29th April

Finding marginals, posterior probabilities

① Given P_X, $P_{Y|X}$. (Prior and likelihood).
X and Y are discrete.

X is Poisson
$Y = X + Z$
Z is Poisson & independent of X

$$P_Y(y) = \sum_{x \in \mathcal{X}_0} P_X(x) P_{Y|X}(y|x).$$

$$P_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) P_X(x)}{P_Y(y)}$$

$$E[Y] = E[E[Y|X]].$$

$$= \sum_x P_X(x) \, E[Y|X=x]$$

$$E[Y|X=x] = \sum_{y \in \mathcal{Y}} y \, P_{Y|X}(y|x).$$

② X is discrete, Y is continuous.

$$P_X(x), \quad f_{Y|X}(y|x)$$

Example:
$Y = XZ$
X is discrete $\in \{+1, -1\}$
Z is normal.

$$f_Y(y) = \sum_{x \in \mathcal{X}_0} f_{Y|X}(y|x) P_X(x)$$

$$P_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) P_X(x)}{f_Y(y)}.$$

$$E[Y] = E[E[Y|X]]$$

$$= \sum_x P_X(x) \, E[Y|X=x]$$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y \, f_{Y|X}(y) \, dy.$$

③ X is continuos, Y is discrete

$X \sim U[0,1]$

$Y | X = x \sim \text{Binomial}(n, x)$

↓ # heads on n tosses for a coin with bias x.

Given $f_X(x) = 1 \quad x \in [0,1]$.

$$P_{Y|X}(y|x) = \binom{n}{y} x^y (1-x)^{n-y}$$

$$P_Y(y) = \int_{-\infty}^{\infty} P_{Y|X}(y|x) f_X(x)\, dx$$

$$f_{X|Y}(x|y) = \frac{P_{Y|X}(y|x) f_X(x)}{P_Y(y)}.$$

$$E[Y] = E\left[E[Y|X=x]\right]$$

$$= \int_{-\infty}^{\infty} f_X(x)\, E[Y|X=x]\, dx$$

$$E[Y|X=x] = \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x)\, y.$$

---

④ X and Y are continuos

X is Gaussian

$Y = X + Z$

Z is Gaussian independent of X.

$$f_X(x)$$

$$f_{Y|X}(y|x) = f_Z(y-x).$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_X(x) f_{Y|X}(y|x)\, dx$$

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

$$E[Y] = \int_{-\infty}^{\infty} f_X(x)\, E[Y|X=x]\, dx$$

$$E[Y|X=x] = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\, dx$$

ML estimator

Y is continuos.

$$\arg\max_{x} f_{Y|X}(y|x), \quad := \hat{x}_{ML}(y).$$

Y is discrete

$$\hat{x}_{ML}(y) = \arg\max_{x} P_{Y|X}(y|x)$$

MAP estimator

Best estimator to minimize decision error Probabili[ty] $P(x \neq \hat{x})$.

X is continuous.

$$\arg\max_{x} f_{X|Y}(x|y) := \arg\max_{x} \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}$$

$$= \arg\max_{x} f_{Y|X}(y|x) f_X(x).$$

$$= \hat{x}_{MAP}(y).$$

MMSE estimator.

$$\hat{x}_{MMSE}(y) = E[X|Y=y]$$

MSE

$$E[(X-\hat{X})^2]$$

$$= E[E[(X-\hat{X})^2 | Y]]$$

Linear MMSE

$$\hat{x}_{LMMSE}(y) = ay+b.$$

$$b = E[X - aY].$$

$$a = \frac{Cov(X,Y)}{Var(Y)}.$$

Made with Goodnotes

Multiple measurements

$$\hat{x}_{ML}(y_1, y_2 \dots y_n) = \arg\max_{x} f_{Y_1 \dots Y_n | X}(y_1 \dots y_n | x)$$

$$\hat{x}_{MAP}(y_1, \dots y_n) = \arg\max_{x} f_{X | Y_1 \dots Y_n}(x | y_1 \dots y_n).$$

$$\hat{x}_{MMSE}(y_1, \dots y_n) = E[X | Y_1 = y_1, \dots Y_n = y_n]$$

$$\hat{x}_{LMMSE}(y_1, \dots y_n) = R_{XY} K_Y^{-1} y.$$

$$R_{XY} = [\mathrm{Cov}(X, Y_1) \ \ \mathrm{cov}(X, Y_2) \ - \cdot \ \ \mathrm{Cov}(X, Y_n)]$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

MSE error of constant estimate $\geq$ MSE error of LMMSE $\geq$ MSE error of MMSE estimate

$\wedge\!\!|$

MSE error of ML, MAP estimates.

Functions of one R.V

Continuos RVs.

$$Y = g(X) \qquad \text{invertible fn.}$$

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}$$

$x : y = g(x)$

$$Y = X^2.$$

$x_1, x_2, \ldots x_n$ result in $g(x_i) = y$.

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}$$

One Function of two R.Vs

$$\underbrace{F_Z(z) = F_X(z) F_Y(z)}_{z = \max(X, Y)}$$

$Z = \min(X, Y)$, $Z = X + Y$, $Z = \max(X, Y)$

$1 - F_Z(z) = (1 - F_X(z))(1 - F_Y(y))$.

$F_Z(z) = P(X + Y \leq z)$.

Two Functions of two RVs

$Z = g_1(X, Y)$
$W = g_2(X, Y)$

$$f_{Z,W}(z, w) = \frac{f_{X,Y}(x, y)}{J(x, y)}$$

(x, y) s.t.
$z = g_1(x, y)$
$w = g_2(x, y)$.

$$= f_{X,Y}(x, y)\, J(z, w)$$

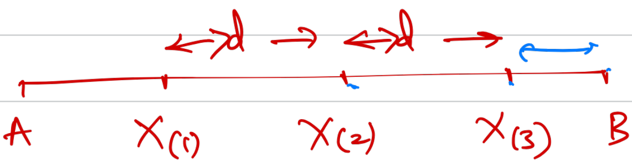
max and min joint density.

Order statistics

$X_{(1)}, X_{(2)} \ldots X_{(n)}$ is the ordered statistics of $X_1, \ldots X_n$

$X_{(1)}$ → smallest, $X_{(2)}$ → second smallest, $X_{(n)}$ → largest

$$f_{X_{(1)} \ldots X_{(n)}}(x_1, \ldots x_n) = n! \prod_{i=1}^{n} f_X(x_i).\quad x_1 < x_2 < \ldots x_n$$



$$X_1,\ X_2,\ X_3$$
$$\Downarrow$$
$$X_{(1)}\quad X_{(2)}\quad X_{(3)}$$

$$P\left(X_{(2)} \geq X_{(1)} + d,\ X_{(3)} \geq X_{(2)} + d \right),$$

$$= \int_0^{1-2d} \int_{x_1+d}^{1-d} \int_{x_2+d}^{1} f_{X_{(1)} X_{(2)} X_{(3)}} (x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1$$

MGF,

Concentration Inequality,.

Convergence of RVs;. Gaussian Vectors