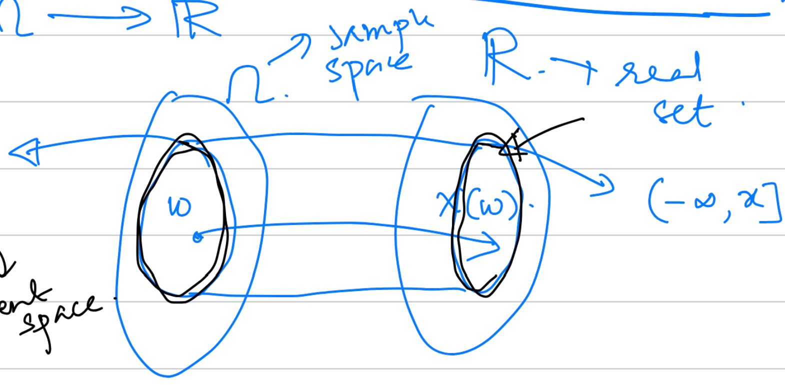
Lec 6 : 17th Jan

Q1 : LHC-6
20th Jan 6-7pm

Recap

Random variable $X : \Omega \to \mathbb{R}$ is a function.

$\forall x \in R$ $\quad X^{-1}((-\infty, x]) \in \mathcal{F}$ ↘ event space



Distribution function of R.V X is

$\omega \in \Omega$

$$F_X(x) = P(X \le x) = P(\{\omega \in \Omega : X(\omega) \le x\})$$

$$= P(X^{-1}((-\infty, x])).$$

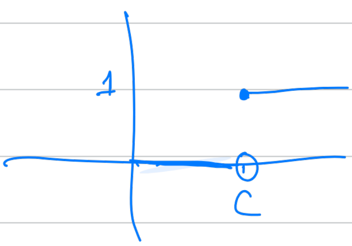
Remark: $X^{-1}(x)$ for $x \in R$ is collection of all $\omega \in \Omega$: $X(\omega) = x$
(it will be a null set if $\forall \omega \in \Omega$, $X(\omega) \neq x$.).
$X^{-1}(S)$ for $S \subseteq R$ is collection of all $\omega \in \Omega$ st $X(\omega) \in S$.

Examples

② Constant random variable.

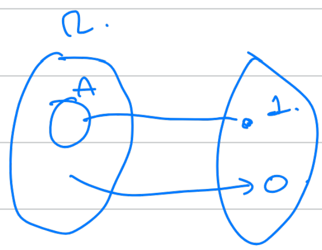
$X(\omega) = c$. $\forall \omega \in \Omega$. for some $c \in \mathbb{R}$

$$F_X(x) = P(X \le x).$$



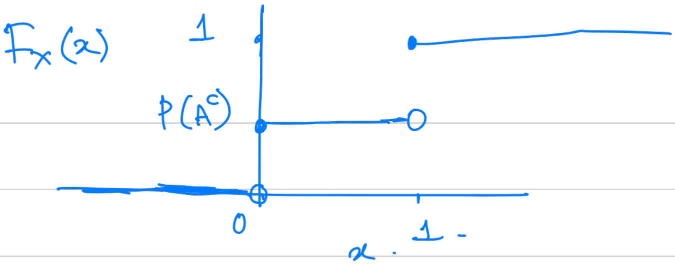
$$= \begin{cases} 0 & x < c. \\ 1. & x \ge c \end{cases}$$

③ Indicator random variable.



Let $A \in \mathcal{F}$.

$$1_A(\omega) = \begin{cases} 1 & \omega \in A. \\ 0. & \end{cases}$$



$$F_X(x) = \begin{cases} 0 & x < 0 \\ P(X \le x) = P(A^c) & 0 \le x < 1 \\ P(A^c) + P(A) = 1 & 1 \le x \end{cases}$$

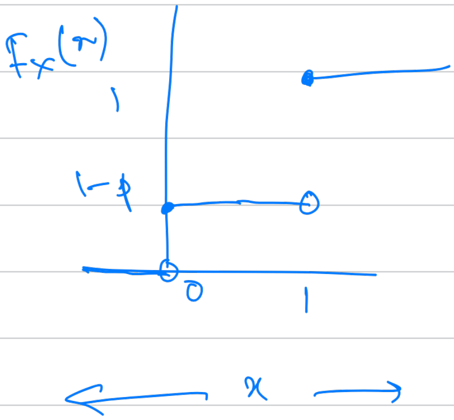
(4) Bernoulli random variable

$\mathcal{F} = 2^\Omega$

$\Omega = \{H, T\}$ $\qquad P(\{H\}) = p$

$P(\{T\}) = 1-p$

$$X(\omega) = \begin{cases} 1 & \omega = H \\ 0 & \omega = T \end{cases}$$



$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1-p & 1 > x \ge 0 \\ 1 & x \ge 1 \end{cases}$$

$$F_X(x) = P(X \le x)$$
$$= P(X \le 0) \qquad \text{for } x = 0$$
$$= P(\{\omega : X(\omega) \le 0\})$$
$$= P(\{T\}) = 1-p$$

$\underline{x = 1}$

$$P(X \le 1) = P(\{\omega : X(\omega) \le 1\})$$
$$= P(\{H, T\})$$
$$= P(\Omega) = 1$$

Made with Goodnotes

(5) Geometric random variable.

$$\Omega = \{ H, TH, TTH, \dots \}.$$

$X(\omega) := $ length of sequence ω.

\# times coin is tossed.

$$P(\{\underbrace{TT\dots T}_{n-1}H\}) = (1-p)^{n-1}p.$$

X takes values in $\{1, 2, 3, 4, \dots\}$ countably infinite.

$$F_X(n) = P(X \le n).$$

n → positive integer

$$= P(\{\omega : X(\omega) \le n\}).$$

$$= \sum_{k=1}^{n} P(\{\omega : X(\omega) = k\})$$

$$= \sum_{k=1}^{n} (1-p)^{k-1}p.$$

$$= \frac{\cancel{p}\left[1-(1-p)^n\right]}{\cancel{p}}$$

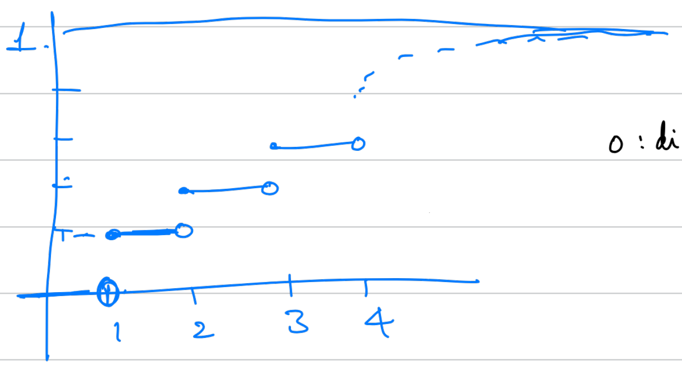
$n \le x < n+1$

$$F_X(x) = F_X(n)$$

$F_X(1) = p$

$F_X(1^-) = 0$

$F_X(1^+) = p.$



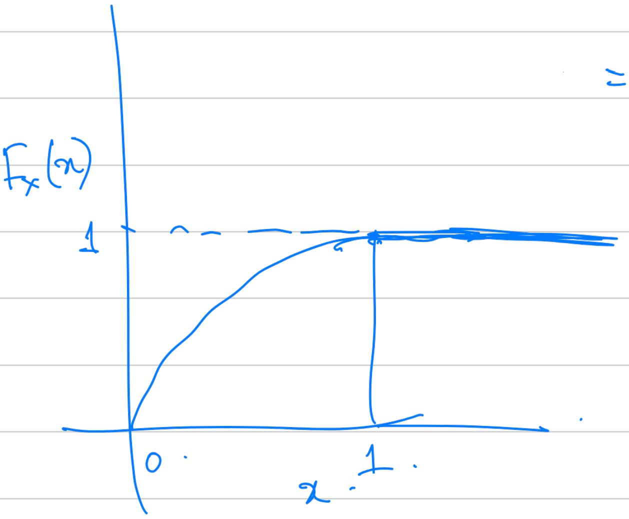
o : discontinuity.

(6) $\Omega = [-1, 1]$ $\qquad P((a,b)) = \frac{b-a}{2}$.

$X(\omega) = \omega^2$

X takes values in $[0, 1]$.

$F_X(x) = P(X \le x) \qquad 0 \le x \le 1$



$= P(\{\omega \in \Omega : X(\omega) \le x\})$

$= P(\{\omega \in \Omega : \underbrace{\omega^2 \le x}_{-\sqrt{x} \le \omega \le \sqrt{x}}\})$.

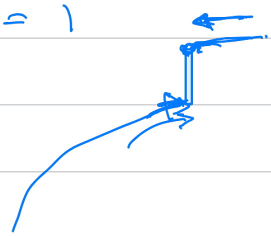
$= P([-\sqrt{x}, \sqrt{x}])$.

$= \frac{\sqrt{x} - (-\sqrt{x})}{2} = \sqrt{x}$.

Properties of the distribution function

① If $x < y$, $F_X(x) \le F_X(y)$ $\rightarrow$ non decreasing function

② $\lim_{x \to -\infty} F_X(x) = 0 \qquad \lim_{x \to \infty} F_X(x) = 1$



③ Right continuity

$F_X(x) = \lim_{\epsilon \to 0^+} F_X(x+\epsilon) = F_X(x^+)$

④ $P(X > x) = 1 - F_X(x)$.

⑤ $P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$

⑥ $P(X = x) = F_X(x) - \lim_{\epsilon \to 0^-} F_X(x+\epsilon)$ $\rightarrow$ $-$ve values increasing towards 0.

$F_X(x) - F_X(x^-)$ $\qquad F_X(x^-)$.

Made with Goodnotes

① $x < y$ $\quad F_X(x) \le F_X(y) \to P(X \le y)$.

$\downarrow$

$P(X \le x)$

$$A_x = \{\omega \in \Omega : X(\omega) \le x\}.$$

$$A_y = \{\omega \in \Omega ; X(\omega) \le y\}.$$

$\omega \in A_x \Rightarrow X(\omega) \le x$

we know $x < y$

$\Rightarrow X(\omega) < y$

$\Rightarrow \omega \in A_y$.

$A_x \subseteq A_y$.

$\Rightarrow \underbrace{P(A_x)}_{P(X \le x)} \le \underbrace{P(A_y)}_{P(X \le y)}$.

② $\lim_{x \to -\infty} F_X(x) = 0$. $\quad \forall x \in \mathbb{R}$

$X^{-1}((-\infty, x]) \in \mathcal{F}$.

$A_n = X^{-1}((-\infty, -n])$

Claim: $\bigcap_{n=1}^{\infty} A_n = \phi$.

$A_1 \supset A_2 \supset A_3 \ldots$

$$0 = P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n).$$

$$= \lim_{n \to \infty} F_X(-n)$$

$$= \lim_{x \to -\infty} F_X(x)$$

④ $P(X > x)$

$$= \overbrace{\{\omega \in \Omega : X(\omega) > x\}}^{A}$$

$P(X \le x)$

$= P(\Omega \setminus A)$

$= 1 - P(A)$

$P(A) = 1 - P(X \le x)$

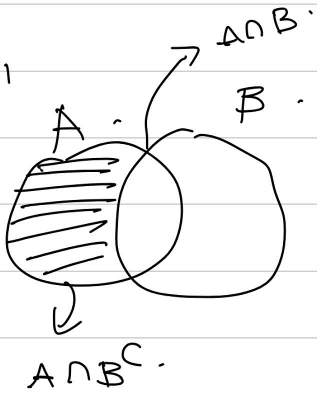
$= 1 - F_X(x)$

⑤ $P(x_1 < X \le x_2)$ $\quad A : X \le x_2$.

$B : X \leq x_1.$

$A \cap B = B \quad \because$ as $B \subseteq A$.

$A \cap B^c \quad : \quad X \leq x_2$ and $X > x_1$



$$P(A) = P(A \cap B^c) + \underbrace{P(B \cap A)}_{= P(B)}.$$

$$P(A \cap B^c) = P(A) - P(B)$$
$$= F_X(x_2) - F_X(x_1).$$

Discrete random variable.

$\mathcal{X} \subseteq \mathbb{R}$.

Any random variable that takes countables values in $\mathbb{R}$ is refered to as discrete random variable.

$\mathcal{X}$: countable.

Probability mass function.

$$P_X(x) = P(X = x)$$
$$= P(\{\omega \in \Omega : X(\omega) = x\})$$

① $\Omega = \{HH, HT, TH, HH\}$.

$X(\omega)$: # heads.

$$P_X(x) = \begin{cases} P(\{TT\}) = 1/4 & x = 0 \\ P(\{HT, TH\}) = 1/2 & x = 1 \\ P(\{HH\}) = 1/4. & x = 2 \end{cases}$$

Properties of PMF.

$$\sum_{x \in \mathcal{X}_0} P_X(x) = 1.$$

$$\sum_{x \in \mathcal{X}_e} P_X(x) = \sum_{x \in \mathcal{X}_e} P\big(\underbrace{\{\omega \in \Omega : X(\omega) = x\}}_{A_x}\big).$$

$A_x \subset \Omega.$

$A_x \in \mathcal{F}.$

$A_x, A_{x'}$ are disjoint

additivity axiom & the fact that $\mathcal{F}$ is closed under union of countable # of sets

$$= P\Big(\bigcup_{x \in \mathcal{X}_e} A_x\Big).$$

$$= P\big(\{\omega \in \Omega : X(\omega) \in \mathcal{X}_e\}\big)$$

$$= P(\Omega) = 1.$$

Function of random variable: $Y = g(X)$ is a R.V if $\forall x \in \mathbb{R}:$ $Y^{-1}((-\infty, x]) \in \mathcal{F}.$ We'll assume the function of RVs are well defined. (i.e., all events we are interested in are part of $\mathcal{F}$).

Probability mass function of R.V. Y.

$$P_Y(y) =. P\big(\underbrace{\{\omega \in \Omega \mid Y(\omega) = y\}}_{A_y}\big).$$

all values y can take and y is countable

$$=. P\big(\{\omega \in \Omega \mid g(X(\omega)) = y\}\big)$$

$$= \sum_{x \in \mathcal{X}_e} P\big(\underbrace{\{\omega \in \Omega \mid g(\cancel{X}) = y, X(\omega) = x\}}_{A_{y,x}}\big)$$

$$A_y = \bigcup_{x \in \mathcal{X}_e} A_{y,x}$$

$$= \sum_{x \in \mathcal{X}_e;\ g(x) = y} P\big(\{\omega \in \Omega \mid X(\omega) = x\}\big)$$

$$\boxed{P_Y(y) = \sum_{x \in \mathcal{X}_e :\ g(x) = y} P_X(x)}$$