EE1080/AI1110/EE2120 Probability, Quiz 1

Solutions by: Dhadheechi Ravva and Pawan Surya

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Max. Marks: 16. **Time:** 1 hour.

Instructions

- Please **write your roll number** and **course id** prominently in the first page of the answer sheet.
- No laptops, mobile devices etc. allowed.
- please write supporting arguments for any of the statements you make. any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer.

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1. (3) Identify the following sets as finite, countably infinite or uncountable as you seem appropriate. Given a set A, the notation A^n is used to indicate vectors of length n with elements taking values from A.

   (a) S^∞ where $S = \{1, 2, \cdots, 6\}$.

   (b) S^{1000}

   (c) Q^2 where Q is the set of rational numbers.

   Solution:

   (a) S^∞ is uncountable. Similar to the argument seen in the class if $S^\infty = \{x_1, x_2, \cdots\}$ then we can construct a sequence y that is not equal to any of the x_i's by setting $y(i) \in S \setminus \{x_i(i)\}$. Therefore it can't be countable

   (b) S^{1000}: set of all vectors of length 1000, each element belongs to $\{1, 2, 3, 4, 5, 6\}$.
   Number of such vectors (sequences) $= 6^{1000}$.
   $\Rightarrow S^{1000}$ is countable (finite).

   (c) Q^2: set of all pairs (two-dimensional vectors), each element $\in Q$.
   Since Q is countable, there exists a mapping from Q to $\mathbb{N}$. This means that there exists a mapping from $Q \times Q$ to $\mathbb{N} \times \mathbb{N}$. And since there exists a mapping from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ (using the same argument used to prove that Q is countable), there exists a mapping from $Q \times Q \Rightarrow \mathbb{N}$. That is, $Q \times Q$ is a countable set.

2. $(3 = 1.5 + 1.5)$ Given an experiment where you roll a six faced die with sample space $\Omega = \{1, 2, \cdots, 6\}$.

   (a) Which of the following can be an event space $\mathcal{F}$:

   i. $\{\phi, \Omega\}$

   ii. $\{\phi, \Omega, \{1, 6\}, \{2, 3\}, \{4, 5\}\}$

   iii. $\{\phi, \Omega, \{1, 2, 3\}, \{4, 5, 6\}\}$

   (b) Assume $\mathcal{F} = 2^\Omega$. Consider probability function $P(\{i\}) = i * x$ for all $i \in \Omega$. Identify the value of x.

   Solution:

   (a) i. $\{\phi, \Omega\}$ is a valid event space.

   ii. $\{\phi, \Omega, \{1, 6\}, \{2, 3\}, \{4, 5\}\}$ is not a valid event space, since it does not contain the set complements and the pairwise unions of the subsets $\{1, 6\}, \{2, 3\}$, and $\{4, 5\}$.

   iii. $\{\phi, \Omega, \{1, 2, 3\}, \{4, 5, 6\}\}$ is a valid event space. All the set complements, unions and intersections of the subsets are also elements of the event space.

   (b) We know,

   $$\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\} = \Omega$$
   $$P(\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}) = P(\Omega)$$

   Since the sets are disjoint, and the sample space is a certain event,

   $$\sum_{i=1}^{6} P(\{i\}) = 1$$
   $$\sum_{i=1}^{6} (i * x) = 1$$
   $$x = \frac{1}{21}$$

   Hence, the value of x is $\frac{1}{21}$.

3. (1) Suppose A and B are events with very high probability: say $P(A) = 0.95$ and $P(B) = 0.85$. Then which of the following are true (check all that apply)

   (a) Either A or B occur with probability at most 0.85

   (b) Both A and B occur with probability at least 0.8

   Given: $P(A) = 0.95$, $P(B) = 0.85$.

(a) $P(A \cup B) \geq P(A)$, $P(A \cup B) \geq P(B)$.
$\Rightarrow P(A \cup B) \geq 0.95$.
$\Rightarrow$ Either A or B occur with probability at least 0.95.
$\Rightarrow$ a) is false.

(b) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ (Principle of Inclusion-exclusion).
$= 1.8 - P(A \cup B)$.
Since $P(A \cup B) \leq 1$ (P(any event) is atmost 1),
$P(A \cap B) \geq 1.8 - 1$.
$P(A \cap B) \geq 0.8$.
$\Rightarrow$ b) is true.

4. (2) Identify if the following statements are true or false

(a) mutual independence of events $A, B, C \implies A, B$ are independent conditioned over C

(b) A, B are independent conditioned over C implies A, B are independent

Solution: True for (a) and False for (b).
(a) Mutual independence implies: $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$, $P(A \cap C) = P(A) \cdot P(C)$, $P(C \cap B) = P(C) \cdot P(B)$, $P(A \cap B) = P(A) \cdot P(B)$.

Using the definition of conditional probability:

$$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)}.$$

Substituting the given values:

$$P(A \cap B \mid C) = \frac{P(A) \cdot P(B) \cdot P(C)}{P(C)} = P(A) \cdot P(B).$$

Since $P(A \mid C) = P(A)$ and $P(B \mid C) = P(B)$ from mutual independence it follows that:

$$P(A \cap B \mid C) = P(A \mid C) \cdot P(B \mid C),$$

which confirms that A and B are independent conditioned on C.

Conclusion: A and B are independent conditioned on C, and the statement is **True**.

(b) Refer to the example from class: A coin is chosen randomly from a set of blue and red coins at the beginning of the experiment and then tossed twice with the chosen coin. C is the event that blue coin is chosen. A, B are the events that the first toss is head, second toss is head respectively. Let $P(B|C) = P(A|C) = 0.9$, $P(B|C^c) = P(A|C^c) = 0.01$. By definition $P(A \cap B|C) = P(A|C)P(B|C)$. Can check that $P(A \cap B) \neq P(A)P(B)$

5. (2) An online store offers three types of products: books, electronics, and clothing. The probability that a book is defective is 0.02, the probability that an electronic item is defective is 0.05, and the probability that a piece of clothing is defective is 0.03. A customer orders one book, one electronic item, and one piece of clothing. What is the probability that at least one of the items is defective?

Solution: Let $P(B)$, $P(E)$, and $P(C)$ denote the probability that the book, electronics and clothing are defective respectively.

$$P(B) = 0.02$$
$$P(E) = 0.05$$
$$P(C) = 0.03$$

$$P(\text{at least one defective}) + P(\text{none\_defective}) = 1$$

$$P(\text{none\_defective}) = P(B^c \cap E^c \cap C^c)$$
$$= P(B^c) \cdot P(E^c) \cdot P(C^c)$$

(since the events are independent).

$$P(\text{none\_defective}) = (1 - P(B))(1 - P(E))(1 - P(C))$$
$$= 0.903$$

The probability that at least one of the items is defective is:

$$P(\text{at least one defective}) = 1 - P(\text{none\_ defective})$$
$$P(\text{at least one defective}) = 1 - 0.903$$
$$P(\text{at least one defective}) = 0.097$$

Therefore, the probability that at least one item is defective is 0.097.

6. (2) We know that a treasure is located in one of two places, with probabilities β and $1 - \beta$, respectively, where $0 < \beta < 1$. We search the first place and if the treasure is there, we find it with probability $p > 0$. Given the treasure is not found in the first place, what is the probability of treasure being in the second place ?
Solution: Let A be the event that treasure is in second place and B be the event that is not found in the first place. We need to find $P(A|B)$. We know that $P(B|A) = 1$ ie., probability that you don't find the treasure at first place, given it is located in second place is 1. $P(A) = (1 - \beta)$. $P(B^c|A^c) = p$ i.e., $P(B|A^c) = (1 - p)$.

Using Bayes' Theorem:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \\ &= \frac{(1-\beta)}{(1-\beta) + \beta(1-p)} \end{aligned}$$

7. (3) Alice and Bob want to choose between the opera and the movies.

   (a) Unfortunately, the only available coin is biased (and the bias is not known to Alice or Bob). How can they use the biased coin to make a decision so that either option (opera/movies) is equally likely to be chosen?

   (b) Now, instead, suppose that Alice and Bob have a fair coin. Alice and Bob want to toss a coin that is predisposed towards the movies - say they would want to pick the opera with probability $p = 1/8$. How can they use the fair coin to make a decision?

   (*Hint: Toss the coin multiple times*)

   (a) Say the coin is biased towards Heads (H) with probability p. Toss the coin twice and make a decision as follows:
   $\Omega = \{HH, HT, TH, TT\}$.

$$\begin{cases} HH, & \text{repeat the experiment, probability} = p^2 \\ HT, & \text{Choose movies, probability} = p(1-p) \\ TH, & \text{Choose Opera, probability} = (1-p)p \\ TT, & \text{repeat the experiment, probability} = (1-p)^2 \end{cases}$$

   $\Rightarrow P(\text{Movies}) = P(\text{Opera}) = p(1-p)$

   (b) Toss the coin three times. Choose Opera only if the result is Heads three consecutive times.
   Then, $P(\text{Opera}) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$.