* Differential lengths, surface area, & volume of Cartesian, cylindrical & spherical coordinate systems.

EE1203: Vector Calculus

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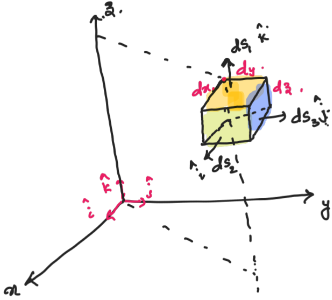
Indian Institute of Technology Hyderabad

Differential length, differential area, differential volume:

In cartesian co-ordinates,



Differential length (Vector tangential to line)

$$\vec{dl} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}.$$

Differential normal surfaces.

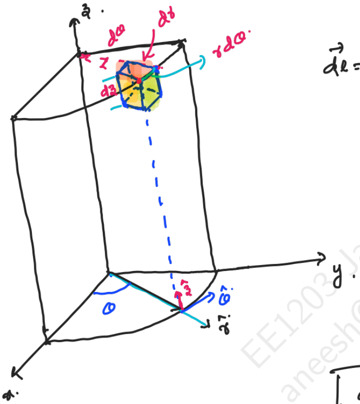
$$\vec{ds} = dy\,dz\,\hat{i},$$
$$dx\,dz\,\hat{j},$$
$$dx\,dy\,\hat{k}.$$

magnitude $= dy\,dz$.
dir $\rightarrow \hat{i}$.

Differential volume is;

$$dv = dx\,dy\,dz, \quad (\text{scalar})$$

Cylindrical co-ordinate systems.



Differential lengths:

$$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,\hat{z}.$$

Differential normal surface area:

$$\vec{ds} = r\,d\theta\,dz\,\hat{r};$$

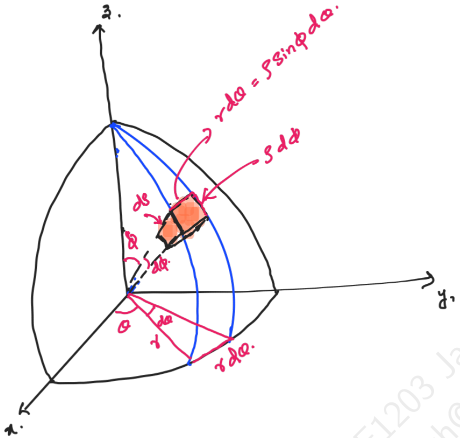
$$= dr\,dz\,\hat{\theta}.$$

$$= dr\;r\,d\theta\;\hat{z},$$

Differential volume:

$$\boxed{dv = r\,d\theta\,dr\,dz}$$

Spherical - co-ordinate systems (ρ, ϕ, θ)



Differential lengths:

$$\vec{dl} = d\rho \, \hat{\rho} + \rho \, d\phi \, \hat{\phi} + \underbrace{\rho \sin\phi}_{r} \, d\theta \, \hat{\theta}.$$

Differential surface area (normal).

$$\vec{ds} = \rho \, d\phi \; \rho \sin\phi \, d\theta \; \hat{\rho}.$$
$$= \rho^2 \sin\phi \, d\phi \, d\theta \, \hat{\rho};$$

$$= (d\rho \times r\,d\theta) \, \hat{\phi}.$$
$$= d\rho \cdot \rho \sin\phi \, d\theta \, \hat{\phi}.$$
$$= \rho \sin\phi \, d\rho \, d\theta \, \hat{\phi};$$

$$= d\rho \times \rho \, d\phi \, \hat{\theta}.$$
$$= \rho \, d\rho \, d\phi \, \hat{\theta}.$$

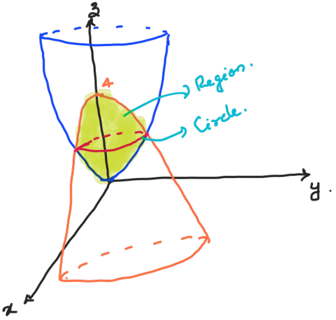
Differential volume:

$$dV = d\rho \times \rho \, d\phi \times \rho \sin\phi \, d\theta.$$
$$= \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Example: Region between

$$\left.\begin{aligned} z &= x^2+y^2 \\ z &= 4-x^2-y^2 \end{aligned}\right\} \to \text{paraboloids.}$$

Volume of this region? $\Rightarrow$ $\text{Volume} = \iiint 1 \cdot dV = \iiint 1 \cdot dz\, dy\, dx.$

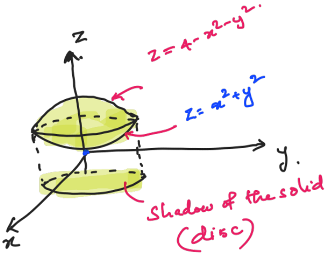


* We select dz as first differential element; because by fixing x & y top & bottom bounds of 'z' is much simpler in this case.

$$V = \iint \int_{x^2+y^2}^{4-x^2-y^2} dz\; dy\; dx.$$

Selecting next as dx or dy is completely random here as it is symmetric in x & y.

Project the solid on to x-y plane; for inner bounds in dx & dy.



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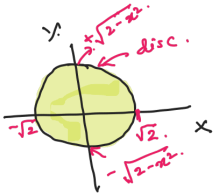

How to find this shadow in x-y plane:

$z_{\text{below (disc)}} < z_{\text{above (disc)}}$

(The circular disc here which separates two halves of the solid)

$x^2 + y^2 < 4 - x^2 - y^2$.

$x^2 + y^2 < 2$ ---> Eqn of a disc. of radius $\sqrt{2}$

$$\Rightarrow \quad V = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{x^2+y^2}^{4-x^2-y^2} 1 \, dz \, dy \, dx \quad \checkmark$$

Try with cylindrical-cordinates : Why.

$\Rightarrow$ Solid is symmetric w.r.t 'z'.

$$V = \iiint 1 \cdot dv = \underbrace{\int_{0}^{2\pi} \int_{0}^{\sqrt{2}} \int_{r^2}^{4-r^2} dz \; \underline{r\,dr} \; \underline{d\theta}}_{\text{Simpler compared to cartesian.}}$$

θ limit $\quad$ r limit $\quad$ z limit

$x^2+y^2 \to r^2$

$4-x^2-y^2 \to 4-r^2$

Try yourself to use spherical-co-ordinate for the same problem!