EE2100: Matrix Analysis

Review Notes - 36

Topics covered :

1. Representation of a Matrix as a sum of Rank-1 Matrices
2. Norm of a Matrix

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1. Representing a real matrix as a summation of Rank-1 matrices: Let $\mathbf{A} \in \mathcal{R}^{m \times n}$ (with $m \geq n$) denote a real matrix. Using singular value decomposition, $\mathbf{A}$ can be represented as

$$\begin{aligned}\mathbf{A} =& \mathbf{UDV}^T = [\mathbf{u}_1, \cdots, \mathbf{u}_m] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^T \\ \cdots \\ \mathbf{v}_n^T \end{bmatrix} = [\mathbf{u}_1, \cdots, \mathbf{u}_m] \begin{bmatrix} \sigma_1 \mathbf{v}_1^T \\ \cdots \\ \sigma_n \mathbf{v}_n^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} \\ =& \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_n \mathbf{u}_n \mathbf{v}_n^T \\ =& \sum_{i=1}^{n} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \end{aligned} \tag{1}$$

Equation (1) indicates that every real matrix can be represented as a summation of Rank 1 matrices.

2. Induced Norm of a Matrix: Let $\mathbf{A}$ be a real matrix. The induced norm of the matrix is defined as

$$\|\mathbf{A}\|_2 = \max_{\mathbf{x}} \frac{\|\mathbf{Ax}\|_2}{\|\mathbf{x}\|_2} \tag{2}$$

In order to compute the induced norm, we compute the maximum possible value of the ratio $\frac{\|\mathbf{Ax}\|_2^2}{\|\mathbf{x}\|_2^2}$.

$$\begin{aligned} \frac{\|\mathbf{Ax}\|_2^2}{\|\mathbf{x}\|_2^2} &= \frac{\mathbf{x}^T \mathbf{A}^T \mathbf{Ax}}{\mathbf{x}^T \mathbf{x}} \\ &= \frac{\mathbf{x}^T \mathbf{VDV}^T \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \text{ (by Spectral Decomposition)} \\ &= \frac{\mathbf{y}^T \mathbf{Dy}}{\mathbf{x}^T \mathbf{x}} \text{ where } \mathbf{y} = \mathbf{V}^T \mathbf{x} \\ &= \frac{\sum_{i=1}^{n} \sigma_i^2 y_i^2}{\mathbf{x}^T \mathbf{x}} \\ &\leq \sigma_1^2 \frac{\mathbf{y}^T \mathbf{y}}{\mathbf{x}^T \mathbf{x}} \end{aligned} \tag{3}$$

It is interesting to note that

$$\|\mathbf{y}\|_2 = \sqrt{< \mathbf{V}^T\mathbf{x}, \mathbf{V}^T\mathbf{x} >} = \sqrt{\mathbf{x}^T \underbrace{\mathbf{V}^T\mathbf{V}}_{\mathbf{I}} \mathbf{x}} = \|\mathbf{x}\|_2 \tag{4}$$

Accordingly (3) be simplified as

$$\frac{\|\mathbf{A}\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} \leq \sigma_1^2 \tag{5}$$

Hence the induced norm of the matrix is the largest singular value i.e.,

$$\|\mathbf{A}\|_2 \;=\; \max_{\mathbf{x}} \frac{\|\mathbf{A}\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \;=\; \sigma_1 \tag{6}$$