① Bi-Weekly on 3th Feb 6pm.

Lecture 12:

31/1/25

Sample mean

$$S = \frac{X_1 + \cdots X_n}{n} \Big\} \text{ sample mean.}$$

$X_1, \ldots X_n$ are all Bernoulli (p) random variables and they are independent

(i.i.d) $\rightarrow$ identical & independent distribution.

PMF of S.

100 samples

$\sqrt{Var(S)}$

$$E[S] = \frac{\sum_{i=1}^{n} E[X_i]}{n}$$

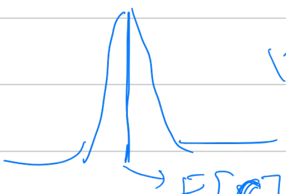
$$= \frac{np}{n} = p$$

$$Var[S] = \frac{Var\left(\sum_{i=1}^{n} X_i\right)}{n^2}$$

as $n \to \infty$

$Var(S) \to 0$.

$$= \frac{\sum_{i=1}^{n} Var(X_i)}{n^2}$$



10^5 samples

$E[S]$.

$$= \frac{np(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n}.$$

Course Outline

① Discrete R.Vs

② Applications of probability

③ Programming session. (involve all the results we have seen so far

④ Continuous Random Variables

Let's say we want to estimate probability of a particular event A.

$$X_i(\omega) = \begin{cases} 1 & \omega \in A. \\ 0 & \omega \notin A. \end{cases}$$

Bernoulli R.V with $p = P(A)$.

$\omega_1 \quad \omega_2 \quad \ldots \quad \omega_n \rightarrow$ experiment outcomes.

$X_1 \quad X_2 \quad \ldots \quad X_n$

$$S = \frac{X_1(\omega_1) \quad X_2(\omega_2) \quad \cdots \quad X_n(\omega_n)}{n}$$

$\downarrow$

giving an estimate of $P(A)$

as $n \to \infty$, the estimate gets better.

$$\boxed{\begin{aligned} &Var(X) E\left[\left(X - E[X]\right)^2\right] = 0 \\ &\Rightarrow X = E[X] \quad \text{w.p } 1. \end{aligned}}$$

$Var(S) \to 0$

$\Rightarrow S = E[S]$

$= p.$

Applications of probability } Examples in Graph Theory.

"Probabilistic Method" Paul Erdos.

(1) Let say that $P(A) > 0$. then A can't be a null set as $P(\phi) = 0$.

$P(X \geq m) > 0 \Rightarrow \{X \geq m\} \neq \phi.$ $\longrightarrow \{\omega \in \Omega : X(\omega) \geq m\}$

$\exists\ \omega \in \Omega$ st $X(\omega) \geq m$.

(2) If $E[X] \geq m$ then

$P(X \geq m) > 0.$

Otherwise $P(X \geq m) = 0$, $P(X < m) = 1$

$E[X] = \sum_{x \in \mathcal{X}} x\, P_X(x) < m \sum_{x \in \mathcal{X}} P_X(x) = m$

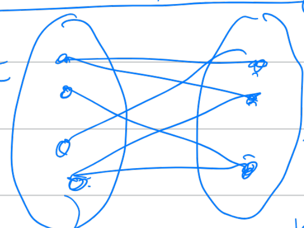
contradicts.

Example 1: Existence of bi-partite subgraph with large number of edges.

Let's say you have a graph with m edges, then there exists a bi-partite subgraph with $m/2$ edges.

7 vertices, 6 edges.

Bipartite graph.



equally likely

Color each of the "n" vertices randomly with blue or green.

$$\Omega = \{b, g\}^n \quad , \quad P(\{\omega\}) = \frac{1}{2^n}.$$

Let the edges in the original graph be $H_1, H_2, \dots H_m$.

From the coloring a bi-partite graph is formed by picking all blue into one set and greens into other and by retaining edges that have different colored vertices and ignoring the rest

$H_i = (1, 2) \rightarrow$ vertices 1 and 2.

(b, b) then H_i is not part of bi-partite

Probability that H_i is retained in the bi-partite subgraph

$$= \underset{(b,g)}{\frac{1}{2} \times \frac{1}{2}} + \underset{(g,b)}{\frac{1}{2} \times \frac{1}{2}} = \frac{1}{2}.$$

$$X_i = \begin{cases} 1 & \text{if } H_i \text{ is not monochromatic} \\ 0 & \text{Otherwise.} \end{cases}$$

$P(X_i = 1) = \frac{1}{2}.$

$E[X_i] = \frac{1}{2}.$

Total # of edges that get included in bi-partite sub graph is $X = \sum_{i=1}^{m} X_i$

$$E[X] = m\, E[X_i] = \frac{m}{2}.$$

$$\Rightarrow P\left(X \geq m/2\right) > 0.$$

$\Rightarrow \exists\ \omega$ st $X(\omega) \geq \frac{m}{2}.$

a coloring

$\Rightarrow \exists$ a bi-partite subgraph with more than $\frac{m}{2}$ edges.

② **Example**: There is generalization to graphs called hypergraphs.

Hyper graphs are described using vertex set S and hyper edge set $E = \{H_1, \dots, H_m\}$ where $H_i \subseteq S$, $|H_i| = l$.

(hyper edge can contain more than two vertices)

2-coloring problem: Can we assign a coloring to the vertices such that each hyper-edge is not monochromatic.

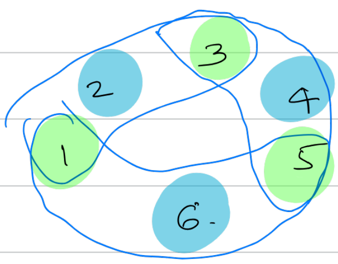
If $m < 2^{l-1}$ then 2-coloring is possible.

l → # vertices in a hyperedge.

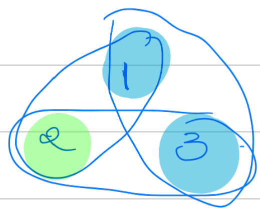
m → # of hyper edges.

Examples.

$H_1 = \{1, 2, 3\}$
$H_2 = \{3, 4, 5\}$
$H_3 = \{1, 5, 6\}$



$m = 3$, $l = 3$.
$3 = m < 2^{l-1} = 4$.



$H_1 = \{1, 2\}$
$H_2 = \{2, 3\}$
$H_3 = \{3, 1\}$

$m = 3$, $l = 2$.
doesn't satisfy.

$\Omega = \{b, g\}^n$ → # vertices

$P(\{w\}) = \frac{1}{2^n}$.

E_i be an event that the edge H_i is monochromatic.

$P\left(\bigcup_{i=1}^{m} E_i\right)$ → event that $\exists$ an edge that is monochromatic.

We want to show $P\left(\left(\bigcup_{i=1}^{m} E_i\right)^c\right) > 0.$

$$\underbrace{P\left(\bigcup_{i=1}^{m} E_i\right) \le \sum_{i=1}^{m} P(E_i)}_{\text{union bound}} \qquad P\left(\underbrace{\bigcap_{i=1}^{m} E_i^c}_{\text{all edges are non-monochromatic}}\right)$$

E_i : event that edge i is mono-chromatic

$$= \{\omega \in \Omega : \forall j \in H_i,\ \omega_j = b\} \cup \{\omega \in \Omega : \forall j \in H_i,\ \omega_j = g\}$$

edge i has vertices $1, 2, 3$.

$(b, b, b, \boxed{})$

$(g, g, g, \underbrace{\boxed{xxxx}}_{n-l})$

$\boxed{|H_i| = l}$

$$P(E_i) = \frac{|E_i|}{2^n} = \frac{2^{n-l} + 2^{n-l}}{2^n}$$

$$= \frac{1}{2^{l-1}}$$

$$P\left(\bigcup_{i=1}^{m} E_i\right) \le \sum_{i=1}^{m} P(E_i) = \frac{m}{2^{l-1}}.$$

$$P\left(\bigcap_{i=1}^{m} E_i^c\right) = 1 - P\left(\bigcup_{i=1}^{m} E_i\right)$$

$$\ge 1 - \frac{m}{2^{l-1}}$$

$$> 0. \qquad \text{If } m < 2^{l-1} \Rightarrow \frac{m}{2^{l-1}} < 1$$

$\Rightarrow \exists$ a coloring such that all the edges are not monochromatic

$\Rightarrow$ existence of 2-coloring.

Probabilistic method by Noga Alon & Spencer.