AI1110 Probability and Random Variables

Lecture 3: Continuous random variables

Feb 5, 2024

Consider a random variable X which takes values in $\mathcal{X} \subseteq \mathbb{R}$.This lecture deals with the scenario when $\mathcal{X}$ is uncountable.

In general for such random variables, the only reasonable way to define $P(X = i)$ is $P(X = i) = 0$. Instead of $\{X = i\}$, we focus on events of the form $\{a < X \leq b\}$.

The CDF (cumulative distribution function), as in the case of discrete random variables is $F_X(x) = P(X \leq x)$. Similar to earlier , it has the properties

Unlike the case when X is a discrete random variable, when X is a continuous random variable, the cdf $F_X(.)$ is a continuous function of x.

1. $0 \leq F_X(x) \leq 1$ for all x.
2. $F_X(x)$ is a non-decreasing function of x.
3. $\lim_{x \to -\infty} F_X(x) = 0$ and $\lim_{x \to \infty} F_X(x) = 1$.

The PDF (probability density function) of X is a function $f_X : \mathbb{R} \mapsto \mathbb{R}$ such that

Recall the parallels to volumetric density or specific mass, etc. Similar to how integrating volumetric mass density gives mass, integrating the pdf gives probability.

1. f is non-negative,i.e. $f_X(x) \geq 0$,
2. The area under f is 1, i.e. $\int_{-\infty}^{\infty} f_X(x)dx = 1$, and
3. $P(a < X \leq b) = \int_a^b f_X(x)dx$ for any a, b.

We say that a random variable X is continuous if it has a PDF.

Given the pdf f_X of X we have

$$f_X(x) = \lim_{\Delta x \to 0+} \frac{P(x < X \leq x + \Delta x)}{\Delta x} = \frac{dF_X}{dx}.$$

Thus the pdf is the derivative of the cdf

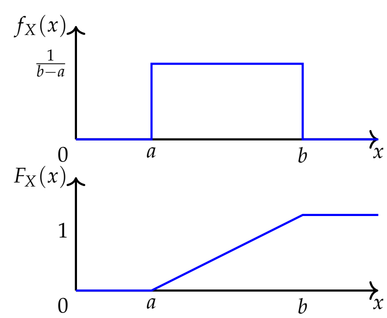


Figure 1: PDF (top) and CDF (bottom) plots for Uniform

Uniform random variable

A random variable X is uniformly distributed between a and b (we write $X \sim U[a, b]$) if the pdf f_X is given by

$$f_X(x) = \begin{cases} 1/b - a \text{ if } a \leq x \leq b \\ 0 \text{ otherwise.} \end{cases}$$

A model where no value of X is preferred over other.

Exponential random variable

We say a random variable X is exponentially distributed if

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} \text{ for } x \geq 0 \\ 0 \text{ otherwise.} \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$[1]. We have $F_X(x) = 1 - e^{-\lambda x}$ and $\bar{F}_X(x) = e^{-\lambda x}$ for $x \geq 0$.

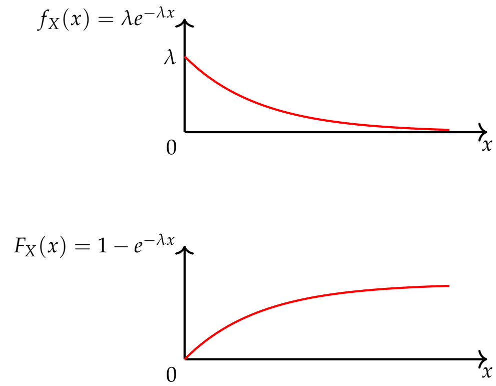


Figure 2: PDF (top) and CDF (bottom) plots for the exponential rv

[1] Exponential random variable is often used to model waiting times.

THE MEMORYLESSNESS property of the exponential random variable refers to the following:

$$Pr(X > t + s | X > t) = P(X > s) \text{ for any t,s} > 0.$$

This property also holds for the geometric random variable (For geometric r.v. t, s in the above expression would be arbitrary natural numbers).

CONNECTION TO GEOMETRIC RV: Consider the following sequence of experiments: toss a coin multiple times, with each coin having a probability of landing heads p (independent of all the other coin tosses). Suppose these coins are tossed every δ seconds (with $\delta <<$ 1).

We have

$$\begin{aligned} P(\text{ first head after } x \text{ seconds}) &= P(\text{no head in } x/\delta \text{ coin tosses}) \\ &= (1-p)^{x\delta} = e^{-\lambda x}, \end{aligned}$$

where $\lambda = -\ln(1-p)/\delta$.

Alternately, we can consider a setup with n coin tosses, and p such that $np = \lambda$ for a fixed $\lambda > 0$. In the limit as $n \to \infty$, we are interested in the time till the first head. We return to this limiting process when we discuss moment generating functions.

Note that here the geometric distribution approaches the exponential distribution, when δ is small enough

Gaussian random variable

We say that X is a Gaussian random variable if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

We write $X \sim N(\mu, \sigma)$.Note that it is not straightforward to check if the above is indeed a valid pdf. For example, we can first show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1,$$

and use this to show that the area under a Gaussian pdf is 1.

The parameter μ controls the centre of the pdf curve, and the parameter σ controls its width.

The Gaussian is a very important and useful analytical model for many phenomena.

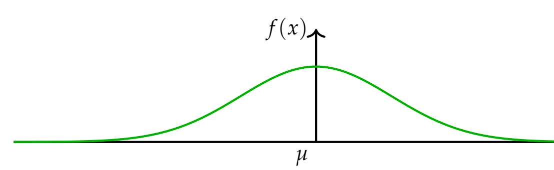
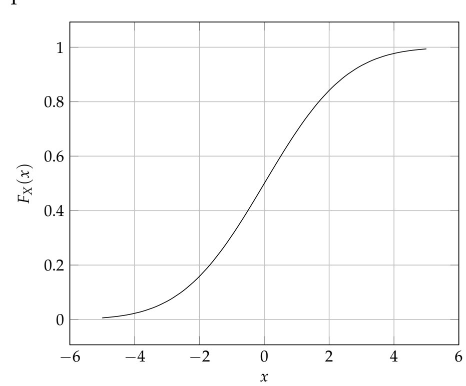


Figure 3: PDF (top) and CDF (bottom) plots for the Gaussian rv



Mean

Similar to discrete random variables, the mean for continuous random variables is defined as

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx.$$

The following can be readily verified

1. If $X \sim U[a, b]$, then $E(X) = (a + b)/2$.
2. If $X \sim \text{Exp}(\lambda)$, then $E(X) = 1/\lambda$,
3. IF $X \sim N(\mu, \sigma)$, then $E(X) = \mu$.

Mapping an arbitrary random variable to uniform

Suppose X is a continuous random variable with a strictly increasing pdf $F_X(x)$. Consider the new random variable Y obtained by applying the function $F_X()$ to X:

$$Y = F_X(X),$$

then $Y \sim U[0, 1]$ is a uniform random variable.

The importance of this result stems from its applicability to random number generation on a computer. If we can design a uniform random number generator, we can use that generator to generate a random variable with any other pdf!

PROOF: Fix $0 \le y \le 1$. Then $P(Y \le y) = P(F_X(X) \le y) = P(X \le F_X^{-1}(y)) = F_X(F_X^{-1}(y)) = y$.

Suggested reading

5.1-5.4 from [2] and 3.1-3.3 from [3]

These notes summarise the lectures, have not been proofread thoroughly, and are not a substitute for the notes you take in class. Please write to the teaching team about any errors you notice.

[2] Sheldon Ross. *First Course in Probability, A*. Pearson Higher Ed, 2019

[3] Dimitri P Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. 2002

References

[1] Dimitri P Bertsekas and John N Tsitsiklis. *Introduction to probability*, volume 1. 2002.

[2] Sheldon Ross. *First Course in Probability, A*. Pearson Higher Ed, 2019.