.

EE1101: Circuits and Network Analysis

Lecture 27: Second-Order Circuits

October 7, 2025

Topics :

1. Solution of Second-Order Homogeneous Differential Equations
2. DC response of Second-Order Circuits

Solution of Second-Order Homogeneous Differential Equations

goal: Solve $\frac{d^2x}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = \underline{0}$ given $x(t_0) = x_0$ and $\left.\frac{dx}{dt}\right|_{t_0} = v_0$

$(\xi > 0)$

fundamental set of solutions (for n^{th} order DE) = $\{x_1(t), \dots x_n(t)\}$ that are linearly independent

for a 2^{nd} order DE, find two functions $x_1(t)$ & $x_2(t)$ that are linearly independent & satisfy the DE

gen sol: $c_1 x_1(t) + c_2 x_2(t)$ where c_1 & c_2 are determined based on initial conditions

How to compute x_1 & x_2 (linearly independent)

x_1 & x_2 are of the form $\underline{e^{st}}$ $s \in \mathbb{C}$ (can be)

Req ①: must satisfy the DE $x = e^{st}$ $\Rightarrow$ $s^2 e^{st} + 2\xi\omega_n s e^{st} + \omega_n^2 e^{st} = 0$

$\Rightarrow (s^2 + 2\xi\omega_n s + \omega_n^2)\, e^{st} = 0 \quad \forall t$

$\Rightarrow s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ → characteristic Eqn.

↓ 2 roots

Solution of Second-Order Homogeneous Differential Equations (contd.)

2 roots are $s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$

a) when $\xi > 1$ (overdamped)

roots of the characteristic eqn are real & distinct

$s_1 = -\xi\omega_n + \omega_n\sqrt{\xi^2-1} \in \mathbb{R}$

$s_2 = -\xi\omega_n - \omega_n\sqrt{\xi^2-1} \in \mathbb{R}$

$\underbrace{x_1(t) = e^{s_1 t} \text{ and } x_2(t) = e^{s_2 t}}_{\text{fundamental set of sol.}}$ satisfy the DE and they are LI.

gen sol $= c_1 x_1(t) + c_2 x_2(t)$

b) when $\xi = 1$ (critical damping)

roots of the characteristic eqn are real & repeat

$s_1, s_2 = -\omega_n$

$x_1(t) = e^{s_1 t}$

$x_2(t) = t e^{s_1 t}$ $\longrightarrow$ Show that x_2 satisfies the DE

gen sol $= c_1 e^{s_1 t} + c_2 t e^{s_1 t}$

c) when $0 < \xi < 1$ (underdamped system)

roots are complex & conjugate of each other

$s_1 = -\xi\omega_n + j\omega_d$ where $\omega_d = \omega_n\sqrt{1-\xi^2}$ $\hookrightarrow$ damped freq.

$s_2 = -\xi\omega_n - j\omega_d$

$s_1 = s_2^*$

Solution of Second-Order Homogeneous Differential Equations (contd.)

when $0 < \xi < 1$ (underdamped system)

$\downarrow$

gen sol $= 2r\, e^{-\xi\omega_n t}\cos(\omega_d t + \theta)$

gen sol: $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$

$\because s_1 = s_2^*$, for $x(t)$ to be real $c_1 = c_2^*$

Proof :- $x(t) = c_1 e^{-\xi\omega_n t}\underbrace{e^{j\omega_d t}}_{\in \mathbb{C}} + c_2 e^{-\xi\omega_n t}\underbrace{e^{-j\omega_d t}}_{\in \mathbb{C}}$

$= e^{-\xi\omega_n t}\left(c_1 e^{j\omega_d t} + \underbrace{c_2}_{c_1^*} e^{-j\omega_d t}\right)$

when $c_1 = c_2^* \Rightarrow$

$x(t) = e^{-\xi\omega_n t}\, Re\{2c\, e^{j\omega_d t}\}$

Let $c = \underline{\underline{r e^{j\theta}}}$ where r & θ need to be determined

$x(t) = e^{-\xi\omega_n t}\, Re\{2r\, e^{j(\omega_d t + \theta)}\}$

gen sol $= 2r\, e^{-\xi\omega_n t}\cos(\omega_d t + \theta)$ where r & θ are determined based on initial cond.

when $\xi = 0$ (undamped system)

$s_1 = j\omega_n$

$s_2 = -j\omega_n$

roots are complex & conjugates of each other

gen sol : $x(t) = c_1 e^{j\omega_n t} + c_2 e^{-j\omega_n t}$

for $x(t)$ to be real $c_1 = c_2^*$

~~Example~~

when $\xi = 0$ (undamped system)

$$x(t) = c_1 e^{j\omega_n t} + c_2 e^{-j\omega_n t}$$

for $x(t)$ to be real $\quad c_1 = c_2^* \Rightarrow x(t) = c\, e^{j\omega_n t} + c^* e^{-j\omega_n t}$

$$= Re\{2c\, e^{j\omega_n t}\}$$

if $c = r e^{j\theta} \Rightarrow x(t) = 2r \cos(\omega_n t + \theta)$

Note that for circuits, $\xi \geq 0$ & hence

$x(t) \longrightarrow 0$ as $t \longrightarrow \infty$ for all scenarios

except when $\xi = 0$.

Steps involved in solving 2nd order ckts: (when the DE is homogeneous)

a) Determine the governing DE and initial conditions

b) based on the value of ξ, pick appropriate general solution & form the gen. sol.

c) evaluate the parameters of the gen sol based on initial conditions