First-Order Circuits

- A first-order circuit contains one energy storage element (capacitor or inductor) and can be described by a first-order differential equation. To begin with, we will introduce a method for solving a first-order differential equation, and then we will apply this method to analyze first-order circuits.

- **Solution of a First-Order Differential Equation:** The general form of a first-order linear differential equation is

$$\frac{dx(t)}{dt} + p(t)x(t) = q(t) \tag{1}$$

where $p(t)$ and $q(t)$ are prescribed functions of time. The objective is to determine the unknown function $x(t)$. If $p(t) \equiv 0$, the equation simplifies to

$$\frac{dx(t)}{dt} = q(t) \tag{2}$$

which can be solved via direct integration. The constant of integration is then determined from the initial condition $x(t_0) = x_0$.

In the more general case where $p(t) \not\equiv 0$, a standard method of solution is the method of integrating factors. The central idea of this method is to multiply both sides of the differential equation by an integrating factor $u(t)$, which is a function of time to be determined. This results in the equation

$$u(t)\frac{dx(t)}{dt} + u(t)p(t)x(t) = u(t)q(t) \tag{3}$$

The goal is to choose $u(t)$ such that the left-hand side of equation (3) can be written as the derivative of the product of two functions, $u(t)x(t)$. For this to hold, $u(t)$ must satisfy

$$\frac{du(t)}{dt} = u(t)p(t) \tag{4}$$

This is a separable differential equation, and separating the variables gives

$$u(t) = e^{\int p(t)\,dt} \tag{5}$$

It is important to note that no constant of integration is necessary here, as we are interested in any function $u(t)$ that satisfies the equation. Multiplying the original differential equation by $u(t)$ and simplifying yields

$$\frac{d}{dt}[u(t)x(t)] = u(t)q(t) \tag{6}$$

Integration of both sides with respect to t results in

$$u(t)x(t) = \int u(t)q(t)\,dt + C \tag{7}$$

where C is the constant of integration. Solving for $x(t)$, we obtain

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + C\right) \tag{8}$$

Thus, the solution for $x(t)$ in terms of the integrating factor $u(t)$ is

$$x(t) = \frac{1}{u(t)}\left(\int u(t)q(t)\,dt + C\right) \tag{9}$$

where $u(t) = e^{\int p(t)\,dt}$, and the constant C is determined from the initial condition $x(t_0) = x_0$.

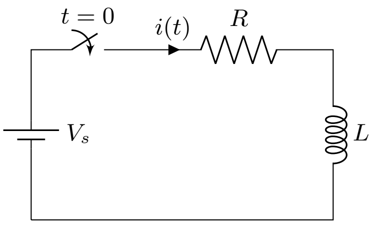


Figure 1: Series RL circuit with a DC voltage source.

- Example 1 - Series RL circuit with a DC voltage source: Compute the current $i(t)$ for the circuit shown in Fig. 1.

Applying KVL around the loop gives

$$V_s - Ri(t) - L\frac{di(t)}{dt} = 0 \tag{10}$$

Note that the equation is valid for $t \geq 0$ since the switch closes at $t = 0$. The general equation that describes the circuit for all t is

$$V_s u(t) - Ri(t) - L\frac{di(t)}{dt} = 0 \tag{11}$$

where $u(t)$ is the unit step function. Rearranging the equation gives

$$\frac{di(t)}{dt} + \frac{R}{L}i(t) = \frac{V_s}{L}u(t) \tag{12}$$

This is a first-order linear differential equation with $p(t) = \frac{R}{L}$ and $q(t) = \frac{V_s}{L}u(t)$. The integrating factor is

$$u(t) = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t} \tag{13}$$

Multiplying both sides of the differential equation by the integrating factor gives

$$\frac{d}{dt}\left(e^{\frac{R}{L}t}i(t)\right) = \frac{V_s}{L}e^{\frac{R}{L}t}u(t) \tag{14}$$

Integrating both sides with respect to t results in

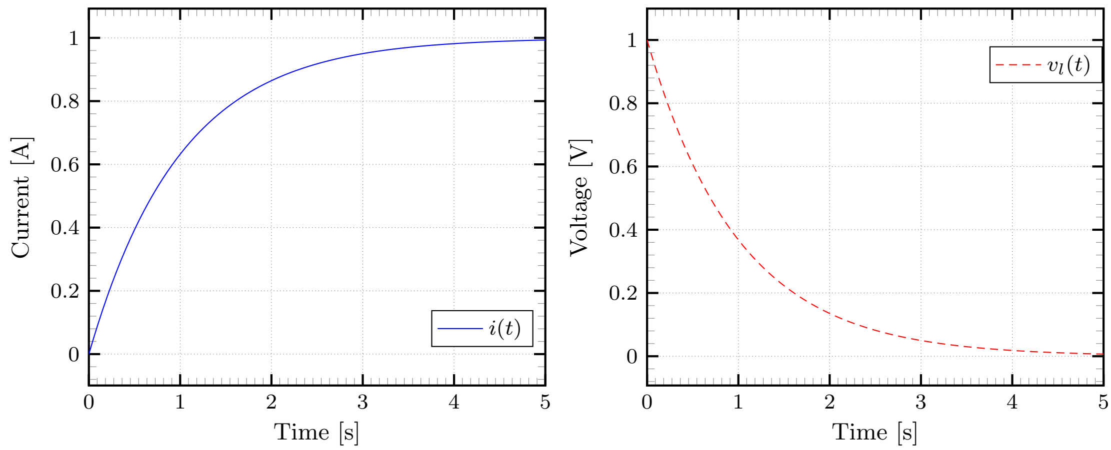
$$e^{\frac{R}{L}t}i(t) = \frac{V_s}{L}\frac{L}{R}e^{\frac{R}{L}t} + C = \frac{V_s}{R}e^{\frac{R}{L}t} + C \tag{15}$$

Solving for $i(t)$ gives

$$i(t) = \frac{V_s}{R} + Ce^{-\frac{R}{L}t} \tag{16}$$

To determine the constant C, we need an initial condition. Assuming the inductor current is zero before the switch closes, we have $i(0) = 0$. Substituting $t = 0$ into the equation gives

$$0 = \frac{V_s}{R} + C \implies C = -\frac{V_s}{R} \tag{17}$$



(a) Current (b) Inductor voltage

Figure 2: Typical variation of $i(t)$ and $v_l(t)$ for a series RL circuit with a step input.

Thus, the current $i(t)$ for $t \geq 0$ is

$$i(t) = \frac{V_s}{R}\left(1 - e^{-\frac{R}{L}t}\right) \tag{18}$$

The current $i(t)$ starts at 0 A when the switch is closed and asymptotically approaches $\frac{V_s}{R}$ as $t \to \infty$. The voltage across the inductor $v_l(t)$ is given by

$$v_l(t) = L\frac{di(t)}{dt} = V_s e^{-\frac{R}{L}t} \tag{19}$$

A typical variation of $i(t)$ and $v_l(t)$ for a series RL circuit with a step input is shown in Fig. 2.

- Example 2 - Parallel RC circuit with a DC current source: Compute the voltage $v(t)$ for the circuit shown in Fig. 3. Applying KCL at node a gives

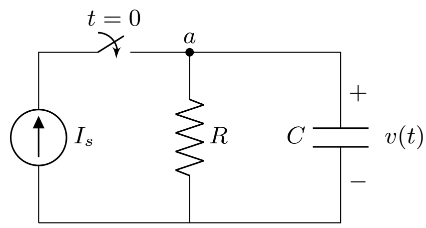


Figure 3: Parallel RC circuit with a DC current source.

$$I_s = \frac{v(t)}{R} + C\frac{dv(t)}{dt} \tag{20}$$

Note that the equation is valid for $t \geq 0$ since the switch closes at $t = 0$. The general equation that describes the circuit for all t is

$$I_s u(t) = \frac{v(t)}{R} + C\frac{dv(t)}{dt} \tag{21}$$

Rearranging the equation gives

$$\frac{dv(t)}{dt} + \frac{1}{RC}v(t) = \frac{I_s}{C}u(t) \tag{22}$$

The integrating factor is

$$u(t) = e^{\int \frac{1}{RC}dt} = e^{\frac{1}{RC}t} \tag{23}$$

Multiplying both sides of the differential equation by the integrating factor gives

$$\frac{d}{dt}\left(e^{\frac{1}{RC}t}v(t)\right) = \frac{I_s}{C}e^{\frac{1}{RC}t} \tag{24}$$

Integrating both sides with respect to t results in

$$e^{\frac{1}{RC}t}v(t) = I_sRe^{\frac{1}{RC}t} + C_1 \tag{25}$$

Solving for $v(t)$ gives

$$v(t) = I_sR + C_1e^{-\frac{1}{RC}t} \tag{26}$$

To determine the constant C_1, we need an initial condition. Assuming the capacitor voltage is zero before the switch closes, we have $v(0) = 0$. Substituting $t = 0$ into the equation gives

$$0 = I_sR + C_1 \implies C_1 = -I_sR \tag{27}$$

Thus, the voltage $v(t)$ for $t \geq 0$ is

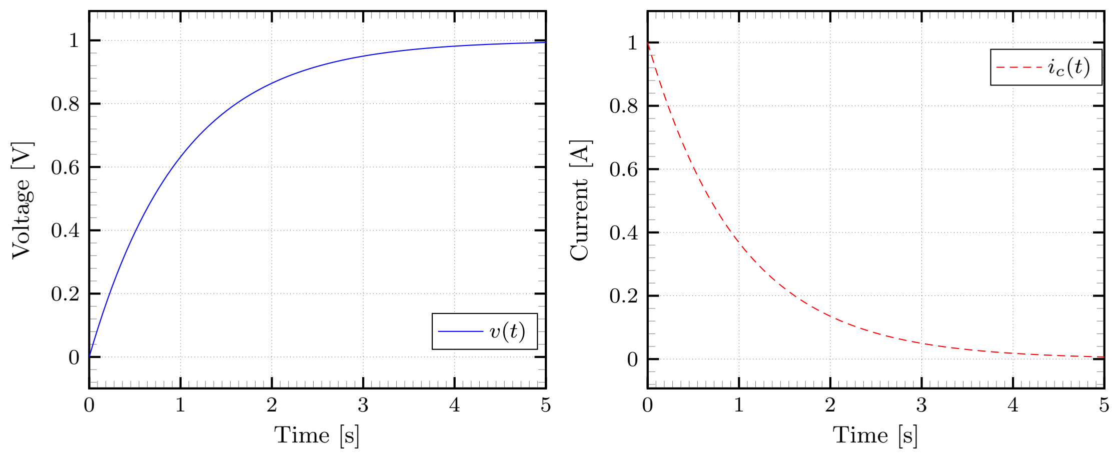
$$v(t) = I_sR\left(1 - e^{-\frac{1}{RC}t}\right) \tag{28}$$

The voltage $v(t)$ starts at 0 V when the switch is closed and asymptotically approaches I_sR as $t \to \infty$. The current through the capacitor $i_c(t)$ is given by

$$i_c(t) = C\frac{dv(t)}{dt} = I_se^{-\frac{1}{RC}t} \tag{29}$$

A typical variation of $v(t)$ and $i_c(t)$ for a parallel RC circuit with a step input is shown in Fig. 4.

- **Steady-State and Transient Response:** The response of a first-order circuit can be expressed as the sum of two components: the steady-state response and



(a) Voltage (b) Capacitor current

Figure 4: Typical variation of $v(t)$ and $i_c(t)$ for a parallel RC circuit with a step input.

the transient response. The steady-state response is the behavior of the circuit as $t \to \infty$, while the transient response describes how the circuit transitions from its initial state to its steady-state condition.

When analyzing the time response of a circuit, we often focus on the response to a step input, as it reveals important insights into the circuit's dynamic behavior. A step input forces the circuit to transition from one steady-state condition to another, and the transient response captures this change. Additionally, it is common practice to examine the responses of state variables, such as inductor currents and capacitor voltages, when quantifying the time-response of a circuit.

The steady-state and transient responses for the two examples considered thus far are as follows:

- Example 1 - Series RL Circuit: The steady-state response is $i(\infty) = \frac{V_s}{R}$, and the transient response is $i(t) - i(\infty) = -\frac{V_s}{R}e^{-\frac{R}{L}t}$.
- Example 2 - Parallel RC Circuit: The steady-state response is $v(\infty) = I_s R$, and the transient response is $v(t) - v(\infty) = -I_s R e^{-\frac{1}{RC}t}$.

The steady-state response of a circuit (to a step-input) is characterized by a constant value that the response approaches as time progresses. Any circuit that comprises of a resistor will eventually reach a steady-state condition when subjected to a step input. This is because resistors dissipate energy, preventing the indefinite accumulation of energy in the circuit.

In contrast, the transient response captures the dynamic behavior of the circuit as it transitions from its initial state to its steady-state condition. It is interesting to note that the transient response of both the examples considered above decays exponentially over time. One way to characterize the transient response is through the concept of a time constant, which provides insight into how quickly the transient response diminishes. For the series RL circuit, the time constant is given by

$$\tau = \frac{L}{R} \tag{30}$$

and for the parallel RC circuit, the time constant is

$$\tau = RC \tag{31}$$

However, when dealing with second-order circuits or circuits of higher order, the transient response may not always exhibit a simple exponential decay. In such cases, the transient response can involve oscillatory behavior or more complex dynamics, depending on the circuit's configuration and components. The metric used to quantify the transient response in these scenarios are

- Rise Time: The time taken for the response to rise from 10% to 90% [1] of its final value.

[1] sometimes also defined as time from 5% to 95%

- Settling Time: The time taken for the response to remain within a certain percentage (commonly 2% or 5%) of its final value.
- Peak overshoot: The maximum value of the response exceeding its final value, expressed as a percentage of the final value.
- Peak time: The time taken to reach the peak overshoot.

These metrics provide a more comprehensive understanding of the transient response in circuits where the behavior is more intricate than a simple exponential decay.

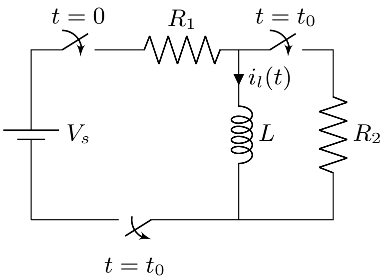


Figure 5: RL circuit with a DC voltage source and two switches.

- Example 3: Compute the time-response of the circuit shown in Fig. 5. Quantify the steady-state and transient responses. Assume that the switches are open for $t<0$ and that the inductor current is zero at $t=0$. Further, assume that t_0 is long enough for the circuit to reach steady-state.

For $t<0$, all switches are open, and the inductor current is zero. At $t=0$, the first switch closes, forming a series RL circuit with a DC source. The current $i_l(t)$ for $0 \le t < t_0$ is

$$i_l(t) = \frac{V_s}{R_1}\left(1 - e^{-\frac{R_1}{L}t}\right) \tag{32}$$

The steady-state current as $t \to t_0^-$ is

$$i_l(t_0^-) = \frac{V_s}{R_1} \tag{33}$$

The transient response is

$$i_l(t) - i_l(t_0^-) = -\frac{V_s}{R_1}e^{-\frac{R_1}{L}t} \tag{34}$$

At $t=t_0$, the first switch opens, and the second closes, changing the circuit to an RL series with R_2. The initial condition at t_0 is

$$i_l(t_0) = \frac{V_s}{R_1} \tag{35}$$

Applying KVL for $t \ge t_0$, we get

$$R_2 i_l(t) + L\frac{di_l(t)}{dt} = 0 \tag{36}$$

Rearranging gives

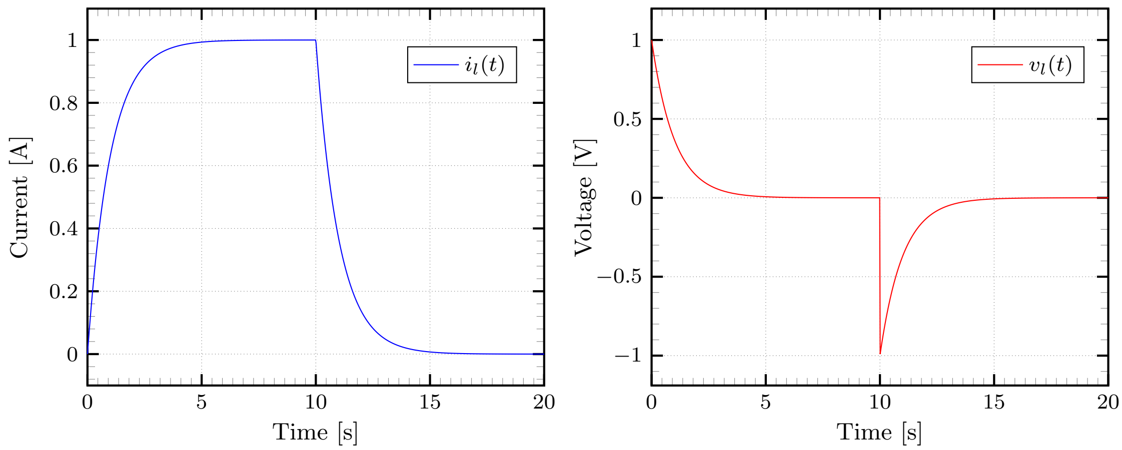
$$\frac{di_l(t)}{dt} + \frac{R_2}{L}i_l(t) = 0 \tag{37}$$

The integrating factor is

$$u(t) = e^{\frac{R_2}{L}t} \tag{38}$$

Multiplying by the integrating factor yields

$$\frac{d}{dt}\left(e^{\frac{R_2}{L}t}i_l(t)\right) = 0 \tag{39}$$



(a) Current (b) Inductor voltage

Figure 6: Typical variation of $i_l(t)$ and $v_l(t)$ for the circuit shown in Fig. 5.

Integrating both sides,

$$e^{\frac{R_2}{L}t} i_l(t) = C_2 \tag{40}$$

Solving for $i_l(t)$,

$$i_l(t) = C_2 e^{-\frac{R_2}{L}t} \tag{41}$$

Using the initial condition at $t = t_0$,

$$C_2 = \frac{V_s}{R_1} e^{\frac{R_2}{L}t_0} \tag{42}$$

Thus, the current for $t \geq t_0$ is

$$i_l(t) = \frac{V_s}{R_1} e^{-\frac{R_2}{L}(t-t_0)} \tag{43}$$

The steady-state response as $t \to \infty$ is

$$i_l(\infty) = 0 \tag{44}$$

The transient response for $t \geq t_0$ is

$$i_l(t) - i_l(\infty) = \frac{V_s}{R_1} e^{-\frac{R_2}{L}(t-t_0)} \tag{45}$$

A typical variation of $i_l(t)$ and $v_l(t)$ is shown in Fig. 6.

- **Zero-input and zero-state response**: Note that the response in the circuit for $t > t_0$ is due to the initial current through the inductor at $t = t_0$. Responses of this nature are often referred to as zero-input responses since they are driven by the initial energy stored in the reactive elements of the circuit, rather than an external source. On the other hand, the response for $0 \leq t < t_0$ is known as the zero-state response because it is driven by the external source (in this case, the voltage source V_s) while assuming that the initial energy stored in the reactive elements is zero.