Second-order Circuits

- A circuit comprising of two energy storage elements (inductors and capacitors) constitute a second-order circuit. Before proceeding with analysis of second-order circuits, we will first look at some examples of circuits leading to second-order differential equations. Subsequently, we will look at methods to solve these differential equations.

- Example 1: Determine the differential equation that govens the response of the circuit shown in Fig. 1. Assume that the circuit is initially relaxed (i.e., the initial currents through the inductors and voltages across the capacitors are zero).

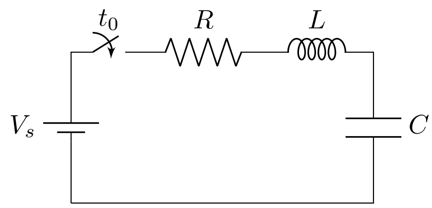


Figure 1: Circuit for Example 1.

  Application of KVL to the loop gives

$$V_s - v_R(t) - v_L(t) - v_C(t) = 0. \quad (t > t_0) \implies L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int i(t)dt = V_s$$

  Taking the derivative of both sides gives

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0.$$

  The governing differential equation is a second-order linear homogeneous differential equation with constant coefficients. In order to solve this differential equation, we need two initial conditions. The first initial condition is $i(t_0) = 0$ (since the circuit is initially relaxed). The second initial condition can be found by evaluating the KVL equation at $t = t_0$:

$$V_s - v_R(t_0) - v_L(t_0) - v_C(t_0) = 0 \implies v_L(t_0) = V_s \implies \left.\frac{di(t)}{dt}\right|_{t=t_0} = \frac{V_s}{L}.$$

- Example 2: Determine the differential equation that governs the response of the circuit shown in Fig. 2. Assume that the circuit is initially relaxed (i.e., the initial currents through the inductors and voltages across the capacitors are zero).

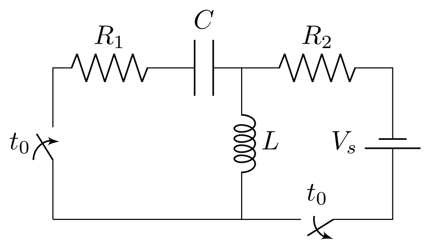


Figure 2: Circuit for Example 2.

  In this example, since the circuit is initially relaxed, the current through the inductor is driven by the voltage source V_s after the switches are closed at $t = t_0$, while the voltage across the capacitor is 0 till $t = t_0$. Let $i(t)$ be the current through the inductor L for $t > t_0$. Applying KVL to the loop containing the voltage source, we have

$$V_s - i(t)R_2 - L\frac{di(t)}{dt} = 0 \implies V_s = i(t)R_2 + L\frac{di(t)}{dt} \quad \text{with} \quad i(0) = 0.$$

  The differential equation governing the response till $t = t_0$ is thus a first-order linear differential equation with constant coefficients.

  For $t > t_0$, the equation that governs the response is given by

$$i(t)R_2 + \frac{1}{C}\int i(t)dt + L\frac{di(t)}{dt} = 0.$$

Taking the derivative of both sides gives

$$L\frac{d^2i(t)}{dt^2} + R_2\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0.$$

Even in this case, the governing differential equation is second-order linear homogeneous differential equation with constant coefficients. Interestingly, the differential equation governing the response for $t > t_0$ is the same as that in Example 1. However, the initial conditions are different. The first initial condition is $i(t_0) = \frac{V_s}{R_2}$ (if t_0 is large enough for the circuit to reach steady state). The second initial condition can be found by evaluating the KVL equation at $t = t_0$

$$i(t_0)R_1 + v_L(t_0) + v_C(t_0) = 0 \implies \left.\frac{di(t)}{dt}\right|_{t=t_0} = -\frac{R_1}{L}i(t_0).$$

- Example 3: As a last example, consider the circuit shown in Fig. 3, which is similar to Example 1, except that the source is now a sinusoidal voltage source. Determine the differential equation that governs the response of the circuit. Assume that the circuit is initially relaxed (i.e., the initial currents through the inductors and voltages across the capacitors are zero). The differential equation governing the response for $t > t_0$ is given by

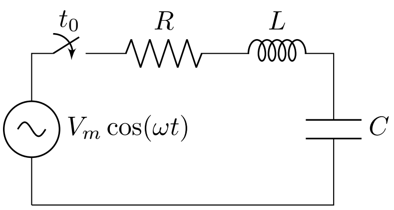


Figure 3: Circuit for Example 3.

$$L\frac{d^2i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{d}{dt}\big(V_m\cos(\omega t)\big) = -\omega V_m\sin(\omega t).$$

The governing differential equation is a second-order linear non-homogeneous differential equation with constant coefficients. The initial conditions are the same as those in Example 1.

- In order to be able to solve the second-order differential equations that arise in the analysis of second-order circuits, it is necessary to first understand the solution of second-order differential equations with constant coefficients. We will first look at the solution of second-order linear homogeneous differential equations with constant coefficients, and then look at the solution of second-order linear non-homogeneous differential equations with constant coefficients.

- **Solution of Second-order Linear Homogeneous Differential Equations with Constant Coefficients:** Consider the second-order linear homogeneous differential equation[1] with constant coefficients given by

$$\frac{d^2x(t)}{dt^2} + 2\zeta\omega_n\frac{dx(t)}{dt} + \omega_n^2x(t) = 0,$$

where ζ and ω_n are constants. The form of the differential equation is chosen to be consistent with the typical analysis of second-order systems in the domain of control systems. In this framework, ζ is called damping ratio and ω_n is called the natural frequency. It is also a common practice to often write the standard

[1] Can you think why we use the terms homogeneous and linear explicitly in the context of differential equations?

form of the second-order differential equation interms of quality factor Q instead of damping ratio ζ. The quality factor is defined as $Q = \frac{1}{2\zeta}$. Thus, the differential equation can also be written as

$$\frac{d^2x(t)}{dt^2} + \frac{\omega_n}{Q}\frac{dx(t)}{dt} + \omega_n^2 x(t) = 0.$$

In general, solving a differential equation of this form requires two initial conditions, which are typically given as $x(t_0)$ and $\frac{dx(t)}{dt}\big|_{t=t_0}$.

Solving the differential equation requires us

1. to find the general solution
2. and subsequently evaluate the parameters in the general solution using the initial conditions.

General Solution: The general solution is typically a linear combination of the fundamental set of solutions [2]. For the type of second-order differential equations that we encounter in circuits, the fundamental set of solutions are of the form e^{st} where s can be complex. Thus, we assume that the solution of the second-order homogeneous differential equation is of the form

$$x(t) = Ce^{st}, \tag{1}$$

where C is a constant to be determined using the initial conditions. Note that the solution in (1) is a family of solutions, one for each value of s. Furthermore, it is required that, for a second order differential equation, we need two linearly independent solutions to form the fundamental set of solutions. Thus, we need to find two values of s that lead to linearly independent solutions or else identify another function that constitutes a linearly independent set.

[2] more on this in the course on differential equations

Since the solution is of the form Ce^{st}, it is necessary that it satisfies the differential equation. Substituting $x(t) = Ce^{st}$ into the differential equation gives

$$s^2Ce^{st} + 2\zeta\omega_n sCe^{st} + \omega_n^2 Ce^{st} = 0 \implies s^2 + 2\zeta\omega_n s + \omega_n^2 = 0. \tag{2}$$

Equation (2) is called the characteristic equation of the differential equation. The characteristic equation is a quadratic equation in s and hence has two roots given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}. \tag{3}$$

The nature of the roots depends on the value of ζ. There are three possible scenarios:

- Overdamped case ($\zeta > 1$): In this case, the roots are real and distinct. The two roots are given by

$$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}. \tag{4}$$

Since the roots are real and distinct, the two solutions $e^{s_1 t}$ and $e^{s_2 t}$ are linearly independent. Thus, the general solution is given by

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}. \tag{5}$$

The constants C_1 and C_2 can be evaluated using the initial conditions.

– Critically damped case ($\zeta = 1$): In this case, the roots are real and repeated. The two roots are given by

$$s_1 = s_2 = -\omega_n. \tag{6}$$

Since the roots are repeated, the two solutions $e^{s_1 t}$ and $e^{s_2 t}$ are not linearly independent. In this case, it can be shown that a second linearly independent solution [3] is given by $te^{s_1 t}$. Thus, the general solution is given by

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t}. \tag{7}$$

The constants C_1 and C_2 can be evaluated using the initial conditions.

– Underdamped case ($0 < \zeta < 1$): In this case, the roots are complex conjugates. The two roots are given by

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1 - \zeta^2} = -\zeta\omega_n \pm j\omega_d, \tag{8}$$

where $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ is called the damped frequency. Since the roots are complex conjugates, the two solutions $e^{s_1 t}$ and $e^{s_2 t}$ are not real-valued. However, the solution that we are interested in is real-valued. For this to be true, the constants C_1 and C_2 in (5) must be complex conjugates. Thus, we can write

$$x(t) = C_1 e^{-\zeta\omega_n t} e^{j\omega_d t} + C_1^* e^{-\zeta\omega_n t} e^{-j\omega_d t} = e^{-\zeta\omega_n t}\left(C_1 e^{j\omega_d t} + C_1^* e^{-j\omega_d t}\right). \tag{9}$$

If we let C, which is complex valued, be defined as $C = re^{j\theta}$, then we have

$$x(t) = e^{-\zeta\omega_n t}\left(re^{j(\omega_d t + \theta)} + re^{-j(\omega_d t + \theta)}\right) = 2re^{-\zeta\omega_n t}\cos(\omega_d t + \theta). \tag{10}$$

The constants r and θ can be evaluated using the initial conditions. When $\zeta = 0$ (undamped), we have $\omega_d = \omega_n$ and the solution in (10) reduces to

$$x(t) = 2re^{-\zeta\omega_n t}\cos(\omega_d t + \theta) = 2r\cos(\omega_n t + \theta). \tag{11}$$

[3] This will be covered in more detail in the course on differential equations. Interested readers can refer to standard texts or the course material available here.

- Example 4: Consider the circuit shown in Fig. 1. Assume that the circuit is initially relaxed (i.e., the initial currents through the inductors and voltages across the capacitors are zero). Determine the response $i(t)$ for $t > t_0$. Consider the following cases

Case A: $R = 6\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$.

Case B: $R = 2\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$.

Case C: $R = 0\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$.

Case D: $R = 6\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 1\,s$.

The differential equation (in standard form) governing the response for $t > t_0$ is given by

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0.$$

Comparing with the standard form, we have

$$2\zeta\omega_n = \frac{R}{L}, \quad \omega_n^2 = \frac{1}{LC} \implies \zeta = \frac{R}{2}\sqrt{\frac{C}{L}}, \quad \omega_n = \frac{1}{\sqrt{LC}}.$$

For the given values of L and C, we have $\omega_n = \frac{1}{\sqrt{0.1}} = 3.162\,rad/s$. The solution for each case is as follows:

  - Case A: $R = 6\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$. For case A, we have $\zeta = \frac{6}{2}\sqrt{\frac{0.1}{1}} = 0.9487 > 0$. Thus, the response is overdamped. The roots of the characteristic equation are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -1.268, -4.732.$$

    The general solution is given by

$$i(t) = C_1 e^{-1.268t} + C_2 e^{-4.732t}.$$

    The initial conditions are given by

$$i(0) = 0, \quad \left.\frac{di(t)}{dt}\right|_{t=0} = \frac{V_s}{L} = 10.$$

    Using the initial conditions, we have

$$i(0) = C_1 + C_2 = 0 \implies C_2 = -C_1 \quad \left.\frac{di(t)}{dt}\right|_{t=0} = -1.268C_1 - 4.732C_2 = 10.$$

    Solving the above equations gives $C_1 = 2.886$ and $C_2 = -2.886$. Thus, we have

$$i(t) = 2.886e^{-1.268t} - 2.886e^{-4.732t}.$$

  - Case B: $R = 2\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$. For case B, we have $\zeta = \frac{2}{2}\sqrt{\frac{0.1}{1}} = 0.3162 < 1$. Thus, the response is underdamped. The roots of the characteristic equation are given by

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -1 \pm j3.$$

The general solution is given by

$$i(t) = 2re^{-t}\cos(3t+\theta).$$

The initial conditions are given by

$$i(0) = 0, \quad \left.\frac{di(t)}{dt}\right|_{t=0} = -2r\sin(\theta) + 2r(-1)\cos(\theta) = \frac{V_s}{L} = 10.$$

Using the initial conditions, we have

$$i(0) = 2r\cos(\theta) = 0 \implies \cos(\theta) = 0 \implies \theta = \frac{\pi}{2} \quad -2r\sin(\theta) = 10.$$

Solving the above equations gives $r = 5$. Thus, we have

$$i(t) = 10e^{-t}\cos(3t + \frac{\pi}{2}) = 10e^{-t}\sin(3t).$$

– Case C: $R = 0\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 0$. For case C, we have $\zeta = \frac{0}{2}\sqrt{\frac{0.1}{1}} = 0$. Thus, the response is undamped. The roots of the characteristic equation are given by

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = 0 \pm j3.162 = \pm j3.162.$$

The general solution is given by

$$i(t) = 2r\cos(3.162t+\theta).$$

The initial conditions are given by

$$i(0) = 0, \quad \left.\frac{di(t)}{dt}\right|_{t=0} = -2r\sin(\theta)\cdot 3.162 = \frac{V_s}{L} = 10.$$

Using the initial conditions, we have

$$i(0) = 2r\cos(\theta) = 0 \implies \cos(\theta) = 0 \implies \theta = \frac{\pi}{2} \quad -2r\sin(\theta)\cdot 3.162 = 10.$$

Solving the above equations gives $r = 1.581$. Thus, we have

$$i(t) = 3.162\cos(3.162t + \frac{\pi}{2}) = 3.162\sin(3.162t).$$

– Case D: $R = 6\,\Omega$, $L = 1\,H$, $C = 0.1\,F$, $V_s = 10\,V$, and $t_0 = 1\,s$. For case D, we have $\zeta = \frac{6}{2}\sqrt{\frac{0.1}{1}} = 0.9487 > 0$. Thus, the response is overdamped. The roots of the characteristic equation are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -1.268, -4.732.$$

The general solution is given by

$$i(t) = C_1 e^{-1.268(t-1)} + C_2 e^{-4.732(t-1)}.$$

The initial conditions are given by

$$i(1) = \frac{V_s}{R} = \frac{10}{6}, \quad \left.\frac{di(t)}{dt}\right|_{t=1} = -1.268C_1 - 4.732C_2 = -\frac{R}{L}i(1) = -3\cdot\frac{10}{6} = -5.$$

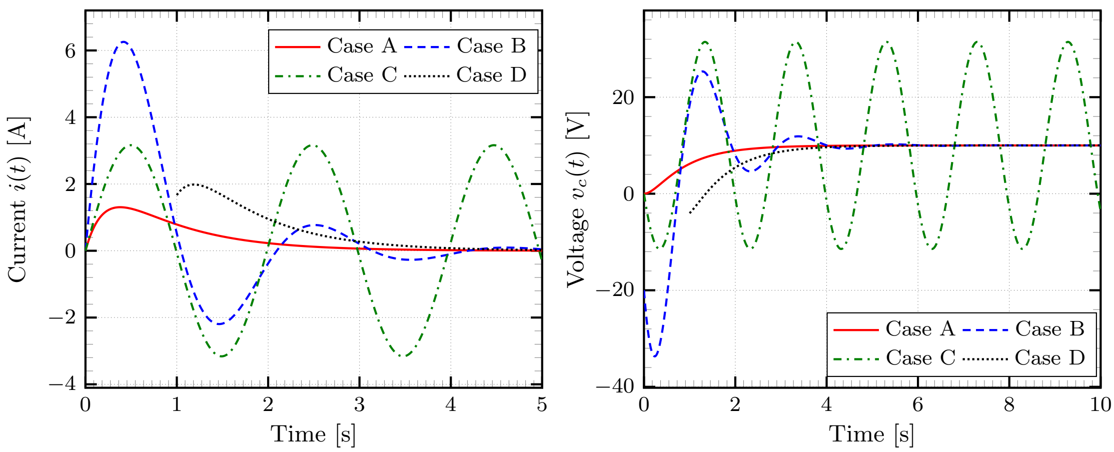
Using the initial conditions, we have

$$i(1) = C_1 + C_2 = \frac{10}{6} \qquad -1.268C_1 - 4.732C_2 = -5.$$

Solving the above equations gives $C_1 = 3.443$ and $C_2 = -1.776$. Thus, we have

$$i(t) = 3.443e^{-1.268(t-1)} - 1.776e^{-4.732(t-1)}.$$

The current responses for the four cases are shown in Fig. 4(a). As a small exercise, compute the voltage across the capacitor for all the four cases. The voltage across the capacitor for all the four cases is shown in Fig. 4(b). Note that as t increases, the capacitor voltage reaches to $V_s = 10\,V$ for all cases except when $R = 0\,\Omega$ (Case C). The current reaches to 0 for all cases except when $R = 0\,\Omega$ (Case C), where it continues to oscillate.



(a) Current (b) Capacitor Voltage

Figure 4: Responses for the four cases in Example 4.