

- ① Lagrange multipliers: Constrained optimization.
- ② Divergence of a vector field

## EE1203: Vector Calculus

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భారతీయ సౌండేటిక విజ్ఞాన పంచ మైదానాల్  
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## Constrained optimization: Lagrange multipliers.

- \* We saw how to find max, min of functions without any constraints on the variables.
  - \* How about having constraints on the variables.

Example 1: For a rectangle whose perimeter is 20 m, find the dimension that will maximize the area?

## Solutions!

$$\text{Maximize } F(x, y) = xy$$

Given  $2x+2y = 20$ .

$$\begin{aligned} \text{width} &= x \\ \text{height} &= y \\ \text{Area: } A &= xy \\ \text{Perimeter} &= \end{aligned}$$

Using single variable calculus;

$$2x+2y = 20 \rightarrow \text{interval } y \Rightarrow [0, 10]$$

$$x = 10 - y$$

$$F(y) = y(10-y) = 10y - y^2$$

$$\begin{aligned} F'(y) &= 10 - 2y = 0 \\ \Rightarrow y &= 5 \quad ; \quad x = 10 - 5 = 5 \end{aligned}$$

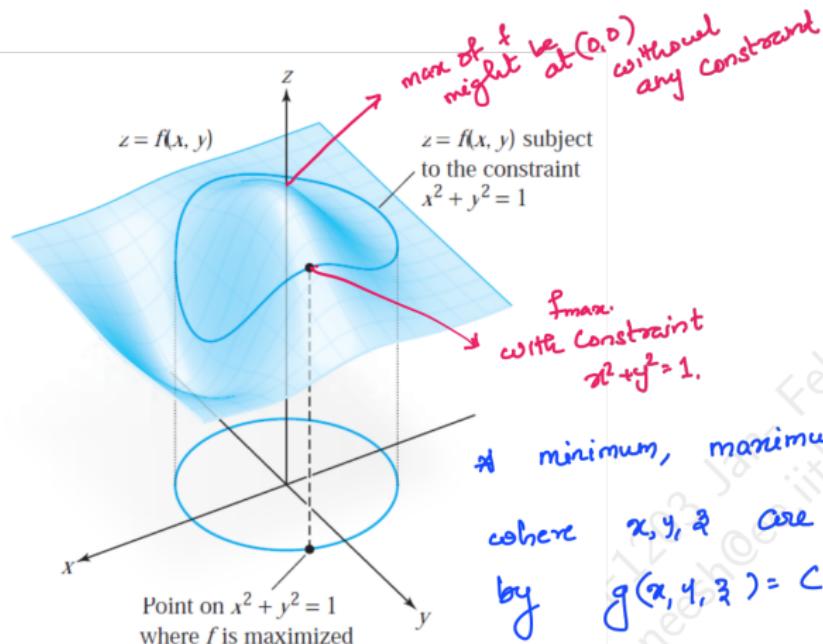
At  $y=5$ ;  $F''(y) = -2 < 0 \Rightarrow y=5$  is local maxima.

$\Rightarrow$  it is global maximum on the interval  $[0, 10]$

$\therefore$  The maximum area occurs for rectangle with

$$x=5m; y=5m$$

Example 2: The geometric meaning of maximizing  $f(x,y)$   
 Subject to the constraint  $x^2+y^2=1$ .

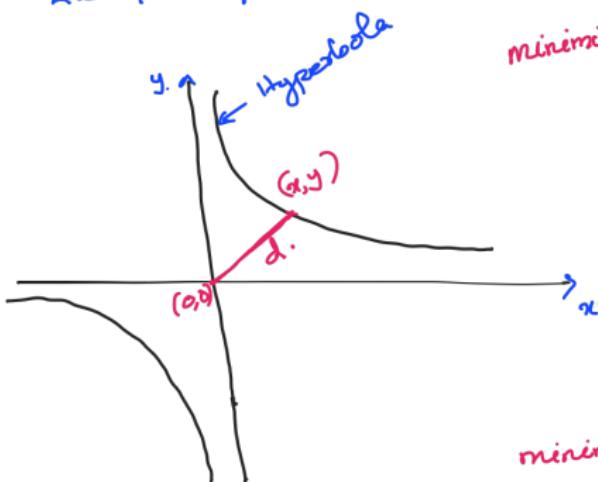


Ref: Vector Calculus (6<sup>th</sup> Ed)

Jerrold E Marsden  
Anthony Tromba

\* First observation: We cannot use our usual method of critical point, as critical point of ' $f$ ' may not satisfy the given constrained condition.  $g = c$

Example: point closest to origin on hyperbola  $xy = 3$ .



minimize distance of point  $(x,y)$  to the origin gives  $xy=3$ . ie; point  $(x,y)$  on  $xy=3$ , which is nearest to origin.

$$d = \sqrt{x^2 + y^2}.$$

minimizing ' $d$ ' is equivalent to minimizing  $d^2 = x^2 + y^2$ .

Problem here is;

$$\Rightarrow \text{minimize } f(x,y) = x^2 + y^2.$$

Given  $g(x,y) = c$   
here " $xy = 3$ ".

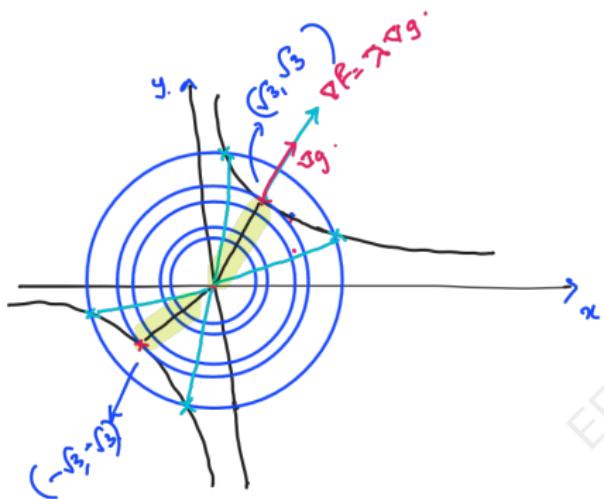
### Observations:

- \* The smallest circle that hit the hyperbola is the tangent circle to hyperbola!

- \* This is the minimum of  $f = x^2 + y^2$ ;  
which has solution on Hyperbola.

- \* below this value of ' $f'$  no solution satisfying  $xy = 3$ .

- \* It is the minimum of ' $f'$ ; gives hyperbola as boundary or constraint.



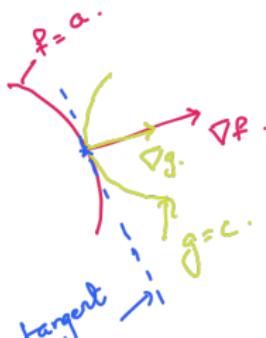
\* When the function is minimum,  
the level curve of ' $f$ ' is tangent

the level curve of  $f$  is tangent to the hyperbola. (In general, at minimum of function  $f$  the level curve of ' $f$ ' will be tangent to the given constraints)

How to find  $(x, y)$  where level curves of  $f(x,y)$  &  $g(x,y)$  are tangent to each other?

\* two level curves are tangent  $\Rightarrow$  they have same tangent line.

$\Rightarrow$  Their normal vectors should be parallel.



$\Rightarrow$  \* Gradients of both the functions should be parallel to each other.

$$\nabla f \parallel^l \nabla g$$

$$\Rightarrow \nabla f = \lambda \nabla g$$

↳ Lagrange multiplier.

$$\begin{aligned} \Rightarrow f_x &= \lambda g_x \\ f_y &= \lambda g_y. \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \text{unknowns } x, y \text{ & } \lambda.$$

Constraint:  $xy = 3$   
 $\Rightarrow g = 3$ .

$$\begin{aligned} f &= x^2 + y^2 \\ g &= xy. \end{aligned} \quad \left. \begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = c \end{array} \right\} \begin{aligned} \Rightarrow 2x &= \lambda y \quad \dots (1) \\ 2y &= \lambda x \quad \dots (2) \\ \Rightarrow xy &= 3 \quad \dots (3) \end{aligned}$$

Three slvs of equations <sup>(1)</sup>  
 3 variables.

$$\begin{aligned} 2x - \lambda y &= 0 \\ \lambda x - 2y &= 0 \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \Rightarrow \begin{bmatrix} 2 & -\lambda \\ \lambda & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda y = 3$       Trivial solution  $(0, 0)$  not satisfied

$$\begin{aligned} \Rightarrow \lambda y &= 3 \\ \Rightarrow 0 &\neq 3. \end{aligned}$$

Other solutions exist when  $\begin{vmatrix} 2 & -\lambda \\ \lambda & -2 \end{vmatrix} = 0$

$$\Rightarrow -4 + \lambda^2 = 0 \Rightarrow \lambda = \pm 2.$$

$$\lambda = +2 \Rightarrow x = 4$$

$$y = x.$$

$$\Rightarrow xy = 3$$

$$\Rightarrow x^2 = 3 \Rightarrow x = \pm \sqrt{3}$$

$(\sqrt{3}, \sqrt{3})$  or  $(-\sqrt{3}, -\sqrt{3})$

$$\underline{\lambda = -2}.$$

$$\Rightarrow x = -y.$$

$$xy = 3$$

$$\Rightarrow -x^2 = 3$$

→ no solutions.

Why this Lagrange multiplier method  
 $\nabla f = \lambda \nabla g$  ← valid for optimization with constraints?

\* At constrained min/max in any direction along level  $g=c$ ; the rate of change of ' $f$ ' should be zero.

Let  $\hat{u}$  be any tangent vector to  $g=c$  (constraint)

$\Rightarrow$  By directional derivative concept

rate of change of function  $f$  along  $\hat{u}$

$$\Rightarrow \nabla f \cdot \hat{u}$$

But here rate of change of  $f$  should be zero.

$\Rightarrow \nabla f \perp^r$  to  $\hat{u}$  (tangent vector in  $g=c$ )

$\Rightarrow \nabla f \perp^r$  to level set ' $g=c$ '

$\nabla g$  also  $\perp^r$  to level set ' $g=c$ '

$$\Rightarrow \boxed{\nabla f = \lambda \nabla g}$$

- \* Drawback: It does not tell whether a solution is maxima / minima.
- \* Check all solutions of Lagrange multiplier equations & find min/max from the corresponding values of functions at each of the solutions!

Divergence operator: ( $\vec{\nabla} \cdot \vec{f}$ )

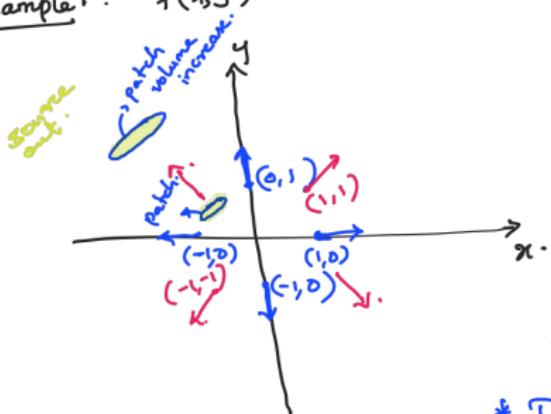
$$\vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \quad (\text{Del operator})$$

$$\vec{f} = (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)) \quad (\text{vector function.})$$

$$\text{div}(\vec{f}) = \vec{\nabla} \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \quad \rightarrow \text{Scalar function.}$$

Physical meaning:  $\operatorname{div}(\vec{f})$  tells how much this vector field  $\vec{f}$  locally sourcing out or sinking in.  
 if  $\operatorname{div}(\vec{f}) = 0 \Rightarrow$  net in/out flow is zero.

Example 1:  $\bar{f}(x,y) = x^{\hat{i}} + y^{\hat{j}}$



\* Intuition is  
the divergence of  $\vec{F}$   
should be +ve ; as  
the field values souring  
out from origin.

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 2$$

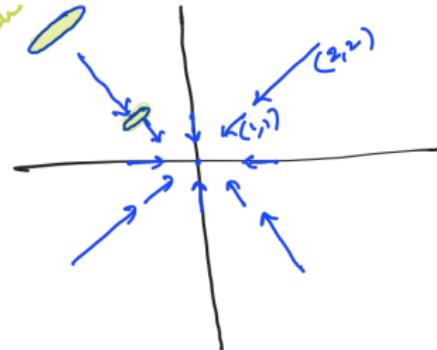
(tve)

\* Divergence +ve  
⇒ Volume grows/expands.

Example 2.

$$\vec{f}(x, y) = -x\hat{i} - y\hat{j}$$

patch sink in.



$$\nabla \cdot \vec{f} = -1 - 1 = -2 < 0.$$

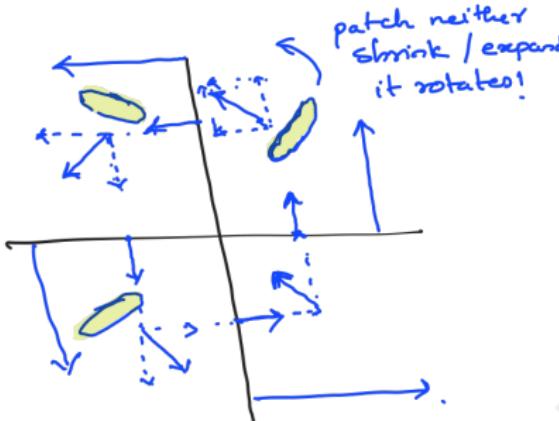
$$\text{div}(\vec{f}) \Rightarrow -\text{ve.}$$

→ Here things are  
sinking is

or flow is converging.

Divergence ne  
⇒ Volume  
shrink.

Example 3:  $\vec{f}(x,y) = -y\hat{i} + x\hat{j}$



$x$	$y$	$\vec{f}$
1	0	$\hat{i}$
0	1	$\hat{j}$
-1	0	$-\hat{j}$
0	-1	$-\hat{i}$
1	1	$\hat{i} + \hat{j}$
-1	1	$-\hat{i} + \hat{j}$
-1	-1	$-\hat{i} - \hat{j}$
1	-1	$\hat{i} - \hat{j}$

$$\nabla \cdot \vec{f} = \frac{\partial(-y)}{\partial x} + \frac{\partial(x)}{\partial y} = 0 \Rightarrow \text{Divergence Free vector field}$$

- \* Characterized by rotation!
- \* All rotational field should satisfy  $\nabla \cdot \vec{f} = 0$   
But it is not sufficient condition for rotation nature of vector fields.

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