Lecture 26 — 18th March

Moment Generating Function (MGF). of a random variable X is defined as:

$$M_X(s) = E[e^{Xs}].$$

similar to z transform of PMF by setting $z = e^{s}$.

$$= \sum_{x \in \mathcal{X}} e^{xs} P_X(x) \quad \text{for } X \text{ discrete}$$

Laplace transform of pdf.

$$= \int_{x \in \mathbb{R}} e^{xs} f_X(x)\, dx. \quad \text{for } X \text{ continuos}$$

May not well-defined at all values of s.

Characteristic function:

$$\phi_X(t) = E[e^{jXt}].$$

similar to fourier transform.

Examples

① $X \sim \text{Bernoulli}(p)$

$$M_X(s) = E[e^{Xs}]$$
$$= (1-p)\, e^{0 \times s} + p\, e^{1 \times s}.$$
$$= (1-p) + p e^{s}.$$

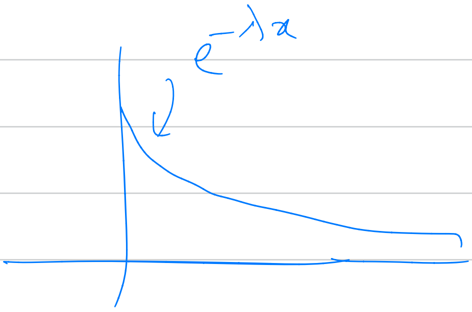
② $X \sim \text{Poisson}(\lambda)$.

$$M_X(s) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} e^{ks}.$$
$$= \sum_{k=0}^{\infty} e^{-\lambda} \frac{(\lambda e^{s})^k}{k!}$$
$$= e^{-\lambda} e^{\lambda e^{s}} = e^{\lambda(e^{s}-1)}.$$

③ $X \sim \text{Exponential}(\lambda)$

$$M_X(s) = E\left[e^{Xs}\right] = \int_0^\infty e^{xs} \underbrace{\lambda e^{-\lambda x}}_{f_X(x)} \, dx.$$

$f_X(x)$ pdf of X.



$$= \int_0^\infty \lambda e^{-x(\lambda - s)} \, dx$$

$$= \begin{cases} \dfrac{\lambda}{\lambda - s} & s < \lambda. \\ \text{not well defined} & \text{otherwise.} \end{cases}$$

④ $X \sim \text{Normal}(0, 1)$

X : standard normal R.V.

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

$$M_X(s) = E\left[e^{Xs}\right] = \int_{-\infty}^{\infty} e^{xs} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x^2 - 2xs + s^2)} e^{s^2/2} \, dx$$

$$= \int_{-\infty}^{\infty} \boxed{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-s)^2}} e^{s^2/2} \, dx$$

$$= e^{s^2/2}$$

pdf of $Y = X + s$

Properties

① Shift and Scaling.

$Y = aX + b.$

$$M_Y(s) = E\left[e^{Ys}\right] = E\left[e^{(aX+b)s}\right]$$

$$= E\left[e^{bs} e^{aXs}\right].$$

$$= e^{bs} E\left[e^{asX}\right] = e^{bs} M_X(as).$$

Example: $X \sim N(0,1)$, $Y \sim N(\mu, \sigma^2)$. $b = \mu$, $a = \sigma$

$$Y = \sigma X + \mu.$$

$$M_Y(s) = e^{bs} M_X(as)$$
$$= e^{\mu s} M_X(\sigma s) = e^{\mu s + \sigma^2 s^2/2}.$$

② Sum of Independent R.Vs: $Y = X_1 + X_2$

$$M_Y(s) = E[e^{Ys}] = E\left[e^{(x_1+x_2)s}\right]$$
$$= E\left[e^{x_1 s} e^{x_2 s}\right] = E\left[e^{X_1 s}\right] E\left[e^{X_2 s}\right]$$
$$= M_{X_1}(s)\, M_{X_2}(s).$$

If $Y = X_1 + X_2 \cdots + X_n$, $M_Y(s) = \prod_{i=1}^{n} M_{X_i}(s)$

Examples: Normal, Poisson, Binomial

Normal case: $X_1 \sim N(\mu_1, \sigma_1^2)$ $X_2 \sim N(\mu_2, \sigma_2^2)$, $Y = X_1 + X_2$

$$M_Y(s) = M_{X_1}(s)\, M_{X_2}(s)$$
$$= e^{\mu_1 s + \sigma_1^2 \frac{s^2}{2}}\; e^{\mu_2 s + \sigma_2^2 s^2/2}$$
$$= e^{(\mu_1+\mu_2)s + (\sigma_1^2+\sigma_2^2)s^2/2}.$$

$\Rightarrow Y$ is a $N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$.

(We have used the inversion property of MGF).

Binomial: $X_1 \sim \text{Binomial}(n, p)$ $X_2 \sim \text{Binomial}(m, p)$.

$Y = X_1 + X_2$ $X_1 = X_{11} + \cdots X_{1n}$

where $X_{11}, \ldots X_{1n}$ are Bernoulli (p) R.Vs.

$$M_{X_1}(s) = \prod_{i=1}^{n} M_{X_{1i}}(s)$$
$$= \left(1 - p + p.e^{s}\right)^n$$

Similarly

$$M_{X_2}(s) = \left(1 - p + p e^{s}\right)^m$$

$$M_Y(s) = M_{X_1}(s)\, M_{X_2}(s) = \left(1-p+pe^{s}\right)^{m+n}$$

Excercise: Sum of two Poisson R.Vs with parameters λ_1, λ_2 respectively is Poisson with parameter $\lambda_1 + \lambda_2$

$\Rightarrow Y \sim \text{Binomial}(m+n, p)$.
(Inversion property).

③ Finding moments from MGF.

$$M_X(s) = E[e^{Xs}]$$

if X is discrete & nonnegative ⓒ $P(X=0) = \lim_{s \to -\infty} M_X(s)$

ⓐ $M_X(0) = 1$ (Excercise)

Moments of RV are $E[X], E[X^2], \ldots E[X^n]$

$E[X^n]$ is n-th moment of R.V X.

ⓑ
$$M_X(s) = E[e^{Xs}]$$

$$= E\left[1 + \sum_{n=1}^{\infty} \frac{(Xs)^n}{n!}\right]$$

$$= 1 + \sum_{n=1}^{\infty} E[X^n]\left(\frac{s^n}{n!}\right).$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \cdots$$

$$\frac{\partial M_X(s)}{\partial s} = E[X] + \frac{2 \times s}{2!} E[X^2] + \frac{3 s^2}{3!} E[X^3] + \cdots \rightarrow ①$$

$$\left.\frac{\partial M_X(s)}{\partial s}\right|_{s=0} = E[X]$$

$$\frac{\partial^2 M_X(s)}{\partial s^2} = E[X^2] + \frac{\cancel{3 \times 2} \times s}{\cancel{3!}} E[X^3] + \cdots$$

$$\left.\frac{\partial^2 M_X(s)}{\partial s^2}\right|_{s=0} = E[X^2].$$

In general,
$$\left.\frac{\partial^n M_X(s)}{\partial s^n}\right|_{s=0} = E[X^n].$$

④ MGF of sum of random # of R.Vs.

Let $X_1, X_2, \ldots$ be iid R.Vs with MGF given by $M_X(s)$ and N be R.V that is independent of X_i's.

$$Y = X_1 + X_2 + \cdots X_N.$$

$$M_Y(s) = E[e^{Ys}]$$

$$= E\left[e^{(X_1 + \cdots X_N)s}\right].$$

total law of expectation

$$= \sum_n P(N=n)\, E\left[e^{(X_1+\cdots X_N)s} \middle| N=n\right]$$

$$= \sum_n P(N=n)\, E\left[e^{(X_1+\cdots X_n)s} \middle| N=n\right]$$

N is independent of $X_1, \ldots X_n, \ldots$

$$= \sum_n P(N=n)\, E\left[e^{(X_1+\cdots X_n)s}\right]$$

$$= \sum_n P(N=n)\left(M_X(s)\right)^n = E\left[M_X(s)^N\right]$$

$$\boxed{a = e^{\log a}}$$

$$= E\left[e^{N \log M_X(s)}\right]$$

$$= M_N(\log M_X(s))$$

Example : $N \sim \text{Poisson}(\lambda)$, $X_i \sim \text{Bernoulli}(p)$ with MGF $M_X(s)$

$$M_X(s) = (1-p+pe^s)$$

$$Y = X_1 + X_2 + \cdots X_N$$

$$M_N(s) = e^{\lambda(e^s-1)}$$

$$M_Y(s) = M_N(\log M_X(s))$$

$$= e^{\lambda(e^{\log M_X(s)}-1)}$$

$$= e^{\lambda(M_X(s)-1)}$$

$$= e^{\lambda(\cancel{1}-p+pe^s-\cancel{1})} = e^{\lambda p(e^s-1)}$$

Y is Poisson(λp) (By inversion property)

5 MGF of "mixture" distributions : R.Vs $X_1, \ldots X_n$ and $Y = X_i$ with probability p_i.

$$f_Y(y) = \sum_{i=1}^{n} p_i\, f_{X_i}(y)$$

$$M_Y(s) = E\left[e^{Ys}\right]$$

$$= \int_{-\infty}^{\infty} e^{ys} f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} e^{ys} \sum_{i=1}^{n} p_i f_{X_i}(y)\, dy.$$

$$= \sum_{i=1}^{n} p_i \int_{-\infty}^{\infty} e^{ys} f_{X_i}(y)\, dy$$

$$\boxed{M_Y(s) = \sum_{i=1}^{n} p_i M_{X_i}(s)}$$

Example: IITH's SBI bank has 3 employees, two of them are too slow, one luckily fast. Time to assist customers is exponentially distributed with parameter at $\lambda = 6$ for fast ones and $\lambda = 2$ for slow ones. A person visits the bank and picks an employee at random with prob. $1/3$. What is the pdf of time it takes to assist this person & what is the MGF? (Y).

$$X_1 \sim \exp(6)$$

$$f_Y(y) = \frac{1}{3} f_{X_1}(x) + \frac{2}{3} f_{X_2}(x). \quad X_2 \sim \exp(2)$$

$$= \frac{1}{3} \times 6 \times e^{-6x} + \frac{2}{3} \times 2 \times e^{-2x} \quad P_1 = 1/3, \; P_2 = 2/3.$$

$$M_Y(y) = \frac{1}{3}\left(\frac{6}{6-s}\right) + \frac{2}{3}\left(\frac{2}{2-s}\right).$$

⑥ Inversion property

The transform $M_X(s)$ uniquely identifies the probability law of R.V X (PMF if X is discrete or p.d.f if X is continuous R.V). In particular if $M_X(s)$ and $M_Y(s)$ are the same $\forall s$, then X and Y are identically distributed.