Solutions are not reviewed. They may have errors/typos.

① $f_{Y|X}(y|x) = \begin{cases} x e^{-yx} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$

$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0. \\ 0 & \text{Otherwise}. \end{cases}$

ⓐ $\hat{x}_{MAP}(y) = \arg\max_x f_{X|Y}(x|y)$

$= \arg\max_x \dfrac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$ → can be ignored not a function of x

$= \arg\max_x f_{Y|X}(y|x) f_X(x).$

$= \arg\max_{x \geq 0} \underbrace{\alpha x e^{-yx} e^{-\alpha x}}$.

$f(x) = x e^{-(\alpha+y)x}.$

$f'(x) = e^{-(\alpha+y)x} - x(\alpha+y) e^{-(\alpha+y)x} = 0$

$\Rightarrow x = \dfrac{1}{\alpha+y}.$

$f''(x) = -2(\alpha+y) e^{-(\alpha+y)x} + x(\alpha+y)^2 e^{-(\alpha+y)x}$

$f''\left(\frac{1}{\alpha+y}\right) = e^{-1}\left[-2(\alpha+y) + \frac{1}{\cancel{\alpha+y}}(\alpha+y)^{\cancel{2}}\right]$

$< 0.$

$\Rightarrow$ $f(x)$ has maximum at $x = \dfrac{1}{\alpha+y}.$

$\Rightarrow$ $\boxed{\hat{x}_{MAP}(y) = \dfrac{1}{\alpha+y}.}$

$\hat{x}_{ML}(y) = \arg\max_x f_{Y|X}(y|x).$

$= \arg\max_{x \geq 0} \underbrace{x e^{-yx}}_{f(x)}$

$f'(x) = e^{-yx} - yx e^{-yx}$

$$= 0$$

$$\Rightarrow x = \frac{1}{y}. \quad \text{\} can show } f''\left(\tfrac{1}{y}\right) < 0.$$

$$\hat{x}_{ML}(y) = \frac{1}{y}$$

$$\hat{x}_{MMSE}(y) = E[X \mid Y = y].$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}.$$

$$f_Y(y) = \int_0^\infty f_{Y|X}(y|x)\, f_X(x)\, dx.$$

$$= \int_0^\infty x e^{-yx} \alpha e^{-\alpha x}\, dx.$$

$$= \frac{\alpha}{y+\alpha} \int_0^\infty x \, \underbrace{\boxed{e^{-(y+\alpha)x}\,(y+\alpha)}}_{f_Z(x)}\, dx.$$

$$= \frac{\alpha}{\alpha+y} E[Z] \quad \text{where}$$

$Z \sim \text{Exponential}(\alpha+y)$

check

$$E[Z] = \frac{1}{\alpha+y}$$

$$E[Z^2] = \frac{2}{(\alpha+y)^2}$$

$$= \alpha / (\alpha+y)^2.$$

$$f_{X|Y}(x|y) \Leftarrow$$

$$= \frac{x e^{-yx} \cancel{\alpha} e^{-\alpha x}}{\frac{\cancel{\alpha}}{(\alpha+y)^2}}.$$

$$= (\alpha+y)^2 x e^{-(\alpha+y)x}.$$

$$E[X \mid Y = y] = (\alpha+y) E[Z^2].$$

$$\boxed{\hat{x}_{MMSE}(y) = \frac{2}{(\alpha+y)}.}$$

(b)

$$\hat{x}_{MAP}(y_1 \dots y_n) = \arg\max_x f_{X|Y_1 \dots Y_n}(x \mid y_1 \dots y_n)$$

$$= \arg\max_x \frac{f_{Y_1 \dots Y_n|X}(y_1 \dots y_n \mid x)\, f_X(x)}{f_{Y_1 \dots Y_n}(y_1 \dots y_n)}$$

can be ignored ← not fn of x

$$= \arg\max_x \prod_{i=1}^{n} f_{Y_i|X}(y_i|x)\, f_X(x).$$

Made with Goodnotes

$$= \arg\max_{x \geq 0} \left[\prod_{i=1}^{n} x e^{-y_i x} \right] \left[\alpha e^{-\alpha x} \right]$$

$$= \arg\max_{x \geq 0} \underbrace{x^n e^{-(\Sigma y_i + \alpha) x}}_{f(x)}$$

$$f'(x) = n x^{n-1} e^{-(\Sigma y_i + \alpha) x} - (\Sigma y_i + \alpha) x^n e^{-(\Sigma y_i + \alpha) x}$$

$$= 0$$

$$\Rightarrow \quad x = \frac{n}{\sum_{i=1}^{n} y_i + \alpha}$$

$$\boxed{\hat{x}_{MAP}(y_1, \ldots y_n) = \frac{n}{\sum_{i=1}^{n} y_i + \alpha}}$$

$$\hat{x}_{ML}(y_1, \ldots y_n) = \arg\max_{x} f_{Y_1 \ldots Y_n}(y_1 \ldots y_n | x)$$

$$= \arg\max_{x} \underbrace{x^n e^{-\sum_{i=1}^{n} y_i x}}_{f(x)}$$

$$f'(x) = n x^{n-1} e^{-\sum_{i=1}^{n} y_i x} - x^n \sum_{i=1}^{n} y_i \, e^{-\sum_{i=1}^{n} y_i x}$$

$$= 0$$

$$\Rightarrow \quad x = \frac{n}{\sum_{i=1}^{n} y_i}$$

$$\boxed{\hat{x}_{ML}(y_1, \ldots y_n) = \frac{n}{\sum_{i=1}^{n} y_i}}$$

$$f_{X|Y_1 \ldots Y_n}(x | y_1 \ldots y_n) = \frac{f_{Y_1 \ldots Y_n | X}(y_1 \ldots y_n | x) f_X(x)}{f_{Y_1 \ldots Y_n}(y_1 \ldots y_n)}$$

$$f_{Y_1\ldots Y_n}(y_1 \ldots y_n) = \int_{x=0}^{\infty} f_{Y_1\ldots Y_n|X}(y_1,\ldots y_n|x)\, f_X(x)\, dx$$

$$= \int_0^{\infty} \left(\prod_{i=1}^{n} x e^{-y_i x}\right) \alpha e^{-\alpha x}\, dx$$

$$= \int_0^{\infty} x^n\, e^{-x\left[\sum_{i=1}^{n} y_i + \alpha\right]}\, dx.$$

$$= \frac{1}{\left(\sum_{i=1}^{n} y_i + \alpha\right)} E[Z^n] \quad \text{where } Z \sim Exp\left(\sum_{i=1}^{n} y_i + \alpha\right).$$

$$= \frac{n!}{\left(\sum_{i=1}^{n} y_i + \alpha\right)^{n+1}}$$

Verify that $Z \sim Exp(\lambda)$ has nth moment. $\frac{n!}{\lambda^n}$

$$f_{X|Y_1\ldots Y_n}(x|y_1\ldots y_n) = \frac{x^n e^{-(\Sigma y_i + \alpha)x}}{n!} (\Sigma y_i + \alpha)^{n+1}.$$

$$E[X|Y_1 = y_1 \ldots Y_n = y_n] = \int x\, f_{X|Y_1\ldots Y_n}(x)\, dx$$

$$= \int_0^{\infty} \frac{x^{n+1} (\Sigma y_i + \alpha)^{n+1}}{n!} e^{-(\Sigma y_i + \alpha)x}\, dx$$

$x(\Sigma y_i + \alpha) = \hat{x}$

$dx = \frac{d\hat{x}}{\Sigma y_i + \alpha}$

$$= \int_0^{\infty} \frac{\hat{x}^{n+1} e^{-\hat{x}}}{(\Sigma y_i + \alpha) n!}\, d\hat{x}$$

using $n+1$th moment of exponential R.V.

$$= \frac{(n+1)!}{n!\,(\Sigma y_i + \alpha)} = \frac{n+1}{\Sigma y_i + \alpha}$$

② ⓐ $E[X] = \int_0^1 x f_X(x)\, dx.$

$$= \int_0^1 \frac{x^{\alpha+1-1} (1-x)^{\beta-1}}{B(\alpha,\beta)} dx.$$

$$= \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = \frac{\frac{\alpha!\, \cancel{(\beta-1)!}}{(\alpha+\beta)!}}{\frac{(\alpha-1)!\,\cancel{(\beta-1)!}}{(\alpha+\beta-1)!}} = \frac{\alpha}{\alpha+\beta}.$$

ⓑ $f_{X|Y}(x|y) = \dfrac{P_{Y|X}(y|x) f_X(x)}{\int P_{Y|X}(y|x) f_X(x)\, dx}.$

$$P_Y(1) = \int_0^1 P_{Y|X}(1|x) f_X(x)\, dx.$$

$$= \int_0^1 x \frac{(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha,\beta)} dx. = \alpha/_{\alpha+\beta}.$$

$$P_Y(0) = \int_0^1 P_{Y|X}(0|x) f_X(x)\, dx.$$

$$= \int_0^1 \frac{(1-x)(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha,\beta)} = \frac{B(\alpha,\beta+1)}{B(\alpha,\beta)} = \frac{\beta}{\alpha+\beta}.$$

$$f_{X|Y}(x|1) = \frac{x \cdot \frac{(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha,\beta)}}{\left(\frac{\alpha}{\alpha+\beta}\right)} = \frac{x^{\alpha+1-1} (1-x)^{\beta-1}}{B(\alpha+1,\beta)}.$$

Similarly $f_{X|Y}(x|0) = \dfrac{x^{\alpha-1} (1-x)^{\beta+1-1}}{B(\alpha,\beta+1)}.$

$\Rightarrow X|Y=1 \sim \text{Beta}(\alpha+1,\beta)$

& $X|Y=0 \sim \text{Beta}(\alpha,\beta+1)$

(c) $$\hat{x}_{MMSE}(y) = E[X \mid Y = y] = \begin{cases} \frac{\alpha+1}{\alpha+1+\beta} & y = 1 \\ \frac{\alpha}{\alpha+\beta+1} & y = 0. \end{cases}$$

(d) $$f_{X|Y_1 \ldots Y_n}(x \mid y_1 \ldots y_n) = \frac{P_{Y_1 \ldots Y_n|X}(y_1 \ldots y_n \mid x)\, f_X(x).}{P_{Y_1 \ldots Y_n}(y_1 \ldots y_n).}$$

$$P_{Y_1 \ldots Y_n|X}(y_1 \ldots y_n \mid x) = \prod_{i=1}^{n} P_{Y_i|X}(y_i \mid x).$$

$$= \prod_{i=1}^{n} \left[x^{y_i} (1-x)^{(1-y_i)} \right]$$

$$= x^{\sum_{i=1}^{n} y_i} (1-x)^{n - \sum_{i=1}^{n} y_i}.$$

$$P_{Y_1 \ldots Y_n}(y_1, \ldots y_n) = \int_{x} P_{Y_1 \ldots Y_n|X}(y_1, \ldots y_n \mid x)\, f_X(x)$$

$$= \int_0^1 \frac{x^{\alpha + \sum_{i=1}^{n} y_i - 1} (1-x)^{\beta + n - \sum_{i=1}^{n} y_i - 1}}{B(\alpha, \beta)}\, dx$$

$$= \frac{B\left(\alpha + \sum_{i=1}^{n} y_i,\ \beta + n - \sum_{i=1}^{n} y_i\right)}{B(\alpha, \beta)}$$

$$f_{X|Y_1 \ldots Y_n}(x \mid y_1 \ldots y_n) = \frac{\dfrac{x^{\alpha + \sum_{i=1}^{n} y_i - 1} (1-x)^{\beta + n - \sum_{i=1}^{n} y_i - 1}}{\cancel{B(\alpha, \beta)}}}{\dfrac{B\left(\alpha + \sum_{i=1}^{n} y_i,\ \beta + n - \sum_{i=1}^{n} y_i\right).}{\cancel{B(\alpha, \beta)}}}$$

i.e., $$X \mid (Y_1 \ldots Y_n) = (y_1 \ldots y_n) \sim \text{Beta}\left(\alpha + \sum_{i=1}^{n} y_i,\ \beta + n - \sum_{i=1}^{n} y_i\right)$$

$$\Rightarrow \quad \hat{x}_{MMSE}(y_1, \ldots y_n) = E\left[X \mid Y_1 = y_1, \ldots Y_n = y_n\right].$$

Using part a for Expectation of Beta R.V. $\swarrow$

$$= \frac{\alpha + \sum_{i=1}^{n} y_i}{\alpha + \beta + n}.$$

③

$$P_{Y|X}(y|x) = \binom{n}{y} x^y (1-x)^{n-y}.$$

$$P_Y(y) = \int_0^1 P_{Y|X}(y|x)\, f_X(x)\, dx.$$

$$= \binom{n}{y} \int_0^1 x^{y+\alpha-1} \frac{(1-x)^{n-y+\beta-1}}{B(\alpha,\beta)}\, dx.$$

$$= \binom{n}{y} \frac{B(\alpha+y, \beta+n-y)}{B(\alpha,\beta)}.$$

$$f_{X|Y}(x|y) = \frac{P_{Y|X}(y|x)\, f_X(x)}{P_Y(y)}$$

$$= \frac{\cancel{\binom{n}{y}}\, x^{y+\alpha-1} \dfrac{(1-x)^{n-y+\beta-1}}{B(\alpha,\beta)}}{\cancel{\binom{n}{y}} \dfrac{B(\alpha+y, \beta+n-y)}{B(\alpha,\beta)}}$$

$$= \frac{x^{y+\alpha-1}(1-x)^{n-y+\beta-1}}{B(\alpha+y, \beta+n-y)}$$

$$\Rightarrow X \mid Y=y \sim \text{Beta}(\alpha+y, \beta+n-y)$$

$$\hat{x}_{MMSE}(y) = \frac{\alpha+y}{\alpha+\beta+n}.$$

this estimator is similar to the earlier estimator as both of use count of "ones"

(4). $$P_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)P_X(x)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = f_Z\left(\frac{y}{x}\right)$$

$$= \begin{cases} f_Z(y) & x=1 \\ f_Z(-y) & x=-1 \end{cases}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$$ since Z is mean 0, f_Z is even function

$$f_Y(y) = P_X(1)f_{Y|X}(y|1) + P_X(-1)f_{Y|X}(y|-1).$$

$$= \frac{e^{-y^2/2}}{\sqrt{2\pi}}.$$

$$P_{X|Y}(x|y) = P_X(x).$$

$$\Rightarrow E[X|Y=y] = E[X] = 0.$$

(c). Estimated value is neither 1, −1

∴ not very useful.

(b) $$\hat{x}_{MAP}(y) = \underset{x\in\{\pm 1\}}{\arg\max}\, P_{X|Y}(x|y)$$

$= 1$ since $P_X(1) = P_X(-1)$ choose 1.

$$\hat{x}_{ML}(y) = \underset{x\in\{\pm 1\}}{\arg\max}\, f_{Y|X}(y|x)$$ → not a function of x.

All the estimators are equally bad here.
In this problem it is very hard to estimate x.

But if Z has non-zero mean this changes. Request you to check this qn for $Z \sim N(2, \sigma^2)$.

(5)

$$f_{Y|X}(y|x) = \frac{1}{x} \qquad y \in [0, x].$$

$$f_X(x) = 1 \qquad x \in [0, 1].$$

$$f_{X|Y}(x|y) = \frac{\frac{1}{x}}{f_Y(y)} \qquad y \le x \le 1$$

$$f_Y(y) = \int_y^1 \frac{1}{x}\, dx = \log x \Big|_y^1 = -\log y$$

≥ 0 as $y \le 1$.

(a) $$\hat{x}_{MAP}(y) = \arg\max_{x \in [y,1]} \frac{1}{x}$$

$$= y.$$

$$\hat{x}_{ML}(y) = y$$

(b) $$\hat{x}_{MMSE}(y) = E[X|Y=y].$$

$$= \int_y^1 \cancel{x} \cdot \frac{1}{\cancel{x}} \frac{1}{(-\log y)}\, dx.$$

$$= \frac{(1-y)}{\log\left(\frac{1}{y}\right)}.$$

(c) $$\hat{x}_{LMMSE}(y) = \frac{Cov(X,Y)}{Var(Y)}\left(y - E[Y]\right) + E[X].$$

$$E[X] = 1/2$$

$$E[Y] = E[\,E[Y|X]\,]$$

$$= E\left[\frac{X}{2}\right] = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

$$\mathrm{Var}(Y) = E[\mathrm{Var}(Y|X)] + \mathrm{Var}(E[Y|X])$$

$$\mathrm{Var}(Y|X=x) = \frac{x^2}{12}.$$

$$E[Y|X=x] = x/2.$$

$$\mathrm{Var}(Y) = E[X^2/12] + \mathrm{Var}(X/2).$$

$$= \frac{1}{12}\left[\mathrm{Var}(X) + \left(\frac{1}{2}\right)^2\right] + \frac{1}{4}\mathrm{Var}(X).$$

$$= \frac{1}{12}\left[\frac{1}{12} + \frac{1}{4}\right] + \frac{1}{4} \times \frac{1}{12}.$$

$$= \left(\frac{1}{3} + \frac{1}{4}\right)\frac{1}{12} = \frac{7}{12^2}.$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = \int_0^1 f_X(x)\, x \int_0^x f_{Y|X}(y|x)\, y\, dy.$$

$$= \int_0^1 x \cdot \int_0^x \frac{1}{x}\, y\, dy.\ dx.$$

$$= \int_0^1 \int_0^x y\, dy\, dx.$$

$$= \int_0^1 \frac{x^2}{2}\, dx = \frac{1}{6}.$$

$$\mathrm{Cov}(X,Y) = \frac{1}{6} - \frac{1}{2} \times \frac{1}{4} = \frac{1}{24}.$$

$$\hat{x}_{LMMSE}(y) = \frac{(1/24)}{(7/12^2)}\left(y - \frac{1}{4}\right) + 1/2.$$

$$= \frac{6}{7}\left(y - \frac{1}{4}\right) + \frac{1}{2}.$$

(d). Error for MMSE estimator

$$= E\left[(X - E[X|Y])^2\right]$$

$$E[X|Y] = \frac{(1-Y)}{\log\left(\frac{1}{Y}\right)}$$

$$E\left[(X - E[X|Y])^2 \,|\, Y = y\right]$$

$$= \int_y^1 \left(x - \frac{(1-y)}{\log(1/y)}\right)^2 f_{X|Y}(x|y)\, dx.$$

$$= \int_y^1 \left(x - \frac{(1-y)^2}{\log(1/y)}\right)^2 \frac{1}{x\log\left(\frac{1}{y}\right)}\, dx.$$

$$= \int_y^1 \frac{x}{\log\left(\frac{1}{y}\right)} - \frac{2(1-y)^2}{\left(\log(1/y)\right)^2} + \frac{(1-y)^4}{x\left(\log\left(\frac{1}{y}\right)\right)^3}\, dx$$

$$= \frac{(1-y)^2}{2\log\left(\frac{1}{y}\right)} - \frac{2(1-y)^3}{\left(\log\left(\frac{1}{y}\right)\right)^2} + \frac{\log\left(\frac{1}{y}\right)(1-y)^4}{\left(\log\left(\frac{1}{y}\right)\right)^3}$$

$$= \frac{(1-y)^2}{\log\left(\frac{1}{y}\right)}\left[\frac{1}{2} - \frac{2(1-y)}{\log\left(\frac{1}{y}\right)} + \frac{(1-y)^2}{\log\left(\frac{1}{y}\right)}\right]$$

$$\text{MSE} = \int_0^1 \underbrace{\log\left(\frac{1}{y}\right)}_{f_Y(y)} E\left[(X - \hat{x}_{MMSE}(y))^2 \,|\, Y=y\right] dy$$

$$= \int_0^1 (1-y)^2 \frac{1}{2} - \frac{2(1-y)^3}{\log\left(\frac{1}{y}\right)} + \frac{(1-y)^4}{\log\left(\frac{1}{y}\right)}\, dy.$$

Error for LMMSE

$$\mathrm{Var}(X) - \frac{\mathrm{Cov}(X,Y)^2}{\mathrm{Var}(Y)}.$$

$$= \frac{1}{12} - \frac{(1/_{2\times 12})^2}{\left(\frac{7}{12}\right)^2}. = \frac{1}{12} - \frac{1}{49\times 4}.$$

⑥. (a) This problem appeared in earlier HW sets

$$Y = Y_1 + \cdots Y_n$$

$E[Y|Y] = Y.$ (as Y is completely determined by Y)

$\Rightarrow$ $E[Y_1 + \cdots Y_n \mid Y] = Y$

$\Rightarrow$ $\sum_{i=1}^{n} E[Y_i|Y] = Y$ by linearity of expectation

$\Rightarrow$ $n\, E[Y_i|Y] = Y$ identical RVs.

$\Rightarrow$ $E[Y_i|Y] = \frac{Y}{n}$

(b). $Y = X + W$ $X \sim N(0, k)$

$W \sim N(0, m)$

Let $Y_1, Y_2, \ldots Y_k$ be standard normal iid R.Vs.

$$X = Y_1 + Y_2 + \cdots Y_k \quad \ldots$$

Similarly $Y_{k+1}, \ldots Y_{k+m}$ be " iid RVs

$$W = Y_{k+1} + Y_{k+2} + \cdots Y_{k+m}$$

$E[Y_i|Y] = \frac{Y}{k+m}$ from part (a)

$$E[X|X+W] = E[X|Y] = E[Y_1 + Y_2 + \cdots Y_k \mid Y]$$

$$= \frac{kY}{k+m}.$$

(c) $E[X] = \lambda, \quad E[W] = \mu.$

$X = Y_1 + Y_2 + \dots Y_\lambda$

where Y_i's are Poisson (1) iid R.Vs.

X will be Poisson(λ)

Similarly $W = Y_{\lambda+1} + \dots + Y_{\lambda+\mu}$ then $Y = Y_1 + \dots Y_{\lambda+\mu}$

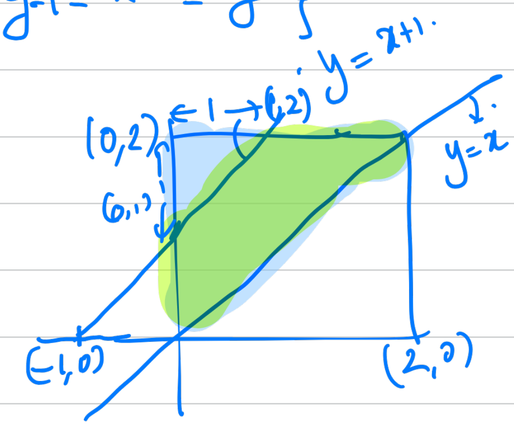
$$E[X \mid Y] = \sum_{i=1}^{\lambda} E[Y_i \mid Y]$$

$$= \lambda \cdot \frac{Y}{\lambda+\mu}.$$

(7) $$f_{Y,X}(y,x) = \begin{cases} c, & (y,x) \in S \\ 0 & \text{otherwise} \end{cases}$$

$$S = \{(y,x) \mid 0 \le y \le 2,\ 0 \le x \le 2,\ y-1 \le x \le y\}$$

(a) (Area of green portion) $c = 1$.



$\|$

$$[2\times 2] - \left[\frac{1}{2}\times 2\times 2\right] - \left[\frac{1}{2}\times 1\times 1\right].$$

$\Rightarrow 3/2 \times c = 1$

$\Rightarrow \underline{c = 2/3}.$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

non zero only when $\max\{0, y-1\} \le x \le y$. Assume $0 \le y \le 2$.

$$= 2/(3 f_Y(y)).$$

$$f_Y(y) = \int_{\max\{0,y-1\}}^{y} f_{X,Y}(x,y)\,dx.$$

$$= \frac{2}{3}\left(y - \max\{0, y-1\}\right) \quad 0 \le y \le 2$$

$$= \begin{cases} \frac{2}{3}. & 2 \ge y \ge 1. \\ \frac{2}{3}y. & 0 \le y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Sanity check

$$\int_0^2 f_Y(y)\,dy$$

$$\int_1^2 \frac{2}{3}\,dy + \int_0^1 \frac{2}{3} y\,dy$$

$$2/3 + 1/3 = 1$$

$$f_{X|Y}(x|y) = \begin{cases} 1 & 1 \le y \le 2,\ y-1 \le x \le y \\ \frac{1}{y} & 0 \le y < 1, \quad 0 \le x \le y \\ 0. & x \notin [0,2] \quad 0 \le y \le 2 \end{cases}$$

defined only for $y \in [0:2]$

$$E[X|Y=y] = \int_0^2 x\, f_{X|Y}(x|y)\,dx.$$

$$= \begin{cases} \int_{y-1}^{y} x\,dx = \frac{1}{2}(2y-1). & 1 \le y \le 2 \\ \int_0^y x\,\frac{1}{y}\,dx = \frac{1}{2}y. & 0 \le y \le 1. \end{cases}$$

$$\hat{x}_{MMSE}(y) = \begin{cases} \frac{1}{2}(2y-1) & 1 \le y \le 2 \\ \frac{1}{2}y & 0 \le y \le 1. \end{cases}$$

(b) $$E[\hat{x}(Y)] = \int_0^2 \hat{x}(y)\, f_Y(y)\, dy.$$

$$= \int_1^2 \frac{2}{3} \times \frac{1}{2}(2y-1)\, dy + \int_0^1 \frac{2}{3}y\, \frac{1}{2}y \cdot dy.$$

$$= \frac{1}{3}\left[y^2\Big|_1^2 - y\Big|_1^2 \right] + \frac{1}{3}\, \frac{y^3}{3}\Big|_0^1.$$

$$= \frac{2}{3} + \frac{1}{9} = 7/9.$$

$$E[(\hat{x}(Y))^2] = \int_0^2 (\hat{x}(y))^2 f_Y(y)\, dy$$

$$\frac{2y-1}{2} = t.$$

$$dy = dt.$$

$$= \int_1^2 \frac{2}{3} \times \left[\frac{1}{2}(2y-1)\right]^2 dy + \int_0^1 \frac{2}{3}y \left(\frac{1}{2}y\right)^2 dy$$

$$= \int_1^{3/2} \frac{2}{3} t^2\, dt + \frac{1}{6}\, \frac{y^4}{4}\Big|_0^1.$$

$$= \frac{2}{3}\, \frac{t^3}{3}\Big|_1^{3/2} + \frac{1}{24}$$

$$= \frac{1}{24} + \frac{2}{9}\left[\frac{27}{8} - 1\right]$$

$$= 1/24 + \frac{\cancel{2}}{\cancel{9}_3}\, \frac{\cancel{19}^5}{\cancel{8}_4} = \frac{11}{24}.$$

$$\mathrm{Var}(\hat{x}(Y)) = E[(\hat{x}(Y))^2] - (E[\hat{x}(Y)])^2$$

$$= 11/24 - (7/9)^2$$

$$E[(X - \hat{x}(Y))^2 | Y = y]$$

$$= \int_x (x - \hat{x}(y))^2 f_{X|Y}(x|y)\,dx$$

For $1 \le y \le 2$

$$= \int_{y-1}^{y} \left(x - \tfrac{1}{2}(2y-1)\right)^2 dx$$

$$= \int_{-1/2}^{1/2} t^2\,dt = \frac{t^3}{3}\Big|_{-1/2}^{1/2}$$

$$= \frac{1}{12}$$

For $0 \le y \le 1$

$$= \int_0^y \left(x - \tfrac{1}{2}y\right)^2 \frac{1}{y}\,dx$$

$$= \int_{-y/2}^{y/2} t^2 \frac{1}{y}\,dt = \frac{t^3}{3}\Big|_{-y/2}^{y/2}\left(\frac{1}{y}\right)$$

$$= \frac{1}{3}\left[\frac{y}{2}\right]^3\left[\frac{1}{y}\right] = \frac{y^2}{24}$$

(c) $E[(X - \hat{x}(Y))]^2 = E\left[E[(X - \hat{x}(Y))^2 | Y]\right]$

$$= \int_0^2 f_Y(y)\, E[(X - \hat{x}(Y))^2 | Y = y]\,dy$$

Made with Goodnotes

$$= \int_0^1 \frac{2}{3} y \times \frac{y^2}{24} dy + \int_1^2 \frac{2}{3} \frac{1}{12} dy.$$

$$= \frac{2}{3} \times \frac{1}{24} \times \frac{1}{4} + \frac{1}{36}.$$

Yes its equal to $E[\mathrm{Var}(X|Y)]$.

ⓓ $$\mathrm{Var}(X) = \underbrace{E[\mathrm{Var}(X|Y)]}_{\substack{\text{MMSE} \\ \text{we found} \\ \text{in part} \\ c}} + \underbrace{\mathrm{Var}(E[X|Y])}_{\substack{\mathrm{Var}(\hat{x}(Y)) \\ \text{that we} \\ \text{found} \\ \text{earlier}}}.$$

ⓔ $$\hat{x}_{LMMSE}(y) = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(Y)} [y - E[Y]] + E[X]$$

$$E[Y] = \int_0^2 y f_Y(y)\, dy.$$

$$= \int_1^2 \frac{2}{3} y\, dy + \int_0^1 \frac{2}{3} y\, y\, dy$$

$$= \cancel{\frac{2}{3}} \times \frac{1}{\cancel{2}} \times \cancel{3} + \frac{2}{3} \times \frac{1}{3} \times 1$$

$$= 1 + \frac{2}{9} = \frac{4}{9}.$$

$$E[X] = E[E[X|Y]]$$

$$= E[\hat{x}(Y)] = 7/9.$$

$$E[Y^2] = \int_1^2 \frac{2}{3} y^2\, dy + \int_0^1 \frac{2}{3} y\, y^2 dy$$

$$= \frac{2}{3} \times \frac{1}{3} \times 7 + \frac{\cancel{2}}{3} \times \frac{1}{\cancel{4}2}$$

$$= 14/9 + \frac{1}{6} = \frac{31}{18}$$

$$\mathrm{Var}(Y) = E[Y^2] - (E[Y])^2$$

$$= \frac{31}{18} - \left(\frac{11}{9}\right)^2.$$

$$= \frac{1}{9}\left[\frac{31}{2} - \frac{121}{9}\right]. > 0.$$

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = \int_0^2 \int_{\max\{0,y-1\}}^{y} \frac{2}{3}\, xy \, dx\, dy.$$

$$= \int_1^2 \int_{y-1}^{y} \frac{2}{3}\, xy\, dx\, dy + \int_0^1 \int_0^y \frac{2}{3} xy\, dx\, dy$$

...

Can find LMMSE error as.

$$E\left[\left(X - \hat{x}_{LMMSE}(Y)\right]^2\right]$$

$$= \mathrm{Var}(X) - \frac{\mathrm{Cov}^2(X,Y)}{\mathrm{Var}(Y)}.$$