1. Discussion on second derivative test for finding extrema / saddle point
2. Chain rule with multivariables.
3. Gradient.

EE1203: Vector Calculus

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భారతీయ సాంకేతిక విజ్ఞాన సంస్థ హైదరాబాద్
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Verify general function Conditions (Second derivative test.) with special quadratic function.

$f = ax^2 + bxy + cy^2$.

$f_{xx} = 2a$; $f_{xy} = f_{yx} = b$; $f_{yy} = 2c$

$\Rightarrow A = 2a$ $B = b$; $C = 2c$.

$$\boxed{AC - B^2 = 4ac - b^2}$$

Why does it work?

Ans: $\Rightarrow$ Taylor's Approximation:

Change in function around a point is by approx. formula

$$\Rightarrow \Delta f \approx \underset{\overset{\parallel}{0}}{f_x} (x - x_0) + \underset{\overset{\parallel}{0}}{f_y} (y - y_0)$$

at critical point.

So we need more terms with quadratic terms, in Taylor series.

$$\Rightarrow \Delta f \approx \underbrace{f_x(x-x_0) + f_y(y-y_0)}_{=0} + \frac{1}{2}\underbrace{f_{xx}}_{a=\frac{1}{2}A}(x-x_0)^2 + \underbrace{f_{xy}}_{B=b}(x-x_0)\cdot(y-y_0)$$

$$+ \frac{1}{2}\underbrace{f_{yy}}_{c=\frac{1}{2}C}(y-y_0)^2 + \cdots$$

→ So the general case reduces to quadratic case. & hence the above explanation with special quadratic function analysis help to derive the second derivative test.

* In degenerate case : We need to conside higher order derivatives, to say function decreases or increases w.r.t critical point.

Total differentials: $f(x, y, z)$

① $df = f_x\,dx + f_y\,dy + f_z\,dz.$

If $x = x(t)$, $y = y(t)$, $z = z(t)$

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt} \leftarrow \text{-- Chain Rule.}$$

Proof:

$\Delta f \approx f_x \Delta x + f_y \Delta y + f_z \Delta z$ (in time Δt)

$$\frac{\Delta f}{\Delta t} \approx f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + f_z \frac{\Delta z}{\Delta t}. \quad ; \Delta t \to 0$$

$\Rightarrow \frac{\Delta f}{\Delta t} \to \frac{df}{dt}.$

$\frac{\Delta x}{\Delta t} \to \frac{dx}{dt}.$

$$\overset{\Delta t \to 0}{\Rightarrow} \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + f_z \frac{dz}{dt}$$

$\hookrightarrow \approx$ becomes in limit $\Delta t \to 0.$

* Application : → Proving product & quotient rules in derivatives.

Try yourself $f = uv$; $u = u(t)$; $v = v(t)$

$g = \frac{u}{v}$,

Using the chain rule in partial derivatives prove product & quotient rules.

Chain rule with more variables:

$\omega = f(x,y)$, where $x = x(u,v)$
$y = y(u,v)$

Goal: $\frac{d\omega}{du}$? $\frac{d\omega}{dv}$

$$d\omega = f_x \, dx + f_y \, dy .$$

$$= f_x \left\{ \underset{\frac{\partial x}{\partial u}}{x_u} du + \underset{\frac{\partial x}{\partial v}}{x_v} dv \right\} + f_y \left\{ \underset{\frac{\partial y}{\partial u}}{y_u} du + \underset{\frac{\partial y}{\partial v}}{y_v} dv \right\} .$$

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$$= \underbrace{(f_x x_u + f_y y_u)}_{\frac{\partial f}{\partial u}} du + (f_x \underbrace{x_v + f_y y_v}_{\partial f / \partial v}) dv$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}.$$

Gradient vector:

Before that let us define scalar field & vector fields.

1. Scalar field: It assigns a scalar value to every point in space.

Example: $T(x, y, z) = x^2 + y^2 + z^2$

Temperature T at each point in space.

2. Vector field : It assigns a vector to every point in space

Example: $\bar{F}(x,y,z) = (y^2, 2xy, -3xy)$

$\uparrow F_1 \quad \uparrow F_2 \quad \uparrow F_3$

$\approx F_1(x,y,z)\,\hat{i} + F_2(x,y,z)\,\hat{j} + F_3(x,y,z)\,\hat{k}$

$\vec{E}(x,y,z) = E_x(x,y,z)\,\hat{i} + E_y(x,y,z)\,\hat{j} + E_z(x,y,z)\,\hat{k}$. ← Electric field, Velocity vector of fluids. ...

Chain Rule: $w = w(x,y,z)$; $x = x(t)$; $y = y(t)$; $z = z(t)$

$$\frac{dw}{dt} = w_x \frac{dx}{dt} + w_y \frac{dy}{dt} + w_z \frac{dz}{dt}.$$

$$= \underline{\underline{\nabla w}} \cdot \frac{d\bar{r}}{dt}$$

Gradient of w

$$\nabla w = (w_x, w_y, w_z) = \left(\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial z}\hat{k}$$ ← vector

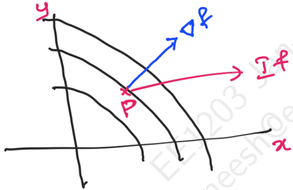
① Gradient of 'w' (scalar field) at some point (x, y, z)

$\Rightarrow$ vector field. Eg: $\vec{E} = -\nabla V$ ← scalar potential. ($\vec{E}$: electric field)

$$\frac{d\bar{r}}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \leftarrow \text{velocity vector.}$$

② Theorem: $\nabla w \perp$ level surface of f^n 'w' $\{w = \text{Constant}\}$. level surface

If 2 variable function → level curves. (Contour plot of function $f(x,y)$)



If we take Gradient of function $f(x,y)$ at point 'P'; ∇f is $\perp^r$ the level curve at that point.

Similarly for 3 variable function $f(x,y,z)$, ∇f is perpendicular to level surface at that point.

Example: proof.

$$w = a_1 x + a_2 y + a_3 z \quad \text{- - - - (1)}$$

$$\nabla w = \frac{\partial}{\partial x} w \, \hat{i} + \frac{\partial}{\partial y} w \, \hat{j} + \frac{\partial}{\partial z} w \, \hat{k}$$

$$= a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}. \quad \text{- - - - (1)}$$

level surface at a fixed function value say $w = c$

$\Rightarrow \quad \underline{a_1} x + \underline{a_2} y + \underline{a_3} z = c \leftarrow$ - - level surface at $w = c$

$\Rightarrow$ is a plane.

$(a_1, a_2, a_3) \leftarrow$ is the normal vector to that plane.

- - - (2)

From (1) & (2) $\Rightarrow$ ∇w is a normal vector to the level surface.

$\Downarrow$ $E = -\nabla V$.

"Electric field between 2 $\parallel^{l}$ plate capacitor".

$\leftarrow$ will see this point later.

$\boxed{\nabla w \rightarrow \text{Always points towards higher value of function } `w'.}$

Example:

Find the tangent plane to surface $x^2+y^2-z^2=4$ at point $(2,1,1)$.

Answer: Say $w(x,y,z) = 4$

$$\Rightarrow \quad w = x^2+y^2-z^2$$

$$\nabla w = \frac{\partial w}{\partial x}\hat{i} + \frac{\partial w}{\partial y}\hat{j} + \frac{\partial w}{\partial x}\hat{k}$$

$$= 2x\,\hat{i} + 2y\,\hat{j} - 2z\,\hat{k}.$$

$$\nabla w_{\text{at }(2,1,1)} = 4\hat{i} + 2\hat{j} - 2\hat{k}.$$

$\Rightarrow \quad (4,2,-2) \rightarrow$ normal to the tangent plane to the surface.

$\Rightarrow \quad$ Equation of tangent plane: $4x+2y-2z=d$

$$d = (4\times 2 + 2\times 1 - 2\times 1) = 8.$$

$$\Rightarrow \quad \boxed{4x+2y-2z=8.}$$

Alternate method: Try thinking to use the concept of Approximation formula Δw?

* $\Delta w = 0$ at the level surfaces.

$$\Rightarrow \quad \Delta w = \overset{\frac{\partial w}{\partial x}}{w_x}\,\Delta x + \overset{\frac{\partial w}{\partial y}}{w_y}\,\Delta y + \overset{\frac{\partial w}{\partial z}}{w_z}\,\Delta z$$

$$0 = 2x\,\Delta x + 2y\,\Delta y - 2z\,\Delta z. \quad \Big|\, \text{at } (2,1,1)$$

$$\Rightarrow \quad 4\Delta x + 2\Delta y - 2\Delta z = 0$$

$$\Rightarrow \quad 4(x-2) + 2(y-1) - 2(z-1) = 0$$

$$\Rightarrow \quad \boxed{4x + 2y - 2z = 8}$$ ✓