EE2100: Matrix Theory

Assignment - 12

Handed out on 04 - Nov - 2023 **Due on 14 - Nov - 2023 (before 5 PM)**

Instructions :

1. Please submit the solutions to the assignment problems to the course page (on the canvas platform). Solutions submitted to the course page will only be evaluated. Refer to the assignment guidelines mentioned on the course page.
2. Submissions received after the deadline will attract negative marks.
3. It is suggested that you attempt all the questions (preferably the ones indicated using *). However, submitting solutions for problems totaling at least 10 points is sufficient.

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1. (5 Points) Show that the eigen values of $\mathbf{G} = \mathbf{A}^T\mathbf{A}$, where $\mathbf{A} \in \mathcal{R}^{m\times n}$ with $m \geq n$ are real and positive.
2. (5 points) Consider a matrix (say $\mathbf{A} \in \mathcal{R}^{m\times n}$, where $m \geq n$) whose rank is r. Prove that the left singular vectors of $\mathbf{A}$, i.e, $\mathbf{u_1}, \mathbf{u_2}, \cdots, \mathbf{u_r}$ are orthogonal.
3. (10 points) Introduction to Differentiation and Gradient: Consider a function $f(\mathbf{x}) : \mathcal{R}^n \rightarrow \mathcal{R}$, where $\mathbf{x} = [x_1, x_2, \cdots x_n]^T$. The derivative of a function (if it exists; more about this aspect in later courses) is defined as

$$\frac{df}{d\mathbf{x}} = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_n}\right] \tag{1}$$

The gradient (typically denoted by $\triangledown f$) is the transpose of the derivative i.e.,

$$\triangledown f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots \frac{\partial f}{\partial x_n}\right]^T \tag{2}$$

For example, the derivative and gradient of a function $f = \mathbf{a}^T\mathbf{x} = \sum_{i=1}^{n} a_i x_i$ is given by $\frac{df}{d\mathbf{x}} = \mathbf{a}^T$ and $\triangledown f = \mathbf{a}$. On the other hand, for a function $\mathbf{g}(\mathbf{x}) : \mathcal{R}^n \rightarrow \mathcal{R}^m = [g_1(\mathbf{x}), g_2(\mathbf{x}), \cdots, g_m(\mathbf{x})]$, the derivative matrix (also referred to as Jacobian) is defined as

$$\mathbf{Dg}(\mathbf{x}) = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} \tag{3}$$

Consider three functions (a) $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x}$ (where $\mathbf{x} \in \mathcal{R}^n$ and $\mathbf{A} \in \mathcal{R}^{n\times n}$, which is symmetric), (b) $\mathbf{g}(\mathbf{x}) = \mathbf{B}\mathbf{x}$ (where $\mathbf{x} \in \mathcal{R}^n$ and $\mathbf{B} \in \mathcal{R}^{m\times n}$) and, (c) $h(\mathbf{x}) = \|\mathbf{b} - \mathbf{A}\mathbf{x}\|^2$ (where $\mathbf{x}, \mathbf{b} \in \mathcal{R}^n$ and $\mathbf{A} \in \mathcal{R}^{n\times n}$).

   (a) (2 Points) Compute the derivative of $f(\mathbf{x})$.

(b) (3 Points) Compute the derivative of $\mathbf{g}(\mathbf{x})$

(c) (5 Points) Compute $\mathbf{x}$ for which the gradient of $h(\mathbf{x})$ is $\mathbf{0} \in \mathcal{R}^n$.