MARKS	8	7	8	6							

Answers

$P(n) \Rightarrow 2^{2n} - 1$ is divisible by 3

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we need to prove $\forall n \in \mathbb{N}$, $P(n)$ is true.

Using Induction to prove this:

Base Case:

$P(1) \Rightarrow 2^{2} - 1$ is divisible by 3

$\Rightarrow$ 3 is divisible by 3

so $P(1)$ is true.

Induction step: $\forall k \in \mathbb{N}$, $P(k) \longrightarrow P(k+1)$

Assuming $P(k)$ to be true for some $k \in \mathbb{N}$

so $3 \mid 2^{2k} - 1$

for $k+1$

$$2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 4 \cdot 2^{2k} - 1$$

$$= 3 \cdot 2^{2k} + 2^{2k} - 1$$

as $3 \mid 3 \cdot 2^{2k}$ & $3 \mid 2^{2k} - 1$

$\therefore 3 \mid 2^{2(k+1)} - 1$

$\therefore$ $P(k+1)$ i.e. $2^{2(k+1)} - 1$ is divisible by 3 is true.

Hence by Induction we proved that $\forall n \in \mathbb{N}$, $2^{2n} - 1$ is divisible by 3.

4 given : $d = \gcd(a,b)$

$\cancel{d}$ $d = sa + tb$ for $s,t \in \mathbb{Z}$

T.P. $\Rightarrow$ d is smallest +ve integer expressed as an integer L.C. of a & b

proof :

~~Let ...~~

Let the smallest +ve integer expressed as L.C. of a & b be r.

given by $\Rightarrow$ $r = s'a + t'b$ for $s', t' \in \mathbb{Z}$

as it is the smallest, $\boxed{r \le d \text{ holds true}}$ — (1)

just take it as $< d$

as d is $\gcd(a,b)$

$d \mid a$ & $d \mid b$ $\therefore$ $d \mid s'a + t'b$

$\therefore$ $d \mid r$ why?

$\Rightarrow$ $\boxed{d \le r}$ — (2)

by (1) & (2)

$\boxed{d = r}$

Hence we proved that d is the smallest +ve integer which can be expressed as an integer linear combination of a & b.

① given "A" a set of n integers,

Let $A = \{a_1, a_2, \ldots, a_n\}$

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where $\forall i, a_i \in \mathbb{Z}$ is an element of set A.

now Let's make another set S. whose elements are $s_j = \sum_{i=1}^{j} a_i$ i.e. $\left\{ \begin{array}{l} a_1, \\ a_1+a_2, \\ a_1+a_2+a_3, \\ \vdots \\ a_1+a_2+a_3 \ldots + a_n \end{array} \right.$

now, two cases.

if. $\exists j \in \{1, \ldots, n\}$ s.t. $n \mid s_j$

then $B = \{a_1, \ldots, a_j\} \subseteq A$

s.t. $\sum_{i=1}^{j} a_i = s_j$ is divisible by n.

Hence proved the required statement

Else $\forall j \in \{1, \ldots, n\}$ $n \nmid s_j$ or $s_j \bmod n \neq 0$

when divided by n the remainders can be

now $\rightarrow \{1, \ldots, n-1\}$ $\hookrightarrow$ $n-1$ possible options.

but the elements in the set S are n

so by Pigeon hole principle

$\exists\ i, j \in \{1, \ldots n\}$, $i \neq j$ s.t. $s_i \bmod n = s_j \bmod n$

$n \mid s_j - s_i$ where $j > i$ without loss of generality.

$\therefore \quad B = \{a_{i+1}, \ldots, a_j\} \subseteq A$ s.t.

$$\sum_{k=i+1}^{j} a_k = \sum_{k=1}^{j} a_k - \sum_{k=1}^{i} a_k$$

$$= s_j - s_i \text{ is divisible by } n$$

Hence proved

$\exists$ a set $B \subseteq A$ & sum of integers in B is divisible by n.

(2) given ⇒ $0 \le l \le k \le n$, $l, k, n \in \mathbb{Z}$

T.P. ⇒ $\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$

proof ⇒ The problem can be interpreted as from a group of n candidates we need to choose some $(k-l)$ as constables and some (l) as senior officers for a police station.

L.H.S. ⇒ from n candidates we first choose k ⇒ total chosen people, from that we choose l candidates to become senior officer

⇒ $\binom{n}{k}\binom{k}{l}$

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R.H.S. ⇒ from n candidates we choose l senior officers directly and $k-l$ constables from remaining candidates.

⇒ $\binom{n}{l}\binom{n-l}{k-l}$

in both these ways we have found all possible combination of people with position for a police station ∴ L.H.S. = R.H.S.

Otherwise mathematically,

$$\binom{n}{k}\binom{k}{l} \Rightarrow \frac{n!}{(n-k)!\,k!} \times \frac{k!}{(k-l)!\,l!}$$

$$= \frac{n!}{l!} \times \frac{1}{(k-l)!\,(n-k)!}$$

multiply & divide by $(n-l)!$

$$= \frac{n!}{l!\,(n-l)!} \times \frac{(n-l)!}{(k-l)!\,(n-k)!}$$

$$= \binom{n}{l}\binom{n-l}{k-l}$$

H.P.