Lec 8 21st Jan

② Binomial random variable (n, p).

Let's say we toss a biased coin with prob of heads being p, n number of times, what is probability that you see a head k # of times

X : # heads.

X takes values in $\mathcal{X} = \{0, 1, 2, \ldots n\}$

H_i : event that ith toss is a head.

$P(H_i) = p$; $H_1, \ldots H_n$ are mutually independent.

$\Omega = \{H, T\}^n \rightarrow$ all n-length sequences with values from H, T.

$$P(X = k) = P(\{w \in \Omega \mid X(w) = k\}).$$

$$= \sum_{w:\ \#\text{heads in } w = k} P(\{w\}).$$

$w = (H\ H \ldots H)$ then

$$P(\{w\}) = P(H_1 \cap H_2 \cap \cdots H_n)$$

$$= \prod_{i=1}^{n} P(H_i) = p^n.$$

Made with Goodnotes

Let w be a sequence with exactly k heads.

$$w = (\underbrace{H \cdots H}_{k} \ \underbrace{T\ T \cdots T}_{n-k}).$$

$$P(\{w\}) = p^k (1-p)^{n-k}$$
$$= P(H_1 \cap \cdots H_k \cap H_{k+1}^c \cdots \cap H_n^c)$$

this is true for any w with exactly k heads.

$$P(X=k) = \sum_{w : \#\text{heads is } k} P(\{w\}).$$
$$= |\{w : \text{heads as } k\}| \ p^k (1-p)^{n-k}$$

PMF

$$\boxed{P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}.}$$

$$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}.$$

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$
$$= (a+b)^n$$
$$= (p+(1-p))^n$$
$$= 1$$

$$= \sum_{k=1}^{n} \cancel{k} \frac{n!}{(n-k)! \, \cancel{k!}} p^k (1-p)^{n-k}$$

$(k-1)!$

$$= np \boxed{\sum_{k=1}^{n} \frac{(n-1)!}{(n-k)! \, (k-1)!} p^{k-1} (1-p)^{n-k}}$$

$\hat{k} = k-1$

$$= np \underbrace{\sum_{\hat{k}=0}^{n-1} \frac{(n-1)!}{(n-1-\hat{k})! \, \hat{k}!} p^{\hat{k}} (1-p)^{n-1-\hat{k}}}$$

$$= np \ (p + (1-p))^{n-1}$$
$$= np.$$

Made with Goodnotes

$$\boxed{Var[X] = E[X^2] - (E[X])^2}$$

$$E[X^2] - E[X] = \sum_{k=0}^{n} (k^2 - k)\binom{n}{k} p^k (1-p)^{n-k}.$$

$$= \sum_{k=2}^{n} k(k-1) \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k}.$$

$$= n(n-1)p^2. \qquad \text{(Excercise)}.$$

$$Var(X) = \underline{n(n-1)p^2 + np} - \underline{n^2p^2}$$

$$= np(1-p).$$

(3) Poisson Random Variable (λ). takes values in the set $\mathcal{X} = \{0, 1, 2, \ldots\}$. and the PMF is defined by

$$P_X(k) = P(X = k) = \frac{e^{-\lambda}}{k!} \lambda^k.$$

$$\sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda}\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots\right)$$

$$= e^{-\lambda} e^{\lambda} = 1.$$

$$E[X] = \sum_{k=0}^{\infty} k\, e^{-\lambda} \frac{\lambda^k}{k!} = \sum_{k=1}^{\infty} e^{-\lambda} \frac{\lambda^{(k-1)}}{(k-1)!} \lambda$$

$$= \lambda \left(\sum_{\hat{k}=0}^{\infty} e^{-\lambda} \frac{\lambda^{\hat{k}}}{\hat{k}!} \right) \qquad \hat{k} = k-1$$

$$= \lambda.$$

$$Var[X] = E[X^2] - (E[X])^2.$$

Compute

$E[x^2] - E[x]$ first.

$= \lambda$ (Exercise).

Poisson R.V approximates Binomial R.V.

Lemma: Let us consider $\lambda = np$ to be a constant and $n \to \infty$. Then the probability of a Binomial R.V being equal to $k \cdot \longrightarrow$ probability of a Poisson R.V being equal to k. (converges) as $n \to \infty$.

$$\binom{n}{k}(1-p)^{n-k} p^k \xrightarrow[\substack{n\to\infty \\ \lambda = np}]{} e^{-\lambda}\frac{\lambda^k}{k!}$$

$$\frac{n!}{(n-k)!\,k!}(1-p)^{n-k} p^k$$

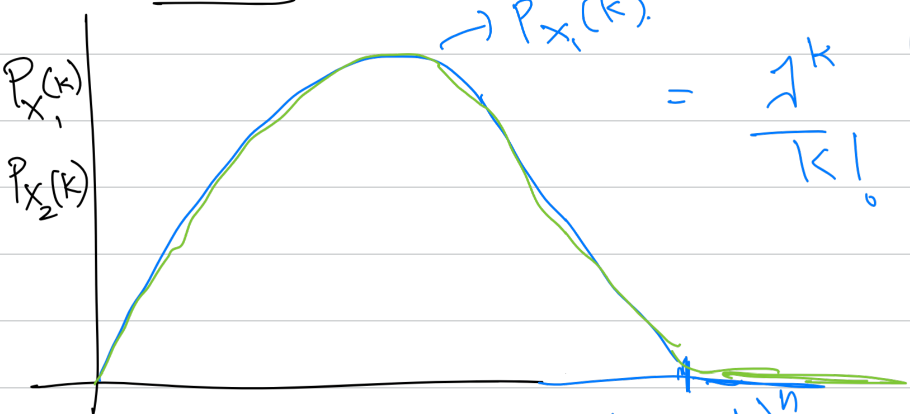
$$= \frac{n(n-1)\cdots(n-k+1)}{k!}\left(1-\frac{\lambda}{n}\right)^{n-k}\left(\frac{\lambda}{n}\right)^k.$$

$X_1 \sim \text{Binomial}(n, p)$

$X_2 \sim \text{Poisson}(\lambda)$

$n = 5000,\ p = 0.001.$

$\lambda = 5$



$$= \frac{\lambda^k}{k!}\left[\frac{n(n-1)\cdots(n-k+1)}{n\ n\ \cdots\ n.}\ \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}\right]$$

$$= \frac{\lambda^k}{k!}\underbrace{\left[\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right)\right]}_{\to 1 \text{ as } n\to\infty}\underbrace{\left[\frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^k}\right]}_{\to 1 \text{ as } n\to\infty}$$

$$\left(1-\frac{\lambda}{n}\right)^n \xrightarrow[n\to\infty.]{} e^{-\lambda}.$$

$$\longrightarrow \frac{e^{-\lambda}\lambda^k}{k!}.$$

Made with Goodnotes

④ Geometric R.V (p) : # times you toss a coin until you see a head.

$$P_X(k) = P(X = k) = (1-p)^{k-1}\, p.$$

Exercise to find $E[X]$ & $Var(X)$.

Joint Probability Mass function

$X : \Omega \to \mathbb{R}$

$Y : \Omega \to \mathbb{R}$

$(X, Y) : \Omega \to \mathbb{R}^2$

$$P_{X,Y}(x, y) = P(\{X = x\} \cap \{Y = y\})$$

$$= P(X = x, Y = y)$$

$$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$$

$$\{Y = y\} = \{\omega \in \Omega \mid Y(\omega) = y\}.$$

Finding PMF of X from joint PMF of (X, Y).

$$P_X(x) = P(\{X = x\}).$$

$$= P(\{X = x\} \cap \Omega).$$

$$\Omega = \bigcup_{y \in \mathcal{Y}} \underbrace{\{Y = y\}}_{A_y}$$

$\longrightarrow$ disjoint sets

A_y and $A_{y'}$ are disjoint

$$= P\Big(\{X = x\} \cap \bigcup_{y \in \mathcal{Y}} \{Y = y\}\Big)$$

Demorgan's law

$$= P\Big(\bigcup_{y \in \mathcal{Y}} \{X = x\} \cap \{Y = y\}\Big)$$

additivity

$$= \sum_{y \in \mathcal{Y}} P(\{X = x\} \cap \{Y = y\})$$

Joint PMF satisfies

$$\underbrace{\sum_x \sum_y}\, P_{X,Y}(x,y) = 1$$

$$= \sum_{y \in \mathcal{Y}} P_{X,Y}(x,y).$$

referred to as marginalization.

$$\boxed{P_Y(y) = \sum_{x \in \mathcal{X}} P_{X,Y}(x,y).}$$

Example: Joint PMF of x, y through a 2D Table

$P_{X,Y}(x,y)$ is element in xth row & yth column in this table

x \ y	1	2	3	4	$P_X(x)$
1	0	1/20	1/20	1/20	3/20
2	1/20	2/20	3/20	1/20	7/20
3	1/20	3/20	2/20	1/20	7/20
4	1/20	1/20	0	1/20	3/20
$P_Y(y)$	3/20	7/20	6/20	4/20	