Jan 10 '25

Lec 4 :

Recap.

① Properties of probability function
continuity property

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right)$$

② Conditional probability

$P(B)>0$. $$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Logistics

Q1 : 20/1
Q2 : 3/2
Q3 : 17/2
} Mondays 6-7pm ?

E1 : 5/3 } Wednesday afternoon slot ?

P1 : 17/3

Q4 : 24/3
Q5 : 7/4
Q6 : 21/4.
} Mondays 6-7pm

P2 : 28/4
E2 : 02/5

HW Set 1 will be posted by eo 13/1

P1 & P2 : Two weeks time.

Multiplication rule.

$$P(A\cap B) = P(A|B)\,P(B)$$

holds for any A, B.

$A \subset B$, $P(A) \leq P(B)$

$A\cap B \subset B$ $P(A\cap B) \leq P(B)$

If $P(B)=0 \Rightarrow P(A\cap B)=0$.

$A_1, A_2, \ldots A_n$.

$$P(A_1\cap A_2 \cdots A_n) = \underbrace{P(A_1)\,P(A_2|A_1)}_{P(A_1\cap A_2)}\,P(A_3|A_1\cap A_2) \cdots P(A_n|A_1 \ldots A_{n-1})$$

$$= \prod_{j=1}^{n} P\left(A_j \,\middle|\, \bigcap_{i=1}^{j-1} A_i\right)$$

$$\cancel{P(A_1)} \quad \frac{P(A_2\cap \cancel{A_1})}{\cancel{P(A_1)}} \quad \frac{P(A_3\cap \cancel{A_2\cap A_1})}{P(\cancel{A_2\cap A_1})} \quad \cdots \quad \frac{P(A_n\cap \cdots A_1)}{P(\cancel{A_{n-1}\cap \cdots A_1})}$$

Made with Goodnotes

13 clubs, 13 ◇ diamo[nds]
13 hearts
13 spades

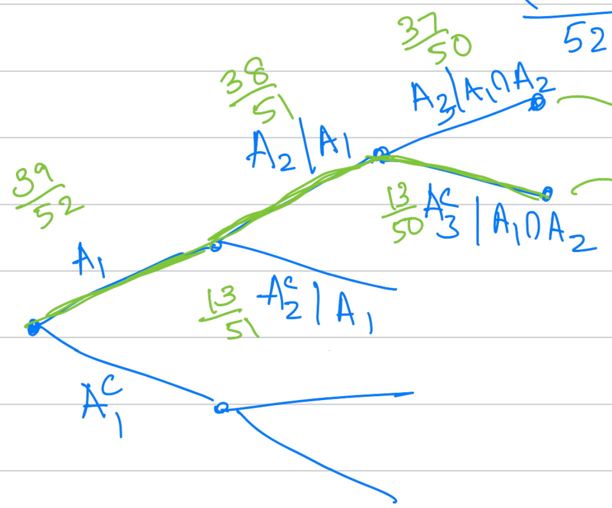
Example: Let's say you have a deck of 52 cards.

Let's say you draw 3 cards one after other where the card is not put back. What is the probability that none of these cards is a heart.

A_i = prob of i-th card not being a heart.

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2).$$

$$= \frac{(52-13)}{52} \frac{(51-13)}{51} \frac{(50-13)}{50}$$



$$P(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}.$$

$$P(A_1 \cap A_2 \cap A_3^c) = \frac{39}{52} \times \frac{38}{51} \cdot \frac{13}{50}$$

Total Probability theorem

Let $A_1, A_2, \ldots A_n \subset \Omega$ that partition the sample space

$A_1, \ldots A_n$ are mutually disjoint $A_i \cap A_j = \phi$ for all $i \neq j$

and $\bigcup_{i=1}^{n} A_i = \Omega$.

To show:

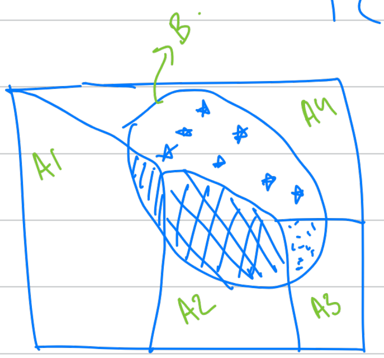
$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$

disjoint union.

$$B = B \cap \Omega. = B \cap \left(\biguplus_{i=1}^{n} A_i \right)$$

$$= \biguplus_{i=1}^{n} (B \cap A_i)$$

Made with Goodnotes



$$P(B) = P\left(\biguplus_{i=1}^{n} (B \cap A_i)\right)$$

$$= \sum_{i=1}^{n} P(B \cap A_i) \qquad (B \cap A_i) \cap (B \cap A_j) = \phi$$

additivity axiom

$$\parallel$$

$$\underline{\underline{A_i \cap A_j}} \cap B.$$

Baye's rule.

Let's say we have events A, B. We'll look at an example first.

Example: RADAR detection.

A: aircraft present.

B: event that alarm is signaled.

given these probabilities

$$\boxed{\begin{aligned} P(B \mid A) &= 0.99 \\ P(B^c \mid A^c) &= 0.9 \\ P(A) &= 0.05. \end{aligned}}$$

how do we find $P(A \mid B)$

Detection error

	B	B^c.
A	$A \cap B$	$A \cap B^c$.
A^c	$B \cap A^c$	$B^c \cap A^c$

False positive.

We are infering about A, given event B

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

A and A^c are disjoint and they partition Ω

$$= \frac{P(B \mid A)\, P(A)}{P(B \cap A) + P(B \cap A^c)}$$

using multiplication rule

total probability theorem

Baye's rule $\Leftarrow$ $$\boxed{P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}}$$

Example: $P(B|A) = 0.99$, $P(B^c|A^c) = 0.9$
$P(A) = 0.05$.

conditional probability also satisfies all the axioms

$$P(A|B) = \frac{0.99 \times 0.05}{0.99 \times 0.05 + 0.1 \times 0.95}$$
$$= 0.3426.$$

$$P(B|A^c) + P(B^c|A^c) = 1$$
$$P(B|A^c) = 1 - P(B^c|A^c)$$
$$= 1 - 0.9 = 0.1$$

$$P(A^c) = 1 - P(A)$$
$$= 0.95$$

Example 2: A Test for a rare disease is assumed to be 95% accurate. Probability that any person has this disease is $0.001 = 10^{-3}$.

A : event a person has rare disease
B : test declares disease.

$$P(B|A) = 0.95 = P(B^c|A^c)$$
$$P(A) = 10^{-3}$$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|A^c)\,P(A^c)}$$

$$P(B|A^c) = 1 - 0.95$$
$$= 0.05$$

$$= \frac{0.95 \times 10^{-3}}{0.95 \times 10^{-3} + 0.05 \times (1 - 10^{-3})} = 0.0187$$

Made with Goodnotes

Excercise: What is the testing accuracy required for $P(A|B) = 0.99$.

Independence of events:

An event A is said to be independent of event B if conditioning doesn't alter its probability

If $P(A|B) = P(A)$ then A is independent of B.

A and B are pairwise independent if $P(A \cap B) = P(A|B)\,P(B) = P(A)\,P(B)$

Example: Two fair dice are rolled.

A: {sum is 7} $P(A) = 6/36 = 1/6$

B: {1st roll is 1} $P(B) = 1/6$.

C: {sum is 8}. $P(C) = 5/36$.

(1, 6)
(6, 1)
(2, 5)
(5, 2)
(3, 4)
(4, 3)

$P(A \cap C) = 0$.

$P(A \cap B) = 1/36 = P(A)\,P(B)$.

A is independent of B.

$P(B \cap C) = 0$.

C and A not pairwise independent

C and B not pairwise independent.

independent ?

$A \perp B$

$A \not\perp C$

$B \not\perp C$.

Mutual Independence

Multiple events i.e., more than two events how do we define independence?

updated from the defn we saw in class.

Any $k < n$ events among $A_1, \dots A_n$ are mutually independant and

$$P(A_1 \cap A_2 \cap \dots A_n) = \prod_{i=1}^{n} P(A_i)$$

Pairwise independence doesn't imply mutual independence

Example: Tossing two coins

A : Prob that 1st roll is a head

B : " 2nd " is a head

C : Prob that both tosses result in different outcome.

$\Omega = \{(HH), (HT), (TH), (TT)\}$

each outcome has probability $1/4$.

$P(A) = P(B) = 1/2$

$P(C) = P(\{HT, HT\}) = \frac{1}{4} + \frac{1}{4}$

$P(A \cap C) = P(A) P(C)$

$P(A \cap B) = P(B) P(A)$.

$P(B \cap C) = P(B) P(C)$

$A = \{(HH), (HT)\}$

$B = \{(TH), (HH)\}$

A, B, C are pairwise independent

$A \cap C = \{HT\}$

$A \cap B = \{HH\}$

$B \cap C = \{TH\}$.

$A \cap B \cap C = \phi$.

$P(A \cap B \cap C) = 0$.

Made with Goodnotes

For $n=3$ events

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3) \\
P(A_1 \cap A_2) &= P(A_1)P(A_2) \\
P(A_1 \cap A_3) &= P(A_1)P(A_3) \\
P(A_2 \cap A_3) &= P(A_2)P(A_3)
\end{aligned}
$$

For $n=4$ events:

$$
\begin{aligned}
P(A_1 \cap A_2 \cap A_3 \cap A_4) &= P(A_1)P(A_2)P(A_3)P(A_4). \\
P(A_1 \cap A_2 \cap A_3) &= P(A_1)P(A_2)P(A_3) \\
P(A_1 \cap A_2) &= P(A_1)P(A_2) \\
P(A_1 \cap A_3) &= P(A_1)P(A_3) \\
P(A_2 \cap A_3) &= P(A_2)P(A_3) \\
P(A_1 \cap A_2 \cap A_4) &= P(A_1)P(A_2)P(A_4) \\
P(A_1 \cap A_3 \cap A_4) &= P(A_1)P(A_3)P(A_4) \\
P(A_2 \cap A_4 \cap A_3) &= P(A_2)P(A_4)P(A_3) \\
P(A_1 \cap A_4) &= P(A_1)P(A_4) \\
P(A_2 \cap A_4) &= P(A_2)P(A_4) \\
P(A_3 \cap A_4). &= P(A_3)P(A_4).
\end{aligned}
$$

any < 4 events are mutually independent.

Made with Goodnotes