① $R = \sqrt{X^2+Y^2}$, $\Theta = \tan^{-1}(Y|X)$

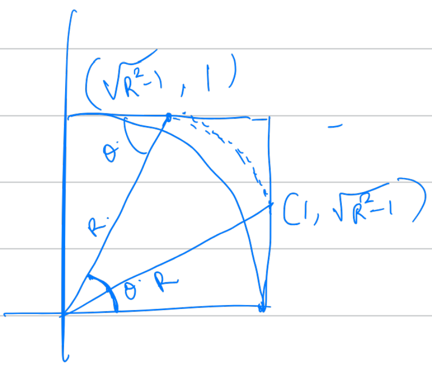
$\Rightarrow$ $X = R\cos\Theta$, $Y = R\sin\Theta$.

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

ⓐ X, Y are independent & uniformly distributed over $(0,1)$.

$0 \le R \le \sqrt{2}$, $\Theta \in [0, \pi/2]$ $\rightarrow$ as $X, Y \ge 0$.

$$f_{R,\Theta}(r,\theta) = f_{X,Y}(r\cos\theta, r\sin\theta)\, r$$



$$= \begin{cases} r & 0 < r\cos\theta < 1 \\ & 0 < r\sin\theta < 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} r & 0 < r < 1, \quad 0 < \theta < \pi/2 \\ r & 1 < r < \sqrt{2}, \quad \frac{\pi}{2} - \cos^{-1}\left(\frac{1}{r}\right) \le \theta \le \cos^{-1}\left(\frac{1}{r}\right) \\ 0 & \text{otherwise} \end{cases}$$

ⓑ $f_{X,Y}(x,y) = \dfrac{1}{\pi}$ if. $x^2+y^2 \le 1$.

$f_{R,\Theta}(r,\theta) = \dfrac{1}{\pi} r$ if $0 \le r \le 1$,
$\theta \in [-\pi, \pi]$.

② $U = X, \quad V = X/Y.$ $\boxed{Y = U/V, \quad X = U}$

$$J(x,y) = \begin{vmatrix} \frac{du}{dx} & \frac{dv}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = \frac{|x|}{y^2}.$$

$$f_{U,V}(u,v) = \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} \frac{e^{-\left(\frac{u}{v}\right)^2 \frac{1}{2}}}{\frac{|u| \cdot v^2}{u^2} \cdot \sqrt{2\pi}}$$

$$= \frac{|u|}{v^2} \frac{1}{2\pi} \left(e^{-\frac{u^2}{2}}\right) \left(e^{-\frac{u^2}{2v^2}}\right).$$

$$f_V(v) = \int_{-\infty}^{\infty} f_{U,V}(u,v)\, du$$

$$= \int_{-\infty}^{\infty} \frac{|u|}{v^2} \frac{1}{2\pi} e^{-\frac{u^2}{2}\left(1+\frac{1}{v^2}\right)} du$$

$$= \int_0^{\infty} \frac{2u}{v^2} \frac{1}{2\pi} e^{-\frac{u^2}{2}\left(1+\frac{1}{v^2}\right)} du$$

$\frac{u^2}{2}\left(1+\frac{1}{v^2}\right) = y$

$u\, du \left(1+\frac{1}{v^2}\right) = dy$

$$= \int_0^{\infty} \frac{e^{-y}}{\pi} \frac{dy}{v^2\left(1+\frac{1}{v^2}\right)} = \frac{1}{\pi(v^2+1)}$$

③ $U \sim \text{uniform}[0, 2\pi]$. $Z \sim \exp(1)$.

$$X = \sqrt{2Z}\cos(U), \quad Y = \sqrt{2Z}\sin(U).$$

$$f_{X,Y}(x,y) = f_{Z,U}(z,u)\, J(x,y).$$

$$= f_{Z,U}(z,u) / J(z,u).$$

$$J(z,u) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial u} \end{vmatrix} = \begin{vmatrix} \frac{\cos u}{\sqrt{2z}} & -\sqrt{2z}\sin u \\ \frac{\sin u}{\sqrt{2z}} & \sqrt{2z}\cos u \end{vmatrix}$$

$$= |\cos^2 u + \sin^2 u| = 1.$$

$$f_{X,Y}(x,y) = f_{Z,U}(z,u) \qquad z, u \text{ st } x = \sqrt{2z}\cos u, \; y = \sqrt{2z}\sin u.$$

$$= f_Z\left(\frac{x^2+y^2}{2}\right) f_U\left(\tan^{-1}\left(\frac{x}{y}\right)\right)$$

$$= e^{-\left(\frac{x^2+y^2}{2}\right)} \cdot \frac{1}{2\pi}$$

$$= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$$

$\Rightarrow$ X, Y are jointly normal

(4)

$$f_{X_{(1)}\cdots X_{(n)}}(x_1, x_2, \ldots x_n) = \begin{cases} n!\, f_{X_1,\ldots X_n}(x_1, \ldots x_n) & x_1 < x_2 < \cdots < x_n \\ 0 & \text{Otherwise} \end{cases}$$

from class

$$f_{X_{(2)}\cdots X_{(n)}}(x_2, \ldots x_n) = \int_{-\infty}^{\infty} f_{X_{(1)}\cdots X_{(n)}}(x_1, \ldots x_n)\, dx_1$$

$$= \int_{-\infty}^{x_2} n!\, f_{X_1\cdots X_n}(x_1, \ldots x_n)\, dx_1$$

$$= \int_{-\infty}^{x_2} n!\, f(x_1) \left(\prod_{i=2}^{n} f(x_i)\right) dx_1$$

$$= n!\, F(x_2)\, f(x_2)\, f(x_3) \cdots f(x_n).$$

$$f_{X_{(3)}\cdots X_{(n)}}(x_3 \ldots x_n) = \int_{-\infty}^{x_3} n!\, F(x_2)\, f(x_2) \left(\prod_{i=3}^{n} f(x_i)\right) dx_2$$

$$y = F(x_2)$$
$$dy = f(x_2)\, dx_2$$

$$= \prod_{i=3}^{n} f(x_i) \int_0^{F(x_3)} n!\, y\, dy.$$

$$= \prod_{i=3}^{n} f(x_i)\; n!\; \frac{y^2}{2}\Big|_0^{F(x_3)}$$

$$= \frac{n!}{2} \left(F(x_3)\right)^2 f(x_3) f(x_4) \cdots f(x_n),$$

Can check now that

$$f_{X_{(4)} \cdots X_{(n)}}(x_4, \ldots x_n) = \frac{n!}{3!} F(x_4)^3 f(x_4) \cdots f(x_n)$$

Similarly

$$f_{X_{(i)} \cdots X_{(n)}}(x_i, \ldots x_n) = \frac{n!}{(i-1)!} F(x_i)^{i-1} f(x_i) \cdots f(x_n).$$

$$f_{X_{(i)}, \cdots X_{(n-1)}}(x_i, \ldots x_{n-1}) = \int_{x_{n-1}}^{\infty} f_{X_{(i)} \cdots X_{(n)}}(x_i, \ldots x_n)\, dx_n$$

$$= \frac{n!}{(i-1)!} F(x_i)^{i-1} \prod_{j=i}^{n-1} f(x_j) \int_{x_{n-1}}^{\infty} f(x_n)\, dx_r$$

$$= \frac{n!}{(i-1)!} F(x_i)^{i-1} \left[\prod_{j=i}^{n-2} f(x_j)\right] f(x_{n-1}) \left(1 - F(x_{n-1})\right).$$

$$f_{X_{(i)}, \ldots X_{(n-2)}}(x_i, \ldots x_{n-2}) = \int_{x_{n-2}}^{\infty} f_{X_{(i)} \cdots X_{(n-1)}}(x_i, \ldots x_{n-1})\, dx_{n-1}$$

$$= \frac{n!}{(i-1)!} F(x_i)^{i-1} \prod_{j=i}^{n-2} f(x_j) \underbrace{\int_{x_{n-2}}^{\infty} f(x_{n-1})\left(1 - F(x_{n-1})\right) dx_{n-1}}$$

$$y = 1 - F(x_{n-1}); \quad dy = -f(x_{n-1})\,dx_{n-1}$$

$$-\int_{1-F(x_{n-2})}^{0} y\,dy = \int_{0}^{1-F(x_{n-2})} y\,dy = \frac{(1-F(x_{n-2}))^2}{2}.$$

$$= \frac{n!}{(i-1)!}\frac{1}{2}F(x_i)^{i-1}\prod_{j=i}^{n-3} f(x_j)\, f(x_{n-2})\,(1-F(x_{n-2}))^2$$

Similarly can show

$$f_{X_{(i)}}(x_i) = \frac{n!}{(i-1)!\,(n-i)!} F(x_i)^{i-1} \cdot f(x_i)\,(1-F(x_i))^{n-i}.$$

Can substitute CDF, PDF of uniform RV

$F(x_i) = x_i, \quad f(x_i) = 1 \quad x_i \in [0,1]$

(b)

$$f_{X_{(n)}|X_{(1)}\dots X_{(n-1)}}(s_n \mid s_1, s_2, \dots s_{n-1})$$

$$= \frac{f_{X_{(1)}\dots X_{(n)}}(s_1, \dots s_{n-1}, s_n)}{f_{X_{(1)}\dots X_{(n-1)}}(s_1, \dots s_{n-1})}.$$

$$f_{X_{(1)}\dots X_{(n-1)}}(s_1 \dots s_{n-1}) = \int_{s_{n-1}}^{\infty} n!\prod_{i=1}^{n} f(s_i)\, ds_n$$

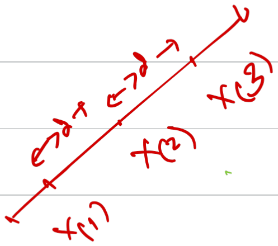
$$= n!\left(\prod_{i=1}^{n-1} f(s_i)\right)(1-F(s_{n-1}))$$

$$\Rightarrow f_{X_{(n)}|X_{(1)}\dots X_{(n-1)}}(s_n \mid s_1, \dots s_{n-1}) = \frac{n!\prod_{i=1}^{n} f(s_i)}{n!\prod_{i=1}^{n-1} f(s_i)\,(1-F(s_{n-1}))}$$

$$= \frac{f(s_n)}{1 - F(s_{n-1})}$$

$$= \frac{1}{1 - s_{n-1}} \quad \text{for uniform } X_i\text{'s}$$

ⓒ $P\left(X_{(k)} - X_{(k-1)} > t \right)$



$$= \int_t^1 \int_0^{x_k - t} f_{X_{(k-1)}, X_{(k)}}(x_{k-1}, x_k)\, dx_{k-1}\, dx_k \quad \rightarrow (1)$$

$$f_{X_{(k-1)}, X_{(k)}}(x_{k-1}, x_k) = \left[\frac{n!}{(k-2)!} F(x_{k-1})^{k-2} f(x_{k-1}) \right] \times$$

$$f(x_k) \times \frac{\left(1 - F(x_k)\right)^{n-k}}{(n-k)!}$$

$x_{(3)} - x_{(2)} \geq d$; $x_{(2)} - x_{(1)} \geq d$.

$x_{(3)} \geq x_{(2)} + d$, $x_{(2)} \geq x_{(1)} + d$, $x_{(1)} \geq 0$

$1 \geq x_{(3)}$, $1-d \geq x_{(2)}$, $1-2d \geq x_{(1)}$

$$= \frac{n!}{(k-2)!\,(n-k)!} \left(x_{k-1}\right)^{k-2} \times \left(1 - x_k\right)^{n-k} \quad \rightarrow (2)$$

Substitute (2) in (1)

$$P\left(X_{(k)} - X_{(k-1)} > t\right) = \int_t^1 \int_0^{x_k - t} \frac{n!}{(k-2)!\,(n-k)!} \left(x_{k-1}\right)^{k-2} \left(1 - x_k\right)^{n-k} dx_{k-1}\, dx_k$$

$$= \int_t^1 \frac{n!}{(k-2)!(n-k)!} \frac{(x_{k-1})^{k-1}}{k-1}\Big|_0^{x_k - t} (1-x_k)^{n-k} \, dx_k$$

$$= \int_t^1 \frac{n!}{(k-1)!(n-k)!} (x_k - t)^{k-1} (1-x_k)^{n-k} \, dx_k$$

$x_k - t = (1-t)\, y.$

$1 - x_k = (1-t)(1-y)$ $\Leftarrow$ $\left\{ \text{let } \dfrac{x_k - t}{1-t} = y, \quad \dfrac{dx_k}{1-t} = dy \right.$

$$= \int_0^1 \frac{n!}{(k-1)!(n-k)!} (1-t)^n y^{k-1} (1-y)^{n-k} \, dy$$

$$= (1-t)^n \underbrace{\int_0^1 \frac{n!}{(k-1)!(n-k)!} y^{k-1} (1-y)^{n-k} \, dy}_{\text{pdf of Beta}(k,\, n-k+1) \text{ distribution}}.$$

$= (1-t)^n$

can be shown that this is 1.

Say $B(\alpha, \beta) = \int_0^1 y^{\alpha-1} (1-y)^{\beta-1} \, dy$

For α, β +ve integers

$$B(\alpha, \beta) = \frac{(\alpha-1)!\,(\beta-1)!}{(\alpha+\beta-1)!}$$

(d) Joint density of $X_{(1)}$ and $X_{(n)}$.

$$f_{X_{(1)},X_{(n)}}(x_1,x_n) = \int_{x_1}^{x_n}\int_{x_1}^{x_{n-1}}\int_{x_1}^{x_{n-2}}\cdots\int_{x_1}^{x_3} n!\prod_{i=1}^{n} f(x_i)\, dx_2\, dx_3 \cdots dx_{n-1}.$$

$$= f(x_1)f(x_n)\,n! \int_{x_1}^{x_n}\int_{x_1}^{x_{n-1}}\cdots\cdots\int_{x_1}^{x_4}\left[\prod_{i=3}^{n-1} f(x_i)\right]\left(F(x_3)-F(x_1)\right)dx_3 \cdots dx_{n-1}$$

$$\int_{x_1}^{x_4}\left(f(x_3)\,F(x_3) - F(x_1)f(x_3)\right)dx_3$$

$$= \frac{F^2(x_3)}{2}\Bigg|_{x_1}^{x_4} - F(x_1)F(x_3)\Big|_{x_1}^{x_4}$$

$$= \frac{\left(F^2(x_4)-F^2(x_1)\right)}{2} - F(x_1)\left[F(x_4)-F(x_1)\right]$$

$$= \frac{\left[F(x_4)-F(x_1)\right]^2}{2}.$$

Induction step

To show
$$\int_{x_1}^{x_{i+1}}\frac{\left(F(x_i)-F(x_1)\right)^{i-2}}{(i-2)!}\,f(x_i)\,dx_i$$
$$= \frac{\left(F(x_{i+1})-F(x_i)\right)^{i-1}}{(i-1)!}$$

Made with Goodnotes

Let $y = F(x_i) - F(x_1)$.

$dy = f(x_i)\, dx_i$.

$$\int_{x_i}^{x_{i+1}} \frac{\left(F(x_i) - F(x_1)\right)^{i-2}}{(i-2)!} f(x_i)\, dx_i = \int_0^{F(x_{i+1}) - F(x_1)} \frac{y^{i-2}}{(i-2)!}\, dy.$$

$$= \left. \frac{y^{i-1}}{(i-1)!} \right|_0^{F(x_{i+1}) - F(x_1)}$$

$$= \frac{\left(F(x_{i+1}) - F(x_1)\right)^{i-1}}{(i-1)!}$$

$$\Rightarrow f_{X_{(1)} X_{(n)}}(x_1, x_n) = \int_{x_1}^{x_n} \int_{x_1}^{x_{n-1}} \cdots \int_{x_1}^{x_3} n! \prod_{i=1}^{n} f(x_i)\, dx_2 \cdots dx_{n-1}$$

$$= n!\, f(x_1) f(x_n) \frac{\left(F(x_n) - F(x_1)\right)^{n-2}}{(n-2)!}.$$

Made with Goodnotes

$$f_{X_{(1)}(X_{(n)})}(x_1, x_n) = \int_{x_1}^{x_n} \cdots \int_{x_1}^{x_4}\int_{x_1}^{x_3} f_{X_{(1)}\cdots X_{(n)}}(x_1 \cdots x_n)\, dx_2 \cdots dx_{n-1}$$

(d) Joint density of

$$R = X_{(n)} - X_{(1)} \quad \} \text{ Range}$$

$$M = \frac{X_{(n)} + X_{(1)}}{2} \quad \} \text{ Mean.}$$

$$f_{X_{(1)}X_{(n)}}(x_1, x_n) = \frac{n!}{(n-2)!} f(x_1) f(x_n) \cancel{[1-F(x_n)]^{n-2}}$$

$$\left(F(x_n) - F(x_1)\right)^{n-2}$$

$$J(x_1, x_n) = \begin{vmatrix} \frac{\partial r}{\partial x_1} & \frac{\partial r}{\partial x_n} \\ \frac{\partial m}{\partial x_1} & \frac{\partial m}{\partial x_n} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}$$

$$= 1.$$

$$f_{R,M}(r, m) = f_{X_{(1)}, X_{(n)}}\left(m - \frac{r}{2}, m + \frac{r}{2}\right)$$

(5) $M_X(s) = e^{s^2/2}$ for $X \sim N(0,1)$

↳ shown in class

$$\left.\frac{\partial^n M_X(s)}{\partial s^n}\right|_{s=0} = E[X^n].$$

$$\frac{\partial M_X(s)}{\partial s} = s e^{s^2/2}$$

$$\frac{\partial^2 M_X(s)}{\partial s^2} = s^2 e^{s^2/2} + e^{s^2/2}.$$

$$\frac{\partial^3 M_X(s)}{\partial s^3} = s^3 e^{s^2/2} + 2s e^{s^2/2} + s e^{s^2/2}$$

$$\frac{\partial^4 M_X(s)}{\partial s^4} = 3s^2 e^{s^2/2} + s^4 e^{s^2/2} + 3 e^{s^2/2} + 3s^2 e^{s^2/2}.$$

$$E[X^3] = \left.\frac{\partial^3 M_X(s)}{\partial s^3}\right|_{s=0} = 0.$$

$$E[X^4] = \left.\frac{\partial^4 M_X(s)}{\partial s^4}\right|_{s=0} = 3.$$

(6) $$M_X(s) = \frac{\lambda}{\lambda - s} \qquad s < \lambda.$$

$$\frac{d M_X(s)}{ds} = \frac{\lambda}{(\lambda - s)^2}$$

$$\frac{\partial^2 M_X(s)}{\partial s^2} = \frac{2\lambda}{(\lambda - s)^3}$$

$$\frac{\partial^3 M_X(s)}{\partial s^3} = \frac{6\lambda}{(\lambda - s)^4} \Rightarrow E[X^3] = \frac{3!}{\lambda^3}$$

$$\frac{\partial^4 M_X(s)}{\partial s^4} = \frac{4!\,\lambda}{(\lambda - s)^5} \Rightarrow E[X^4] = \frac{4!}{\lambda^4}$$

$$\frac{\partial^5 M_X(s)}{\partial s^5} = \frac{5!\,\lambda}{(\lambda - s)^6} \Rightarrow E[X^5] = \frac{5!}{\lambda^6}$$

7. Need $M_X(s)$ to satisfy $M_X(0) = 1$.

(a) $M(s) = e^{2(e^{e^s - 1} - 1)}$

$M(0) = 1$

(b) $M(s) = e^{2(e^{e^s} - 1)}$

$M(0) = e^{2(e-1)}$, not an MGF.

$P(X=0)$ using (a)

$$P(X=0) = \lim_{s \to -\infty} e^{2(e^{e^s - 1} - 1)}$$

$$= e^{2(e^{-1} - 1)} < 1.$$

8. $$f_X(x) = \frac{1}{3} 2e^{-2x} + \frac{2}{3} \times 3e^{-3x}.$$

using exponential MGF & Mixture distribution

(9) $M_X(0) = 1.$

$$c\frac{(3+4+2)}{(3-1)} = 1$$

$$\Rightarrow c = \frac{2}{9}.$$

(10) $X \sim Ber(1/3)$, $Y \sim exp(2)$

$Z \sim Poisson(3).$

(a) $U = XY + (1-X)Z.$

$$M_U(s) = E[e^{sU}]$$

$$= E[e^{sXY}\, e^{s(1-X)Z}].$$

Used total law of expectation & the fact that Z, Y are independent of X.

$$= P(X=0)\, E[e^{sZ}] + P(X=1)\, E[e^{sY}]$$

$$= \frac{2}{3} M_Z(s) + \frac{1}{3} M_Y(s).$$

$$M_Y(s) = \frac{2}{2-s}$$

$$M_Z(s) = e^{3(e^s-1)}.$$

(b) $M_{2Z+3}(s) = E\left[e^{(2Z+3)s}\right]$

$= e^{3s} M_Z(2s)$

$= e^{3s} e^{3(e^{2s}-1)}$

$= e^{3(e^{2s}+s-1)}$

(c) $M_{Y+Z}(s) = M_Y(s) M_Z(s)$

as Y, Z are independent

$= \left(\frac{2}{2-s}\right) e^{3(e^{s}-1)}$