# KINEMATIC PROGRAMMING ALTERNATIVES FOR REDUNDANT MANIPULATORS*

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## Abstract

In the growing literature on redundant manipulator control, a number of techniques have been proposed for solving the inverse kinemetics problem. Some of these techniques are surveyed with a discussion of strengths and weaknesses of each. A new approach, called the extended Jacobian technique, is also presented. It is argued that because this technique may be expected to lift closed end effector paths to closed joint angle paths, it provides a promising approach for the control of kinematically redundant industrial manipulators. It is further shown that this technique may be implemented as a suitably parameterized generalized inverse method.

## 1. Introduction

New developments in kinematically redundant manipulator technology have been the focus of considerable interest and excitement in the engineering community during the last several years. Both academic and industrial researchers have become intrigued with the problems and benefits associated with the addition of extra degrees of freedom to robot manipulator designs. Indeed, robot manufacturers are now at the point of announcing the first generation of redundant manipulator products, while in the robotics engineering literature, one finds increasing numbers of reports on new techniques for control of such devices. (See e.g., Liegeois, 1977, Hanafusa et al., 1981, Konstantinov et al., 1982, Klein and Huang, 1983, Yoshikawa, 1984 a,b and Baillieul et al., 1984 a,b).

Together with special advantages come additional engineering problems. Any design of a redundant manipulator must take account of the extra weight of additional actuators and drive mechanisms. Moreover, the ever present problems of friction and backlash may be more severe when more degrees of freedom are added. As we shall see in the next section, the development of control software for redundant manipulators required the use of new techniques for solving the inverse kinematics problem.

The remainder of this paper is devoted to a discussion of these techniques. In Section 2, we describe the inverse kinematics problem for kinematically redundant mechanisms. Section 3 describes a number of approaches that have been proposed to solve this problem. Among the most widely discussed of these are the various generalized inverse and modified generalized inverse techniques. It is pointed out that without modification, the generalized inverse technique does <u>not</u> generate joint space trajectories which avoid kinematically singular configurations, and in general these techniques do not generate closed joint space trajectories corresponding to closed end effector trajectories. The practical consequences of these problems are briefly discussed. In Section 4, a new method called the <u>extended Jacobian technique</u> is announced, and it is indicated how this method may be expected to solve the various practical problems of redundant manipulator control described in Section 3. Section 5 provides a synthesis of the various approaches, and it is shown that the extended Jacobian technique may be implemented as a suitably parameterized generalized inverse method.

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## 2. The Inverse Kinematics Problem for Redundant Manipulators

For any robot, each possible configuration of the joints defines a unique position and orientation of the end effector. This is expressed mathematically by an equation of the form $f(\theta) = x$, where $x$ is a vector defining position and orientation in a suitable coordinate system, and $\theta$ is a vector defining the joint configuration. For kinematically redundant manipulators dim $\theta$<dim $x$, and because of this there are, for a given $x$, typically infinitely many points in the set $[\ \theta: f(\theta) = x]$, which is called the fiber over $x$. For any given $x$ there are a number of heuristic criteria which might be employed to select $\theta$ from this set (e.g. choose $\theta$ to make torque requirements for some future motion as small as possible). Most of the references we have cited on redundant manipulators deal with the more refined problem of choosing a trajectory $\theta(t)$ in joint space to generate a given trajectory $x(t)$ of the end effector.

The purpose of the present paper is to compare several approaces to the inverse kinematics problem, and to advocate one approach (which we call the extended Jacobian technique) that is felt to offer some significant advantages. The point of view we shall adopt in the remainder of this paper is that kinematically singular configurations ** are to be avoided, and thus we wish to define algorithms to steer the linkages under consideration so as to follow 'maximally nonsingular' trajectories. It should be emphasized, however, that in certain applications other objectives may be more appropriate.

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** Recall that kinematically singular configurations are points $\theta_s$ in joint space where the Jacobian $\frac{\partial f}{\partial \theta}$ is rank deficient. These are points at which the ordinary relation between end effector velocity and joint angle velocity breaks down in the sense that infinite joint velocities could be called for in order to maintain a given finite velocity of the end effector.

## 3. Functional Constraints and Generalized Inverse Solutions to the Inverse Kinematics Problem

By differentiating both sides of the equation $x(t) = f(\theta(t))$, we obtain the kinematic relation

$$\dot{x}(t) = \frac{\partial f}{\partial \theta}(\theta(t))\ \dot{\theta}(t) \qquad (3.1)$$

from which we can compute $\theta(.)$ in terms of the prescribed end effector trajectory $x(.)$. For redundant linkages $\frac{\partial f}{\partial \theta}$ is not a square matrix, and there are for each velocity vector $\dot{x}(t)$ an affine variety of joint velocities $\dot{\theta}(t)$ which satisfy (3.1). One way to uniquely specify a joint velocity vector for each $\dot{x}(t)$ is to use the Moore-Penrose inverse solution

$$\dot{\theta}_0(t) = \frac{\partial f}{\partial \theta}(\theta(t))^+\ \dot{x}(t), \qquad (3.2)$$

where for any mxn matrix A of rank m (m<n) the Moore-Penrose inverse is given by

$$A^+ = A^T\,(AA^T)^{-1}$$

Recall (see e.g., Noble and Daniel, 1977) that if $\dot{\theta}_0$ is the solution to (3.1) given by (3.2), then for any other solution $\dot{\theta}$ of (3.1), $\|\dot{\theta}_0\| < \|\dot{\theta}\|$.

As was pointed out in Baillieul et al (1984 b), this solution to the inverse kinematics problem does not generate joint angle trajectories which avoid singular configurations in any practical sense. This is because for almost any point $x_0$ in the workspace (in the measure theoretic sense) and any point $\theta^*$ in a neighborhood of a singular configuration, there is an initial configuration $\theta(0) = \theta_0$ and a workspace path $x(t)$, with $x(0) = x_0$, such that the trajectory $\theta(t)$ as defined by (3.2) passes through $\theta^*$.

This problem can be addressed by adding a 'null-space vector' to the right-hand side of (3.2). Indeed, if A is an mxn matrix (m<n) of rank m, and $A^+$ is the Moore-Penrose inverse, then it is easy to see that $I-A^+A$ is the orthogonal projection of $R^n$ onto the null space of A. By using this projection operator, it is possible to specify the following alternative to (3.2) for defining joint angle trajectories.

$$\dot{\theta} = \frac{\partial f}{\partial \theta}(\theta)^+\dot{x} + \left[I - \frac{\partial f}{\partial \theta}(\theta)^+\frac{\partial f}{\partial \theta}(\theta)\right] v \qquad (3.3)$$

Here v is a (time varying) vector of the same dimension as $\theta$ which remains to be specified. With this modification of the Moore-Penrose technique we can show that trajectories which avoid singular configurations may be generated by appropriate choice of $v(\cdot)$ in 3.3). This is made precise in the following theorem.

Theorem 3.1 (Baillieul et al., 1984b): Let $x(\cdot)$ be an arbitrary $C^1$ curve in the workspace. If $\theta(\cdot)$ is any $C^1$ joint angle trajectory which corresponds to $x(\cdot)$ (in the sense that $f(\theta(t)) = x(t)$) and which does not pass through a singular configuration, then there exists a continuous $v(\cdot)$ which generates $\theta(\cdot)$ via (3.3).

While these results show how any given joint angle trajectory avoiding singularities may be generated, the problem of algorithmically specifying trajectories in an *a priori* fashion is more delicate. Several choices for $v(\cdot)$ have been proposed. Liegeois (1977) has proposed we let $v(t) = \alpha\frac{\partial g}{\partial\theta}$ where $g(\theta)$ is any criterion or index which we wish to maximize (instantaneously at each point on the trajectory) and $\alpha$ is a scalar weighting function. Yoshikawa (1984 a,b) has pursued this approach wherein he has taken g to be a certain 'manipulability' index. In Section 5, we shall present an alternative way to define $v(\cdot)$ which not only addresses the problem of singularities but deals with the following somewhat subtle but practically important issue.

This further problem, as noted by Klein and Huang (1983), is that the relation (3.2) between end effector and joint angle derivatives is not integrable in the sense that a closed path in x-space will not generally yield a closed path in $\theta$-space. This is also a problem with trajectories generated via (3.3). The practical implication of this observation is that after a robot moves its end effector a number of cycles around a closed path (as it would in carrying out the repetitions of any industrial task), the state $\theta$ corresponding to a given x will assume a value which could not have been predicted in advance. This may be undesirable, and hence for practical manipulator control we cannot advocate these generalized inverse techniques without further refinement.

A conceptually simpler way to resolve the redundancy and address the problem of avoiding singularities is to impose a functional relation on the joint coordinates. Suppose there is a mapping

$\phi \longmapsto \theta(\phi)$ from $T^m = \{(\phi_1,\ldots,\phi_m) : -\pi < \phi_i \leq \pi\}$ to

$T^n = \{(\theta_1,\ldots,\theta_n): -\pi < \theta_i \leq \pi\}$ such that $\hat{f} = f\circ\theta$ maps $T^m$ onto a subset of $f(T^n)$ having nonempty interior. Suppose further that this mapping has been chosen such that the image $\theta(T^m)$ does not intersect the set of avoidable singular configurations

$$S = \{\theta_s \varepsilon T^n: \text{rank } \frac{\partial f}{\partial\theta}(\theta_s) < m,\ f(\theta_s) = x_s, \text{ and there exists } \theta_r \varepsilon T^n \text{ s.t. } f(\theta_r) = f(\theta_s) \text{ and rank } \frac{\partial f}{\partial\theta}(\theta_r) = m\}.$$

By means of this mapping, the redundancy is resolved, and no trajectory $\theta(\phi(t))$ corresponding to a curve $\phi(\cdot)$ in $T^m$ ever passes through a singularity of f.

Consider, for example, the problem of positioning the wrist point (=4th joint) in the redundant manipulator depicted in Figure 3.1.

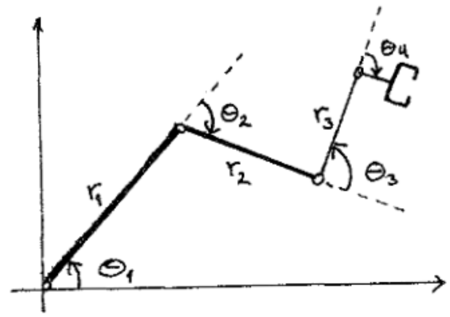


Figure 3.1. A redundant planar linkage.

The set of avoidable singular configurations is the one dimensional subset S =

$$\{\theta\varepsilon T^3: (\theta_2, \theta_3) = (0,\pi) \text{ or } (\theta_2,\theta_3) = (\pi,\pi)\}.$$

Suppose we constrain $\theta_2,\theta_3$ by $\theta_2 + \theta_3 = C$. Provided C is not an integral multiple of $\pi$, it is easy to see that for all $(\theta_1, \theta_2)\varepsilon T^2$, $(\theta_1, \theta_2, C-\theta_2)$ does not intersect the set S. The modified jointspace-to-workspace transformation we shall consider is given by

$$f(\theta_1\theta_2) = \begin{pmatrix} r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+C) \\ r_1\sin\theta_1 + r_2\sin(\theta_1+\theta_2) + r_3\sin(\theta_1+C) \end{pmatrix}$$

and from this formula it is easy to see that for a fixed vlaue of C the workspace is an annular region whose radial width is $2r_2$. Notice also that the Jacobain of the function f loses rank whenever $r_1\sin\theta_2 + r_3\sin(\theta_2 - C) = 0$. Hence, although the constrained mechanism never passes through the previously identified avoidalbe singularities, there are other singular configurations which are both intrinsic and unavoidalbe in the constrained mechanism.

This example is indicative of the type of problem one encounters using an imposed functional relationship on the joint variables to resolve redundancy and avoid singular configurations. Any such relationship should be chosen so that the topology of the domain of f is different from $T_m$, since whenever the domain of the jointspace-to-workspace transformation is an n-torus for some n, it can be shown there will be singularities. In the next two sections we shall show how a very special functional constraint may be defined which can be shown in some cases to introduce no extraneous singularities.

## 4. The Extended Jacobian Technique for Inverse Kinematics Problems in Redundant Manipulators

As has been mentioned in the preceding sections, there are (except for singular configurations) infinitely many joint configurations $\theta$ corresponding to a given end effector configuration via the basic equation $f(\theta)=x$. One way to impose practical limits on the number of choices of $\theta$-configurations for each x is to insist that $\theta$ optimize (at least locally) some objective function g. More precisely, suppose f maps the n-torus $T^n$ onto a nonempty subset of $R^m$ where $m=n-1$. (More genearl situations in which $m\leq n-1$ can be considered in a conceptually straightforward way, but for simplicity of the exposition we shall not treat situations here.) Let $g(\theta)$ be some criterion which we wish to optimize. g could be, for example, the average torque required from the joint actuators to produce a desired force/moment vector at the end effector position x, or g could be the manipulatability index discussed by Yoshikawa (1984 a, b).

Example 4.1: Consider the mechanism depicted in Figure 3.1. If we consider just the problem of positioning the wrist point, the joint space to end effector transformation is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3) \\ r_1\sin\theta_1 + r_2\sin(\theta_1+\theta_2) + r_3\sin(\theta_1+\theta_2+\theta_3) \end{pmatrix}$$

the Jacobian transformation $J = \frac{\partial f}{\partial \theta}$ is easily computed, and the manipulatability index is given by

$$w = \sqrt{\det J^T J} \quad = \quad \sqrt{g_1^2 + g_2^2 + g_3^2}$$

where $g_1 = r_2 r_3 \sin\theta_3$, $g_2 = r_1 r_3 \sin(\theta_2+\theta_3) + r_2 r_3 \sin\theta_3$, and $g_3 = r_1 r_2 \sin\theta_2 + r_1 r_3 \sin(\theta_2+\theta_3)$. One criterion by which to move the joint angles is to follow a trajectory $\theta(t)$ such that w is maximized subject to the constraint of following the desired end effector trajectory $x(t)$. Notice that $w=0$ if and only if $g(\theta_2,\theta_3) = \sin^2\theta_2+\sin^2\theta_3=0$. Hence, $g(\theta_2,\theta_3)$ is another (perhaps simpler) criterion that could be maximized.

Baillieul et al. (1984a) have recently studied a number of engineering issues related to a few specific redundant manipulator designs -- including several seven degree-of-freedom PUMA-like robots and a four degree-of-freedom spherical wrist which can avoid singular configurations. Yoshikawa (1984a) has computed the manipulatbility index for this device.

In general, once an objective function $g(\theta)$ has been selected, the extended Jacobian technique for solving the inverse kinematics problem may be described as follows. If J denotes the Jacobian matrix, as above, let the rows of J be $\partial f_i/\partial\theta$, $i=1,\ldots, m-n-1$. When J is of full rank (i.e. when the mechanism is not in a singular configuration), there is a one dimension null space. This is spanned by the vector

$$n_J(\theta) = \frac{\partial f_1}{\partial\theta} \wedge \cdots \wedge \frac{\partial f_m}{\partial\theta} = (J_1, J_2, \ldots, J_n)^T$$

where

$$J_i = (-1)^{i+1} \cdot \det \begin{bmatrix} \frac{\partial f_1}{\partial\theta_1} & \cdots & \widehat{\frac{\partial f_1}{\partial\theta_1}} & \cdots & \frac{\partial f_1}{\partial\theta_n} \\ \vdots & & & & \vdots \\ \frac{\partial f_m}{\partial\theta_1} & \cdots & \frac{\partial f_m}{\partial\theta_1} & \cdots & \frac{\partial f_m}{\partial\theta_n} \end{bmatrix}$$

with the circumflex denoting omission of the i-th column. When m=2, for example, $f_1/\partial\theta \wedge \partial f_2/\partial\theta$ reduces to the ordinary vector cross product in $R^3$. The following proposition provides useful information about the constrained extrema of the objective function.

Proposition 4.1: Let x be a given position/orientation of the end effector, and let $\theta=\theta_o$ be a joint angle setting at which $g(\theta)$ is extremized subject to the constraint $f(\theta)=x$. Then the inner product $\frac{\partial g}{\partial\theta}(\theta_o) \cdot n_J(\theta_o)$ vanishes.

Proof: Suppose $\theta=\theta_o$ is a constrained local extremum as in the hypothesis. Let $\theta(t)$ be a joint angle trajectory generating a 'self-motion' of the manipulator at x and satisfying $-\varepsilon<t<\varepsilon$, $(\varepsilon>0)$ and $\theta(0) = \theta_o$ This is expressed mathematically as

$$f(\theta(t)) \equiv x \qquad (4.1)$$

for $-\varepsilon<t<\varepsilon$.

Differentiating both sides of (4.1) at t=0 we obtain

$$\frac{\partial f}{\partial \theta}(\theta_o) \cdot (\dot{\theta}(0) = 0.$$

Hence $\dot{\theta}(0) = \lambda\, n_J(\theta_o)$ for some real number $\lambda$.

It follows from the hypothesis that the function $h(t) = g(\theta(t))$ has a local extremum at t=0. Hence

$$0 = h'(0) = \frac{\partial g}{\partial \theta}(\theta_o) \cdot \dot{\theta}(0). \qquad (4.2)$$

But if $\lambda \neq 0$, then this implies the desired conclusion $\partial g/\partial\theta\ (\theta_o) \cdot n_J(\theta_o) = 0$. If $\lambda = 0$, then the same type of argument may be applied using higher derivatives of $\theta(t)$ to arrive at the same conclusion.

Now define the function $G(\theta) = \partial g/\partial\theta \cdot n_J$. If a linkage is positioned with its end effector at x so that G is extremized, then the equation

$$\begin{pmatrix} f(\theta) \\ G(\theta) \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

is satisfied. If the end effector traces a trajectory x(t) along which the corresponding joint configuration $\theta(t)$ extremizes g(.) at each point, then we have

$$\begin{bmatrix} F(\theta(t)) \\ G(\theta(t)) \end{bmatrix} = \begin{bmatrix} x(t) \\ 0 \end{bmatrix}.$$

Differentiating both sides provides a (forward) kinematic relationship between the joint angle velocities and end effector velocity.

$$\begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \hline \frac{\partial G}{\partial \theta} \end{bmatrix} \cdot \dot{\theta}(t) = \begin{bmatrix} \dot{x}(t) \\ \\ 0 \end{bmatrix}$$

$\partial f/\partial\theta$ is the (mxn) Jacobian matrix of f and $\partial G/\partial\theta = (\partial G/\partial\theta_1, \ldots, \partial G/\partial\theta_n)$ is a row vector consisting of partial derivatives of G. Thus

$$\begin{bmatrix} \frac{\partial f}{\partial \theta} \\ \hline \frac{\partial G}{\partial \theta} \end{bmatrix}$$

is a square matrix, and we shall call this the <u>extended Jacobian</u>, $J_e$.

Provided that this extended Jacobian is nonsingular along a trajectory of interest, we may solve the inverse kinematics problem by writing

$$\dot{\theta}(t) = J_e^{-1}\begin{pmatrix} \dot{x}(t) \\ 0 \end{pmatrix} \qquad (4.3)$$

It is important to note here that this technique will propagate joint configurations that extremize $g(\theta)$ provided $J_e$ does not become singular, and <u>provided our initial configuration $\theta(0)$ has been chosen to extremize g</u>. We shall discuss the occurence of singularities below, but before doing this we shall apply the extended Jacobian technique to the linkage of Example 4.1.

<u>Example 4.3</u>: Consider the planar linkage of Example 4.1, and consider trajectories which maximize the objective function $g(\theta_2,\theta_3) = \sin^2\theta_2 + \sin^2\theta_3$. The extended Jacobian is then given by

$$J_e = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ 0 & G_1 & G_2 \end{pmatrix}$$

where the entries in this matrix are listed in Table 4.1. For this example we have chosen unit link lengths, $r_1=r_2=r_3=1$. One easily verifies that

$$n_J(\theta_1,\theta_2,\theta_3) = \begin{pmatrix} \sin\theta_3 \\ -\sin\theta_3 - \sin(\theta_2+\theta_3) \\ \sin\theta_2 + \sin(\theta_2+\theta_3) \end{pmatrix},$$

and the condition $\partial g/\partial\theta \cdot n_J = 0$ is satisfied whenever $\theta_2=\theta_3$. By a direct computation it can be verified that for any given position of the wrist-point of the manipulator, $g(\theta_2,\theta_3)$ is (locally) maximized by setting $\theta_2=\theta_3$. We have performed a number of simulations wherein the wrist point of this planar linkage traces a closed curve. In all cases, the joint variables returned to their starting configurations when the wrist point traversed the closed curve. This behavior stands in sharp contrast to the generalized inverse techniques described in Section 3.

Whether linkages in general will return to their starting configurations when the end effector traces a closed path is not known at present. It can be claimed in general, however, that by employing the extended Jacobian technique as we have outlined it, there are only a (small)

$$f_{11} = -r_1\sin\theta_1 - r_2\sin(\theta_1+\theta_2) - r_3\sin(\theta_1+\theta_2+\theta_3)$$

$$f_{12} = -r_2\sin(\theta_1+\theta_2) - r_3\sin(\theta_1+\theta_2+\theta_3)$$

$$f_{13} = -r_3\sin(\theta_1+\theta_2+\theta_3)$$

$$f_{21} = r_1\cos\theta_1 + r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3)$$

$$f_{22} = r_2\cos(\theta_1+\theta_2) + r_3\cos(\theta_1+\theta_2+\theta_3)$$

$$f_{23} = r_3\cos(\theta_1+\theta_2+\theta_3)$$

$$\begin{aligned} G_1 = {} & r_1r_3(\sin^2\theta_2-\cos^2\theta_2)\ \sin(\theta_2+\theta_3) \\ & -r_1r_3\sin\theta_2\ \cos\theta_2\ \cos(\theta_2+\theta_3) \\ & +r_2r_3(\sin^2\theta_2-\cos^2\theta_2)\sin\theta_3 \\ & +r_1r_2\cos\theta_2\ \sin\theta_3\ \cos\theta_3 \\ & +r_1r_2\ \sin\theta_3\ \cos\theta_3\ \cos(\theta_2+\theta_3) \end{aligned}$$

$$\begin{aligned} G_2 = {} & -r_1r_2\sin\theta_2\ \cos\theta_2\ \cos(\theta_2+\theta_3) \\ & -r_2r_3\ \sin\theta_2\ \cos\theta_2\ \cos\theta_3 \\ & -r_1r_2\sin\theta_2\ (\sin^2\theta_3-\cos^2\theta_3) \\ & +r_1r_2\ \sin\theta_3\ \cos\theta_3\ \cos(\theta_2+\theta_3) \\ & -r_1r_2\ (\sin^2\theta_3-\cos^2\theta_3)\ \sin(\theta_2+\theta_3) \end{aligned}$$

Table 4.1. Entries in the extended Jacobian matrix of Example 4.3.

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finite number of possible configurations corresponding to each position/orientation of the end effector. Work now in progress is aimed at better understanding this feature of the extended Jacobian technique.

The question of when the extended Jacobian becomes singular may be addressed as follows. It is obviously singular at any joint angle setting at which $\partial f/\partial\theta$ is singular. Avoidable singularities of $\partial f/\partial\theta$ probably do not present any problems, however, since solving the inverse kinematics problem using the extended Jacobian technique automatically avoids them. (See Baillieul et al. (1984 b) for a discussion of avoidable and unavoidable singularities.) Singularities of $J_e$ which are not also singularities of $\partial f/\partial\theta$ are characterized by the following.

Proposition 4.2: Suppose $\theta_s$ is a joint angle configuration at which $J_e$ is singular but where $\partial f/\partial\theta\,(\theta_s)$ has full rank. Then (and only then) $n_J(\theta_s) \neq 0$ and $\partial G/\partial\theta(\theta_s)\cdot n_J(\theta_s) = 0$.

Proof: Suppose $n_J(\theta_s) \neq 0$ and $\frac{\partial G}{\partial\theta}(\theta_s)\cdot n_J(\theta_s) = 0$. The first of these assumptions implies $\partial f/\partial\theta(\theta_s)$ has full rank $n-1$, and the second implies $J_e$ is singular.

Conversely, if $J_e$ is singular then there exists a nonzero vector v such that $J_e\cdot v=0$. Since $\partial f/\partial\theta(\theta_s)$ is assumed to be of full rank, v must be a scalar multiple of $n_J(\theta_s)$. Hence, $n_J(\theta_s) \neq 0$ and $\frac{\partial G}{\partial\theta}(\theta_s)\ \cdot\ n_J(\theta_s) = 0$, proving the proposition.

## 5. The Extended Jacobian Technique Viewed as a Generalized Inverse Technique

In this Section we show how to realize the extended Jacobian technique as a generalized inverse technique.

This represents a synthesis of all the methods we have discussed, and includes the method of functional constraint where the given constraint is $G(\theta) = 0$.

It is clear that an equation of the following form is valid.

$$\begin{bmatrix} \frac{\partial f}{\partial\theta} \\ \frac{\partial G}{\partial\theta} \end{bmatrix} \begin{bmatrix} \frac{\partial f}{\partial\theta}^{+} & \vdots & n_J \end{bmatrix} = \begin{bmatrix} I & \vdots & 0 \\ \cdots & \cdots & \cdots \\ \lambda_1\ldots\lambda_{n-1} & \cdot & \lambda_n \end{bmatrix} \tag{5.1}$$

where

$$\lambda_n = \frac{\partial G}{\partial\theta}\ \cdot\ n_J$$

and

$$\lambda_j = \frac{\partial G}{\partial\theta}\ \cdot\ F_j \qquad (j=1,\ldots,n-1)$$

where $F_j$ is the j-th column of $(\frac{\partial f}{\partial\theta})^+$.

From this equation we obtain a simple expression for the inverse of $J_e$

$$\begin{bmatrix} \frac{\partial f}{\partial\theta} \\ \frac{\partial G}{\partial\theta} \end{bmatrix}^{-1} = \begin{bmatrix} \frac{\partial f}{\partial\theta}^{+} & \vdots & n_J \end{bmatrix} \begin{bmatrix} I & \vdots & 0 \\ \cdots & \cdots & \cdots \\ -\frac{\lambda_1}{\lambda_n} \ldots -\frac{\lambda_{n-1}}{\lambda_n} & \vdots & \frac{1}{\lambda_n} \end{bmatrix},$$

and it follows that equation (4.3) may be rewritten as

$$\dot\theta(t) = \left(\frac{\partial f}{\partial\theta}\right)^{+}\dot x(t) - v(t)\ n_J(\theta) \tag{5.2}$$

where the scalar v(t) is given by

$$v(t) = \frac{\frac{\partial G}{\partial \theta} \cdot \left(\frac{\partial f}{\partial \theta}\right)^{+} \cdot \dot{x}(t)}{\frac{\partial G}{\partial \theta} \cdot n_J} \tag{5.3}$$

Equation (5.3) makes it clear that the simgularities of concern in applying the extended Jacobian technique occur where the entries of the row vector

$$\frac{\frac{\partial G}{\partial \theta} \cdot \left(\frac{\partial f}{\partial \theta}\right)^{+}}{\frac{\partial G}{\partial \theta} \cdot n_J} \tag{5.4}$$

vanish. It will be seen in the next example that zeros may occaisionally be cancelled between numerator and denominator in this expression, and hence the singularities or poles of v(t) will in some applications be a proper subset of the points where $J_e$ actually becomes singular. In practice, there will be cases where we can use (5.2) to generate trajectories even though these trajectories may pass through points where $J_e$ becomes singular.

Example 5.1: Consider the redundant planar mechanism of example 4.3. A straightforward calculation shows that if the entries of (5.4) are expressed as the ratios of analytic functions in lowest terms and if we assume in accordance with Example 4.3 that $\theta_3 = \theta_2$, the (common) denominator is $(\sin \theta_2 + \sin 2\theta_2) \cdot |n_J|^2$. This vanishes at the singularities of $\partial f/\partial \theta$ and also when $\theta_2 = \pm 2\pi/3$. If $\theta_2 = \theta_3 = \pm 2\pi/3$, the wrist point of the manipulator is at the origin of the workspace coordinate system, producing in effect a closed kinematic chain. There are, in this configuration, no self motions involving $\theta_2$, $\theta_3$, and hence the constrained optimization of $g(\theta_2,\theta_3)$ which at the basis of our method is formally degenerate. Thus, if one wishes to apply the extended Jacobian technique to solve the inverse kinematics problem in this example, trajectories bringing the wrist point into coincidence with the origin of the workspace coordinate system should be avoided. Other trajectories which evolve in the interior of the workspace, however, do not produce kinematically singular configurations. Indeed, if one follows trajectories of (5.2) from initial conditions $-2\pi/3 < \theta_2 = \theta_3 < 2\pi/3$, then provided that the wrist point is never commanded to reach the workspace boundary or to pass through the origin of the workspace coordinate system it can be shown that each wrist point position $(x,y)^T$ is associated to a unique joint configuration $\theta_1$, $\theta_2 = \theta_3$.

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