

# From Fair Decision Making To Social Equality

Hussein Mouzannar  
American University of Beirut  
hssein.mzannar@gmail.com

Mesrob I. Ohannessian  
Toyota Technological  
Institute at Chicago  
mesrob@ttic.edu

Nathan Srebro  
Toyota Technological  
Institute at Chicago  
nati@ttic.edu

## ABSTRACT

The study of fairness in intelligent decision systems has mostly ignored long-term influence on the underlying population. Yet fairness considerations (e.g. affirmative action) have often the implicit goal of achieving balance among groups within the population. The most basic notion of balance is eventual equality between the qualifications of the groups. How can we incorporate influence dynamics in decision making? How well do dynamics-oblivious fairness policies fare in terms of reaching equality? In this paper, we propose a simple yet revealing model that encompasses (1) a selection process where an institution chooses from multiple groups according to their qualifications so as to maximize an institutional utility and (2) dynamics that govern the evolution of the groups' qualifications according to the imposed policies. We focus on demographic parity as the formalism of affirmative action.

We first give conditions under which an unconstrained policy reaches equality on its own. In this case, surprisingly, imposing demographic parity may break equality. When it doesn't, one would expect the additional constraint to reduce utility, however, we show that utility may in fact increase. In real world scenarios, unconstrained policies do not lead to equality. In such cases, we show that although imposing demographic parity may remedy it, there is a danger that groups settle at a worse set of qualifications. As a silver lining, we also identify when the constraint not only leads to equality, but also improves all groups. These cases and trade-offs are instrumental in determining when and how imposing demographic parity can be beneficial in selection processes, both for the institution and for society on the long run.

## CCS CONCEPTS

• **Social and professional topics** → *Government technology policy*; • **Applied computing** → *Law, social and behavioral sciences; Decision analysis*;

## KEYWORDS

selection processes; fairness; demographic parity; affirmative action; influence on society; dynamics; social equality

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## 1 INTRODUCTION

As many aspects of human society become increasingly automated, questions of ethical nature migrate into the technological sphere. Admittedly, no amount of formalization can capture all the complexity and subtlety of these issues. Yet, one is compelled to mathematize them and integrate them with familiar frameworks. The result of this process is the design of purportedly ethically-minded technology. An important component of this process, then, is to understand just how well such technology interacts with society and whether its use does indeed result in the intended effect.

Automated decision and policy making, particularly the data-driven case embodied by machine learning, is one such technology that has recently alloted considerable attention to these ethical questions. In particular, guaranteeing non-discriminatory behavior has been at the forefront of research. Various formalizations of non-discrimination have been proposed, relationships between these notions have been studied [6, 9], and fundamental trade-offs have been identified [15]. Yet the influence of the resulting decision systems on society has only garnered modest treatment [17]. One place where this question has been thoroughly considered is in studying the evolution of negative stereotypes in labor markets, in economics [5, 12]. The dynamics there, however, are very carefully crafted to mimic the labor market. To relate the formalism of the current paper with these prior work, see Section 2.

This paper starts by proposing a selection process where an institution chooses from multiple groups according to their qualifications so as to maximize an institutional utility. In this context, non-discrimination often takes the form of constraining the type of selection policies. Demographic parity, for example, enforces groups to have equal selection rates. This is a simple yet rich model that has been well studied [9] [15], and as such this paper chooses it as the archetype of non-discrimination, referring to it with the more colloquial name of affirmative action (AA). But non-discrimination is rarely a goal in and of itself. Ultimately, one tacitly expects a benefit to society. What kind of expected benefit is implicit in AA? The fact that a fully transformed society does not require non-discrimination hints at a possible answer. This is in particular true if the groups themselves are indistinguishable. If one accepts that such social equality is the intended effect of non-discrimination, then the question becomes: does AA lead to it? To answer this, modeling how the populations change in response to policies is not simply a luxury, but a necessity. This paper starts with some natural tenets, and derives from them a model of dynamics that

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govern the evolution of the groups' qualifications according to the imposed policies. These dynamics are general enough to model behavior beyond the specific ones in prior work.

The paper then proceeds by first giving conditions under which an unconstrained policy reaches equality on its own. In this case, surprisingly, imposing AA may break equality. When it doesn't, one would expect the additional constraint to reduce utility, however, it can be shown that utility may in fact increase. In practice, AA acts in two possible ways. Either selection rates of the privileged are reduced or those of the underprivileged are increased [7]. This dichotomy is at the heart of the starkly different ways in which non-discrimination manifests itself. In real world scenarios, unconstrained policies do not lead to equality. In such cases, it can be shown that although imposing AA may remedy it, there is a danger that groups settle at a worse set of qualifications. As a silver lining, one can identify exactly when the constraint not only leads to equality, but also improves all groups.

To summarize, the contributions of this paper are as follows:

- A simple yet flexible model for both a selection process by an institution and dynamics describing the change in the population due to selection policies
- A characterization of the policies resulting from institutional utility maximization without and with imposing affirmative action as non-discrimination
- Conditions under which society equalizes for each type of policy, along with the impact on the long-term institutional utility

These cases and trade-offs are instrumental in determining when and how imposing demographic parity can be beneficial in selection processes, both for the institution and for society on the long run.

The rest of the paper is organized as follows. Section 2 covers the most relevant related work. Section 3 formulates the problem and establishes the models. Section 4 derives the policies that arise from the models and characterizes the conditions under which they lead to social equality. Sections 6 and 7 study the contrast between imposing affirmative action or not, in the case when not imposing it naturally leads to equality or not, respectively. Section 8 concludes.

## 2 RELATED WORK

On the economics side, a large literature exists examining statistical discrimination, dating back to Arrow [2]. Prior work has attempted to study the effect of affirmative action in the labor market to remove stereotypes [3, 5, 8, 12, 18], where stereotypes are defined as misrepresenting the quality of a worker from a disadvantaged group, when groups are assumed to be ex-ante equal. The models of [5] and [12] are the most related to the present work: they both set up a game in a market between workers and employers, where workers have the choice to invest in themselves to become qualified, and emit a test signal that employers threshold as their policy. The dynamics that this game generates on the distribution of qualified individuals per group is very specific. In particular, it is restricted to be unimodal. Both papers look at sufficient conditions for equality and propose interventions to actively reach it. The current paper trades off the concreteness of this line of work for more domain generality, through less specific dynamics. This helps garner more insight about the fundamental interplay between the constraints

and changes in society. Moreover, examining dynamics and tracking the populations through time instead of only looking at equilibria as in [8, 18], allows comparing cumulative utility which plays a role in short term considerations.

A recent work [17] raised very similar questions to the ones considered here. The main difference in that setup is that individuals are characterized by a potentially non-binary score and are studied over only a single step of the selection process. The influence of the selection is modeled by a social utility function  $\Delta$  that can be interpreted as the expected change in score for an individual due to selection. The authors do not suggest that this change is due to population shifts, though it could be. Without tailoring  $\Delta$  specifically to have such an interpretation, it is not possible to track distributions of scores over long time horizons. The dynamics proposed here implements precisely that aspect and enables the study of both institutional utility and social equality that emerge from fairness constraints. Another somewhat related work is that of [10], which models the retention rate of users of a particular service as a function of that service's quality. There, this is thought of as the error of a learning algorithm and the authors propose a robust learning approach to avoid amplifying the difference in retention rate by group.

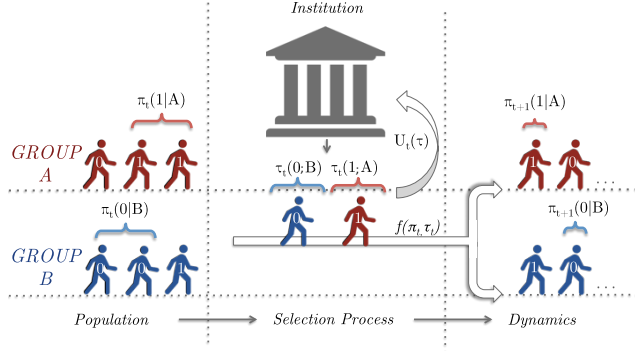
The present formulation also bears some similarity to reinforcement learning due to the dynamics being essentially Markovian. As such, it can be thought of as an optimal control problem. The goal here however is to analyze policies that are unaware of the dynamics, and do not make an explicit effort to learn it. Also, the focus is not simply on utility, but the secondary objective of social equality too. Previous work [13] has explored fairness with learning, guaranteeing that actions are preferentially selected only if they effectively generate more reward. However, there are many key differences, such as the state and action space in the current model being continuous as opposed to the discrete versions treated in [13] and having deterministic transitions. Work in online decision making [14] has also explored fairness constraints, but in a bandit setting that lacks the dynamical aspect.

## 3 PROBLEM FORMULATION

### 3.1 Society: groups, qualification profiles

Throughout this paper, the total population is divided into two groups indexed by the variable  $G$ : group  $G = A$  which constitutes the  $g_A$  fraction and group  $G = B$  the  $g_B = 1 - g_A$  fraction. The limitation to only two groups simplifies the exposition yet maintains the essence of the problem.

One can conceptually think of each individual in either group as having some attribute  $\theta \in \Theta$  that bears information about qualification. For example in a college admission scenario the attribute  $\theta$  can be: [GPA, SAT, Letters of recommendation]. This attribute is thought to provide a complete description in this context. In what follows,  $\theta$  is implicitly mapped through an estimator  $F : \theta \rightarrow \{0, 1\}$  to a crisp evaluation of qualification: say  $v = 1$  if qualified and  $v = 0$  otherwise. The assumption is that this binary classification is sufficient to characterize the utility that the individual would bring. This is compatible with the work on statistical discrimination [5, 12]. An alternative way to reach this model is to first consider a more general multiclass characterization and then show that a



**Figure 1: Selection process and influence dynamics – Two subgroups of the population have each publicly known qualification profiles  $\pi$ . An institution performs a selection by maximizing utility  $U$ , possibly subject to a non-discrimination constraint. The selection policies  $\tau$  influence the profiles at the next round, through dynamics  $f$ .**

threshold policy is optimal with respect to utility and societal improvement, as was done in [17]. This again indicates that a binary characterization, i.e. above and below the threshold, is often sufficient. For the example of admissions, one can think of  $v = 1$  as indicating an individual likely to be successful in college.

Let the distribution of evaluations  $V = v$  in group  $G = j$  be denoted by  $\pi(\cdot|G = j)$ . This probability distribution is the *qualification profile* of group  $G = j$ . Instead of tracking individuals, it is more natural to study changes in the population in terms of the qualification profiles of groups. For example, in college admissions it is not the same students that would change their qualifications in response to admission policies, but rather those of the next application cycle.

### 3.2 Institution: policy, utility, selection rates

The population undergoes a selection process. An *institution*, referred to also by *policy maker*, designs a *policy*  $\tau(V = v; G = j): \{0, 1\} \times \{A, B\} \rightarrow [0, 1]$  that maps each individual to a probability of selection, possibly depending on the group identity. The institution also places a *utility* on each possible evaluation  $v$ , through a map  $u: \{0, 1\} \rightarrow \mathbb{R}, v \mapsto u(v)$ . Since  $v = 1$  and  $v = 0$  are assumed to be beneficial and detrimental respectively, and to avoid trivial policies where none/all are selected in what follows, assume  $u(0) \leq 0 \leq u(1)$ . The (average) *institutional utility* of a policy can then be defined as follows:

$$U(\tau) = \sum_{j \in \{A, B\}} g_j \sum_{v \in \{0, 1\}} u(v) \cdot \tau(V = v; G = j) \cdot \pi(v|G = j). \quad (1)$$

Any given policy also defines (per group) *selection rates*:  $\beta(G = j) = \sum_{v \in \{0, 1\}} \tau(V = v; G = j) \cdot \pi(V = v|G = j)$ . For college admission, this corresponds to acceptance rates per group. Note that each term in the sum is the rate corresponding to a particular evaluation. Denote these by  $\beta(V = v; G = j) = \tau(V = v; G = j) \cdot \pi(V = v|G = j)$ . In what follows, for notational convenience, the explicit ' $V =$ ' and ' $G =$ ' are dropped from the expressions of the qualification profiles, policies, and selection rates, since the context of the argument is clear.

### 3.3 Influence Dynamics

The execution of a selection policy can be thought of as demarcating time  $t$ . The main premise of this paper is that the selection process affects the population, namely by changing the qualification profiles of either group at time  $t + 1$ . This change can be interpreted in different ways. It could be due to genuine change within society: the institutions' policies can then be thought of as having a secondary effect as social incentive or deterrent. But change could also be due to self-selection in the pool of individuals available to the institution on the next time step: policies then have the effect of filtering through time, possibly without society itself changing. Although the model is not specific to an interpretation, when the perspective moves to social equality in Section 4, the tacit assumption is that dynamics are due to genuine change.

Dynamics are in general very complicated. They may vary in time, be stochastic, and depend on many parameters of the problem. Some of these aspects, however, are more relevant than others for capturing the fundamental effects. In what follows, dynamics are taken to be time-invariant and deterministic. Time-invariance is a convenient assumption, which can be supported by the slow change in the way in which society responds to stimuli. Determinism does not mean that there is no stochasticity within society, but that any such change is summarized through the evolution of the qualifications. Moreover, one can interpret these deterministic dynamics as the expectation of a more general stochastic dynamics. As such it is a Markovian assumption, with the state consisting of the qualification profiles.

Finally, what parameters of the problem should govern change and in what manner should they do so? There are three tenets that lead to this paper's proposal:

- Within each group, there is an (a priori) distribution of *potential* qualifications, identical to the qualification profiles at the previous time step.
- Upon observing the previous time step's policy, individuals respond by instantiating with some probability as (a posteriori) qualified or unqualified.
- The response probabilities, which summarize how policies influence individuals, depend only on the selection rates within the group of the individual and their potential qualification.

Tenet (a) describes society's inertia in the absence of influencing policies. Tenet (b) could be thought of as a result of adequate or lacking individual effort in response to policies. In tenet (c), the fact that potentially qualified and potentially unqualified individuals may be influenced differently is necessary for the history of qualifications to have relevance. The main restriction thus is that the only influence of policies is through selection rates within a group. Lack of influence across groups is an idealization of such influence arguably being weaker than within a group. As for selection rates being the key variable rather than the selection policy itself, one can motivate it by considering extremes: indeed, if no qualified (or unqualified) individuals exist then selection policy for qualified (or unqualified) individuals is never observed and cannot have influence.

With these tenets, dynamics induced by the influence of policies can be defined as follows.

**Assumption 3.1 (Dynamics).** For a given group  $j$ , let  $\pi_t(1|j) =: \pi_t(1)$  (the group index dropped here and later, for clarity) denote the qualification profile of group  $j$  for  $v = 1$  at time  $t$  and let the policies  $\tau_t$  at that time step induce the selection rates  $\beta_t$ . Then the qualification profiles at time  $t + 1$  are given by

$$\pi_{t+1}(1) = \pi_t(1) f_1(\beta_t(0), \beta_t(1)) + \pi_t(0) f_0(\beta_t(0), \beta_t(1)) \quad (2)$$

where  $f_0$  and  $f_1$  are two arbitrary continuously differentiable functions  $[0, 1] \times [0, 1] \rightarrow [0, 1]$ . The pair  $(f_0, f_1)$  is referred to as the *dynamics*.

These dynamics make the tenets concrete. For each group  $G = j$ ,  $\pi_t(\cdot|j)$  describes the potential qualification profile. The function  $f_1$  represents the *retention at the top*: the rate of retention of the sub-population with potential  $v = 1$  due to the current policy. The function  $f_0$  represents *change for the better*: the fraction of the potential  $v = 0$  sub-population that progresses to have evaluation 1. Equivalently, one can pair these with their counterparts to describe full conditional distributions. The pair  $1 - f_1$  (*change for the worse*) and  $f_1$  are the conditional distribution describing the response of an individual with potential  $v = 1$  into actual evaluations 0 or 1 respectively, while the pair  $1 - f_0$  (*retention at the bottom*) and  $f_0$  are the same respectively for an individual with potential  $v = 0$ .

It is worth making a couple of remarks about these dynamics. First, the two groups' responses are identical since the choice of  $f_1$  and  $f_0$  does not depend on the group. To justify this, note that in fairness considerations such as demographic parity one makes the inherent assumption that the groups are ex-ante equal in all respect except for their qualification profiles, as otherwise there may be other reasons to treat them differently based only on their identity. Second, the groups do not interact through the functional form of the dynamics. Differences between groups and any potential coupling between groups can only happen through the different and interacting selection rates induced by the policies.

Finally, leaving  $f_0$  and  $f_1$  entirely arbitrary is good for generality but may lead to esoteric behavior. It may be more reasonable to restrict them in certain natural ways. One assumption to make, for example, is that change is harder than retention.

**Assumption 3.2 (Status quo bias).** For all  $x, y \in [0, 1]$ :

$$f_1(x, y) \geq f_0(x, y).$$

### 3.4 Utility Maximization and Non-discrimination

The primary tendency of the institution is to maximize its average utility. Rarely, if ever, are institutions aware of the underlying dynamics. Thus *dynamics-oblivious* policies are the appropriate ones, where the utility is maximized over a single time step. Next, two approaches taken in practice are modeled: unconstrained maximization without any non-discrimination considerations and affirmative action via demographic parity.

**3.4.1 Unconstrained.** In the simplest of settings, institutional utility maximization amounts to solving the following linear program (LP) at every time step  $t$ :

$$\max_{\tau} U_t(\tau), \quad (3)$$

where  $U_t$  is as in (1), with all quantities defined at time step  $t$ .

**3.4.2 Affirmative Action.** If the evaluation  $v$  is perfect, then the unconstrained policy is trivially *identity blind*, thus one could argue that it is fair in the sense of equalized opportunity [9]. However, as distributions of qualified individuals differ per group, the policy will have different selection rates per group. *Affirmative action* (AA), at an intuitive level, attempts to bridge such inequalities between different groups. More precisely, AA forces equal aggregate selection rates between groups [5, 12], and it is more formally known as demographic parity [4, 17]. This paper adheres to this as the archetype of non-discrimination.

**Definition 3.3 (Affirmative Action).** The *affirmative action* constraint forces the policy to select at an equal rate between both groups:

$$\beta(A) = \beta(B). \quad (4)$$

Thus in this case, the policy maker solves (3) with the additional constraint (4). Since the constraint is linear in  $\tau$ , the optimal policy is also found via a LP. This is elaborated in Section 4.

## 4 ROAD TO EQUALITY

With the models of selection process, dynamics, and affirmative action in place, one can now ask the main question: when and how can resulting policies lead to social equality? Perhaps the simplest notion of equality within the population can be defined as the groups becoming indistinguishable in their qualification profiles. This is formalized as follows.

**Definition 4.1 (Social Equality).** Under given dynamics  $(f_0, f_1)$ , a policy is said to be *equalizing* if for all starting  $\pi_0(1|A)$  and  $\pi_0(1|B)$

$$\lim_{t \rightarrow \infty} |\pi_t(1|A) - \pi_t(1|B)| = 0$$

When this happens, say that the policy/population *reaches equality*, which is understood to be in an asymptotic sense.

This section characterizes the policies that result in both the unconstrained and affirmative action settings, and studies the relationships between these policies and the conditions under which they become equalizing.

### 4.1 Unconstrained

**4.1.1 Resulting policy.** First consider the case when the institution does not impose non-discrimination, that is the utility is maximized as in (3) without further constraints. Denote this case by UN. It is then straightforward to see that the optimal policies are  $\tau_t(1; \cdot) = 1$  and  $\tau_t(0; \cdot) = 0$  for all time  $t$ .

Note that the policy only depends on the distribution at time  $t$ . Any dynamics for group  $j$  is thus simply of the form:

$$\pi_{t+1}(1|j) = f(\pi_t(1|j)),$$

where  $f$  is defined in terms of  $(f_0, f_1)$  according to Equation (2) as:

$$f(\pi) = \pi f_1(0, \pi) + (1 - \pi) f_0(0, \pi).$$

For convenience denote the  $t^{\text{th}}$  iterate of  $f$  by  $f^t$ .

**4.1.2 Condition for equality.** Can an unconstrained policy be equalizing? The following result from classical stability analysis, proved for completeness in Appendix A, characterizes exactly when this is possible.

PROPOSITION 4.2. *Dynamics of  $\pi_{t+1} = f(\pi_t)$  are equalizing if and only if  $f$  has a unique globally attracting equilibrium point.*

Thus under the unconstrained policy, the dynamics  $f$  must have a unique attracting equilibrium point  $\pi^e$ . It is worth noting that if  $f$  has an attracting point from all but a finite number of points, then these must be unstable fixed points of  $f$ . A point  $\pi'$  is an unstable fixed point if  $f(\pi') = \pi'$  yet for all  $\pi_0 \neq \pi'$  and all small enough  $\epsilon > 0$ , for large enough  $t$  (depending on  $\pi_0$  and  $\epsilon$ )  $|f^t(\pi_0) - \pi'| > \epsilon$ . Since in the real world any reasonable dynamics would have noise, these unstable points won't have an influence, and one could alternatively assume that  $f$  has a unique stable fixed point except at finitely many points.

If  $f$  has a globally attracting fixed point then  $f^t$  converges to  $\pi^e$  as  $t \rightarrow \infty$ , a constant function which is trivially a contractive mapping. If this convergence is uniform, then there would be a large enough  $T$  such that  $f^T$  is contractive. One could instead consider the following stronger condition in the form that  $f$  itself be a contractive mapping.

Definition 4.3 (Contractivity).  $f : [0, 1] \rightarrow [0, 1]$  is a contractive mapping if  $\exists L \in [0, 1]$  such that  $f$  is  $L$ -Lipschitz:  $\forall x, y \in [0, 1]$ ,

$$|f(x) - f(y)| \leq L|x - y|. \quad (5)$$

If  $f$  is differentiable, then this condition is equivalent to having the derivative of  $f$  bounded by  $L$  on  $[0, 1]$ . By an application of the Banach fixed-point theorem [16], this shows that the condition of Definition 4.3 is indeed sufficient for  $f$  to have a globally attractive fixed point. Recall the following.

THEOREM 4.4. (Banach fixed-point theorem) Let  $(X, d)$  be a non-empty complete metric space and let  $f : X \rightarrow X$  be a contractive mapping. Then  $f$  admits a unique fixed-point  $x^e \in X$  where  $x^e$  can be found by choosing any arbitrary element  $x_0 \in X$  and iterating the function  $f$ , i.e. define  $x_t = f^t(x_0)$ , then  $x_t \rightarrow x^e$ .

To stress again, contractivity is not a necessary condition to reach equality, but it is a simple condition to place on the dynamics that captures the intuition of reaching equality. More importantly, since the dynamics are defined in terms of the functions  $f_0$  and  $f_1$ , it is more useful to give conditions on the latter to obtain a contractive  $f$ . For  $\pi$  and  $\pi'$  such that  $\pi - \pi' = \Delta > 0$ , rewrite the requirement that  $|f(\pi) - f(\pi')| \leq L \cdot \Delta$  for  $L \in [0, 1]$  as:

$$\begin{aligned} |f(\pi) - f(\pi')| &= |\pi f_1(0, \pi) + f_0(0, \pi) - \pi f_0(0, \pi) \\ &\quad - (\pi' f_1(0, \pi') + f_0(0, \pi') - \pi' f_0(0, \pi'))| \\ &= |\pi(f_1(0, \pi) - f_1(0, \pi - \Delta)) \\ &\quad + (1 - \pi)(f_0(0, \pi) - f_0(0, \pi - \Delta)) \\ &\quad + \Delta(f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta))| < L \end{aligned} \quad (6)$$

A first step is to take both  $f_1$  and  $f_0$  to be Lipschitz.

Assumption 4.5.  $f_1$  and  $f_0$  are  $\ell_1$  Lipschitz continuous with constants  $L_1$  and  $L_0$  respectively. That is  $\forall x_1, x_2, y_1, y_2 \in [0, 1]$ :

$$|f_1(x_1, x_2) - f_1(y_1, y_2)| \leq L_1(|x_1 - y_1| + |x_2 - y_2|)$$

and

$$|f_0(x_1, x_2) - f_0(y_1, y_2)| \leq L_0(|x_1 - y_1| + |x_2 - y_2|).$$

This assumption is reasonable as one would expect the jump in the rate of retention ( $f_1$ ) or rate of change ( $f_0$ ) to be bounded as a function of the jump in the selection rates. But looking at Equation (6), it is not sufficient. How large should  $L_0$  and  $L_1$  be?

$$\begin{aligned} |f(\pi) - f(\pi')| &\leq |\pi(f_1(0, \pi) - f_1(0, \pi - \Delta))| \\ &\quad + |(1 - \pi)(f_0(0, \pi) - f_0(0, \pi - \Delta))| \\ &\quad + |\Delta(f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta))| \quad (\text{triangle inequality}) \\ &< \Delta(\pi L_1 + (1 - \pi)L_0 + |f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)|) \end{aligned} \quad (7)$$

Thus a sufficient condition for  $f$  to be a contractive mapping is:

$$L_{\text{UN}} := \max_{\pi \in [0, 1], \Delta < \pi} \pi L_1 + (1 - \pi)L_0 + |f_1(0, \pi - \Delta) - f_0(0, \pi - \Delta)| < 1 \quad (8)$$

4.1.3 *Speed of convergence.* Contractivity is a restrictive condition but it has the obvious benefit of easily describing the rate at which the population reaches equality. In particular, if

$$\max_{j \in \{A, B\}} |\pi_0(1|j) - \pi^e| \leq \Delta,$$

then

$$|\pi_t(1|A) - \pi_t(1|B)| \leq 2\Delta L^t.$$

Equality is reached at a linear rate (in the exponent): to equalize to within an  $\epsilon$  difference, about  $\frac{\log \epsilon / \Delta}{\log L}$  time steps are sufficient.

## 4.2 Affirmative Action

4.2.1 *Two resulting policies.* The following characterizes the solution of maximizing the utility as in (3) under the additional constraint (4). The proof can be found in Appendix A.

PROPOSITION 4.6. Assume that at time  $t$ ,  $\pi_t(1|j) \geq \pi_t(1|\neg j)$  and say that  $j$  is advantaged. Then the optimal policy is one of two cases, depending only on  $g_j$ ,  $u(0)$ , and  $u(1)$ :

$$\begin{aligned} \text{AA}^- \text{ If } g_j u(1) + (1 - g_j)u(0) \leq 0, \text{ then} \\ \tau_t(1; j) &= \frac{\pi_t(1|\neg j)}{\pi_t(1|j)}, \quad \tau_t(0; j) = 0 \text{ (under-acceptance),} \\ \tau_t(1; \neg j) &= 1, \quad \tau_t(0; \neg j) = 0. \end{aligned}$$

$$\begin{aligned} \text{AA}^+ \text{ If } g_j u(1) + (1 - g_j)u(0) \geq 0, \text{ then} \\ \tau_t(1; j) &= 1, \quad \tau_t(0; j) = 0, \\ \tau_t(1; \neg j) &= 1, \quad \tau_t(0; \neg j) = \frac{\pi_t(1|j) - \pi_t(1|\neg j)}{1 - \pi_t(1|\neg j)} \text{ (over-acceptance).} \end{aligned}$$

There are two possible cases through which affirmative action impacts the policy, denoted by  $\text{AA}^-$  and  $\text{AA}^+$ . These represent two drastically different approaches to fairness.  $\text{AA}^-$  (under-acceptance) accepts fewer qualified individuals from the advantaged group so as to equalize the selection rates for qualified individuals between both groups. On the other hand,  $\text{AA}^+$  (over-acceptance) accepts unqualified individuals from the disadvantaged group in order to equalize the aggregate selection rates. One could think of  $\text{AA}^-$  as increasing the standard for the advantaged group and as such reducing total selection rates. As a peek at the intuition to develop, this could possibly reduce the motivation for individuals to become qualified. As for  $\text{AA}^+$ , one could think of it as reducing the standard for the disadvantaged group. It is not as evident whether this could potentially increase or decrease the motivation to become qualified.

One one hand, it does present the possibility of leading to a more qualified overall society by increasing the total selection rates and training more unqualified people. On the other hand, selecting unqualified corresponds to the condition of the so-called mismatch hypothesis [21], which claims that this could actually reduce the number of qualified individuals over time. This mismatch effect has been amply debated in discussions on public policy with arguments on both sides [1, 11, 21]. One perspective of the current investigation is to understand how the dynamics tip the scales in this debate.

**4.2.2 Conditions for equality.** Under AA, the analysis of equality gets complicated. This is mainly due to the fact that in Proposition 4.6 there is a specific advantaged group. If from one time step to the next the advantage changes, the identities of the groups swap. This in itself is not an issue, unless the swap also causes a switch between the cases  $AA^-$  and  $AA^+$ . This last situation will not be considered for two reasons: first, because it is too unwieldy, and second, because it does not arise if the dynamics are slow enough, as formalized in Section 5. The conditions for equality given here are thus provisional on remaining in  $AA^-$  or  $AA^+$ , with swaps or without. Here is a characterization to remain in one case at all times, given with no proof as it follows from Proposition 4.6.

**PROPOSITION 4.7.** *Let  $C_{j,t} := [\pi_t(1|j) \geq \pi_t(1|\neg j)]$  be the  $j$ -advantage condition. Resolving equality either way, note that  $C_{\neg j,t} = \neg C_{j,t}$ . Let  $C_{-|j} := [g_j u(1) + (1 - g_j)u(0) \leq 0]$  be the  $AA^-$  condition under  $j$ -advantage and let  $C_{+|j} := [g_j u(1) + (1 - g_j)u(0) \geq 0] = \neg C_{-|j}$  be the  $AA^+$  condition under  $j$ -advantage.*

*Then to remain in  $AA^-$  at all times, it is necessary that for all  $t$ :*

$$(C_{A,t} \wedge C_{-|A}) \vee (C_{B,t} \wedge C_{-|B}),$$

*and in particular it suffices that  $C_{-|A} \wedge C_{-|B}$ . Similarly, to remain in  $AA^+$  at all times, it is necessary that for all  $t$ :*

$$(C_{A,t} \wedge C_{+|A}) \vee (C_{B,t} \wedge C_{+|B}),$$

*and in particular it suffices that  $C_{+|A} \wedge C_{+|B}$ .*

Another reason for the complication is because, under AA, the dynamics for one group depends on the other group: in Proposition 4.6 there always exists one  $j$  for which the policy depends on the qualification profiles of  $\neg j$ . The dynamics therefore have to be tracked jointly, unlike in UN. Write

$$\begin{aligned}\pi_{t+1}(1|A) &= f_A(\pi_t(1|A), \pi_t(1|B)) \\ \pi_{t+1}(1|B) &= f_B(\pi_t(1|A), \pi_t(1|B)),\end{aligned}$$

where  $f_A$  and  $f_B$  are two components of a joint dynamics function  $f : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ . These components can be written in terms of  $f_0$  and  $f_1$  according to Equation (2) and Proposition 4.6, but have a complicated expression that accounts for all four cases (who is advantaged and which AA case it is). Nevertheless, one could say that given dynamics of this form, equality will be reached if as  $t \rightarrow \infty$   $|f_A^t(\pi_0(1|B), \pi_0(1|A)) - f_B^t(\pi_0(1|B), \pi_0(1|A))| \rightarrow 0$ , for all  $\pi_0(1|A)$  and  $\pi_0(1|B)$ .

Similarly to the unconstrained case, since this would otherwise give no guarantees on the speed of convergence, one can require  $f$  to satisfy the following contraction condition. There exists  $L \in [0, 1]$  such that  $f$  satisfies:

$$|f_A(\pi, \pi') - f_B(\pi, \pi')| \leq L|\pi - \pi'| \quad \forall \pi, \pi' \in [0, 1] \quad (9)$$

This does not quite fit the classical Banach fixed-point theorem, but it is easy to show sufficiency. That is, if the dynamics satisfy this condition then the population will reach equality. Indeed, let  $\pi_0, \pi'_0 \in [0, 1]$ :

$$\begin{aligned}|f_A(\pi_0, \pi'_0) - f_B(\pi_0, \pi'_0)| &\leq L|\pi_0 - \pi'_0| \\ |f_A(\pi_1, \pi'_1) - f_B(\pi_1, \pi'_1)| &\leq L|\pi_1 - \pi'_1| \leq L^2|\pi_0 - \pi'_0| \\ &\vdots \\ |f_A(\pi_{t-1}, \pi'_{t-1}) - f_B(\pi_{t-1}, \pi'_{t-1})| &\leq L^t|\pi_0 - \pi'_0|\end{aligned}$$

Since  $L < 1$  and  $|\pi_0 - \pi'_0| \leq 1$  then  $\lim_{t \rightarrow \infty} L^t = 0$  and  $|f_A(\pi_{t-1}, \pi'_{t-1}) - f_B(\pi_{t-1}, \pi'_{t-1})| \geq 0 \quad \forall t$  therefore  $\lim_{t \rightarrow \infty} |f_A(\pi_t, \pi'_t) - f_B(\pi_t, \pi'_t)| = 0$  and the population will reach equality.

As in the unconstrained case, this contractivity requirement on  $f$  can be reduced to sufficient conditions on  $f_0$  and  $f_1$ .

**PROPOSITION 4.8.** *Assume that  $AA^-$  holds at all times, by the necessary or sufficient conditions of Proposition 4.7. Then if*

$$L_{AA^-} := \max_{\pi \in [0, 1]} |f_1(0, \pi) - f_0(0, \pi)| < 1 \quad (10)$$

*holds, then society equalizes.*

**PROPOSITION 4.9.** *Assume that  $AA^+$  holds at all times, by the necessary or sufficient conditions of Proposition 4.7. Then if*

$$\begin{aligned}L_{AA^+} := \\ \max_{0 \leq \Delta \leq \pi \leq 1} 2[\pi L_1 + (1 - \pi)L_0] + |f_1(\Delta, \pi - \Delta) - f_0(\Delta, \pi - \Delta)| < 1\end{aligned} \quad (11)$$

*holds, then society equalizes.*

The proofs can be found in Appendix A. Note that while social equality conditions for UN and  $AA^-$  can be related, the conditions for  $AA^+$  is not comparable to either of them in the sense that there exist dynamics where one holds but the other does not.

### 4.3 Effect of Stereotypes

Everywhere in this paper, except in this subsection, the policy maker is assumed to know exactly the distribution  $\pi$  of qualifications of both groups. But the institution relies on its estimator  $F$  to identify qualified individuals, errors in  $F$  translate to errors  $\epsilon$  in estimates of group qualifications and thus  $\hat{\pi}$ . As a result of these errors, it becomes hard to properly impose fairness constraints, as even selection rates per true groups are unknown. But the framework allows one to investigate the effect of certain types of errors on the dynamics.

At each time step  $t$ , define the errors in the qualification's estimates as  $\epsilon_t^A$  and  $\epsilon_t^B$  such that  $-\pi_t(1|A) \leq \epsilon_t^A \leq 1 - \pi_t(1|A)$  and  $-\pi_t(1|B) \leq \epsilon_t^B \leq 1 - \pi_t(1|B)$ . Then the institution has the following qualification estimates:

$$\begin{aligned}\hat{\pi}_t(1|A) &= \pi_t(1|A) + \epsilon_t^A \\ \hat{\pi}_t(1|B) &= \pi_t(1|B) + \epsilon_t^B\end{aligned}$$

Assume the errors are small enough that one can still identify which group truly is more qualified. This translates to having  $\epsilon_t^B - \epsilon_t^A \leq \pi_t(1|A) - \pi_t(1|B)$ . Furthermore, restrict errors to be of the form of positive or negative stereotypes. Concretely, if  $\epsilon_t^B \leq 0$ , call

this negative stereotype, the institution identifies only  $\frac{\pi_t(1|B)+\epsilon_t^B}{\pi_t(1|B)}$  of qualified individuals and considers the rest unqualified. If  $\epsilon_t^A \geq 0$ , call this positive stereotype, then it identifies all qualified individuals correctly and mistakes  $\frac{\epsilon_t^A}{\pi_t(0|A)}$  of unqualified individuals as qualified.

The institution bases its policy on these estimates, however in terms of the actual distribution the implemented policy actually differs.

If the policy acted upon is UN, meaning  $\tau(1) = 1$  and  $\tau(0) = 0$ , with a positive stereotype  $\hat{\pi} = \pi + \epsilon \geq \pi$ , the actual policy is:

$$\tau(1) = 1, \tau(0) = \frac{\epsilon}{1 - \pi}$$

And with a negative stereotype  $\hat{\pi} = \pi + \epsilon \leq \pi$ :

$$\tau(1) = \frac{\pi + \epsilon}{\pi}, \tau(0) = 0$$

These policies are those that influence the next step distribution. Note that in both cases the unconstrained condition for equality might fail to hold as the difference in selection rates between the two groups can be exaggerated.

Now consider that the institution applies affirmative action. In the case of AA<sup>-</sup>, surprisingly, given negative stereotype against group B, no matter if group A's qualification is overestimated or underestimated, equality is reached under the same cases as if the true distribution been known (no stereotypes). However, given a positive stereotype for group B, then the conditions for equality under AA<sup>-</sup> derived previously are no longer sufficient. Thus the AA<sup>-</sup> policy is robust to negative stereotypes against disadvantaged groups.

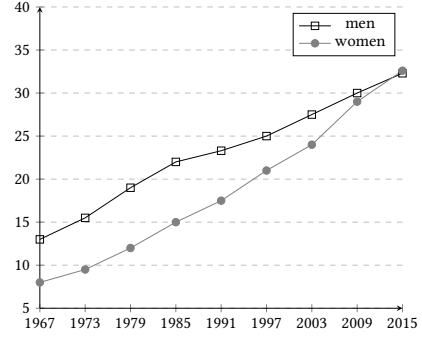
On the other hand with AA<sup>+</sup>, with a negative bias against group A and a positive bias for group B one can guarantee equality under the same cases as without stereotypes. A further treatment of the actual policies and analysis of the above conclusions is left to Appendix B.

## 5 DYNAMICS IN CONTINUOUS TIME

While discrete time (DT) describes successive selection-response steps naturally, the arbitrary nature of the dynamics can lead to sudden jumps and oscillatory behavior. These do not quite correspond to how populations evolve in the real world, where one expects change to happen gradually and slowly through time. Figure 2 shows the evolution of the percentage of bachelor degree holders by sex in the United States of America. In 1967, 13% of men 25 years and older held a bachelor's degree or higher while 8% of women did [20]; this gap has decreased through time as degree attainment increased for both sexes. In 2015, 33% of women held a college degree compared to 32% of men, thus the proportion of degree holders equalized between both sexes in 47 years time. One of the reported reasons behind this increase has been attributed to Title IX [19], a civil rights law passed in 1972 aiming to eliminate discrimination in educational programs and the removal of quotas against disadvantaged groups.

A change that is smooth and gradual can be modeled by transforming the discrete time dynamics into continuous time (CT). The transformation is standard, starting from dynamics (2) and dropping group indices, one can rewrite it as a difference equation:

$$\pi_{t+1}(1) - \pi_t(1) = \pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1))$$



**Figure 2: Percentage of the US population 25 years and older holding a bachelor degree by sex from 1967 till 2015, adapted from [20].**

Now assume that the difference from  $t$  to  $t + \Delta t$  for all  $\Delta t > 0$  is proportional to that from  $t$  to  $t + 1$  (linear interpolation):

$$\begin{aligned} & \pi_{t+\Delta t}(1) - \pi_t(1) \\ &= \Delta t \cdot (\pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1))) \end{aligned}$$

dividing by  $\Delta t$ :

$$\frac{\pi_{t+\Delta t}(1) - \pi_t(1)}{\Delta t} = \pi_t(1)(f_1(\beta_t(0), \beta_t(1)) - 1) + \pi_t(0)f_0(\beta_t(0), \beta_t(1))$$

Now taking the limit as  $\Delta t \rightarrow 0$ :

$$\frac{d\pi_t}{dt} = \pi_t(f_1(\beta_t(0), \beta_t(1)) - 1) + (1 - \pi_t)f_0(\beta_t(0), \beta_t(1)) \quad (12)$$

The CT dynamics thus derived (12) can be motivated in two ways. As is shown next, it maintains the properties of the DT dynamics regarding conditions leading to social equality. Moreover, it possesses additional properties that greatly facilitate the analysis.

The proof of this and all the following lemmas can be found in Appendix A.

**LEMMA 5.1.** *Assume the unconstrained policy is implemented in both discrete time (DT) and continuous time (CT). If the sufficient condition of Definition 4.3 for social equality holds for the DT dynamics with equilibrium point  $\pi^e$ , then the CT dynamics also converges to  $\pi^e$  starting from any initial condition.*

The following shows that populations in CT, unlike in DT, maintain their order.

**LEMMA 5.2.** *Following either an unconstrained or affirmative action policy in CT will preserve the initial advantage. For example, if at  $t = 0$ ,  $\pi_0(1|B) \leq \pi_0(1|A)$  then for all  $t \geq 0$   $\pi_t(1|A) \leq \pi_t(1|B)$ .*

With Lemma 5.2, reaching equality with AA is straightforward in CT if it does in DT.

**LEMMA 5.3.** *If an AA policy in DT reaches equality, then in CT equality is likewise reached.*

Thus to summarize, transitioning to CT does not break the conditions previously derived for DT but in fact facilitates them. One can additionally give a bound on the difference of the distributions between groups at each time step:

LEMMA 5.4. *If the DT dynamics is contractive with constant  $L \in [0, 1)$ , then under CT the difference between the distributions of both groups  $\Delta_t = \pi_t(1|A) - \pi_t(1|B)$  at any time  $t$  obeys:*

$$\Delta_0 e^{-t(1+L)} \leq \Delta_t \leq \Delta_0 e^{-t(1-L)} \quad (13)$$

Motivated by these properties, in what follows the remainder of the analysis of the behavior of the selection process on the groups' distributions due to the different policies is performed under CT dynamics. The treatment could have initially started with CT dynamics, however the setup is more intuitive in DT and motivates the transition better.

## 6 AFFIRMATIVE ACTION UNDER NATURAL EQUALITY

This section is concerned with the following scenario: assume that using the *unconstrained* policy equality is reached, what can happen if one instead follows an affirmative action policy? Even though affirmative action may seem unnecessary, as without requiring further intervention equality is reached, but can there be any additional benefit to applying fairness constraints and is one guaranteed that in this setting equality will also be reached? This is relevant since the policy maker may not be aware of operating in this regime.

Formally, make the assumption that the dynamics under the unconstrained policy (UN) is contractive with constant  $L_{UN} < 1$ , implying that there exists a globally attracting point  $\pi^e$ . The precise questions are then: Are the conditions for equality with affirmative action also met? And how does the cumulative utility obtained under AA policies compare to that under the UN policy?

Because in CT advantage is maintained (Lemma 5.2) and Proposition 4.8 applies (Lemma 5.3), if  $AA^-$  holds initially it continues to hold and the only remaining condition, which is therefore sufficient, is Equation (10). It can be restated to require that the following doesn't hold for any  $\pi \in [0, 1]$ :

$$(f_1(0, \pi) = 1 \wedge f_0(0, \pi) = 0) \vee (f_1(0, \pi) = 0 \wedge f_0(0, \pi) = 1).$$

The following then holds.

THEOREM 6.1. *If equality is reached with an unconstrained (UN) policy then it is necessarily reached by an  $AA^-$  policy, however with no more, and possibly less, utility at each step.*

PROOF. Since equality is reached under the UN dynamics then  $\forall \pi \in [0, 1]$ :

$$|f'(\pi)| \leq L_{UN}$$

where  $f'$  is the derivative of the dynamics under the UN policy. Expanding in terms of the proposed dynamics:

$$|f_1(0, \pi) - f_0(0, \pi) + \pi f'_1(0, \pi) + (1 - \pi) f'_0(0, \pi)| \leq L_{UN} \quad (14)$$

where  $f'_1$  and  $f'_0$  are the partial derivatives of  $f_1(0, \pi)$  and  $f_0(0, \pi)$  with respect to  $\pi$ .

Suppose  $|f_1(0, \pi^*) - f_0(0, \pi^*)| = 1$  for some  $\pi^* \in [0, 1]$ . If  $\pi^* \in (0, 1)$  then since both  $f_1$  and  $f_0$  take values in  $[0, 1]$ , then at  $\pi^*$  one attains its maximum and the other its minimum, thus  $f'_1(0, \pi^*) = 0$  and  $f'_0(0, \pi^*) = 0$ . This contradicts condition (14) as one would have  $1 \leq L_{UN}$ .

Now suppose  $\pi^* = 1$ , if  $f_1(0, \pi^*) = 1$  then  $f'_1(0, \pi^*) \geq 0$  and thus the condition can't be satisfied as one will have  $1 + f'_1(0, \pi^*) \geq 1$ , else if  $f_1(0, \pi^*) = 0$  then  $f'_1(0, \pi^*) \leq 0$  and similarly the condition

can't be satisfied as  $|-1 + f'_1(0, \pi^*)| \geq 1$ ; the argument is similar if  $\pi^* = 0$ .

As the sufficient conditions for both  $AA^-$  and UN to reach equality are satisfied, in CT the equilibrium point  $\pi^e$  will be reached by both policies, however their trajectories toward  $\pi^e$  are different and thus each achieves a different utility.

At each time-step  $t$ , to differentiate between the distributions due to different policies denote by  $\pi_t(1|A, UN)$  the distribution of  $\pi(1|A)$  at time  $t$  due to the UN policy starting from  $t = 0$ , and similarly for  $AA^-$ . The utility of the UN policy at time  $t$  is:

$$U_t(UN) = g_A \cdot u(1) \cdot \pi_t(1|A, UN) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, UN)$$

and that of  $AA^-$  at time  $t$  is:

$$U_t(AA^-) = g_A \cdot u(1) \cdot \pi_t(1|B, AA^-) + (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, AA^-)$$

Consider the difference of the utilities:

$$\begin{aligned} U_t(UN) - U_t(AA^-) &= \\ g_A \cdot u(1) \cdot \pi_t(1|A, UN) &+ (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, UN) \\ - (g_A \cdot u(1) \cdot \pi_t(1|B, AA^-) &+ (1 - g_A) \cdot u(1) \cdot \pi_t(1|B, AA^-)) \end{aligned}$$

Since under  $AA^-$  and UN the trajectory of  $\pi(1|B)$  is identical then  $\pi_t(1|B, AA^-) = \pi_t(1|B, UN)$ , thus:

$$U_t(UN) - U_t(AA^-) = g_A \cdot u(1) \cdot (\pi_t(1|A, UN) - \pi_t(1|B, UN))$$

Following Proposition 5.2, one has that  $\forall t \geq 0$ ,  $\pi_t(1|A, UN) - \pi_t(1|B, UN) \geq 0$ , and thus the utility of the UN policy is always no less than that of  $AA^-$ .  $\square$

In light of this, the policy maker has no gain in enforcing affirmative action in this case.

Consider next how  $AA^+$  acts on the population in this setting. The sufficient condition (11) for equality with the  $AA^+$  policy cannot be directly implied from the UN equality. However, under  $AA^+$  group A's distribution trajectory will reach the equilibrium point  $\pi^e$  as the group's policy is identical to the UN policy in CT. On the other hand, group B's distribution evolution through time is unclear. The dynamics governing it is, with  $\Delta(t) = \pi_t(1|A, AA^+) - \pi_t(1|B, AA^+)$ :

$$\begin{aligned} \frac{d\pi_t(1|B, AA^+)}{dt} &= \pi_t(1|B, AA^+) f'_1(\Delta(t), \pi_t(1|B, AA^+) - \Delta(t)) \\ &+ (1 - \pi_t(1|B, AA^+)) f'_0(\Delta(t), \pi_t(1|B, AA^+) - \Delta(t)) \\ &- \pi_t(1|B, AA^+). \end{aligned}$$

Since  $\pi_t(1|A, AA^+)$  converges to  $\pi^e$  as  $t \rightarrow \infty$ , then  $\Delta(t)$  will only become a function of  $\pi_t(1|B, AA^+)$  and  $\pi^e$ . Therefore, the dynamics will approximately become an ODE with one variable, and oscillatory behavior cannot exist in first order one dimensional ODEs [22], then  $\pi_t(1|B, AA^+)$  will converge to some fixed point of the dynamics necessarily below  $\pi^e$ . Looking closely, if  $\pi^e$  was initially below  $\pi_0(1|B, AA^+)$ , then the trajectory of  $\pi_t(1|A, AA^+)$  will force group B to converge toward the equilibrium point. On the other hand, if  $\pi^e \geq \pi_0(1|B, AA^+)$  then nothing can be said, thus unfortunately only worse off equilibria are guaranteed.

Now let's assume the conditions for reaching equality with the  $AA^+$  policy are met to see what could be the consequence. Specifically there exists  $L_{AA^+} \in [0, 1)$  such that for all  $\pi, \pi'$  in  $[0, 1]$  one has:

$$|f_A(\pi, \pi') - f_B(\pi, \pi')| \leq L_{AA^+} |\pi - \pi'|$$



Similarly to the discussion in the  $AA^-$  section, by comparing the cumulative utility with UN one surprisingly finds a lifeline for enforcing affirmative action.

**THEOREM 6.2.** *Let  $\alpha = (1 - g_A)u(1)/((1 - g_A)u(1) + |u(0)|)$ . Assume UN and  $AA^+$  both satisfy sufficient conditions (8) and (11) to reach equality in DT, assuming remaining in the same case. Then if  $L_{UN}$  and  $L_{AA^+}$  satisfy the following:*

$$L_{UN} \geq 1 - \alpha$$

and

$$L_{AA^+} \leq 1 + (L_{UN} - 1)/\alpha,$$

Then  $AA^+$  provides more utility in CT over an infinite time horizon.

The proof can be found in Appendix A. This statement can be roughly restated to be that if under  $AA^+$  equality is reached faster than under the unconstrained policy, then  $AA^+$  results in a utility gain. This implies that rather than trading off, speed of convergence and utility go hand in hand.

## 7 AFFIRMATIVE ACTION UNDER DISPARATE EQUILIBRIA

### 7.1 Multiple Equilibria

It is more realistic to expect that myopically maximizing utility is unlikely to equalize the qualifications of groups. That is, one expects following an unconstrained policy to not lead to equality. To model this scenario, the dynamics should not have a unique attracting equilibrium point. In general one could have arbitrarily many, even infinitely many, fixed points for the dynamics. In this section this is simplified to assuming that the dynamics under UN has a finite number of attracting fixed points, each with its own basin of attraction, and delimited by two repelling unstable fixed points. Thus the UN policy would not equalize the groups unless they are initially close enough to fall in the same basin. The notion of such dynamics can be formalized as follows.

**Definition 7.1 ( $k$ -Equilibrium Dynamics).** A  $k$ -equilibrium mapping  $s$  a continuously differentiable function  $f : [0, 1] \rightarrow [0, 1]$  has  $k$  stable attracting points  $\pi_1^e \leq \dots \leq \pi_k^e$  where  $f(\pi_i^e) = \pi_i^e$  for all  $i \in [k]$  and  $k + 1$  limiting points  $\delta_0 = 0, \delta_1 \in (\pi_1^e, \pi_2^e), \delta_2 \in (\pi_2^e, \pi_3^e), \dots, \delta_{k-1} \in (\pi_{k-1}^e, \pi_k^e)$  and  $\delta_k = 1$  such that for all  $\pi \in (\delta_{i-1}, \delta_i)$  one has  $f^n(\pi) \rightarrow \pi_i^e$  as  $n \rightarrow \infty$ . When  $i = 1$ , then the corresponding interval becomes  $[0, \delta_1)$ , when  $i = k$ , it is  $(\delta_{k-1}, 1]$ , and finally if  $k = 1$  the basin is  $[0, 1]$ .

Figure 3 shows 3-equilibrium dynamics under an UN dynamics with the direction of the gradient illustrated at each point. See Appendix C for the specific pair  $(f_0, f_1)$ .

For CT dynamics of the form:

$$\frac{d\pi}{dt} = f(\pi_t) - \pi_t$$

to be  $k$ -equilibrium dynamics, necessary and sufficient conditions on the function  $f$  are the following,  $\forall i \in [k]$ :

$$f(\pi_i^e) = \pi_i^e$$

$$\forall \pi \in (\delta_{i-1}, \pi_i^e) : f(\pi) > \pi, \forall \pi \in (\pi_i^e, \delta_i) : f(\pi) < \pi \quad (15)$$

This directly implies that  $\pi_i^e$  for  $i \in [k]$  are asymptotically stable fixed points and as a consequence  $\delta_i$ 's are unstable fixed points.

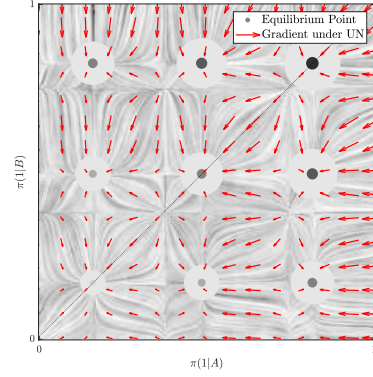


Figure 3: 3-equilibrium mapping under UN policy

To parallel the fact that conditions for CT thus far have mirrored that in DT, note that these conditions are indeed necessary in DT but not sufficient, because oscillations may result (an example is a single equilibrium at 0.5 with  $f(\pi) = 1 - \pi$ .) To make it work, it suffices to modify condition (15) to:

$$\forall \pi \in (\delta_{i-1}, \pi_i^e) : \pi < f(\pi) < \pi_i^e, \forall \pi \in (\pi_i^e, \delta_i) : \pi_i^e < f(\pi) < \pi \quad (16)$$

and additionally require that there exists a neighborhood of radius  $r^i > 0$  around every equilibrium point  $\pi_i^e$  such that:

$$\frac{df}{d\pi} \leq L_i \quad (17)$$

for some  $L_i \in [0, 1]$  for all  $\pi \in [\pi_i^e - r^i, \pi_i^e + r^i]$ . These are local Lipschitz conditions that allow to control the rate of convergence to the equilibrium points.

### 7.2 Eliminating Disparate Equilibria

The following shows that similarly to the single equilibrium UN, the  $AA^-$  policy is equalizing even under a  $k$ -equilibrium UN. Refer to Appendix A for the proofs of this section,

**THEOREM 7.2.** *Assume status quo bias, i.e. Assumption 3.2. If  $f$  is a  $k$ -equilibrium dynamics, as in Definition 7.1 that satisfies conditions (16) and (17) under an UN policy in DT, then following  $AA^-$  with the DT dynamics reaches equality from any starting distribution. Since necessary conditions for DT are sufficient for CT, the same holds also for the CT dynamics.*

The above theorem states that following  $AA^-$  is a guaranteed way to reach equality, however equality comes at a price.

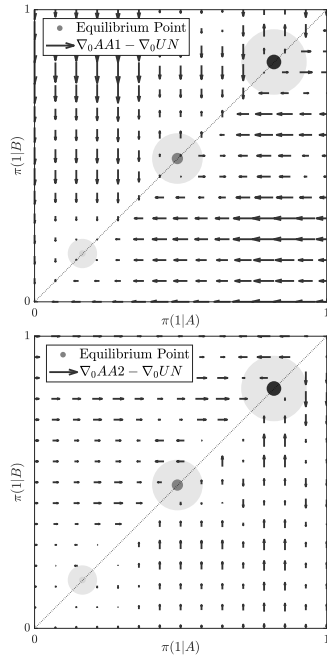
With only the necessary condition for a  $k$ -equilibrium dynamics in CT (15) and assuming that  $AA^-$  holds  $AA^-$  leads to an eventual population with utility no better than under UN.

On the other hand when following  $AA^+$ , equality will always be beneficial. As before, under the conditions of a  $k$ -equilibrium dynamics, one cannot deduce that the conditions for  $AA^+$  hold. Assuming that these conditions do hold, it would actually guarantee a utility no worse than following the UN policy. The following Theorem characterizes this.

**THEOREM 7.3.** *If  $f$  is a  $k$ -equilibrium dynamics under UN in CT, enforcing  $AA^-$  results in an equilibrium where populations have a*

utility value no higher (and possibly lower), while if the conditions for equality under  $AA^+$  hold, enforcing  $AA^+$  will result in an equilibrium where populations have a utility value no lower (and possibly higher) compared to the limiting population under UN.

Figure 4 shows the difference between the gradient of  $\pi_t$  under either  $AA^-$  and  $AA^+$  and UN, for the dynamics which resulted in Figure 3 in a 3-equilibrium UN. This can be shown to satisfy the conditions to make the  $AA^-$  and  $AA^+$  policies equalizing. Note how the gradient difference for  $AA^-$  points generally downward and leftward toward states with lower utility as opposed to that of  $AA^+$  which generally points upward and rightward toward equilibrium points with higher utility. These illustrate qualitatively the kind of behavior characterized in Theorem 7.3.



**Figure 4: Illustration of Theorem 7.3 for the  $AA^-$  and  $AA^+$  policies respectively under the same dynamics as that of Figure 3**

## 8 CONCLUSIONS

Imposing fairness considerations on decision making systems requires understanding the influence that they will have on the population at hand. Without knowledge of this influence, seemingly fair constraints might in fact exacerbate existing differences between groups of the population.

This paper proposed a simple but expressive model of a selection process to concretely consider these concerns. Namely, affirmative action was studied in terms of its ability to equalize the qualifications of different groups. Imposing one case of affirmative action was shown to guarantee equality at the cost of worse institutional utility and possibly decreasing the population's overall qualification level. In another case, however, affirmative action was shown to lead to a policy with different characteristics: equality cannot be

directly guaranteed to hold but when it does, it results, in equilibria where the population becomes more qualified.

This paper is certainly not the last word on the role of dynamics in non-discrimination. The hope is to spur many new lines of investigation. The proposed model and analysis could serve as a framework to evaluate other non-discrimination constraints. More ambitiously, it could be used to motivate and derive new constraints with explicit awareness of their long term impact.

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