SLT/Concentration Inequalities **2025**

Exam 1: August 2025

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Instructions: *This is a closed-book exam. You are not permitted to refer to any material or discuss the problem with anyone. Malpractice will be severely punished. Please mention your ROLL Number and name clearly in the answer sheet.*

Justify all your statements clearly. You may the concentration inequalities derived in class without proof (but state which inequality you use), but everything else needs to be proved.

Question 1.1 (3+3+4+5pts). Consider a random variable X with probability density function

$$f_X(x) = \frac{e^{-|x|/b}}{2b}$$

Derive the mean, variance and moment generating function from first principles. Use the Chernoff bound to obtain an upper bound on $\Pr[|X - \mathbb{E}[X]| \geqslant \epsilon]$, for any $\epsilon > 0$, and find the value of $t > 0$ that minimizes the bound.

Question 1.2 (5pts). Let X be a random variable and $p > 1$. Prove that

$$\mathbb{E}[|X|^p] = \int_0^\infty p u^{p-1} \Pr[|X| > u] du$$

PS >0, <0

Question 1.3 (5+10pts). Suppose that we want to estimate the mean of a distribution from iid samples $X_1, \ldots, X_n$. We want to get estimates that have an error at most ϵ, i.e., if $f(X_1, \ldots, X_n)$ is one estimate, then we want $f(X_1, \ldots, X_n) \in (\mu - \epsilon, \mu + \epsilon)$ with probability at least $1 - \delta$. Here, ϵ, δ are parameters that decide the quality of the estimate.

Design an estimator that achieves $\delta = 3/4$ and arbitrary $\epsilon > 0$ using $n = c_1\sigma^2/\epsilon^2$ samples, where σ^2 is the variance of X_1 and c_1 is a universal constant.

Design an estimator that achieves arbitrary $\delta > 0, \epsilon > 0$ this using

$$n = c\frac{\sigma^2}{\epsilon^2} \log \frac{1}{\delta},$$

samples, where σ^2 is the variance of X_1 and c is a universal constant.

Question 1.4 (3+3+5+5). Let $\mathcal{G}$ be an Erdös-Rényi random graph on n vertices, where for every $i \neq j$, vertex i is connected to vertex j (independently of all other pairs of vertices) with probability $p \in [0, 1]$.

Find the expected number of edges, and average degree of each vertex in $\mathcal{G}$. Let D_i denote the degree of vertex i in $\mathcal{G}$. Show that for any fixed $\delta > 0$

$$\Pr[|D_i - \mathbb{E}[D_i]| \leqslant \delta\mathbb{E}[D_i] \text{ for all } i = 1, 2, \ldots, n] \to 1 \text{ as } n \to \infty$$

as long as $p \leqslant \frac{c \log n}{n}$, for some universal constant $c > 0$.

Also show that if $p = 1/n$ then for any i,

$$Pr\left[D_i \geqslant c' \frac{\log n}{\log\log n}\right] \geqslant \frac{1}{en}(1 + o(1))$$

for some universal constant $c' > 0$ and $o(1) \to 0$ as $n \to \infty$.

You may use any of the following results without proof (but state which result you use):

- Any of the concentration inequalities or bounds proved in class
- (B1) If $X_1, \ldots, X_n$ are iid Bernoulli(p) random variables, then for $0 \leqslant \delta \leqslant 1$

$$\Pr\left[\left|\frac{1}{n}\sum_{i=1}^{n} X_i - p\right| \geqslant \delta\right] \leqslant 2e^{-np\delta^2/3}$$

- (B2) If $X_1, \ldots, X_n$ are iid random variables where each X_i is 1 with probability 0.5 and -1 with probability 0.5, then for any $\delta > 0$ and $\underline{a} \in \mathbb{R}^n$,

$$\Pr\left[|\sum_{i=1}^{n} X_i| \geqslant \delta\right] \leqslant 2\exp\left(-\frac{\delta^2}{2\|\underline{a}\|_2^2}\right)$$

- (B3) For positive integers $n \geqslant k$,

$$\left(\frac{n}{k}\right)^k \leqslant \binom{n}{k} \leqslant \left(\frac{ne}{k}\right)^k$$

$$\Gamma(n) = \int_0^\infty x^n e^{-x}\,dx = (n-1)!$$

Chernoff : $\dfrac{\mathbb{E}[e^{tX}]}{e^{t(\mu-\delta)}}$