CS1010: Discrete Mathematics for Computer Science

(Exam-1 Answer Key. Total: 30 marks.)
(Duration: 45 minutes, 26 Aug 2024)

1. Let G represent the set of all T.V. game shows. Let P be the set of people in your neighbourhood. Let $C(p, g)$ be the predicate that the person p appeared on game show g and $D(p)$ be the predicate that p is a doctor. Using the above definitions,

   (a) Translate: "There is a person in your neighbourhood who has been a contestant on a game show but is not a doctor" into symbolic notation.
   Answer:
   $$\exists\, p \in P,\ g \in G\ C(p, g) \wedge \sim D(p)$$

   (b) Negate the symbolic expression of (a)
   Answer: Negating the expression:
   $$\begin{aligned} \sim [\exists\, p \in P, \exists\, g \in G\ C(p, g) \wedge \sim D(p)] && (1)\\ \forall p \in P, \sim [\exists\, g \in G, C(p, g) \wedge \sim D(p)] && (2)\\ \forall p \in P, \forall g \in G \quad \sim [C(p, g) \wedge \sim D(p)] && (3)\\ \forall p \in P, \forall g \in G \quad \sim C(p, g) \vee D(p) && (4) \end{aligned}$$

   In (1) we take the negation of the original statement.
   In (2) we negate the quantifier in $\exists\, p \in P$, using the first rule of Theorem 2 (Lo-15).
   In (3) we negate the quantifier in $\exists\, g \in G$, using the first rule of Theorem 2 (Lo-15).
   In (4) we use DeMorgan's law to negate the predicates $C(p, g)$ and $\sim D(p)$.

   (c) Finally, translate the expression from (b) into English.
   Answer: "Every person in your neighbourhood either is a doctor or has never been a contestant on a game show."

   1 x 3 = 3 marks.

2. Using the algebraic rules for boolean functions, show that $\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p$. At each step, identify the rule used.
   Answer:
   $$\begin{aligned} \sim (\sim p \wedge q) \wedge (p \vee q) &\equiv (p \vee (\sim q)) \wedge (p \vee q) (DeMorgan'sLaw)\\ &\equiv p \vee ((\sim q) \wedge q) \qquad (DistributiveLaw)\\ &\equiv p \vee 0\\ &\equiv p \end{aligned}$$

   2 marks.

3. Let **I** $=$ Interest rates go down; **S** $=$ Stock Market goes up. For the implications below, write the implication, its converse and its contrapositive in words.
   $I \Rightarrow S$
   Answer: If interest rates go down, then Stock Market goes up. (implication)
   If Stock Market goes up, then the Interest rates go down. (converse)
   If Stock Market doesn't go up, then Interest rates won't go down. (contrapositive) 1 x 3 = 3 marks.

4. Rewrite the following statements in if-then form:

   (a) Either you get to work on time or you are fired.

   (b) Kamala's attaining age 35 is a necessary condition for her being the president of India.

   (c) Pedro's birth on U.S. soil is a sufficient condition for him to be a U.S. citizen.

   (d) Arun will go to school unless it rains.

   Answer:

   (a) If you don't get to work on time, then you are fired.

   (b) If Kamala has not attained the age of 35, then she is not eligible for being the president of India.

   or

   If Kamala is the president of India, then she has attained the age of 35.

   (c) If Pedro's birth was on U.S. soil, then he is eligible to be a U.S. citizen.

   (d) If it doesn't rain, then Arun will go to school.

   1 x 4 = 4 marks.

5. Suppose that p and q are statements so that $p \Rightarrow q$ is false. Find the truth values of each of the following:

   (a) $\sim p \Rightarrow q$.

   (b) $p \vee q$.

   (c) $q \Rightarrow p$.

   Answer: We know that $p \Rightarrow q$ is false if and only if p is true and q is false. Thus, p is true and q is false.

   (a) $\sim p \Rightarrow q$: We know that $\sim p$ is false and therefore this implication is always **true**.

   (b) $p \vee q$: We know that p is true and therefore this expression is **true**

   (c) $q \Rightarrow p$: We know that q is false and therefore this implication is always **true**.

   1 x 3 = 3 marks.

6. Prove that if x is irrational, then $1/x$ is irrational.
   Answer: We prove by contradiction. Suppose x is irrational and $1/x$ is rational. Let $1/x = a/b$, for some $a, b \in \mathbb{Z}$ with $b \neq 0$. Thus, $x = 1/(a/b) = b/a$. For x to be valid and well-defined, $a \neq 0$. This means x is rational which is a contradiction to our assumption that x is irrational.
   3 marks.

7. Let T be the set of all infinite 0-1 bit sequences. We know that T is *uncountable*. For any non-negative integer k, the k-th element of a sequence $\alpha \in T$ is denoted by $\alpha(k)$. Prove the countability/uncountability of the subsets of T given in sub-questions (a) and (b). Solve the two sub-questions independently. That is, do not use the result of any sub-question in the other.

   (a) $T_1 = \{\alpha \in T \ : \ \alpha(k) = 1 \text{ and } \alpha(k+1) = 0 \text{ for } \underline{\text{some}}\ k \geq 0\}$

   Answer: Uncountable.

   $$T_3 = \{\alpha \in T \ : \ \alpha(0) = 1 \text{ and } \alpha(1) = 0\}.$$

   We have $T_3 \subseteq T_1$, and so $|T_3| \leq |T_1| \leq |T|$. On the other hand, take any $\alpha = (1, 0, a_2, a_3, a_4, \ldots, a_n, \ldots) \in T_3$. The map taking $\alpha \rightarrow (a_2, a_3, a_4, \ldots, a_n, \ldots) \in T$ is clearly a bijection from $T_3 \rightarrow T$, implying $|T_3| = |T|$. We therefore conclude that $|T_1| = |T_3| = |T|$.

(b) $T_2 = \{\alpha \in T \;:\; \alpha(k) = 1 \text{ and } \alpha(k+1) = 0 \text{ for } \underline{\text{no}}\ k \geq 0\}$

Answer: Countable. Each sequence of T_2 either starts with a finite (may be empty) sequence of 0's followed by an infinite sequence of 1's, or consists only of 0's. Consider the function $T_2 \to \mathbb{N}$ that maps $(0, 0, 0, \ldots, 0)$ to 1, and $(0, 0, \ldots, 0, 1, 1, 1, \ldots 1, \ldots)$ with $n \geq 0$ number of initial 0's to $n + 2 \in \mathbb{N}$. This function is clearly bijective.

6+6 = 12 marks.

ALL THE BEST