EE2100: Matrix Theory

Assignment - 10

Handed out on 15 - Oct - 2023 **Due on 03 - Nov - 2023 (before 5 PM)**

Instructions :

1. Please submit the solutions to the assignment problems to the course page (on the canvas platform). Solutions submitted to the course page will only be evaluated. Refer to the assignment guidelines mentioned on the course page.
2. Submissions received after the deadline will attract negative marks.
3. It is suggested that you attempt all the questions (preferably the ones indicated using *). However, submitting solutions for problems totalling at least 10 points is sufficient.

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1. (2 Points) Let $\mathbf{A} \in \mathcal{R}^{n\times n}$ denote an skew symmetric matrix. Show that the Eigen values of $\mathbf{A}$ are purely imaginary.
2. (2 Points) Let $\mathbf{A} \in \mathcal{R}^{m\times n}$ denote a matrix whose column vectors are linearly independent. Show that the Eigen values of $\mathbf{A}^T\mathbf{A}$ are purely real and positive.
3. (4 Points) Compute the Eigen values and Eigen vectors of the following matrices.
$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{A}_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{1}$$
4. (5 points) Prove or disprove the following statement: If the eigenvectors of a matrix are linearly independent, then their corresponding eigenvalues are distinct. [Recollect the first property related to Eigen values and Eigen vectors made in the class. You need to prove this statement.].
5. (5 points) An Introduction to generalized eigenvectors: You have learned in class that the eigenvectors of any given matrix $\mathbf{A}$ satisfy the equation $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$. Now, consider the following matrix:
$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}$$

$det(\mathbf{A} - \lambda\mathbf{I}) = (1-\lambda)^2$. This gives $\lambda = 1$ as the only eigenvalue. It has an algebraic multiplicity of 2. Thus, we get

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Such matrices are not diagonalizable and are called defective. This gives the only eigenvector $\mathbf{v}$ as

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Thus, the geometric multiplicity of the eigenvalue 1 is 1. Clearly, it doesn't form a basis for $\mathbf{R}^2$. Why is this a problem? In some cases, it's a minor inconvenience. Whereas, for some cases, such as solving Linear Ordinary Differential Equations with constant coefficients, this becomes a major issue. Thus, to resolve this issue, the following method is important.

If we had some vector $\mathbf{u} = [0\ 1]^T$, then we would've had a basis. This can be obtained from $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{v}$. On substituting $\mathbf{v}$ as $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u}$ in the equation $(\mathbf{A} - \lambda\mathbf{I})^2\mathbf{u} = 0$, we get the desired results.

Therefore, generalized eigenvectors are defined as solutions to the equation:

$$(\mathbf{A} - \lambda\mathbf{I})^p\mathbf{v} = 0$$

This means that these vectors lie in the null space of $(\mathbf{A} - \lambda\mathbf{I})^p$. They are used to enlarge a set of eigenvectors to form a basis. Notice that if an eigenvalue has algebraic multiplicity k, then the generalized eigenvectors belong to the null space of $(\mathbf{A} - \lambda\mathbf{I})^k$.

Find the generalized eigenvectors for:

$$\mathbf{A} = \begin{bmatrix} 6 & -2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix}$$