EE2100: Matrix Analysis

Review Notes - 29

Topics covered :

1. Characteristic Polynomial of a Matrix
2. Eigen Values and Eigen Vectors of a Real Symmetric Matrix

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1. Characteristic Polynomial of a Matrix: Let $\mathbf{A} \in \mathcal{R}^{n\times n}$. The characteristic polynomial of $\mathbf{A}$ (commonly denoted by $P_{\mathbf{A}}(\lambda)$) is defined as

$$P_{\mathbf{A}}(\lambda) = \mathbf{det}(\lambda\mathbf{I} - \mathbf{A}) \tag{1}$$

It is straight forward to note that the roots of the characteristic polynomial $P_{\mathbf{A}}(\lambda) = 0$ are the Eigen values of the matrix $\mathbf{A}$. Given the nature in which the determinant is defined, it can be easily shown that the order of the characteristic polynomial of $\mathbf{A}$ is n. In general the characteristic polynomial associated with a $n \times n$ matrix can be written as

$$P_{\mathbf{A}}(\lambda) = \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0 \tag{2}$$

2. Since the order of the characteristic polynomial of a $n \times n$ matrix is n, there are n values of λ (some of which can be repeated) that satisfy $P_{\mathbf{A}}(\lambda) = 0$). Consequently, the number of Eigen values of a $n \times n$ matrix is n.

3. The Eigen values of a matrix are distinct if its characteristic polynomial has no repeated roots. If the roots of the characteristic polynomial are repeated, the number of distinct Eigen values of the matrix is less than n.

4. We say that the Eigen value of a matrix has algebraic multiplicity m, if λ_i is a root of the characteristic polynomial that is repeated m times.

5. The product of the Eigen Values (including the repeated ones) is the determinant of the matrix.

   Proof: Without any loss of generality, assume that the Eigen values of $\mathbf{A} \in \mathcal{R}^{n\times n}$ are $\lambda_1, \cdots, \lambda_m, \underbrace{\lambda_{m+1}, \lambda_{m+1}, \cdots, \lambda_{m+1}}_{\text{repetitions}}$.

   Accordingly the characteristic polynomial of the matrix can be written as

$$P_{\mathbf{A}}(\lambda) = \mathbf{det}(\lambda\mathbf{I} - \mathbf{A}) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_m)(\lambda - \lambda_{m+1})^{n-m} \tag{3}$$

   Substituting $\lambda = 0$ in (3) results in

$$\begin{aligned} \mathbf{det}(-\mathbf{A}) &= (-1)^n\lambda_1\cdots\lambda_m\lambda_{m+1}^{n-m} \\ (-1)^n\mathbf{det}(\mathbf{A}) &= (-1)^n\lambda_1\cdots\lambda_m\lambda_{m+1}^{n-m} \\ \mathbf{det}(\mathbf{A}) &= \lambda_1\cdots\lambda_m\lambda_{m+1}^{n-m} \end{aligned} \tag{4}$$

6. The sum of the Eigen Values (including the repeated ones) is the Trace of the matrix. (can be proved using Vieta's Formula).

7. Some of the properties of the Eigen Values and Eigen Vectors (along with proof) are

(a) There is only one Eigen value associated with any Eigen Vector. As a consequence, any Eigen Vector can only be an element of one and only one Eigen space.

Proof: Let $\mathbf{v} \in \mathcal{R}^n$ be an Eigen vector of $\mathbf{A} \in \mathcal{R}^{n \times n}$. Since $\mathbf{v}$ is an Eigen Vector, it satisfies the equation $\mathbf{Av} = \lambda \mathbf{v}$. Since the matrix vector product is defined uniquely, it is not possible to have $\mathbf{Av} = \lambda_1 \mathbf{v}$ and $\mathbf{Av} = \lambda_2 \mathbf{v}$. Hence the Eigen value associated with an Eigen Vector is always unique.

(b) The Eigen Vectors corresponding to distinct Eigen Values are linearly independent.

Proof: Let λ_1 and λ_2 denote two distinct Eigen values (i.e., $\lambda_1 \neq \lambda_2$) of $\mathbf{A}$, with the corresponding Eigen Vectors being $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Accordingly, $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy

$$\begin{aligned} \mathbf{Av}_1 &= \lambda_1 \mathbf{v}_1 \\ \mathbf{Av}_2 &= \lambda_2 \mathbf{v}_2 \end{aligned} \tag{5}$$

To prove by contradiction, assume that $\mathbf{v}_2$ is linearly dependent on $\mathbf{v}_1$ i.e.,

$$\mathbf{v}_2 = \alpha \mathbf{v}_1 \tag{6}$$

Notice that the coefficient α is unique. Pre-multiplying (6) with $\mathbf{A}$ results in

$$\begin{aligned} \mathbf{Av}_2 &= \mathbf{A}\alpha \mathbf{v}_1 \\ \lambda_2 \mathbf{v}_2 &= \alpha \lambda_1 \mathbf{v}_1 \\ \mathbf{v}_2 &= \alpha \frac{\lambda_1}{\lambda_2} \mathbf{v}_1 \end{aligned} \tag{7}$$

Since α is unique or stated alternatively, for (7) and (6) to be consistent, $\lambda_1 = \lambda_2$. This contradicts the distinct nature of Eigen Values.

(c) If λ is a complex Eigen Value of $\mathbf{A} \in \mathcal{R}^{n \times n}$ with the Eigen Vector $\mathbf{v}$, than, λ^* is also an Eigen Value of the matrix with Eigen Vector $\mathbf{v}^*$.

Proof: Since λ is a complex Eigen Value of $\mathbf{A} \in \mathcal{R}^{n \times n}$ with the Eigen Vector $\mathbf{v}$, it satisfies

$$\mathbf{Av} = \lambda \mathbf{v} \tag{8}$$

Taking complex conjugate on both sides of (8) results in

$$\mathbf{A}^* \mathbf{v}^* = \lambda^* \mathbf{v}^* \implies \mathbf{A} \mathbf{v}^* = \lambda^* \mathbf{v}^* \text{ since } \mathbf{A} \text{ is real} \tag{9}$$

Equation (9) indicates that λ^* is also an Eigen Value of $\mathbf{A}$ with Eigen Vector $\mathbf{v}^*$.

(d) If $\mathbf{A}$ is a Real Symmetric Matrix (i.e., $\mathbf{A} \in \mathcal{R}^{n \times n}$, with $\mathbf{A}^T = \mathbf{A}$), then all Eigen Values of $\mathbf{A}$ are Real.

Proof: Let λ be an Eigen value of $\mathbf{A}$ with associated Eigen vector of $\mathbf{x}$. Note that there is no assumption on the nature (i.e., whether it has real/complex entries) of λ and $\mathbf{x}$. Accordingly,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{10}$$

The norm of the vector $\mathbf{A}\mathbf{x}$, which is a real number, can be computed as (allowing for the possibility of $\mathbf{x}$ to be a complex vector)

$$\begin{aligned}
\|\mathbf{A}\mathbf{x}\|_2^2 &= \left((\mathbf{A}\mathbf{x})^*\right)^T \mathbf{A}\mathbf{x} \\
&= (\mathbf{x}^*)^T \mathbf{A}^T\mathbf{A}\mathbf{x} \\
&= (\mathbf{x}^*)^T \mathbf{A}^2\mathbf{x} \text{ (since } \mathbf{A}^T = \mathbf{A}) \\
&= (\mathbf{x}^*)^T \lambda^2\mathbf{x} \text{ (since } \mathbf{A}\mathbf{x} = \lambda\mathbf{x}) \\
&= \lambda^2\|\mathbf{x}\|_2^2 \\
\implies \lambda^2 &= \frac{\|\mathbf{A}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2}
\end{aligned} \tag{11}$$

Equation 11 indicates that λ^2 (which is the ratio to square of the norms of two vectors) is always real and positive. As a result, λ is always a real value.

(e) If $\mathbf{A}$ is a Real Symmetric Matrix, then the Eigen vectors corresponding to distinct Eigen values are orthogonal.

Proof: Let λ_1 and λ_2 denote two distinct Eigen values (i.e., $\lambda_1 \neq \lambda_2$) of $\mathbf{A}$, with the corresponding Eigen Vectors being $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. Accordingly, $\mathbf{v}_1$ and $\mathbf{v}_2$ satisfy

$$\begin{aligned}
\mathbf{A}\mathbf{v}_1 &= \lambda_1\mathbf{v}_1 \\
\mathbf{A}\mathbf{v}_2 &= \lambda_2\mathbf{v}_2
\end{aligned} \tag{12}$$

Applying the inner product with $\mathbf{v}_1$ on both sides of $\mathbf{A}\mathbf{v}_2 = \lambda_2\mathbf{v}_2$ results in

$$\begin{aligned}
\mathbf{v}_2^T\mathbf{A}^T\mathbf{v}_1 &= \lambda_2\mathbf{v}_2^T\mathbf{v}_1 \\
\mathbf{v}_2^T\mathbf{A}\mathbf{v}_1 &= \lambda_2\mathbf{v}_2^T\mathbf{v}_1 \text{ (since } \mathbf{A}^T = \mathbf{A}) \\
\lambda_1\mathbf{v}_2^T\mathbf{v}_1 &= \lambda_2\mathbf{v}_2^T\mathbf{v}_1 \text{ (since } \mathbf{A}\mathbf{v}_1 = \lambda_1\mathbf{v}_1) \\
\implies \mathbf{v}_2^T\mathbf{v}_1 &= 0 \text{ (since } \lambda_1 \neq \lambda_2)
\end{aligned} \tag{13}$$

Equation 13 indicates that $\mathbf{v}_1$ is orthogonal to $\mathbf{v}_2$.