# Lec 11 — Jan 28

**Recap**

$$E[X] = E_Y\big[\underbrace{E[X|Y]}_{\text{function of random variable } Y}\big]$$

$$E[g(X)] = E_Y\big[E[g(X)|Y]\big]$$

Total law of Expectation

## Total law of Variance

$$Var(X) = E_Y[Var(X|Y)] + Var_Y[E[X|Y]]$$

$Var(X|Y=y)$, $E[X|Y=y]$

Example:
Consider that there are N users who ~~rate~~ rate Netflix app on playstore. We want to compute mean & variance of these ratings.

Suppose user $i$ rates as $r_i$. We want to find $E[R] = \frac{\sum_{i=1}^{n} r_i}{n}$

$$Var[R] = E[R^2] - (E[R])^2$$

$$= \frac{\sum_{i=1}^{n} r_i^2}{n} - (E[R])^2$$

However,
These users are located in m different countries.
$A_1, A_2, \ldots A_m$ as partitioning of $\Omega = \{1, 2, \ldots N\}$.
$A_i$ is the set of users from country $i$.

$P(\{i\}) = \frac{1}{N}$.

$R(i) = r_i$

Users are accumulated from each country to get mean & variance per that country.

$Y$ is the random variable that indicates the country of user.

Country of user. $Y(i) = j$ if $i \in A_j$.

User index. $X(i) = i$

user rating. $R$

$$P(Y=j) = P(\{i \in \Omega \mid i \in A_j\}).$$

$|A_j| = n_j$

Let's say there are $n_j$ users in country $j$.

$$\sum_{j=1}^{m} n_j = N.$$

$$= \frac{|A_j|}{N}.$$

$\left.\begin{array}{l} E[R \mid Y=j] \\ Var[R \mid Y=j]. \end{array}\right\}$ is given

$$E[R \mid Y=j] = \sum_{x \in A_j} R(x) \cdot P_{X|Y}(x|j).$$

$$P_{X|Y}(x|j) = \frac{P_{X,Y}(x,j)}{P_Y(j)}.$$

$$= \begin{cases} \dfrac{1/N}{n_j/N} = \dfrac{1}{n_j} & x \in A_j \\ 0 & x \notin A_j \end{cases}$$

$$E[R \mid Y=j] = \sum_{x \in A_j} \frac{r_x}{n_j}.$$

Apply total law of expectation

$$\sum_{j=1}^{m} P(Y=j)\, E[R \mid Y=j] = \sum_{j=1}^{m} \boxed{\left( \sum_{x \in A_j} \frac{r_x}{n_j} \right)} \frac{n_j}{N}$$

local mean / conditional expectation

weights

$$= \frac{\sum_{x=1}^{N} r_x}{N} = E[R].$$

$$\text{Var}(X \mid Y=y) = E[X^2 \mid Y=y] - \left(E[X \mid Y=y]\right)^2.$$

$$= \frac{\sum_{x \in A_j} r_x^2}{n_j} - \left( \frac{\sum_{x \in A_j} r_x}{n_j} \right)^2$$

$$\text{Var}(X) = \text{Var}(E[X \mid Y]) + \underbrace{E[\text{Var}(X \mid Y)]}$$

$$E[\text{Var}(X \mid Y)] = \sum_{j=1}^{m} P_Y(j)\, \text{Var}(X \mid Y=y)$$

Excercise.

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Recap on Independence

$X$ and $Y$ are independent.

$$P_{X,Y}(x,y) = P_X(x)\, P_Y(y) \quad \forall x, y.$$

$$P_{X_1, \ldots X_n}(x_1, \ldots x_n) = \prod_{i=1}^{n} P_{X_i}(x_i) \quad \forall x_1, \ldots x_n$$

If $X_1, \ldots X_n$ are independent then any sub-collection of them are independent.

Excercise

If $X$ and $Y$ are independent then $g(X)$ and $h(Y)$ are also independent.

$$E[g(X)\,h(Y)] = E[g(X)]\,E[h(Y)].$$

If $X$ and $Y$ are independent

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y).$$

$$'' \quad E\left[\left(X+Y-\underbrace{E[X+Y]}_{E[X]+E[Y]}\right)^2\right].$$

$$E[g(X)+h(Y)] = E[g(X)] + E[h(Y)]$$

$$= E\left[\left(X-E[X]+Y-E[Y]\right)^2\right]$$

$$= E\left[(X-E[X])^2 + (Y-E[Y])^2 + 2(X-E[X])(Y-E[Y])\right]$$

$$= E\left[(X-E[X])^2\right] + E\left[(Y-E[Y])^2\right] + 2E\left[XY + E[X]E[Y] - E[X]Y - E[Y]X\right]$$

$$= \mathrm{Var}(X) + \mathrm{Var}(Y).$$

$$2\left[E[XY] + E[X]E[Y] - E[X]E[Y] - E[Y]E[X]\right]$$

$$0 = 2\left[E[XY] - E[X]E[Y]\right]$$

if $X$ and $Y$ are independent

Example

\# heads in $n$ coin tosses : $X$ is Binomial $(n, p)$ random variable.

$X_1, \ldots X_n$ are independent.

$$X = X_1 + X_2 + \cdots X_n.$$

$$X_i = \begin{cases} 1 & \text{if } i\text{th toss is a hea} \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = E\left[\sum_{i=1}^{n} X_i\right]$$

$$= \sum_{i=1}^{n} E[X_i]$$

linearity of expectation

$$= np.$$

$E[X_i] = p.$

$Var[X_i] =$

$E[X_i^2] - (E[X_i])^2$

$1^2 \cdot p + 0^2 \cdot (1-p) - p^2$

$= p - p^2$

$= p(1-p)$

$$Var(X) = Var\left(\sum_{i=1}^{n} X_i\right)$$

as $X_1, \ldots X_n$ are independent

$$= \sum_{i=1}^{n} Var(X_i)$$

$$= n\, p(1-p).$$

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