1. Double integrals:

2. Change of variables in multiintegrals.

3. Triple integral.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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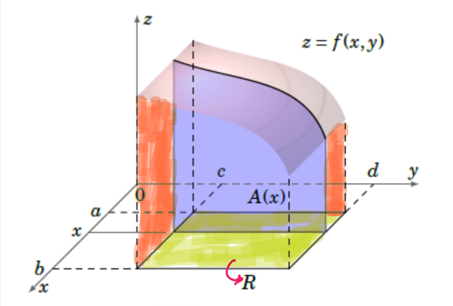
Double integrals: Review.

Def: The double integral of a nonnegative real-valued function $f(x,y) \geq 0$ represents the volume under the surface $z = f(x,y)$



* The expression what are called iterated integrals.
* first function $f(x,y)$ is integrated as a function of 'y' treating 'x' as a constant
* Then perform integration w.r.t 'y'. Thus the name double integrals.

$$V = \int_c^d \left[\int_a^b f(x,y)\, dx \right] dy$$

* Order of integration does not matter.

Ref: Vector Calculus by – Michael Corral

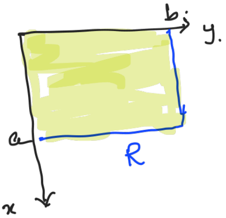
$$A(x) = \int_c^d f(x,y)\, dy.$$

Volume 'V' is then,

$$V = \int_a^b A(x)\, dx.$$

$$= \int_a^b \left[\int_c^d f(x,y)\, dy \right] dx.$$

(a) Understanding finding the bounds:

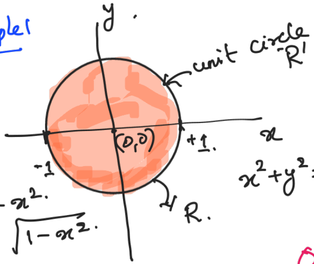


'R' Region over which we integrate $f(x,y)$

$$\Rightarrow \int_{x=0}^{a} \left[\int_{y=0}^{b} f(x,y)\, dy \right] dx.$$

Complicated Region 'R': To find the integrating limits.

Example1



$x^2+y^2=1.$

$$\int_{x=-1}^{+1} \left[\int_{y=-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} f(x,y)\, dy \right] dx.$$

$y^2 = 1-x^2.$

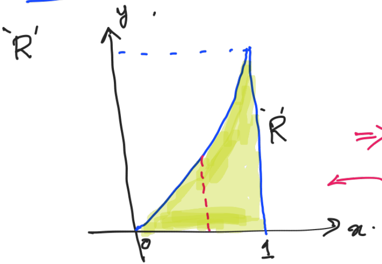
$y = \pm\sqrt{1-x^2}.$

OR

$$\int_{y=-1}^{+1} \left[\int_{x=\sqrt{1-y^2}}^{+\sqrt{1-y^2}} f(x,y)\, dx \right] dy.$$

OR $x = \pm\sqrt{1-y^2}$

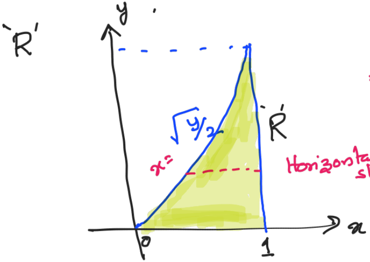
Example 2



$$R = \{(x,y) : 0 \le x \le 1,\ \ 0 \le y \le 2x^2.$$

vertical slice.

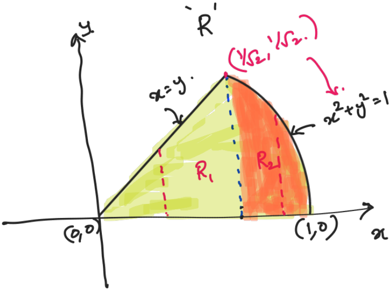
$$\Rightarrow \int_{x=0}^{1} \left[\int_{0}^{2x^2} f(x,y)\, dy \right] dx.$$



Horizontal slice.

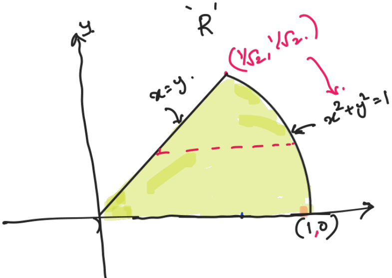
$$\Rightarrow \;=\; \int_{y=0}^{2} \left[\int_{x=\sqrt{y/2}}^{1} f(x,y)\, dx \right] dy.$$

Example 3



$$x^2+y^2=1,\ x=y \Rightarrow x=1/\sqrt{2},\ y=1/\sqrt{2}.$$

$$\Rightarrow \underbrace{\int_{x=0}^{1/\sqrt{2}}\left[\int_{y=0}^{x} f(x,y)\,dy\right]dx}_{R_1} + \underbrace{\int_{x=1/\sqrt{2}}^{1}\left[\int_{y=0}^{\sqrt{1-x^2}} f(x,y)\,dy\right]dx}_{R_2}$$



$$\Rightarrow \int_{y=0}^{1/\sqrt{2}}\left[\int_{x=y}^{\sqrt{1-y^2}} f(x,y)\,dx\right]dy.$$

iterated integral.

Change of variables in multiple integrals.

Example: $\int_1^2 x^3 \sqrt{x^2-1}\, dx.$; $u = x^2-1$

$x^2 = u+1$; $x = (u+1)^{1/2}$

$du = 2x\,dx.$

$x=1 \Rightarrow u=0$

$x=2 \Rightarrow u=3$

$$\Rightarrow \int_1^2 x^3\sqrt{x^2-1}\,dx = \int_0^3 (1+u)^{3/2}\, u^{1/2} \frac{du}{2(1+u)^{1/2}}$$

$$= \int_0^3 \frac{1}{2}(1+u)\, u^{1/2}\, du.$$

What is happening here?

Let $u = x^2-1$;

Inverse function is; $x = g(u) = \sqrt{u+1}$

If $f(x) = x^3\sqrt{x^2-1}$

$$f(x) = f(gu) = (u+1)^{3/2}\ u^{1/2}$$

$$\frac{dx}{du} = g'(u) \implies dx = g'(u)\,du$$

$$dx = \frac{1}{2}\underline{(u+1)^{-1/2}}\,du.$$

$$g(0) = 1 \implies 0 = g^{-1}(1)$$

$$g(3) = 2 \implies 3 = g^{-1}(2)$$

$$\implies \int_1^2 f(x)\,dx = \int_1^2 x^3\sqrt{x^2-1}\,dx$$

$$= \int_0^3 \frac{1}{2}(u+1)\,u^{1/2}\,du,$$ which can be written as,

$$= \int_0^3 \underline{(u+1)^{3/2}\ u^{1/2}}\ \underbrace{\frac{1}{2}(u+1)^{-1/2}}_{g'(u)}\,du,$$

$$= \int_{g^{-1}(1)}^{g^{-1}(2)} f(g(u))\,g'(u)\,du.$$

EE1203 Jan- Feb 2025 aneesh@ee.iith.ac.in

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In general, if $x = g(u)$ is a differentiable one-to-one function from an interval $[c, d]$ (assume it be on u-axis) onto an interval $[a,b]$ (on the x-axis); then $g'(u) \neq 0$ on the interval $[c,d]$ so that $a = g(c)$ & $b = g(d)$, then $c = g^{-1}(a)$ & $d = g^{-1}(b)$ &

$$\boxed{\int_a^b f(x)\,dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(u))\, g'(u)\, du}$$

Similarly if $x = x(u,v)$ and $y = (u,v)$ define a one-to one mapping of a region R' in the uv-plane onto a region R in the xy-plane such that

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$ is never 0 in R', then

$$\iint_R f(x,y)\, dA(x,y) = \iint_{R'} f(x(u,v), y(u,v))\, |J(u,v)|\, dA(u,v)$$

$dA(x,y)$ and $dA(u,v)$ denote Area element in (x,y) and (u,v) Co-ordinates respectively

$$\boxed{dA(x,y) = |J(u,v)|\, dA(u,v)}$$ is similar to $\boxed{dx = g'(u)\,du}$ in single variable case.

$J(u,v) \Rightarrow$ Jacobian of x and y w.r.t u & v

$J(u,v)$ is also denoted as;

$$\boxed{J(u,v) = \frac{\partial(x,y)}{\partial(u,v)}}.$$

Example: Evaluate $\iint_R e^{x-y/x+y}\, dA$, where $R = \{(x,y) : x \geq 0,\ y \geq 0,\ x+y \leq 1\}$.

Solution: $u = x - y$ ✓

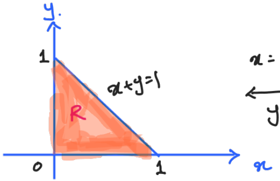
$v = x + y$

We need to write x & y in terms of u, v.

$x = \dfrac{u+v}{2}$; $y = \dfrac{v-u}{2}$

$x = x(u,v) = \frac{1}{2}u + v$; $y = y(u,v) = \frac{1}{2}v - u$.

$(0,1)$
$\Rightarrow u = -1$
$v = 1$
$\Rightarrow u = -v$.

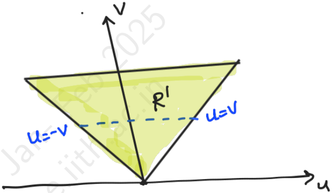


$(1,0) \rightarrow$ $u = 1$, $v = 1$ $\Rightarrow u = v$

$x = \frac{1}{2}(u+v)$

$\longleftarrow$

$y = \frac{1}{2}(v-u)$



$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}. \Rightarrow |J(u,v)| = \frac{1}{2}.$$

If use horizontal slizes, (ie $v =$ Constant)

$$\Rightarrow \iint_R e^{\frac{x-y}{x+y}} \, dA = \iint_{R'} f(x(u,v), y(u,v)) \, |J(u,v)| \, dA.$$

$$= \int_{v=0}^{1} \left[\int_{u=-v}^{v} e^{\frac{u}{v}} \frac{1}{2} \, du \right] dv$$

Example 2:

Change of variable can be used to evaluate double integral in polar co-ordinates

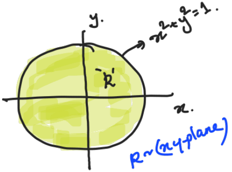
$$x = x(r,\theta) = r\cos\theta, \quad y = y(r,\theta) = r\sin\theta.$$

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$|J(u,v)| = |r| = r$$

Double integral in polar-co-ordinate.

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$



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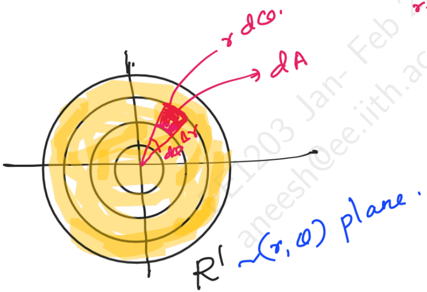
$$\int_{\substack{r=0\\ R'}}^{1}\int_{\theta=0}^{2\pi} f(r\cos\theta, r\sin\theta)\,\underbrace{dA}_{\downarrow\, r\,dr\,d\theta}.$$

$$= \int_{r=0}^{1}\int_{\theta=0}^{2\pi} f(r,\theta)\, r\,dr\,d\theta.$$



Triple integral: $R^3 \to R$.

$$\iiint_S f(x,y,z)\, dV = \lim \sum\sum\sum f(x,y,z)\, \underbrace{\Delta x\, \Delta y\, \Delta z}_{dV}.$$

$dV \downarrow$ volume element.

* It can be thought of representing a hyper volume under a 3-D hypersurface $w = f(x,y,z)$ whose graph lies in R^4.

* Volume 'V' of a solid in R^3 is

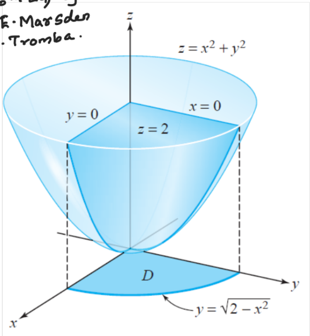
$$\boxed{V = \iiint_S 1\, dV.}$$

In general:

$$\int_{x=a}^{b} \left[\int_{y=h(x)}^{h_2(x)} \left[\int_{z=g_1(x,y)}^{g_2(x,y)} f(x,y,z)\, dz \right] dy \right] dx.$$

Example: Let W be the region bounded by the planes $x=0$, $y=0$ and $z=2$, and the surface $z=x^2+y^2$ and lying in the quadrant $x\ge 0$, $y \ge 0$. Compute $\iiint_W x\,dx\,dy\,dz$ and sketch the region.

Ref: Vector Calculus (6th Ed) by J.E. Marsden, A. Tromba.



'W' is the region below the plane $z=2$, above the paraboloid $z=x^2+y^2$ and on the +ve side of planes $x=0$, $y=0$.

$$\iiint_W x\,dx\,dy\,dz = \int_{x=0}^{\sqrt{2}} \left[\int_{y=0}^{\sqrt{2-x^2}} \left(\underbrace{\int_{z=x^2+y^2}^{2} x\,dz}_{} \right) dy \right] dx.$$

$$= \int_{x=0}^{\sqrt{2}} \left[\int_{y=0}^{\sqrt{2-x^2}} x(2-x^2-y^2)\,dy \right] dx.$$

$$= \int_{x=0}^{\sqrt{2}} x\left[(2-x^2)^{3/2} - \frac{(2-x^2)^{3/2}}{3} \right] dx.$$

$$= \int_{x=0}^{\sqrt{2}} \frac{2x}{3}(2-x^2)^{3/2}\,dx = \frac{8\sqrt{2}}{15}.$$