Convergence of Random Variables

Ref: Principles of Math. Analysis by Rudin

① Convergence of sequence of real numbers.

For a sequence of real numbers $a_1, a_2, \dots = (a_n)_{n\geq 1}$, $(a_n)_{n\geq 1}$ is said to converge to a if the following holds:

$$\forall \epsilon > 0, \ \exists N \text{ st: }$$

$$|a_n - a| \leq \epsilon \quad \forall n \geq N.$$

Example: ① $a_n = \frac{1}{n}$, $\lim_{n\to\infty} a_n = 0$.

Let $a = 0$,

for $\epsilon > 0$.

$$|a_n| \leq \epsilon \quad \forall n \geq \left\lceil \frac{1}{\epsilon} \right\rceil$$

$$n \geq \left\lceil \frac{1}{\epsilon} \right\rceil \geq \frac{1}{\epsilon} \Rightarrow n \geq \frac{1}{\epsilon}$$

$$\Rightarrow \frac{1}{n} \leq \epsilon.$$

② $a_n = (-1)^n$ doesn't converge.

$$1 \quad -1 \quad 1 \quad -1 \quad \dots$$

Let: $\epsilon = \frac{1}{100}$, N be some number

Suppose $|1-a| < \frac{1}{100}$
then $|-1-a| > \frac{1}{100}$

We'll show that $\exists \epsilon$ st $\forall N \in \mathbb{N}$, $\exists n \geq N$ st

$$|a_n - a| > \epsilon.$$

"there are infinitely many n st $|a_n - a| > \epsilon$" for some ϵ.

$(\cup A_n)^c = \cap A_n^c$

(a) Some properties

(1) $a_n \to a$, $b_n \to b$ then $a_n + b_n \longrightarrow a+b$
and $a_n - b_n \longrightarrow a-b$.

(2) $l_n \le a_n \le u_n$ then $\lim l_n \le \lim a_n \le \lim u_n$

(3) Any monotonic sequence that is bounded converges.

Example: $a_n = \frac{1}{n} \ge 0$ (a) Let $a_1 \ge a_2 \ge \dots \ge l$. then $\lim_{n\to\infty} a_n = a^* = \inf_{n\to\infty} a_n$ exists

a^*, $a^*+\epsilon$, ..., a_2, a_1

$\exists a_N \in (a^*, a^*+\epsilon) \Rightarrow \forall n \ge N, |a_n - a^*| < \epsilon$

$a_n = 1 - \frac{1}{n} \le 1$. (b) Let $a_1 \le a_2 \le \dots \le u$ then $\lim_{n\to\infty} a_n$ exists.

(b) convergence of series (sum sequence).

$$S_n = \sum_{i=1}^{n} a_i \quad , \quad \sum_{i=1}^{\infty} a_i \text{ is finite or not.}$$

$S_1 \quad S_2 \quad S_3 \; ----.$

If $S_n \longrightarrow s = \sum_{i=1}^{\infty} a_i < \infty$.

Lemma 1: Suppose S_n converges to s, then
$$\hat{S}_n = \sum_{i=n}^{\infty} a_i \longrightarrow 0.$$

Proof:
$$\hat{S}_n = S - S_{n-1}$$
$$\lim_{n\to\infty} \hat{S}_n = \lim_{n\to\infty} (S - S_{n-1}) = S - \lim_{n\to\infty} S_{n-1}$$
$$= s - s = 0.$$

Lemma 2: Suppose S_n converges to s, then $a_n \to 0$.

Proof:
$$a_n = s_n - s_{n-1}$$
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0.$$

Suppose $S_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$. S_n converges for $p > 1$ and diverges for $p \le 1$.

We'll use the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$.

$$\sum \frac{1}{n^2} \le \sum \frac{1}{n(n-1)} = 1 + \sum_{i=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) \le 2.$$

(2) Pointwise convergence of random variables

(1) Bertsekas, Chapter 5

② Random Processes for Engineers, Bruce Hayek.

Let $X_1, X_2, \cdots$ be random variables defined over the same sample space $(\Omega, \mathcal{F}, P)$. $X_i : \Omega \to \mathbb{R}$.

For every $\omega \in \Omega$ if $X_n(\omega) \longrightarrow X(\omega)$. i.e.,

$$\lim_{n\to\infty} X_n(\omega) = X(\omega)$$

then $X_1, X_2 \cdots$ converge to R.V X pointwise.

Example: $X_n(\omega) = \omega^n \quad \forall \omega \in [0,1] = \Omega$.

$$\lim_{n\to\infty} X_n(\omega) = \lim_{n\to\infty} \omega^n = \begin{cases} 0 & \omega \in [0,1) \\ 1 & \text{if } \omega = 1 \end{cases}$$

Say $X(\omega) = \begin{cases} 0 & \omega \in [0,1) \\ 1 & \text{otherwise} \end{cases}$

then $(X_n)_{n\geq 1}$ converges to X pointwise.

③ Almost - Sure Convergence :

$X_1, X_2, \cdots, X$ are R.Vs defined over same $(\Omega, \mathcal{F}, P)$ then

$X_n \longrightarrow X$ almost surely iff

$$P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = 1.$$

Example ⓐ $X_n(\omega) = \omega^n$, $X(\omega) = 0$. $\forall \omega \in [0,1]$.

$$\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\} = [0,1).$$

$$P\left(\left\{\omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)\right\}\right) = P(\{\omega \in [0,1)\}) = 1.$$

ⓑ min of uniform R.Vs. Let $X_1, X_2, \cdots$ be iid uniform(0,1) R.Vs and $Y_n = \min\{X_1, X_2, \ldots X_n\}$

(Bertsekas).

$\Downarrow$

$Y_n(\omega) = \min\{X_1(\omega) \cdots X_n(\omega)\}$

then $Y_n \longrightarrow 0$ almost surely.

Can define R.V Y as $\quad Y(\omega) = \lim_{n\to\infty} Y_n(\omega)$. $\quad \{X_i > \epsilon\}\ \forall i$ - $\bigcap_{i=1}^{\infty} \{X_i > \epsilon\}$

$$= \inf_n Y_n(\omega)$$

The limit exist because $Y_1(\omega) \geq Y_2(\omega) \geq \cdots \geq 0$

$Y(\omega) \leq Y_n(\omega)\ \forall n$ $\quad$ ie., $(Y_n(\omega))$ is a monotonically decreasing sequence lower bounded by 0.

If $Y > \epsilon \Rightarrow Y_n > \epsilon$

$$P(Y > \epsilon) \leq P(Y_n > \epsilon) = P\left(\bigcap_{i=1}^{n} \{X_i > \epsilon\}\right)$$

$$= \prod_{i=1}^{n} P(X_i > \epsilon) = (1-\epsilon)^n$$

$$P(Y > \epsilon) \leq \lim_{n\to\infty} (1-\epsilon)^n = 0.$$

$$\Rightarrow P(Y > \epsilon) = 0 \text{ for } \epsilon > 0.$$

$\Rightarrow P(Y = 0) = 1$. as Y is non-negative R.V.

$$P\left(\left\{\omega : \lim_{n\to\infty} Y_n(\omega) = 0\right\}\right) = P(\{\omega : Y(\omega) = 0\})$$

as we defined $Y(\omega)$ to be $\lim_{n\to\infty} Y_n(\omega)$

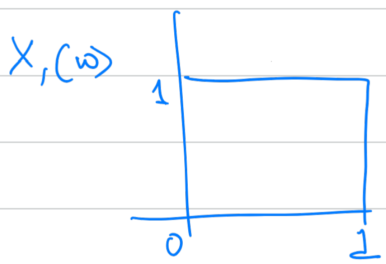
$= 1$.

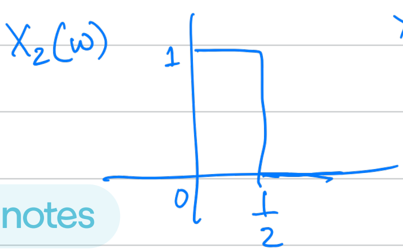
$\Rightarrow Y_n \to 0$ a.s. (almost surely).

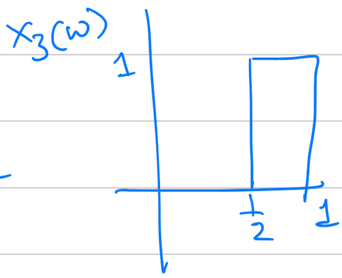
c) Moving, shrinking rectangles

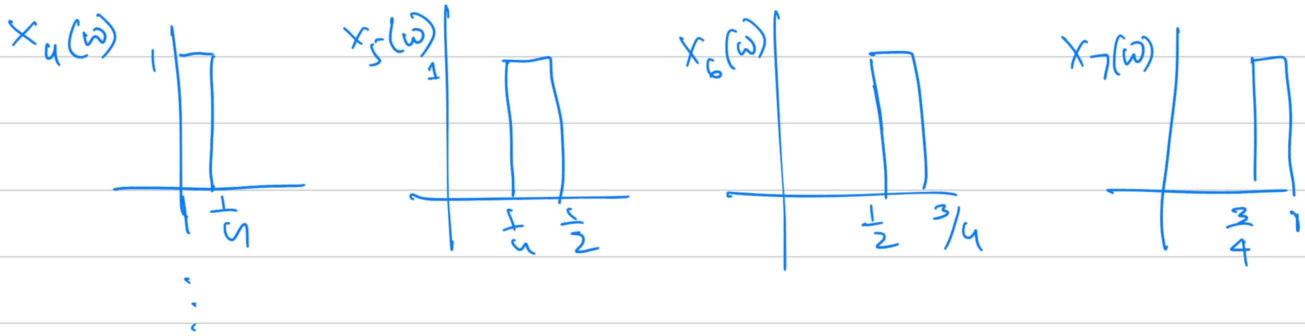
Lecture 31

$\Omega = [0, 1]$.





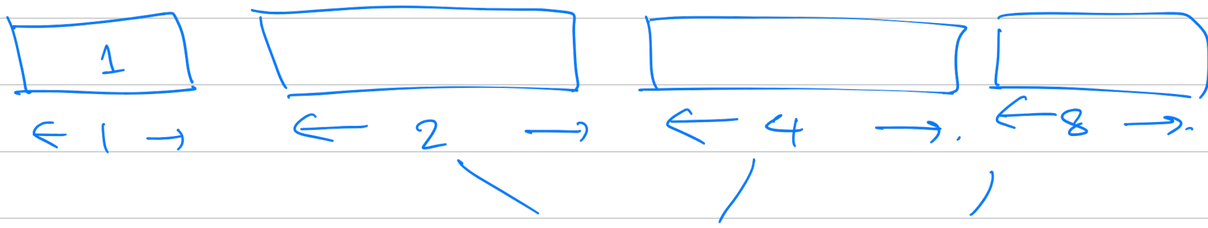




$$(X_n(0))_{n\geq 1} = (1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad \ldots).$$

This sequence doesn't converge as 1 appears "infinitely often"

$(X_n(\omega))_{n\geq 1}$ $X_1(\omega)$ $X_2(\omega)$ $X_3(\omega)$ $X_4(\omega) \ldots X_7(\omega) \ldots$



exactly one element is 1 and rest are zeroes.

This sequence doesn't converge as 1 appears infinitely often.

$\Rightarrow$ X_n doesn't converge to 0 a.s

" " any R.V a.s.

Made with Goodnotes

④ **Convergence in probability.** Let $X_1, X_2, \ldots, X$ be R.Vs defined over sample space Ω, then $X_n \to X$ in probability if $\forall \epsilon > 0$.

$$\lim_{n\to\infty} P\left(|X_n - X| > \epsilon \right) = 0 .$$

Examples ⓐ Moving, Shrinking Rectangles.

Let $k = \lfloor \log_2 n \rfloor$, $i = n - 2^k + 1 . \Rightarrow n = 2^k + i - 1$

$$X_n(\omega) = \begin{cases} 1 & \omega \in \left[\frac{(i-1)}{2^k} ; \frac{i}{2^k} \right] \\ 0 & \text{Otherwise} \end{cases}$$

If $n = 1$, $k = 0$, $i = 1$

$X_1(\omega) = 1 \quad \omega \in [0, 1]$

$n = 2$, $k = 1$, $i = 1$

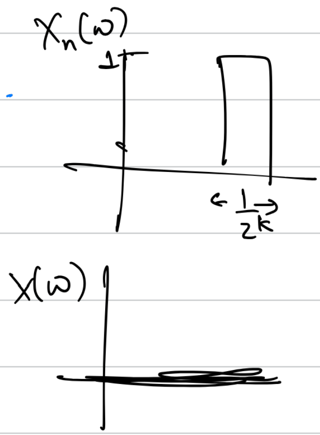
$X_2(\omega) = 1 \quad \omega \in \left[0 ; \frac{1}{2}\right]$

$n = 3$, $k = 1$, $i = 2$

$X_2(\omega) = 1 \quad \omega \in \left[\frac{1}{2}, 1\right].$

$\vdots$

$X(\omega) = 0 \quad \omega \in [0, 1] .$



For some $\epsilon > 0$, let $k = \lfloor \log_2 n \rfloor$

$$P\left(|X_n - X| > \epsilon \right) = \frac{1}{2^k} .$$

$$\leq \frac{2}{n} .$$

$\log_2 n - 1 \leq k \leq \lfloor \log_2 n \rfloor$

$\frac{n}{2} \leq 2^k \leq n .$

$$\lim_{n\to\infty} P\left(|X_n - X| > \epsilon \right) \leq \lim_{n\to\infty} \frac{2}{n}$$

$$= 0 .$$

$$\Rightarrow \lim_{n\to\infty} P\left(|X_n - X| > \epsilon \right) = 0$$

$X_n \to X$ in probability.

(b) Weak Law of Large numbers: X_i's are iid with mean μ & variance σ^2

$$S_n = \sum_{i=1}^{n} X_i, \quad E[S_n] = \mu, \quad Var(S_n) = \frac{\sigma^2}{n}$$

$$P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \frac{Var\left(\frac{S_n}{n}\right)}{\epsilon^2}$$

$$= \frac{\sigma^2}{n\epsilon^2}$$

$\lim_{n\to\infty} a_n \leq 0$ and $a_n \geq 0$

$\Rightarrow \lim_{n\to\infty} a_n \geq 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0.$

$$\lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) \leq \lim_{n\to\infty} \frac{\sigma^2}{n\epsilon^2}$$

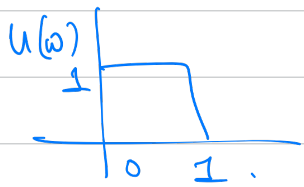
$$= 0.$$

$$\Rightarrow \lim_{n\to\infty} P\left(\left|\frac{S_n}{n} - \mu\right| > \epsilon\right) = 0.$$

$\Rightarrow \frac{S_n}{n} \longrightarrow \mu$ in probability sense.

(c) $X_n(\omega) = U(\omega) \quad \omega \in [0, 1]$

$X(\omega) = 1 - U(\omega).$



does X_n converge to X in probability sense?

$$P(|X_n - X| > \epsilon) = P(|2U-1| > \epsilon).$$

$$= P(2U-1 > \epsilon) + P(2U-1 < -\epsilon)$$

$$= P\left(U > \frac{1+\epsilon}{2}\right) + P\left(U < \frac{1-\epsilon}{2}\right).$$

$$= \left[1 - \left(\frac{1+\epsilon}{2}\right)\right] + \left(\frac{1-\epsilon}{2}\right)$$

$$= (1-\epsilon).$$

$\Rightarrow$ X_n doesn't converge to X in probability.

" We'll define convergence in distribution which is further weaker form of convergence".

(d) Strong law of large numbers (SLLN)

$$\frac{S_n}{n} \xrightarrow{a.s.} \mu$$

where $S_n = \sum_{i=1}^{n} X_i$ and X_i's are iid with mean μ & variance σ^2.

We'll also assume $E[S_n^4] < \infty$.

We want to show (Bertsekas)

$$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z)$$

$\Uparrow$ Follows from inversion property.

Moment Generating function

$$\lim_{n\to\infty} M_{Z_n}(s) = M_Z(s)$$

$$M_{Z_n}(s) = E\left[e^{sZ_n}\right] \qquad Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma} = \frac{\sum_{i=1}^{n}(X_i-\mu)}{\sqrt{n}\,\sigma}$$

$$= E\left[e^{\frac{s}{\sqrt{n}\sigma}\left(\sum_{i=1}^{n}(X_i-\mu)\right)}\right]$$

$\bar{X}_i = X_i - \mu$

$$= E\left[\prod_{i=1}^{n} e^{\frac{s}{\sqrt{n}\sigma}\bar{X}_i}\right]$$

independence

$$= \prod_{i=1}^{n} E\left[e^{\frac{s}{\sqrt{n}\sigma}\bar{X}_i}\right]$$

MGF defn

$$= \prod_{i=1}^{n} M_{\bar{X}_i}\left(\frac{s}{\sqrt{n}\,\sigma}\right).$$

identical $\bar{X}_i$'s.

$$= M_{\bar{X}_1}\left(\frac{s}{\sqrt{n}\,\sigma}\right)^n$$

$$\log M_{Z_n}(s) = n \log M_{\bar{X}_1}\left(\frac{s}{\sqrt{n}\,\sigma}\right).$$

$$\lim_{n\to\infty} \log M_{Z_n}(s) = \lim_{n\to\infty} \frac{\log M_{\bar{X}_1}\left(\frac{s}{\sqrt{n}\sigma}\right)}{(1/n)}.$$

Let $L(s) = \log M_{\bar{X}_1}(s)$. $\qquad = \lim_{n\to\infty} \frac{L\left(\frac{s}{\sqrt{n}\sigma}\right)}{1/n}.$

$$L'(s) = \frac{1}{M_{\bar{X}_1}(s)} M'_{\bar{X}_1}(s) \qquad = \lim_{n\to\infty} \frac{\left(-\frac{1}{2}\right) L'\left(\frac{s}{\sqrt{n}\sigma}\right)\frac{s}{\sigma}\, n^{-3/2}}{(-1)\, n^{-2}}$$

$$L''(s) = -\frac{1}{\left(M_{\bar{X}}(s)\right)^2}\left(M'_{\bar{X}}(s)\right)^2 + \frac{1}{M_{\bar{X}}(s)} M''_{\bar{X}}(s).$$

$$L''(0) = -\frac{\left(M'_{\bar{X}}(0)\right)^2}{\left(M_{\bar{X}}(0)\right)^2} + \frac{M''_{\bar{X}}(0)}{M_{\bar{X}}(0)}$$

$M'_{\bar{X}}(0) \to E[\bar{X}] = 0.$ $M_{\bar{X}}(0) = 1$

$M''_{\bar{X}}(0) \to E[\bar{X}^2] = \sigma^2$

$$= \sigma^2$$

$$= \lim_{n\to\infty} \frac{L'\left(\frac{s}{\sqrt{n}\sigma}\right)\frac{s}{\sigma}}{2\left(1/\sqrt{n}\right)}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{s}{\sigma}\right)^2 L''\left(\frac{s}{\sqrt{n}\sigma}\right)\left(\frac{-1}{2}\right) n^{-3/2}}{2 \times \left(\frac{-1}{2}\right) n^{-3/2}}$$

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{s}{\sigma}\right)^2 L''\left(\frac{s}{\sqrt{n}\sigma}\right)$$

$$= \frac{1}{2} s^2.$$

$$\lim_{n\to\infty} \log M_{Z_n}(s) = \frac{s^2}{2}.$$

$$\Rightarrow \lim_{n\to\infty} M_{Z_n}(s) = e^{s^2/2}.$$

$$= M_Z(s)$$ i.e., MGF of standard normal R.V.

$\#$

$$\lim_{n\to\infty} F_{Z_n}(z) = F_Z(z).$$

Example: "Exit Poll"

Let p be the fraction of voters that support Trump for office. n "randomly selected" voters are interviewed & M_n fraction of them support him.

"randomly selected" $\longrightarrow$ chosen independently & uniformly from the pool.

Each voter's response is $X_i \sim Ber(p)$.

$$M_n = \frac{X_1 + X_2 + \cdots X_n}{n}.$$

X_i's are iid.

How accurately does M_n predict "p"?

We want to say with "high confidence" that $M_n \in [p-\epsilon, p+\epsilon]$.

High confidence $\Rightarrow$

$$P(|M_n - p| \le \epsilon) \ge 1-\delta.$$

$$P(|M_n - p| \ge \epsilon) \le \frac{Var(M_n)}{\epsilon^2} = \frac{p(1-p)}{n\epsilon^2} \le \frac{1}{4n\epsilon^2}$$

$$E[M_n] = E\left[\frac{X_1 + \cdots X_n}{n}\right]$$

$$= \frac{n\,E[X_1]}{n}$$

$$= p.$$

$$Var[M_n] = \frac{1}{n^2} Var(X_1 + \cdots X_n)$$

$$= \frac{1}{n} Var(X_1).$$

$$= \frac{1}{n} p(1-p)$$

If $\boxed{\frac{1}{4n\epsilon^2} < \delta.}$

then $P(|M_n - p| \ge \epsilon) < \delta.$

$$P(|M_n - p| \le \epsilon)$$

$$= 1 - P(|M_n - p| > \epsilon)$$

$$\ge 1-\delta.$$

Suppose we want to declare that exit poll is 95% confident that the estimate M_n is accurate upto 10^{-2} error. What should the sample size of poll be?

Made with Goodnotes

$\epsilon = 10^{-2}$, $\quad 1-\delta = \frac{95}{100}$, $\delta = \frac{5}{100}$

Pick n st

$$\frac{1}{4n\epsilon^2} \le \frac{5}{100}$$

$$\Rightarrow \quad n \ge \frac{100}{4 \times 5 \times \epsilon^2} = 5 \times 10^4.$$

You'll need 50K sample size to be 95% Confident within 10^{-2} error.

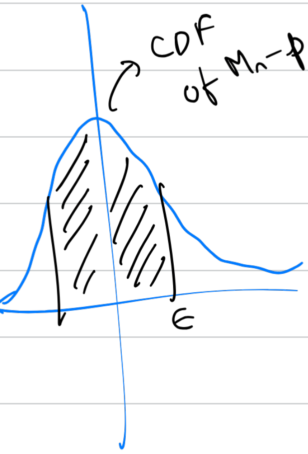
How does n value look like using CLT based approximation.

$$M_n = \frac{S_n}{n}$$

$\mu = p$

$\sigma^2 = p(1-p)$

$$Z_n = \frac{S_n - np}{\sqrt{n}\,\sigma} = \frac{n M_n - n\mu}{\sqrt{n}\,\sigma} = \left(\frac{M_n - \mu}{\sigma/\sqrt{n}}\right)$$



If $Z_n \approx N(0,1)$

$M_n \approx N\left(\mu, \frac{\sigma^2}{n}\right)$, $\quad M_n - p \approx N\left(0, \frac{\sigma^2}{n}\right)$

$$P(|M_n - p| > \epsilon) = 2P(M_n - p > \epsilon)$$

normal RV with mean 0. Symmetric

$$= 2P\left(\frac{M_n - p}{(\sigma/\sqrt{n})} > \frac{\epsilon}{\sigma/\sqrt{n}}\right)$$

$$= 2\left[1 - \Phi\left(\frac{\epsilon}{\sigma/\sqrt{n}}\right)\right].$$

$\sigma^2 = p(1-p).$

$\sigma^2 \le 1/4$

$\sigma \le \frac{1}{2}$, $\quad \frac{1}{\sigma} \ge 2.$

CDF is monotonically increasing

$$\Phi\left(\frac{\epsilon\sqrt{n}}{\sigma}\right) \ge \Phi\left(2\epsilon\sqrt{n}\right)$$

$$\le 2\left[1 - \Phi\left(2\epsilon\sqrt{n}\right)\right]$$

$\epsilon = 10^{-2}$, $\quad \delta = 5/100 \qquad \le \delta.$

$$2\left(1 - \Phi\left(2\times 10^{-2}\sqrt{n}\right)\right) \le \frac{5}{100}.$$

$$\Rightarrow \boxed{n \ge 9604}$$

<u>Berry Esseen Theorem</u>: Let $X_1, X_2, \ldots$ are iid

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\,\sigma}.$$

$$\left|F_{Z_n}(z) - F_Z(z)\right| \le \frac{C\rho}{\sigma^3\sqrt{n}} \quad \text{where}$$

$\rho = E\left[|X|^3\right] < \infty$ and $\sigma^2 = E\left[X_1^2\right] < \infty$

and $E[X_1] = 0$.

$$C \ge \frac{1}{\sqrt{2\pi}} + 0.011.$$

Why look at "infinitely-often" sets?

To show $X_n \xrightarrow{a.s} X$, it is enough to show $P(A)=0$

where $A = \{\omega : \lim_{n\to\infty} X_n(\omega) \ne X(\omega)\}$

$= \{\omega : \exists \epsilon > 0, \forall N;\ \exists n > N \text{ st}, |X_n(\omega) - X(\omega)| > \epsilon\}$

Let $A_n = \{\omega : |X_n(\omega) - X(\omega)| > \epsilon\}$.

$$A_\epsilon = \bigcap_{N=1}^{\infty} \bigcup_{n \ge N} A_n = \{\omega : |X_n(\omega) - X(\omega)| > \epsilon \text{ i.o}\}.$$

Also refered as $\limsup_{n\to\infty} A_n$

$\bigcup_{\epsilon>0,\, \epsilon\in\mathbb{Q}} A_\epsilon$. Enough to show $P(A_\epsilon) = 0$ for any $\epsilon > 0$.

$$\Rightarrow \quad P\left(\limsup_{n\to\infty} A_n\right) \leq 0$$

Lemma 2: If $A_1, A_2, \ldots$ are independent events such that

$$\sum_{n=1}^{\infty} P(A_n) = \infty$$

Then $P\left(\limsup_{n\to\infty} A_n\right) = 1$.

Proof:

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \underbrace{\bigcup_{n=N}^{\infty} A_n}_{B_N}$$

$$B_1 \supset B_2 \supset B_3 \ldots$$

$$P\left(\limsup_{n\to\infty} A_n\right) = P\left(\bigcap_{N=1}^{\infty} B_N\right).$$

$$= \lim_{N\to\infty} P(B_N).$$

$$P(B_N) = P\left(\bigcup_{n=N}^{\infty} A_n\right).$$

Let us define $B_{N,M} = \bigcup_{n=N}^{M} A_n$

$$B_{N,N} = A_N$$

$$B_{N,N+1} = A_N \cup A_{N+1}$$

$$\vdots$$

$$B_{N,N} \subseteq B_{N,N+1} \subset B_{N,N+2} \ldots$$

$$P(B_N) = P\left(\bigcup_{M=N}^{\infty} B_{N,M}\right) = \lim_{M\to\infty} P(B_{N,M}).$$

$$P(B_{N,M}) = P\left(\bigcup_{n=N}^{M} A_n\right) = 1 - P\left(\bigcap_{n=N}^{M} A_n^c\right).$$

$$= 1 - \prod_{n=N}^{M}\left(1 - P(A_n)\right)$$

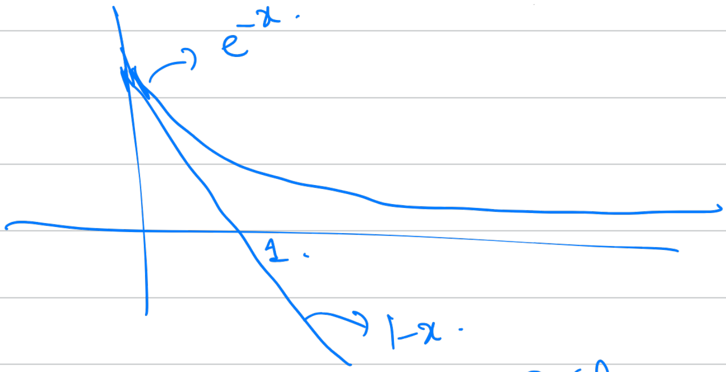
as A_n's are independent.

$$P\left(\limsup_{n\to\infty} A_n\right) = \lim_{N\to\infty} P(B_N).$$

$$= \lim_{N\to\infty}\lim_{M\to\infty} P(B_{N,M}).$$

$e^{-x} \geq 1-x.$

$$= \lim_{N\to\infty}\lim_{M\to\infty} 1 - \prod_{n=N}^{M}\left(1-P(A_n)\right)$$



$$\Rightarrow -e^{-P(A_n)} \leq -\left(1 - P(A_n)\right)$$

$$P\left(\limsup_{n\to\infty} A_n\right) \geq \lim_{N\to\infty}\lim_{M\to\infty} 1 - e^{-\sum_{n=N}^{M} P(A_n)}$$

$$= \lim_{N\to\infty} 1 - e^{-\sum_{n=N}^{\infty} P(A_n)}$$

as $\sum_{n=1}^{\infty} P(A_n) = \infty$

$\Rightarrow \sum_{n=N}^{\infty} P(A_n) = \infty$ as $\sum_{n=1}^{N} P(A_n)$ is finite

$$= \lim_{N\to\infty} 1 - e^{-\infty}$$

$$= 1.$$

$$P\left(\limsup_{n\to\infty} A_n\right) \geq 1.$$

$$\Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

(d) m.s convergence $\not\Rightarrow$ a.s convergence X_i's are independent R.V.s defined as

$$P(X_n = x) = \begin{cases} 1 - \frac{1}{n} & x = 0 \\ \frac{1}{n} & x = 1. \end{cases}$$

$$E[X_n^2] = \frac{1}{n} \cdot 1^2 + \left(1 - \frac{1}{n}\right) \cdot 0^2 = \frac{1}{n}.$$

$\Rightarrow X_n \to 0$ in m.s sense

Let $A_n = \{\omega : |X_n(\omega)| > \epsilon\}$, $P(A_n) = \frac{1}{n}$.

$$P(A_1 \cap A_2) = P(X_1 > \epsilon, X_2 > \epsilon)$$
$$= P(X_1 > \epsilon)\, P(X_2 > \epsilon) \quad \text{as } X_1 \text{ \& } X_2 \text{ are independent}$$
$$= P(A_1)\, P(A_2)$$

$A_1, A_2, \ldots$ are independent events.

$$\sum_n P(A_n) = \sum_n \frac{1}{n} = \infty$$

Borel Cantelli lemma 2 $\Rightarrow P(\limsup_{n\to\infty} A_n) = 1 \Rightarrow X_n \not\to 0$ in a.s sense.

(e) a.s convergence $\not\Rightarrow$ m.s convergence.

$X_n \to 0$?

$$P(X_n = x) = \begin{cases} 1 - \frac{1}{n^2} & x = 0 \\ \frac{1}{n^2} & x = n^2. \end{cases}$$

$$E[X_n^2] = \left(1 - \frac{1}{n^2}\right) 0^2 + (n^2)^2 \frac{1}{n^2}$$
$$= n^2.$$

$\Rightarrow X_n \not\to 0$ in m.s sense.

$A_n = \{\omega : |X_n(\omega)| > \epsilon\}$, $P(A_n) = \frac{1}{n^2}$.

$$\Rightarrow \sum_n P(A_n) = \sum \frac{1}{n^2} < \infty$$

Borel Cantelli Lemma 1. $\Rightarrow P\left(\limsup_{n\to\infty} A_n\right) = 0$

$\Rightarrow X_n \to X$ in a.s sense.

⑧ Hierarchies of convergence notions

$$\begin{tikzcd}
\text{almost sure convergence} \arrow[r, Rightarrow, "\text{Lemma 3}\ \checkmark"] & \text{convergence in probability} \arrow[d, Rightarrow, "\text{Lemma 4}"] \\
\text{mean square convergence} \arrow[ur, Rightarrow, "\checkmark"'] & \text{convergence in distribution}
\end{tikzcd}$$

Lemma 3: If $X_n \to X$ in almost sure sense then $X_n \to X$ in probability.

Proof: Let $A_n = \{ \omega : |X_n(\omega) - X(\omega)| > \epsilon \}$.

To show $X_n \to X$ in probability we need to prove

$$\lim_{n\to\infty} P\left(\{|X_n - X| > \epsilon\}\right) = 0 .$$

$$\Updownarrow$$

To show $\displaystyle\lim_{n\to\infty} P(A_n) = 0$

We have that $P\left(\limsup_{n\to\infty} A_n\right) = 0$

as $X_n \to X$ in almost sure sense.

$$\limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \underbrace{\bigcup_{n=N}^{\infty} A_n}_{B_N}.$$

$$B_1 \supset B_2 \supset \cdots$$

$$A_N \subseteq B_N .$$

$$P(A_N) \le P(B_N).$$

$$\Rightarrow \lim_{N\to\infty} P(A_N) \le \lim_{N\to\infty} P(B_N)$$

$$B_1 \supset B_2 \supset \cdots\cdots$$

Continuity property of probability.

$$= P\left(\bigcap_{N=1}^{\infty} B_N\right).$$

$$= P\left(\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n\right)$$

$$= P\left(\limsup_{n\to\infty} A_n\right).$$

$$= 0$$

$$\Rightarrow \lim_{N\to\infty} P(A_N) \leq 0.$$

$$\Rightarrow \lim_{N\to\infty} P(A_N) = 0.$$

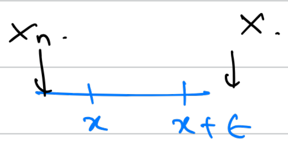
$$\Rightarrow X_n \to X \text{ in probability.}$$

Lemma 4: If $X_n \to X$ in probability then $X_n \to X$ in distribution.

To show $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$.

given that

$$\lim_{n\to\infty} P(|X_n - X| > \epsilon) = 0.$$



$$F_{X_n}(x) = P(X_n \leq x).$$

$$= P(\{X_n \leq x\} \cap \{X > x+\epsilon\}).$$

$$+ \underbrace{P(\{X_n \leq x\} \cap \{X \leq x+\epsilon\})}_{\leq P(\{X \leq x+\epsilon\}) = F_X(x+\epsilon)}.$$

$$\boxed{\begin{aligned} P(A) &= P(A \cap \Omega) \\ &= P(A\cap B) \\ &\quad + P(A \cap B^c) \end{aligned}}$$

If $w \in \{X_n \leq x\} \cap \{X > x+\epsilon\}$

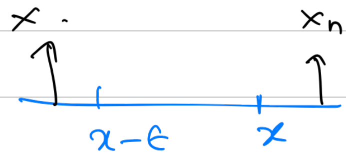
$$\Rightarrow w \in \{ X - X_n > \epsilon\}$$

$$F_{X_n}(x) \le P(X - X_n > \epsilon) + F_X(x+\epsilon).$$

$$\le P(|X - X_n| > \epsilon) + F_X(x+\epsilon).$$

$$\lim_{n\to\infty} F_{X_n}(x) \le \lim_{n\to\infty} P(|X - X_n| > \epsilon) + F_X(x+\epsilon)$$

$$= F_X(x+\epsilon).$$



$$1 - F_{X_n}(x) = P(X_n > x).$$

$$= \underbrace{P(X_n > x,\ X > x-\epsilon)}_{\le P(X \ge x-\epsilon)} + \underbrace{P(X_n > x,\ X \le x-\epsilon)}_{\le P(X_n - X \ge \epsilon)}.$$

$$\le P(X \ge x-\epsilon) + P(X_n - X \ge \epsilon).$$

$$\le 1 - F_X(x-\epsilon) + P(|X_n - X| \ge \epsilon).$$

$$F_X(x-\epsilon) - P(|X_n - X| \ge \epsilon) \le F_{X_n}(x).$$

$$F_X(x-\epsilon) - \lim_{n\to\infty} P(|X_n - X| \ge \epsilon) \le \lim_{n\to\infty} F_{X_n}(x)$$

$$\Rightarrow \quad F_X(x-\epsilon) \le \lim_{n\to\infty} F_{X_n}(x)$$

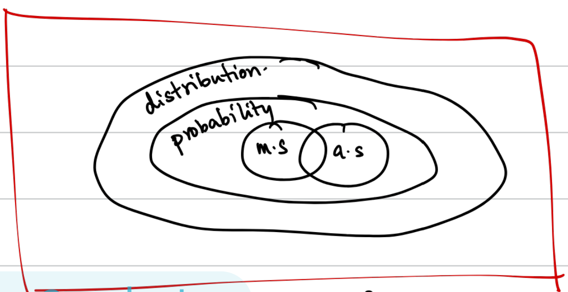
$$\Rightarrow \quad F_X(x-\epsilon) \le \lim_{n\to\infty} F_{X_n}(x) \le F_X(x+\epsilon). \quad \forall \epsilon > 0.$$

$$\Rightarrow \quad \lim_{n\to\infty} F_{X_n}(x) = \lim_{\epsilon\to 0} F_X(x+\epsilon)$$

$$= F_X(x)$$

for $x \in \mathcal{C}_{F_X}$

continuos points of F_X.



Convergence of R.Vs
Hierarchies.

9. **Strong Law of Large numbers:** Let $X_1, X_2, \ldots$ are iid R.Vs

$$S_n = \sum_{i=1}^{n} X_i$$

$$\frac{S_n}{n} \xrightarrow{a.s} \mu \quad \text{given} \quad E[X^4] < \infty.$$

For simplicity we'll show this proof for X_i's with mean 0.

It is equivalent to showing

$$\frac{S_n - \mu n}{n} \xrightarrow{a.s} 0.$$

$$\frac{\sum_{i=1}^{n}(X_i - \mu)}{n}$$

We'll show $\frac{S_n}{n} \xrightarrow{a.s} 0$ and assume X_i's to be of 0 mean.

$$A_n = \left\{ w : \left|\frac{S_n}{n}\right| > \epsilon \right\}.$$

$$\sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} P\left(\left|\frac{S_n}{n}\right| > \epsilon \right).$$

$$P\left(\left|\frac{S_n}{n}\right| > \epsilon \right) = P\left(\left(\frac{S_n}{n}\right)^4 > \epsilon^4 \right).$$

$$\leq \frac{E[S_n^4]}{n^4 \epsilon^4}.$$