EE1080/AI1110/EE2120 Probability, Midterm Exam

5th March, 2025

Max. Marks: 40 **Time:** 2 hours.

Instructions

- You are welcome to answer all questions. The maximum marks will be capped at 40. That is, if your answers are evaluated to x marks then your score will be $\min\{40, x\}$.
- Please **write your roll number**, prominently in the first page of the answer sheet. Mark your **serial number** at the top right corner of the answer sheet.
- No laptops, mobile devices, cheat sheets allowed. You may use calculators
- Please write supporting arguments for any of the statements you make. any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer.

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1. *(3=1+1+1)* In the game of bridge, the entire deck of 52 cards in dealt out to 4 players. What are the total number of possible outcomes of this game ? What is the probability that

   (a) each player receives 1 ace.

   (b) one of the players receives all 13 spades.

   Note that a deck of 52 cards has 13 spades, 13 diamonds, 13 hearts and 13 clubs. Each of these 13 cards correspond to ace, queen, king, jack and the numbers 2 till 10. So each number/ace/queen /jack/king appear exactly 4 times.

2. *(6=1+2+3)* Given sequence of events $A_1, A_2, \cdots$, assume continuity property of the probability function i.e., $P(\cup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} P(\cup_{i=1}^{n} A_i))$ and the union bound for the finite case i.e.,

   $$P(\cup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i).$$

   (a) *Boole's inequality*: Prove the following union bound for the infinite case:

   $$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i)$$

   (b) Show that if $P(B_i) = 1$ for all $i \geq 1$ then $P(\cap_{i=1}^{\infty} B_i) = 1$. *(Hint: Use part (a)).*

   (c) Infinite sequence of independent trials is to be performed. Each trial results in a success with probability p and a failure with probability $(1-p)$.

      i. What is the probability that first n trails result in success ?

      ii. What is the probability that all the trials result in success ?

3. *(6=1+1+1+1+2)* Identify true or false for each of the statement below. Please given thorough proof/justification for each of your claims.

   (a) If $P(A \cap B) > 0$, then $P(A \cup B | A \cap B) \geq P(A \cap B | A \cup B)$.

   (b) Given three events A, B, C with $P(B) > 0$,

   $$P(A|B) \geq P(C|B) \implies P(A^c|B) \geq P(C^c|B)$$

   (c) If A and B are disjoint then they are independent.

   (d) There can exist disjoint events A, B such that $P(A) = P(B) = 0.6$.

   (e) For any two events A, B $P(A|A \cup B) \geq P(A|B)$ *(Hint: $A \cup B = A \cup (B \setminus A)$, $B = (B \cap A) \cup (B \setminus A)$)*

4. (7=3+4) Alice and Bob flip coins. Alice starts and continues flipping until a tail occurs, at which point Bob starts flipping and continues until there is a tail. Then Alice takes over and so on. Let p be the probability of coin landing on heads when Alice flips and let q when Bob flips. The winner of the game is the first one to get:

   (a) 2 heads in a row

   (b) a total of 2 heads.

   In both the cases find the probability that Alice wins. (*Hint for Part (b): Let $p_{i,j}$ be the probability of Alice winning conditioned over the event that she has already seen i heads and Bob has already seen j heads. Find $p_{0,0}$*)

5. (9=1+3+2+1+2) Let $Y = ZX$ where $Z \sim N(\mu = 2, \sigma^2 = 4)$ and X is a discrete random variable independent of Z, that takes values $1, -1$ with probabilities $p = 1/4, (1-p) = 3/4$ respectively.

   (a) What is the conditional density $f_{Y|X}(y|x)$ for $x = +1, -1$?

z	+ 0.00	+ 0.01	+ 0.02	+ 0.03	+ 0.04	+ 0.05	+ 0.06	+ 0.07	+ 0.08	+ 0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56360	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91308	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408

Table 1: Shown below are the values of $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$. First row implies $\phi(0) = 0.5$, $\phi(0.01) = 0.50399, \cdots$. Second row implies $\phi(0.1) = 0.53983$, $\phi(0.11) = 0.54380, \cdots$

(b) Find $E[Y], Var(Y)$ without using the p.d.f of Y.

(c) Are X, Y uncorrelated ? What is the correlation value, $\rho(X, Y)$?

(d) What is the p.d.f of Y ? Is it normal ?

(e) Find $P(X = +1|Y = y)$ and $P(X = -1|Y = y)$ as a function of y.

$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{z-\mu}{\sigma})^2}$

6. (8=2+2+3+1) In the earlier problem, suppose we would like to estimate $\hat{x}(Y)$ of X from Y:

   (a) What is the Maximum Aposteriori Probability (MAP) detection rule $\hat{x}_{\text{MAP}}(y)$?

   (b) What is the Maximum Likelihood (ML) detection rule $\hat{x}_{\text{ML}}(y)$?

   (c) What are detection error probabilities

   $$P(\hat{x}_{\text{MAP}}(Y) \neq X), \quad P(\hat{x}_{\text{ML}}(Y) \neq X).$$

   Which one is better ?

   (d) Can you find a detection rule with error smaller the values seen in part (c) ?

7. (5=3+1+1) Let X, Y be uniform($[0, 1]$) random variables that are independent.

   (a) Find CDF of $Z = |X - Y|$.

   (b) Find PDF of Z.

   (c) Find $E[|X - Y|]$.

8. (5=2+3)(Poisson Splitting) A transmitter sends out either a 1 with probability p or 0 with probability $1 - p$ independent of earlier transmissions. The number of bits, N transmitted in a slot has a Poisson PMF with parameter λ. Let Y be the number of bits transmitted in a slot.

   (a) Find the $E[Y]$

   (b) Find the PMF of Y

   $P[N = n] = e^{-\lambda} \frac{\lambda^n}{n!}$ for $n = 0, 1, 2, \cdots$