CS1010: Discrete Mathematics for Computer Science

(Exam-1. Total: 30 marks.)
(Duration: 45 minutes. Date: 13 Oct 2025)

Instructions: ★ You may not get time to answer all the questions unless you have prepared really well. The exam is designed like that. ★ If your mobile phone is found with you during the exam, you will lose one grade. ★ Anybody found copying will get an F grade for the course straight away. ★ It is a no-break exam. You cannot take a break in between. The exam is only for 45 minutes. If you want to go out of the exam hall, you will have to submit your answer paper. ★ You should sit far apart from each other. The halls are big enough. If we see two students sitting close to each other, both the students will lose one grade.

Questions

1. TRUE or FALSE (One word answer. No need to give any explanation). Assume a, b, c, n are positive integers.

   (a) If $gcd(a, b) \neq 1$ and $gcd(b, c) \neq 1$, then $gcd(a, c) \neq 1$.

   (b) If $a|bc$ and $gcd(a, b) = 1$, then $a|c$.

   (c) $gcd(a^n, b^n) = (gcd(a, b))^n$.

   (d) $gcd(ab, ac) = a \cdot gcd(b, c)$.

   (e) $gcd(1 + a, 1 + b) = 1 + gcd(a, b)$.

   (f) If an integer linear combination of a and b equals 1, then so does some integer linear combination of a and b^2.

   (g) If no integer linear combination of a and b equals 2, then neither does any integer linear combination of a^2 and b^2.

   (h) One **cannot** obtain the integer 5 as an integer linear combination of 256 and 81.

   (i) If $gcd(a, b) = 1$ and $gcd(a, c) = 1$, then $gcd(a, bc) = 1$.

   (j) $gcd(a, b) = gcd(b, a \bmod b)$. 1 x 10 = 10 marks.

2. Suppose that S is a set of n integers. Show that one can always find a nonempty subset T of S such that the sum of all the elements of T is divisible by n.
   Example: Let $n = 4$ and $S = \{3, 7, 21, -5\}$. Since $(3 + 21)$ is divisible by 4, we can take $T = \{3, 21\}$. 10 marks.

3. Let $\mathcal{F}$ be a family (or collection) of m subsets of a finite set X. For any $x \in X$, let $p(x)$ be the number of pairs (A, B) of sets $A, B \in \mathcal{F}$ such that either $x \in A \cap B$ or $x \notin A \cup B$. Prove that $p(x) \geq \frac{m^2}{2}$, for every $x \in X$. (You may use the fact that for any two positive real numbers a and b, $a^2 + b^2 \geq \frac{(a+b)^2}{2}$.)
Example: Let $X = \{1, 2, 3\}, \mathcal{F} = \{\{1, 2\}, \{1\}, \{2\}\}$. So, here $m = |\mathcal{F}| = 3$ and $\frac{m^2}{2} = 4.5$. Let $A := \{1, 2\}, B := \{1\}, C := \{2\}$.
Then, $p(1) = |\{(A, A), (A, B), (B, A), (B, B), (C, C)\}| = 5 \geq 4.5$. Similarly, one can see that both $p(2)$ and $p(3)$ are at least 4.5. 10 marks.

—————— ALL THE BEST ——————