Best 2 out of 3 qns will be used for marking due to typo in the third qn.

① $E[X] = 75$,

ⓐ $P(X > 85) \overset{0.5}{\leq} \frac{E[X]}{85} \overset{0.5}{=} \frac{15}{17}$.

ⓑ $Var(X) = 25$

$P(65 \leq X \leq 85) \overset{0.5}{=} P(|X - 75| \leq 10)$.

$P(|X - 75| > 10) \overset{0.5}{\leq} \frac{Var(X)}{100} = \frac{1}{4}$.

$P(|X - 75| \leq 10) \geq 3/4$.

Will be given marks if you use CLT based estimate for b part as it doesn't mention lower bound in the qn.

ⓒ $X_1, X_2, \ldots X_n$ iid RVs. $E[X_1] = 75$, $Var(X_1) = 25$.

Find n st $P\left(70 \leq \frac{\sum_{i=1}^{n} X_i}{n} \leq 80\right) \geq 0.9$. } 0.5 (problem formulation).

$P\left(70 \leq \frac{\sum_{i=1}^{n} X_i}{n} \leq 80\right) \overset{0.5}{=} P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - 75\right| \leq 5\right)$

$P\left(\left|\frac{\sum_{i=1}^{n} X_i}{n} - 75\right| > 5\right) \overset{0.5}{\leq} \frac{Var\left(\frac{\sum X_i}{n}\right)}{5^2} = \frac{Var(X_1)}{n \cdot 25}$

$= \frac{1}{n}$.

If $\frac{1}{n} \overset{0.5}{\leq} 0.1$ i.e., $n \geq 10$.

then $P\left(70 \leq \frac{\sum_{i=1}^{n} X_i}{n} \leq 80\right) = 1 - P\left(\left|\frac{\sum X_i}{n} - 75\right| > 5\right)$

$\geq 1 - 0.1 = 0.9$.

(d) Using central limit theorem.

$$P\left(70 \leq \frac{\sum_{i=1}^{n} X_i}{n} \leq 80\right)$$

$$= P\left(-\sqrt{n} \leq \frac{\sum_{i=1}^{n}(X_i - 75)}{5\sqrt{n}} \leq \sqrt{n}\right).$$

0.5

By central limit theorem

0.5 $$\left\{ Z_n = \frac{\sum_{i=1}^{n}(X_i - \mu)}{\sigma\sqrt{n}} \xrightarrow[\text{distribution}]{\text{in}} Z \sim N(0,1) \right.$$ standard normal distribution

$$\Rightarrow P\left(70 \leq \frac{\sum_{i=1}^{n} X_i}{n} \leq 80\right)$$

$$= \Phi(\sqrt{n}) - \Phi(-\sqrt{n}).$$

0.5 $$\left\{ = 2\Phi(\sqrt{n}) - 1. \right.$$

$$2\Phi(\sqrt{n}) - 1 \geq 0.9.$$

$$\Rightarrow \Phi(\sqrt{n}) \geq 0.95.$$

$$\Rightarrow \sqrt{n} \geq \Phi^{-1}(0.95)$$

0.5 $$\left\{ \Rightarrow \sqrt{n} \geq 1.65 \Rightarrow n \geq (1.65)^2. \right.$$

$$\Rightarrow n \geq 3.$$ [as n should also be a integer].

② $P(X>7) = P\left(\frac{X-4}{2} > \frac{7-4}{2}\right) = 1-\Phi(1.5)$ $\to 0.5$

$0.5 \checkmark = 1-0.93319 \approx 0.067$

ⓐ $P(X>7) = P(X-4>3)$

$0.5\checkmark \quad \leq P(|X-4|>3)$

$0.5\checkmark \quad 1 \leq \frac{Var(X)}{9} = \frac{4}{9} = 0.44$

full marks even if intermediate steps are not provided if it is correct.

Will be given marks if you used one sided Chebyshev from HW to get $\frac{4}{4+9} = \frac{4}{13} = 0.307$

ⓑ $P(X>7) \leq \inf_{t>0} E[e^{Xt}] \cdot e^{-7t}$ — 1 mark for correct Chernoff expression.

Z be standard normal R.V.

$X = \sigma Z + \mu$ where $\sigma=2$, $\mu=4$.

$M_X(t) = E[e^{Xt}] = E[e^{(\sigma Z+\mu)t}]$

$= e^{\mu t} M_Z(\sigma t)$

$= e^{\mu t + \sigma^2 t^2/2} = e^{4t+2t^2}$ — 1 mark

$P(X>7) \leq \inf_{t>0} e^{4t+2t^2-7t}$

$= e^{-\sup_{t>0} \underbrace{(3t-2t^2)}_{f(t)}}$ — 2 marks.

$f'(t) = 3-4t = 0 \qquad f''(t) = -4.$

$\Rightarrow f(t)$ has sup at $t=3/4$ $\Rightarrow \sup_{t>0} f(t) = 3\times\left(\frac{3}{4}\right) - 2\left(\frac{3}{4}\right)^2$

$= \frac{9}{4} - 2\times\frac{9}{16} = 9/8.$

Other way

$P\left(\frac{X-4}{2} > 1.5\right)$

$= P(Z>1.5)$

$\leq \inf_{t>0} E[e^{Zt}] e^{-1.5t}$

$= \inf_{t>0} e^{t^2/2 - \frac{3}{2}t}$

$= e^{-\frac{1}{2}\sup_t (3t-t^2)}$

$= e^{-9/8}$.

$$P(X > 7) \leq e^{-9/8} \approx 0.324.$$

Surprising that one sided chebyshev gives better bound than Chernoff.

(3) Convergence $E[X_n] = 0$, $Var(X_n) = \frac{1}{n^2}$.

(a)

$$P(|X_n| > \epsilon) \leq \frac{Var(X_n)}{\epsilon^2} = \frac{1}{n^2\epsilon^2}$$

$$\Rightarrow \lim_{n\to\infty} P(|X_n| > \epsilon) \leq \lim_{n\to\infty} \frac{1}{n^2\epsilon^2} = 0$$

1 mark

1 mark for defn.

$$\Rightarrow \lim_{n\to\infty} P(|X_n| > \epsilon) = 0$$

$\therefore$ X_i's converge to 0 in probability.

(b) $A_n = \{w : |X_n(\omega)| > \epsilon\}$.

1 mark $P(A_n) \leq \frac{1}{n^2\epsilon^2}$ by chebyshev inequality

1 mark $\Rightarrow \sum_{n=1}^{\infty} P(A_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2\epsilon^2} < \infty$ → finite sum

From Borel Cantelli lemma

1 mark

$$\Rightarrow P(\limsup_{n\to\infty} A_n) = 0.$$

i.e., $P(\{w : |X_n(\omega)| > \epsilon \ i.o\}) = 0$

i.e., $X_n \to X$ in a.s sure.

(c)

$$E[X_n^2] = Var(X_n) + (E[X_n])^2$$

$$= \frac{1}{n^2}$$

0.5.

0.5 defn

$$\lim_{n\to\infty} E[X_n^2] = \lim_{n\to\infty} \frac{1}{n^2} = 0$$

(d) Converges in distribution as they converge in probability. } 1 mark

Proving (c) $\Rightarrow$ (d), (b).

Proving (a) $\Rightarrow$ (d), (b).

For d, b enough to show c or a and say it is implied or do the proof independently

Please give full marks if the implications are clearly stated and the proof for c or a is provided.