Resonance in Circuits

- Consider the following problem: A sinusoidal voltage source is connected to an electrical circuit whose equivalent impendance is given by $\mathbf{z}_{eq} = R_{eq} + jx_{eq}$ (as shown in Fig. 1). Assume that the frequency of the voltage source can be varied while the magnitude and phase of the source remain fixed. The goal is to find the conditions under which the magnitude of the current [1] drawn from the source is maximum? (in steady state). The two parameters that can be varied in this problem are the frequency of the source and the equivalent impedance of the circuit. The equivalent impedance is inturn a function of frequency. Thus, we are interested in finding the frequency at which the magnitude of the current drawn from the source is maximum. The current drawn from the source is given by

[1] RMS value

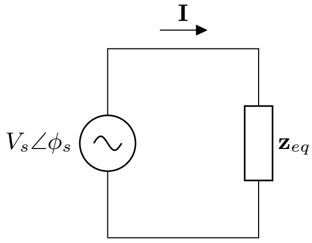


Figure 1: Circuit with variable frequency source

$$\mathbf{I} = \frac{V_s\angle\phi_s}{\mathbf{z}_{eq}} = \frac{V_s\angle\phi_s}{R_{eq} + jx_{eq}} = \frac{V_s}{R_{eq} + jx_{eq}} e^{j\phi_s}$$

The magnitude of current attains its maximum value when the magnitude of impedance attains its minimum value. The magnitude of the equivalent impedance is given by

$$|\mathbf{z}_{eq}| = \sqrt{R_{eq}^2 + x_{eq}^2}$$

The minimum possible value of $|\mathbf{z}_{eq}|$ is R_{eq} which occurs when $x_{eq} = 0$ [2]. Thus, the magnitude of current drawn from the source is maximum when the equivalent reactance of the circuit is zero. This condition is known as **resonance**.

[2] Note that it is not gauranteed that x_{eq} can be made 0 for every possible circuit.

Before we proceed further, lets consider some simple examples to understand the concept of resonance better.

  - Example 1: Series RL circuit: Consider a series RL circuit connected to a variable frequency sinusoidal voltage source as shown in Fig. 2. It is known that the maximum possible value of current drawn from the source occurs when the equivalent reactance of the circuit is zero. This occurs when $\omega = 0$ (i.e., a DC source). This raises an interesting question - do we consider this circuit to be in resonance at $\omega = 0$? Before we answer this question, lets look at another example and see if we can come up with a defintion of resonance that makes sense in both cases.

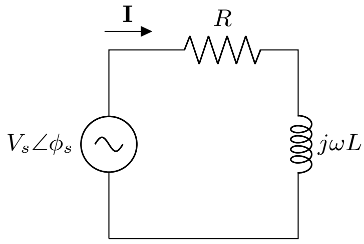


Figure 2: Series RL circuit with variable frequency source

  - Example 2: Series RLC circuit: Consider a series RLC circuit connected to a variable frequency sinusoidal voltage source as shown in Fig. 3. It is known that the maximum possible value of current drawn from the source occurs when the equivalent reactance of the circuit is zero. This occurs when

$$x_{eq} = \omega L - \frac{1}{\omega C} = 0 \implies \omega = \frac{1}{\sqrt{LC}} \tag{1}$$

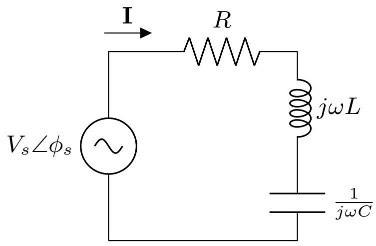


Figure 3: Series RLC circuit with variable frequency source

It is a classic result that the series RLC circuit is said to be in resonance at the frequency $\omega = \frac{1}{\sqrt{LC}}$.

An important question that we need to address now is - if the definition of resonance is simply based on the condition that $x_{eq} = 0$ (in which case, the series RL circuit is also at resonance at $\omega = 0$), or is there some additional condition that needs to be satisfied? To answer this question, lets make the following important observations

– Observation 1: For circuits comprising of only inductors and resistors (like the series RL circuit), the equivalent reactance x_{eq} can be made zero only at $\omega = 0$. However, at $\omega = 0$, the source is a DC source and the inductor behaves like a short circuit. On the other hand, for circuits comprising of only capacitors and resistors (like a series RC circuit), the equivalent reactance x_{eq} can be made zero only at $\omega \to \infty$. However, at $\omega \to \infty$, the source behaves like a very high frequency AC source and the capacitor behaves like a short circuit. In both these cases, there is no energy stored in the magnetic field of the inductor or the electric field of the capacitor respectively. Thus, we do not consider these circuits to be in resonance at these frequencies.

– Observation 2: For circuits comprising of both inductors and capacitors (like the series RLC circuit), it is possible that the equivalent reactance x_{eq} is zero at some non-zero frequency ω_0. At this frequency, there is energy stored in both the magnetic field of the inductor and the electric field of the capacitor. Further, there is continuous exchange of energy between the inductor and the capacitor. Thus, we consider these circuits to be in resonance at these frequencies.

Based on these observations, a suitable defintion of resonance can be stated as follows:

Resonance in Circuits: A circuit comprising of inductors and capacitors (in addition to other linear elements like resistors) is said to be in resonance if there exists a frequency $\omega_0 \in \mathbb{R}$ such that $x_{eq}(\omega_0) = 0$. The frequency at which the circuit is in resonance is called the resonant frequency of the circuit. Some of the properties that a circuit exhibits under resonance conditions are

– Property 1: For circuits at resonance i.e., when $x_{eq} = 0$, the voltage and current are in phase with each other. This is because, when $x_{eq} = 0$, the equivalent impedance of the circuit is purely resistive (i.e., $\mathbf{z}_{eq} = R_{eq}$). Thus if $v(t) = V_m \cos(\omega t + \phi)$, then the current drawn from the source is given by

$$i(t) = \frac{V_m}{R_{eq}} \cos(\omega t + \phi) \text{ (Note } V_m = \sqrt{2}V_s)$$

– Property 2: For circuits at resonance i.e., when $x_{eq} = 0$, the net power drawn from the source is given by

$$\mathbf{S} = \mathbf{VI}^* = \frac{V_s\angle\phi_s \cdot V_s\angle-\phi_s}{R_{eq}} = \frac{V_s^2}{R_{eq}}\angle 0$$

Thus the source supplies only real power to the circuit when $x_{eq} = 0$ or the reactive power drawn from the source is zero.

– Property 3: Consider the series RLC circuit shown in Fig. 3. At resonance, the current through the inductor and capacitor are given by

$$i_l(t) = i_c(t) = i(t) = I_m \cos(\omega_0 t + \phi_s)$$

where $I_m = \frac{\sqrt{2}V_s}{R}$. The voltage across the capacitor is given by

$$v_c(t) = \frac{1}{\omega_0 C} I_m \sin(\omega_0 t + \phi_s)$$

The energy stored in the inductor and the capacitor at any time t are given by

$$\begin{aligned} e_l(t) =& \frac{1}{2} L i_l^2(t) = \frac{1}{2} L I_m^2 \cos^2(\omega_0 t + \phi_s) \text{ and} \\ e_c(t) =& \frac{1}{2} C v_c^2(t) = \frac{1}{2} C \left(\frac{1}{\omega_0 C} I_m\right)^2 \sin^2(\omega_0 t + \phi_s) \end{aligned}$$

The total energy stored in the circuit ($e_s(t)$) at any time t is given by

$$e_s(t) = e_l(t) + e_c(t) = \frac{1}{2} L I_m^2 \cos^2(\omega_0 t + \phi_s) + \frac{1}{2} C \left(\frac{1}{\omega_0 C} I_m\right)^2 \sin^2(\omega_0 t + \phi_s)$$

Using the relation $\omega_0 = \frac{1}{\sqrt{LC}}$, it can be shown that $e_s(t) = \frac{1}{2} L I_m^2$ (a constant). This implies that the total energy stored in the circuit remains constant over time at resonance. Note that the energy stored in the inductor and capacitor vary with time, but their sum remains constant. This implies that there is continuous exchange of energy between the inductor and the capacitor at resonance.

Further, note that the energy dissipated by the resistor is given by

$$e_d(t) = \int_{-\infty}^{t} R i^2(\tau) d\tau = \int_{-\infty}^{t} R I_m^2 \cos^2(\omega_0 \tau + \phi_s) d\tau$$

This is the same as the energy supplied by the source. Thus, at resonance, the energy supplied by the source is completely dissipated by the resistor. The energy disspiated through the resistor per cycle is given by

$$E_d = \int_{t}^{t+T} R I_m^2 \cos^2(\omega_0 \tau + \phi_s) d\tau = \frac{R I_m^2 T}{2}$$

where $T = \frac{2\pi}{\omega_0}$ is the period of the source. Thus, at resonance, the energy stored in the circuit remains constant over time, while the energy dissipated per cycle is given by $\frac{R I_m^2 T}{2}$.

One important characteristic feature that distinguishes a circuit at resonance from a circuit whose $x_{eq} = 0$ at some frequency is that, in the former case, there is continuous exchange of energy between the inductors and capacitors in the circuit, while in the latter case, there is no energy stored in the magnetic field of the inductor or the electric field of the capacitor. In addition, in the former case, the energy stored in the inductors and capacitors are not drawn from the source (in steady-state).

- Property 4: Consider the series RLC circuit shown in Fig. 3. At resonance, the voltage across the inductor and capacitor (in phasor form) are given by

$$\mathbf{V}_L = j\omega_0 L\mathbf{I} = j\omega_0 L\frac{V_s\angle\phi_s}{R} \quad \text{and} \quad \mathbf{V}_C = \frac{1}{j\omega_0 C}\mathbf{I} = \frac{1}{j\omega_0 C}\frac{V_s\angle\phi_s}{R}$$

The voltage across the inductor and the capacitor are equal in magnitude and opposite in phase. Thus the total voltage drop across the LC combination is zero. However, the magnitudes of $\mathbf{V}_L$ and $\mathbf{V}_C$ can be significantly higher than the source voltage $\mathbf{V}_s$. This phenomenon is known as **voltage magnification** at resonance. The voltage magnification factor (VMF) is defined as the ratio of the magnitude of voltage across the inductor (or capacitor) to the magnitude of source voltage, i.e.,

$$\text{VMF} = \frac{|\mathbf{V}_L|}{|\mathbf{V}_s|} = \frac{|\mathbf{V}_C|}{|\mathbf{V}_s|} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Thus, at resonance, the voltage across the inductor and capacitor can be significantly higher than the source voltage.

- **Resonance in Series RLC circuit**: Going forward, we will primarily focus on the series RLC circuit shown in Fig. 3 to illustrate various concepts related to resonance. The equivalent impedance of the series RLC circuit is given by

$$\mathbf{z}_{eq} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The resonant frequency of the circuit is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

To reiterate, at resonance, the following properties hold true:

  - The magnitude of current drawn from the source is maximum and is given by
$$I_{max} = \frac{V_s}{R}$$
  - The voltage and current are in phase with each other.
  - The source supplies only real power to the circuit.
  - The total energy stored in the circuit remains constant over time.

- The voltage across the inductor and capacitor can be significantly higher than the source voltage.

The impedance offered by the circuit at frequencies other than the resonant frequency is given by

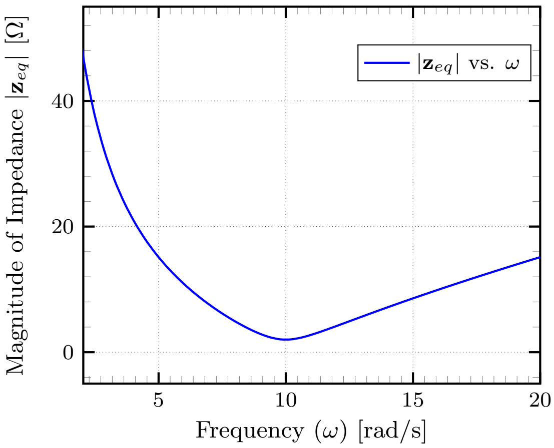
$$\mathbf{z}_{eq} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The magnitude of the impedance $z_{eq} = |\mathbf{z}_{eq}|$ and the phase $\theta_z = \angle \mathbf{z}_{eq}$ are given by

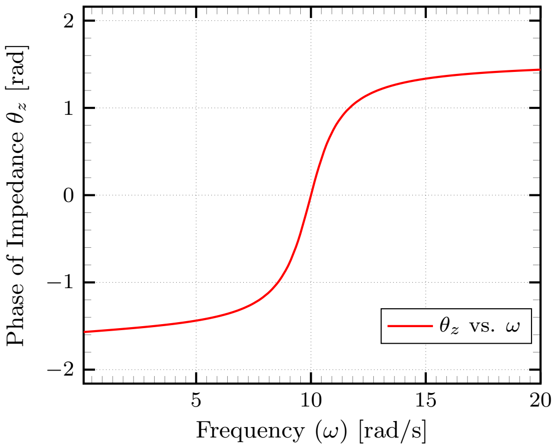
$$|\mathbf{z}_{eq}| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} \quad \text{and} \quad \theta_z = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)$$

Thus, at frequencies other than the resonant frequency, the following properties hold true:

- The magnitude of impedance is greater than R. More specifically, at low frequencies ($\omega < \omega_0$), the circuit behaves like a capacitive circuit (i.e., $x_{eq} < 0$) and at high frequencies ($\omega > \omega_0$), the circuit behaves like an inductive circuit (i.e., $x_{eq} > 0$). The typical variation of $|\mathbf{z}_{eq}|$ with frequency is shown in Fig. 4 (a). On the other hand, the phase of the impedance θ_z is negative at low frequencies and positive at high frequencies. The typical variation of θ_z with frequency is shown in Fig. 4 (b).



(a) Magnitude of impedance



(b) Phase of impedance

Figure 4: Variation of impedance with frequency

- The magnitude of current drawn from the source is less than I_{max} and is given by

$$I = \frac{V_s}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

The real power drwan from the source is given by

$$P = \frac{V_s^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} R$$

Thus the real power drawn from the source is maximum at resonance and decreases as we move away from the resonant frequency. The typical variation of real power with frequency is shown in Fig. 5 (a).

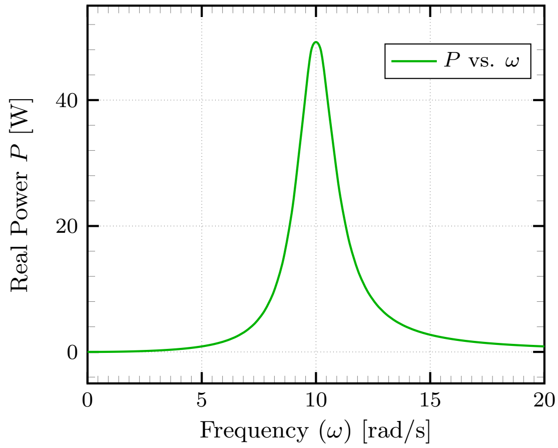


Figure 5: Variation of real power with frequency

– The quality factor (Q-factor) of the circuit is defined as the ratio of energy stored in the circuit to the energy dissipated per cycle at resonance, i.e.,

$$Q = (2\pi)\frac{\text{Energy stored in the circuit at resonance}}{\text{Energy dissipated per cycle at resonance}} = (2\pi)\frac{\frac{1}{2}LI_m^2}{\frac{RI_m^2 T}{2}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}$$

Higher the value of Q-factor, lower is the energy dissipated per cycle relative to the energy stored in the circuit. On the other hand, a higher Q-factor implies that the damping in the circuit is low. Thus, the Q-factor is a measure of the sharpness of resonance in the circuit.

– The range of frequencies for which the power drawn from the source is at least half of the maximum power is called the **bandwidth** of the circuit. In the case of series RLC circuit, the bandwidth is thus the range of frequencies for which

$$P \geq \frac{P_{max}}{2} \implies \frac{V_s^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} R \geq \frac{1}{2} \cdot \frac{V_s^2}{R} \tag{2}$$

For (2) to hold true, it is necessary that

$$\left(\omega L - \frac{1}{\omega C}\right)^2 \leq R^2 \tag{3}$$

In terms of the Q-factor, (3) can be rewritten as

$$\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2 \leq \frac{1}{Q^2} \tag{4}$$

The two frequencies ω_1 and ω_2 at which the equality holds in (4) are called the **half-power frequencies** or **3dB frequencies**. Solving for ω in (4), we get

$$\omega_1 = \omega_0\left(\sqrt{1 + \frac{1}{4Q^2}} - \frac{1}{2Q}\right) \quad \text{and} \quad \omega_2 = \omega_0\left(\sqrt{1 + \frac{1}{4Q^2}} + \frac{1}{2Q}\right)$$

The bandwidth B of the circuit is given by

$$B = \omega_2 - \omega_1 = \frac{\omega_0}{Q}$$

Thus, the bandwidth of the circuit is inversely proportional to the Q-factor of the circuit.

- **Resonance in Parallel RLC circuit**: Consider a parallel RLC circuit connected to a variable frequency sinusoidal current source as shown in Fig. 6. The equivalent admittance of the parallel RLC circuit is given by

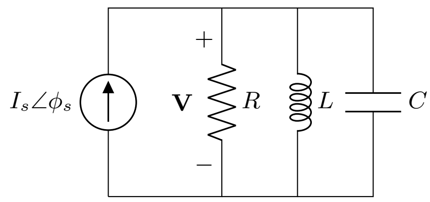


Figure 6: Parallel RLC circuit with variable frequency source

$$\mathbf{y}_{eq} = \frac{1}{R} + j\left(\omega C - \frac{1}{\omega L}\right)$$

The frequency at which the circuit is in resonance is given by

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

In case of the parallel RLC circuit, at resonance, the following properties hold true [3]:

[3] These properties can be derived in a manner similar to that of the series RLC circuit.

  – Since the admittance is maximum at resonance, the magnitude of voltage across the circuit is maximum at resonance and is given by

$$V_{max} = I_s R$$

  – Since the equivalent admittance is purely real at resonance, the voltage and current are in phase with each other. This implies that the source supplies only real power to the circuit.

  – The energy stored in the inductor and capacitor vary with time, but their sum remains constant. Thus, the total energy stored in the circuit remains constant over time.

  – The current through the inductor and capacitor are equal in magnitude and opposite in phase. However, the magnitudes of these currents can be significantly higher than the source current. This phenomenon is known as **current magnification** at resonance.

  – The ratio of energy stored in the circuit to the energy dissipated per cycle at resonance (i.e., the Q-factor) is given by

$$Q = (2\pi)\frac{\text{Energy stored in the circuit at resonance}}{\text{Energy dissipated per cycle at resonance}} = \frac{R}{\omega_0 L} = \omega_0 CR$$

  – The range of frequencies for which the power drawn from the source is at least half of the maximum power (i.e., the bandwidth) is given by

$$B = \omega_2 - \omega_1 = \frac{\omega_0}{Q}$$