1. A space probe in the shape of the ellipsoid

$$4x^2 + y^2 + 4z^2 = 16$$

enters Earth's atmosphere and its surface begins to heat. After 1 hour, the temperature at the point (x, y, z) on the probe's surface is given by

$$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$$

Find the hottest point on the surface of the probe.
Solution: To find the hottest point, we need to maximize the temperature function $T(x, y, z)$ subject to the constraint given by the ellipsoid equation. We use the method of Lagrange multipliers.

Define the constraint function as:

$$g(x, y, z) = 4x^2 + y^2 + 4z^2 - 16 = 0. \tag{1}$$

Compute the gradients:

$$\nabla T = (16x, 4z, 4y - 16), \tag{2}$$
$$\nabla g = (8x, 2y, 8z). \tag{3}$$

Using the method of Lagrange multipliers, we set

$$\nabla T = \lambda \nabla g. \tag{4}$$

This gives the system of equations:

$$16x = \lambda(8x), \tag{5}$$
$$4z = \lambda(2y), \tag{6}$$
$$4y - 16 = \lambda(8z). \tag{7}$$

For the first equation, if $x \neq 0$, we solve for λ:

$$\lambda = 2. \tag{8}$$

For the second equation,

$$4z = 2\lambda y = 4y \Rightarrow z = y. \tag{9}$$

For the third equation,

$$4y - 16 = 8\lambda z = 16z \Rightarrow 4y - 16 = 16z. \tag{10}$$

Substituting $z = y$,

$$4y - 16 = 16y \Rightarrow 12y = 16 \Rightarrow y = \frac{4}{3}, \quad z = \frac{4}{3}. \tag{11}$$

Substituting $y = \frac{4}{3}, z = \frac{4}{3}$ into the constraint equation:

$$4x^2 + \left(\frac{4}{3}\right)^2 + 4\left(\frac{4}{3}\right)^2 = 16. \tag{12}$$

Simplifying,

$$4x^2 + \frac{16}{9} + \frac{64}{9} = 16 \Rightarrow 4x^2 + \frac{80}{9} = 16. \tag{13}$$

Multiplying by 9,

$$36x^2 + 80 = 144 \Rightarrow 36x^2 = 64 \Rightarrow x^2 = \frac{16}{9} \Rightarrow x = \pm\frac{4}{3}. \tag{14}$$

Thus, the points are:

$$\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right), \quad \left(-\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right). \tag{15}$$

Evaluating T at these points,

$$T\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) = 8\left(\frac{4}{3}\right)^2 + 4\left(\frac{4}{3}\right)\left(\frac{4}{3}\right) - 16\left(\frac{4}{3}\right) + 600, \tag{16}$$

which simplifies to

$$T = 616. \tag{17}$$

Similarly, $T(-\frac{4}{3}, \frac{4}{3}, \frac{4}{3}) = 616$.

Thus, the hottest points on the probe are:

$$\left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right), \quad \left(-\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right). \tag{18}$$

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2. Find the volume of the largest closed rectangular box in the first octant having three faces in the coordinate planes and a vertex on the plane

$$x/a + y/b + z/c = 1.$$

where $a > 0, b > 0$ and $c > 0$.

Solution: Let the vertex of the box be (x, y, z) in the first octant, so that the volume of the box is given by:

$$V = xyz. \tag{19}$$

From the given constraint:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{20}$$

Using the method of Lagrange multipliers, define:

$$F(x, y, z, \lambda) = xyz + \lambda\left(1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c}\right). \tag{21}$$

Taking partial derivatives and setting them to zero:

$$\frac{\partial F}{\partial x} = yz - \lambda\left(-\frac{1}{a}\right) = 0 \Rightarrow yz = \frac{\lambda}{a}, \tag{22}$$

$$\frac{\partial F}{\partial y} = xz - \lambda\left(-\frac{1}{b}\right) = 0 \Rightarrow xz = \frac{\lambda}{b}, \tag{23}$$

$$\frac{\partial F}{\partial z} = xy - \lambda\left(-\frac{1}{c}\right) = 0 \Rightarrow xy = \frac{\lambda}{c}. \tag{24}$$

Multiplying the three equations:

$$xyz = \left(\frac{\lambda}{a}\right)^{1/3}\left(\frac{\lambda}{b}\right)^{1/3}\left(\frac{\lambda}{c}\right)^{1/3} = \frac{\lambda}{(abc)^{1/3}}. \tag{25}$$

Using the constraint,

$$x = \frac{a}{3}, \quad y = \frac{b}{3}, \quad z = \frac{c}{3}. \tag{26}$$

Thus, the maximum volume is:

$$V_{\max} = \left(\frac{a}{3}\right)\left(\frac{b}{3}\right)\left(\frac{c}{3}\right) = \frac{abc}{27}. \tag{27}$$

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3. Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution: For any x between 0 and 1, y may vary from $y = 0$ to $y = x$. Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2}\right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2}\right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed , the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy\right]_{x=y}^{x=1} dy$$

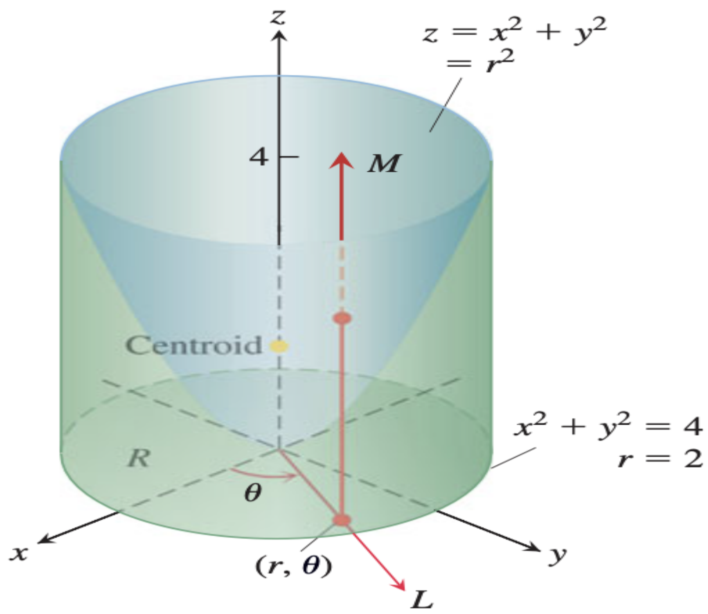
$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2\right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3y^2}{2}\right) dy = \left[\frac{5}{2}y - 2y^2 + \frac{y^3}{2}\right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

4. Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.

Solution: We sketch the solid, bounded above by the paraboloid $z = r^2$ and below by the plane $z = 0$. Its base R is the disk $0 \le r \le 2$ in the xy-plane.



The solid's centroid $(\bar{x}, \bar{y}, \bar{z})$ lies on its axis of symmetry, here the z-axis. This makes $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we divide the first moment M_{xy} by the mass M.

To find the limits of integration for the mass and moment integrals, we continue with the four basic steps. We completed our initial sketch. The remaining steps give the limits of integration.

- **The z-limits**: A line M through a typical point (r, θ) in the base parallel to the z-axis enters the solid at $z = 0$ and leaves at $z = r^2$.
- **The r-limits**: A ray L through (r, θ) from the origin enters R at $r = 0$ and leaves at $r = 2$.
- **The θ-limits**: As L sweeps over the base like a clock hand, the angle θ it makes with the positive x-axis runs from $\theta = 0$ to $\theta = 2\pi$.

The value of M_{xy} is

$$M_{xy} = \int_0^{2\pi} \int_0^2 \int_0^{r^2} z\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 \frac{r^5}{2}\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^6}{12}\right]_0^2 d\theta$$
$$= \int_0^{2\pi} \frac{16}{3}\,d\theta = \frac{32\pi}{3}.$$

The value of M is

$$M = \int_0^{2\pi} \int_0^2 \int_0^{r^2} dz\,r\,dr\,d\theta$$
$$= \int_0^{2\pi} \int_0^2 r^2\,dr\,d\theta$$
$$= \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^2 d\theta$$
$$= \int_0^{2\pi} 4\,d\theta = 8\pi.$$

Therefore,

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies on the z-axis, outside the solid.

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5. Find the potential and field everywhere between the spheres in a spherical capacitor, which is having two concentric spheres of radii R_1, R_2, with the inner one maintained at a potential V_0 and the outer at zero.

Solution:
We use spherical co-ordinate (ρ, ϕ, θ) due to the symmetry in the problem. The Laplace's equation between the spheres has the imposing form;

$$\nabla^2 V = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \frac{\partial V}{\partial \rho}) + \frac{1}{\rho^2 \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta \frac{\partial V}{\partial \theta}) + \frac{1}{\rho^2 \sin\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Here V- the potential can only be a function of ρ, since there is no way to distinguish a point (ρ, ϕ, θ) from another (ρ, ϕ', θ') with same ρ, but different θ and ϕ. Thus;

$$\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial \theta} = 0$$

, and Laplace's equation reduces to;

$$\frac{1}{\rho^2}\frac{d}{d\rho}(\rho^2\frac{dV}{d\rho}) = 0$$

The interested region in this problem is $R_1 < \rho < R_2$, which satisfies the boundary conditions:

$$V(\rho) = \begin{cases} V_0, & \text{at } \rho = R_1, \\ 0, & \text{at } \rho = R_2. \end{cases}$$

Multiplying the reduced Laplace's equation by ρ^2 and substituting $\psi = \frac{dV}{d\rho}$, we get;

$$\frac{d}{d\rho}(\rho^2\psi) = 0$$

and so,

$$\rho^2\psi = c_1$$

where c_1 is a constant, hence;

$$\psi = \frac{dV}{d\rho} = \frac{c_1}{\rho^2}$$

$$\implies V = -\frac{c_1}{\rho} + c_2$$

, c_2 another constant. Now imposing boundary conditions,

$$-\frac{c_1}{R_1} + c_2 = V_0$$

;

$$-\frac{c_1}{R_2} + c_2 = 0$$

Therefore;

$$c_1 = \frac{V_0 R_1 R_2}{R_1 - R_2}$$

$$c_2 = \frac{V_0 R_1}{R_1 - R_2}$$

Substituting in the expression for potential V;

$$V(\rho) = \frac{V_0 R_1}{R_1 - R_2}(1 - \frac{R_2}{\rho})$$

; in $R_1 < \rho < R_2$

Now to get the electric filed we need to take gradient of V.

$$E_\rho = -\frac{dV}{d\rho} = -\frac{V_0 R_1 R_2}{R_1 - R_2}\frac{1}{\rho^2}$$

Again, since in this case V depends only on ρ we get only radial component;$E_\theta = E_\phi = 0, R_1 < \rho < R_2$

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6. Find the divergence of these vector fields:

   (a) $Q = r\sin\theta\hat{r} + r^2 z\hat{\theta} + z\cos\theta\hat{z}$; using cylindrical co-ordinates (r, θ, z)
   Solution:

Divergence in Cylindrical Coordinates

We know that the divergence of the vector field is given as

$$\nabla \cdot \mathbf{A}$$

Here, ∇ is the del operator and $\mathbf{A}$ is the vector field.
If we take the del operator in cylindrical coordinates and dot it with $\mathbf{A}$ written in cylindrical form, then we would get the divergence formula in the cylindrical coordinate system.
In cylindrical coordinates, any vector field is represented as follows:

$$\mathbf{A} = A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_z\hat{a}_z$$

The cylindrical del operator is given by:

$$\nabla = \hat{a}_r\frac{\partial}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{a}_z\frac{\partial}{\partial z}$$

So let's take the dot product.

$$\nabla \cdot \mathbf{A} = \left(\hat{a}_r\frac{\partial}{\partial r} + \hat{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{a}_z\frac{\partial}{\partial z}\right) \cdot (A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_z\hat{a}_z)$$

You have to take into account the derivative.
For example, let us consider the first term,

$$(\nabla \cdot \mathbf{A})_r = \left(\hat{a}_r\frac{\partial}{\partial r}\right) \cdot (A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_z\hat{a}_z)$$

Let us put the dot product outside and consider the derivative first.

$$(\nabla \cdot \mathbf{A})_r = \hat{a}_r \cdot \left[\frac{\partial}{\partial r}(A_r\hat{a}_r + A_\theta\hat{a}_\theta + A_z\hat{a}_z)\right]$$

Now consider the first derivative,

$$\frac{\partial}{\partial r}(A_r\hat{a}_r)$$

It is simplified using the product rule of the derivative.

$$\frac{\partial}{\partial r}\left(A_r\hat{a}_r\right) = \frac{\partial A_r}{\partial r}\hat{a}_r + A_r\frac{\partial \hat{a}_r}{\partial r}$$

cylindrical and spherical unit vectors are not universally constant. Though their magnitude is always 1, they can have different directions at different points of consideration.
therefore,

$$\frac{\partial}{\partial r}\left(\hat{a}_r\right) = 0, \quad \frac{\partial}{\partial r}\left(\hat{a}_\theta\right) = 0, \quad \frac{\partial}{\partial r}\left(\hat{a}_z\right) = 0$$
$$\frac{\partial}{\partial \theta}\left(\hat{a}_r\right) = \hat{a}_\theta, \quad \frac{\partial}{\partial \theta}\left(\hat{a}_\theta\right) = -\hat{a}_r, \quad \frac{\partial}{\partial \theta}\left(\hat{a}_z\right) = 0$$
$$\frac{\partial}{\partial z}\left(\hat{a}_r\right) = 0, \quad \frac{\partial}{\partial z}\left(\hat{a}_\theta\right) = 0, \quad \frac{\partial}{\partial z}\left(\hat{a}_z\right) = 0$$

So let us consider the complete terms together,

$$\nabla\cdot\vec{A} = \left(\vec{a}_r\frac{\partial}{\partial r} + \vec{a}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \vec{a}_z\frac{\partial}{\partial z}\right)\cdot\left(A_r\vec{a}_r + A_\theta\vec{a}_\theta + A_z\vec{a}_z\right)$$

$$= \vec{a}_r\cdot\left[\left(A_r\frac{\partial}{\partial r} + \vec{a}_r\frac{\partial A_r}{\partial r}\right) + \left(A_\theta\frac{\partial}{\partial r} + \vec{a}_\theta\frac{\partial A_\theta}{\partial r}\right) + \left(A_z\frac{\partial}{\partial r} + \vec{a}_z\frac{\partial A_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\vec{a}_\theta\cdot\left[\left(A_r\frac{\partial}{\partial \theta} + \vec{a}_r\frac{\partial A_r}{\partial \theta}\right) + \left(A_\theta\frac{\partial}{\partial \theta} + \vec{a}_\theta\frac{\partial A_\theta}{\partial \theta}\right) + \left(A_z\frac{\partial}{\partial \theta} + \vec{a}_z\frac{\partial A_z}{\partial \theta}\right)\right]$$

$$+\vec{a}_z\cdot\left[\left(A_r\frac{\partial}{\partial z} + \vec{a}_r\frac{\partial A_r}{\partial z}\right) + \left(A_\theta\frac{\partial}{\partial z} + \vec{a}_\theta\frac{\partial A_\theta}{\partial z}\right) + \left(A_z\frac{\partial}{\partial z} + \vec{a}_z\frac{\partial A_z}{\partial z}\right)\right]$$

$$= \vec{a}_r\cdot\left[\left(A_r(0) + \vec{a}_r\frac{\partial A_r}{\partial r}\right) + \left(A_\theta(0) + \vec{a}_\theta\frac{\partial A_\theta}{\partial r}\right) + \left(A_z(0) + \vec{a}_z\frac{\partial A_z}{\partial r}\right)\right]$$

$$+\frac{1}{r}\vec{a}_\theta\cdot\left[\left(A_r(\vec{a}_\theta) + \vec{a}_r\frac{\partial A_r}{\partial \theta}\right) + \left(A_\theta(-\vec{a}_r) + \vec{a}_\theta\frac{\partial A_\theta}{\partial \theta}\right) + \left(A_z(0) + \vec{a}_z\frac{\partial A_z}{\partial \theta}\right)\right]$$

$$+\vec{a}_z\cdot\left[\left(A_r(0) + \vec{a}_r\frac{\partial A_r}{\partial z}\right) + \left(A_\theta(0) + \vec{a}_\theta\frac{\partial A_\theta}{\partial z}\right) + \left(A_z(0) + \vec{a}_z\frac{\partial A_z}{\partial z}\right)\right]$$

Now, we know that,

$$\vec{a}_r\cdot\vec{a}_r = \vec{a}_\theta\cdot\vec{a}_\theta = \vec{a}_z\cdot\vec{a}_z = 1$$

$$\vec{a}_r\cdot\vec{a}_\theta = \vec{a}_\theta\cdot\vec{a}_z = \vec{a}_z\cdot\vec{a}_r = 0$$

$$\therefore \nabla\cdot\vec{A} = \frac{\partial A_r}{\partial r} + \frac{A_r}{r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\therefore \nabla \cdot \vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \cdot \vec{Q} = \frac{1}{r}\frac{\partial}{\partial r}(rQ_r) + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{\partial Q_z}{\partial z}$$

Substituting the components:

$$Q_r = r\sin\theta, \quad Q_\theta = r^2 z, \quad Q_z = z\cos\theta$$

Computing the partial derivatives:

$$\frac{1}{r}\frac{\partial}{\partial r}(rQ_r) = \frac{1}{r}\frac{\partial}{\partial r}(r^2\sin\theta) = \frac{1}{r}(2r\sin\theta) = 2\sin\theta$$

$$\frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}(r^2 z) = 0$$

$$\frac{\partial Q_z}{\partial z} = \frac{\partial}{\partial z}(z\cos\theta) = \cos\theta$$

Thus, the divergence of **Q** is:

$$\nabla \cdot \mathbf{Q} = 2\sin\theta + \cos\theta$$

(b) $T = \frac{1}{\rho^2}\cos\phi\hat{\rho} + \rho\sin\phi\cos\theta\hat{\phi} + \cos\phi\hat{\theta}$; using spherical -cordinates (ρ, ϕ, θ)
Solution:
Follow the similar procedure to derive the Divergence formula for spherical co-ordinates:

Divergence in Spherical Coordinates

The divergence of a vector field $\mathbf{V} = V_\rho\hat{\rho} + V_\phi\hat{\phi} + V_\theta\hat{\theta}$ in spherical coordinates (ρ, ϕ, θ) is:

$$\nabla \cdot \mathbf{V} = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 V_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial \phi}(V_\phi\sin\phi) + \frac{1}{\rho\sin\phi}\frac{\partial V_\theta}{\partial \theta}$$

For $\mathbf{T} = \frac{1}{\rho^2}\cos\phi\hat{\rho} + \rho\sin\phi\cos\theta\hat{\phi} + \cos\phi\hat{\theta}$:

$$V_\rho = \frac{1}{\rho^2}\cos\phi, \quad V_\phi = \rho\sin\phi\cos\theta, \quad V_\theta = \cos\phi$$

Now, calculating each term:

$$\frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 V_\rho) = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\rho^2 \cdot \frac{\cos\phi}{\rho^2}) = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}(\cos\phi) = 0$$

$$\frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(V_\theta\sin\phi) = \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\rho\sin\phi\cos\theta\cdot\sin\phi)$$

$$= \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(\rho\sin^2\phi\cos\theta)$$

$$= \frac{1}{\rho\sin\phi}(\rho\cos 2\phi\cdot\cos\theta)$$

$$= \frac{\cos 2\phi\cos\theta}{\rho\sin\phi}$$

$$\frac{1}{\rho\sin\phi}\frac{\partial V_\theta}{\partial\theta} = \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\theta}(\cos\theta) = 0$$

Thus, the divergence is:

$$\nabla\cdot\mathbf{T} = 0 + \frac{\cos 2\phi\cos\theta}{\rho\sin\phi} + 0$$
$$= \frac{\cos 2\phi\cos\theta}{\rho\sin\phi}$$