Lec 13 :

Recap:

Random variable X over $(\Omega, \mathcal{F}, P)$ is a function on Ω taking values in $\mathbb{R}$

$$X : \Omega \longrightarrow \mathbb{R}.$$

$\Omega \to$ sample space.

st $X^{-1}((-\infty, x]) \in \mathcal{F}$ $\forall x \in \mathbb{R}$.

$\mathcal{F} \to$ event space

Example : $\Omega = [-1, 1]$, $P([a,b]) = \frac{b-a}{2}$ for

$X(\omega) = \omega^2$.

$F_X(x) = P(\{\omega : \omega^2 \le x\}) = P([-\sqrt{x}, \sqrt{x}]) = \sqrt{x}$.

If you define prob events in event space then CDF is well defined $F_X(x) = P(\{X \le x\})$.

We define RV X to be "continuous" if the distribution function $F_X(x)$ can be written as

$$F_X(x) = \int_{-\infty}^{x} \underbrace{f_X(t)}\, dt.$$

$f_X(t)$ is called probability density function or the p.d.f.

From calculus it follows that

$$f_X(x) = \frac{d F_X(x)}{dx}.$$

Properties of p.d.f.

For a function $f_X(x)$ to be a p.d.f it has to satisfy.

① $f_X(x) \ge 0 \quad \forall x \in \mathbb{R}. \longrightarrow \quad P_X(x) \ge 0.$

② $\int_{-\infty}^{\infty} f_X(x)\, dx = 1 \quad \xrightarrow{\text{equivalent}} \quad \sum_{x \in \mathcal{X}_e} P_X(x) = 1$

③ $P(a < X \le b) = \int_{t=a}^{b} f_X(t)\, dt$

$$f_X(x) = \lim_{\delta \to 0} \frac{F_X(x+\delta) - F(x)}{\delta}$$

If $\delta > 0$ $\quad F_X(x+\delta) \ge F(x)$.

If $\delta < 0$ $\quad F_X(x+\delta) \le F(x)$.

$$\Rightarrow \quad f_X(x) \geq 0. \qquad \forall x \in \mathbb{R}.$$

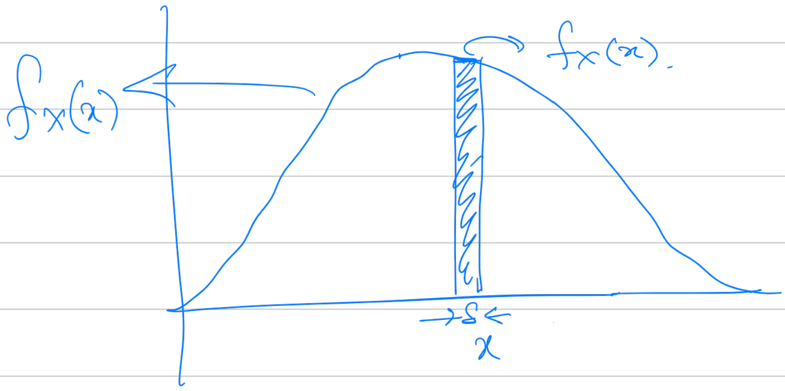
$$\int_{-\infty}^{\infty} f_X(x)\, dx = \lim_{x \to \infty} \int_{-\infty}^{x} f_X(t)\, dt$$

$$= \lim_{x \to \infty} F_X(x) = 1.$$

$$P(a < X \leq b) = F_X(b) - F_X(a)$$

$$= \int_a^b f_X(x)\, dx.$$

$$P(x < X \leq x+\delta) = \int_x^{x+\delta} \underbrace{f_X(t)}_{\text{can assume function in that interval}}\, dt.$$



$$= f_X(x)\,\delta.$$

Density function at x gives you "probability per interval length" in the vicinity of x

$$\boxed{P(X \in B) = \int_{x \in B} f_X(x)\, dx. \qquad B \subseteq \mathbb{R}.}$$

4th Feb.

Examples: ① Uniform $\sim [a, b]$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = \int_a^b \frac{1}{b-a}\, dx.$$

$$= \left. \frac{x}{b-a} \right|_a^b = \frac{b-a}{b-a} = 1.$$

Given samples from Uniform $[0,1]$: how do you generate samples for R.V X with pmf

$(P_1, P_2, \ldots P_n)$.
$(1 \quad 2 \quad \ldots \quad n)$

$\sum P_i = 1$.

$y_1 \quad y_2 \quad y_3 \quad \cdots \quad y_N$ samples.

Define interval $I_i = \left[\sum_{j=1}^{i-1} P_j \, , \, \sum_{j=1}^{i} P_j \right]$

$x_1, \quad \ldots\ldots \quad x_N$

$I_1 = [0, P_1]$
$I_2 = [P_1, P_1 + P_2$
$\vdots$
$I_n = [P_1 + \cdots P_{n-1}; 1]$.

$x_i = j$ if $y_i \in I_j$

② Exponential (λ) R.V.

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0. \\ 0 & \text{Otherwise} \end{cases}$$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx.$$

$$= \lambda e^{-\lambda x} \left(\frac{-1}{\lambda}\right) \Big|_0^{\infty}$$

$$= -e^{-\lambda x} \Big|_0^{\infty} = 1 - 0.$$

$$= 1.$$

Memoryless property

$P(X > m + t \mid X > t)$

$= P(X > m) = e^{-\lambda m}$

(Check that exponential R.V also satisfies this property).

Connections between exponential and Geometric R.V.

X : Geometric (p) R.V and

Y : Exponential (λ) R.V.

$$F_X(n) = P(X \le n)$$

$$= 1 - P(X > n)$$

$$= 1 - (1-p)^n$$

$$F_Y(y) = \int_{-\infty}^{y} f_Y(t)\, dt$$

$$= \int_0^x \lambda e^{-\lambda t}\, dt$$

$$= \left(\cancel{\lambda} e^{-\lambda t}\right)\left(\frac{1}{-\cancel{\lambda}}\right) \Big|_0^x$$

$$= -e^{-\lambda t} \Big|_0^x$$

$$= -e^{-\lambda x} + e^{-\lambda(0)}$$

$$= 1 - e^{-\lambda x}$$

$x = \delta n$ and equate $(1-p)^n = e^{-\lambda \delta n}$

i.e., $(1-p) = e^{-\lambda \delta}$

$$\Rightarrow \boxed{\delta = \frac{1}{\lambda} \log\left(\frac{1}{1-p}\right)}$$