Lec 10 : 27th Jan

X : Geometric random variable (p) : # of Tosses until you see a head.

Prob of seeing a head = p.

$$P(X = k) = (1-p)^{k-1} p.$$

Memoryless property : RV X is said to have memoryless property if

$$P(X > m+t \mid X > t) = P(X > m). \quad \forall m, t.$$

Upon seeing tails for first t instances, Expected # tosses to see a head is $t + E[X]$

1				t	t+1				m+t
0	-	-	.	0	0	0	-	- -	0
T	-	-	.	T					
					1	2	- -	.	m

$$P(X > m) = \sum_{x=m+1}^{\infty} P_X(x) = \sum_{x=m+1}^{\infty} (1-p)^{x-1} p$$

$$= (1-p)^m$$

$$= (1-p)^m p + (1-p)^{m+1} p \dots = p(1-p)^m \left[1 + (1-p) + \dots \right]$$

$$P(X > m+t \mid X > t) = \frac{P(X > m+t, X > t)}{P(X > t)} = (1-p)^m.$$

$A \cap B$ where

$A = \{X > m+t\}$

$B = \{X > t\}$

$A \subseteq B$

$$= \frac{P(X > m+t)}{P(X > t)}$$

$$= \frac{(1-p)^{m+t}}{(1-p)^t} = (1-p)^m = P(X > m).$$

Check that $P(X = m+t \mid X > t) = P(X = m)$.

Example: Using total law of expection to find mean of Geometric R.V.

X is Geometric(p) R.V.

$$E[X] = P(A)\, E[X|A] + P(A^c)\, E[X|A^c]$$

$A = \{X > 1\}$

$A^c = \{X \le 1\} = \{X = 1\}$

$$P(A) = (1-p)$$
$$P(A^c) = p.$$

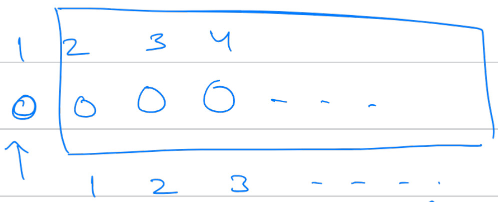
$$E[X|A^c] = E[X \mid X \le 1] = 1.$$

$P_{X|A^c}(x) = 1$ if $x=1$; 0 o.w

$$E[X|A] = E[X \mid X > 1] = 1 + E[X].$$

$E[X|A^c] = 1.$

Confirming this using conditional probabilities



$$E[X \mid X > 1] = \sum_{x=1}^{\infty} x\, P_{X|A}(x) = \sum_{x=2}^{\infty} x \cdot P_X(x-1)$$

Conditional probability

$$P_{X|A}(x) = \frac{P(\{X=x\} \cap A)}{P(A)}$$

$$= \begin{cases} 0 & x = 1 \\ \dfrac{P(\{X=2\})}{(1-p)} = P(\{X=1\}) & x = 2 \\ P_X(k-1). & x = k. \end{cases}$$

Comes from Memoryless property

$$P(X = x \mid X > 1) = P(X = x-1).$$

$$E[X \mid X > 1] = \sum_{x=2}^{\infty} x\, P_X(x-1)$$

$k = x-1$

$$= \sum_{k=1}^{\infty} (k+1)\, P_X(k)$$

$$= \underbrace{\sum_{k=1}^{\infty} k\, P_X(k)}_{E[X]} + \underbrace{\sum_{k=1}^{\infty} P_X(k)}_{1}$$

$$= E[X] + 1$$

$$E[X] = P(A)\, E[X|\underbrace{A}_{X>1}] + P(A^c)\, E[X|\underbrace{A^c}_{X \le 1}]$$

$$E[X] = (1-p)\left[E[X] + 1\right] + p \cdot 1.$$

$$\Rightarrow p\,E[X] = (1-p) + p = 1 \Rightarrow E[X] = 1/p.$$

$$E[X] = \sum_{i=1}^{n} P(A_i)\, E[X \mid A_i]$$

$$E[g(X)] = \sum_{i=1}^{n} P(A_i)\, E[g(X) \mid A_i].$$

$$E[X^2] = P(A)\, E[X^2 \mid A] + P(A^c)\, E[X^2 \mid A^c]$$

$A : X > 1$

$$E[(X+1)^2] \qquad 1.$$

$$E[X^2 \mid A] = \sum_{x=1}^{\infty} x^2\, \underline{\underline{P_{X|A}(x)}}$$

$$= \sum_{x=1}^{\infty} x^2\, P_X(x-1)$$

$E[X^2] + 2E[X] + 1.$

$$= \sum_{k=1}^{\infty} (k+1)^2\, P_X(k) = E[(X+1)^2]$$

$$E[X^2] = \underbrace{(1-p)}_{P(A)}\, E[X^2 \mid A] + p\, \underbrace{E[X^2 \mid A^c]}_{=1}$$

$$= (1-p)\left[E[X^2] + 2E[X] + 1 \right] + p \cdot 1$$

$$E[X^2]\, p = 2(1-p)\, E[X] + (1-p) + p.$$

$$= 2(1-p)\frac{1}{p} + 1.$$

$$E[X^2] = \frac{2(1-p)}{p^2} + \frac{1}{p}.$$

$$\mathrm{Var}(X) = E[X^2] - (E[X])^2$$

$$= \frac{2}{p^2} - \frac{2}{p} + \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{1}{p^2} - \frac{1}{p} = \frac{(1-p)}{p^2}.$$

Conditional Variance and law of total Variation

100 values	100 values	300 values	400 values
$x_1 \ldots x_{100}$	$x_{101} \ldots x_{200}$	$x_{201} \ldots x_{500}$	$x_{501} \ldots x_{900}$

$Y =$	1	2	3	4
$P_Y(y)$	$\frac{100}{900}$	$\frac{100}{900}$	$\frac{300}{900}$	$\frac{400}{900}$

$$E[X] = E\Big[\underbrace{E[X|Y]}_{\text{local mean value}}\Big]$$

each node y is providing you $E[X|Y=y]$ and it is also providing variance given by

Conditional Variance

$$\boxed{Var(X|Y=y) = E\left[\left(X - E[X|Y=y]\right)^2 \Big| Y=y\right]}$$

How do we find $Var(X)$?

$$E[X] = \sum_y P_Y(y)\, E[X|Y=y]$$

Total Law of Variance

$$\boxed{Var(X) = E[\underbrace{Var(X|Y)}] + Var(E[X|Y])}$$

Proof:

$$E[Var(X|Y)] = \sum_y P_Y(y)\, Var(X|Y=y)$$

$$= \underbrace{\sum_y P_Y(y) \left[E[X^2|Y=y]\right.}\left. - (\underbrace{E[X|Y=y]}_{f(y)})^2\right]$$

$$E[g(x)] = \sum P(A_i)\, E[g(x)|A_i]$$

$$E[g(x)] = \sum P_Y(y)\, E[g(x)|Y=y]$$

$$= E[E[g(x)|Y]]$$

$$= \underbrace{E[E[X^2|Y]]}_{=E[X^2]} - \underbrace{\sum_y P_Y(y)\,(f(y))^2}_{E[(f(Y))^2]}$$

$$= E[X^2] - E[(f(Y))^2]$$

$$Var(E[X|Y]) = Var(f(Y))$$
$$= E[(f(Y) - E[f(Y)])^2].$$
$$= E[(f(Y))^2] - (E[f(Y)])^2$$

$$E[f(Y)] = E[E[X|Y]] = E[X].$$

$$Var(E[X|Y]) = \underline{E[(f(Y))^2]} - (E[X])^2$$

$$\Rightarrow \quad Var(E[X|Y]) + E[Var(X|Y)] = Var(X)$$

Independence of random variables.

Random variable X is said to be independant of event A if

$$P_{X|A}(x) = P_X(x) \quad \text{for all } x \in \mathcal{X}.$$

or equivalently.

$$\frac{P(\{X=x\} \cap A)}{P(A)} = P_X(x)$$

$$P(\{X=x\} \cap A) = P_X(x)\, P(A).$$

Independence of two R.Vs.

X and Y are said to be independant if for every x, y.

$$P(X=x, Y=y) = P_X(x)\, P_Y(y).$$

Excercise: X, Y are R.Vs taking values in $\{0, 1\}$ and $P_{X,Y}(1,1) = P_X(1)\, P_Y(1)$.

are X, Y independant?

Independence of more than 2 R.V's

$X_1, X_2, \ldots X_n$ are said to be independent if

$$P(X_1 = x_1, X_2 = x_2, \ldots X_n = x_n) = \prod_{i=1}^{n} P_{X_i}(x_i)$$

for all $(x_1, \ldots x_n) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \cdots \mathcal{X}_n$.

If $X_1, \ldots X_n$ are independent then any sub collection of them are independent.

For example: Let X_1, X_2, X_3 be independent.

$\Rightarrow$ X_1, X_2 are independent.

$$P_{X_1,X_2,X_3}(x_1, x_2, x_3) = \prod_{i=1}^{3} P_{X_i}(x_i)$$

$$P_{X_1,X_2}(x_1, x_2) = \sum_{x_3 \in \mathcal{X}_3} P_{X_1,X_2,X_3}(x_1, x_2, x_3)$$

$$= \sum_{x_3 \in \mathcal{X}_3} P_{X_1}(x_1) P_{X_2}(x_2) P_{X_3}(x_3)$$

$$= P_{X_1}(x_1) P_{X_2}(x_2) \underbrace{\boxed{\sum_{x_3 \in \mathcal{X}_3} P_{X_3}(x_3)}}_{=1}$$

If X and Y are independent then

$$E[XY] = E[X]\,E[Y].$$

$$E[XY] = \sum_{x \in \mathcal{X}, y \in \mathcal{Y}} x \cdot y \; P_{X,Y}(x, y).$$

$$= \sum_{x} \sum_{y} x \cdot y \; P_X(x) P_Y(y)$$

$$= \left[\sum_{x} x P_X(x)\right] \left[\sum_{y} y P_Y(y)\right]$$

$= E[X]\,E[Y]$

If $E[XY] = E[X]\,E[Y]$ then the random variables X, Y are said to be "uncorrelated".

$\boxed{X, Y \text{ uncorrelated does not imply } X, Y \text{ are independent}}$

Example:

$X \in \mathcal{X} = \{-1, 0, 1\}$ $\qquad P_X(x) = \frac{1}{3}\ \forall\, x \in \mathcal{X}$

$$Y(\omega) = \begin{cases} 1 & \text{if } X(\omega) = 0. \\ 0 & \text{otherwise.} \end{cases}$$

$E[XY] = 0$, $\qquad E[X] = \frac{1}{3}(-1) + 0 \cdot \left(\frac{1}{3}\right) + 1 \cdot \left(\frac{1}{3}\right)$

X, Y are uncorrelated. $\qquad = 0.$

$P_{X|Y}(0 \mid 1) = 1$

$P_X(0) = \frac{1}{3}$. $\qquad \therefore$ not independent.

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