

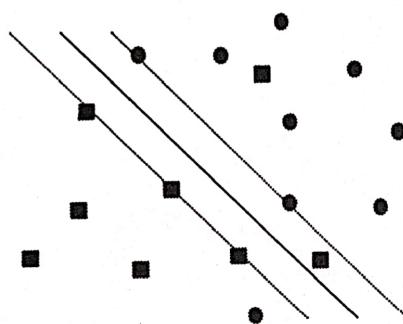
## Exam 2: November 2024

Instructor: Shashank Vatedka

**Instructions:** This is a closed-book exam. You are not permitted to refer to any material or discuss the problem with anyone. Malpractice will be severely punished. Please mention your ROLL Number and name clearly in the answer sheet.

State your final answer clearly. Justify all your statements. You may use any result proved in class (but clearly state which results you are using), but everything else needs to be proved.

**Question 2.1** (5pts). Consider the following figure. Suppose that the solid line indicates the soft-SVM solution and the dotted line are marginal hyperplanes. Label all points that cannot be valid support vectors.



**Question 2.2** (6+6pts). For each of the following functions, find the differential set at all points in the domain. You must verify that these are valid differential sets.

$$\underline{1.} \ f : \mathbb{R}^2 \rightarrow \mathbb{R} \text{ with } f(\underline{x}) = \|\underline{x}\|_1$$

$$\underline{2.} \ f : \mathbb{R} \rightarrow \mathbb{R} \text{ with } f(x) = e^{|x|}$$

**Question 2.3** (6+6+6pts). Justify whether the following kernels are PDS or not:

$$1. \ \mathcal{X} = \mathbb{R}^d \text{ and } K(\underline{x}_1, \underline{x}_2) = 1_{\{\|\underline{x}_1 - \underline{x}_2\|_2 \leq 1\}}$$

$$2. \text{ For a finite set } \mathcal{A}, \text{ define } \mathcal{X} \text{ to be the collection of all subsets of } \mathcal{A}. \text{ The kernel, } K(A, B) = e^{-|A \Delta B|}$$

$$3. \ \mathcal{X} = [0, 1], \text{ and } K(x_1, x_2) = \min\{x_1, x_2\} - x_1 x_2.$$

**Question 2.4** (10pts). Consider the PCA problem:

$$\arg \min_{P \in \mathbb{P}_k} \|PX - X\|_F^2$$

where  $X$  is an arbitrary  $d \times N$  matrix, and  $k < d < N$ . Here,  $\mathbb{P}_k$  is the set of all orthogonal projections onto  $k$ -dimensional subspaces of  $\mathbb{R}^d$ .

Prove that the minimizing  $P$  is of the form  $UU^T$ , where the columns of  $U$  correspond to the eigenvectors with  $k$  largest eigenvalues of  $XX^T$ . You may use any fundamental result in linear algebra, but clearly state which result you use in each step.

$$e^{-1 + AB^T + (A^T A)} \quad 2-1$$

$$(AB^T)^2 = (A^T A)^2 B^T - (A^T A)(B^T A)$$

$$e^{-1 + A^T A} e^{-1 + B^T B} e^{-2(A^T A)(B^T A)}$$

$$0.1 - \frac{0.1 \times 0.1}{0.9 \times 0.9} = 0.09$$

$$0.9 - \frac{0.9 \times 0.9}{0.1 \times 0.1} = -0.09$$

$$0.1 - 0.09 = 0.01$$