exploring Quarto and Latex

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4.1 ANTIDIFFERENTIATION AND INDEFINITE INTEGRALS

Integration by Substitution

4.1.2(Substitution Rule)

If $u = g(x)$ is a differentiable function whose range is an interval I and f is a continuous on I then,

$$\int f(g(x)) \cdot g'(x) dx = \int f(u) du$$

Example 4.1.12.

1.$\int (1-4x)^{\frac{1}{2}}\, dx$

If we let $u = 1 - 4x$, then $du = -4dx$. We multiply the integrand $\frac{-4}{-4}$. Thus,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \int (1-4x)^{\frac{1}{2}} \cdot \frac{-4}{-4} dx = \int u^{\frac{1}{2}} \left(-\frac{du}{4}\right) = -\frac{1}{4} \int u^{\frac{1}{2}} du = -\frac{1}{4} \cdot \frac{2u^{\frac{3}{2}}}{3} + C.$$

We put the final answer in terms of x by substituting $u = 1 - 4x$. Therefore,

$$\int (1-4x)^{\frac{1}{2}}\, dx = \frac{(1-4x)^{\frac{3}{2}}}{6} + C.$$

2.$\int x^2 \left(x^3 - 1\right)^{10} dx$

Let $u = x^3 - 1$. Then $du = 3x^2 dx$, or $\frac{du}{3} = x^2 dx$. By substitution,

$$\int x^2 \left(x^3 - 1\right)^{10} dx = \int u^{10} \cdot \frac{du}{3} = \frac{1}{3} \int u^{10} du = \frac{u^{11}}{33} + C = \frac{\left(x^3 - 1\right)^{11}}{33} + C.$$

3.$\int \frac{x}{\left(x^2+1\right)^3} dx$

Let $u = x^2 + 1$. Then $du = 2xdx$, or $\frac{du}{2} = xdx$. By substitution,

$$\int \frac{x}{\left(x^2+1\right)^3} dx = \frac{1}{2} \int u^{-3} du = \frac{1}{2} \cdot \frac{u^{-2}}{-2} + C = -\frac{1}{4\left(x^2+1\right)^2} + C.$$

4.$\int \cos^4 x \sin x dx$

Let $u = \cos x$. Then $du = -\sin x dx$, or $-du = \sin x dx$. By substitution,

$$\int \cos^4 x \sin x dx = -\int u^4 du = -\frac{u^5}{5} + C = -\frac{\cos^5 x}{5} + C.$$

5.$\int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx$

Let $u = \sec\left(x^2\right)$. Then $du = \sec\left(x^2\right) \tan\left(x^2\right) \cdot 2xdx$, or $\frac{du}{2} = \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx$. By substitution,

$$\begin{aligned} \int x \sec^3\left(x^2\right) \tan\left(x^2\right) dx &= \int \sec^2\left(x^2\right) \sec\left(x^2\right) \tan\left(x^2\right) \cdot xdx \\ &= \int u^2 du = \frac{1}{2} \cdot \frac{u^3}{3} + C \\ &= \frac{\sec^3\left(x^2\right)}{6} + C. \end{aligned}$$

6.$\int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2\cos\frac{1}{s}} ds$ Let $u = \frac{1}{s}$. Then $du = -\frac{1}{s^2} ds$ or $-du = \frac{ds}{s^2}$. By substitution,

$$\begin{aligned} \int \frac{\tan\frac{1}{s} + \tan\frac{1}{s}\sin\frac{1}{s}}{s^2\cos\frac{1}{s}} ds &= -\int \frac{\tan u + \tan u \sin u}{\cos u} du \\ &= -\int (\sec u \tan u + \tan^2 u) du \\ &= -\int (\sec u \tan u + \sec^2 u - 1) du \\ &= -(\sec u + \tan u - u) + C \\ &= -\sec\frac{1}{s} - \tan\frac{1}{s} + \frac{1}{s} + C. \end{aligned}$$

7.$\int t\sqrt{t-1} dt$

Let $u = t - 1$. Then $u = dt$. Also, $t = u + 1$. By substitution,

$$\int t\sqrt{t-1}dt = \int (u+1)\,u^{\frac{1}{2}}du = \int \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right)du = \frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3} + C$$
$$= \frac{2\,(t-1)^{\frac{5}{2}}}{5} + \frac{2\,(t-1)^{\frac{3}{2}}}{3} + C.$$

8.$\int \frac{t^3}{\sqrt{t^2+3}}dt$

Let $u = t^2 + 3$. Then $du = 2tdt$, or $\frac{du}{2} = tdt$. Also, $t^2 = u - 3$. By substitution,

$$\int \frac{t^3}{\sqrt{t^2+3}}dt = \int \frac{t^2 \cdot t}{\sqrt{t^2+3}}dt = \int u^{\frac{-1}{2}}(u-3)\frac{du}{2}$$
$$= \frac{1}{2}\int \left(u^{\frac{1}{2}} - 3u^{\frac{-1}{2}}\right)du = \frac{1}{2}\left(\frac{2u^{\frac{3}{2}}}{3} - 6u^{\frac{1}{2}}\right) + C$$
$$= \frac{\left(t^2+3\right)^{\frac{3}{2}}}{3} - 3\left(t^2+3\right)^{\frac{1}{2}} + C.$$

9.$\int \sqrt{4+\sqrt{x}}dx$

Let $u = 4 + \sqrt{x}$. Then $du = \frac{1}{2\sqrt{x}}dx$ or $2du = \frac{dx}{\sqrt{x}}$. By substitution,

$$\int \sqrt{4+\sqrt{x}}dx = \int \sqrt{4+\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}}dx$$
$$= \int \sqrt{4+\sqrt{x}} \cdot \sqrt{x} \cdot \frac{dx}{\sqrt{x}}(\sqrt{x} = u - 4)$$
$$= \int u^{\frac{1}{2}} \cdot (u-4) \cdot 2du$$
$$= \int (2u^{\frac{3}{2}} - 8u^{\frac{1}{2}})du$$
$$= \frac{2 \cdot 2u^{\frac{5}{2}}}{5} - \frac{2 \cdot 8u^{\frac{3}{2}}}{3} + C$$
$$= \frac{4\,(4+\sqrt{x})^{\frac{5}{2}}}{5} - \frac{16\,(4+\sqrt{x})^{\frac{3}{2}}}{3} + C.$$

Particular Antiderivatives

Now suppose that given a function $f(x)$, we wish to find a particular antiderivative $F(x)$ of $f(x)$ that satisfies a given condition. Such a condition is called an initial or boundary condition.

1. Given that $F'(x) = 2x$ and $F(2) = 6$, find $F(x)$.

   Solution.

   Since $F'(x) = 2x$, we have

   $$F(x) = \int 2x dx = x^2 + C.$$

   The initial condition $F(2) = 6$ implies that $F(2) = 2^2 + C = 6$. We get $C = 2$. Therefore, the particular antiderivative that we wish to find is

   $$F(x) = x^2 + 2.$$

2. The slope of the the tangent line at any point (x, y) on a curve is given by $3\sqrt{x}$. Find an equation of the curve if the point $(9, 4)$ is on the curve.

   Solution.

   Let $y = F(x)$ be an equation of the curve. The slope of the tangent line m_{TL} at a point (x, y) on the graph of the curve is given by $F'(x) = 3\sqrt{x}$. We have

   $$F(x) = \int 3x^{\frac{1}{2}} dx = 2x^{\frac{3}{2}} + C.$$

   The initial condition that $(9, 4)$ is on the curve implies that $F(9) = 2 \cdot 9^{\frac{3}{2}} = 4$. We obtain $C = -50$. Thus, an equation of the curve is

   $$y = 2x^{\frac{3}{2}} = -50.$$

Indeterminate Forms and L'Hospital's Rule

Indeterminate Forms of Type

$\frac{0}{0}$ and $\frac{\infty}{\infty}$

We began this course with the concept of the limit: the behavior of a function as the independent variable approaches a certain value, or as it increases or decreases without bound. We saw tangent lines, rates of change, and areas of plane regions, as limits of certain quantities. Indeed, the concept of the limit is the central idea about which the entire calculus revolves.

Now, we conclude this course by revising this fundamental idea. We shall see that, with the aid of derivatives, certain limits can be evaluated more conveniently. Before proceeding, we first recall some terminology defined in the early part of this course. We also recall here some techniques in evaluating limits we have previously encountered.

Definition

The $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is an indeterminate form of type 1. $\frac{0}{0}$ if $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$. 2. $\frac{\infty}{\infty}$ if $\lim\limits_{x\to a} f(x)$ and $\lim\limits_{x\to a} g(x)$ are both $+\infty$ or $-\infty$. Of course, "$x \to a$" may be replaced by "$x \to a^+$", "$x \to a^-$", "$x \to +\infty$" or "$x \to -\infty$".

Evaluate the following limits.

1. $\lim\limits_{x\to 0}\dfrac{x^2-3x}{2x^2+x}$ $\left(\frac{0}{0}\right)$ Solution.

$$\lim_{x\to 0}\frac{x^2-3x}{2x^2+x} = \lim_{x\to 0}\frac{x(x-3)}{x(2x+1)} = \lim_{x\to 0}\frac{x-3}{2x+1} = -3$$

2. $\lim\limits_{x\to 0}\dfrac{\sin 5x}{\sin 3x}$ $\left(\frac{0}{0}\right)$ Solution.

$$\lim_{x\to 0}\frac{\sin 5x}{\sin 3x} = \lim_{x\to 0}\left(\frac{\sin 5x}{5x}\right)\left(\frac{3x}{\sin 3x}\right)\left(\frac{5}{3}\right) = 1\cdot 1\cdot\frac{5}{3} = \frac{5}{3}$$

3. $\lim\limits_{x\to 2}\dfrac{x^2+3x-10}{x^2-4x+4}$ $\left(\frac{0}{0}\right)$ Solution.

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+3x-10}{x^2-4x+4} &= \lim_{x\to 2}\frac{(x+5)(x-2)}{(x-2)^2}\\ &= \lim_{x\to 2}\frac{x+5}{x-2}\\ &= -\infty\end{aligned}$$

4. $\lim\limits_{x\to +\infty}\dfrac{3x-1}{7-6x}$ $\left(\frac{+\infty}{-\infty}\right)$ Solution.

$$\lim_{x\to +\infty}\frac{3x-1}{7-6x} = \lim_{x\to +\infty}\frac{3x-1}{7-6x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to +\infty}\frac{3-\frac{1}{x}}{\frac{7}{x}-6} = -\frac{1}{2}$$

L'Hospital's Rule

The following theorem tells us how derivatives can be used to evaluate limits that are indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$. It usually referred to as L'Hospital's Rule, after the French mathematician Guillaume Francois Marquis de L'Hospital.

Let f and g be functions differentiable on an open interval I containing a except possibly at a and $g'(x) \neq 0$ for all $x \in I \quad a$. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or $\lim_{x\to a} \frac{f'(x)}{g'(x)} = \pm\infty$.

Remarks L'Hospital's Rule, with suitable modifications, is valid if "$x \to a$" is replaced by "$x \to a^+$", "$x \to a^-$", "$x \to +\infty$" or "$x \to -\infty$".

Example Evaluate the following limits. 1. $\lim_{x\to 0} \frac{x^2-3x}{2x^2+x}$ $\frac{0}{0}$ Solution.

$$\lim_{x\to 0} \frac{x^2-3x}{2x^2+x} = \lim_{x\to 0} \frac{D_x\left(x^2-3x\right)}{D_y\left(2x^2+x\right)} = \lim_{x\to 0} \frac{2x-3}{4x+1} = -3$$

2. $\lim_{x\to 0} \frac{\sin 5x}{\sin 3x}$ $\left(\frac{0}{0}\right)$ Solution.

$$\lim_{x\to 0} \frac{\sin 5x}{\sin 3x} = \lim_{x\to 0} \frac{5\cos 5x}{3\cos 3x} = \frac{5}{3}$$

3. $\lim_{x\to 2^-} \frac{x^2+3x-10}{x^2-4x+4}$ $\left(\frac{0}{0}\right)$ Solution.

$$\lim_{x\to 2^-} \frac{x^2+3x-10}{x^2-4x+4} = \lim_{x\to 2^-} \frac{2x+3}{2x-4}$$

$$= -\infty$$

4. $\lim_{x\to +\infty} \frac{3x-1}{7-6x}$ $\left(\frac{\pm\infty}{-\infty}\right)$ Solution.

$$\lim_{x\to +\infty} \frac{3x-1}{7-6x} = \lim_{x\to +\infty} \frac{3}{-6} = -\frac{1}{2}$$

5. $\lim_{x\to 1} \frac{x^3-3x+2}{1-x+\ln x}$ $\left(\frac{0}{0}\right)$ Solution.

$$\begin{aligned} \lim_{x\to 1} \frac{x^3-3x+2}{1-x+\ln x} &= \lim_{x\to 1} \frac{3x^2-3}{-1+\frac{1}{x}} \\ &= \lim_{x\to 1} \frac{6x}{-\frac{1}{x^2}} \\ &= -6 \end{aligned}$$