Time-varying Signals

- The circuit analysis covered so far has dealt with scenarios where voltage and current do not vary over time (i.e., they are DC in nature). In practice, however, circuits are often subjected to time-varying voltages and currents (referred to as AC signals[1]). Analyzing such circuits requires familiarity with (a) time-varying signals and their characterization, (b) circuit elements and their behavior under time-varying conditions, (c) Circuit laws that apply to time-varying signals, (d) deriving circuit equations using these laws, and (e) methods for solving the resulting equations.

[1] Quite often, AC signals are interpreted as sinusoidal. In this course, AC signals refer to any time-varying signal

- In mathematics, a function is typically defined as a relationship between a set of inputs and outputs, often denoted as $f(x)$, where x is the independent variable. In the context of circuits, functions where time t is the independent variable are referred to as signals. If the signal's value remains unchanged over time, it is termed a DC signal; if it varies over time, it is termed an AC signal. When studying functions in mathematics, we often focus on their graphs, which visually represent the relationship between inputs and outputs. Similarly, when dealing with signals, we focus on the waveform, which is a graphical representation of how the signal changes over time.

- The time-varying signals that typically appear in circuits include:
  - Step Signal $u(t)$: A step can be considered as a signal that represents a sudden yet finite change in value at a time instant. A unit step signal is defined as

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \tag{1}$$

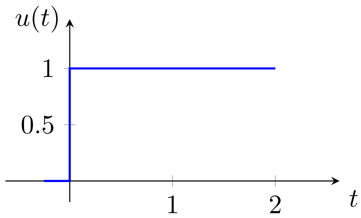


Figure 1: Unit step signal

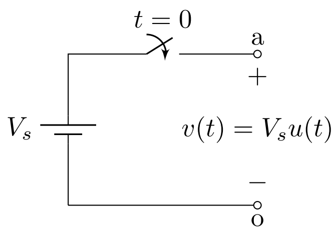


Figure 2: Example showing a unit step signal in a circuit

The unit step signal (shown in Fig. 1) represents a change in signal value from 0 to 1 at time instant $t = 0$. Step signals often appear in circuits with switching elements. A simple example is shown in Fig. 2. The voltage $v(t)$ between the terminals a and o can be represented as $V_s u(t)$. A shifted unit step signal represents a sudden yet finite change in value occurring at a time instant other than zero. It is defined as

$$u(t-\tau) = \begin{cases} 0 & \text{for } t < \tau \\ 1 & \text{for } t \geq \tau \end{cases} \tag{2}$$

where τ is the time at which the step change occurs. The shifted step signal is useful for modeling events or switching actions that happen at arbitrary times within a circuit. For example, if a switch in the circuit shown in Fig. 2 closes at $t = \tau$, the voltage between a and o is given by $V_s u(t-\tau)$. The graph of a shifted unit step signal is shown in Fig. 3.

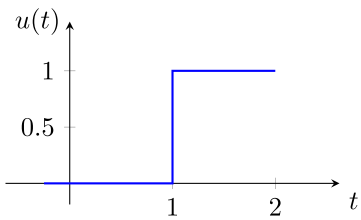


Figure 3: Shifted unit step signal $u(t-1)$

Step signals are often used to benchmark the transient performance of circuits, which will be studied later.

- Impulse signal $\delta(t)$: An impulse can be considered as a signal that, when passed through an integrator, results in a unit output[2]. Since an integrator typically represents an area operator, the impulse signal can be defined as the limiting case of a signal with unit area i.e. $\delta(t) = \lim_{\epsilon \to 0} s(t)$ where $s(t)$ has unit area.

[2] More precisely, an impulse is a distribution and not a function in the strict mathematical sense. However, for the purposes of this course, the stated definition suffices.

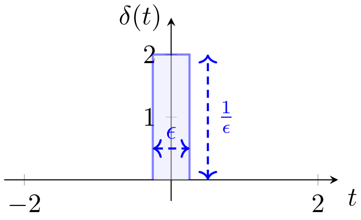


Figure 4: Impulse function (rectangle approximation)

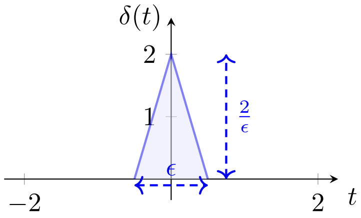


Figure 5: Impulse function (triangle approximation)

Two commonly used signals $s(t)$ for representing the impulse are the rectangular pulse and the triangular pulse (as shown in Fig.4 and Fig.5, respectively). Among these, the rectangular pulse is the most commonly used basis signal.

Alternatively, an impulse function can be defined by the property

$$\int_a^b f(t)\delta(t)\partial t \;=\; f(0) \tag{3}$$

This reflects the sifting (or sampling) property of the impulse. A more general form, incorporating time-shifting, is given by

$$\int_{t=-\infty}^{t=\infty} x(t)\delta(t-\tau)\; dt \;=\; x(\tau) \tag{4}$$

This shifting property allows the impulse to "pick out" the value of a function at a specific time τ, making it a powerful tool in system analysis. **Note** that in this definition, the impulse is characterized by how it operates on other signals, rather than by its explicit form as a function or signal. This operator-based perspective is what underlies the concept of a distribution.

In the context of circuits, impulse signals are often used to study system stability. Additionally, impulse signals can be used to model initial conditions that arise in circuit elements.

One approach for computing the response of a circuit to an impulse signal is to first determine the circuit's response to a pulse signal. This pulse can be represented using step functions as

$$s(t) = \frac{1}{\epsilon}\left[u\left(t+\frac{\epsilon}{2}\right) - u\left(t-\frac{\epsilon}{2}\right)\right] \tag{5}$$

Then, by letting $\epsilon \to 0$, the pulse signal approaches an ideal impulse[3].

[3] More advanced and elegant techniques for computing the impulse response will be introduced in later courses.

- Periodic Signals: Another important class of signals that commonly appear in circuits are periodic signals. A signal $v(t)$ is said to be periodic with period T if $v(t+T) = v(t)\forall t$. Some common examples of periodic signals include

  - $\sin(n\omega t)$, where n is an integer: This signal is periodic with a period $T = \frac{2\pi}{n\omega}$. However, it is also periodic with period $\frac{2\pi}{\omega}$.

– A signal composed as a sum of sinusoids takes the form

$$v(t) \;=\; \sum_{n=1} mV_n \sin(n\omega t) \tag{6}$$

In this case, the term with is called the fundamental component and the terms with $n > 1$ are called the harmonic components[4]. The period of the entire signal is $T = \frac{2\pi}{\omega}$, which corresponds to the period of the fundamental component.

– Consider the circuit shown in Fig. 6. Assume that the base terminal of the BJT is driven by a periodic signal $v_{bb}(t)$, as shown in Fig. 7, where the voltage level V_1 is sufficiently large to drive the BJT into the ON state. In this scenario, the output voltage also exhibits a periodic behavior, as illustrated in Fig.7. The period of the output voltage is equal to that of v_{bb}.

[4] In Fourier analysis, periodic signals are often decomposed into their fundamental and harmonic parts, enabling frequency-domain analysis.

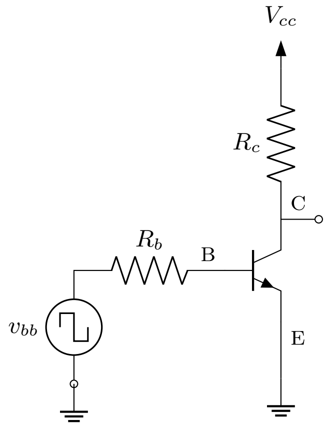


Figure 6: Switching circuit resulting in a periodic signal

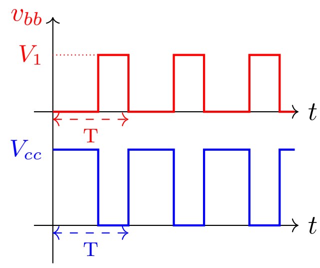


Figure 7: Base and output signals for the circuit shown in Fig. 6

- Sinusoidal Signals: Sinusoidal signals constitute a large fraction of periodic signals that appear in the domain of electrical engineering. In the context of this course, signals of the form given in equation (7) are called sinusoidal signals. Sinusoidal signals can be expressed using either the sine (sin) or cosine (cos) functions.

$$v(t) \;=\; V_m \cos(\omega t + \phi) \tag{7}$$

For simplicity and consistency throughout this course, we will specifically use the cosine (cos) function to represent and analyze these sinusoidal signals. This choice does not limit the generality of sinusoidal signals, as any sinusoidal signal can be transformed between sine and cosine forms using phase shifts. For instance, a signal given by $v(t) = V_m \sin(\omega t + \gamma)$ can alternatively be written as $v(t) = V_m \cos(\omega t + \phi)$, where $\phi = \gamma - \frac{\pi}{2}$.

- Average and root mean square value of a periodic signal: Two important measures that are often used to characterize periodic signals are the average and the root mean square (RMS) value. The average value[5] of a periodic signal $v(t)$ with period T is defined as

$$V_{avg} = \frac{1}{T}\int_0^T v(t)\,dt \tag{8}$$

The average value represents the mean or DC component of the signal.

The Root Mean Square (RMS) Value[6] is a measure of its effective magnitude, especially important in power calculations. It is defined as

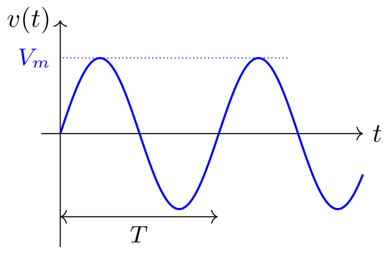
$$V = \sqrt{\frac{1}{T}\int_0^T [v(t)]^2\,dt}. \tag{9}$$

Physically, the RMS value corresponds to the equivalent DC value that would deliver the same power to a resistive load as the time-varying signal.

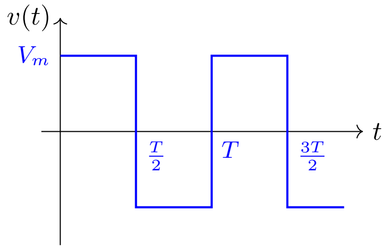
[5] defined over one complete period

[6] Note that RMS values are denoted in upper case

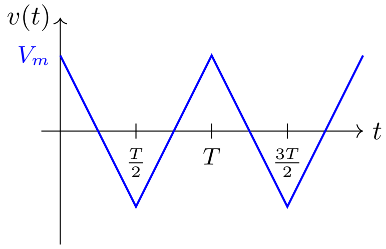
- Examples for Common Waveforms: The average and RMS values for commonly



(a) sinusoidal signal

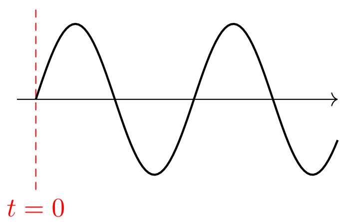


(b) square signal with 50% duty ratio

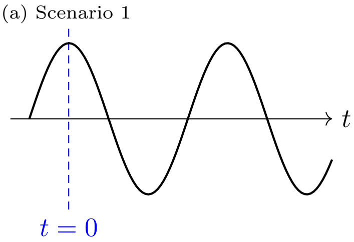


(c) Triangular signal

Figure 8: Waveforms of common periodic signals



(a) Scenario 1



(b) Scenario 2

Figure 9: Dependence of phase on time reference

encountered periodic signals are summarized below.

1. Sinusoidal Signal: For a sinusoidal waveform of the form $v(t) = V_m \sin(\omega t)$ (shown in Fig. 8(a)), the average and RMS values are given by

$$V_{avg} = 0, \quad V = \frac{V_m}{\sqrt{2}}. \tag{10}$$

2. Square Signal: The average and RMS values of a square wave with peak amplitude V_m and 50% duty cycle (shown in Fig. 8(b)) are

$$V_{avg} = 0, \quad V = V_m. \tag{11}$$

3. Triangular Wave: The average and RMS values of a symmetric triangular wave (shown in Fig. 8(c)) with peak amplitude V_m yields

$$V_{avg} = 0, \quad V = \frac{V_m}{\sqrt{3}}. \tag{12}$$

- **Sinusoidal signals and phasor representation**: A sinusoidal signal is characterized by three parameters: (a) peak value or amplitude (V_m), (b) phase (ϕ), and (c) angular frequency (ω), where $\omega = 2\pi f$ and f is the frequency in hertz (Hz). The time period of the sinusoidal signal is given by $T = \frac{2\pi}{\omega}$, which is the reciprocal of the frequency f. The phase of the signal depends on the time reference (i.e., $t = 0$). To illustrate this, consider the two scenarios shown in Fig.9. These scenarios represent the same signal but with different choices of time reference. In Fig.1(a), the phase of the signal is $-\frac{\pi}{2}$, while in Fig. 9(b), the phase is 0.

- Equation (7) illustrates that the phase of the signal can be calculated as [7] $\phi = \cos^{-1}\left(\frac{v(0)}{V_m}\right)$, where V_m denotes the positive peak value. Alternatively, if $t_p \in \mathbb{R}$ denotes the time of the nearest positive peak relative to $t = 0$, the phase of the signal can be computed as $\phi = -\omega t_p$, where ω is the angular frequency of the signal (in radians per second).

- Using Euler's identity, a sinusoidal signal of the form (7) can be represented as

$$v(t) \;=\; \mathrm{Re}\{V_m e^{j\omega t + \phi}\} \;=\; \mathrm{Re}\{V_m e^{j\phi} e^{j\omega t}\} \tag{13}$$

In general[8], voltages and currents in a given circuit operate at the same frequency. Therefore, they can be effectively described using just two parameters: amplitude and phase. A phasor for a sinusoidal signal $v(t) = V_m \cos(\omega t + \phi)$, which is represented as a complex number (denoted by $\mathbf{V}$), is defined as:

$$\mathbf{V} = \frac{V_m}{\sqrt{2}} e^{j\phi} \quad \text{or} \quad V\angle\phi \tag{14}$$

where $V = \frac{V_m}{\sqrt{2}}$ denotes the RMS value of signal. It will be shown later that the

[7] recollect the minor ambiguities that were discussed in the class

[8] considering the type of circuits discussed in this course

using the concept of phasors greatly simplifies certain aspects of circuit analysis (mainly, computing steady state response of AC circuits).

- Given the phasor of voltage (or current) along with the frequency, it is possible to generate the time-domain signal as

$$v(t) \;=\; \mathrm{Re}\{\sqrt{2}\mathbf{V}e^{j\omega t}\} \tag{15}$$

- Phasors and Uniform circular motion: Consider an object undergoing uniform circular motion in a 2D plane. The object moves counterclockwise around a circle with a radius r at an angular velocity of ω (in radians per second), as illustrated in Fig. 10. Additionally, let the initial angular displacement of the object be represented by θ_0.

  The angular position of the object at any instant t is given by

$$\theta = \omega t + \theta_0 \tag{16}$$

  Further, the position of the object (in 2D plane) at any instant t is given by

$$p(t) \;=\; (r\cos\theta, r\sin\theta) \;=\; (r\cos(\omega t+\theta_0), r\sin(\omega t+\theta_0)) \tag{17}$$

  Equation (17) illustrates that the object's position at any time t corresponds to a standard sinusoidal signal as described by Equation (7). To visualize this, consider a vector rotating with an angular velocity ω in the complex plane. The sinusoidal signal $v(t)$ can be interpreted as the projection of this rotating vector onto the real axis. This approach aids in understanding complex topics such as negative frequency and phase shifts in signal processing.

  In this analogy, the phasor represents the position of the vector at $t = 0$ and can be expressed either in polar coordinates or as a complex exponential. The phase of the signal is the angular position of this vector, measured counterclockwise from the positive real axis. Importantly, if ϕ denotes the phase of the signal, then $\phi \pm 2\pi$ is also a valid representation of the phase. In this course, we define the phase within the range $[-\pi, \pi]$.

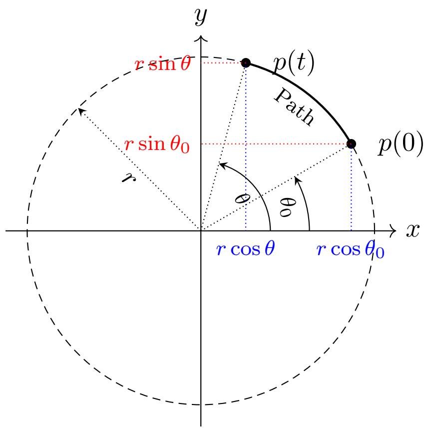


Figure 10: Uniform circular motion with radius r

- In circuit analysis, phasors are often represented on a single complex plane to simplify understanding of voltage and current relationships. This visual representation is called a phasor diagram. When constructing a phasor diagram, it's important to note that only phasors with the same frequency can be compared or depicted together. The phase difference ($\Delta\phi$) between two phasors is the angular difference between their respective phase angles (see Fig. 11). The phase is defined within the range $[-\pi, \pi]$. A phasor $\mathbf{V}_1$ is said to lead another phasor $\mathbf{V}_2$ if the phase angle of $\mathbf{V}_1$ is greater than that of $\mathbf{V}_2$ (considering the phase range $[-\pi, \pi]$). Conversely, $\mathbf{V}_1$ is said to lag $\mathbf{V}_2$ if its phase angle is smaller (see Fig. 11).

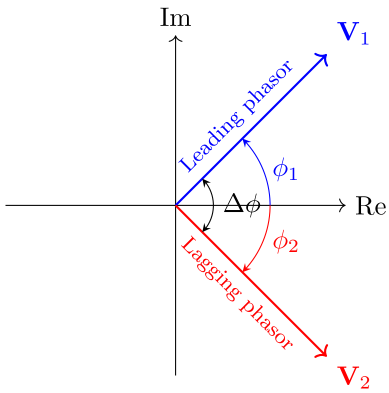


Figure 11: Sample phasor diagram