Curvilinear Co-ordinates ...

1. Cylindrical co-ordinates
2. Spherical co-ordinates.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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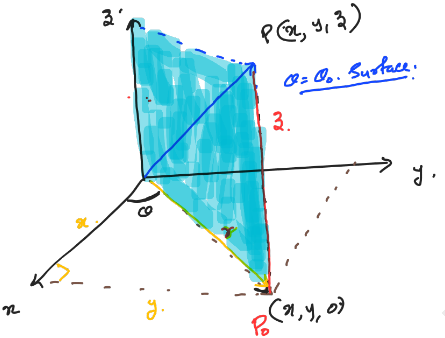
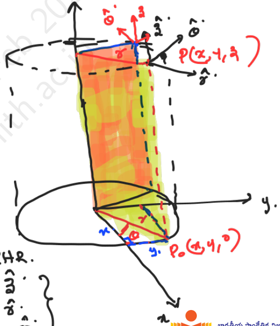
Cylindrical co-ordinates



$x = r\cos\theta$.
$y = r\sin\theta$.
$z = z$.

$r = \sqrt{x^2 + y^2}$.
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z$.

$0 \le \theta \le \pi$ if $y \ge 0$; $\pi < \theta < 2\pi$ if $y < 0$.



→ Unit vectors $(\hat{r}, \hat{\theta}, \hat{z})$

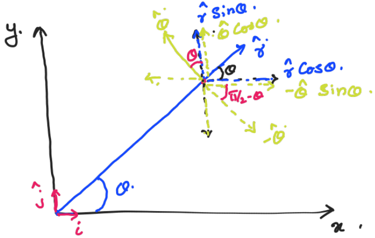
Orthonormal basis.

$$\Rightarrow \left.\begin{array}{l} \hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = \hat{z}\cdot\hat{z} = 1. \\ \hat{r}\cdot\hat{\theta} = \hat{r}\cdot\hat{z} = \hat{\theta}\cdot\hat{z} = 0 \end{array}\right\}.$$

By RHR.

$$\left\{\begin{array}{l} \hat{r}\times\hat{\theta} = \hat{z}. \\ \hat{\theta}\times\hat{z} = \hat{r}. \\ \hat{z}\times\hat{r} = \hat{\theta}. \end{array}\right\}.$$

The relationship between $\hat{r}, \hat{\theta}, \hat{z}$ & $\hat{i}, \hat{j}, \hat{k}$. $(\hat{k} = \hat{z})$



$$\left.\begin{aligned} \hat{i} &= \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} \\ \hat{j} &= \sin\theta\,\hat{r} + \cos\theta\,\hat{\theta}. \end{aligned}\right\}$$ Using geometry.

$\hat{k} = \hat{z}$ (or $\hat{k}$)

OR.

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}.$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$\hat{z} = \hat{k}.$$

Transformation matrix.

$$\begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix}$$

OR

$$\begin{bmatrix} \hat{r} \\ \hat{\theta} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

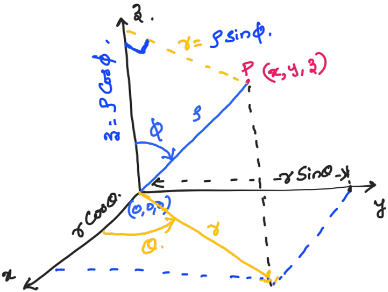
If

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_z\hat{z}$$

$$|\vec{A}| = \sqrt{A_r^2 + A_\theta^2 + A_z^2}.$$

Spherical co-ordinate systems.

$(\rho, \phi, \theta) \rightarrow$ RHS ✓ Right Handed systems.
$(\rho, \theta, \phi) \rightarrow$ LHS



* More similar to polar co-ordinate than cylindrical co ordinates.

① ρ = distance from the origin.
$0 \le \rho \le \infty$.

② ϕ = angle measured clockwise from 'z' axis.
$0 \le \phi \le \pi$.

③ θ = angle measured anticlock wise from +ve - x axis.
$0 \le \theta \le 2\pi$.

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$\rho = \sqrt{x^2+y^2+z^2}$.

$\phi = \cos^{-1}\left\{\dfrac{z}{(x^2+y^2+z^2)^{1/2}}\right\}$.

$\theta = \tan^{-1}(y/x)$

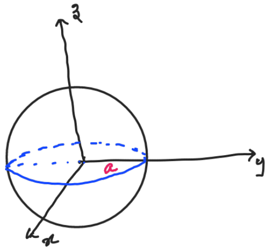
ρ = constant, varying θ & ϕ $\Rightarrow$ spherical surface.
Centered at origin.

$\therefore$ Spherical-coordinates.

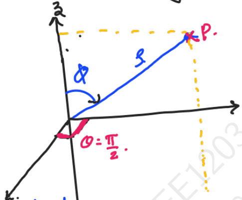
$\rho = a$

$\phi \rightarrow$ latitude.

$\theta \rightarrow$ Longitude.



If we consider a slice along 'z' axis in cylindrical co-ordinate.



Spherical- co-ordinate

when $\theta = \frac{\pi}{2}$.

$$\boxed{\begin{aligned} z &= \rho \cos\phi. \\ r &= \rho \sin\phi. \end{aligned}}$$

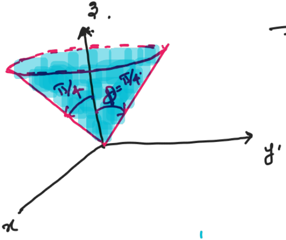
Spherical & Cylindrical Conversion.

$$\rho = \sqrt{r^2 + z^2} \; ; \; \phi = \tan^{-1}\left(\frac{r}{z}\right)$$

* May be thought as polar co-ordinates in r-z plane.

* $\rho = a$ → sphere of radius 'r' centered at origin.

* $\phi = \frac{\pi}{4}$.



→ Cone

Equation: $z = \rho \cos \pi/4$

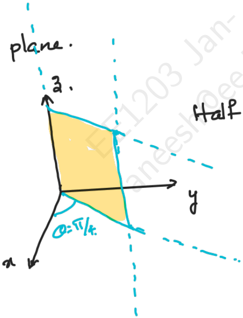
$r = \rho \sin \pi/4$.

$\Rightarrow z = \rho/\sqrt{2}\,;\ r = \rho/\sqrt{2}$.

$\Rightarrow \boxed{z = r}$.

* $\phi = \frac{\pi}{2}$ → xy plane.

* $\theta = \pi/4 \Rightarrow$



Half plane: where 'r' is $0 \leq r < \infty$.

$(\rho, \phi, \theta) \to$ orthonormal basis.

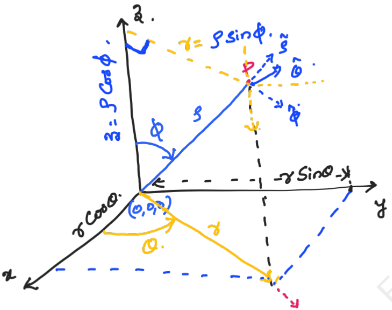
$$\hat{\rho}\cdot\hat{\rho} = \hat{\phi}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\theta} = 1$$

$$\hat{\rho}\cdot\hat{\phi} = \hat{\phi}\cdot\hat{\theta} = \hat{\rho}\cdot\hat{\theta} = 0$$

$$\hat{\rho}\times\hat{\phi} = \hat{\theta}\ ; \quad \hat{\phi}\times\hat{\theta} = \hat{\rho}\ ; \quad \hat{\theta}\times\hat{\rho} = \hat{\phi}.$$

Proof will be discussed later.

" You can try yourself too! "



Spherical to cartesian.

$$\left\{\begin{aligned} \hat{i} &= \sin\phi\cos\theta\,\hat{\rho} + \cos\phi\cos\theta\,\hat{\phi} - \sin\theta\,\hat{\theta}. \\ \hat{j} &= \sin\phi\sin\theta\,\hat{\rho} + \cos\phi\sin\theta\,\hat{\phi} + \cos\theta\,\hat{\theta}. \\ \hat{k} &= \cos\phi\,\hat{\rho} - \sin\phi\,\hat{\phi}. \end{aligned}\right.$$

OR. Cartesian to spherical.

$$\hat{\rho} = \sin\phi\cos\theta\,\hat{i} + \sin\phi\sin\theta\,\hat{j} + \cos\phi\,\hat{k}.$$

$$\hat{\phi} = \cos\phi\cos\theta\,\hat{i} + \cos\phi\sin\theta\,\hat{j} - \sin\phi\,\hat{k}.$$

$$\hat{\theta} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$