EE1080/AI1110/EE2120 Probability, Quiz 4

27th March, 2025

Max. Marks: 21. **Time:** 1 hour.

Instructions

- Please **write your roll number**, **serial number** (used for attendance) and **course id** prominently in the first page of the answer sheet.
- No laptops, mobile devices etc. allowed.
- please write supporting arguments for any of the statements you make. any result proved in the lectures can be used by stating it clearly, no need to prove them in your answer.

Indicate the range values for all the densities.

1. *Joint density: (2+2)* Let X and Y be uniform $(0,1)$ random variables.
   (a) Find the joint density of $U = X$, $V = X+Y$
   (b) Use the result obtained above to compute the density function of V.

2. *Sum of Normal RVs and Conditional Densities: (4 marks = 2+2)* Let $Z_1, Z_2, \cdots, Z_n$ be independent standard normal random variables. Let:
$$S_j = \sum_{i=1}^{j} Z_i$$
   for $j = 1, 2, \cdots, n$.
   (a) Find the conditional density of S_n given that $S_k = y$ for $k = 1, 2, \cdots, n-1$ i.e, find $f_{S_n|S_k}(s|y)$.
   (b) Use the above part to find $f_{S_k|S_n}(s|x)$ for $k = 1, \cdots, n$.

3. *Ordered Statistics (5 marks = 2+3)* Along a road 1 mile long are 3 people "distributed at random" (assume identical, independent and uniform).
   (a) Let $X_{(1)}, X_{(2)}, X_{(3)}$ be placements of the three people in order. Find the joint density of $X_{(1)}, X_{(2)}$
   (b) Find the probability that no two people are less than a distance d apart. Assume $d \leq 1/2$.

4. *Transform (3 marks = 2+1)* Find the PMF of random variable X whose transform is given by
$$\begin{aligned} M_X(s) &= \frac{1}{2}e^s + \frac{1}{2^2}e^{2s} + \frac{1}{2^3}e^{3s} + \cdots \\ &= \sum_{i=1}^{\infty} \frac{1}{2^i}e^{is}. \end{aligned}$$
   Can you identify what random variable X is ?

5. *Transforms (5marks = 3+2)* A defective coin minting machine produces coins whose probability of heads is a random variable P that is uniform in $[0,1]$. A coin produced by this machine is tossed n times. Let X be the number of heads seen in these n coin tosses.
   (a) Find $M_X(s)$ *(Hint: Use total law of expectation $E[f(X)] = \int_y E[f(X)|P=p] f_Y(y)dy$)*
   (b) What is the PMF of X. *(Hint: Use MGF)*

- Pdf of standard normal random variable Z is $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.
- Joint density of $X_{(1)}, X_{(2)}, \cdots, X_{(n)}$ ordered statistics of $X_1, X_2, \cdots, X_n$ is given by:
$$\begin{aligned} &f_{X_{(1)},X_{(2)},\cdots,X_{(n)}}(x_1, x_2, \cdots, x_n) \\ &\quad = n! f_{X_1,X_2,\cdots,X_n}(x_1, \cdots, x_n) \end{aligned}$$
  for $x_1 < x_2 < \cdots x_n$ and 0 otherwise.
- $\int_a^b x^i = x^{i+1}/(i+1)|_a^b$
- $\frac{x^{n+1}-1}{x-1} = x^n + x^{n-1} + \cdots + 1$