Lec 9: Jan 24

Function of multiple random variables

$$Z = g(X, Y)$$

$$P_Z(z) = \sum_{(x,y) : g(x,y) = z} P_{X,Y}(x, y) \quad \text{(check)}$$

Say $W = \hat{g}(X)$

for function of single R.V.

$$P_W(w) = \sum_{x : \hat{g}(x) = w} P_X(x)$$

$$E[W] = \sum_x \hat{g}(x) P_X(x)$$

for multiple RVs

$$\boxed{E[Z] = \sum_{x,y} g(x,y) P_{X,Y}(x,y) \quad \text{(check)}}$$

Consider $Z = aX + bY + c$

① $E[Z] = aE[X] + bE[Y] + c.$

Proof:

$$\sum_x \sum_y (ax + by + c) P_{X,Y}(x, y)$$

$$= a \sum_x x \underbrace{\sum_y P_{X,Y}(x,y)}_{P_X(x)} + b \sum_y y \underbrace{\sum_x P_{X,Y}(x,y)}_{P_Y(y)} + c \underbrace{\sum_x \sum_y P_{X,Y}(x,y)}_{=1}$$

$$= a \sum_x x P_X(x) + b \sum_y y P_Y(y) + c$$

$$= a E[X] + b E[Y] + c.$$

More than two random variables.

$$P_{X_1, X_2, \ldots X_n}(x_1, \ldots x_n) = P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n).$$

For example

$$P_{X_1, X_2}(x_1, x_2) = \sum_{x_3, \ldots x_n} P_{X_1, \ldots X_n}(x_1, \ldots x_n).$$

$$P_{X_i}(x_i) = \sum_{\substack{x_1 \ldots x_{i-1} \\ x_{i+1}, \ldots x_n}} P_{X_1, \ldots X_n}(x_1, \ldots x_n).$$

→ marginalize over everything except x_i

Linearity of Expectation

$$\boxed{E\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i E[X_i]}$$

(check)

Example ①:

Mean of Binomial (n, p) random variable X.

$X = X_1 + \ldots X_n$ where $X_i \sim \text{Bernoulli}(p)$.

$X_i = 1$ wp p

$= 0$ wp $1-p$

$E[X_i] = 1 \cdot p + 0(1-p) = p.$

$$E[X] = E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E[X_i]$$

$$= \sum_{i=1}^{n} p$$

$$= np.$$

Example 2: (Hat problem) Consider there are n individuals each with a hat. They all drop their hat and "randomly" pick a hat. Each hat to individual pairing is equally likely.

$\Omega = S_n \longrightarrow$ permutations of $\{1, 2, \ldots n\}$.

$S_3 = \{ (1,2,3), (1,3,2),$
$(2,1,3), (2,3,1), (3,1,2)$

$(3,2,1)\}$

$$|S_3| = 3! = 6.$$
$$|S_n| = n!$$

For any $\omega \in \Omega$ $\quad P(\{\omega\}) = \frac{1}{n!}$

What is the expected # of people who get their hat back.

X is a R.V that represent the # of people who get their back.

X_i is 1 if i-th individual gets his hat back
0 if he/she doesn't

$$P(X_i = 1) = \sum_{\substack{\omega \in \Omega \\ \omega_i = i}} P(\{\omega\})$$

$(\_\_\ i\ \_\_)$

$$= \frac{|\omega \in \Omega \mid \omega_i = i|}{n!} = \frac{(n-1)!}{n!}$$

$$= \frac{1}{n}.$$

$$= 1 - P(X_i = 0)$$

$$X = X_1 + X_2 + \dots X_n.$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

Conditioning. Let X be a r.v and A be an event such that $P(A) > 0$

$$P_{X|A}(x) = P(\{X = x\} \mid A).$$

$$= P(\{X = x\} \cap A) / P(A).$$

$$\sum_{x \in \mathcal{X}_0} P_{X|A}(x) = 1. \qquad \text{(check)}.$$

$$= \sum_{x \in \mathcal{X}_0} \frac{P(\overbrace{\{X = x\} \cap A}^{A_x})}{P(A)}$$

A_x and $A_{x'}$ are disjoint

$$= \frac{1}{P(A)} \sum_{x \in \mathcal{X}_0} P(A_x \cap A).$$

$\bigcup_{x \in \mathcal{X}_0} A_x = \Omega$

countable

$$= \frac{P\left(\bigcup_{x \in \mathcal{X}_0} (A_x \cap A)\right)}{P(A)} = \frac{P\left(\overbrace{\left(\bigcup_{x \in \mathcal{X}_0} A_x\right)}^{\Omega} \cap A\right)}{P(A)}$$

$$= 1.$$

Example : Rolling a six faced die

$\Omega = \{1, 2, 3, 4, 5, 6\}$

$X(\omega) = \omega.$

A : event that the experiment outcome is even $= \{2, 4, 6\}$

$P(A) = 3/6.$

$P_X(x) = 1/6. \qquad x \in \{1, 2, \dots 6\}$

$$P_{X|A}(x) = \begin{cases} \dfrac{P(\{X = x\} \cap A)}{P(A)} = 0 & x \text{ is odd.} \\[2ex] \dfrac{1/6}{3/6} = \dfrac{1}{3}. & x \text{ is even.} \end{cases}$$

Conditioning over a R.V.

$$P_{X|Y}(x|y) = P(\{X = x\} | \{Y = y\}).$$

event you are conditioning over is "$Y = y$"

i.e, $A = \{Y = y\}.$

$$= \frac{P(\{X = x\} \cap \{Y = y\})}{P(\{Y = y\})}$$

$$\boxed{P_{X|Y}(x|y) = P_{X,Y}(x, y) / P_Y(y)}$$

Similarly

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}.$$

Multiplication rule:

$$P_{X,Y}(x,y) = P_X(x)\, P_{Y|X}(y|x).$$

$$= P_Y(y)\, P_{X|Y}(x|y).$$

To find PMF of R.V Y from PMF of R.V X and conditional PMF $Y|X$.

$$P_Y(y) = \sum_{x \in \mathcal{X}} P_{X,Y}(x,y)$$

$$= \sum_{x \in \mathcal{X}} P_X(x)\, P_{Y|X}(y|x)$$

Bayes' rule.

We are given $P_X(x)$, $P_{Y|X}(y|x)$ and we are interested in inferring about X conditioned over Y. (i.e., on observing Y)

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P(y)}$$

$$= \frac{P_X(x)\, P_{Y|X}(y|x)}{\sum_x P_X(x)\, P_{Y|X}(y|x)}.$$

Total probability theorem.

$$P(A) = \sum_{i=1}^{n} P(A \cap A_i)$$

$A_1, \ldots A_n$ that partition Ω.

& $P(A_i) > 0$

$$= \sum_{i=1}^{n} P(A_i)\, P(A|A_i)$$

Say $A = \{X = x\}$

$$P_X(x) = P(\{X = x\})$$

$$= \sum_{i=1}^{n} P(A_i)\, P(\{X = x\} \mid A_i)$$ → well defined if $P(A_i) > 0$

$$= \sum_{i=1}^{n} P(A_i)\, P_{X|A_i}(x).$$

Conditional expectation

Let A_i be some event s.t $P(A_i) > 0$, then

$$E[X | A_i] = \sum_{x \in \mathcal{X}_0} x\, P_{X|A_i}(x)$$

Assume that the event we are conditioning over is $A_i = \{Y = y\} = \{\omega : Y(\omega) = y\}$

$$E[X | Y = y] = \sum_{x \in \mathcal{X}_0} x\, P_{X|Y}(x|y).$$

$$E[g(X) | A_i] = \sum_{x \in \mathcal{X}_0} g(x)\, P_{X|A_i}(x)$$

(Check this).

Remark :

$f(y) = E[X | Y = y]$ is a function of y.

$E[X | Y] = f(Y)$ is a random variable.

$F = f(Y) = E[X|Y]$ is a random variable that takes values f with probability $\sum_{y : f(y) = f} P_Y(y)$.

Total expectation theorem

: $A_1, \dots, A_n$ that partition Ω and $P(A_i) > 0$.

$$E[X] = \sum_{i=1}^{n} P(A_i)\, E[X | A_i]$$

Proof:

$$E[X] = \sum_{x \in \mathcal{X}_0} x\, P_X(x)$$

$$= \sum_{x \in \mathcal{X}_0} x \sum_{i=1}^{n} P(A_i)\, P_{X|A_i}(x)$$

total probability theorem

$$= \sum_{i=1}^{n} P(A_i) \underbrace{\sum_{x \in \mathcal{X}} x \, P_{X|A_i}(x)}_{E[X|A_i]}.$$

$$= \sum_{i=1}^{n} P(A_i) \cdot E[X|A_i].$$

Let $A_y = \{Y = y\}$ $\quad A_y,\ y \in \mathcal{Y}$ partition Ω.

$$E[X] = \sum_{y \in \mathcal{Y}} P(A_y) \, E[X|A_y]$$

$$= \sum_{y \in \mathcal{Y}} P_Y(y) \, \underbrace{E[X|Y=y]}_{:= f(y)}.$$

$$= E[f(Y)]$$

$$= E[\underbrace{E[X|Y]}_{\text{random variable}}].$$