$X_1, X_2, \ldots X_n$ are iid

$$\hat{S}_n = \frac{(X_1 + \cdots + X_n) - \sum_{i=1}^{n} E[X_i]}{\sqrt{n \, Var(X_i)}}$$

$$\left.\begin{aligned} E[X_i] &= E[X_1] \quad \forall i \\ Var[X_i] &= Var(X_1) \end{aligned}\right\} \text{ because of iid condition}$$

$$E[\hat{S}_n] = \frac{E\left[\sum_{i=1}^{n} X_i\right] - \sum_{i=1}^{n} E[X_i]}{\sqrt{n Var(X_1)}} = 0.$$

$E[aX+b] = aE[X]+b$

$$Var[\hat{S}_n] = \frac{Var\left(\sum_{i=1}^{n} X_i\right)}{n \, Var(X_1)}$$

$Var[aX+b] = a^2 Var(X)$

$$= \frac{\sum_{i=1}^{n} Var(X_i)}{n \, Var(X_1)} = \frac{n \, Var(X_1)}{n \, Var(X_1)} = 1.$$

$$P(\hat{S}_n \leq s) \xrightarrow[\text{as } n \to \infty]{} \Phi(s)$$

$$P(a \leq \hat{S}_n \leq b) \approx \Phi(b) - \Phi(a).$$

Remark: Can use CDF of standard normal random variable to approximate CDF of $\hat{S}_n$.

Suppose $X_1, \ldots X_n$ are Bernoulli

$$P\left(\sum_{i=1}^{n} X_i \geq m\right) = P\left(\frac{\sum_{i=1}^{n} X_i - nE[X_1]}{\sqrt{n \, Var(X_1)}} \geq \frac{m - nE[X_1]}{\sqrt{n Var(X_1)}}\right)$$

$$\approx 1 - \Phi\left(\frac{m - nE[X_1]}{\sqrt{n Var(X_1)}}\right).$$

$$P\left(m_2 \leq \sum_{i=1}^{n} X_i \leq m_1\right) \approx \Phi\left(\frac{m_1 - nE[X_1]}{\sqrt{n Var(X_1)}}\right) - \Phi\left(\frac{m_2 - nE[X_1]}{\sqrt{n Var(X_1)}}\right)$$

Dec 17: 14th Feb.

① Joint CDF : $F_{X,Y}(x,y) = P(X \le x, Y \le y)$

ⓐ Properties of joint CDF.

① $\lim_{x \to \infty} F_{X,Y}(x,y) = F_Y(y)$

$\lim_{y \to \infty} F_{X,Y}(x,y) = F_X(x)$

$\lim_{x \to \infty} \lim_{y \to \infty} F_{X,Y}(x,y) = \lim_{x \to \infty} F_X(x) = 1.$

② $\lim_{x \to -\infty} F_{X,Y}(x,y) = 0$

$\lim_{y \to -\infty} F_{X,Y}(x,y) = 0.$

③ Monotonic property. If $x_1 \le x_2$, $y_1 \le y_2$

$F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2).$

④ Right continuity:

$$\lim_{\substack{\Delta x \to 0^+ \\ \Delta y \to 0^+}} F_{X,Y}(x+\Delta x, y+\Delta y) = F_{X,Y}(x,y).$$

⑤. $P(x_1 < X \le x_2, \; y_1 < Y \le y_2).$

$$= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_2) + F_{X,Y}(x_1,y_1).$$

③ Proof

Monotonic property: $x_1 \le x_2$ and $y_1 \le y_2$.

$$F_{X,Y}(x_1, y_1) = P(\underbrace{\{X \le x_1\} \cap \{Y \le y_1\}}_{A_{x_1,y_1}}).$$

$$F_{X,Y}(x_2, y_2) = P(\underbrace{\{X \le x_2\} \cap \{Y \le y_2\}}_{A_{x_2,y_2}}).$$

If $\omega \in A_{x_1,y_1} \Rightarrow X(\omega) \le x_1$ & $Y(\omega) \le y_1$

If $X(\omega) \le x_1 \le x_2$.

$\Rightarrow X(\omega) \le x_2$

& Similarly $Y(\omega) \le y_2$.

$\Rightarrow \omega \in A_{x_2,y_2}$.

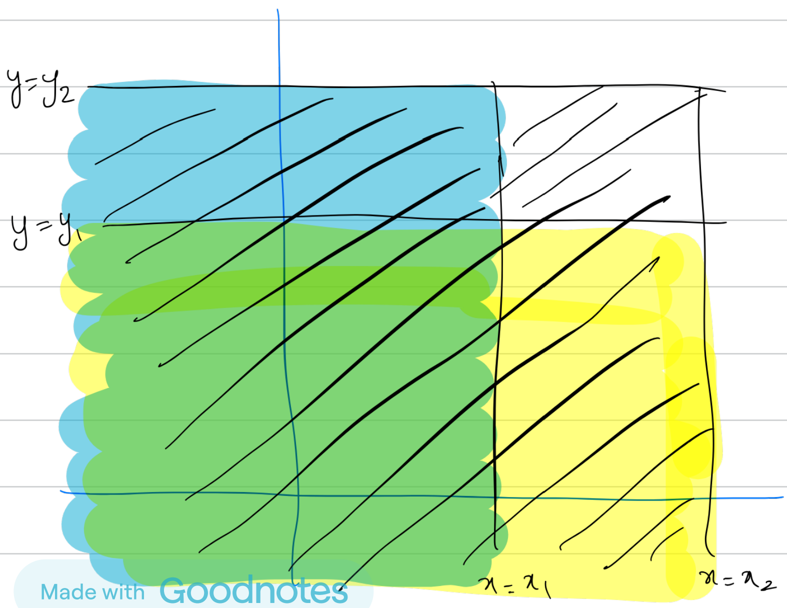
$A_{x_1,y_1} \subseteq A_{x_2,y_2}$

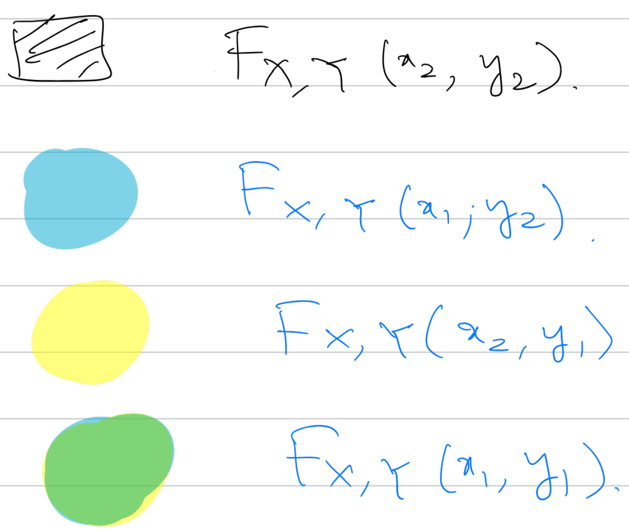
$\Rightarrow P(A_{x_1,y_1}) \le P(A_{x_2,y_2})$

$\Rightarrow F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$.

⑤ $P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$.

$$= F_{X,Y}(x_2, y_2) - F_{X,Y}(x_1, y_2) - F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_1).$$





1a $\lim_{x \to \infty} F_{X,Y}(x, y) = F_Y(y)$.

$F_{X,Y}(n, y) = P(X \leq n, Y \leq y)$.

$A_n = \{X \leq n \,\&\, Y \leq y\}$.

$A_1 \subseteq A_2 \subseteq A_3 \ldots$,

Using continuity corollary a we have

$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} P(A_n)$.

LHS.

$= \lim_{n \to \infty} F_{X,Y}(n, y)$

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \{X \leq n \,\&\, Y \leq y\}$.

$\bigcup_{i=1}^{\infty} (S_i \cap P)$
$= \left(\bigcup_{i=1}^{\infty} S_i\right) \cap P$.

as

$= \left[\underbrace{\bigcup_{n=1}^{\infty} \{X \leq n\}}_{\Omega}\right] \cap \{Y \leq y\}$

$= \Omega \cap \{Y \leq y\}$

$= \{Y \leq y\}$.

$\lim_{n \to \infty} F_{X,Y}(a, y) = P\left(\bigcup_{n=1}^{\infty} A_n\right) = F_Y(y)$.

Made with Goodnotes

② Joint PDF.

Given two random variables X, Y with Joint CDF $F_{X,Y}(x,y)$ such that

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv.$$

Then $f_{X,Y}(x,y)$ is the joint p.d.f of X, Y.

and it follows that $f_{X,Y}(x,y) = \dfrac{\partial^2 F_{X,Y}(x,y)}{\partial x\,\partial y}$.

2a Marginal PDF from Joint PDF.

$$F_X(x) = \lim_{y\to\infty} F_{X,Y}(x,y).$$

$$= \lim_{y\to\infty}\int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv$$

$$= \int_{-\infty}^{x} \underbrace{\left[\int_{-\infty}^{\infty} f_{X,Y}(u,v)\,dv\right]}_{f_X(u)} du.$$

Therefore by definition of p.d.f of X.

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx.$$

Feb 17 : lecture 18:

2c Probability of the rectangle region (like interval for two R.Vs). in terms of PDF.

$$P(x_1 < X \le x_2,\ y_1 < Y \le y_2)$$

$$= \underbrace{[F(x_2,y_2)^{①} - F(x_1,y_2)^{②}]}_{} - [F(x_2,y_1)^{③} - F(x_1,y_1)^{④}]$$

① : $\displaystyle\int_{-\infty}^{y_2}\int_{-\infty}^{x_2} f_{X,Y}(x,y)\,dx\,dy$

② $\displaystyle\int_{-\infty}^{y_2}\int_{-\infty}^{x_1} f_{X,Y}(x,y)\,dx\,dy.$

$$= \int_{-\infty}^{y_2}\left(\underbrace{\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx}_{①-②}\right)dy - \underbrace{\int_{-\infty}^{y_1}\int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy}_{③-④}$$

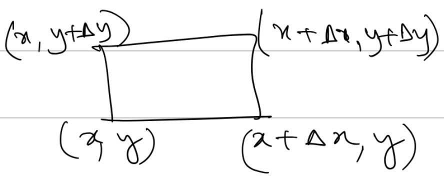
$$= \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{X,Y}(x,y)\,dx\,dy.$$

(2d) Probability of any subset B of $\mathbb{R}^2$.

$$P\big((X,Y) \in B\big) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy.$$

(2e) What does joint PDF mean?

$$P\big(x < X \le x+\Delta x,\ y < Y \le y+\Delta y\big)$$



$$= \int_{y}^{y+\Delta y} \int_{x}^{\Delta x + x} f_{X,Y}(u,v)\,du\,dv.$$

(function stays constant in that region) $\approx f_{X,Y}(x,y)\,\Delta x\,\Delta y.$

Joint PDF is
Probability per unit area at the vicinity of (x,y).

Remarks: When is $f_{X,Y}(x,y)$ valid?

$f_{X,Y}(x,y) \geq 0.$

Normalization $\lim_{y\to\infty} \lim_{x\to\infty} F_{X,Y}(x,y) = 1.$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

Made with Goodnotes

Eg Example

2D uniform R.V.

① $f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$

find c

$$\int_0^1 \int_0^1 f_{X,Y}(x,y)\, dx\, dy$$

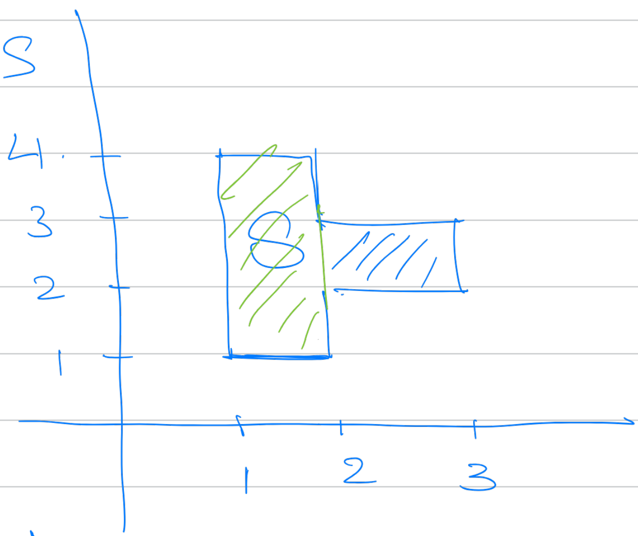
$$= c \int_0^1 \int_0^1 dx\, dy$$

$$= c = 1$$

$\Rightarrow c = 1.$

② Uniform in a region S.

$f_{X,Y}(x,y) = \begin{cases} c & (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$



Find c:

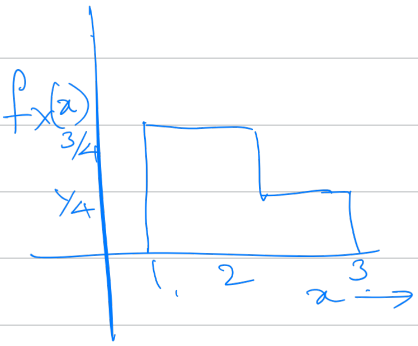
$$\int_1^4 \int_1^2 c\, dx\, dy + \int_2^3 \int_2^3 c\, dx\, dy.$$

$\Rightarrow c = 1/4.$

$= c \times 3 + c \times 1 = 1.$

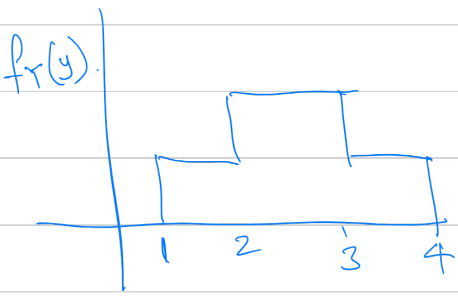
What is the marginal PDF of X, Y ?

$$f_X(x) = \int_{y=1}^{4} f_{X,Y}(x,y)$$



$$= \begin{cases} \int_{y=1}^{4} c = 3/4 & 1 \le x \le 2. \\ \int_{y=2}^{3} c = 1/4 & 2 \le x \le 3 \end{cases}$$

$$f_Y(y) = \int_{x=1}^{3} f_{X,Y}(x,y).$$



$$= \begin{cases} \int_{x=1}^{2} c\, dx = \frac{1}{4}. & 1 \le y \le 2. \text{ or } 3 \le y \le 4. \\ \int_{x=1}^{3} c\, dx = \frac{1}{2}. & 2 \le y \le 3 \end{cases}$$

Theorem 6.4 in Papoulli Pillai
Theorem 5.1 (Single R.V)

③ Expectation of function of two R.Vs

$$E[g(X,Y)] = \int_x \int_y g(x,y)\, f_{X,Y}(x,y)\, dx\, dy.$$

Linearity of Expectation

$$E[aX+bY+c] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by+c)\, f_{X,Y}(x,y)\, dx\, dy$$

$$= a\int_{-\infty}^{\infty} x \boxed{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy}\, dx + \cdots$$

$$= aE[X] + bE[Y] + c.$$

④ Conditional density w.r.t an event.

Conditional CDF w.r.t an event A.

$$F_X(x) = P(X \le x), \qquad F_{X|A}(x) = \frac{P(\{X \le x\} \cap A)}{P(A)}.$$

Now suppose event A : $\{X \in B\}$.

$$F_{X|A}(x) = \frac{P(\{X \le x\} \text{ and } \{X \in B\})}{P(X \in B)}$$

$$= \int_{t=-\infty}^{x} \underbrace{\frac{f_X(t) \cdot 1_{\{t \in B\}}}{P(X \in B)}}_{\text{Conditional PDF}} dt.$$

$$f_{X|\{X \in B\}}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in B)} & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases}$$

Lec 19 : Feb 18

⑤ Total Probability theorem

$$F_X(x) = P(X \le x).$$

Let $A_1, \ldots, A_n$ be the sets that partition Ω.

$$= \sum_{i=1}^{n} P(\{X \le x\} \cap A_i)$$

$$= \sum_{i=1}^{n} P(\{X \le x\} \mid A_i)\, P(A_i).$$

$$= \sum_{i=1}^{n} \int_{-\infty}^{x} f_{X|A_i}(t)\, dt \;\; P(A_i)$$

$$= \int_{-\infty}^{x} \left(\sum_{i=1}^{n} f_{X|A_i}(t)\, P(A_i) \right) dt$$

$$\Rightarrow \quad f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\, P(A_i). \quad \{\text{by defn of p.d.f of } X\}.$$

5a) **Example:** A metro train arrives once every 15 mins starting at 6 am. You are equally likely to walk in to station at any time between 7.10 am – 7.30 am with time interval being uniform R.V. What is the pdf of your wait time before first train arrives?

X : arrival time of the person.

$X \sim \text{Uniform}(7.10\text{am}, 7.30\text{am})$

A_1: $X \le 7.15$ am.

$P(A_1) = 5/20 = 1/4.$

$A_2 = A_1^c$

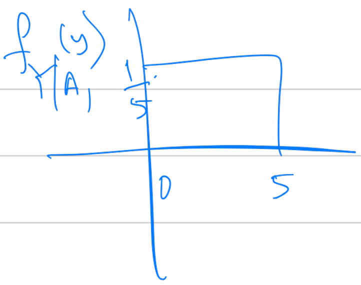
$P(A_2) = 3/4.$

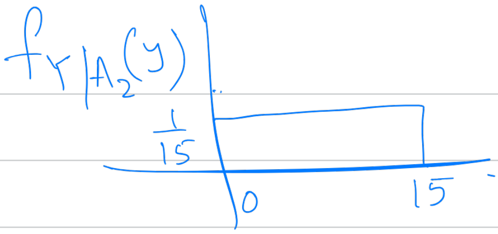
Y is the R.V representing the wait time.

$Y|A_1 = 7.15 - X. \sim \text{Uniform}(0, 5).$

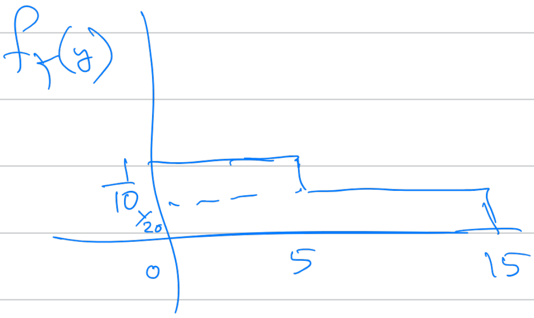
$Y|A_2 = 7.30 - X. \sim \text{Uniform}(0, 15).$

$$f_Y(y) = P(A_1)\, f_{Y|A_1}(y) + P(A_2)\, f_{Y|A_2}(y)$$





$$= \begin{cases} \frac{1}{4} \times \frac{1}{5} + \frac{1}{\cancel{15}_{5}} \times \frac{\cancel{3}}{4} = \frac{1}{10} & 0 \le y \le 5 \\ \frac{3}{4} \times \frac{1}{\cancel{15}_{5}} = \frac{1}{20} & 5 \le y \le 15 \\ 0 & \text{otherwise} \end{cases}$$



Conditioning over R.V

$P(Y = y) = 0$ for continuos R.V. So condition density is defined as following.

$$\lim_{\Delta y \to 0} f_{X|\{y < Y \le y + \Delta y\}}(x) = f_{X|Y}(x|y).$$

We'll show That $f_{X|Y}(x|y) = \dfrac{f_{X,Y}(x,y)}{f_Y(y)}$.

$$f_{X|Y}(x|y) = \lim_{\Delta y \to 0} f_{X|\underbrace{\{y < Y \le y + \Delta y\}}_{A}}(x)$$

$$= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{F_{X|A}(x + \Delta x) - F_{X|A}(x)}{\Delta x}.$$

$$F_{X|A}(x+\Delta x) = \frac{P(\{X \le x+\Delta x\}, \{y \le Y \le y+\Delta y\})}{P(y \le Y \le y+\Delta y)}$$

$$= \frac{F_{X,Y}(x+\Delta x, y+\Delta y) - F_{X,Y}(x+\Delta x, y)}{F_Y(y+\Delta y) - F_Y(y)}.$$

$$= \frac{\left[\dfrac{F_{X,Y}(x+\Delta x, y+\Delta y) - F_{X,Y}(x+\Delta x, y)}{\Delta y}\right]}{\left[\dfrac{F_Y(y+\Delta y) - F_Y(y)}{\Delta y}\right]}$$

$$\lim_{\Delta y \to 0} F_{X|A}(x+\Delta x). = \frac{\dfrac{\partial F_{X,Y}(x+\Delta x, y)}{\partial y}}{\dfrac{\partial F_Y(y)}{\partial y}}.$$

$$\lim_{\Delta y \to 0} F_{X|A}(x) = \frac{\dfrac{\partial F_{X,Y}(x, y)}{\partial y}}{f_Y(y)}$$

$$f_{X|Y}(x|y) = \lim_{\Delta y \to 0} f_{X|\{y < Y \le y+\Delta y\}}(x).$$

$$= \lim_{\Delta y \to 0} \lim_{\Delta x \to 0} \frac{F_{X|A}(x+\Delta x) - F_{X|A}(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\dfrac{\partial F_{X,Y}(x+\Delta x, y)}{\partial y} - \dfrac{\partial F_{X,Y}(x,y)}{\partial y}}{\Delta x\,(f_Y(y))}.$$

$$= \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y} \Big/ f_Y(y).$$

$$= \frac{f_{X,Y}(x,y)}{f_Y(y)}.$$

Properties of conditional density function

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx$$

$$= \frac{f_Y(y)}{f_Y(y)} = 1.$$

66 Examples.