

EE1080/AI1110/EE2102 Probability: HW2

28th Jan 2025

Assume discrete random variables throughout.

1. The sample space Ω of an experiment consists of all 8-dimensional binary vectors, e.g., every member of Ω has the form $\omega = (\omega_0, \dots, \omega_7)$, where ω_i is 0 or 1. The probability law P assigns a probability of $1/2^8$ to each of the 2^8 elements in Ω .

Find the pmfs describing the following random variables:

- (a) $W(\omega)$ = the number of 1's in the binary vector ω .
 - (b) $Z(\omega) = \max(\omega_i)$
2. Gukesh and Pragya play a chess match in which first player to win a game wins the match. After 10 successive draws, the match is declared drawn. Each game is won by Pragya with probability 0.4, is won by Gukesh with probability 0.3 and declared a draw with probability 0.3, independent of the previous game.
 - (a) What is the probability that Pragya wins the match ?
 - (b) What is the PMF of the duration of the match ?
 3. Consider a hierarchical storage system composed of n subsystems, each comprising k servers. Each subsystem can tolerate a single server failure, and the overall system can tolerate a single subsystem failure. Thus, in order for the overall system to fail, there has to be at least two subsystems that each have at least two server failures. Suppose servers fail independently with probability p .
 - (a) Let X be the number of servers that failed in a particular subsystem. Find the pmf of X in terms of k and p . What is the probability p_0 that a particular subsystem fails?
 - (b) Let Y be the number of subsystems that failed. Find the pmf of Y in terms of n and p_0 . What is the probability p_1 that the overall system fails?
 - (c) Give upper bounds on p_0 and p_1 using the union of events bound (Let A_{ij} be the event both server i and j fail for $1 \leq i < j \leq k$ and use the fact that $X \geq 2 = \cup_{1 \leq i < j \leq k} A_{ij}$).

4. Binary Symmetric Channel: Repetition Coding: A communications channel transmits the digits 0 and 1. However, due to static, the digit transmitted is incorrectly received with probability 0.2. Suppose that we want to transmit an important message consisting of one binary digit. To reduce the chance of error, we transmit 00000 instead of 0 and 11111 instead of 1. If the receiver of the message uses "majority" decoding, what is the probability that the message will be wrong when decoded? What independence assumptions are you making?
5. Binomial and Poisson PMFs: Show that the PMF $P_X(x)$ of Binomial (n, p) is monotonically non-decreasing as long as $x \leq (n+1)p$ and is monotonically decreasing for $x \geq (n+1)p$. Similarly Poisson PMF $P_Y(y)$ is non-decreasing for $y \leq \lambda$ and decreasing for $y > \lambda$
6. A total of 4 buses carrying 148 students from the same school arrives at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus.
 - (a) Which of $E[X]$ or $E[Y]$ do you think is larger? Why?
 - (b) Compute $E[X]$ and $E[Y]$
 - (c) Find $Var(X)$ and $Var(Y)$
7. If $E[X] = 1$ and $Var(X) = 5$, find $E[(2+X)^2]$ and $Var(4+3X)$.
8. Let X be a non-negative random variable ie., $P(X \geq 0) = 1$. Show that

$$E[X] = \sum_{k=0}^{\infty} P(X > k)$$

Similarly show that for any random variable X :

$$E[X] = \sum_{k=0}^{\infty} P(X > k) - \sum_{k=0}^{\infty} P(X < -k)$$

9. A product that is sold seasonally yields a net profit of b dollars for each unit sold and a net loss of l dollars for each unit left unsold when the season ends. The number of units of the product that are ordered at a specific department store during any season is a random variable having probability mass function $p(i)$, $i \geq 0$. If the store must stock this product in advance, determine the number of units the store should stock so as to maximize its expected profit.
10. Quiz Problem: In an examination a student can choose the order in which two questions(Ques A and Ques B) must be attempted.

- (a) If first question is answered wrong, the student gets zero marks.
- (b) If the first question is answered correctly and the second question is not answered correctly, the student gets the marks only for the first question.
- (c) If both questions are answered correctly, the student gets the sum of the marks of the two questions.

The probability of answering Question A and B correctly are 0.8 and 0.5 respectively. And the marks for each question A and B are 10 and 20 respectively. Assuming that the student always wants to maximize her expected marks in the examination, in which order should she attempt the questions and what is the expected marks for that order.

11. Extension of Quiz problem: Consider there are n questions to be answered. Question i will be answered correctly with probability p_i and the person will receive a reward v_i on correctly answering the question. The Quiz terminates on a wrong answer. The problem here is to choose the best ordering for the answers. Show that it is optimal to answer questions in non-decreasing order of $p_i v_i / (1 - p_i)$
12. Minimum of random variables:
 - (a) Consider you have three identical, independent random variables X_1, X_2, X_3 with cumulative distribution function $F_{X_1}(\cdot)$ and $Y = \min(X_1, X_2, X_3)$. Show that $F_Y(y) = 1 - (1 - F_{X_1}(y))^3$.
 - (b) On a given day, your golf score takes values in 101 to 110 with probability 0.1, independent of the other days. Determined to improve your score, you decide to play on three different days and declare as your score the score the minimum Y , of scores X_1, X_2, X_3 on the different days.
 - i. Calculate the PMF of Y
 - ii. By how much has your expected score improved as a result of playing on these days ?
13. Poisson R.V:
 - (a) (Combining) Let X, Y be Poisson random variables with means λ_1, λ_2 respectively. Find the PMF of random variable $Z = X + Y$.
 - (b) (Splitting) A transmitter sends out either a 1 with probability p or 0 with probability $1 - p$ independent of earlier transmissions. The number of packets transmitted in a slot has a Poisson PMF with parameter λ . Show that the number of 1's transmitted in a slot has a Poisson PMF with parameter $n\lambda$
14. Tossing a coin till a pattern appears: What is the PMF of number of tosses, expected number of tosses, variance for the following scenarios ?

- (a) Toss a coin until you see two heads.
- (b) Toss a coin until you see two successive heads.
- (c) Toss a coin until you see two successive heads or tails.