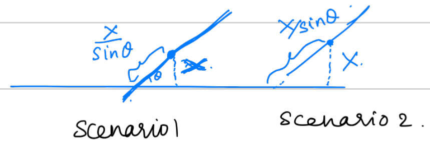
Disclaimer: These solutions may have errors / typos. If you find any kindly post in the public chat so that others know as well.

① Buffon's needle problem: $X \sim \text{Uniform}\left(0, \frac{D}{2}\right)$

$\Theta \sim \text{uniform}\left(0, \frac{\pi}{2}\right)$



The needle touches a line

iff $\dfrac{X}{\sin\theta} \le \dfrac{l}{2}$.

$$P\left(X \le \frac{l}{2}\sin(\Theta)\right) = \int_{\theta=0}^{\pi/2}\int_{x=0}^{\frac{l}{2}\sin\theta} \frac{4}{d\pi}\, dx\, d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{4}{d\pi}\left(\frac{l}{2}\sin\theta\right) d\theta$$

$$= -\frac{2}{d\pi}\cos\theta \Big|_0^{\pi/2} = \frac{2}{d\pi}$$

② $P(X > Y+10) + P(Y > X+10)$ $\qquad$ $X, Y \sim \text{uniform}[0, 60]$, X is independent of Y

$$= 2P(X > Y+10) \quad \text{(due to symmetry)}$$

$$= 2\int_0^{50}\int_{y+10}^{60} \frac{1}{60} \times \frac{1}{60}\, dx\, dy$$

$$= 2\int_0^{50} \frac{1}{60} \times \frac{1}{60}(50-y)\, dy$$

$$= \left(\frac{50}{60}\right)^2 = \frac{25}{36}.$$

③ $$P(V > 1) = \int_1^2 \frac{1}{2}\, dv = \frac{1}{2}.$$

$$f_{V|\{V>1\}}(v) = \begin{cases} \dfrac{f_V(v)}{P(V>1)} & v>1 \\ 0 & \text{otherwise} \end{cases}$$

ⓐ $$f_{V|\{V>1\}} = \begin{cases} 1 & 1 \le V \le 2 \\ 0 & \text{Otherwise} \end{cases}$$

(b) $P\left(\frac{1}{2} < V < \frac{3}{2}\right) = \int_{\frac{1}{2}}^{3/2} \frac{1}{2}\, dv = \frac{1}{2}.$

$$f_{V|\{\frac{1}{2}<V<3/2\}}(v) = \begin{cases} 1 & \frac{1}{2} < v < 3/2 \\ 0 & \text{otherwise.} \end{cases}$$

(c) $$F_{V|\{\frac{1}{2}<V<\frac{3}{2}\}}(v) = \begin{cases} 0 & v < 1/2 \\ v - \frac{1}{2} & 1/2 \le v < 3/2 \\ 1 & v \ge 3/2 \end{cases}$$

(4) $$f_X(x) = \begin{cases} \frac{c}{x^2} & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) $$\int_1^2 f_X(x)\,dx = \int_1^2 \frac{c}{x^2}\,dx = -cx^{-1}\Big|_1^2$$

$$= -c\left[\frac{1}{2} - 1\right]$$

$$= 1. \Rightarrow c = 2.$$

(b) $A = \{X > 1.5\}$

$$P(A) = \int_{1.5}^{2} \frac{2}{x^2} = -\frac{2}{x}\Big|_{1.5}^{2} = 2\left[\frac{2}{3} - \frac{1}{2}\right] = \frac{1}{3}$$

$$f_{X|A}(x) = \begin{cases} \frac{6}{x^2} & 2 \ge x > 1.5 \\ 0 & \text{otherwise.} \end{cases}$$

(5) $Y \sim \text{Uniform}[0, l]$.

$X | Y = y \sim \text{Uniform}[0, y]$.

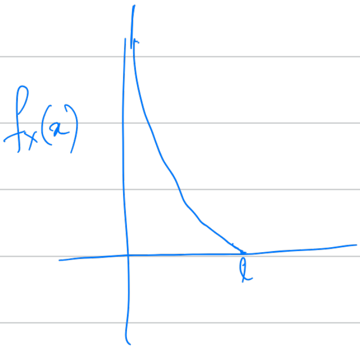
(a) $$f_{X,Y}(x, y) = f_{X|Y}(x|y)\, f_Y(y). \quad 0 \le y \le l$$

$$= \begin{cases} \frac{1}{yl} & 0 \le x \le y,\ 0 \le y \le l. \\ 0 & \text{otherwise} \end{cases}$$

Made with Goodnotes

(b)

$$f_X(x) = \int_0^l f_{X,Y}(x,y)\, dy$$



$$= \int_x^l \frac{1}{yl}\, dy = \left.\frac{\ln y}{l}\right|_x^l$$

$$= \begin{cases} \dfrac{\ln l - \ln x}{l} & 0 \le x \le l \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$E[X] = \int_0^l x f_X(x)\, dx$$

$$= \int_0^l x \left(\frac{\ln l - \ln x}{l}\right) dx$$

$$= \frac{l^2}{2l} \ln l - \int_0^l \frac{x \ln x}{l}\, dx$$

$$= \frac{l}{4} + \lim_{x \to 0} (\ln x) \frac{x^2}{2}$$

$$= \frac{l}{4}$$

Integration by parts:

$$\int u v' = uv - \int u' v$$

$$\int \ln x \, x = \ln x \frac{x^2}{2} - \int \frac{1}{x} \frac{x^2}{2}$$

$$= \ln x \left(\frac{x^2}{2}\right) - \frac{x^2}{4}$$

(d)

$$E[X] = E\left[E[X|Y]\right]$$

$$E[X|Y=y] = \frac{y}{2} \quad \text{as } X|Y=y \sim \text{Uniform}[0,y]$$

$$E[X] = \int E[X|Y=y]\, f_Y(y)\, dy$$

$$= \int_0^l \frac{y}{2} \frac{1}{l}\, dy = \left.\frac{y^2}{4l}\right|_0^l = \frac{l}{4}$$

⑥ $X \sim N(0,1)$. $A : \{X > 0\}$.

$$P(A) = P(X>0) = \frac{1}{2} \quad \text{(standard normal r.v)}.$$

$$f_{X|A}(x) = \begin{cases} \dfrac{e^{-x^2/2} \frac{1}{\sqrt{2\pi}}}{\frac{1}{2}} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X|A] = \int_0^\infty \frac{2e^{-x^2/2}}{\sqrt{2\pi}} x \, dx.$$

let $y = x^2$, $dy = 2x\,dx$.

$$= \int_0^\infty \frac{e^{-y}}{\sqrt{2\pi}} dy = \left. \frac{-e^{-y}}{\sqrt{2\pi}} \right|_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}}.$$

⑦ $E[X_1 + \dots X_n \mid X_1 + X_2 + \dots X_n = 1] = 1$

$\parallel$

linearity of expectation $\sum_{i=1}^{n} E[X_i \mid X_1 + \dots X_n = 1] = 1$

iid $\Rightarrow$ $n\,E[X_1 \mid X_1 + \dots X_n = 1] = 1$

$\Rightarrow$ $E[X_1 \mid X_1 + \dots X_n = 1] = \frac{1}{n}$.

$$E[Y_{10} \mid Y_5] = E[Y_5 + X_6 + \dots X_{10} \mid Y_5] = Y_5 + \sum_{i=6}^{10} E[X_i]$$

$$= Y_5.$$

$E[Y_{10} \mid Y_5, Y_2] = Y_5$ [as Y_2 doesn't reveal anything extra].

(8) Let $Y = X_1 + \dots X_N$.

$$E[Y] = E[E[Y|N]]$$

$$E[Y|N=n] = E[X_1 + \dots X_n | N=n]$$

$$= E[X_1 + \dots X_n] \quad \text{[since } X_i\text{'s are independent of } N\text{]}$$

$$= n\,E[X_1].$$

$$E[Y] = E[N\,E[X]] = E[X]\,E[N].$$

$$Var(Y|N=n) = Var(X_1 + \dots X_n | N=n).$$

$$= Var(X_1 + \dots X_n). \quad X_i\text{'s independent of } N.$$

$$= \sum_{i=1}^{n} Var(X_i) = n\,Var(X_1). \quad X_i\text{'s iid}.$$

$$Var(Y) = E[Var(Y|N)] + Var[E[Y|N]].$$

$$= E[N\,Var(X_1)] + Var[N\,E[X_1]].$$

$$= E[N]\,Var(X_1) + Var(N)(E[X_1])^2.$$

as $Var(aX) = a^2 Var(X)$.

$N \sim Poisson(\lambda)$, $E[X_i] = \mu$, $Var(X_1) = \sigma^2$.

$E[N] = \lambda$ $\quad Var(N) = \lambda$.

$$E[Y] = E[N]\,E[X_1] = \mu\lambda.$$

$$Var(Y) = E[N]\,Var(X_1) + Var(N)(E[X_1])^2.$$

$$= \lambda\sigma^2 + \lambda\mu^2 = \lambda(\sigma^2 + \mu^2).$$

(9) R.V N is a "Stopping time" if it is described through some condition on sequence of random variables $X_1, X_2, \dots$

$$E[X_1 + \dots X_N] = E\left[\sum_{i=1}^{\infty} X_i \, 1_{\{N \geq i\}}\right].$$

Linearity of expectation $\leftarrow$ $= \sum_{i=1}^{\infty} E[X_i \, 1_{\{N \geq i\}}]$

Let $I = 1_{\{N \geq i\}}$.

$$= 1_{\{X_1 < x, X_2 < x, \dots X_{i-1} < x\}}.$$

ie; I is a function of $X_1, X_2, \dots X_{i-1}$ and therefore independent of X_i

$$\Rightarrow E[X_1 + \dots X_N] = \sum_{i=1}^{\infty} E[X_i] \, E[1_{\{N \geq i\}}].$$

$$= E[X_1] \sum_{i=1}^{\infty} E[1_{\{N \geq i\}}]$$

$$E[1_{\{N \geq i\}}] = P(N \geq i).$$

$$= P(N > i-1).$$

$$E[Y] = E[X_1] \underbrace{\sum_{i=0}^{\infty} P(N > i)}$$

$$= E[X_1] . E[N]$$

(10) Let Y: Wait time of Fintin.

$X \sim$ Uniform$[0:2]$.

$A = \{X \leq 1\}$: event that Prof Calculus arrives before 10am.

$P(A) = 1/2$.

$Y | A = 0$.

$Y | A^c = X - 1$.

Can check that $X | A^c \sim$ uniform$[1, 2]$.

(a) $E[Y] = P(A)E[Y|A] + P(A^c)E[Y|A^c]$

$= \frac{1}{2} \times 0 + \frac{1}{2} E[X-1 | A^c]$.

$= \frac{1}{2}\left[E[X|A^c] - 1\right] = \frac{1.5-1}{2} = 1/4$.

(b) Z: duration of meeting.

$Z|A = 3$

$Z|A^c \sim \text{uniform}(0, 3-X)$

$E[Z] = 3 \times P(A) + P(A^c)E[Z|A^c]$.

$= \frac{3}{2} + \frac{1}{2} E[Z|A^c]$.

$E[Z|X=x, A^c] = \left(\frac{3-x}{2}\right)$,

$E[Z|A^c] = \int_1^2 E[Z|X=x, A^c] f_{X|A^c}(x)dx$

$= \int_1^2 \left(\frac{3-x}{2}\right) dx = \frac{3}{2} - \frac{x^2}{4}$

$= 3/2$.

$E[Z] = \frac{3}{2} + \frac{1}{2} \times \frac{3}{2} = \frac{9}{4} = 2.25$.

(c) M = # meetings by which Prof Calculus exceeds meeting time by 45 mins twice.

Let $p = P(X > 1.75) = \frac{2-1.75}{2} = 0.125$.

M_1 = # meetings by which Prof. Calculus exceeds meeting time once.

$= \text{Geom}(p)$.

$E[M_1] = 1/p$.

$M = M_1 + M_2$ where M_1 & M_2 are Geometric. p

M_2 := From the first time prof calculus exceeds his time till the next time he does so.

$$E[M] = \frac{2}{p} = \frac{2}{\left(\frac{1}{8}\right)} = 16.$$

(11) (a) $Var(X+3Y) = Var(X-3Y)$.

$$Var\left(\sum_{i=1}^{n} a_i x_i\right) = E\left[\left(\sum_{i=1}^{n} a_i \underbrace{(X_i - E[X_i])}_{\overline{X}_i}\right)^2\right].$$

$$= E\left(\sum_{i=1}^{n} a_i^2 \overline{X}_i^2 + 2\sum_{1\le i<j\le n} a_i a_j \overline{X}_i \overline{X}_j\right).$$

$$\boxed{Var\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2\sum_{1\le i<j\le n} a_i a_j Cov(X_i, X_j).}$$

$$Var(X+3Y) = Var(X) + 9Var(Y) + 6Cov(X,Y).$$

$$Var(X-3Y) = Var(X) + 9Var(Y) - 6Cov(X,Y)$$

Subtracting both we get $\Rightarrow$ $12 Cov(X,Y) = 0$.

$\Rightarrow$ X, Y are uncorrelated.

(b) $Var(X) = Var(Y)$ need not mean they are uncorrelated. Consider $Y = X$. st $Var(X) \neq 0$.

$$Cov(X,Y) = Var(X). \neq 0.$$

$\Rightarrow$ X, Y are not correlated.

(c) $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$.

$$\rho(X_1, X_2) = \frac{Cov(X_1, X_2)}{\sqrt{Var(X)}\sqrt{Var(Y)}} > 0.$$

$\Rightarrow$ $Cov(X_1, X_2) > 0$.

$\Rightarrow$ $Var(X_1 + X_2) > Var(X_1) + Var(X_2)$.

(d) $$\rho(X, aY+b) = \frac{Cov(X, aY+b)}{\sqrt{Var(X)}\sqrt{Var(aY+b)}}$$

$$= \frac{E\left[(X-E[X])(aY+b-\underbrace{E[aY+b]}_{aE[Y]+b})\right]}{\sqrt{Var(X)}\; a\sqrt{Var(Y)}}$$

$$= \frac{\cancel{a}\, E[(X-E[X])(Y-E[Y])]}{\cancel{a}\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$= \rho(X,Y).$$

(12) if $\rho(X,Y) = 1$

$$E\left[(\hat{X} - c\hat{Y})^2\right] \geq 0.$$

$$E\left[\hat{X}^2 + c^2\hat{Y}^2 - 2c\hat{X}\hat{Y}\right] \geq 0.$$

Let $c = \frac{E[\hat{X}\hat{Y}]}{E[\hat{Y}^2]}$

$$\Rightarrow E[\hat{X}^2] + \frac{E[\hat{X}\hat{Y}]^2}{E[\hat{Y}^2]^{\cancel{2}}}\cancel{E[\hat{Y}^2]} - 2\frac{E[\hat{X}\hat{Y}]^2}{E[\hat{Y}^2]} \geq 0.$$

$$\Rightarrow E[\hat{X}^2] - \frac{E[\hat{X}\hat{Y}]^2}{E[\hat{Y}^2]} \geq 0.$$

[gives a different proof to $-1 \leq \rho(X,Y) \leq 1$.

$$\Rightarrow (\rho(X,Y))^2 \leq 1.$$

$\rho(X,Y) = 1$ iff the inequality we start off with is an equality i.e.;

$$E\left[(\hat{X} - c\hat{Y})^2\right] = 0.$$

$\Rightarrow (\hat{X} - c\hat{Y})^2 = 0$ with probability 1

$\Rightarrow \quad \hat{Y} = \frac{1}{c}\hat{X}$ with prob 1.

(13)

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{1\le i<j\le n} \mathrm{Cov}(X_i, X_j)$$

$$-\sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)} \le \mathrm{Cov}(X_i, X_j) \le \sqrt{\mathrm{Var}(X_i)\,\mathrm{Var}(X_j)}$$

$$\mathrm{Var}(Y) \le \sum_{i=1}^{n} \mathrm{Var}(X_i) + 2\sum_{1\le i<j\le n} \sqrt{\mathrm{Var}(X_i)\mathrm{Var}(X_j)}$$

(14)

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\underbrace{\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2r\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)\right]}_{T(x,y).}}$$

$$T(x,y) = \underbrace{\left[\frac{y-\mu_y}{\sigma_y} - r\left(\frac{x-\mu_x}{\sigma_x}\right)\right]^2}_{T_1(x,y)} + \underbrace{\left(\frac{x-\mu_x}{\sigma_x}\right)^2(1-r^2)}_{T_2(x).}.$$

$$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\, e^{-\frac{1}{2(1-\rho^2)}\left[T_1(x,y)+T_2(x)\right]}\,dy.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x}\, e^{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2\frac{1}{2}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_y-r\frac{\sigma_y}{\sigma_x}(x-\mu_x)}{\sigma_y}\right]^2}}{\sqrt{2\pi}\,\sigma_y\sqrt{1-\rho^2}}\,dy$$

Let $\hat{\mu}_y = \mu_y + r\frac{\sigma_y}{\sigma_x}(x-\mu_x)$.

$$\hat{y} = \frac{y - \hat{\mu}_y}{\sigma_y \sqrt{1-\rho^2}}$$

$$d\hat{y} = \frac{\partial y}{\sigma_y \sqrt{1-\rho^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\left(\frac{x-\mu_x}{\sigma_x}\right)^2 \frac{1}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\hat{y}^2/2}}{\sqrt{2\pi}} d\hat{y}}_{1}$$

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2}$$

$\Rightarrow X \sim N(\mu_x, \sigma_x^2)$ Similarly can show

$Y \sim N(\mu_y, \sigma_y^2)$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{1}{(\sqrt{2\pi})^2 \sigma_x \cancel{\sigma_y} \sqrt{1-\rho^2}} e^{-\frac{T(x,y)}{2(1-\rho^2)}}}{\dfrac{1}{\cancel{\sqrt{2\pi}}\,\cancel{\sigma_y}} e^{-\left(\frac{y-\mu_y}{\sigma_y}\right)^2 \frac{1}{2}}}$$

$$\frac{T(x,y)}{(1-\rho^2)} - \left(\frac{y-\mu_y}{\sigma_y}\right)^2 = \frac{1}{1-\rho^2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \left[\frac{1}{1-\rho^2} - 1\right]$$

$$- \frac{2\rho}{1-\rho^2}\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right)$$

$$= \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right) - \rho\left(\frac{y-\mu_y}{\sigma_y}\right)\right]^2.$$

$$f_{X|Y}(x|y) = \frac{e^{-\frac{1}{2(1-\rho^2)\sigma_x^2}\left[x-\mu_x-\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right]^2}}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}}$$

$$\Rightarrow X|Y=y \sim N\left(\hat{\mu}_x,\ \sigma_x^2(1-\rho^2)\right)$$

where $\hat{\mu}_x = \mu_x + \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)$

$$\mathrm{Cov}(X,Y) = E\left[(X-\mu_x)(Y-\mu_y)\right]^2.$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_x)(y-\mu_y)\,f_{X,Y}(x,y)\,dx\,dy.$$

$$\hat{x} = \frac{x-\mu_x}{\sigma_x}, \quad \hat{y} = \frac{y-\mu_y}{\sigma_y}.$$

$$d\hat{x}\,d\hat{y} = \begin{vmatrix} \frac{\partial\hat{x}}{\partial x} & \frac{\partial\hat{y}}{\partial x} \\ \frac{\partial\hat{x}}{\partial y} & \frac{\partial\hat{y}}{\partial y} \end{vmatrix} dx\,dy. = \frac{1}{\sigma_x}\frac{1}{\sigma_y}\,dx\,dy.$$

$$\mathrm{Cov}(X,Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sigma_y\sigma_x\,\hat{x}\,\hat{y}\,\frac{1}{2\pi\sqrt{1-\rho^2}}\,e^{-\frac{1}{2(1-\rho^2)}\left(\hat{x}^2+\hat{y}^2-2\rho\hat{x}\hat{y}\right)}\,d\hat{x}\,d\hat{y}$$

$$\hat{x}^2+\hat{y}^2-2\rho\hat{x}\hat{y}$$
$$= (\hat{x}-\rho\hat{y})^2 + (1-\rho^2)\hat{y}^2$$

$$= \int_{-\infty}^{\infty}\sigma_x\sigma_y\,\hat{y}\,\frac{e^{-\frac{\hat{y}^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}}\,\hat{x}\,e^{-\frac{1}{2(1-\rho^2)}}e^{-(\hat{x}-\rho\hat{y})^2}\,d\hat{x}\,d\hat{y}$$

$$= \int_{-\infty}^{\infty} \sigma_x \sigma_y \frac{\hat{y}}{\sqrt{2\pi}} e^{-\frac{\hat{y}^2}{2}} \underbrace{\rho \hat{y}}_{\text{i.e., } \rho\hat{y}} \, d\hat{y}$$

Expectation of a normal R.V with mean $\rho \hat{y}$ and variance $1-\rho^2$.

$$= \sigma_x \sigma_y \rho \underbrace{\int_{-\infty}^{\infty} \frac{\hat{y}^2}{\sqrt{2\pi}} e^{-\hat{y}^2/2} \, d\hat{y}}_{\text{Variance of standard normal R.V i.e., 1}}$$

$$= \sigma_x \sigma_y \rho.$$

(d) If X and Y are uncorrelated

$\Rightarrow \operatorname{Cov}(X,Y) = 0$ i.e., $\rho = 0$.

then

$$f_{X,Y}(x,y) = \frac{1}{2\pi \sigma_x \sigma_y} e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= f_X(x) \, f_Y(y) \quad \forall x, y$$

$\Rightarrow X, Y$ are independent.

(15) (a) $Y = 2X$, $X \sim N(0,1)$.

$$Z = \begin{cases} +1 & \text{w.p } 1/2 \\ -1 & \text{w.p } 1/2 \end{cases}$$

Let $A_1 = \{Z = +1\}$, $A_2 = \{Z = -1\}$

$Y \mid A_1 = X \mid A_1 = X$ as X is independent of Z.

$Y \mid A_2 = -X \mid A_2 = -X$

$Y \mid A_1 \sim N(0,1)$ and $Y \mid A_2 \sim N(0,1)$.

$$f_Y(y) = f_{Y|A_1}(y)\,P(A_1) + f_{Y|A_2}(y)\,P(A_2)$$

$$= \frac{1}{2}\,\frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} + \frac{1}{2}\,\frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

$$= \frac{e^{-y^2/2}}{\sqrt{2\pi}}$$

i.e., $Y \sim N(0,1)$.

(b) $$E[XY] = \frac{1}{2}E[XY|A_1] + \frac{1}{2}E[XY|A_2]$$

$$= \frac{1}{2}E[X^2|A_1] + \frac{1}{2}E[-X^2|A_2]$$

as X is independent of Z. $$= \frac{1}{2}E[X^2] - \frac{1}{2}E[X^2] = 0.$$

$$E[X] = E[Y] = 0.$$

$\therefore E[XY] = E[X]\cdot E[Y]$ i.e., X, Y are uncorrelated.

(c) "Intuitively we believe they are not independent. So we'll use contradiction method to show".

If X and Y are independent then $g(X)$ and $h(Y)$ also should be independent.

But for $g(x) = x^2$, $h(y) = y^2$.

X^2 and $Y^2 = X^2Z^2 = X^2$ are not independent.

Made with Goodnotes

(d) No.

We can write

$$f_{Y|X}(y|x) = f_{Y|X,A_1}(y|x)\frac{1}{2} + \frac{1}{2}f_{Y|X,A_2}(y|x)$$

$$= \frac{1}{2}\delta(y-x) + \frac{1}{2}\delta(y+x)$$

$\delta(t)$.

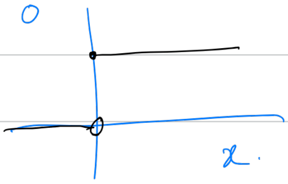
dirac-delta function is used to represent the density function of discrete random variables.

For example if W is a discrete R.V taking values w_1, w_2, w_3 with probabilities w_1, w_2, w_3 the density function is given by.

$\delta(t)$

$$f_W(w) = P_1\,\delta(w-w_1) + P_2\,\delta(w-w_2) + P_3\,\delta(w-w_3).$$

$\int_0^x \delta(t)\,dt$



$$P(W=w_1) = \lim_{\Delta w\to 0} P(w_1 < W \leq w_1+\Delta w)$$

$$= \lim_{\Delta w\to 0} \int_{w_1}^{w_1+\Delta w} f_W(w)\,dw$$

$$= \lim_{\Delta w\to 0} P_1 = P_1.$$

" The integral of dirac delta function is 1 ".

$$f_{X,Y}(x,y) = f_X(x).\left[\frac{1}{2}\delta(y-x) + \frac{1}{2}\delta(y+x)\right]$$

Clearly doesn't have joint Gaussian form.

Above expression can also be used to show that X, Y are not independent.

(16) $$\mathrm{Cov}(Z, Z^2) = E\left[(Z - E[Z])(Z^2 - E[Z^2])\right]$$

$$= E\left[Z(Z^2 - 1)\right] = \underbrace{E[Z^3] - E[Z]}_{\text{integrating odd fns}}$$

$$= 0.$$

(17)

(a) $$f_{X,Y}(x, y) = f_Y(y)\, f_{X|Y}(x|y)$$

$$X | \{Y = y\} \sim N(y, 1).$$

$$\therefore Y \sim N(0, 1).$$

$$f_{X,Y}(x, y) = \left(\frac{e^{-y^2/2}}{\sqrt{2\pi}}\right)\left(\frac{e^{-(x-y)^2 \frac{1}{2}}}{\sqrt{2\pi}}\right) \rightarrow (1)$$

$$W = Y + Z, \quad Z \sim N(0, 1).$$

$$W | \{Y = y\} = y + Z.$$

$$\sim N(y, 1).$$

$$\therefore f_{W,Y}(w, y) = f_{W|Y}(w|y)\, f_Y(y).$$

$$= f_{X,Y}(w, y)$$

(b) X, Y are clearly joint normal from equation (1).

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\left[2y^2 + x^2 - 2xy\right] \frac{1}{2}}$$

$$= \frac{1}{2\pi} e^{-\frac{1}{2 \times \left(\frac{1}{2}\right)}\left[y^2 + \frac{x^2}{2} - xy\right]}$$

$\sigma_x = \sqrt{2}$

$\sigma_y = 1.$ $\rho = \frac{1}{\sqrt{2}}$

$X, Y \sim$ Joint Normal with

$$(\mu_x, \sigma_x^2, \mu_y, \sigma_y^2, \rho) = \left(0, 2, 0, 1, \tfrac{1}{\sqrt{2}}\right).$$

(c) $E[X] = 0$, $Var(X) = 2$, $Cov(X, Y) = \rho \sqrt{Var(X)}\sqrt{Var(Y)}$

$$= \frac{1}{\sqrt{2}} \sqrt{2} \times 1 = 1.$$

(e) $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{1}{2\pi} e^{-\left(\frac{x^2}{2} - xy + y^2\right)}}{\frac{e^{-x^2/4}}{\sqrt{2\pi}\sqrt{2}}} \quad \text{as } X \sim N(0,2)$$

$$= \frac{1}{\sqrt{2\pi}\left(\frac{1}{\sqrt{2}}\right)} e^{-\left(\frac{x^2}{4} - xy + y^2\right)}$$

$$= \frac{1}{\sqrt{2\pi}\,\frac{1}{\sqrt{2}}} e^{-\frac{1}{2}\left(\frac{y - \frac{x}{2}}{\frac{1}{\sqrt{2}}}\right)^2}$$

(d) $E[Y \mid X = x] = \frac{x}{2}$ as $Y \mid X = x \sim N\left(\frac{x}{2}, \frac{1}{2}\right)$.
as seen above.

Different way to show the same:
Suppose

$$X = Y + Z.$$

$$E[Y + Z \mid X = x] = x.$$

$$E[Y \mid X = x] + E[Z \mid X = x] = x$$

$$2E[Y \mid X = x] = x \quad \text{Since } Y, Z \text{ are iid;}$$

$$\Rightarrow E[Y \mid X = x] = x/2.$$

(18) $f_P(p) = \begin{cases} p e^p & p \in \{0,1\} \\ 0 & \text{otherwise} \end{cases}$

Sanity check
$$\int_0^1 pe^p\,dp = pe^p\Big|_0^1 - e^p\Big|_0^1 = e^1 - (e^1 - 1) = 1$$

$B \mid \{P = p\} \sim \text{Bernoulli}(p)$

(a) $$P(B=1) = \int_p P(\{B=1\} \mid \{P=p\})\, f_P(p)\, dp$$

$$= \int_0^1 p \cdot p e^p \, dp$$

$$= \int_0^1 p^2 e^p \, dp.$$

Using integration by parts

$$= p^2 e^p \Big|_0^1 - 2\int_0^1 p e^p \, dp$$

$$= (e^1 - 2).$$

(b) $$f_{P \mid \{B=1\}}(p) = \frac{P(B=1 \mid P=p)\, f_P(p)}{P(B=1)}$$

$$= \frac{p^2 e^p}{(e-2)}$$

(c) $P(B_2 = 1 \mid B_1 = 1).$

$$= \int_0^1 P(B_2 = 1 \mid B_1 = 1, \{P=p\})\, f_{P \mid \{B_1=1\}}(p)\, dp.$$

B_2 is independent of B_1 conditioned over P. by definition.

$$= \int_0^1 P(B_2 = 1 \mid \{P=p\})\, \frac{p^2 e^p}{(e-2)}\, dp.$$

$$= \frac{\int_0^1 p^3 e^p \, dp}{e-2}$$

$$= \frac{p^3 e^p \Big|_0^1 - 3\int_0^1 p^2 e^p \cdot dp}{e-2}$$

$$= \frac{e^1 - 3(e-2)}{e-2} = \frac{6-2e}{e-2} \approx 0.78$$

(19) $Y = \frac{X-a}{b-a}$ has density.

$$f_Y(y) = f_X((b-a)y+a).(b-a) \quad \text{[from class]}$$

$$= \begin{cases} \frac{1}{b-a} \times b-a = 1 & a \le (b-a)y + a \le b \\ & \text{i.e., } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

20. (a) $X \sim \text{Uniform}(-1, 1)$.

$$P(|X| > \tfrac{1}{2}) = P(X > \tfrac{1}{2}) + P(X < -\tfrac{1}{2}).$$

$$= \frac{1}{2\times 2} + \frac{1}{2\times 2} = \frac{1}{2}.$$

(b) $Y = |X|$

$$F_Y(y) = P(|X| \le y)$$

$$= P(-y \le X \le y). = F_X(y) - F_X(-y).$$

$$f_Y(y) = f_X(y) + f_X(-y).$$

$$= \begin{cases} \frac{1}{2} + \frac{1}{2} = 1 & \text{if } y \in [0, 1) \\ 0 & \text{otherwise} \end{cases}$$

i.e., $Y \sim \text{Uniform}[0, 1]$.

21. X, Y independent.

$$X \sim \exp(\lambda_1), \quad Y \sim \exp(\lambda_2).$$

$$Z = X + Y.$$

$$F_Z(z) = P(X + Y \leq z).$$

$$= \int_{y=0}^{z} \int_{x=0}^{z-y} f_X(x) f_Y(y) \, dy \, dx$$

$$\frac{\partial F_Z(z)}{\partial z} = \int_{x=0}^{z-z} f_X(x) f_Y(z) \, dx \;\to 0$$

$$+ \int_{y=0}^{z} \left[f_X(z-y) f_Y(y) \right] dy.$$

$$= \int_{y=0}^{z} \lambda_1 e^{-\lambda_1(z-y)} \lambda_2 e^{-\lambda_2 y} \, dy$$

$$= \lambda_1 \lambda_2 e^{-\lambda_1 z} \left[\int_{y=0}^{z} e^{(\lambda_1 - \lambda_2) y} \, dy. \right]$$

$$= \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)} e^{-\lambda_1 z} \left[e^{(\lambda_1 - \lambda_2) z} - 1 \right].$$

$$f_Z(z) = \lambda_1 \lambda_2 \left(\frac{e^{-\lambda_2 z} - e^{-\lambda_1 z}}{\lambda_1 - \lambda_2} \right).$$

$$W = \min\{X, Y\}$$

$$1 - F_W(w) = P(W \geq w) = (1 - F_X(w))(1 - F_Y(w)).$$

$\rightarrow P(X \geq w, Y \geq w)$. $\xrightarrow{\text{independence}}$

$$\Rightarrow \quad -f_W(w) = -f_X(w)\left(1-F_Y(w)\right) - f_Y(w)\left(1-F_X(w)\right).$$

$$f_W(w) = \lambda_1 e^{-\lambda_1 w}\left(e^{-\lambda_2 w}\right) + \lambda_2 e^{-\lambda_2 w} e^{-\lambda_1 w}.$$

$$= (\lambda_1+\lambda_2)\, e^{-(\lambda_1+\lambda_2)w}.$$

$\Rightarrow$ W is $\exp(\lambda_1+\lambda_2)$

Q2. $X_i \sim \text{uniform}[0,1]$ $\quad X_1, X_2, \dots X_n$ iid.

say $Y_1 = X_1$

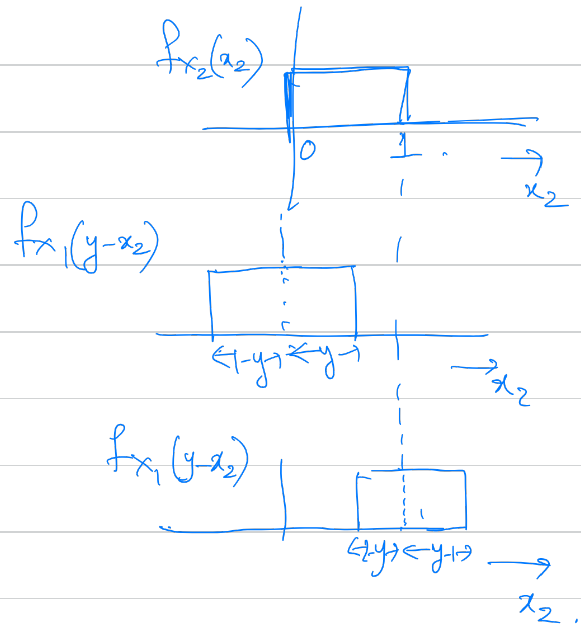
$Y_2 = X_1 + X_2$.

$\vdots$

$Y_n = X_1 + X_2 + \dots X_n$

$$f_{Y_2}(y) = \int_0^y f_{X_1}(y-x_2)\, f_{X_2}(x_2)\, dx_2. \qquad 0 \le y \le 2$$

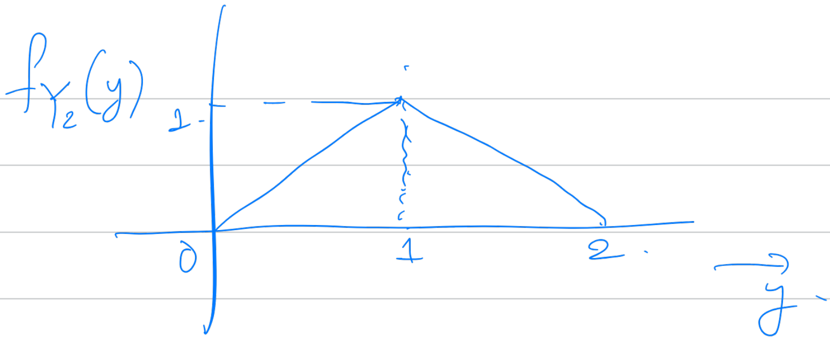
$y - x_2$



If $0 \le y \le 1$

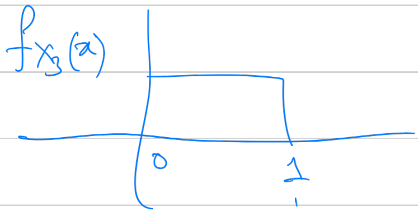
If $1 \le y \le 2$.

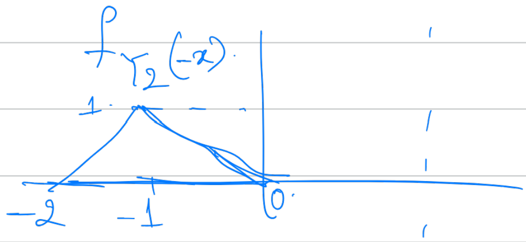
$$f_{Y_2}(y) = \begin{cases} \int_0^y dx_2 = y. & 0 \le y \le 1 \\ \int_{y-1}^{1} dx_2 = 2-y. & \end{cases}$$

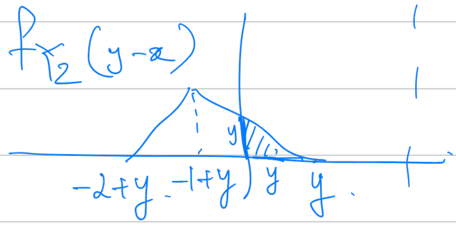


$Y_3 = Y_2 + X_3.$ Y_3 takes values in $[0:3]$.

$$f_{Y_3}(y) = \int_0^y f_{Y_2}(y-x)\, f_{X_3}(x)\, dx.$$







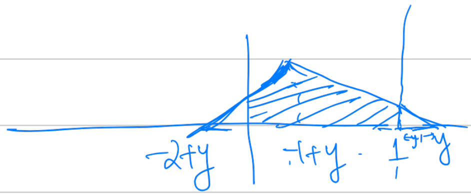
If $0 \le y \le 1$

$$f_{Y_3}(y) = y^2/2.$$

$1 \le y \le 2$

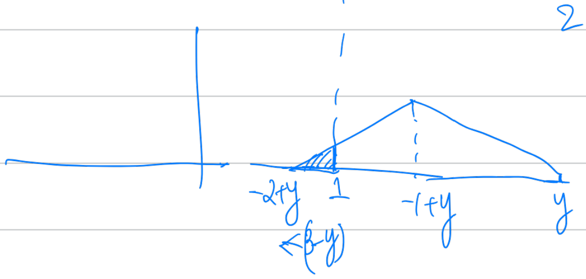
$$f_{Y_3}(y) = 1 - (y-1)^2.$$

$$f_{Y_3}(y) = \begin{cases} y^2/2 & 0 \le y \le 1. \\ 1-(y-1)^2 & 1 \le y \le 2 \\ \dfrac{(3-y)^2}{2} & 2 \le y \le 3. \end{cases}$$



$2 \le y \le 3$

$$f_{Y_3}(y) = \frac{(3-y)^2}{2}$$



Ignore the general sum as we have not covered moment generating functions.

23. $Z = X + Y$, $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sqrt{2\pi}\,\sigma_1} \frac{e^{-\frac{1}{2}\left(\frac{\overbrace{z-x-\mu_2}^{(z-(x-\mu_1)-\mu_2-\mu_1)^2}}{\sigma_2}\right)^2}}{\sqrt{2\pi}\,\sigma_2}\, dx.$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\overbrace{\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x-\mu_1}{\sigma_2}\right)^2 + \left(\frac{z-\mu_2-\mu_1}{\sigma_2}\right)^2 - 2\left(\frac{x-\mu_1}{\sigma_2}\right)\left(\frac{z-\mu_2-\mu_1}{\sigma_2}\right)\right]}^{T(x,z)}}}{(\sqrt{2\pi})^2\,\sigma_1\sigma_2}\, dx.$$

$$T(x,z) = \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left[(x-\mu_1)^2 + \left(\frac{z-\mu_1-\mu_2}{\cancel{\sigma_2}}\right)^2 \frac{\sigma_1^2\cancel{\sigma_2^2}}{\sigma_1^2+\sigma_2^2} - 2\frac{\sigma_1^2\cancel{\sigma_2^2}}{\sigma_1^2+\sigma_2^2}(x-\mu_1)\frac{(z-\mu_1-\mu_2)}{\cancel{\sigma_2^2}}\right]$$

$$= \frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left[\left\{(x-\mu_1) - \left(\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)(z-\mu_1-\mu_2)\right]^2 + \boxed{\cancel{\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}}\left(1 - \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\right)\frac{\cancel{\sigma_1^2}}{\sigma_1^2+\sigma_2^2}(z-\mu_1-\mu_2)^2}\right.$$

$$\frac{1}{\sigma_1^2+\sigma_2^2}$$

$$f_Z(z) = \frac{e^{-\frac{1}{2}\cdot\frac{(z-\mu_1-\mu_2)^2}{\sigma_1^2+\sigma_2^2}}}{\sqrt{2\pi}\sqrt{\sigma_1^2+\sigma_2^2}} \underbrace{\int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}\left[x-\mu_1-\frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}(z-\mu_1-\mu_2)\right]^2}}{\sqrt{2\pi}\cdot\frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}}\, dx}_{1.}$$

$$\Rightarrow Z \sim N(\mu_1+\mu_2,\ \sigma_1^2+\sigma_2^2).$$

Made with Goodnotes

(24)

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{x^2y^2} & x \geq 1,\ y \geq 1 \\ 0 & \text{Otherwise} \end{cases}$$

$$U = XY, \quad V = X/Y.$$

$$J(x,y) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ \dfrac{1}{y} & \dfrac{-x}{y^2} \end{vmatrix}$$

$$= \left| -\frac{x}{y} - \frac{x}{y} \right| = \frac{2x}{y}$$

$$f_{U,V}(u,v) = \frac{f_{X,Y}(x,y)}{J(x,y)}$$

st $yx = u$
$\frac{x}{y} = v$
$x \geq 1,\ y \geq 1$

$$= \frac{\frac{1}{x^2y^2}}{\frac{2x}{y}} = \frac{1}{2u^2v}$$

$\boxed{u = xy \geq 1}$

$V = \frac{x}{y}$
$= \frac{u}{y^2}$
$= \frac{x^2}{u}$

$\Rightarrow \boxed{\frac{1}{u} \leq V \leq u}$

(b)

$$f_U(u) = \int_{\frac{1}{u}}^{u} f_{U,V}(u,v)\,dv = \int_{\frac{1}{u}}^{u} \frac{1}{2u^2v}\,dv = \frac{\ln v}{2u^2}\Big|_{\frac{1}{u}}^{u}$$

$$= \frac{\ln u}{u^2}$$

$$f_V(v) = \begin{cases} \displaystyle\int_{v}^{\infty} \frac{1}{2u^2v}\,du & v \geq 1 \\ \displaystyle\int_{1/v}^{\infty} \frac{1}{2v^2u}\,du & v \leq 1 \end{cases}$$

$$= \begin{cases} \frac{-1}{2uv} \Big|_{v}^{\infty} = \frac{1}{2v^2} & \text{if } v \geq 1 \\ \frac{-1}{2uv} \Big|_{\frac{1}{v}}^{\infty} = \frac{1}{2}. & \text{if } v \leq 1 \end{cases}$$

$$= \begin{cases} \frac{1}{2} & 0 \leq v \leq 1. \\ \frac{1}{2v^2} & v \geq 1. \end{cases}$$