1. CROSS PRODUCT
2. Volume of solids
3. Area.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
भारतीय प्रौद्योगिकी संस्थान हैदराबाद
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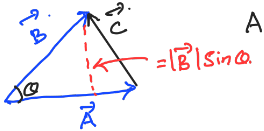
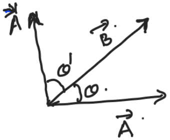
CROSS PRODUCT:

Area of a triangle:



$$\text{Area} = \frac{1}{2}\,\text{base} \times \text{height}.$$

$$= \frac{1}{2} |\vec{A}|\,|\vec{B}| \sin\theta.$$



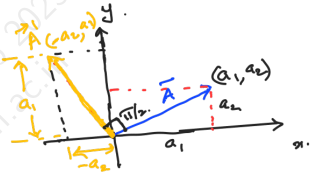
$$\theta' = \frac{\pi}{2} - \theta.$$

$$\Rightarrow |\bar{A}|\,|\bar{B}| \sin\theta.$$

$$= |A'|\,|\bar{B}| \sin\left(\frac{\pi}{2} - \theta'\right)$$

$$= |\vec{A}'|\,|\bar{B}| \cos\theta'$$

$$= \vec{A}' \cdot \vec{B}$$



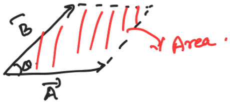
$$\Rightarrow (-a_2, a_1) \cdot (b_1, b_2)$$

$$\Rightarrow a_1 b_2 - a_2 b_1$$

How to calculate $\bar{A}'$?

$$\vec{A} = (a_1, a_2) \Rightarrow \bar{A}' = (-a_2, a_1) \Rightarrow \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

determinant of $\bar{A}$ and $\bar{B}$ $= \pm$ Area of parallellogram



$$\text{Area}(\square) = |\bar{A}||\bar{B}|\sin\theta = \det(\bar{A},\bar{B})$$

$$\text{Area}(\triangle) = \frac{1}{2}|\bar{A}||\bar{B}|\sin\theta = \frac{1}{2}\det(\bar{A},\bar{B})$$

Volume of solid :

* Determinant in space ($\mathbb{R}^3$).

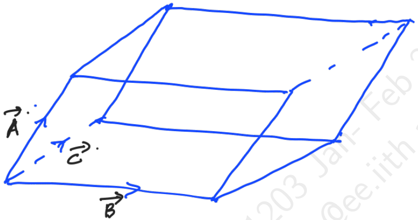
If we have $\bar{A}, \vec{B}, \vec{C}$,

Def:

$$\det(\bar{A},\bar{B},\bar{C}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Theorem: $\det(\bar{A}, \bar{B}, \bar{C}) = \pm$ Volume of parallelepiped.

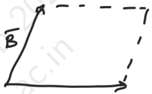


Cross product of 2 vectors in R^3- space.

Def: $\underbrace{\vec{A} \times \vec{B}}_{\text{A vector.}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \hat{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \hat{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \hat{k}.$$

Theorem: $|\bar{A} \times \bar{B}|$ = Area of parallelogram



$= |\vec{A}||\vec{B}| \sin\theta.$

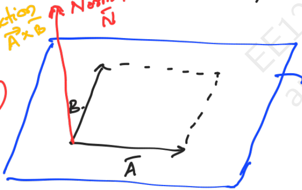
direction $(\bar{A} \times \bar{B})$ is $\perp^r$ to the plane of the parallelogram

check $\Rightarrow \left(\begin{array}{l} (\vec{A} \times \vec{B}) \cdot \bar{A} = 0 \\ (\bar{A} \times \bar{B}) \cdot \bar{B} = 0 \end{array} \right)$

Verify $|\vec{A} \times \bar{B}| = (a_1 b_2 - a_2 b_1)$

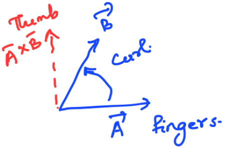
if $\bar{A} = (a_1, a_2, 0)$

$\vec{B} = (b_1, b_2, 0)$



→ plane *

The direction of $\vec{A} \times \vec{B}$ goes with the right hand rule.

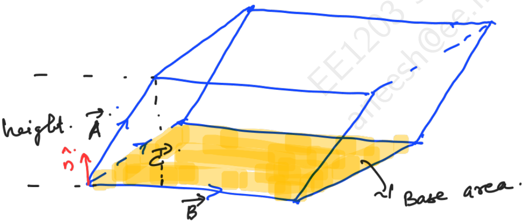


Check with $\hat{i} \times \hat{j} = \hat{k}$?

$$\hat{i}\times\hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \hat{i}\cdot 0 - \hat{j}\cdot 0 + \hat{k}\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \hat{k} \checkmark$$

Volume of parallelepiped. $(\vec{A}, \vec{B}, \vec{C})$.

Volume = Area of base × height.



$$\text{Volume} = \underbrace{|\vec{B} \times \vec{C}|}_{\text{area of base}} \; \underbrace{\vec{A} \cdot \hat{n}}_{\text{Height}} \qquad \hat{n} = \frac{\vec{B} \times \vec{C}}{|\vec{B} \times \vec{C}|}$$

$$= \cancel{|\vec{B} \times \vec{C}|} \; \vec{A} \cdot \frac{\vec{B} \times \vec{C}}{\cancel{|\vec{B} \times \vec{C}|}}$$

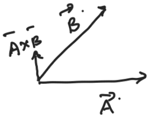
$$= \vec{A} \cdot (\vec{B} \times \vec{C})$$

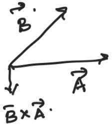
$$= \det(\vec{A}, \vec{B}, \vec{C})$$

$$a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \underbrace{(a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})}_{(\vec{A})} \cdot \left(\underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \hat{i} - \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \hat{j} + \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}}_{\vec{B} \times \vec{C}} \hat{k} \right)$$

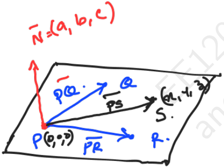
$* \quad \vec{A} \times \vec{B} = - \vec{B} \times \vec{A}.$





$* \quad \vec{A} \times \vec{A} = 0.$

Goal : (a) If given three non-collinear points:
How to define the plane?



$\bar{PQ} = (x_1, y_1, z_1)$

$\bar{PR} = (x_2, y_2, z_2)$

$\vec{PS} = (x, y, z)$

$\det(\vec{PQ}, \vec{PR}, \vec{PS})$ = Volume of parallelepiped.

If all vectors are coplanar

$\Rightarrow$ Volume $= 0$.

$\Rightarrow \det(\vec{PS}, \vec{PR}, \vec{PQ}) = 0$.

$$\Rightarrow \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$$ (Equation of plane)

$$x \underbrace{\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}}_{a} - y \underbrace{\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}}_{b} + z \underbrace{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}}_{c} = 0.$$

$$\Rightarrow \boxed{ax + by + cz = 0.}$$

Alternate way: $\vec{PS} \cdot \vec{N} = 0$ --- Eqn of plane.

$\bar{N}$ is in the direction of $\underline{\overrightarrow{PR} \times \overrightarrow{PQ}}$.

Equation of the plane: $\overrightarrow{PS} \cdot \bar{N} = 0$.

$\Rightarrow \overrightarrow{PS} \cdot (\overrightarrow{PR} \times \overrightarrow{PQ}) = 0$

$\Rightarrow \underline{\underline{\det(\overline{PS}, \overline{PR}, \overline{PQ})}} = 0$.

Example:

The vector $\bar{v} = (2, 4, -2)$ and the plane $x + y + 3z = 5$ are

(a) Parallel ✓

(b) Perpendicular

(c) Neither.

Soln: Eqn of the plane $ax + by + cz = d \Rightarrow \vec{N} = (a, b, c)$

$\therefore \vec{N} = (1, 1, 3)$

Given $\bar{v} = (2, 4, -2)$

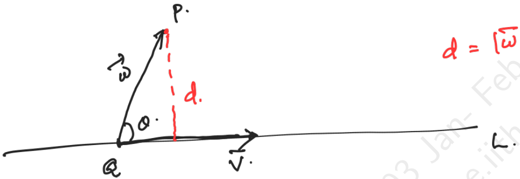
(a) $\bar{N}$ not $\parallel^{l}$ to $\vec{v}$.

$\Rightarrow \vec{v}$ is not perpendicular to the given plane

$$\bar{N}\cdot\bar{U} = (1,1,3)\cdot(2,4,-2) = 2+4-6 = 0.$$

$\Rightarrow \bar{N} \perp^r \bar{U} \Rightarrow \vec{U}$ is $\parallel^l$ to the given plane

Distance between a point (not on the line) and a Line.



$$d = |\vec{w}|\sin\theta.$$

$$|\vec{U}\times\vec{w}| = |\vec{U}|\underbrace{|\vec{w}|\sin\theta}_{d}.$$

$$\Rightarrow d = \frac{|\vec{U}\times\vec{w}|}{|\vec{U}|}$$