$$

$$P(X \geq 30) = P(X - E[X] \geq 30 - E[X]).$$

$$\leq \frac{Var(X)}{(30 - E[X])^2} \quad \left\{ \text{using chebyshev inequality} \right.$$

$$P(X \geq 30) \leq \frac{Var(X)}{Var(X) + (30 - E[X])^2}$$

one sided chebyshev.

(7) $$P\left(\prod_{i=1}^{100} X_i \leq a^{100}\right)$$

$$= P\left(\sum_{i=1}^{100} \log X_i \leq 100 \log a\right)$$

Let $Y_i = \log X_i$

$$\mu_y = E[Y_i] = E[\log X_i]$$

$$= \sum_{i=1}^{6} \frac{1}{6} \log i = \frac{1}{6}\left(\sum_{i=1}^{6} \log i\right).$$

$$E[Y_i^2] = \frac{1}{6} \sum_{i=1}^{6} (\log i)^2.$$

$$\sigma_y^2 = E[Y_i^2] - (E[Y_i])^2.$$

$$P\left(\frac{\sum_{i=1}^{100} Y_i - 100\mu_y}{\sqrt{100}\,\sigma_y} \le \frac{100(\log a - \mu_y)}{\sqrt{100}\,\sigma_y} \right)$$

By Central Limit theorem

$$\approx \Phi\left(\frac{\log a - \mu}{(\sigma_y / \sqrt{100})} \right).$$

(8) (a) $Y_n = X_n / n$

claim: $Y_n \to 0$.

Let $\epsilon > 0$.

$$P(|Y_n| > \epsilon) = P(X_n > n\epsilon) + P(X_n < -n\epsilon)$$

$$= 2(1 - \min(n\epsilon, 1)).$$

$$\lim_{n\to\infty} P(|Y_n| > \epsilon) = 0.$$

(b) $Y_n = (X_n)^n$: claim $Y_n \to 0$ in probability

$$P(|Y_n| > \epsilon) = P(|X_n^n| > \epsilon).$$

$$= P(X_n^n > \epsilon) + P(X_n^n < -\epsilon).$$

using symmetry for if even this is a loose bound.

$$\le 2P(X_n > \epsilon^{1/n}) = 2(1 - \epsilon^{1/n}).$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n| > \epsilon) < \lim_{n\to\infty} 2(1 - \epsilon^{1/n}).$$

$$= 0$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n| > \epsilon) = 0.$$

ⓒ $Y_n = X_1 X_2 \dots X_n$.

Claim: $Y_n \to 0$ in probability.

$$P(|Y_1| > \epsilon) = P(|X_1| > \epsilon).$$

$$= (1-\epsilon).$$

$$P(|Y_n| > \epsilon) = P(|X_1 X_2 \dots X_n| > \epsilon)$$

$$= P\left(\prod_{i=1}^{n} |X_i| > \epsilon\right)$$

markov inequality

$$\le \frac{E\left[\prod_{i=1}^{n} |X_i|\right]}{\epsilon}.$$

Check That
$|X_1| \sim$ Uniform $[0,1]$.
$E[X_1] = 1/2$.

$$P(|Y_n| > \epsilon) \le \frac{\prod_{i=1}^{n} E[|X_i|]}{\epsilon}$$

$|X_i|$ are independent

$$= \frac{1}{2^n \epsilon}.$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n| > \epsilon) \le \lim_{n\to\infty} \frac{1}{2^n \epsilon}$$

$$= 0.$$

$$\Rightarrow \lim_{n\to\infty} P(|Y_n| > \epsilon) = 0.$$

ⓓ $Y_n = \max(X_1, X_2, \dots X_n)$.

Claim: $Y_n \to 1$.

$$P(|Y_n - 1| > \epsilon). = P(Y_n > 1+\epsilon) + P(Y_n < 1-\epsilon).$$

$= 0$ can't be > 1.

$$P(Y_n \le y) = \prod_{i=1}^{n} P(X_i \le y)$$

Made with Goodnotes

$$= \frac{(y+1)^n}{2^n} \quad \text{for } -1 \le y \le 1.$$

$$P(Y_n < 1-\epsilon) = \left(\frac{2-\epsilon}{2}\right)^n = \left(1-\frac{\epsilon}{2}\right)^n.$$

$$\lim_{n\to\infty} P(|Y_n - 1| > \epsilon) = \lim_{n\to\infty} \left(1-\frac{\epsilon}{2}\right)^n$$

$$= 0.$$

(a) $E[X_n] = 5$, $Var(X_n) = 9$.

$$P\left(\sum_{i=1}^{100} X_i \le 440\right) = P\left(\underbrace{\frac{\sum_{i=1}^{100}(X_i - 5)}{10 \times 3}}_{\text{mean } 0,\ \text{variance } 1} \le \frac{-60}{10 \times 3}\right)$$

$$\approx \Phi(-2)$$

by central limit theorem.

(b) need $P(X_1 + \cdots X_n \ge 200 + 5n) \le 0.05$

$$P\left(\frac{\sum_{i=1}^{n}(X_i - 5)}{\sqrt{n}\,3} \ge \frac{200}{\sqrt{n}\,3}\right) \approx 1 - \Phi\left(\frac{200}{\sqrt{n}\,3}\right).$$

Need: $1 - \Phi\left(\frac{200}{\sqrt{n}\,3}\right) \le 0.05$

$$\Phi\left(\frac{200}{\sqrt{n}\,3}\right) \ge 0.95.$$

$$\frac{200}{\sqrt{n}\,3} \ge \Phi^{-1}(0.95).$$

$$n \le \frac{200}{3\,\Phi^{-1}(0.95)}.$$

(c) $N = \min\left\{ n : \sum_{i=1}^{n} X_i \geq 1000 \right\}$.

$$P(N \geq 220) = P\left(\sum_{i=1}^{219} X_i < 1000 \right).$$

$$= P\left(\frac{\sum_{i=1}^{219} (X_i - 5)}{\sqrt{219} \times 3} < \frac{1000 - 219 \times 5}{\sqrt{219} \times 3} \right)$$

$$= \Phi\left(\frac{-95}{3\sqrt{219}} \right).$$

(10) $\Omega = [0, 1]$

$$X_n(\omega) = \begin{cases} 1 & \omega \in [0, \frac{1}{n}] \\ 0 & \text{otherwise} \end{cases}$$

Claim: $X_n \longrightarrow 0$ in almost sure sense.

Can see that $\lim_{n \to \infty} X_n(\omega) = 0 \quad \forall \omega \in (0, 1]$.

For any $\omega \in (0, 1]$. let $N^* = \left\lceil \frac{1}{\omega} \right\rceil$

$$X_n(\omega) = 0, \quad \forall n \geq N^*.$$

$(X_n(\omega))$ sequence will look like the following

1 1 1 1 1 . . . 1 0 0 0 0 - - - -

Also converges in mean-squared sense

$E[|X_n - 0|^2] = 1^2 \cdot \frac{1}{n}$

$\lim_{n \to \infty} E[|X_n|^2] = 0.$

$$P\left(\left\{ \omega : \lim_{n \to \infty} X_n(\omega) = 0 \right\} \right) \geq P((0, 1]) = 1.$$

$$\Rightarrow \boxed{P\left(\left\{ \omega : \lim_{n \to \infty} X_n(\omega) = 0 \right\} \right) = 1.}$$

as probability is also upper bounded by 1.

It also follows that $X_n \to X$ in probability & $X_n \to X$ in distribution.

(11) $X_n \sim N(0, \frac{1}{n})$.

$$P(|X_n| > \epsilon) \le \frac{Var(X_n)}{\epsilon^2} = \frac{1}{n\epsilon^2}.$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n| > \epsilon) \le \lim_{n\to\infty} \frac{1}{n\epsilon^2} = 0.$$

$$\Rightarrow X_n \longrightarrow 0 \text{ in probability.}$$

Does it also converge in a.s sense? (It does)

Using symmetry of normal distribution

$$P(|X_n| > \epsilon) = 2P(X_n > \epsilon).$$

using Chernoff

$$\le 2 \inf_{t>0} E[e^{X_n t}] e^{-\epsilon t}$$

$$= 2 e^{-\left[\sup_{t>0} \epsilon t - \frac{t^2}{n^2}\right]}.$$

$\epsilon - \frac{2t}{n^2}$.

$$= 2 e^{-n^2\epsilon^2/4}.$$

$\frac{n^2\epsilon^2}{2} - \left(\frac{n^2\epsilon}{2}\right)^2 \frac{1}{n^2}$.

$e^{-\frac{n^2\epsilon^2}{4}}$.

$$P(|X_n| > \epsilon) \le \frac{2}{e^{n^2\epsilon^2/4}}.$$

$$\sum_n P|X_n| > \epsilon) \le \underbrace{\sum \frac{2}{e^{n^2\epsilon^2/4}}}_{\text{Sequence that converges}} < \infty.$$

We know $\sum \frac{1}{n^2}$ does therefore this should

From Borel-Cantelli Lemma (Yet to be covered in class)

$$\Rightarrow P(|X_n| > \epsilon \text{ i.o}) = 0.$$

$$\Rightarrow X_n \to 0 \text{ a.s.}$$

Mean square sense

$$E[(X_n - 0)^2] = Var(X_n) = \frac{1}{n}$$

$$\lim_{n\to\infty} E[|X_n|^2] = 0. \Rightarrow X_n \to 0 \text{ in mean square sense}$$

(12) $P(X_n = 1) = \frac{1}{n^2} = 1 - P(X_n = 0)$

$$P(|X_n| > \epsilon) \leq \frac{1}{n^2}.$$

$$\Rightarrow \sum_{n=1}^{\infty} P(|X_n| > \epsilon) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

From Borel Cantelli Lemma I $\Rightarrow P(|X_n| > \epsilon \text{ i.o}) = 0.$

$$\Rightarrow X_n \to 0 \quad a.s.$$

$\Rightarrow X_n \to 0$ in probability, distribution.

$$E[|X_n - 0|^2] = E[X_n^2] = 1^2 \cdot \frac{1}{n^2}.$$

$\Rightarrow \lim_{n\to\infty} E[|X_n|^2] = 0. \Rightarrow X_n \to 0$ in m.s sense