Inductors

- **Inductors**: From the point of view of circuits, an inductor can be defined as a two-terminal element whose voltage is proportional to time rate of change of flux linkage. Mathematically, this is expressed as

$$e(t) = \frac{d\lambda(t)}{dt} \tag{1}$$

The flux linkage $\lambda(t)$ is related to the current $i(t)$ flowing through the inductor by

$$\lambda(t) = Li(t) \tag{2}$$

where L is the inductance of the inductor, measured in henrys (H). The inductance is typically a positive quantity, and it depends on the physical characteristics of the inductor, such as the number of turns in the coil, the cross-sectional area, and the permeability of the core material. For the circuits we will be considering, the inductance L is assumed to be constant. Thus, the voltage across an inductor (also referred to as the induced voltage) is given by

$$e(t) = v_l(t) = L\frac{di(t)}{dt} \tag{3}$$

The circuit symbol for an inductor is shown in Fig. 1. Note that the voltage-current relation of an inductor (given by 3) is true when a passive sign convention is used.

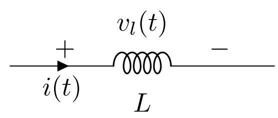

Figure 1: Circuit symbol for an inductor.

- **Flux linkage within the inductor and in the circuit**: When dealing with circuit comprising of inductors, a common source of confusion is the distinction between the flux linkage within the inductor and the flux linkage in the circuit. The flux linkage within the inductor is given by $\lambda(t) = Li(t)$, where L is the inductance of the inductor and $i(t)$ is the current flowing through it. This flux linkage is directly related to the magnetic field generated by the current in the inductor. The flux linkage within the inductor will be significant and results in an induced voltage across the inductor when there is a change in current. On the other hand, it is common practice to assume that the flux linkage associated with the loops of the circuit is still negligible and hence KVL remains valid.

To understand this clearly let us consider a simple circuit with a single inductor, switch and a voltage sournode (as shown in Fig. 2). Note that the inductor current is 0 for $t < 0$. At $t = 0$, the switch is closed and the voltage source $v(t)$ is applied to the circuit. This sets up a current $i(t)$ in the circuit, which increases from 0 to some final value. As the current $i(t)$ changes, the flux linkage within the inductor $\lambda(t) = Li(t)$ also changes, resulting in an induced voltage across the inductor given by $v_l(t) = L\frac{di(t)}{dt}$. Note that although the same current flows through the loop formed by the circuit elements, the flux linkage assocaited with this loop is

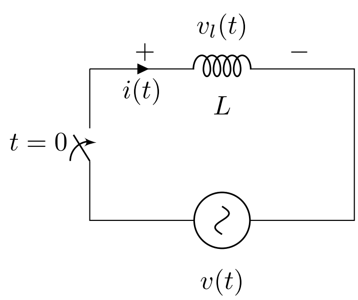

Figure 2: A simple circuit with a single inductor, switch and a voltage source.

typically negligible. This is because the magnetic field generated by the current in the circuit is usually very weak compared to the magnetic field within the inductor. As a result, we can still apply KVL to the circuit, which gives us

$$v(t) = v_l(t) \tag{4}$$

This equation relates the voltage source $v(t)$ to the induced voltage across the inductor $v_l(t)$, allowing us to analyze the circuit using standard circuit analysis techniques. In summary, the flux linkage within the inductor is significant and results in an induced voltage across the inductor, while the flux linkage associated with the loops of the circuit is typically negligible, allowing us to apply KVL to the circuit.

- **Computing indcutor current when the induced voltage is known**: The current through an inductor can be determined if the voltage across the inductor is known. Rearranging Eq. 3, we have

  $$\frac{di(t)}{dt} = \frac{v_l(t)}{L} \tag{5}$$

  Integrating both sides with respect to time (note the limits on either side)

  $$\int_{i(t_0)}^{i(t)} di = \frac{1}{L}\int_{t_0}^{t} v_l(\tau)d\tau \tag{6}$$

  where t_0 is some initial time and τ is a dummy variable of integration. Evaluating the integrals, we get

  $$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v_l(\tau)d\tau \tag{7}$$

- Example 1: Consider the circuit shown in Fig. 2. The switch is closed at $t = 0$ and the voltage source $v(t) = V$ (a constant) is applied to the circuit. If the inductor current is 0 at $t = 0$, determine the expression for the inductor current $i(t)$ for $t > 0$.

  When the switch is closed at $t = 0$, the voltage across the inductor (by KVL) can be expressed as

  $$v_l(t) = Vu(t) \tag{8}$$

  where $u(t)$ is the unit step function. The current through the inductor for $t > 0$ is thus given by

  $$\begin{aligned} i(t) &= i(0) + \frac{1}{L}\int_{0}^{t} v_l(\tau)d\tau \\ &= \frac{V}{L}t, \quad t > 0 \end{aligned} \tag{9}$$

  Note that the inductor current increases linearly with time when a constant voltage is applied across the inductor. Specifying the inductor current at $t = 0$ ensures that the complete solution is obtained. Interestingly, note that the specified initial condition is $i(0) = 0$, which is same as the current through the inductor just before the switch is closed (i.e., for $t < 0$). This is not a coincidence, and is a direct consequence of the continuity property of inductor current, which we will discuss next.

- **Continuity of flux linkage and current**: From the voltage-current relation of an inductor (Eq. 3), we can see that an instantaneous change (like a step jump) in the flux linkage $\lambda(t)$ would require an impulse voltage across the inductor. This is because the derivative of a step function is an impulse function. However, in practical circuits, it is not possible to have an impulse voltage across an inductor, as this would require an infinite amount of energy. Therefore, the flux linkage $\lambda(t)$ cannot change instantaneously in a real circuit. This implies that the current $i(t)$ through an inductor cannot change instantaneously, since $\lambda(t) = Li(t)$.

  In other words, the current through an inductor is continuous, and any sudden change in current would require an infinite voltage across the inductor, which is not physically realizable. This property of inductors is often referred to as the inductor current continuity property. It is an important consideration in circuit analysis and design, as it affects the transient response of circuits containing inductors.

- Example 2: Consider the circuit shown in Fig. 2. The switch is closed at $t = 0$ and the voltage source $v(t) = V_m \cos(\omega t + \phi_v)$ is applied to the circuit.

  When the switch is closed at $t = 0$, the voltage across the inductor (by KVL) can be expressed as

$$v_l(t) = V_m \cos(\omega t + \phi_v)\, u(t) \tag{10}$$

  The current through the inductor for $t > 0$ is thus given by [1]

$$\begin{aligned} i(t) &= \frac{V_m}{L} \int \cos(\omega t + \phi_v)\, u(t) d\tau \\ &= \frac{V_m}{\omega L} \sin(\omega t + \phi_v) + c, \quad t > 0 \end{aligned} \tag{11}$$

  where c is a constant. To determine the value of c, we use the current continuity principle. Since the inductor current cannot change instantaneously, we have

$$i(0) = 0 \implies c = -\frac{V_m}{\omega L} \sin(\phi_v) \tag{12}$$

  Substituting this value of c back into the expression for $i(t)$, we get

$$i(t) = \frac{V_m}{\omega L} \left[\sin(\omega t + \phi_v) - \sin(\phi_v)\right], \quad t > 0 \tag{13}$$

  Unlike Example 1, where the initial current through the inductor was explicitly given, we were still able to determine the complete expression for the inductor current in this case by using the continuity property of inductor current. **Sinusoidal (steady-state) response of an inductor**: The steady-state sinusoidal

[1] Note that in Example 2, we are using indirect integration, where we treat the constant c as an unknown and evaluate it later using the initial condition. This approach involves integrating the voltage over time and adding the constant to the solution. In contrast, in Example 1, we used direct integration, where the limits of integration were chosen to account for the initial condition automatically, eliminating the need for an explicit constant. Both methods are equivalent, but the handling of the constant differs between the two.

response of an inductor can be determined from the voltage-current relation of the inductor. Consider a sinusoidal voltage applied across an inductor, given by

$$v_l(t) = V_m \cos(\omega t + \phi_v) \tag{14}$$

The current through the inductor is given by

$$i(t) = \frac{1}{L}\int v_l(t)dt = \frac{V_m}{\omega L}\sin(\omega t + \phi_v) \tag{15}$$

Note that in this case, the constant of integration is not explicitly considered. This is because we are primarily interested in the sinusoidal steady-state response of the inductor, which describes the behavior after all transient effects have vanished.[2]

[2] To be clearly defined at a later point.

Only when focusing on the steady-state response can we assume that the inductor current is purely sinusoidal. Note that, since the voltage and current are now sinusoidal, we can express them in phasor form as

$$\mathbf{V} = \frac{V_m}{\sqrt{2}}\angle\phi_v \quad \text{and} \quad \mathbf{I} = \frac{I_m}{\sqrt{2}}\angle\phi_i \tag{16}$$

where $I_m = \frac{V_m}{\omega L}$ and $\phi_i = \phi_v - 90°$. This shows that in the sinusoidal steady-state, the current through an inductor lags the voltage across it by $\frac{\pi}{2}$.

When dealing with sinusoidal steady-state analysis of circuits, it is often more convenient to work with phasors and complex impedances rather than time-domain signals. The impedance of a circuit element is defined as the ratio of the phasor voltage across the element to the phasor current through the element. For an inductor, the impedance is given by

$$\mathbf{z} = \frac{\mathbf{V}}{\mathbf{I}} = j\omega L \tag{17}$$

The impedance of an inductor is purely imaginary and positive. The impedance of an inductor increases with frequency while the phase angle is always $\frac{\pi}{2}$. This means that at higher frequencies, the magnitude of the current for the same voltage decreases. This property of inductors is often utilized in various applications, such as filters and tuning circuits. The current through an inductor in sinusoidal-steady state is given by

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{z}} = \frac{\mathbf{V}}{j\omega L} \tag{18}$$

- **Power associated with an Inductor**: The complex power assocaited with an inductor (in sinusoidal steady-state) is given by

$$\mathbf{S} = \mathbf{V}\mathbf{I}^* = (V\angle\phi_v)\left(\frac{V\angle\phi_v}{j\omega L}\right)^* = j\frac{V^2}{\omega L} \tag{19}$$

The active power associated with an inductor is zero ($P = 0$), while the reactive power is positive ($Q = \frac{V^2}{\omega L} > 0$). This indicates that an inductor does not dissipate any real power; instead, it stores energy in its magnetic field during one half of the AC cycle and releases it back to the circuit during the other half. The instantaneous power associated with an inductor is given by

$$s(t) = p(t) + q(t) = P(1 + \cos(2\omega t)) + Q\sin(2\omega t) \;=\; \frac{V^2}{\omega L}\sin(2\omega t) \tag{20}$$

The power factor of the inductor ($\cos\theta = \frac{P}{|\mathbf{S}|}$) is zero. It is often a common practice to write the power factor of an inductor as 0 lagging, since the current lags the voltage by $\frac{\pi}{2}$.

- **Energy stored in the inductor**: To compute the energy associated with an inductor, consider a simplified scenario, where in $v(t) = V_m \cos\omega t$. The current in steady-state is given by $i(t) = \frac{V_m}{\omega L}\sin\omega t$. The energy stored (since v and i are referred using passive sign convention) in the inductor at any time t is given by

$$\begin{aligned} E(t) &= \int_{-\infty}^{t} s(\tau)d\tau = \int_{-\infty}^{t} \frac{V^2}{\omega L}\sin(2\omega\tau)d\tau \\ &= -\frac{V^2}{2\omega^2 L}\cos(2\omega t)\Big|_{-\infty}^{t} = \frac{V^2}{2\omega^2 L}\left[1 - \cos(2\omega t)\right] \\ &= \underbrace{\frac{V^2}{\omega^2 L}}_{LI^2}\sin^2(\omega t) = \frac{1}{2}LI_m^2\sin^2(\omega t) \\ &= \frac{1}{2}Li^2(t) \end{aligned} \tag{21}$$

Note that the energy associated with the inductor is always non-negative. This should not be confused with the fact that energy is always getting stored in the inductor's magnetic field. The energy stored in the inductor increases when the current increases and when the current decreases, the energy stored in the inductor decreases, indicating that the energy is being released back to the circuit. The energy stored in the inductor is maximum when the current through the inductor is at its peak value I_m and is zero when the current is zero. The maximum energy assocaited with the inductor is given by $E_{max} = \frac{1}{2}LI_m^2$