2014.3.4

第五讲. 偏相关系数

1. 偏相关系数
2. 偏相关系数矩阵的计算公式

“不相关化”（“正交”化/对角化）

任意随机向量x_1，x_2，记$\Sigma = \mathrm{cov}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}$

令$x_1^{\perp} = x_1 - \Sigma_{12}\Sigma_{22}^{-1}x_2$，即$\begin{pmatrix} x_1^{\perp} \\ x_2 \end{pmatrix} = \begin{pmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$则

$$\mathrm{cov}\begin{pmatrix} x_1^{\perp} \\ x_2 \end{pmatrix} = \begin{pmatrix} I & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I \end{pmatrix} = \begin{pmatrix} \Sigma_{11\bullet 2} & 0 \\ 0 & \Sigma_{22} \end{pmatrix}$$

所以$x_1^{\perp}$与x_2不相关，且$\mathrm{var}(x_1^{\perp}) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \overset{\text{记为}}{=} \Sigma_{11\bullet 2}$

对角化公式两边求逆$\Rightarrow$

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I_k & 0 \\ -\Sigma_{22}^{-1}\Sigma_{21} & I_{n-k} \end{pmatrix}\begin{pmatrix} \Sigma_{11\bullet 2}^{-1} & 0 \\ 0 & \Sigma_{22}^{-1} \end{pmatrix}\begin{pmatrix} I_k & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{n-k} \end{pmatrix}$$

$$= \begin{pmatrix} \Sigma_{11\bullet 2}^{-1} & -\Sigma_{11\bullet 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\bullet 2}^{-1} & \Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\bullet 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \end{pmatrix}$$

注意：由对称性知右下角的$\Sigma_{22}^{-1} + \Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\bullet 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1}$
实际上等于$\Sigma_{22\bullet 1}^{-1}$，同样左下角$-\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11\bullet 2}^{-1} = -\Sigma_{22\bullet 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1}$

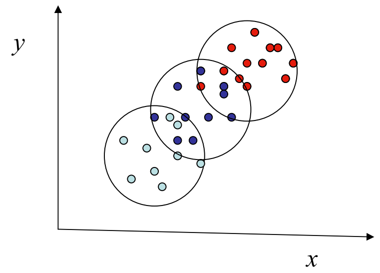
引理1（分块矩阵的逆）：$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} > 0$ （正定）

则$\Sigma^{-1} = \begin{pmatrix} \Sigma_{11\bullet 2}^{-1} & -\Sigma_{11\bullet 2}^{-1}\Sigma_{12}\Sigma_{22}^{-1} \\ -\Sigma_{22\bullet 1}^{-1}\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22\bullet 1}^{-1} \end{pmatrix}$

其中$\Sigma_{11\bullet 2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$，$\Sigma_{22\bullet 1} = \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$，

1. 偏相关系数

例**1.** 调查**7，8，9**岁儿童，发现阅读能力**(y)**与身高**(x)**正相关，相关系数r_{xy}**=0.56.**



年龄 **z** 是一个与**x，y**都相关的量：

$r_{xz}=0.8,\ r_{yz}=0.7$

事实上，从图中看住：给定年龄**z** (同一种颜色)时，**x**与**y**不相关！即**x, y** 在给定**z**时不相关。我们以偏相关系数度量这种“条件”相关性。

偏相关系数的定义

设y,x为1维随机变量，**z**为随机向量或变量，

$$\Sigma=\operatorname{cov}\begin{pmatrix}y\\x\\\mathbf{z}\end{pmatrix}=\begin{pmatrix}\Sigma_{yy}&\Sigma_{yx}&\Sigma_{y\mathbf{z}}\\\Sigma_{xy}&\Sigma_{xx}&\Sigma_{x\mathbf{z}}\\\Sigma_{\mathbf{z}y}&\Sigma_{\mathbf{z}x}&\Sigma_{\mathbf{zz}}\end{pmatrix}$$

$$y^{\perp}=y-\Sigma_{yz}\Sigma_{zz}^{-1}z,$$

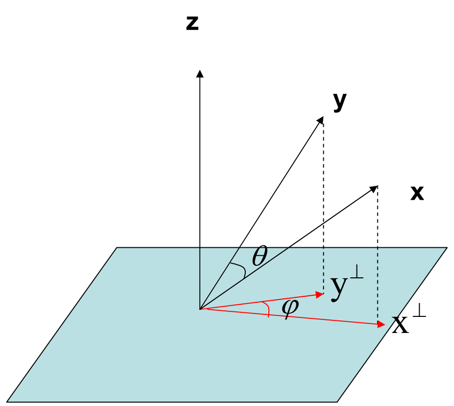
$$x^{\perp}=x-\Sigma_{xz}\Sigma_{zz}^{-1}z$$

定义：偏相关系数$\rho_{yx\bullet\mathbf{z}}=\rho_{y^{\perp}x^{\perp}}=\dfrac{\operatorname{cov}(y^{\perp},x^{\perp})}{\sqrt{\operatorname{var}(x^{\perp})\operatorname{var}(y^{\perp})}}$

即 $$\rho_{yx\bullet\mathbf{z}}=\frac{\Sigma_{xy}-\Sigma_{x\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}y}}{\sqrt{\Sigma_{xx}-\Sigma_{x\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}x}}\sqrt{\Sigma_{yy}-\Sigma_{y\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}y}}}$$

$$\cos(\varphi)=\rho_{xy\bullet z}$$

$$\cos(\theta)=\rho_{xy}$$



特例：当z是一个一维随机变量时，容易验证：

$$\rho_{xy\bullet z}=\frac{\rho_{xy}-\rho_{xz}\rho_{yz}}{\sqrt{1-\rho_{xz}^2}\sqrt{1-\rho_{yz}^2}}$$

其中ρ_{ab}代表随机变量a,b的Pearson相关系数.

例**1**（续）已知**3**个变量的相关系数矩阵，求偏相关系数。

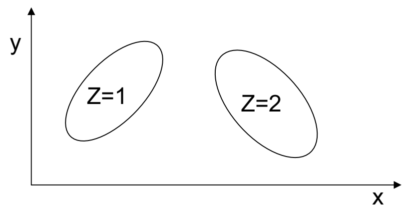
$$\begin{array}{cc} & \begin{array}{ccc}\text{x身高} & \text{y阅读} & \text{z年龄}\end{array}\\ \begin{array}{c}\text{x}\\\text{y}\\\text{z}\end{array} & \begin{pmatrix}1&0.56&0.8\\0.56&1&0.7\\0.8&0.7&1\end{pmatrix}\end{array}$$

$$\rho_{xy\bullet z}=\frac{\rho_{xy}-\rho_{xz}\rho_{yz}}{\sqrt{1-\rho_{xz}^2}\sqrt{1-\rho_{yz}^2}}=\frac{0.56-0.8\times0.7}{\sqrt{1-0.8^2}\sqrt{1-0.7^2}}=0$$

$$\rho_{yz\bullet x}=\frac{\rho_{yz}-\rho_{yx}\rho_{zx}}{\sqrt{1-\rho_{yx}^2}\sqrt{1-\rho_{zx}^2}}=\frac{0.7-0.56\times0.8}{\sqrt{1-0.56^2}\sqrt{1-0.8^2}}=0.5$$

注意：

偏相关系数$\rho_{yx\bullet \mathbf{z}}$ 与z的具体取值无关,即不论z取不同值的时候, $\rho_{yx\bullet \mathbf{z}}$都是一样的,所以下图数据不宜计算其偏相关系数



样本偏相关系数:

(1) 将协方差矩阵换成样本协方差矩阵, 计算得样本相关系数，记为 $\mathrm{r}_{xy\bullet z}$

(2) 以后会看到， $\mathrm{r}_{xy\bullet z}$ **= y,x**分别对**z**线性回归所得残差向量的样本回归系数。

检验偏相关系数的显著性 $H_0:\rho_{xy\bullet z}=0$

记样本偏相关系数为 $\mathrm{r}_{xy\bullet z}$，设变量总个数为 p （z的长度为$p-2$），样本量为n。假设数据正态，在原 假设下：

$$\sqrt{n-p}\,\frac{\mathrm{r}_{xy\bullet z}}{\sqrt{1-\mathrm{r}_{xy\bullet z}^2}}\sim t_{n-p}$$

不假设正态，当 n较大时，在原假设下近 似地有

$$\sqrt{n-p}\times \mathrm{r}_{xy\bullet z}\sim N(0,1)$$

注：***p*=2**时，即为相关系数的检验

例**2**： 下表给出了**2001**年美国**48**个州或地区的汽油消耗量**(y)**、税率**(x)**、持驾照人数**(z)**的相关系数矩阵。 关心的问题是税率是否与汽油消耗有关**.**

$$\begin{array}{cc} & \begin{array}{ccc} \text{y} & \text{x} & \text{z} \end{array} \\ \begin{array}{c} \text{y} \\ \text{x} \\ \text{z} \end{array} & \begin{pmatrix} 1 & -0.17 & 0.98 \\ -0.17 & 1 & -0.08 \\ 0.98 & -0.08 & 1 \end{pmatrix} \end{array}$$

$$\sqrt{n-2}\,|\,\mathrm{r}_{xy}\,|=7*0.17=1.15,\ \mathrm{p}=0.25$$

$$\rho_{xy\bullet z}=\frac{\rho_{xy}-\rho_{xz}\rho_{yz}}{\sqrt{1-\rho_{xz}^2}\sqrt{1-\rho_{yz}^2}}=\frac{-0.17-(-0.08)*0.98}{\sqrt{1-0.08^2}\sqrt{1-0.98^2}}=-0.46$$

$$\sqrt{n-3}\,|\,\mathrm{r}_{xy\bullet z}\,|=\sqrt{45}*0.46=3.1,\ \mathrm{p}=0.002$$

2. 偏相关系数矩阵的计算公式

回忆：Σ为协方差矩阵，则相关系数矩阵
$R = C^{-1/2}\Sigma C^{-1/2}$，其中$C = diag(\Sigma)$

定理：设Σ为协方差矩阵，$\Omega = \Sigma^{-1}, D = diag(\Omega)$，
则偏相关系数矩阵$R_{偏} = 2I_n - D^{-1/2}\Omega D^{-1/2}$，其中
$R_{偏}$的(i,j)元素为第i个变量与第j个变量的偏相关系数。

引理：设x, y是一维随机变量，$\mathbf{z}$为随机向量，

$$\Sigma = \operatorname{cov}\begin{pmatrix} x \\ y \\ \mathbf{z} \end{pmatrix} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{x\mathbf{z}} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{y\mathbf{z}} \\ \Sigma_{\mathbf{z}x} & \Sigma_{\mathbf{z}y} & \Sigma_{\mathbf{zz}} \end{pmatrix}。$$

记 $\Omega = \Sigma^{-1} = \begin{pmatrix} \omega_{xx} & \omega_{xy} & * \\ \omega_{yx} & \omega_{yy} & * \\ * & * & * \end{pmatrix}$，

则$\rho_{xy\bullet\mathbf{z}} = -\dfrac{\omega_{xy}}{\sqrt{\omega_{xx}}\sqrt{\omega_{yy}}}$

$$证明:记\Sigma = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} & \Sigma_{x\mathbf{z}} \\ \Sigma_{yx} & \Sigma_{yy} & \Sigma_{y\mathbf{z}} \\ \Sigma_{\mathbf{z}y} & \Sigma_{\mathbf{z}x} & \Sigma_{\mathbf{zz}} \end{pmatrix} \triangleq \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}, \qquad (\Sigma_{\mathbf{zz}}记成\Sigma_{22})$$

其中$\Sigma_{11} = \operatorname{cov}\begin{pmatrix} x \\ y \end{pmatrix}, \Sigma_{12} = \operatorname{cov}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \mathbf{z}\right), \Sigma_{22} = \operatorname{cov}(\mathbf{z})$

由分块矩阵求逆公式，知 $\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{11} \end{pmatrix} = \Sigma_{11\bullet 2}^{-1}$

另一方面，$\Sigma_{11\bullet 2} = \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix} - \begin{pmatrix} \Sigma_{x\mathbf{z}} \\ \Sigma_{y\mathbf{z}} \end{pmatrix} \Sigma_{\mathbf{zz}}^{-1} (\Sigma_{\mathbf{z}x}, \Sigma_{\mathbf{z}y})$

$$= \begin{pmatrix} \Sigma_{xx} - \Sigma_{x\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}x} & \Sigma_{xy} - \Sigma_{x\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}y} \\ \Sigma_{yx} - \Sigma_{y\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}x} & \Sigma_{yy} - \Sigma_{y\mathbf{z}}\Sigma_{\mathbf{zz}}^{-1}\Sigma_{\mathbf{z}y} \end{pmatrix} \triangleq \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

由偏相关系数的定义，$\rho_{xy\bullet\mathbf{z}} = \dfrac{b}{\sqrt{ac}}$

由于$\begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{11} \end{pmatrix} = \Sigma_{11\bullet 2}^{-1} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}^{-1} = \dfrac{1}{ac - b^2}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$

所以$\dfrac{b}{\sqrt{ac}} = -\dfrac{\omega_{12}}{\sqrt{\omega_{11}\omega_{11}}}$

$\Rightarrow \rho_{xy\bullet\mathbf{z}} = -\dfrac{\omega_{12}}{\sqrt{\omega_{11}\omega_{11}}}$，证毕。

例**2**。随机向量**x**满足：$\mathrm{cov}(\mathrm{x}_i, \mathrm{x}_j) = \rho_{\mathrm{x}_i, \mathrm{x}_j} = \rho^{|i-j|}$，即协方差矩阵

$$\Sigma = \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & 1 & \rho & \dots & \rho^{n-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{n-3} \\ \dots & & & & \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \dots & 1 \end{pmatrix},$$

$$\Sigma^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} 1 & -\rho & 0 & \dots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & \dots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+\rho^2 & -\rho \\ 0 & 0 & 0 & \dots & -\rho & 1 \end{pmatrix}$$

$$R_{\text{偏}} = \begin{pmatrix} 1 & \frac{\rho}{\sqrt{1+\rho^2}} & 0 & \dots & 0 \\ \frac{\rho}{\sqrt{1+\rho^2}} & 1 & \frac{\rho}{1+\rho^2} & \dots & 0 \\ 0 & \frac{\rho}{1+\rho^2} & 1 & \dots & 0 \\ \dots & & & & \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$