EE2100: Matrix Analysis

Review Notes - 31

Topics covered :

1. Quadratic Forms

2. Positive Definite and Positive Semidefinite Matrices

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1. A quadratic form in n variables (say $x_1,\ x_2,\ \cdots,\ x_n$; can alternately be denoted by a vector $\mathbf{x}$) is a degree-2 homogeneous polynomial, which is typically of the form

$$\begin{aligned} f(x_1, \cdots, x_n) &= \sum_{i=1}^{n} c_i x_i^2 + \sum_{i\neq j,\ i=1,\ j=1}^{n,n} c_{ij} x_i x_j \\ \text{i.e., } f(x_1, \cdots, x_n) &= \sum_{i=1}^{n} c_i x_i^2 + \sum_{i>j} c_{ij} x_i x_j \end{aligned} \tag{1}$$

In the context of matrices, quadratic forms can be represented as

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} \tag{2}$$

where $\mathbf{A} \in \mathcal{R}^{n\times n}$. The matrix $\mathbf{A}$ corresponding to a given quadratic form is given by

$$A_{ii} = c_i \text{ and } A_{ij} + A_{ji} = c_{ij} \tag{3}$$

2. The matrix representation of a given quadratic from is not unique. However, the choice of $\mathbf{A}$ that is commonly adopted to represent a given quadratic form is the one that is symmetric i.e.,

$$A_{ii} = c_i \text{ and } A_{ij} = A_{ji} = \frac{1}{2} c_{ij} \tag{4}$$

3. Let $\mathbf{A} \in \mathcal{R}^{n\times n}$ be a symmetric matrix. Then $\mathbf{A}$ is a positive definite matrix if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0\ \forall\ \mathbf{x} \neq \mathbf{0}$. A common way to represent positive definite matrices is by using the notation $\mathbf{A} > 0$ or $\mathbf{A} \succ 0$. Similarly, $\mathbf{A}$ is a positive semidefinite matrix if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0\ \forall\ \mathbf{x}$. A common way to represent negative definite matrices is by using the notation $\mathbf{A} \geq 0$ or $\mathbf{A} \succeq 0$. (Recollect the arguments covered in the class related to negative definite/semidefinite and indefinite matrices).

4. In Lecture 30, it is shown that

$$\lambda_{min}\|\mathbf{x}\|_2^2 \leq \mathbf{x}^T\mathbf{A}\mathbf{x} \leq \lambda_{max}\|\mathbf{x}\|_2^2 \tag{5}$$

Accordingly, for $\mathbf{A}$ to be positive definite, its eigen values must be strictly positive. Similarly, for $\mathbf{A}$ to be positive semi definite, the eigen values must be positive or zero.

5. Properties of Positive Definite/Semidefinite matrices:

(a) If $\mathbf{A}$ is a positive definite matrix, then $\mathbf{Tr}(\mathbf{A}) > 0$.

Proof: Since, $\mathbf{A}$ is a positive definite matrix, all its eigen values are > 0. Accordingly, $\mathbf{Tr}(\mathbf{A}) = \sum_i \lambda_i > 0$.

(b) If $\mathbf{A}$ is a positive definite matrix, then $\mathbf{Det}(\mathbf{A}) > 0$.

Proof: Since, $\mathbf{A}$ is a positive definite matrix, all its eigen values are > 0. Accordingly, $\mathbf{Det}(\mathbf{A}) = \prod_i \lambda_i > 0$.

(c) If $\mathbf{A}$ is a positive definite matrix, then $A_{ii} > 0$.

Proof: Since, $\mathbf{A}$ is a positive definite matrix, $\mathbf{x}^T\mathbf{A}\mathbf{x}$ must be $> 0 \; \forall \mathbf{x} \neq \mathbf{0}$. For $\mathbf{x}^T\mathbf{A}\mathbf{x}$ to be > 0 when $\mathbf{x} = \mathbf{e}_i$, it is necessary that $A_{ii} > 0$.

A more generic property related to diagonal entries of a positive definite matrix is the following: $\lambda_{min} \leq A_{ii} \leq \lambda_{max}$.

(d) If $\mathbf{A}$ is a positive semidefinite matrix, then, $\mathbf{P}^T\mathbf{A}\mathbf{P}$ is also positive semidefinite for every $\mathbf{P} \in \mathcal{R}^{n\times m}$.

Proof: Let $\mathbf{C} = \mathbf{P}^T\mathbf{A}\mathbf{P} \in \mathcal{R}^{m\times m}$. The quadratic form $\mathbf{x}^T\mathbf{C}\mathbf{x}$ can be written as

$$\begin{aligned} \mathbf{x}^T\mathbf{C}\mathbf{x} &= \mathbf{x}^T\mathbf{P}^T\mathbf{A}\mathbf{P}\mathbf{x} \\ &= \mathbf{y}^T\mathbf{A}\mathbf{y} \text{ where } \mathbf{y} = \mathbf{P}\mathbf{x} \\ &\geq 0 \text{ since } \mathbf{y}^T\mathbf{A}\mathbf{y} > 0 \;\forall\; \mathbf{y} \end{aligned} \tag{6}$$

Thus, $\mathbf{C}$ is positive semidefinite.

(e) Every positive semidefinite matrix ($\mathbf{A}$) can be expressed as $\mathbf{A} = \mathbf{P}^T\mathbf{P}$.

Proof: Since $\mathbf{A}$ is symmetric, it can be expressed as

$$\mathbf{A} = \mathbf{Q}^T\mathbf{D}\mathbf{Q} \tag{7}$$

where $\mathbf{D}$ is a diagonal matrix whose entries are eigen values. Further, since $\mathbf{A} \succeq 0$, its eigen values are either positive or zero. Thus, the matrix $\mathbf{D}$ can be expressed as $\mathbf{D} = \mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}$ where $\mathbf{D}^{\frac{1}{2}} = \mathbf{Diag}(\sqrt{\lambda_1}, \cdots, \sqrt{\lambda_n})$. Accordingly,

$$\begin{aligned} \mathbf{A} &= \mathbf{Q}^T\mathbf{D}\mathbf{Q} \\ &= \mathbf{Q}^T\mathbf{D}^{\frac{1}{2}}\mathbf{D}^{\frac{1}{2}}\mathbf{Q} \\ &= \mathbf{P}^T\mathbf{P} \text{ where } \mathbf{P} = \mathbf{D}^{\frac{1}{2}}\mathbf{Q} \end{aligned} \tag{8}$$

(f) Every positive semidefinite matrix ($\mathbf{A}$) can be expressed as $\mathbf{A} = \mathbf{L}\mathbf{L}^T$ where $\mathbf{L}$ is a lower triangular matrix.

Proof: Since $\mathbf{A}$ is positive semi definite, $\mathbf{A} = \mathbf{P}^T\mathbf{P}$.

An extension of Grahm-Schmidt Algorithm to matrices gives a matrix factorization technique commonly referred to as $\mathbf{QR}$ factorization (will be covered in a few lectures from now). Accordingly, any matrix $\mathbf{A}$ can be represented as $\mathbf{A} = \mathbf{QR}$ where $\mathbf{Q}$ is an orthonormal matrix and $\mathbf{R}$ is an upper triangular matrix.

Thus, $\mathbf{P}$ can be expressed as $\mathbf{P} = \mathbf{QR}$. The positive semidefinite matrix can in turn be expressed as

$$\begin{aligned} \mathbf{A} &= \mathbf{P}^T\mathbf{P} = (\mathbf{QR})^T\mathbf{QR} \\ &= \mathbf{R}^T\mathbf{R} = \mathbf{LL}^T \text{ where } \mathbf{L} = \mathbf{R}^T \text{ is a lower triangular matrix} \end{aligned} \tag{9}$$

(g) If $\mathbf{A}$ is positive semidefinite, then $\mathbf{A}^k$ is also positive semidefinite.

Proof: In general, for any symmetric matrix $\mathbf{A}$ it can be shown that $\mathbf{A}^k = \mathbf{QD}^k\mathbf{Q}^T$. Since $\mathbf{A}$ is positive semidefinite, the matrix $\mathbf{D}^k = \mathbf{Diag}(\lambda_1^k, \cdots, \lambda_n^k)$ is also positive semidefinite. Thus, by property (d), it can be inferred that $\mathbf{A}^k$ is also positive semidefinite.