Lecture-13

1. Line integrals:

(a) Introduction

(b) Path dependence of line integrals.

(c) Line integrals of vector fields.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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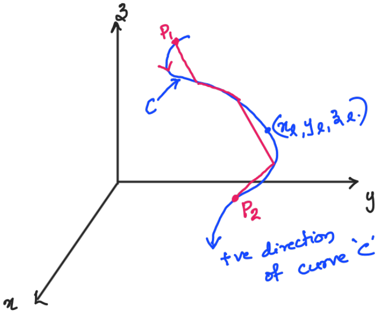
Line integrals:

In one dimension if a force $F(x)$ acts from $x = a$ to $x = b$ the work done is;

$$W = \int_a^b F(x)\, dx .$$

$\because$ Elementary definition of work = force × distance.

For handling more general situations, we introduce the concept of line integral.



- Suppose curve 'C' in three dimensions is directed
- If we devide curve into 'N' subdivisions; and if $f(x_l, y_l, z_l)$ is the functional value at (x_l, y_l, z_l) in l^{th} subdivision with length Δs_l

Line integral of $f(x, y, z)$ along the curve 'C' is;

$$\int_C f(x,y,z)\,ds = \lim_{\substack{N \to \infty \\ \approx \Delta S_\ell \to 0}} \sum_{\ell=1}^{N} f(x_\ell, y_\ell, z_\ell)\, \Delta S_\ell .$$

* To evaluate the line integral we need to know the path 'C'.
* Most convenient way to specify the path is parametrically interms of arclength parameter 's'.

$x = x(s), \quad y = y(s), \quad z = z(s)$

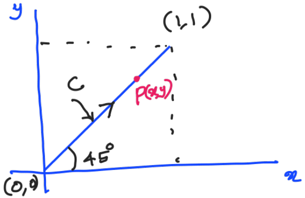
$\Rightarrow$ Line integral reduce to ordinary definite integral.

$$\int_C f(x, y, z)\,ds = \int_{s_1}^{s_2} f\left[x(s), y(s), z(s)\right] ds .$$

Example of line integral.

$\int_C (x+y)\,ds$, where 'C' is the straight line from origin to the point whose co-ordinates are $(1,1)$.

Solution:



$x = s\cos\theta$
$= s\cos 45 = s/\sqrt{2}$

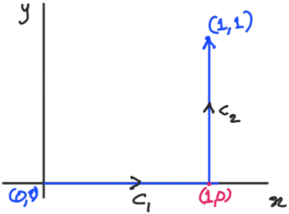
$y = s\sin 45 = s/\sqrt{2}$

If (x,y) are the co-ordinates of any point 'P' on C and if 's' is the arc length measured from origin, then $x = \frac{s}{\sqrt{2}}$; $y = \frac{s}{\sqrt{2}}$. Hence, $x+y = \frac{2s}{\sqrt{2}} = \sqrt{2}\,s$,

$$\Rightarrow \int_C (x+y)\,ds = \int_0^{\sqrt{2}} \sqrt{2}\,s\,ds = \sqrt{2}\,\frac{s^2}{2}\Big|_0^{\sqrt{2}} = \underline{\underline{\sqrt{2}}}.$$

x, y	$s = (x+y)/\sqrt{2}$
0 0	0
1 1	$\sqrt{2}$

Let us integrate the same function $(x+y)$ from $(0,0)$ to $(1,1)$ along another path:



Solution:

Now $\displaystyle\int_C (x+y)\,ds = \int_{C_1} (x+y)\,ds + \int_{C_2} (x+y)\,ds.$

In $C_1: x = s;\ y = 0$ $\qquad$ $C_2: x = 1;\ y = s$

$$= \int_0^1 s\,ds + \int_0^1 (s+1)\,ds$$

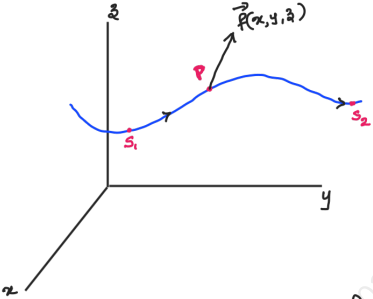
$$= \frac{1}{2} + \frac{3}{2} = \underline{\underline{2}}.$$

* Lesson: The value of a line integral can depend on the path

Line integral involving vector function:

Work done = Force (vector) times displacement (vector).

Consider some path 'C' in three dimensions:



Assume under the action of a force on object moves on this path s_1 to s_2.

At any point 'p' on the curve let the force be $\vec{f}(x,y,z)$.

The component of $\vec{F}$ that does work is,
the component which acts along the curve.
i.e. the tangential component.

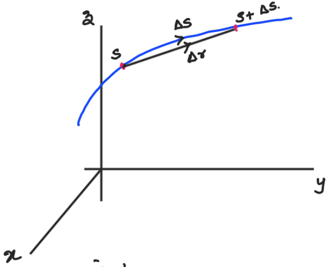
Let $\hat{t}(x,y,z)$ be unit vector that is tangential to the curve at 'p'. The work done by the force in moving the object from s_1 to s_2 along curve 'C' is,

$$W = \int_C \underbrace{\vec{F} \cdot \hat{t}} \, ds \quad ; \quad s \to s_1 \text{ to } s_2 \text{ on 'C'}.$$

Integrand is the dot product of 2 vector functions.

How to find $\hat{t}$?

Let us consider an arbitrary curve 'C' parametrised by its arc length.



* At some point 's' on 'C'

$x = x(s); \; y = y(s); \; z = z(s)$

* At another point 's+Δs' on C

$x + \Delta x = x(s+\Delta s); \; y + \Delta y = y(s+\Delta s)$

$z + \Delta z = z(s+\Delta s)$

* The chord joining s to s+Δs

$\Delta \bar{r} = i\,\Delta x + j\,\Delta y + k\,\Delta z \; \text{---} (1)$

$\Delta x = x(s+\Delta s) - x(s)$

$\Delta y = y(s+\Delta s) - y(s);$

$\Delta z = z(s+\Delta s) - z(s).$

Devide (1) by 'Δs'

$$\frac{\Delta \bar{r}}{\Delta s} = i\,\frac{\Delta x}{\Delta s} + j\,\frac{\Delta y}{\Delta s} + k\,\frac{\Delta y}{\Delta s}$$

$$\Delta s \to 0 \Rightarrow \quad \hat{t}(s) = i\,\frac{dx}{ds} + j\,\frac{dy}{ds} + k\,\frac{dz}{ds}.$$

Why?

* As $\Delta s \to 0$ $\Delta \bar{r}$ becomes tangent to the curve at 's'.
* As $\Delta s \to 0$ $|\Delta \bar{r}| \to \Delta s$ $\Rightarrow$ in the limit $\Delta s \to 0 \left|\frac{\Delta \bar{r}}{\Delta s}\right| = 1.$

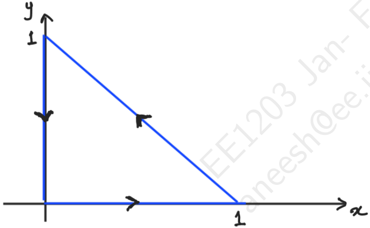
Now

$$W = \int_C \bar{F} \cdot \hat{t}\, ds = \int_C \left[(F_x i + F_y j + F_z k) \cdot \left(i \frac{dx}{ds} + j \frac{dy}{ds} + k \frac{dz}{ds} \right) \right] ds$$

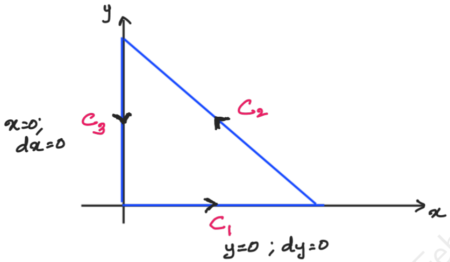
$$\boxed{W = \int_C F_x dx + F_y dy + F_z dz}$$ ← Formal expression.

* It is useful sometime to restore 'ds' to carry out integration.

Example: Calculate $\int_C \bar{F} \cdot \hat{t}\, ds$ for $\bar{F} = iy - jx$ in 'C'



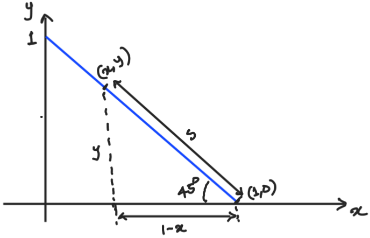
<u>Solution:</u> Break the path C into three parts C_1, C_2, C_3.



$$\int_C \bar{F}\cdot\hat{t}\, ds = \int_C F_x dx + F_y dy = \int_C y\,dx - x\,dy$$

$$\int_{C_1} y\,dx - x\,dy = 0; \qquad \int_{C_3} y\,dx - x\,dy = 0.$$

$$\int_C \left(y\frac{dx}{ds} - x\frac{dy}{ds}\right) ds = \int_{C_2}\left(y\frac{dx}{ds} - x\frac{dy}{ds}\right) ds.$$



$$\frac{1-x}{s} = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

$$\frac{y}{s} = \sin 45^\circ = \frac{1}{\sqrt{2}}$$

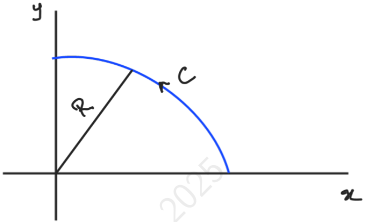
$$\left.\begin{aligned} x &= 1 - \frac{s}{\sqrt{2}}\,; \quad \frac{dx}{ds} = \frac{-1}{\sqrt{2}} \\ y &= \frac{s}{\sqrt{2}}\,; \quad \frac{dy}{ds} = \frac{1}{\sqrt{2}} \end{aligned}\right\} 0 \le s \le \sqrt{2}.$$

$$\Rightarrow \int_{C_2} \left(y\frac{dx}{ds} - x\frac{dy}{ds} \right) ds = \int_0^{\sqrt{2}} \left[\frac{s}{\sqrt{2}}\left(\frac{-1}{\sqrt{2}}\right) - \left(1 - \frac{s}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} \right] ds$$

$$= \int_0^{\sqrt{2}} \frac{-1}{\sqrt{2}}\, ds = \underline{\underline{-1}}.$$

Example 2

Find $\int_C \bar{F}\cdot\hat{t}\, ds$ of $\bar{F} = ix^2 - jxy$; take C to be quarter circle of radius R



Solution:

$$\int_C \bar{F}\cdot\hat{t}\, ds = \int_C \left(x^2\,dx - xy\,dy.\right)$$

Let $x = R\cos\theta$; $y = R\sin\theta$.

$$= \int_0^{\pi/2} \left[R^2\cos^2\theta\,(-R\sin\theta) - R^2\cos\theta\sin\theta\,(R\cos\theta)\right] d\theta$$

$$= -2R^3 \int_0^{\pi/2} \cos^2\theta\sin\theta\, d\theta = \frac{-R^3}{3}.$$

$\cos\theta = x.$

$dx = -\sin\theta\, d\theta.$

$$\Rightarrow 2R^3 \int_1^0 x^2\,dx$$

$$2R^3 \times \left.\frac{-x^3}{3}\right|_0^1.$$

$$= 2R^3 \times \frac{-1}{3}$$

$$= \frac{-2R^3}{3}.$$