Lecture 3 : Jan 7, 25.

Bi-Weekly Exam. Thursdays 5-30 - 7 pm

Continuity of Probability

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right).$$

Let $B_1, B_2, \ldots$

$B_1 = A_1$

$B_1, B_2, \ldots$ are disjoint sets. $\Leftarrow$ $B_2 = A_2 \setminus A_1$

$\vdots$

$$B_j = A_j \setminus \bigcup_{i=1}^{j-1} A_i .$$

$\vdots$

Claim 1 : $\displaystyle\bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j$.

Claim 2 : $\displaystyle\bigcup_{j=1}^{\infty} B_j = \bigcup_{j=1}^{\infty} A_j$

Let $\displaystyle C_k = \bigcup_{j=1}^{k} A_j$ and $\displaystyle D_k = \bigcup_{j=1}^{k} B_j$

$A_1 = C_1 = D_1 = B_1$

Let us assume $C_k = D_k$

$C_{k+1} = C_k \cup A_{k+1} = C_k \cup \underbrace{(A_{k+1} \setminus C_k)}_{B_{k+1}}$.

$$= C_K \cup B_{K+1}$$
$$= D_K \cup B_{K+1} = D_{K+1}.$$

$$\bigcup_{j=1}^{\infty} A_j = \{ x \in \Omega \mid \exists n \in \mathbb{N} \text{ st } x \in A_n \}.$$

$$B_K \subset A_K \quad \forall K \Rightarrow \bigcup_{K=1}^{\infty} B_K \subseteq \bigcup_{K=1}^{\infty} A_K$$

Suppose $x \in \bigcup_{K=1}^{\infty} A_K$

To show $A \subset B$
if $x \in A \Rightarrow x \in B$
then $A \subset B$.

$$\exists n \in \mathbb{N} \text{ st } x \in A_n$$
$$\Rightarrow x \in C_n = \bigcup_{j=1}^{n} A_j = D_n = \bigcup_{j=1}^{n} B_j$$
$$\Rightarrow x \in \bigcup_{j=1}^{\infty} B_j$$

$$\Rightarrow \bigcup_{K=1}^{\infty} A_K \subseteq \bigcup_{j=1}^{\infty} B_j$$

$$\Rightarrow \bigcup_{K=1}^{\infty} A_K = \bigcup_{K=1}^{\infty} B_K$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = P\left(\bigcup_{i=1}^{\infty} B_i\right) \to \text{claim 2.}$$

additivity

$$= \sum_{i=1}^{\infty} P(B_i)$$

$$= \lim_{n\to\infty} \underbrace{\sum_{i=1}^{n} P(B_i)}$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} B_i\right) \to \text{additivity}$$

$$= \lim_{n\to\infty} P\left(\bigcup_{i=1}^{n} A_i\right) \to \text{claim 1}$$

__Corollary b:__ $B_1 \supset B_2 \supset B_3 \cdots$

$$P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n)$$

__Example:__ $\Omega = [0,1]$

$$P((a,b]) = b-a.$$

$$B_n = \left(x-\frac{1}{n}, x+\frac{1}{n}\right) = \left\{ y \in [0,1] \text{ st } x-\frac{1}{n} < y < x+\frac{1}{n} \right\}$$

Can see that these are decreasing sets

$B_1 \supset B_2 \supset B_3 \cdots$

$$\boxed{\bigcap_{i=1}^{\infty} B_i = \{x\}.}$$

ⓐ $x \in B_n \ \forall n \in \mathbb{N}$ by defn.

ⓑ $\cancel{(\;(\;\underset{x}{\cdot}\;)\;)}$

Consider $x+\epsilon$ for some $\epsilon>0$

Find large enough n st

$x+\epsilon > x+\frac{1}{n}$

$$\boxed{n > \frac{1}{\epsilon}.}$$

$\Rightarrow x+\epsilon \notin B_n$. for $n > \frac{1}{\epsilon}$

For any $\epsilon>0$ can show $x+\epsilon \notin \bigcap_{i=1}^{\infty} B_i$

Can similarly show $x-\epsilon \notin \bigcap_{i=1}^{\infty} B_i$

$$P(\{x\}) = P\left(\bigcap_{i=1}^{\infty} B_i\right) = \lim_{n\to\infty} P(B_n)$$

from Corollary b.

$$= \lim_{n\to\infty} \frac{2}{n}$$

$$= 0.$$

Conditional Probability.

$\Omega = \{(1,1) \;\; \ldots \;\; (1,6) \;\; \ldots \;\; (6,6)\}$

① What is the probability that first die saw an out come 1 given that the sum of two ~~R~~olls is 9.

B : event that sum is 9.

$= \{ (3,6) \; (6,3) \; , \; (4,5) \; \ldots \}$

$P(A|B) = 0.$

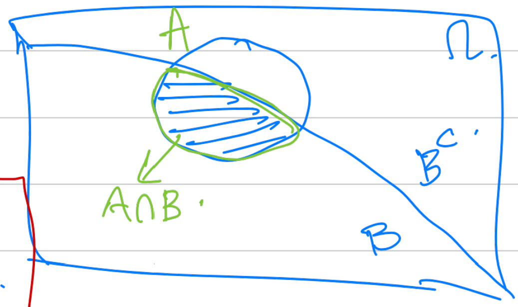
as $P(A \cap B) = 0$.

A : event that first die sees 1.

$= \{(1,1) \;\; \ldots \;\; (1,6)\}.$

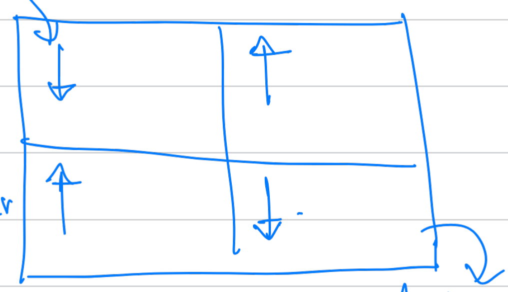
$P(A|B) \propto P(A \cap B).$

$P(B|B) = 1.$



Defn of conditional Probability.

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}.$$



For any given event $B \in \mathcal{F}$ st $P(B) > 0$.

$$P_B(A) = P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Does it satisfy all the probability axioms ?

$$P_B(A) = \frac{P(A \cap B)}{P(B)} \geq 0.$$

Non negativity

$$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)}$$

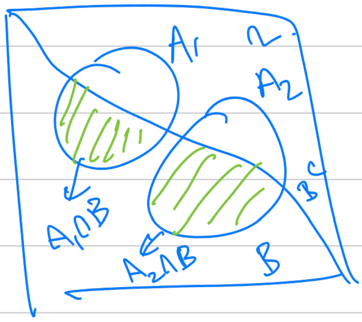
normalization $= 1$

Let A_1, A_2 be disjoint subsets.

$$P_B(A_1 \cup A_2) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)}$$

distributive law

$$= \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)}$$



$$= \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)}$$

$$= P_B(A_1) + P_B(A_2)$$

Examples: $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

A : more heads than tails come up $= \{HHH, HHT, HTH, THH\}$ ← 2 or more heads

B : 1st toss is head
$= \{HHH, HHT, HTH, HTT\}$.

$P(A) = 4/8 = 1/2$

$P(B) = 4/8 = 1/2$.

$P(\{\omega\}) = \frac{1}{8}$
$\omega \in \Omega$.

$P(A \cap B) = 3/8$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{3/8}{1/2}$$

$= 3/4$.

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