Inference.

$$X \longrightarrow \boxed{Y|X} \xrightarrow{Y} \boxed{\text{Estimation}} \longrightarrow \hat{X} = \underbrace{\hat{x}(Y)}_{\text{function}}$$

① Minimize the error probability

$$P(\hat{X} \neq X) = P(\hat{x}(Y) \neq X).$$

$\hat{x}_{MAP}(Y)$ has the smallest probability of error

$$\Rightarrow P(\hat{x}_{MAP}(Y) \neq X) \leq P(\hat{x}(Y) \neq X)$$

for any estimator $\hat{x}(Y)$.

② If X is a continuos R.V then it doesn't make sense to look at $P(\hat{X} \neq X)$.

So we'll look at $E[(\hat{X} - X)^2]$.

as it maybe "enough that $\hat{X}$ is close to X"

Mean Square Error (MSE) of an estimator $\hat{x}(Y)$ is given by

$$MSE = E[(\hat{x}(Y) - X)^2].$$

① Suppose $\hat{x}(Y)$ is a "constant" estimator, what constant should it pick to estimate X.

Let $\hat{x}(y) = c \quad \forall\, y \in \mathbb{R}$.

$$\min_c E[(X-c)^2] = \min_c E[X^2 - 2Xc + c^2]$$

$$= \min_c \underbrace{E[X^2] - 2c\,E[X] + c^2}_{f(c)}.$$

$$\frac{d f(c)}{dc} = 2c^* - 2E[X] = 0$$

$$\Rightarrow c^* = E[X]$$

The best constant estimator for X is $E[X]$.

i.e., $\hat{x}(Y) = E[X]$.

Example: If $X \sim$ Uniform $[4, 10]$ what is the best constant MSE estimator?

$$\hat{x}_c = E[X] = \frac{4+10}{2} = 7.$$

MSE error $= E[(X-7)^2] = \int_4^{10} \frac{(x-7)^2}{6}\,dx$

$$= \int_{-3}^{3} \frac{y^2}{6}\,dy = \frac{\cancel{2}\times \cancel{27}^{3}}{\cancel{3}\times \cancel{6}_{3}} = 3$$

MSE for constant estimator

$$E[(X-\hat{x}_c)^2] = E\{(X-E[X])^2\} = \mathrm{Var}(X).$$

If X has large variance then this is a poor estimate.

② Estimating X from Y. (Minimum MSE estimator).

$$\min_{\hat{x}(\cdot)} E[(X-\hat{x}(Y))^2]$$

$$E\left[(X-\hat{x}(Y))^2\right] = E\left[E\left[(X-\hat{x}(Y))^2 \mid Y\right]\right]$$

$$= \int f_Y(y)\, E\left[(X-\hat{x}(y))^2 \mid Y=y\right].$$

mean is the best constant estimator

$$\geq \int f_Y(y)\, E\left[(X - E[X|Y=y])^2 \mid Y=y\right]$$

$$\Rightarrow \underbrace{\hat{x}(Y)}_{MMSE} = E[X|Y=y] \text{ is the estimator}$$

Minimum MSE — with least amount of error.

ⓐ Example:

$X \sim$ Uniform $[4, 10]$.

$W \sim$ Uniform $[-1, 1]$.

$$Y = X + W$$

$$\hat{x}_{MMSE}(y) = E[X|Y=y]$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Tentative

Quiz 6 time. Thursday 6 pm (24/4/25)

$$(X, W) \Longleftrightarrow (X, Y)$$

$$g_1(x, w) = x$$
$$g_2(x, w) = x + w$$

X and W are iid

$$f_{X,Y}(x,y) = \frac{f_{X,W}(x, y-x)}{J(x,w)} = f_X(x)\, f_W(y-x).$$

$$J(x,w) = \begin{vmatrix} \frac{\partial g_1(x,w)}{\partial x} & \frac{\partial g_1(x,w)}{\partial w} \\ \frac{\partial g_2(x,w)}{\partial x} & \frac{\partial g_2(x,w)}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1.$$

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_W(y-x)}{\int_x f_X(x)\, f_W(y-x)\, dx}.$$

$$f_X(x) = \frac{1}{6} \qquad 4 \le x \le 10.$$

$$f_W(y-x) = \frac{1}{2} \qquad -1 \le y-x \le 1 \Rightarrow y-1 \le x \le y+1$$

$$x-1 \le y \le x+1$$

$$f_Y(y) = \int_x f_X(x)\, f_W(y-x)\, dx.$$

$\Rightarrow y \in [3, 11].$

$$= \int_{\max\{4,\, y-1\}}^{\min\{10,\, y+1\}} \frac{1}{6} \cdot \frac{1}{2}\, dx.$$

$$= \frac{1}{12}\left(\min\{10, y+1\} - \max\{4, y-1\}\right)$$

for $y \in [3, 11]$.

$$= \begin{cases} \frac{1}{12}\left[10 - (y-1)\right] & 9 \le y \le 11. \\ \frac{1}{12}\left[(y+1)-(y-1)\right] & 5 \le y \le 9. \\ \frac{1}{12}\left[(y+1) - 4\right] & 3 \le y \le 5 \end{cases}$$

$$= \begin{cases} \frac{11-y}{12} & 9 \le y \le 11. \\ 1/6. & 5 \le y \le 9. \\ \frac{y-3}{12} & 3 \le y \le 5 \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_W(y-x)}{f_Y(y)}.$$

$$= \begin{cases} \frac{1}{11-y} & y-1 \le x \le 10. \quad 9 \le y \le 11 \\ \frac{1/2 \times 1/6}{1/6} = \frac{1}{2}. & y-1 \le x \le y+1 \quad 5 \le y \le 9 \end{cases}$$

Made with Goodnotes

$\Big(\frac{1}{y-3}$ $\qquad 3 \le y \le 5$

$4 \le x \le y+1$

For $9 \le y \le 11$

$$E[X|Y=y] = \int x f_{X|Y}(x|y)\, dx$$

$X|Y=y \sim \text{Uniform}[y-1, 10]$ $\qquad = \int_{y-1}^{10} \frac{x}{11-y}\, dx.$

$$= \frac{9+y}{2}.$$ (check !).

For $5 \le y \le 9$

$$E[X|Y=y] = y.$$ (check).

$X|Y=y \sim \text{Uniform}[y-1, y+1]$

For $3 \le y \le 5$

$X|Y=y \sim \text{Uniform}[4, y+1].$ $\Big\}$ $E[X|Y=y] = \frac{y+5}{2}.$

(check).

MMSE estimator for this example is

$$\hat{x}(y) = \begin{cases} y & 5 \le y \le 9 \\ \frac{9+y}{2} & 9 \le y \le 11. \\ \frac{5+y}{2}. & 3 \le y \le 5 \end{cases}$$

$$\hat{x}(y) = y\, \mathbb{1}_{\{5 \le y \le 9\}} + \left(\frac{9+y}{2}\right) \mathbb{1}_{\{9 \le y \le 11\}}$$

$$+ \left(\frac{5+y}{2}\right) \mathbb{1}_{\{3 \le y \le 5\}}$$

(b) MMSE Error Analysis.

Let $\tilde{X} = X - \hat{X}$ error RV.

and $\hat{X} = E[X|Y]$.

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad$ $\| \; E[\tilde{X}^2] = Var(\tilde{X})$.

Goal: To show that $\underbrace{E[(X-\hat{X})^2]}_{\text{error seen by MMSE estimator.}} \leq \underbrace{Var(X)}_{\text{error seen by constant estimator}}$.

$$E[\tilde{X}] = E[X - \hat{X}]$$

$$= E[X] - E[\hat{X}]$$

total law of expectation

$$= E[X] - E[E[X|Y]] = E[X] - E[X]$$

$$= 0$$

$E[\tilde{X}|Y] = 0$ as well.

linearity of expectation

$$E[\tilde{X}|Y] = E[X - \hat{X}|Y] = E[X|Y] - E[\hat{X}|Y]$$

This is because $\hat{X}$ is a function of Y

$$= E[X|Y] - \hat{X}.$$

$$= 0 \quad \Big\} \text{ by defn } \hat{X} = E[X|Y]. \text{ is the MMSE estimator}$$

$E[\hat{X}|Y] = \hat{X}$

as $E[Y|Y] = Y$

$E[g(Y)|Y] = g(Y)$.

$\hat{X} = \hat{x}(Y)$

$E[\hat{x}(Y)|Y] = \hat{x}(Y)$

$= \hat{X}$

MMSE Error $= Var(\tilde{X})$

$$= E[(X - \hat{X})^2].$$

Using total law of expectation

$$= E\{(X - E[X|Y])^2\}.$$

$$= E\Big[\underbrace{E\big[(X - E[X|Y])^2 \mid Y\big]}_{Var(X|Y).}\Big].$$

$$= E[Var(X|Y)].$$

From total law of variance, we know

$$Var(X) = E[Var(X|Y)] + Var(E[X|Y]).$$

$$Var(E[X|Y]) \geq 0$$

$$\Rightarrow Var(X) \geq E[Var(X|Y)]$$

$$= \text{MMSE error}.$$

What if Y is independent of X ?

$$f_{Y|X}(y|x) = f_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x)$$

$$E[X|Y=y] = \int x\, f_{X|Y}(x|y)\, dx$$

$$= \int x\, f_X(x)\, dx.$$

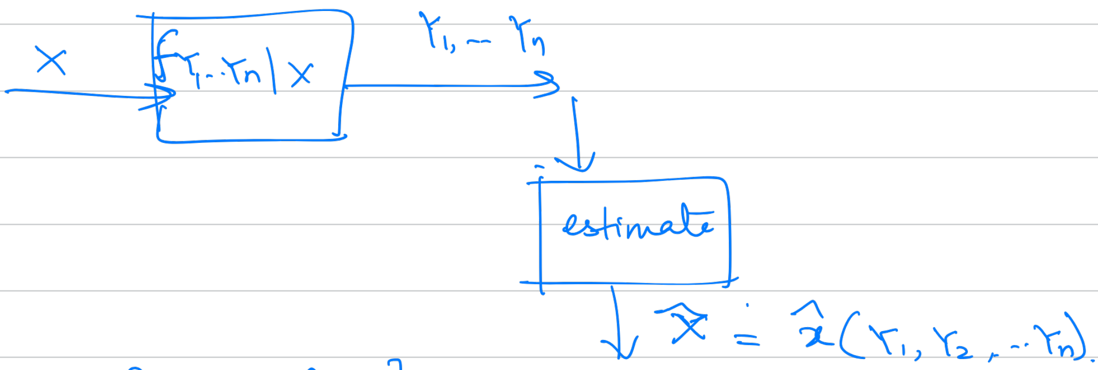
$$= E[X].$$

$$\Rightarrow E[X|Y] = E[X] \quad \text{w.p } 1.$$

$$\underline{\mathrm{Var}\left(E[X \mid Y] \right)} = 0 .$$

$$\Rightarrow \underbrace{\mathrm{Var}(X)}_{\substack{\text{MSE with} \\ \text{best constant} \\ \text{estimator.}}} = \underbrace{E\left[\mathrm{Var}(X \mid Y) \right]}_{\text{MMSE error}}$$

What if you have multiple observations ?



$$MSE = E\left[(X - \hat{X})^2 \right].$$

$$\text{and} \quad \hat{X}_{MMSE} = \hat{x}(Y_1, \ldots Y_n)$$

$$= E\left[X \mid Y_1, \ldots Y_n \right].$$

We assume observations are independent if

$$f_{Y_1 \ldots Y_n \mid X}(y_1 \ldots y_n \mid x) = \prod_{i=1}^{n} f_{Y_i \mid X}(y_i \mid x)$$

③ Linear MSE estimator : linear estimator is a function of Y of the form $aY + b$.

$$\text{LMMSE} = \min_{(a,b)} E\left[\left(X - (aY + b) \right)^2 \right]$$

↓

linear minimum MSE.

$$E\left[\left[(X - aY) - b \right]^2 \right].$$

↓

this is minimized by

$$b = E[X - aY].$$

$$= \min_a \; E\left[\left((X - aY) - E[X - aY]\right)^2\right].$$

$$= \min_a \; E\left[(\bar{X} - a\bar{Y})^2\right]$$

$$\bar{X} = X - E[X]$$
$$\bar{Y} = Y - E[Y].$$

$$= \min_a \; E\left[\bar{X}^2 + a^2\bar{Y}^2 - 2a\bar{X}\bar{Y}\right]$$

$$= \min_a \; E[\bar{X}^2] + a^2 E[\bar{Y}^2] - 2a\, E[\bar{X}\bar{Y}].$$

$$2a\, E[\bar{Y}^2] = 2\, E[\bar{X}\bar{Y}]$$

$$\Rightarrow a = \frac{E[\bar{X}\bar{Y}]}{E[\bar{Y}^2]}$$

$$= \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}.$$

Linear MSE estimator is

$$\hat{x}_{LMMSE}(Y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}\left(Y - E[Y]\right) + E[X]$$

(a) LMMSE error analysis. **Lecture 36**

$E[(X - \hat{X})^2]$. where $\hat{X} = \hat{x}_{LMMSE}(Y)$

$= E[(X - aY - b)^2]$ where $b = E[X] - aE[Y]$, $a = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)}$.

$= E[(\bar{X} - a\bar{Y})^2].$

$$= E[\bar{X}^2] + a^2 E[\bar{Y}^2] - 2a \underbrace{E[\bar{X}\bar{Y}]}_{Cov(X,Y)}$$

$$= E[\bar{X}^2] + \frac{(Cov(X,Y))^2}{Var(Y)^{\cancel{2}}} \cancel{Var(Y)} - 2\frac{Cov^2(X,Y)}{Var(Y)}$$

$$= Var(X) - \frac{Cov(X,Y)^2}{Var(Y)}.$$

$$\leq Var(X).$$

MSE error seen by linear MMSE estimator is larger than the error seen by MMSE estimator

$$Var(X) - \frac{Cov(X,Y)^2}{Var(Y)} \geq Var(X) - Var(\hat{X})$$

where $\hat{X} = E[X|Y]$
$= \hat{x}_{MMSE}(Y).$

To show $\frac{Cov(X,Y)^2}{Var(Y)} \leq Var(\hat{X}).$

We already know that $\rho^2(\hat{X}, Y) \leq 1$

$$\Rightarrow \frac{Cov(\hat{X},Y)^2}{Var(Y)\,Var(X)} \leq 1.$$

We'll now show that $Cov(X,Y) = Cov(\hat{X}, Y).$

$Cov(X,Y) - Cov(\hat{X}, Y)$

$$= E\left[(Y - E[Y])\left(X - E[X] - \hat{X} + E[\hat{X}]\right)\right]$$

$$= E\left[(Y - E[Y])(X - \hat{X})\right].$$

$$= E\left[E\left[(Y - E[Y])(X - \hat{X}) \middle| Y \right] \right]$$

$$= E\left[(Y - E[Y]) \underbrace{E\left[(X - \hat{X}) \middle| Y \right]}_{=0} \right]$$

$$= 0.$$

$$\Rightarrow Cov^2(X, Y) \le Var(\hat{X})\, Var(Y)$$

Continuation of the example

$X \sim U[4, 10]$, $W \sim U[-1, 1]$

Finding MMSE error.

X and W are independent

$Y = X + W$

$$Y | X = x \sim \text{Uniform}\,[x-1, x+1]$$

$$X | Y = y \sim \begin{cases} \text{Uniform}\,[4, y+1] & 3 \le y \le 5 \\ \text{Uniform}\,[y-1, y+1] & 5 \le y \le 9 \\ \text{Uniform}\,[y-1, 10] & 9 \le y \le 11 \end{cases}$$

$$f_{X,Y}(x, y) = f_{X,W}(x, y-x) = \frac{1}{12} \quad \boxed{\begin{matrix} 4 \le x \le 10 \\ x-1 \le y \le x+1 \end{matrix}}$$

$$\hat{x}_{MMSE}(y) = \begin{cases} \frac{y+5}{2} & 3 \le y \le 5 \\ y & 5 \le y \le 9 \\ \frac{y+9}{2} & 9 \le y \le 11 \end{cases}$$

MMSE :

$$E[(X - \hat{x}_{MMSE}(Y))^2].$$

$$= \int_3^{11} f_Y(y) E\left[(X - \hat{x}_{MMSE}(Y))^2 \middle| Y \right] dy.$$

$$= \int_3^5 f_Y(y) \underbrace{\int_4^{y+1} f_{X|Y}(x|y) \left[x - \left(\frac{y+5}{2}\right) \right]^2 dx}_{T1}\, dy$$

$$+\int_5^9 f_Y(y)\int_{y-1}^{y+1} f_{X|Y}(x|y)\,\underbrace{(x-y)^2}_{T2}\,dx\,dy.$$

$$+\int_9^{11} f_Y(y)\int_{y-1}^{10} f_{X|Y}(x|y)\,\underbrace{\left(x-\left(\frac{9+y}{2}\right)\right)^2}_{T3}\,dx\,dy.$$

T1 : $\displaystyle\int_3^5 \frac{1}{12}\int_4^{y+1}\left(x-\left(\frac{y+5}{2}\right)\right)^2 dx\,dy.$

$$\int_3^5 \frac{1}{12}\int_{\frac{3-y}{2}}^{\frac{y-3}{2}} t^2\,dt\,dy.$$

$x-\frac{y+5}{2}=t$

$dx=dt.$

$$\int_3^5 \frac{1}{12}\,\frac{\left(\frac{y-3}{2}\right)^3}{3}\times 2\,dy.$$

$\frac{y-3}{2}=z$

$dy=2dz$

$$=\frac{1}{6\times 3}\int_0^1 2z^3\,dz.$$

$$=\frac{1}{6\times 3}\times\cancel{2}\times\frac{z^4}{\cancel{4}_2}\Big|_0^1=1/36.$$

Can show that T3 is 1/36 as well.

T2 : $\displaystyle\int_5^9 \frac{1}{12}\int_{y-1}^{y+1}(x-y)^2\,dx\,dy.$

$$=\int_5^9 \frac{1}{12}\int_{-1}^{1} t^2\,dt\cdot dy.$$

$$\frac{1}{12}\;\frac{2}{3}\times 4=8/36.$$

Total error $= 1/36+1/36+8/36 = \underline{10/36.}$

$\approx 0.27.$

MSE error for

Linear MMSE estimator

$X \sim \text{Unif}[4, 10]$.

$Var(X) = (10-4)^2/12 = 3$.

$$\hat{x}_{LMMSE}(y) = \frac{Cov(X,Y)}{Var(Y)}\left(Y - E[Y]\right) + E[X]$$

$$Y \mid X = x \sim \text{Uniform}[x-1, x+1]$$

$$E[Y|X] = X, \quad Var(Y|X) = \frac{((\cancel{x}+1)-(\cancel{x}-1))^2}{12} = 1/3.$$

$$E[Y] = E\{E[Y|X]\} = E[X] = 7.$$

$$Var[Y] = Var(E[Y|X]) + E[Var(Y|X)]$$

$$= Var(X) + E\left[\frac{1}{3}\right].$$

$$= 3 + \frac{1}{3} = 10/3$$

$$Cov(X,Y) = E[XY] - \underbrace{E[X]E[Y]}_{= 7^2}$$

$$E[XY] = \int_4^{10}\int_{x-1}^{x+1} f_{X,Y}(x,y)\, xy\, dy\, dx.$$

$$= \int_4^{10} x\left(\int_{x-1}^{x+1} \frac{y\, dy}{12}\right) dx.$$

$$= \int_4^{10} \frac{x}{12} \left.\frac{y^2}{2}\right|_{x-1}^{x+1} dx$$

$$= \int_4^{10} \frac{\cancel{2}x^2}{\cancel{12}\,6} dx.$$

$$= \frac{1}{6}\left.\frac{x^3}{3}\right|_4^{10} = \frac{1000-64}{6\times 3}$$

$$= 52$$

$$\text{Cov}(X, Y) = 52 - 7^2 = 52 - 49 = 3.$$

MSE for linear MMSE estimate is

$$\text{Var}(X) - \frac{\text{Cov}(X,Y)^2}{\text{Var}(Y)}.$$

$$= 3 - \frac{3^2 \times 3}{10} = \frac{30-27}{10} = 0.3.$$

(c) **Linear MMSE under Multiple observations**

$$\hat{x}(Y_1, Y_2, \ldots Y_n). = a_1 Y_1 + a_2 Y_2 + \ldots a_n Y_n + b$$

$$\min_{(a_1, a_2, \ldots a_n, b)} E\left[\left(X - \sum_{i=1}^{n} a_i Y_i - b\right)^2\right].$$

$\Downarrow$

minimized by $b = E[X] - \sum_{i=1}^{n} a_i E[Y_i]$

Orthogonality condition

Let $\hat{X}$ is linear MMSE estimator of X from Y where X and Y are mean 0. then

$$E[(\hat{X} - X)Y] = 0.$$

Proof: $\min_a E[(X - aY)^2]$

$$\min_a \underbrace{E[X^2 + a^2 Y^2 - 2aXY]}_{f(a)}.$$

$$f'(a) = 2a\, E[Y^2] - 2E[XY] = 0.$$

$$E[aY^2 - XY] = 0.$$

$$= E[(aY - X)Y] = 0$$

$$\rightarrow \quad E\left[(\hat{X}-X)\,Y\right]=0.$$

Orthogonality Condition for multiple measurements: Lecture 37

Suppose you have multiple measurements $Y_1, Y_2, \ldots Y_n$

$$\min_{(a_1, a_2 \ldots a_n)} E\left[\left(X - \sum a_i Y_i - b\right)^2\right]$$

$$\text{"} \quad E\left[\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right)^2\right]$$

as

$$b = E[X] - \sum a_i E[Y_i]$$

and

$$\bar{X} = X - E[X]$$

$$\bar{Y}_i = Y_i - E[Y_i]$$

$$f(a_1, \ldots a_n) = E\left[\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right)^2\right].$$

$$\frac{\partial f(a_1, \ldots a_n)}{\partial a_j} = E\left[2\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right) \cdot \frac{\partial}{\partial a_j}\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right)\right]$$

can swap derivative & expectation

$$= E\left[2\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right)\left(-\bar{Y}_j\right)\right] = 0$$

$$\Rightarrow E\left[\left(\bar{X} - \sum_{i=1}^{n} a_i \bar{Y}_i\right) \bar{Y}_j\right] = 0 \quad \forall j = 1, 2, \ldots n \quad \rightarrow (1)$$

Orthogonality condition for n measurements case.

$$E\left[\left(\bar{X} - \hat{X}\right) \bar{Y}_j\right] = 0 \quad \forall j = 1, 2, \ldots n.$$

Rewriting (1) as

$$E\left[\bar{X}\,\bar{Y}_j\right] - \sum_{i=1}^{n} a_i E\left[\bar{Y}_i \bar{Y}_j\right] = 0$$

$$\mathrm{Cov}(X, Y_j) = \sum_{i=1}^{n} a_i \,\mathrm{Cov}(Y_i, Y_j)$$

Varying $j = 1, 2, \dots n$

$$\begin{bmatrix} \mathrm{Cov}(Y_1,Y_1) & \mathrm{Cov}(Y_1,Y_2) & \cdots & \mathrm{Cov}(Y_1,Y_n) \\ \vdots & & & \\ \mathrm{Cov}(Y_n,Y_1) & - & - & \mathrm{Cov}(Y_n,Y_n) \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \mathrm{Cov}(X,Y_1) \\ \mathrm{Cov}(X,Y_2) \\ \vdots \\ \mathrm{Cov}(X,Y_n) \end{bmatrix}$$

Covariance matrix K_Y corresponding to the random vector $Y = (Y_1, \dots Y_n)$.

Cross covariance across X and random vector $Y = (Y_1, \dots Y_n)$ $= R_{Y,X}$.

$$\underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = K_Y^{-1} R_{Y,X}.$$

(If K_Y is non-singular / invertible).

$$\Rightarrow \quad \hat{x}_{LMMSE}(Y_1, Y_2, \dots Y_n) = \sum_{i=1}^{n} a_i Y_i + b$$

$$= a^T Y + b$$

$$= R_{Y,X}^T \left(K_Y^{-1}\right)^T Y + b.$$

K_Y is a symmetric matrix i.e., $K_Y = K_Y^T$

define $R_{Y,X}^T = R_{X,Y}$.

Then $\hat{x}_{LMMSE}(Y) = R_{X,Y} K_Y^{-1} Y + b.$

④ MMSE estimator and Linear MMSE estimator are the same if X, Y are jointly Gaussian.

$$(X, Y) \sim N(0, \sigma_x^2, 0, \sigma_y^2, \rho).$$

$$f_{X,Y}(x, y) = \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{2\rho x y}{\sigma_x \sigma_y}\right]}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

$$= \frac{1}{\sqrt{2\pi}\, \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2} - \frac{(1-\rho^2) y^2}{\sigma_y^2} - \frac{2\rho x y}{\sigma_x \sigma_y}\right]}.$$

$$= \frac{1}{\sqrt{2\pi}\, \sigma_y \sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)\sigma_x^2}\left[x^2 + \frac{\rho^2 y^2 \sigma_x^2}{\sigma_y^2} - 2\rho x\left(y \frac{\sigma_x}{\sigma_y}\right)\right]}$$

$$\Rightarrow X | Y = y \sim N\left(\frac{\sigma_x \rho y}{\sigma_y}, (1-\rho^2)\sigma_x^2\right).$$

$$\hat{x}_{MMSE}(Y) = E[X|Y] = \frac{Cov(X, Y)}{Var(Y)} \cdot Y. = \hat{x}_{LMMSE}(Y)$$