1. Find the line integral of $f(x, y, z) = 2xy + \sqrt{z}$ over the helix

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \pi.$$

Solution:

For the helix, we find

$$\mathbf{v}(t) = \mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}, \quad \text{and} \quad |\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}.$$

Evaluating the function f along the path, we obtain

$$f(\mathbf{r}(t)) = f(\cos t, \sin t, t) = 2\cos t \sin t + \sqrt{t} = \sin 2t + \sqrt{t}.$$

The line integral is given by

$$\int_C f(x, y, z)\, ds = \int_0^{\pi} (\sin 2t + \sqrt{t})\sqrt{2}\, dt.$$

$$= \sqrt{2}\left[-\frac{1}{2}\cos 2t + \frac{2}{3}t^{3/2}\right]_0^{\pi}.$$

$$= \frac{2\sqrt{2}}{3}\pi^{3/2} \approx 5.25.$$

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2. A fluid's velocity field is

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}.$$

Find the flow along the helix

$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq \frac{\pi}{2}.$$

Solution:

We evaluate $\mathbf{F}$ on the curve:

$$\mathbf{F} = x\mathbf{i} + z\mathbf{j} + y\mathbf{k} = (\cos t)\mathbf{i} + t\mathbf{j} + (\sin t)\mathbf{k}.$$

Then we find $\frac{d\mathbf{r}}{dt}$:

$$\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k}.$$

Next, we integrate $\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}$ from $t = 0$ to $t = \frac{\pi}{2}$:

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = (\cos t)(-\sin t) + (t)(\cos t) + (\sin t)(1)$$

$$= -\sin t \cos t + t \cos t + \sin t.$$

So,

$$\text{Flow} = \int_{t=a}^{t=b} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^{\pi/2} (-\sin t \cos t + t \cos t + \sin t)\, dt.$$

$$= \left[\frac{\cos^2 t}{2} + t \sin t \right]_0^{\pi/2}.$$

$$= \left(0 + \frac{\pi}{2}\right) - \left(\frac{1}{2} + 0\right) = \frac{\pi}{2} - \frac{1}{2}.$$

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3. Calculate the outward flux of the vector field

$$\mathbf{F}(x, y) = 2e^{xy}\mathbf{i} + y^3\mathbf{j}$$

across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.

Solution

Calculating the flux with a line integral would take four integrations, one for each side of the square. With Green's Theorem, we can change the line integral to one double integral. With $M = 2e^{xy}$, $N = y^3$, C the square, and R the square's interior, we have

$$\text{Flux} = \oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy.$$

Using Green's Theorem, we compute:

$$= \int_{-1}^{1} \int_{-1}^{1} (2ye^{xy} + 3y^2)\, dx\, dy.$$

Evaluating the inner integral:

$$= \int_{-1}^{1} \left[2e^{xy} + 3xy^2\right]_{x=-1}^{x=1} dy.$$

$$= \int_{-1}^{1} \left(2e^{y} + 6y^2 - 2e^{-y}\right) dy.$$

Evaluating the outer integral:

$$= \left[2e^{y} + 2y^3 + 2e^{-y}\right]_{-1}^{1} = 4.$$

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4. Find the circulation of the field $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the curve C in which the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise as viewed from above.

Solution

Stokes' Theorem enables us to find the circulation by integrating over the surface of the cone. Traversing C in the counterclockwise direction viewed from above corresponds to taking the inner normal $\mathbf{n}$ to the cone, the normal with a positive $\mathbf{k}$-component.

We parametrize the cone as

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \quad 0 \le r \le 2, \quad 0 \le \theta \le 2\pi.$$

We then have

$$\mathbf{n} = \frac{\mathbf{r}_r \times \mathbf{r}_\theta}{|\mathbf{r}_r \times \mathbf{r}_\theta|} = \frac{-(r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}}{r\sqrt{2}} = \frac{1}{\sqrt{2}}\left(-(\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} + \mathbf{k}\right).$$

The surface area element is

$$d\sigma = r\sqrt{2}\, dr\, d\theta.$$

The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k} = -4\mathbf{i} - 2r\cos\theta\mathbf{j} + \mathbf{k}.$$

Accordingly,

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \frac{1}{\sqrt{2}}\left(4\cos\theta + 2r\cos\theta\sin\theta + 1\right).$$

Thus, the circulation is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma = \int_0^{2\pi} \int_0^2 \frac{1}{\sqrt{2}}\left(4\cos\theta + r\sin 2\theta + 1\right)\left(r\sqrt{2}\, dr\, d\theta\right).$$

Simplifying,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \int_0^2 (4\cos\theta + r\sin 2\theta + 1)\, r\, dr\, d\theta = 4\pi.$$

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5. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

Given vector field:

$$\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

And the parameterized surface:

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k},$$

where $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

Solution:

Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

Given vector field:

$$\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$$

And the parameterized surface:

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k},$$

where $0 \le r \le 1$, $0 \le \theta \le 2\pi$.

First, compute the partial derivatives:

$$\frac{\partial \mathbf{r}}{\partial r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + \mathbf{k}$$

$$\frac{\partial \mathbf{r}}{\partial \theta} = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}$$

Taking the cross product:

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$$

$$= (-r\cos\theta)\mathbf{i} + (-r\sin\theta)\mathbf{j} + (r)\mathbf{k}$$

Next, compute the curl of $\mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix}$$

$$= (-2y^3)\mathbf{i} + 0\mathbf{j} + (-x^2)\mathbf{k}$$

Now compute the surface integral:

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$$

$$\iint_S (-2y^3\mathbf{i} - x^{\mathbf{2}}) \cdot (-r\cos\theta\mathbf{i} - r\sin\theta\mathbf{j} + r\mathbf{k}) dS$$

$$= \iint_S (2ry^3\cos\theta + rx^2\sin\theta) dS$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$:

$$= \int_0^{2\pi} \int_0^1 (2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta\sin\theta) dr d\theta$$

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Thus, the flux of the curl of $\mathbf{F}$ across S is:

$$-\frac{\pi}{4}$$

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6. (a) Calculate the flux of the vector field

$$\mathbf{F} = x\mathbf{i} + 4xyz\mathbf{j} + ze^x\mathbf{k}$$

out of the box-shaped region $D : 0 \le x \le 3, 0 \le y \le 2, 0 \le z \le 1$.

(b) Integrate $\nabla \cdot \mathbf{F}$ over this region and show that the result is the same value as in part (a), as asserted by the Divergence Theorem.

Solution

(a)

The region D has six sides. We calculate the flux across each side in turn. Consider the top side in the plane $z = 1$, having outward normal $\mathbf{n} = \mathbf{k}$. The flux across this side is given by

$$\mathbf{F} \cdot \mathbf{n} = ze^x.$$

Since $z = 1$ on this side, the flux at a point (x, y, z) on the top is $e^1 = e$. The total outward flux across this side is given by the surface integral:

$$\int_0^3 \int_0^2 e\, dx\, dy = 2e^3 - 2.$$

The outward flux across the other sides is computed similarly, and the results are summarized in the following table:

Side	Unit normal $\mathbf{n}$	$\mathbf{F} \cdot \mathbf{n}$	Flux across side
$x = 0$	$-\mathbf{i}$	$-x^2 = 0$	0
$x = 3$	$\mathbf{i}$	$x^2 = 9$	18
$y = 0$	$-\mathbf{j}$	$-4xyz = 0$	0
$y = 2$	$\mathbf{j}$	$4xyz = 8xz$	18
$z = 0$	$-\mathbf{k}$	$-ze^z = 0$	0
$z = 1$	$\mathbf{k}$	$ze^z = e$	$2e^3 - 2$

The total outward flux is obtained by adding the terms for each of the six sides, giving:

$$18 + 18 + 2e^3 - 2 = 34 + 2e^3.$$

(b)

We first compute the divergence of $\mathbf{F}$, obtaining:

$$\nabla \cdot \mathbf{F} = 2x + 4xz + e^z.$$

The integral of the divergence of $\mathbf{F}$ over D is:

$$\iiint_D (\nabla \cdot \mathbf{F})\, dV = \int_0^1 \int_0^2 \int_0^3 (2x + 4xz + e^z)\, dx\, dy\, dz.$$

Simplifying:

$$\int_0^1 \int_0^2 \left[\int_0^3 (2x + 4xz + e^z)\, dx\right] dy\, dz = \int_0^1 \int_0^2 (8 + 18z + e^z)\, dy\, dz = \int_0^1 (16 + 36z + 2e^z)\, dz.$$

Finally:

$$\int_0^1 (16 + 36z + 2e^z)\, dz = 34 + 2e^3.$$

As asserted by the Divergence Theorem, the integral of the divergence over D equals the outward flux across the boundary surface of D.

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7. Find the net outward flux of the field

$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\rho^3}, \quad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D : 0 < b^2 \leq x^2 + y^2 + z^2 \leq a^2$.

Solution:

The flux can be calculated by integrating $\nabla \cdot \mathbf{F}$ over D. Note that $\rho \neq 0$ in D.

We compute:

$$\frac{\partial \rho}{\partial x} = \frac{1}{2}(x^2 + y^2 + z^2)^{-1/2}(2x) = \frac{x}{\rho}$$

$$\frac{\partial M}{\partial x} = \frac{\partial}{\partial x}(x\rho^{-3}) = \rho^{-3} - 3x\rho^{-4}\frac{\partial \rho}{\partial x} = \frac{1}{\rho^3} - \frac{3x^2}{\rho^5}$$

Similarly,

$$\frac{\partial N}{\partial y} = \frac{1}{\rho^3} - \frac{3y^2}{\rho^5}, \quad \frac{\partial P}{\partial z} = \frac{1}{\rho^3} - \frac{3z^2}{\rho^5}$$

Thus, the divergence is:

$$\nabla \cdot \mathbf{F} = \frac{3}{\rho^3} - \frac{3}{\rho^5}(x^2 + y^2 + z^2) = \frac{3}{\rho^3} - \frac{3\rho^2}{\rho^5} = 0.$$

So the net outward flux of F across the boundary of D is zero by the corollary to the Divergence Theorem. There is more to learn about this vector field F, though. The flux leaving D across the inner sphere S_b, is the negative of the flux leaving D across the outer sphere S_a (because the sum of these fluxes is zero). Hence, the flux of F across S_b, in the direction away from the origin equals the flux of F across S_a in the direction away from the origin. Thus, the flux of F across a sphere centered at the origin is independent of the radius of the sphere. What is this flux? To find it, we evaluate the flux integral directly for an arbitrary sphere Sa. The out- ward unit normal on the sphere of radius a is

For a sphere S_a of radius a, the outward unit normal is:

$$\mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a}.$$

On the sphere,

$$\mathbf{F} \cdot \mathbf{n} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a^3} \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{a} = \frac{x^2 + y^2 + z^2}{a^4} = \frac{a^2}{a^4} = \frac{1}{a^2}.$$

Thus, the flux integral is:

$$\iint_{S_a} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \frac{1}{a^2} \iint_{S_a} d\sigma = \frac{1}{a^2}(4\pi a^2) = 4\pi.$$

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