→ Continue discussion on planes & lines

→ Introduce Curvilinear Co-ordinates.

(a) Cylindrical Co-ordinates.

(b) Spherical co-ordinates.

EE1203: Vector Calculus

Aneesh Sobhanan
Department of Electrical Engineering
IIT Hyderabad, India

9^{th} January 2025

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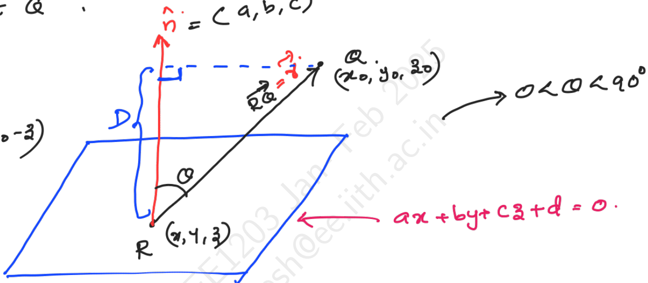
Distance between a point and a plane.

Theorem: Let $Q = (x_0, y_0, z_0)$ be a point in R^3 and let $ax + by + cz + d = 0$ be a plane not passing through point 'Q'.



$$\vec{RQ} = (x_0 - x,\ y_0 - y,\ z_0 - z)$$

$$D = |\vec{r}| \cos\theta. = |\vec{r}| \frac{\hat{n} \cdot \vec{r}}{|\hat{n}||\vec{r}|} = \frac{\hat{n} \cdot \vec{r}}{|\hat{n}|}$$

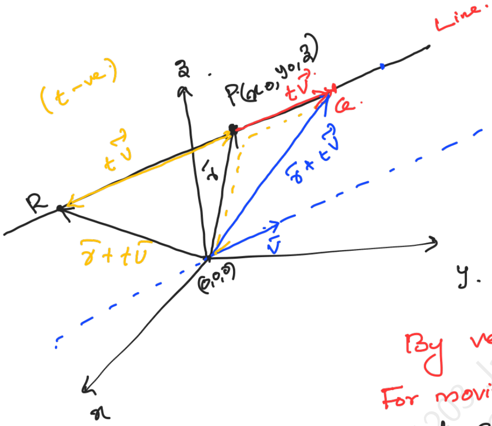
$$= \frac{a(x_0 - x) + b(y_0 - y) + c(z_0 - z)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_0 + by_0 + cz_0 - (ax + by + cz)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_0 + by_0 + cz_0 - (-d)}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{ax_0 + by_0 + cz_0 + d}{\sqrt{a^2 + b^2 + c^2}}.$$

Line through a point, parallel to a vector



$t \in \mathbb{R}$

$-\infty < t < \infty$

$\vec{r} = (x_0, y_0, z_0)$

$\vec{v} = (a, b, c)$

By vector addition,

For moving from point P to point Q along 'L'

$\Rightarrow \quad \vec{r} + t\vec{v}$

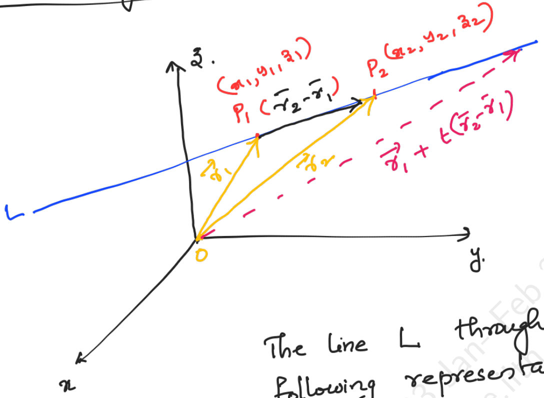
So any point in line is represented with respect to a given point (with position vector $\vec{r}$) and direction vector (or a vector parallel to the line L) $\vec{v}$ is;

$$\vec{r} + t\vec{v} \quad \text{for } -\infty < t < \infty \quad t \in \mathbb{R}$$

So parametric representation of the line with parameter 't'

$$\left.\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}\right\} \text{ for } -\infty < t < \infty$$

Line through two points.



$\vec{r}_1 = (x_1, y_1, z_1)$

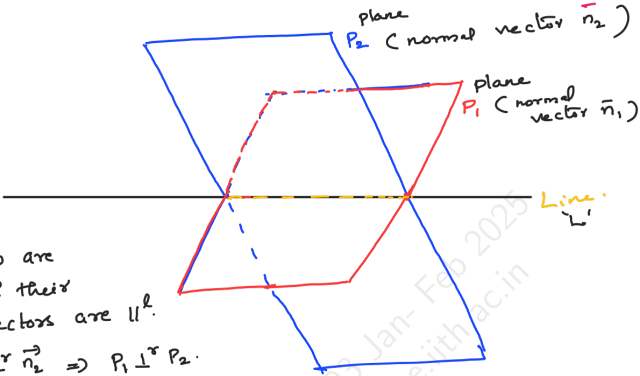
$\vec{r}_2 = (x_2, y_2, z_2)$

The line L through P_1 and P_2 has the following representations:

Vector: $\vec{r}_1 + t(\vec{r}_2 - \vec{r}_1)$

Parametric:
$$x = x_1 + (x_2 - x_1)t$$
$$y = y_1 + (y_2 - y_1)t$$
$$z = z_1 + (z_2 - z_1)t.$$

Line of intersection of two planes:



* Two planes are parallel if their normal vectors are $\parallel^{l}$.

* If $\vec{n}_1 \perp^{r} \vec{n}_2 \Rightarrow P_1 \perp^{r} P_2$.

* $(\vec{n}_1 \times \bar{n}_2) \perp^{r} \vec{n}_1$
  $\Rightarrow (\vec{n}_1 \times \vec{n}_2) \parallel^{l}$ to P_1

* $(\vec{n}_1 \times \vec{n}_2) \perp^{r} \vec{n}_2 \Rightarrow (\vec{n}_1 \times \vec{n}_2) \parallel^{l}$ to P_2.

$\Rightarrow (\vec{n}_1 \times \vec{n}_2)$ is parallel to intersection of P_1 & P_2

ie; $(\vec{n}_1 \times \vec{n}_2)$ is parallel to the line 'L'

Parametric Equation:

$$L: \quad \underbrace{\vec{r}}_{\substack{\text{position}\\ \text{vector}\\ \text{of a}\\ \text{given}\\ \text{point}\\ \text{in 'L'}}} + \overset{\text{parameter.}}{t}\,\underbrace{(\vec{n}_1 \times \vec{n}_2)}_{\substack{\text{parallel/ directional}\\ \text{vector}}}$$

It is going to be a point in (x, y, z) both the planes

$\Rightarrow$ It is found from the solution of two plane equations.

Example: Find line of intersection 'L' of the planes

$$5x - 3y + z - 10 = 0$$
$$2x + 4y - z + 3 = 0$$

$$\left.\begin{aligned} \vec{n}_1 &= (5, -3, 1) \\ \vec{n}_2 &= (2, 4, -1) \end{aligned}\right\}$$

not scalar multiples.

$\Rightarrow$ $\vec{n}_1$ is not $\parallel$ to $\vec{n}_2$

$\Rightarrow$ They intersect in a line in R^3.

Set $y = 0$.

$$5x + z - 10 = 0$$
$$2x - z + 3 = 0$$

$$\Rightarrow \quad 7x = 7 \quad \Rightarrow \quad x = 1.$$
$$z = 2x + 3 = 2 + 3 = 5$$

So the point on 'L' is $(1, 0, 5)$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 5 & -3 & 1 \\ 2 & 4 & -1 \end{vmatrix} = i(-1) - j(-7) + k(26)$$
$$= (-1, 7, 26)$$

$$L: \vec{r} + t(\bar{n}_1 \times \bar{n}_2)$$
$$: (1, 0, 5) + t(-1, 7, 26)$$

$$\Rightarrow \left.\begin{aligned} x &= 1 - t \\ y &= 7t \\ z &= 5 + 26t \end{aligned}\right\} \text{ for } -\infty < t < \infty.$$

$\rightarrow$ importance of parametric form: $\rightarrow$ extendable to 'n' dimension. to represent any curve.

Curvilinear Co-ordinate systems.

1. Cylindrical ; 2. Spherical co-ordinates.

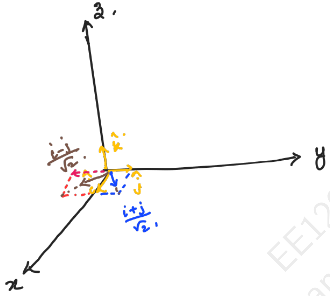
Orthonormal basis.

$$\left.\begin{aligned}\bar{v} &= 4\hat{i} + 3\hat{j} + 2\hat{k}. \\ &= (\bar{v}\cdot\hat{i})\,\hat{i} + (\vec{v}\cdot\hat{j})\,\hat{j} + (\vec{v}\cdot\hat{k})\,\hat{k}.\end{aligned}\right\}.$$

$(\hat{i}, \hat{j}\ \hat{k})$ $\Rightarrow$ orthonormal co-ordinates.

$i\cdot j = j\cdot\hat{k} = \hat{i}\cdot\hat{k} = 0$

& $i\cdot i = j\cdot j = \hat{k}\cdot\hat{k} = 1.$



$\hat{i}, \hat{j}, \hat{k}$

$$\underbrace{\underbrace{\frac{i+j}{\sqrt{2}}, \frac{i-j}{\sqrt{2}}}_{\frac{\pi}{2}}, \hat{k}.}_{\text{orthogonal co-ordinates}}$$

$\left(\frac{i+j}{\sqrt{2}}\right)\cdot\left(\frac{i+j}{\sqrt{2}}\right) = \frac{2}{2} = 1.$

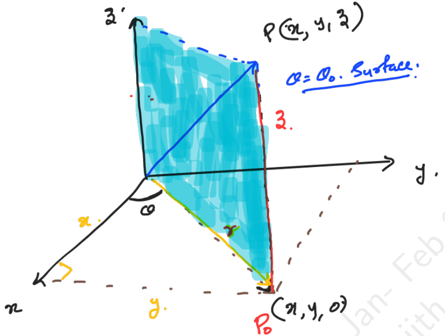
$\left(\frac{i-j}{\sqrt{2}}\right)\cdot\left(\frac{i-j}{\sqrt{2}}\right) = \underline{1}$ $\quad \hat{k}\cdot\hat{k} = \underline{1}$

$\Rightarrow \left(\frac{i+j}{\sqrt{2}}, \frac{i-j}{\sqrt{2}}, \hat{k}.\right)$ orthonormal. co-ordinate s/m.

$\Rightarrow$ orthonormal.

Cylindrical Co-ordinate systems.

$(\hat{r}, \hat{\theta}, \hat{z})$



$x = r\cos\theta.$
$y = r\sin\theta.$
$z = z.$

$r = \sqrt{x^2 + y^2}.$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$z = z.$

$0 \leq \theta \leq \pi$ if $y \geq 0$; $\pi < \theta < 2\pi$ if $y < 0$.

unit vector along $\hat{z} \rightarrow \hat{z}$.
unit vector along 'r' direction $\rightarrow$?

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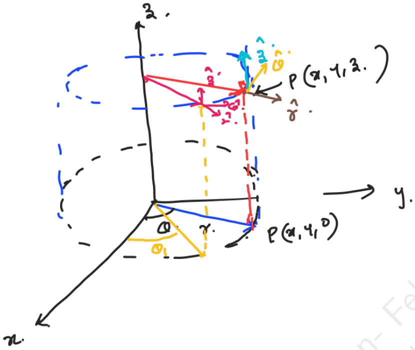
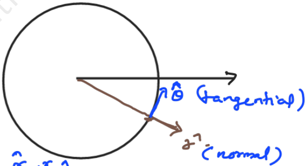


Clearly $\hat{r}$, $\hat{\theta}$ are orthogonal to $\hat{z}$.

→ What about between $\hat{r}$ & $\hat{\theta}$?

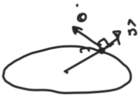
For that look from top.
Cross section is circle.

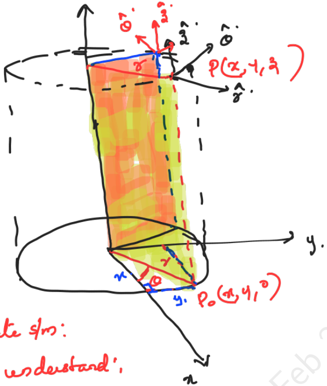


⇒ $\hat{r} \perp^r \hat{\theta}$.

⇒ $(\hat{r}, \hat{\theta}, \hat{z})$ also orthonormal system.

⇒ This is the co-ordinate s/m which depends on the point we are looking at.





$$z = z$$
$$\left.\begin{aligned} x &= r\cos\theta. \\ y &= r\sin\theta. \end{aligned}\right\}.$$

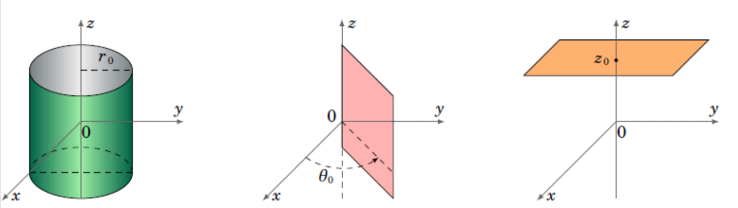
$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z,$$

In cylindrical Co-ordinate s/m:
→ Need to understand:

* Represent a position vector in (r, θ, z)
* Characterize path, surface & volume.

Will discuss in the next lecture!

Cylindrical co-ordinate surfaces.



(a) $r = r_0$ (b) $\theta = \theta_0$ (c) $z = z_0$

Ref: Vector Calculus By Michael Corral.