Lecture-14

1. Line integral

(a) Path independence.

(b) Conservative fields.

2. Green's Theorem.

EE1203: Vector Calculus

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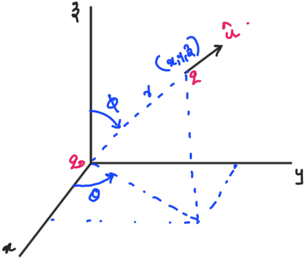
Path Independence:

* In line integral the path of integration:
  One of the ingredients that determines the very function we integrate.

* Under some condition value of integral does not depend on the path.

Let us show this path independence in Coulomb's force.

Let a charge q_0 be fixed at the origin;
another charge q be situated at (x, y, z).



Coulomb force on 'q' is

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q q_0}{r^2}\hat{u}$$

$$r = (x^2+y^2+z^2)^{1/2}$$ (distance b/w q & q_0)

$\hat{u}$ = unit vector pointing from q_0 to q

$$\hat{u} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r} = \frac{x\,i + y\,j + z\,k}{r}$$

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$$\bar{F} = \frac{1}{4\pi\epsilon_0} \frac{q q_0}{r^3}(xi + yj + zk)$$

$\bar{F} = \langle F_x, F_y, F_z \rangle = \frac{1}{4\pi\epsilon_0} \frac{q q_0}{r^3} \langle x, y, z \rangle$

x-component

$$\bar{F} \cdot \hat{t}\, ds = F_x\, dx + F_y\, dy + F_z\, dz$$

$$= \frac{q q_0}{4\pi\epsilon_0} \frac{x\,dx + y\,dy + z\,dz}{r^3}$$

The trick is: use the relation

$$r^2 = x^2 + y^2 + z^2$$

Taking differentials in the equation

$$r\,dr = x\,dx + y\,dy + z\,dz$$

$$\Rightarrow \bar{F} \cdot \hat{t}\, ds = \frac{q q_0}{4\pi\epsilon_0} \frac{r\,dr}{r^3} = \frac{q q_0}{4\pi\epsilon_0} \frac{dr}{r^2}$$

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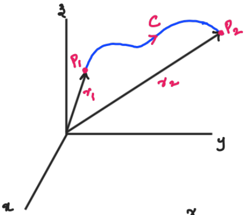
If the charge 'q' moves from a point P_1 at a distance r_1 from origin, to a point P_2 at a distance r_2 over some path C connecting P_1 & P_2; then.



$$\int_C \bar{F} \cdot \hat{t}\, ds = \frac{q\, q_0}{4\pi\epsilon_0} \int_{r_1}^{r_2} \frac{dr}{r^2} = \frac{q\, q_0}{4\pi\epsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$\Longrightarrow$ not depending on the path 'C'.

ie; $\int_C \bar{F} \cdot \hat{t}\, ds$ with $\vec{F}$ (Coulomb force) is path independent

But $\bar{F} = q\vec{E}$

$$\underbrace{\int_C \bar{F} \cdot \hat{t}\, ds}_{\text{path independent}} = q \underbrace{\int_C \bar{E} \cdot \hat{t}\, ds}_{\text{path independent}}$$

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The path independence solely depends on two properties of Coulomb's force;

(1) It depends only on the distance between two charges

(2) It acts along the line joining them.

Any $\vec{F}$ with these two properties $\Rightarrow$ Central force.

$\int_C \vec{F}\cdot\hat{t}\, ds$ is independent of path for any central force.

* Many other functions which are not central forces also have path independent line integral.

Theorem: $\int_C \vec{F}\cdot\hat{t}\, ds$ is independent of path, then

$$\int_{C_1} \vec{F}\cdot\hat{t}\, ds = \int_{C_2} \vec{F}\cdot\hat{t}\, ds.$$

C_1 & C_2 : two different arbitrary paths connecting P_1 & P_2

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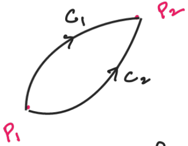


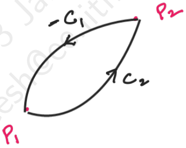
If we integrate from P_2 to P_1 through C_1

$$\Rightarrow \int_{-C_1} \bar{F}\cdot\hat{t}\,ds = -\int_{C_1} \bar{F}\cdot\hat{t}\,ds \quad ; \quad \text{Thus}$$

$$\int_{C_2} \bar{F}\cdot\hat{t}\,ds = -\int_{-C_1} \bar{F}\cdot\hat{t}\,ds$$

or
$$\int_{-C_1+C_2} \bar{F}\cdot\hat{t}\,ds = 0 .$$

$-C_1+C_2$ is just the closed loop from P_1 to P_2 & back.



Thus if $\int \bar{F}\cdot\hat{t}\,ds$ is independent of path then,

$$\boxed{\oint \bar{F}\cdot\hat{t}\,ds = 0 .}$$

$\oint \Rightarrow$ line integral around a closed path.

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If $\bar{E}$ is an electrostatic field,

$$\oint \bar{E}\cdot\hat{t}\cdot ds = 0 \Rightarrow$$ The circulation of the electrostatic field is zero.

$\Rightarrow$ Circulation law:

Theorem: In a region 'R' the line integral $\int_C \bar{F}\cdot\hat{t}\,ds$ is independent of path between any two points in R if and only if $\oint_C \bar{F}\cdot\hat{t}\,ds = 0$, for every closed curve C which is contained in R.

Proof: Suppose $\oint_C \bar{F}\cdot\hat{t}\,ds = 0$ for every closed curve C which is contained in R. P_1, P_2 be two distinct points in R and C_1 be a curve from P_1 to P_2 in R. & C_2 be another from P_1 to P_2

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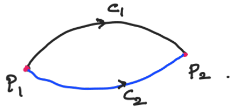

Then $C = C_1 \cup -C_2$ is closed curve in R (P_1 to P_1)

and so $\oint_C \bar{F}\cdot\hat{t}\,ds = 0$

$$0 = \oint \bar{F}\cdot\hat{t}\,ds$$
$$= \int_{C_1} \bar{F}\cdot\hat{t}\,ds + \int_{-C_2} \bar{F}\cdot\hat{t}\,ds$$
$$= \int_{C_1} \bar{F}\cdot\hat{t}\,ds - \int_{C_2} \bar{F}\cdot\hat{t}\,ds.$$

$$\Rightarrow \boxed{\int_{C_1} \bar{F}\cdot\hat{t}\,ds = \int_{C_2} \bar{F}\cdot\hat{t}\,ds.}$$

* The theorem does not give a practical way to verify the path independence, as it is almost impossible to check the line integrals around all the closed curves in a region.

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Practical method to verify path independence.

Theorem: Let $\vec{f}(x,y) = P(x,y)\,\hat{i} + Q(x,y)\,\hat{j}$ be a vector field in some region 'R' with P & Q continuously differentiable functions on R. Let C be the smooth curve in R parametrized by $x = x(s)$; $y = y(s)$; $a \le s < b$. Suppose there exist a real-valued function $F(x,y)$ such that

$\nabla F = \vec{f}$ on R, Then

$$\int_C \vec{f} \cdot \hat{t}\, ds = F(B) - F(A)$$

where $A = (x(a), y(a))$; $B = (x(b), y(b))$ are end points of C. Thus the line integral is independent of the path between its endpoints, since it depends only on the values of F at these points.

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Proof:

$$\int_C \bar{f}\cdot\hat{t}\,ds = \int_a^b \left[P(x(s),y(s))\,\frac{dx}{ds} + Q(x(s),y(s))\,\frac{dy}{ds}\right] ds$$

Given $\nabla F = \vec{f}$

$$\Rightarrow \bar{f} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} \qquad \Rightarrow \frac{\partial F}{\partial x} = P; \quad \frac{\partial F}{\partial y} = Q.$$

$$\Rightarrow \int_C \vec{f}\cdot\hat{t}\,ds = \int_a^b \underbrace{\left(\frac{\partial F}{\partial x}\frac{dx}{ds} + \frac{\partial F}{\partial y}\frac{dy}{ds}\right)}_{F'(x(s),\,y(s)) \,=\, \frac{dF}{ds}} ds.$$

Recall chain rule: $z = f(x,y)$ $\quad x = x(s);\ y = y(s)$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}.$$

$$= f'(x(s),y(s))$$

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$$\Rightarrow \int_C \vec{f}\cdot\hat{t}\,ds = \int_a^b F'(x(s), y(s))\,ds.$$

$$= F(x(s), y(s))\Big|_{s=a}^{s=b} = F(x(b), y(b)) - F(x(a), y(b))$$

$$= F(B) - F(A) \quad \rightarrow \text{Fundamental theorem of calculus.}$$

* A real valued function $F(x,y)$ such that $\nabla F(x,y) = \vec{f}(x,y)$ is called potential of $\vec{f}$.

* A ~~conservative~~ field which has potential.

$\vec{f} = \nabla F$ $(\nabla\times\vec{f} = 0)$?

also $\oint \vec{f}\cdot\hat{t}\,ds = 0$

We connect these two using Green's theorem!

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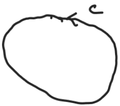
Green's Theorem.

If we have a closed curve;



$$\oint_C (\bar{F}\cdot\hat{t})\,ds = ?$$

(1) Direct method.

(2) Using Green's theorem.

Green's Theorem: If C is a closed curve, enclosing the region R, counterclockwise, and $\bar{F}$ is a vector field defined & differentiable in region 'R' enclosed, then the line integral (work done)

$$\boxed{\oint_C (\bar{F}\cdot\hat{t})\,ds. = \iint_R \text{Curl } \bar{F}\; dA.}$$

If $\bar{F}(x,y) = P(x,y)\,i + Q(x,y)\,j$ be vector field defined in Region R and C then.

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$$\oint_C (\vec{F}\cdot\hat{t})\,ds = \iint_R \underbrace{|\nabla\times\vec{F}|}\,dA = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

* Green's theorem : Only for closed curve.

If $\text{Curl}\,\vec{F} = 0 \Rightarrow \oint_C (\vec{F}\cdot\hat{t})\,ds = \iint_R \text{Curl}\,\vec{F}\,dA = 0.$

(Proof for conservative field $\vec{F}$)

$\Rightarrow$ If $\vec{F}$ is conservative : $\text{Curl}\,\vec{F} = 0.$

$\oint_C (\vec{F}\cdot\hat{t})\,ds = 0.$ ✓

* If $\vec{F}$ is <u>defined everywhere</u> in the plane & $\nabla\times\vec{F} = 0$ everywhere $\Rightarrow \vec{F}$ is Conservative.

* Important : If $\vec{F}$ not defined in some point in region 'R' then Green's theorem does not hold!

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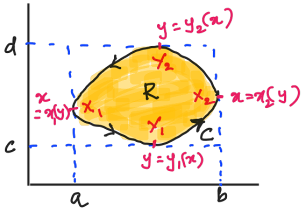
Proof: Consider a simple region R where boundary curve

$C = C_1 \cup C_2$; Also,



C_1 = the curve $y = y_1(x)$ from point X_1 to X_2

C_2 = the curve $y = y_2(x)$ from X_2 to X_1

where X_1, X_2 ; points on C farthest on left & right ;

Also,

C_1 = the curve $x = x_1(y)$ from point Y_2 to Y_1

C_2 = the curve $x = x_2(y)$ from Y_1 to Y_2

where Y_1, Y_2 ; lowest & highest points on 'C'.

Green's theorem:

$$\oint_C P(x,y)\,dx + Q(x,y)\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Let us see when $Q = 0$; Let us proove the following ;

$$\Rightarrow \quad \oint_C P(x,y)\,dx = \iint_R -\frac{\partial P}{\partial y}\,dA \quad \text{-----} \quad (1)$$

* Integrate $P(x,y)$ around C using $C = C_1 \cup C_2$;

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$$\oint_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx + \int_{C_2} P(x,y)\,dx.$$

$$= \int_a^b P(x, y_1(x))\,dx + \int_b^a P(x, y_2(x))\,dx. = \int_a^b P(x, y_1(x))\,dx - \int_a^b P(x, y_2(x))\,dx.$$

$$= -\int_a^b \left[P(x, y_2(x)) - P(x, y_1(x))\right] dx. = -\int_a^b \left(P(x,y)\Big|_{y=y_1(x)}^{y=y_2(x)} \right) dx.$$

$$= -\int_{x=a}^{x=b} \int_{y=y_1(x)}^{y_2(x)} \frac{\partial P(x,y)}{\partial y}\,dy\,dx. = -\iint_R \frac{\partial P}{\partial y}\,dA.$$ (By Fundamental theorem of Calculus)

Similarly integrate $Q(x,y)$ around 'C'. $C = C_1 \cup C_2$

$$\oint Q(x,y)\,dy = \int_{C_1} Q(x,y)\,dy + \int_{C_2} Q(x,y)\,dy.$$

$$= \int_{y=d}^{c} Q(x_1(y), y)\,dy + \int_{y=c}^{d} Q(x_2(y), y)\,dy. = -\int_c^d Q(x_1(y), y)\,dy + \int_c^d Q(x_2(y), y)\,dy.$$

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$$= \int_c^d \left[Q(x_2(y), y) - Q(x_1(y), y) \right] dy = \int_c^d Q(x,y) \Big|_{x_1(y)}^{x_2(y)} dy.$$

$$= \int_c^d \int_{x=x_1(y)}^{x_2(y)} \frac{\partial Q(x,y)}{\partial x} \, dx \, dy. = \iint_R \frac{\partial Q}{\partial x} dA.$$

$$\therefore \oint_C (\bar{F} \cdot \hat{t}) ds = \oint_C P(x,y) dx + \oint_C Q(x,y) dy$$

$$= -\iint_R \frac{\partial P}{\partial y} dA + \iint_R \frac{\partial Q}{\partial x} dA.$$

$$= \iint_R \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)}_{\nabla \times \bar{F}} dA.$$

(QED) ('meaning' nicely concluded)

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