Lecture 2 :

updated: 7/1 11am

6th Jan '25

(Bertsekas)

Logistics.

EE24, EE ICDT 24, BT 24. }

Any one other than these three batches please drop the course

Probability space

$(\Omega, \mathcal{F}, P)$

sample space

event space

probability function.

$\mathcal{F}$: event space is collection of subsets of Ω satisfy the following conditions

σ-algebra

1. $\emptyset \in \mathcal{F}$ should contain null set.
2. If $A \in \mathcal{F}$, $A^c \in \mathcal{F}$
3. If $A_1, A_2, \ldots \in \mathcal{F}$ then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

$A_1, A_2 \in \mathcal{F}$

$A_1 \cup A_2 \in \mathcal{F}$

$A_1 \cap A_2 \in \mathcal{F}$

$\shortparallel$

$(A_1^c \cup A_2^c)^c$

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$

Examples:

1. $\mathcal{F} = \{\phi, \Omega\}$
2. $\mathcal{F} = \{\phi, A, A^c, \Omega\}$ For some $A \subset \Omega$.

σ-algebra generated by set A.

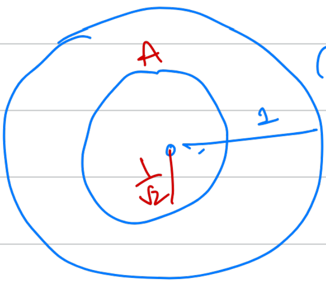
3. $\mathcal{F}$ = all subsets of Ω.
4. $\mathcal{F} = \{\phi, A, A^c, \Omega\}$.

Excercise

Find a minimal cardinality $\mathcal{F}$ that has A, B as elements.

$A = \{(x, y) \in \mathbb{R}^2, \; x^2 + y^2 \le \frac{1}{2}\}$

5. $\mathcal{F}$ = all subsets of Ω.



Ω : $\Omega = \{(x, y) \in \mathbb{R}^2, \; x^2 + y^2 \le 1\}$.

General additivity axiom

$A_1, A_2, \ldots$ are disjoint

$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

Probability function $P : \mathcal{F} \to \mathbb{R}$

① Non-negativity: for any $A \in \mathcal{F}$,
$$P(A) \geq 0$$

② Normalization
$$P(\Omega) = 1.$$

③ Additivity: $A, B \in \mathcal{F}$ that are disjoint i.e., $A \cap B = \phi$.

then $P(A \cup B) = P(A) + P(B)$.

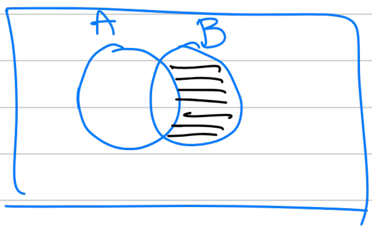
① $P(\phi) = 0.$ $\rightarrow$ $1 \overset{\text{normalization}}{=} P(\Omega) = P(\phi \cup \Omega) \overset{\text{additivity}}{=} P(\phi) + P(\Omega)$
$= P(\phi) + 1$
$\Rightarrow P(\phi) = 0$

② $P(A \cup B) \leq P(A) + P(B)$

Union bound

ⓐ $(A \cup B) = (A \cup B \setminus A)$

$P(A \cup B) = P(A) + P(B \setminus A) \rightarrow ①$



ⓑ $B = (B \setminus A) \cup (B \cap A)$

$B \cap A^c = B \setminus A$

$P(B) = \underline{P(B \setminus A)} + P(A \cap B) \rightarrow ②$

A and $B \setminus A$ are disjoint

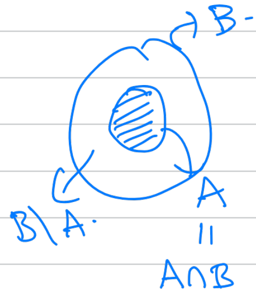
From ① and ②

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\leq P(A) + P(B) + 0 \quad \text{(non-negativity).}$$

③ $A \subset B$, $P(A) \leq P(B)$

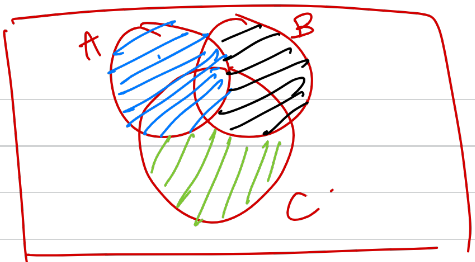
$B = (B \cap A) \cup (B \setminus A).$

$P(B) = P(B \cap A) + P(B \setminus A)$
$= P(A) + P(B \setminus A)$
$\geq P(A).$



④ $P(A \cup B \cup C) = P(A) + P(B \cup C \setminus A)$

$= P(A) + P(B \setminus A) + P(C \setminus A \cup B).$



$$(A\cup B\cup C) = (A)\cup(B\setminus A)\cup(C\setminus A\cup B).$$

disjoint union

Union bound for three sets

$$P(A\cup B\cup C) \le P(A)+P(B\cup C) \le P(A)+P(B)+P(C)$$

General union bound.

$$P\left(\bigcup_{i=1}^{n} A_i\right) \le \sum_{i=1}^{n} P(A_i)$$

By induction assume true for $n-1$ set collection.

$$P\left(A_n \cup \bigcup_{i=1}^{n-1} A_i\right).$$

by induction assumption

$$\le P(A_n) + P\left(\bigcup_{i=1}^{n-1} A_i\right)$$

$$\le P(A_n) + \sum_{i=1}^{n-1} P(A_i)$$

by induction assumption

Examples

① $\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F}$ = all subsets of Ω.

$P(\{i\}) = \frac{1}{6} \quad \forall i\in\Omega.$

$P(\Omega) = 1.$

$A \subseteq \Omega.$

$P(A) = \sum_{i\in A}\frac{1}{6} = \frac{|A|}{6}.$

$P(\{1,2\}) = \frac{2}{6} = \frac{1}{3}.$

② $\Omega = [0,1]$

$\mathcal{F}$: all subsets of Ω. } vague.

Borel σ-algebra

consider all open intervals in Ω.

$\mathcal{O} = \{(a,b)\cap[0,1],\ a,b\in\mathbb{R}\}$

$\mathcal{F}$ is σ-algebra generated by the open sets

$P([a,b]) = (b-a)$ for any interval $[a,b]$.

$P(\Omega) = 1.$

Suppose we asign probability p to every element $x\in[0,1]$

Pick a finite set S comprising of $>\frac{1}{p}$ distinct elements from Ω.

(Can always find such S as Ω is uncountable + infinite)

$P(S) = \sum_{x\in S} P(\{x\})$

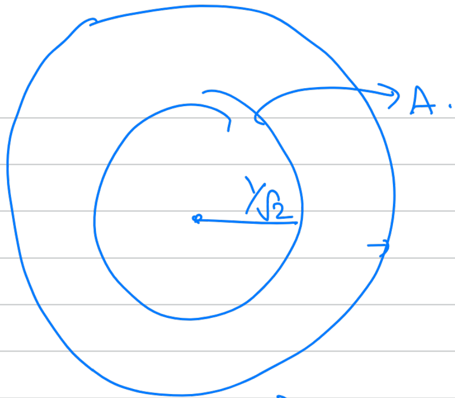
$= |S|\,p > \frac{1}{p}\cdot p = 1.$

Size of S

$P(\Omega) \ge P(S) > 1 \Leftarrow$

contradicts normalization axiom.

③ $\Omega = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \le 1\}$.



$$P(A) = \frac{\text{area } A}{\pi} = \frac{1}{2}.$$

$A \subseteq \Omega$.

$$P(\Omega) = \frac{\text{area } \Omega.}{\pi} = 1$$

$A := \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \le \frac{1}{2}\}$

Continuing on Properties of probability function

Inclusion - Exclusion Principle

① $P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}$.

$-$ve when interaction of two sets.

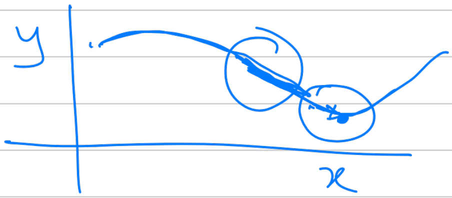
$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{1 \le i < j \le n} P(A_i \cap A_j)$$
$$+ \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3})$$
$$- \cdots (-1)^{n+1} P(A_1 \cap A_2 \cdots \cap A_n)$$

Excercise: prove using induction.

② Continuity property of probability function.

Let $A_1, A_2, \ldots$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$



$x_n \to x$

$\lim_{n\to\infty} x_n = x$

$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$
$= f(x)$.

Corollaries:

ⓐ $A_1 \subset A_2 \subset \ldots$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = P(A_n).$$

$$\boxed{P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n).}$$

ⓑ $B_1 \supset B_2 \supset \cdots$

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n).$$

$B_1^c \subset B_2^c \ldots\ldots$

From Corollary (a)

$$\Rightarrow P\left(\bigcup_{i=1}^{\infty} B_i^c\right) = \lim_{n\to\infty} P(B_n^c)$$

$$\Rightarrow 1 - P\left(\bigcup_{i=1}^{\infty} B_i^c\right) = 1 - \lim_{n\to\infty} P(B_n^c)$$

$$\Rightarrow P\left(\left(\bigcup_{i=1}^{\infty} B_i^c\right)^c\right)$$

De-Morgan's laws

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n)$$

EE : Niketh
EE1CDT : Pranavi
BT : Rahul.

Tentative Bi-Weekly Slot.

After Jan 15. 5-30 – 7 pm. Thursday