EE1080/AI1110/EE2102 Probability: HW 7

25th April, 2025

1. Suppose that Y is a normal random variable with mean 0 and variance 1, and suppose also that the conditional distribution of X, given that $Y = y$, is normal with mean y and variance 1.
   (a) Argue that the joint distribution of X,Y is the same as that of Y + Z,Y when Z is a standard normal random variable that is independent of Y.
   (b) Use the result of part (a) to argue that X,Y has a bivariate normal distribution.
   (c) Find $E[X]$, $Var(X)$, and $Cov(X, Y)$.
   (d) Find $E[Y|X = x]$.
   (e) What is the conditional distribution of Y given that $X = x$?

2. Let Y, N_1, N_2 be zero mean, unit variance, independent random variables, and suppose we observe, for some constant $\alpha > 0$

$$X_1 = Y + N_1 + \alpha N_2 \quad X_2 = Y + 3N_1 + \alpha N_2.$$

   (a) Find the LMMSE estimate of Y from X_1 and X_2
   (b) What is the corresponding MSE ?
   (c) At what value of α does the MSE from part (b) become zero ?

3. Suppose $X \sim N(\mu, K_X)$ be Gaussian random vector with

$$\mu = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}, \; K_X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

   (a) Find the pdf of X_1.
   (b) Find the pdf of (X_2, X_3) given X_1.
   (c) Find the pdf of $2X_1 + X_2 + X_3$.
   (d) Find the pdf of $Y = AX$ where $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

4. Plot the points at which the joint pdf value is exactly 0.8 of the maximum possible value for the 2-Gaussian vector defined with following covariance matrices.

   (a) $K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   (b) $K = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix}$

   (c) $K = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

5. Can the following matrix be a covariance matrix ? $K = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix}$

6. Let Q be an orthonormal matrix. Show that the squared distance between any two vectors z and y is equal to the squared distance between Qz and Qy.

7. Let $K = \begin{bmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{bmatrix}$.

   (a) Show that 1 and 1/2 are eigenvalues of K and find the normalized eigenvectors. Express K as QDQ^{-1}, where D is diagonal and [Q] is orthonormal.

   (b) Let $K' = \alpha K$ for $\alpha \neq 0$ Find the eigenvalues and eigenvectors of K'. Do not use brute force

   (c) Find the eigenvalues and eigenvectors of K^m, where K^m is the mth power of K.

8. Let X and Y be zero-mean jointly Gaussian with variances σ_X^2 and σ_Y^2 and normalized covariance ρ.

   (a) Let $V = Y^3$. Find the conditional density $f_{X|V}(x|v)$. Hint: This requires no computation.

   (b) Let $U = Y^2$ and find the conditional density of $f_{X|U}(x|u)$. Hint: First understand why this is harder than (a).