# UNIT 5 VECTORS AND VECTOR SPACES

**Structure**





## 5.0 AIMS AND OBJECTIVES

This Unit is the first of three units in Block 3; this Block is concerned with a branch of mathematics called linear algebra. The present unit lays the foundation for the study of linear algebra, and deals with mathematical objects called vectors. After going through the unit, you will be able to:

- Define a vector and distinguish a vector and a scalar;
- Show how vectors can be represented geometrically;
- Carry out algebraic operations on vectors like addition, subtraction, and multiplication of a vector by a scalar;
- Describe the inner product of two vectors;
- Explain the ideas of linear combination and linear dependence of vectors;

- Describe norm and orthogonality of vectors; and
- Discuss the concept of vector spaces.

## 5.1 INTRODUCTION

This unit begins the study of linear algebra, that is, the study of linear equations, particularly simultaneous linear equations. Linear algebra is the topic of study for you in this block. To study linear algebra we make use of certain mathematical concepts like matrices. Matrices you will study in detail in the next unit. At this stage let us just say that a matrix (the plural of matrix is matrices) is a rectangular array, usually of real numbers. These numbers are arranged in rows (horizontally) and columns (vertically). If we take a single row or a single column of numbers, it is called a vector. That is, a matrix consisting of only a single row or a single column is called a vector.

Let us leave aside matrices for a moment (you just learnt the relationship of vectors and matrices) and study vectors independently. What is a vector? What algebraic operations can be performed on vectors? How do we depict a vector diametrically? What do we understand by a 'vector space'? Furthermore, how are vectors useful in the study of economics? What applications do vectors find in economics? All these we study in this unit.

In the next section, we begin our study of vectors by discussing the definition and meaning of vectors. We mention some specific types of vectors and also see how a vector can be represented diagrammatically. In section 5.3, we study algebraic operations on vectors like addition and subtraction of vectors. We also see what happens when a vector is multiplied by a scalar. In the subsequent section, section 5.4, the unit discusses the idea of linear combination; the idea of linear combination is used to study the linear dependence and independence of vectors. Section 5.5 discusses the product of vectors. This is different from the multiplication of a vector by a scalar, which you study in section 5.3. Section 5.5 discusses the multiplication of a vector by another vector. This is called the inner product of vectors. The next section discusses the concept of a vector 'space'. You will study what is meant by a 'space' in this context, and what a space of vectors means. The final section discusses the idea of the 'length' of a vector (called the norm of a vector) and the situation when two vectors (when depicted geometrically) are perpendicular to each other (called orthogonal vectors). The unit discusses throughout applications of vectors in economics, and relates vectors to concepts that you come across in your study of other courses in economics, like microeconomics and macroeconomics

## 5.2 VECTORS

Let us go directly to the definition of a vector. A vector is an ordered set of numbers. A vector is characterized not only by the elements contained in it but also by the order in which the elements appear.

### 5.2.1 Meaning of Vectors

Suppose a consumer is consuming two goods: apple and banana. Suppose we depict the various combination of the consumption of two goods by a pair of numbers, with the first number showing the number of apples and the second number the number of bananas. Suppose the consumer consumes 5 apples and 3 bananas, we can show this as (5,3). So we have shown this by a pair of numbers separated by a comma within parentheses. If we see a pair of numbers like (3,7) in this context, then that would tell us that the consumer has a combination of 3 apples and 7 bananas. A vector is defined as an ordered set of numbers, parameters, or variables. For our purposes, we will consider vectors as ordered set of real numbers. Vectors are usually put in parentheses or square brackets, and we shall use square brackets [ ] to denote vectors. The individual real numbers in vectors are called the components of the vector. Within the brackets, the components are not usually separated by commas, unlike when we depict an ordered pair or ordered n-tuple. We can use a simple example to elucidate this definition of a vector. Assume that a consumer buys $x_1$ and $y_1$ units of two goods $x$ and $y$, respectively. We can write these units either in a column as

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$$

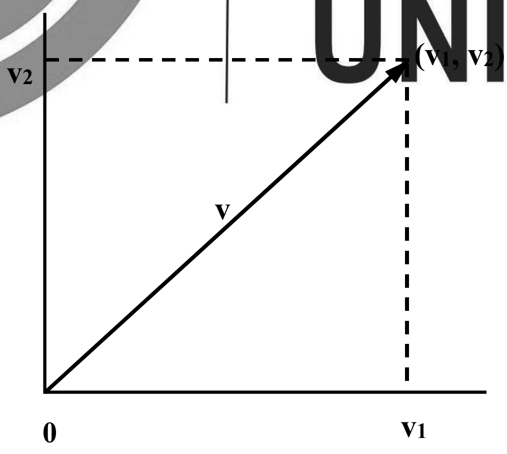
or in a row as

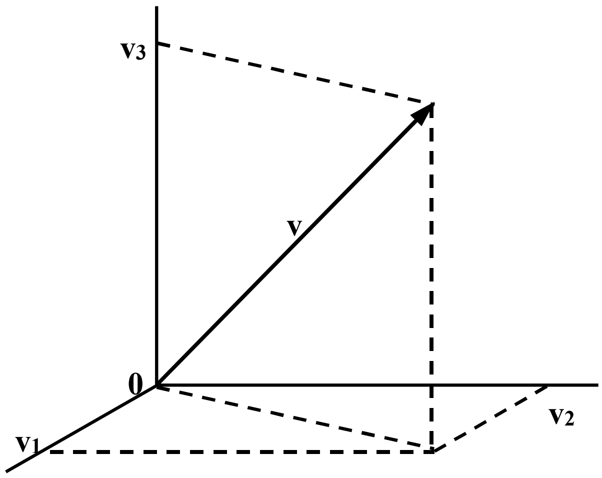
$$[x_1 \; y_1]$$

. This column or row of numbers is called a vector. If the vector appears in the form of a column, it is called a *column vector*; if it appears in the form of a row, it is called a *row vector*. Each entry inside the brackets is called a *component* of the vector. Column vectors are generally denoted by bold, lowercase letters such as **u**, **v**, etc. And row vectors are denoted by notations **u'**, **v'**, etc. Please note that the symbol $'$ is also used to denote derivatives in the context of calculus. Please do not confuse between the two. Both of the above vectors have two components each, and we say that each is a *2-vector* or each has *dimension* 2. If a vector has three components it would be called a 3-vector, and if a vector has $n$ components, then we can say that it is an *nvector* or it has dimension $n$. Suppose there are n goods that the consumer is consuming, denoted $(x_1, x_2, \ldots, x_n)$. This is an n-vector. We can denote it as $\mathbf{x}'$. Here $\mathbf{x}' = (x_1, x_2, \ldots, x_n)$. It can also be denoted, as a column vector as $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Similarly, if the prices of the n goods are $[p_1, p_2, \ldots, p_n]$ where $p_i$ is the price of the i[th] good (where i = 1,2,3,…,n), then $[p_1, p_2, \ldots, p_n]$ can be represented as a (row) vector $\mathbf{p}' = [p_1, p_2, \ldots, p_n]$. It can also be represented as a column vector.

Those of you who were acquainted with science in school may recall the definition that vectors are objects that have magnitude as well as direction, while scalars have only magnitudes. We will discuss vectors on these lines when we come to the geometric representation of vectors.






### 5.2.2 Types of Vectors

There are various types of vectors. One type is what is called a *unit* vector. A unit 2-vector is of the form $\mathbf{u}' = [1\ 0]$ or $\mathbf{v}' = [0\ 1]$. Geometrically they coincide with the x-axis and y-axis of the Cartesian plane. We will discuss this further in the next sub-section.

Another type of vectors is called *null* vector ,or *zero* vector. This has only zeroes as components for a 2-vector, a null vector would be $\mathbf{u}' = [0, 0]$

A third type of vectors is called *equal* vectors. Two vectors are said to be equal vectors only if they have the same dimension and if their corresponding components are equal. For example,

the two row 2-vectors $\mathbf{u'} = [u_1\ u_2]$ and $\mathbf{v'} = [v_1\ v_2]$ are said to be equal vectors only if $u_1 = v_1$ and $u_2 = v_2$.

The fourth type consists of vectors called *like* vectors and *unlike* vectors. Vectors having the same direction are called like vectors, and vectors with opposite directions are called unlike vectors. Another type of vector is *collinear* vectors. If two vectors lie on the same line or on parallel lines, they are called collinear vectors. Finally, *coplanar* vectors are vectors which lie on the same or parallel planes. All these types of vectors you can appreciate when you do the geometric representation of vectors

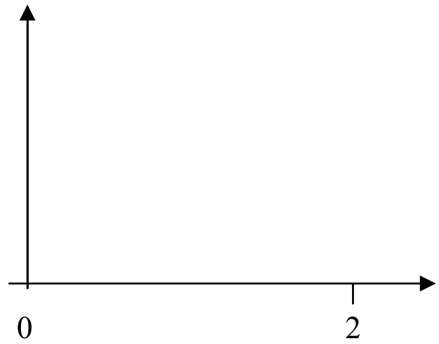
### 5.2.3 Geometric Representation of Vectors

In this subsection we look at how vectors can be represented geometrically. We have to limit ourselves to 2-vectors or 3-vectors since we can draw diagrams of these. In fact, for vectors with four or more components, it is not possible to draw diagrams, and you have to 'visualise' the geometric representation of these in an abstract manner.

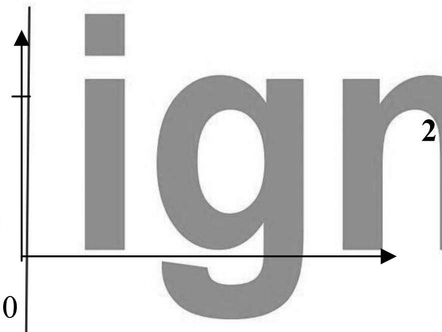
Let us begin with 2-vectors. Suppose we have vectors $\mathbf{u}'$ =[$x_1$ $y_1$] and $\mathbf{v}'$ = [$x_2$ $y_2$]. Let $x_1 = 2$,

$y_1 = 0$, $x_2 = 0$ and $y_2 = 0$. Then we can write the vectors as $\mathbf{u}'$ =[2 0] and $\mathbf{v}'$ = [0 2]. Then the two vectors can be depicted graphically as

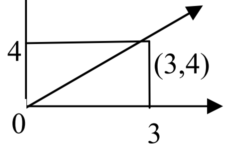
**u/**




**v/**




Suppose we have a vector w = [3 4]. This is a 2-vector so we can make use of the x axis and y-axis to depict the vector. So start at the origin of the Cartesian plane (point (0,0)). Count off three units to the right to reach point 3 on the x-axis. From this point, count off 4 units in the vertical direction (parallel to the y-axis). You will reach the point (3,4). This point on the Cartesian plane can be considered the vector. Alternatively, an arrow originating at (0,0) and going till (3,4) can also be considered the vector. In the above, to reach the point (3,4), you could have gone first in an upward direction for four units on the y-axis, and then moved rightward for three units parallel to the x-axis. You would still have reached the point (3, 4)




So any 2-vector can also be seen as a point in the Cartesian plane (an ordered pair). Similarly, a 3-vector can be viewed as an ordered triple. Extending the analogy, an n-vector can be viewed as an n-tuple.

## 5.3 ALGEBRAIC OPERATIONS ON VECTORS

In the previous section, we understood the concept of vectors. We also looked at the types of vectors and saw how vectors can be represented geometrically. In this section, we take a look how we can undertake algebraic operations on vectors like addition, subtraction, multiplication, and so on.

### 5.3.1 Addition of vectors

We shall addition of vectors here. here. Suppose that we have two row 2vectors: **u'** = [1 2] and **v' = [**2 1]. Then the sum **u'** + **v'**, called the *addition of vectors*, is obtained by adding each

component of **u'** to the corresponding component of **v'.** We have **u'** + **v'** = [1+2 2+1] = [3 3]. Please bear in mind that for addition of two vectors, the number of components in the given vectors (or the dimension of the vectors) has to be the same. We cannot add, say, a 3-vector with a 2-vector.

There are certain properties of vector addition

**Property I.** Vector addition is commutative. For any two vectors **u** and **v**, **u** + **v** =**v** + **u**.

**Property II.** Vector addition is associative. For any three vectors **u**, **v**, and **z**, (**u** + **v**) + **z** = **u** + (**v** + **z**).

**Property III.** Existence of *additive identity*. For any vector **u**, there exists a zero vector **0**, called the additive identity, such that **u** + **0** = **u**.

**Property IV.** Existence of *additive inverse*. For any vector **u**, there exists a vector called −**u**, such that **u** + (−**u**) = **0**.

### 5.3.2 Subtraction of Vectors

Assume that we have two vectors: **u'** = [ 1 4] and **v'** =[ 3 2]. The difference of these two vectors is called the subtraction of vectors. This is obtained by subtracting each component of **v'** from the corresponding component of **u'**. One thing you have to remember is that subtracting one vector from another vector is the same as adding the negative of the second vector to the first vector. Thus **u' - v' = u'** +(- **v')**

In the above example, **u' - v' = u'** +(- **v')**

[1 4] – [3 2] = [ 1 4] + [ -3 -2] = [ 1-3 2- 4] = [-2 2]

Note that **u'** + (−**u'**) or **v'** + (−**v'**) will give rise to a zero vector

### 5.3.3 Multiplication of a Vector by a Scalar

The last algebraic operation we shall deal with here is the multiplication of a vector by a scalar. A scalar, as we have seen, is a single real number. When we multiply a vector by the scalar, we multiply each component of the vector by the scalar. Suppose $r$ is a scalar and **u'** is a vector. Multiplying this vector by $r$, we get a new vector $r$**u'** which is r times the old vector. Suppose we have a vector [3 2] and we have a scalar 3, the multiplication of the vector with this scalar gives rise to a new vector [9 6]. Note that the direction of the newly generated vector will be reversed if the scalar r is negative. Moreover, scalar multiplication can be performed in combination with vector addition and vector subtraction. This means that given two vectors of the same dimension **u'**and **v'** and the scalar $r$, $r(\mathbf{u'} \pm \mathbf{v'}) = r\,\mathbf{u'} \pm r\mathbf{v'}$.

We now state some properties of scalar multiplication (multiplication of vectors with scalars).

**Property 1.** Scalar multiplication is associative. If there are scalar $r1$ and $r2$ and if **u** is a vector, then $(r1 \times r2)\,\mathbf{u} = r1 \times (r2\times \mathbf{u})$.

**Property 2.** Scalar multiplication is distributive. If $r1$ and $r2$ are two scalars and **u** and **v** are two vectors of the same direction, then $(r1 + r2)\,\mathbf{u} = r1\times \mathbf{u} + r2\times \mathbf{u}$, and $r1\times (\mathbf{u} + \mathbf{v}) = r1\times\mathbf{u} + r1\times \mathbf{v}$.

**Property 3.** Existence of *multiplicative identity*. For any vector **u**, $1\times \mathbf{u} = \mathbf{u}$.

**Check Your Progress 1**

1) What is the difference between a scalar and a vector? Can you think of the relationship between a set (which you studied in BECC 102) and a vector?

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2) If $u^{/} = [3,8,7]$, $v^{/} = [1, 6, 5]$, compute:

i) $u^{/} + 3v$ ii)

5u -2v

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