1

$$E[X^n] = \int_{t=0}^{\infty} P(X^n > t)\, dt \qquad \begin{aligned} t &= x^n. \\ dt &= n x^{n-1} dx. \end{aligned}$$

$$= \int_{x=0}^{\infty} P(X^n > x^n)\, n x^{n-1}\, dx.$$

$$= \int_{x=0}^{\infty} P(X > x)\, n x^{n-1}\, dx.$$

2

$$P_Y(y) = \begin{cases} 1/3 & y = 1 \\ 2/3 & y = 2 \end{cases}$$

$$E[Y] = \frac{1}{3} + \frac{4}{3} = \frac{5}{3}.$$

$$= \int_0^1 g(x) f_X(x)\, dx$$

$$= \int_0^{1/3} 1\, dx + \int_{1/3}^{1} 2\, dx.$$

$$= \frac{1}{3} + 2 \times \frac{2}{3} = \frac{5}{3}.$$

3

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|}$$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 2\int_0^{\infty} \frac{\lambda}{2} e^{-\lambda x}\, dx.$$

$$= 2 \frac{\lambda}{2} \left(\frac{1}{-\lambda}\right) e^{-\lambda x} \Big|_0^{\infty}.$$

$$= 1.$$

$$E[X] = \int_{-\infty}^{\infty} \underbrace{x\, f_X(x)}_{\text{odd function}}\, dx.$$

$$= 0$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx.$$

$$= 2\int_0^\infty \frac{\lambda}{2} x^2 e^{-\lambda x} dx.$$

$$= \int_0^\infty \lambda x^2 e^{-\lambda x}.$$

$$= \frac{2}{\lambda^2}.$$

(4) $$f_X(x) = \begin{cases} c\left(x - \frac{3}{x^2}\right) & 2 < x < 4 \\ 0 & \text{o.w.} \end{cases}$$

$$\int_2^4 f_X(x)\,dx = c\int_2^4 \left(x - \frac{3}{x^2}\right)$$

$$= c\left[\left.\frac{x^2}{2}\right|_2^4 - 3\left.\frac{x^{-2+1}}{(-1)}\right|_2^4\right]$$

$$= c\left[6 + 3\left[\frac{1}{4} - \frac{1}{2}\right]\right].$$

$$= c\left[6 - \frac{3}{4}\right] = \frac{21}{4}c.$$

$$\Rightarrow c = 4/21.$$

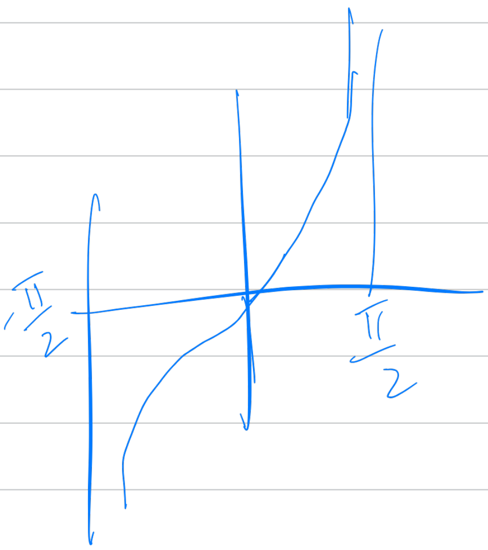
$$\int_2^x f_X(x)\,dx = \left[\frac{4}{21}\right]\left(\frac{x^2}{2} + \frac{3}{x}\right).$$

$$- \frac{4}{21}\left(\frac{4}{2} + \frac{3}{2}\right).$$

$$= \left(\frac{4}{21}\right)\left(\frac{x^2}{2} + \frac{3}{x}\right) - \frac{2}{3}.$$

(5) $f_X(x) = \dfrac{1}{\pi(1+x^2)}$.

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\pi(1+x^2)}\, dx.$$



$x = \tan\theta$.

$1+x^2 = \dfrac{1}{\cos^2\theta}$

$$= \int_{-\pi/2}^{\tan^{-1}(x)} \frac{\cancel{\cos^2\theta}}{\pi} \cdot \frac{d\theta}{\cancel{\cos^2\theta}}$$

$dx = \dfrac{\cancel{\cos\theta}}{\cancel{\cos\theta}} + \dfrac{\sin^2\theta}{\cos^2\theta} = \dfrac{1}{\cos^2\theta}\,d\theta$

$$= \frac{\left(\tan^{-1}(x) + \frac{\pi}{2}\right)}{\pi}$$

$$\lim_{x\to\infty} F_X(x) = 1.$$

(c) $f_X(x) = \begin{cases} c x e^{-\sqrt{x}} & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$

$$P(X > 10) = 1 - \int_0^{10} c x e^{-\sqrt{x}}\, dx.$$

$$F_X(x) = \int_0^x c x e^{-\sqrt{x}}\, dx$$

$$= \int_0^x c y^2 e^{-y}\, 2y\, dy.$$

$\sqrt{x} = y.$

$x = y^2$

$$= \int_0^x 2c\, y^3 e^{-y}\, dy.$$

$dx = 2y\,dy$

$$= -2c\, y^3 e^{-y} \Big|_0^x + \int_0^x 6c\, y^2 e^{-y}\, dy$$

$$\int y^2 e^{-y}\, dy = -y^2 e^{-y}\Big|_0^x + \int_0^x 2y\, e^{-y}\, dy$$

$$\int_0^x y\, e^{-y}\, dy = -y e^{-y}\Big|_0^x + \int_0^x e^{-y}\, dy$$

$$= 1 - e^{-x} - x e^{-x},$$

$$F_X(x) = -2c\, x^3 e^{-x} + 6c\left[-x^2 e^{-x} + 2\left(1 - e^{-x} - x e^{-x}\right)\right]$$

$$\lim_{x\to\infty} F_X(x) = 12c \Rightarrow c = 1/12.$$

$$P(X > 10) = 1 - F_X(10).$$

$$= \frac{1}{12}\left[-2 \times 10^3 e^{-10} - 6 \times 10^2 e^{-10} - 22 e^{-10} + 2\right]$$

$$= \frac{1}{6} - (2622)\, e^{-10}.$$

(7) $P(U \le u) = P(F(X) \le u).$

assuming F is monotonic in [0,1] $\leftarrow$ $= P(X \le F^{-1}(u)).$

$= F(F^{-1}(u)) = u.$

since F is CDF of X

$f_U(u) = 1 \qquad u \in [0,1].$

(b) $F(x) = 1 - e^{-\lambda x}$.

$1 - e^{-\lambda x} = u$.

$X = F^{-1}(u)$.

$(1-u) = e^{-\lambda x}$.

$\frac{1}{\lambda}\log\left(\frac{1}{1-u}\right) = x$.

i.e., $X = \frac{1}{\lambda}\log\left(\frac{1}{1-U}\right)$.

given U is uniform in $[0,1]$

X is exponential (λ) R.V.

(c) let $X = k$ if $F_X(k-1) < U \le F_X(k)$.

(We saw this in class)

$$P(X=k) = P\left(F_X(k-1) < U \le F_X(k)\right)$$

$$= F_X(k) - F_X(k-1).$$

$$= P_X(k).$$

$\therefore$ We get desired R.V.

(8) A : 0 customers ahead

A^c : 1 "

$P(A) = P(A^c) = \frac{1}{2}$.

X : wait time

$$P(X \le x) = P(X \le x \mid A)\,P(A) + \underbrace{P(X \le x \mid A^c)}_{\text{CDF of exp}(\lambda)}P(A^c)$$

$$= 0 \qquad x < 0$$

$$= 1 \cdot \frac{1}{2} + \left(1 - e^{-\lambda x}\right)\frac{1}{2} \qquad x \ge 0$$

9. $X \sim \text{Uniform}([0, L])$.

$$P\left(\frac{\min(X, L-X)}{\max(X, L-X)} > \frac{1}{4}\right).$$

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A: $X \geq L - X$. i.e., $2X \geq L$

A^c: $X < L - X$

$$P(A) = P(A^c) = \frac{1}{2}.$$

$$P\left(\frac{\min(X, L-X)}{\max(X, L-X)} > \frac{1}{4} \,\middle|\, A\right) P(A)$$

$$+ P\left(\frac{\min(X, L-X)}{\max(X, L-X)} > \frac{1}{4} \,\middle|\, A^c\right) P(A^c).$$

$$= \frac{1}{2} P\left(L - X > \frac{1}{4}X \,\middle|\, X \geq \frac{L}{2}\right)$$

$$+ \frac{1}{2}\left(X > \frac{1}{4}(L-X) \,\middle|\, X \leq \frac{L}{2}\right).$$

$$P\left(X \leq \frac{4}{5}L \,\middle|\, \left(X \geq \frac{L}{2}\right)\right) = \frac{P\left(\frac{L}{2} \leq X \leq \frac{4}{5}L\right)}{P\left(X \geq \frac{L}{2}\right)}$$

$$= \frac{\left(\frac{4}{5} - \frac{1}{2}\right)}{\frac{1}{2}} = \frac{3}{5}.$$

$$P\left(X \geq \frac{1}{5}L \,\middle|\, X \leq \frac{L}{2}\right) = \frac{\frac{1}{2} - \frac{1}{5}}{\frac{1}{2}} = \frac{3}{5}.$$

$$\therefore P\left(\frac{\min(X, L-X)}{\max(X, L-X)} \geq \frac{1}{4}\right) = 3/5.$$

10. $F_X(x) = P(X \leq x)$.

A: event that $X = Y$

A^c: " " $X = Z$

also assuming that these are disjoint events which is not clear in the q.

$$F_X(x) = P(X \leq x \mid A) P(A) + P(X \leq x \mid A^c) P(A^c)$$

$$= p\, P(Y \le x) + (1-p)\, P(Z \le x)$$

$$= p\, F_Y(x) + (1-p)\, F_Z(x).$$

$$\frac{\partial}{\partial x} F_X(x) = p \frac{\partial}{\partial x} F_Y(x) + (1-p) \frac{\partial F_Z(x)}{\partial x}$$

$$= p\, f_Y(x) + (1-p) f_Z(x).$$

Remark:

$$P(X \le x \mid A).$$

$$= \frac{P(\{X \le x\} \cap A)}{P(A)}.$$

Assume event A is independent of $\{Z \le x\}$.

$$= \frac{P\{\omega \in A \mid Z(\omega) \le x\}}{P(A)}.$$

$$= \frac{\cancel{P(A)}\, P(Z \le x)}{\cancel{P(A)}} = P(Z \le x).$$

(b)

$$F_X(x) = \begin{cases} \int_{-\infty}^{x} p \lambda e^{\lambda x} & x \le 0 \\ \int_{-\infty}^{0} p \lambda e^{\lambda x} + \int_{0}^{x} (1-p) \lambda e^{-\lambda x} & x \ge 0 \end{cases}$$

$$= \begin{cases} p e^{\lambda x} & x \le 0 \\ p + (1-p)(1 - e^{-\lambda x}) & x \ge 0 \end{cases}$$

(11) A: event taxi is waiting ; $P(A) = \frac{2}{3}$, $P(A^c) = \frac{1}{3}$,

$$F_X(x) = P(A)P(X \le x \mid A) + P(X \le x \mid A^c)\,P(A^c)$$

B: event taxi arrives < 5 mins.

$$P(X \le x \mid A^c) = P(B \mid A^c)\,P(X \le x \mid B \cap A^c)$$

$$+ P(B^c \mid A^c) \cdot P(X \le x \mid B^c \cap A^c)$$

$$P(X \le x \mid A) = \begin{cases} 0 & x < 0 \\ 1 & x \ge 0 \end{cases}$$

$U \sim \text{Uniform}[0, 10]$, $B: 0 \le U \le 5$.

$$P(B \mid A^c) = P(B) = P(0 \le U < 5)$$

$$= 1/2$$

$$P(B^c \mid A^c) = \frac{1}{2}$$

$$P(X \le x \mid B \cap A^c) = \frac{P(U \le x, \overbrace{B}^{U \le 5} \cap A^c)}{P(B \cap A^c)}$$

Wait time in this case is time to get a car i.e., U.

$$= \frac{P(U \le x)\,\cancel{P(A^c)}}{P(U \le 5)\,\cancel{P(A^c)}}$$

$$= \frac{x/10}{5/10} = x/5$$

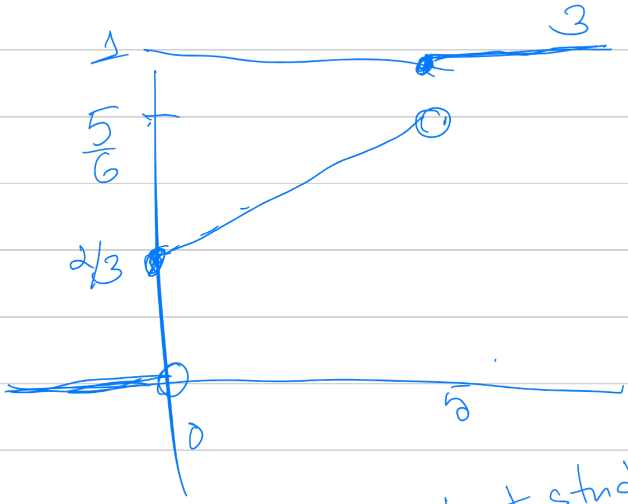
$$P(X \le x \mid B^c \cap A^c) = \begin{cases} 0 & x < 5 \\ 1 & x = 5 \end{cases}$$

$$F_X(x) = 0 \qquad x < 0$$

$$= \frac{2}{3} + \frac{1}{3}\left[\frac{1}{2} \cdot \frac{x}{5}\right] \qquad x < 5$$

$$= \frac{2}{3} + x/30$$

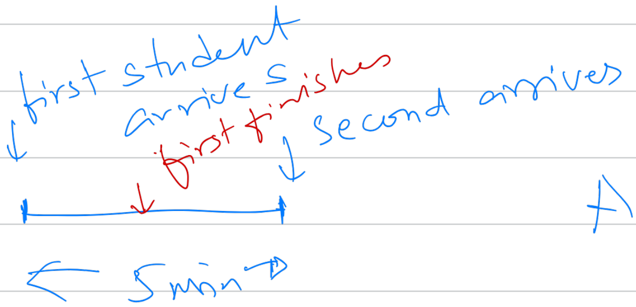
$$= \frac{2}{3} + \frac{1}{3}\left[\frac{1}{2} + \frac{1}{2}\right] = 1. \qquad x = 5.$$



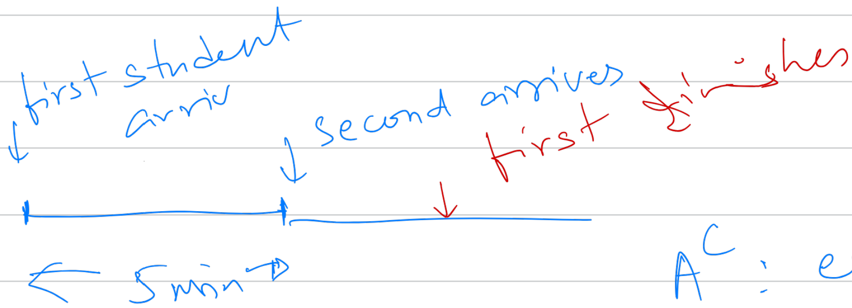
12.

$X_1 \sim \exp\left(\frac{1}{30}\right)$

$X_2 \sim \exp\left(\frac{1}{30}\right)$



A: event that $x_1 < 5$.



A^c: event that $x_1 \geq 5$.

$$X \mid A = 5 + X_2$$

$$X \mid A^c = X_1 + X_2.$$

$$P(A) = \left(1 - e^{-5/30}\right)$$

$$P(A^c) = e^{-1/6}.$$

$$E[X] = E[X \mid A]\, P(A) + E[X \mid A^c]\, P(A^c)$$

$$= (5 + 30)\left(1 - e^{-1/6}\right) + (60)\, e^{-1/6}.$$

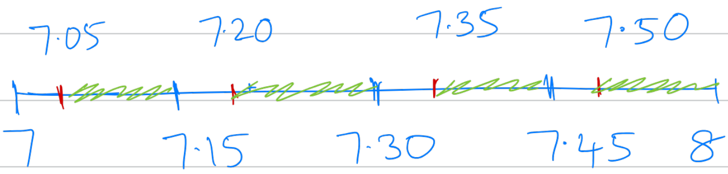
Made with Goodnotes

$$= 35 - 30\, e^{-1/6}.$$

13.

$$P_1 \sim \text{Uniform}\ [7 : 8]$$

Proportion of time passenger goes to destination B.



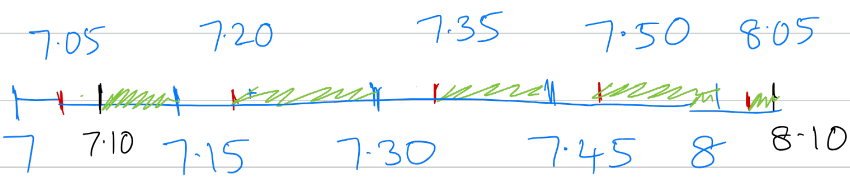
$$P\big(P_1 \in [7.05, 7.15] \cup (7.20, 7.30] \cup (7.35, 7.45] \cup (7.50, 8.00]\big).$$

$$= \frac{40}{60} = 2/3.$$

$$P_2 \sim \text{Uniform}\ [7.10 : 8.10].$$

$$P\big(P_2 \in [7.10 : 7.15] \cup [7.20, 7.30] \cup [7.35, 7.45] \cup [7.5, 8] \cup [8.05 : 8.10]\big).$$

$$= \frac{40}{60} = \frac{2}{3}.$$



(14)

$$P(X_i = 1) = 0.65.$$

(a)

$$P\left(\sum_{i=1}^{100} X_i \geq 50\right) = P\left(\frac{\sum_{i=1}^{100} X_i}{100} \geq \frac{1}{2}\right)$$

$$P\left(\frac{\sum_{i=1}^{100}(X_i-0.65)}{100} \geq \frac{1}{2}-0.65\right).$$

$$= P\left(\frac{\sum_{i=1}^{100}(X_i-0.65)}{10\sqrt{Var(X_i)}} \geq \frac{-1.5}{\sqrt{Var(X_1)}}\right).$$

$Var(X_1) = 0.65 \times 0.35$.

$$= 1-\Phi\left(\frac{-1.5}{\sqrt{Var(X_1)}}\right)$$

Uses the standard normal approximation of distribution for:

$$= \Phi\left(\frac{1.5}{\sqrt{Var(X_1)}}\right).$$

$$\hat{S}_n = \frac{\sum_{i=1}^{n}(X_i - E[X_i])}{\sqrt{n Var(X_i)}}.$$

$$= \Phi\left(\frac{1.5}{\sqrt{0.65\times 0.35}}\right).$$

X_i's are iid.
In this case Bernoulli($p=0.65$).

$n = 100$ here
$E[X_i] = 0.65$
$Var(X_i) = 0.65 \times 0.35$

(b)

$$P\left(60 \leq \sum X_i \leq 70\right).$$

$$= P\left(6 \leq \frac{\sum_{i=1}^{100} X_i}{10} \leq 7\right).$$

$$= P\left(-0.5 \leq \frac{\sum_{i=1}^{100}(X_i-0.65)}{10} \leq 0.5\right),$$

$$= P\left(\frac{-0.5}{\sqrt{Var(X_i)}} \leq \frac{\sum_{i=1}^{100}(X_i-0.65)}{10\sqrt{Var(X_i)}} \leq \frac{0.5}{\sqrt{Var(X_i)}}\right)$$

$$= \Phi\left(\frac{0.5}{\sqrt{Var(X_i)}}\right) - \Phi\left(\frac{-0.5}{\sqrt{Var(X_i)}}\right)$$

$$= 2\,\Phi\left(\frac{0.5}{\sqrt{Var(X_i)}}\right) - 1.$$

(c) Similarly $P\left(\sum X_i \le 75\right)$.

(15) $Var(X) = E[X^2] - (E[X])^2$.

$$E[X^2] \le c\,E[X]$$

$$\text{"} \int_0^c x^2 f_X(x)\,dx \;\le\; c\int_0^x x\, f_X(x)\,dx$$

$$= c\,E[X].$$

$$\Rightarrow Var(X) \le c\,E[X] - (E[X])^2$$

$$= E[X]\,(c - E[X]).$$

maximized when $E[X] = c/2$.

$$f(y) = y\,(c-y)$$

$$\frac{\partial f}{\partial y} = c - 2y \quad \Rightarrow \quad \frac{\partial f}{\partial y} = 0.$$

if $y = c/2$.

$$\Rightarrow Var(X) \le c^2/4.$$

16. To show $E[X^k] = \frac{k!}{\lambda^k}$, show this by induction.

We know $E[X] = 1/\lambda$

Assume true for $n < k$ that $E[X^n] = \frac{n!}{\lambda^n}$.

$$E[X^k] = \int_0^\infty x^k e^{-\lambda x} \lambda \, dx$$

$$= \boxed{-x^k e^{-\lambda x} \Big|_0^\infty} = 0$$

$$+ \int_0^\infty k x^{k-1} e^{-\lambda x} dx$$

$$= \frac{k}{\lambda} \int_0^\infty x^{k-1} \lambda e^{-\lambda x} dx$$

$$= \frac{k}{\lambda} E[X^{k-1}]$$

$$= \frac{k}{\lambda} \frac{(k-1)!}{\lambda^{k-1}} = \frac{k!}{\lambda^k}.$$

17. (a) $$F_X(x) = \int_a^x \frac{1}{(b-a)} dx.$$

$$= \frac{x-a}{b-a}$$

$$\frac{x-a}{b-a} = \frac{1}{2} \Rightarrow x = a + \left(\frac{b-a}{2}\right).$$

$$= \frac{a+b}{2}.$$

$$\Rightarrow m = \frac{a+b}{2}.$$

(b) $$F_X(x) = \int_{-\infty}^x \frac{e^{-\left(\frac{\cancel{x}-\mu}{\sigma}\right)^2 \frac{1}{2}}}{\sqrt{2\pi}\,\sigma} d\cancel{t}$$

Made with Goodnotes

$$y = \frac{t-\mu}{\sigma} \qquad = \int_{-\infty}^{\frac{x-\mu}{\sigma}} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\, dy$$

$$dt = \sigma\, dy \qquad = \Phi\left(\frac{x-\mu}{\sigma}\right).$$

We know that $\Phi(0) = 1/2$.

$\Rightarrow$ $F_X(x) = 1/2$
when $\frac{x-\mu}{\sigma} = 0$ i.e., for $x = \mu$.

(c) $$F_X(x) = \int_0^x \lambda e^{-\lambda x}\, dx$$

$$= 1 - e^{-\lambda x}.$$

$$F_X(m) = 1/2$$

$$\Rightarrow 1 - e^{-\lambda m} = 1/2.$$

$$\Rightarrow 1/2 = e^{-\lambda m}$$

$$m = \frac{\log 2}{\lambda}.$$