A Definition of the Extended Jacobian Inverse Kinematics Algorithm for Mobile Robots

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Abstract— **Using control theoretic concepts we present a definitional procedure of extended Jacobian inverse kinematics algorithms for mobile robots. As a point of departure we assume a representation of the mobile robot kinematics as the end point map of a driftless control system with outputs. The space of admissible control functions of this system plays the role of the configuration space of the mobile robot, the system output space corresponds to the taskspace, so that the mobile robot kinematics transform the configuration space into the taskspace. The extended Jacobian inverse kinematics algorithm is obtained by means of the continuation method, and is based on the extended Jacobian inverse. The main step of its derivation consists in extending the original mobile robot kinematics to a map of the configuration space into itself. To this aim the configuration space is decomposed into a finite dimensional subspace, isomorphic to the taskspace, and the remaining quotient subspace. In compliance with this decomposition an augmenting kinematics map is introduced. The original kinematics and the augmenting kinematics constitute the extended kinematics. In a region free from singularities the inverse of the derivative of the extended kinematics defines the extended Jacobian inverse. By design, the extended Jacobian inverse kinematics algorithm has the property of repeatability. In the paper, the general procedure is exemplified by a derivation of the extended Jacobian inverse for a chained form system that is feedback equivalent to the kinematics of the kinematic car type mobile robot. An examination of algorithmic singularities of this algorithm is carried out. Computer simulations illustrate the performance of the algorithm.**

I. INTRODUCTION

In this paper a mobile robot will be referred to as a mobile platform whose motion obeys nonholonomic velocity constraints. In the case of a wheeled mobile platform such constraints come from an assumption that the wheels are not permitted to slip laterally or longitudinally. Making use of the analytic form of the constraints, the kinematics of the mobile robot can be represented by a driftless control system. The taskspace coordinates of the mobile robot are determined by an output function.

Relying on the control system formulation, we shall address the inverse kinematics problem for the mobile robot, that consists in determining a control such that the system output reaches a desirable point of the taskspace at a prescribed time instant. Various control theoretic approaches to this problem have been reviewed in [1]. Here we shall present a robotics-oriented approach that will be based on the method of the endogenous configuration space [2], [3]. The essence of our approach lies in defining the mobile robot kinematics as the end point map of the control system representation, and then proceeding with this map in exactly the same way as with the kinematics of a stationary manipulator. This results in the derivation of Jacobian inverse kinematics algorithms for mobile robots by means of the continuation method, and in the association of an inverse kinematics algorithm with every right inverse of the Jacobian. The inverse widely used in mobile robotics is the Jacobian pseudoinverse [3]–[6]. An alternative, advocated in this paper, is the extended Jacobian inverse. For stationary manipulators its construction is well known [7]–[9]. Recently, the extended Jacobian inverse has been introduced for mobile manipulators [10], [11] as well as for mobile robots [1]. A general definitional procedure of the extended Jacobian inverse kinematics algorithm will be described following the last reference. The procedure entails the following steps. From the control system representation we obtain the mobile robot kinematics as a transformation of a configuration space into a taskspace. The configuration space is decomposed into a finite dimensional subspace, isomorphic to the taskspace, and the remaining, infinite dimensional, quotient space. Subordinated to this decomposition is an augmenting kinematics map that along with the original kinematics constitutes the extended kinematics. In a region free from singularities the inverse of the derivative of the extended kinematics defines the extended Jacobian inverse. Plugged into an underlying dynamic system this inverse produces the extended Jacobian inverse kinematics algorithm. By design, the extended Jacobian inverse kinematics algorithm has the property of repeatability [12]. General developments will be exemplified with the derivation of an extended Jacobian algorithm for a chained form system feedback equivalent to the kinematics of the kinematic car type mobile robot [13]. The extended Jacobian algorithm applied to the chained form system will further be subject to a thorough examination with respect to the algorithmic singularities. The performance of the algorithm will be illustrated with computer simulations.

This paper is composed in the following way. Section 2 introduces basic concepts. In section 3 we present the definitional procedure of the extended Jacobian inverse kinematics algorithm. Section 4 includes a derivation of the extended Jacobian inverse for a chained form system. Section 5 presents results of computer simulations. Algorithmic singularities of the algorithm are examined in section 6. The paper is concluded with Section 7.

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II. BASIC CONCEPTS

In the consideration of kinematic constraints imposed on the motion of a mobile robot we shall restrict to the constraints linear in velocity, that can be expressed in the Pfaffian form $A(q)\dot{q} = 0$. This form of the constraints will allow us to represent the mobile robot kinematics as a driftless control system equipped with an output function

$$\dot{q} = G(q)u = \textstyle\sum_{i=1}^{m} g_i(q)u_i, \quad y = k(q). \tag{1}$$

In the system (1) $q \in R^n$ denotes the vector of generalized coordinates of the platform, and $y \in R^r$ is a vector of taskspace coordinates usually including the platform position and orientation. Admissible control functions $u(\cdot)$ acting on the system (1) will be chosen Lebesgue square integrable on a time interval $[0, T]$, for a certain $T > 0$, with values in R^m. The Hilbert space $\mathcal{X} = L^2_m[0, T]$ of admissible control functions with inner product

$$< u(\cdot), v(\cdot) >= \int_0^T u^T(t)v(t)dt, \tag{2}$$

will be referred to as the endogenous configuration space of the mobile robot.

Suppose that for a fixed initial state q_0 of the mobile robot and every endogenous configuration $u(\cdot) \in \mathcal{X}$, the trajectory $q(t) = \varphi_{q_0,t}(u(\cdot))$ exists for every $t \in [0, T]$. Then, the end point map

$$K_{q_0,T}(u(\cdot)) = k(\varphi_{q_0,T}(u(\cdot))) \tag{3}$$

of the control system (1) will be identified with the kinematics of the mobile robot. Consequently, we shall regard the derivative

$$D\, K_{q_0,T}(u(\cdot))v(\cdot) = J_{q_0,T}(u(\cdot))v(\cdot)$$

of the kinematics as the mobile robot Jacobian. The Jacobian, that can be expressed as

$$J_{q_0,T}(u(\cdot))v(\cdot) = C(T)\int_0^T \Phi(T, s)B(s)v(s)ds, \tag{4}$$

represents the input-output map of the linear approximation

$$\dot{\xi} = A(t)\xi + B(t)v, \quad \eta = C(t)\xi, \tag{5}$$

to the system (1) along a control-trajectory pair $(u(t), q(t))$, initialized at $\xi_0 = 0$. The matrices appearing in (4) and (5) are defined in the standard way

$$A(t) = \frac{\partial(G(q(t))u(t))}{\partial q}, B(t) = G(q(t)),$$
$$C(t) = \frac{\partial k(q(t))}{\partial q}, \tag{6}$$

and the transition matrix $\Phi(t, s)$ satisfies the evolution equation

$$\frac{\partial}{\partial t}\Phi(t, s) = A(t)\Phi(t, s), \tag{7}$$

with initial condition $\Phi(s, s) = I_n$.

It is easily observed that at a fixed configuration $u(\cdot) \in \mathcal{X}$ the Jacobian is a map

$$J_{q_0,T}(u(\cdot)) : \mathcal{X} \longrightarrow R^r$$

that transforms velocities of motion from the endogenous configuration space into the taskspace. This being so, an endogenous configuration $u(\cdot)$ is called regular, if the Jacobian map $J_{q_0,T}(u(\cdot))$ is surjective, otherwise the configuration is singular. The singularity of a configuration means that the control underlying this configuration is a singular control for the system (1) [4], [14]. At regular configurations the kinematic map (3) is an open map, that means that, if $u(\cdot)$ produces at T a task vector $y = K_{q_0,T}(u(\cdot))$ then, by choosing appropriate configurations from an open neighbourhood U of $u(\cdot)$, we can reach at T all platform positions and orientations belonging to an open neighbourhood V of y. In control theoretic terminology the system (1) is referred to as locally output controllable. A necessary and sufficient condition for regularity of a configuration asserts that the rank of the mobility matrix

$$\mathcal{M}_{q_0,T}(u(\cdot)) = C(T)\int_0^T \Phi(T, s)B(s)B^T(s)\Phi^T(T, s)ds\, C^T(T)$$

of the mobile robot should be equal to the taskspace dimension r.

III. EXTENDED JACOBIAN ALGORITHM

Given the kinematics (3) of a mobile robot and a desirable taskspace point $y_d \in R^r$, the inverse kinematic problem amounts to computing a configuration $u_d(\cdot)$ such that $y_d = K_{q_0,T}(u_d(\cdot))$. Suppose that a solution $u_d(\cdot)$ of the problem exists. Then, the solution can be found by means of the following reasoning originating from the continuation method [4], [6], [15]. We begin with any initial configuration $u_0(\cdot) \in \mathcal{X}$. If $K_{q_0,T}(u_0(\cdot)) = y_d$, the problem is solved, otherwise we choose in the configuration space a differentiable curve $u_\theta(\cdot)$, passing through $u_0(\cdot)$, and compute the taskspace error

$$e(\theta) = K_{q_0,T}(u_\theta(\cdot)) - y_d \tag{8}$$

along this curve. Next, we require that the error decreases exponentially along with θ, with a certain decay rate $\gamma > 0$, i.e.

$$\frac{d\, e(\theta)}{d\theta} = -\gamma\, e(\theta). \tag{9}$$

A differentiation of the error formula (8) results in the following equation

$$J_{q_0,T}(u_\theta(\cdot))\frac{d\, u_\theta(\cdot)}{d\theta} = -\gamma\, e(\theta),$$

involving the Jacobian (4). Finally, after applying a right inverse $J^{\#}_{q_0,T}(u(\cdot))$ of the Jacobian, we obtain a Ważewski equation [6], [16] that yields the dynamic system

$$\frac{d\, u_\theta(\cdot)}{d\theta} = -\gamma\, J^{\#}_{q_0,T}(u_\theta(\cdot))(K_{q_0,T}(u_\theta(\cdot)) - y_d). \tag{10}$$

The equilibrium points of (10) satisfy the equation $K_{q_0,T}(u(\cdot)) - y_d = 0$. A solution to the inverse kinematics problem provided by the Jacobian inverse kinematics algorithm determined by the system (10) can be found as the limit

$$\lim_{\theta\to+\infty} u_\theta(\cdot) = u_d(\cdot).$$

Different right inverses of the Jacobian employed in (10) will produce different inverse kinematics algorithms. In this paper we shall focus on the development of the extended Jacobian inverse, following [1]. The first step in the definitional procedure of the extended Jacobian inverse kinematics algorithm consists in extending the original kinematics (3) to a map of the endogenous configuration space into itself. To this aim we need to represent the endogenous configuration space as a direct sum of two components

$$\mathcal{X} \cong R^r \oplus \mathcal{X}/R^r, \tag{11}$$

of whose the quotient space is infinite dimensional. Next, we introduce an augmenting kinematics map

$$H_{q_0,T} : \mathcal{X} \longrightarrow \mathcal{X}/R^r. \tag{12}$$

The original kinematics together with the map (12)

$$L_{q_0,T} = (K_{q_0,T}, H_{q_0,T}) : \mathcal{X} \to \mathcal{X}$$

form the extended kinematics of the mobile robot, with derivative

$$D\,L_{q_0,T}(u(\cdot)) = (J_{q_0,T}(u(\cdot)), D\,H_{q_0,T}(u(\cdot))) = \bar{J}_{q_0,T}(u(\cdot))$$

called the extended Jacobian of the mobile robot. It is desirable that the extended kinematics be a local diffeomorphism of the endogenous configuration space, however, the extension procedure usually adds to singularities of the original kinematics certain extra singular configurations called algorithmic singularities [7].

In a singularity free region of the endogenous configuration space the extended Jacobian inverse can be defined in the following way

$$J^{E\,\#}_{q_0,T}(u(\cdot))\eta = \bar{J}^{-1}_{q_0,T}(u(\cdot))(\eta, 0(\cdot)), \tag{13}$$

where $\eta \in R^r$, and $0(\cdot) \in \mathcal{X}/R^r$ denotes the zero element of the quotient space. Two basic properties of the inverse (13) will be used in the sequel [1], namely the identity property

$$J_{q_0,T}(u(\cdot))J^{E\,\#}_{q_0,T}(u(\cdot)) = I_r, \tag{14}$$

and the annihilation property

$$D\,H_{q_0,T}(u(\cdot))J^{E\,\#}_{q_0,T}(u(\cdot)) = 0(\cdot). \tag{15}$$

The first property means that the extended Jacobian inverse is a right inverse of the Jacobian, whereas the second one implies that the augmenting map is the first integral of the dynamic system underlying the extended Jacobian inverse kinematics algorithm, and results in repeatability of this algorithm.

IV. CHAINED FORM SYSTEM

In this section the general procedure described in section 3 will be specified to the chained form system that is feedback equivalent to the kinematics of the kinematic car type mobile platform moving without the lateral slip of its wheels. The equations of the chained form system assume the following form [13]

$$\begin{aligned} &\dot{q}_1 = u_1, \quad \dot{q}_2 = u_2, \quad \dot{q}_3 = u_1 q_2, \quad \dot{q}_4 = u_1 q_3, \\ &y = k(q) = (q_1, q_2, q_3, q_4). \end{aligned} \tag{16}$$

The endogenous configuration space of the chained form system $\mathcal{X} = L^2_2[0,T]$, its taskspace is equal to R^4. Let $J_{q_0,T}(u(\cdot))$ denote the Jacobian (4) of the chained form system. The control functions will be expanded into the orthogonal series

$$u_i(t) = \sum_{k=0}^{\infty} \lambda_{i\,k}\phi_k(t), \tag{17}$$

where $i = 1, 2$. The functions $\phi_k(t), k \geq 0$ form an orthogonal basis of the endogenous configuration space $\mathcal{X}$. This means that $\int_0^T \phi_0(t)dt = 1$, $\int_0^T \phi_i(t)dt = 0$ for $i \geq 1$, and for $i \neq j$ the inner product $< \phi_i, \phi_j >= \int_0^T \phi_i(t)\phi_j(t)dt = 0$. For every pair $(u_1(t), u_2(t))$ of bounded controls there exists a trajectory $q(t) = \varphi_{q_0,t}(u(\cdot))$ of the system (16) that defines the kinematics

$$K_{q_0,T}(u(\cdot)) = q(T)$$

of the chained form system. Singular endogenous configurations of the chained form system (16) can be derived from those of the kinematic car kinematics, using the fact that the feedback equivalence in the form established in [13] preserves singular controls. This means that the singular configuration of the chained form system consist of the pairs $u_1(t), u_2(t)$, with $u_1(t)$ vanishing almost everywhere in $[0, T]$, [4].

In order to accomplish the decomposition (11), we select coordinating coefficients $(\lambda_{10}, \lambda_{20}, \lambda_{21}, \lambda_{22}) \in R^4$, and include in the quotient space $\mathcal{X}/R^4$ pairs of functions $(\tilde{u}_1(\cdot), \tilde{u}_2(\cdot))$ such that

$$\tilde{u}_1(t) = \sum_{k=1}^{\infty} \lambda_{1\,k}\phi_k(t), \quad \tilde{u}_2(t) = \sum_{k=3}^{\infty} \lambda_{2\,k}\phi_k(t). \tag{18}$$

The augmenting kinematics map (12) we shall choose in the following way

$$H_{q_0,T}(u(\cdot))(t) = \left(\frac{\tilde{u}_1(t)}{\sqrt{\lambda_{10}^2 + \epsilon}}, \frac{\tilde{u}_2(t)}{\sqrt{\lambda_{20}^2 + \epsilon}}\right), \tag{19}$$

where the regularization coefficient $\epsilon > 0$ makes this map well defined for all $\lambda_{10}, \lambda_{20}$. The derivative of (19) is equal to

$$(D\,H_{q_0,T}(u(\cdot))v(\cdot))(t) = \begin{pmatrix} \frac{(\lambda_{10}^2+\epsilon)\tilde{v}_1(t) - \lambda_{10}\mu_{10}\tilde{u}_1(t)}{(\lambda_{10}^2+\epsilon)^{3/2}} \\ \frac{(\lambda_{20}^2+\epsilon)\tilde{v}_2(t) - \lambda_{20}\mu_{20}\tilde{u}_2(t)}{(\lambda_{20}^2+\epsilon)^{3/2}} \end{pmatrix}, \tag{20}$$

where

$$v(t) = \begin{bmatrix} \phi_0(t) & 0 & 0 & 0 \\ 0 & \phi_0(t) & \phi_1(t) & \phi_2(t) \end{bmatrix} \mu + \tilde{v}(t), \tag{21}$$

moreover, the function $\tilde{v}(\cdot) \in \mathcal{X}/R^4$ and $\mu = (\mu_{10}, \mu_{20}, \mu_{21}, \mu_{22})$. It is easily seen from (20) that the augmenting kinematics map (19) is non-singular.

In order to compute the extended Jacobian inverse (13), we shall invoke the identity and the annihilation properties.

To begin with, imitating the decomposition (21), let us write the extended Jacobian inverse as the sum

$$(J^{\#\,E}_{q_0,T}(u(\cdot))\eta)(t) = \begin{bmatrix} \phi_0(t) & 0 & & 0 \\ 0 & \phi_0(t) & \phi_1(t) & \phi_2(t) \end{bmatrix} \mu(\eta) + \left(\tilde{J}^{\#\,E}_{q_0,T}(u(\cdot))\eta\right)(t) \quad (22)$$

A substitution of (20) and (22) into the annihilation property yields first

$$\left(\tilde{J}^{\#\,E}_{q_0,T}(u(\cdot))\eta\right)(t) = \begin{pmatrix} \frac{\lambda_{10}\tilde{u}_1(t)}{\lambda_{10}^2+\epsilon}\mu_{10}(\eta) \\ \frac{\lambda_{20}\tilde{u}_2(t)}{\lambda_{20}^2+\epsilon}\mu_{20}(\eta) \end{pmatrix}$$

and then results in

$$(J^{\#\,E}_{q_0,T}(u(\cdot))\eta)(t) = F_{q_0,T}(u(\cdot))(t)\mu(\eta), \quad (23)$$

where

$$F_{q_0,T}(u(\cdot))(t) = \begin{bmatrix} \phi_0(t) + \frac{\lambda_{10}\tilde{u}_1(t)}{\lambda_{10}^2+\epsilon} & 0 & 0 & 0 \\ 0 & \phi_0(t) + \frac{\lambda_{20}\tilde{u}_2(t)}{\lambda_{20}^2+\epsilon} & \phi_1(t) & \phi_2(t) \end{bmatrix}. \quad (24)$$

Eventually, using (23) and the definition of the Jacobian (we have $C(T) = I_4$), we deduce from the identity property

$$J_{q_0,T}(u(\cdot))J^{\#\,E}_{q_0,T}(u(\cdot))\eta = \int_0^T \Phi(T,t)B(t)F_{q_0,T}(u(\cdot))(t)dt\mu(\eta) = E_{q_0,T}(u(\cdot))\mu(\eta) = \eta.$$

Wherever the matrix

$$E_{q_0,T}(u(\cdot)) = \int_0^T \Phi(T,t)B(t)F_{q_0,T}(u(\cdot))(t)dt \quad (25)$$

is invertible, we can compute $\mu(\eta)$, substitute the result into (23), and obtain the extended Jacobian inverse

$$\left(J^{\#\,E}_{q_0,T}(u(\cdot))\eta\right)(t) = F_{q_0,T}(u(\cdot))(t)E^{-1}_{q_0,T}(u(\cdot))\eta. \quad (26)$$

The extended Jacobian inverse kinematics algorithm associated with the map (19) is obtained by inserting (26) into the right hand side of (10). By design, the algorithm is well defined on condition that $\det E_{q_0,T}(u(\cdot)) \neq 0$.

V. COMPUTER SIMULATIONS

The extended Jacobian inverse kinematics algorithm derived in the previous section was tested using 8 initial values $q_0 = (q_{0\,1}, q_{0\,2}, q_{0\,3}, q_{0\,4})$ of system (16) coordinates, with pairs $q_{0\,1}, q_{0\,4}$ lying on a circle of radius $r = 10$, shown in Figure 1. This choice is a reminiscent of the fact that in the feedback transformation of the kinematic car kinematics to the chained form the coordinates q_1 and q_4 correspond to the positions x, y of the car in the plane [13]. The initial values of remaining coordinates $q_{0\,2}, q_{0\,3} = 0$. The desirable taskspace point $y_d = (0, 0, 0, 0)$. Band-limited control functions

$$u_1(t) = \lambda_{10} + \lambda_{11}\sin\omega t + \lambda_{12}\cos\omega t = \lambda_{10} + \tilde{u}_1(t),$$
$$u_2(t) = \lambda_{20} + \lambda_{21}\sin\omega t + \lambda_{22}\cos\omega t + \lambda_{23}\sin 2\omega t +$$
$$\lambda_{24}\cos 2\omega t = \lambda_{20} + \lambda_{21}\sin\omega t + \lambda_{22}\cos\omega t + \tilde{u}_2(t) \quad (27)$$

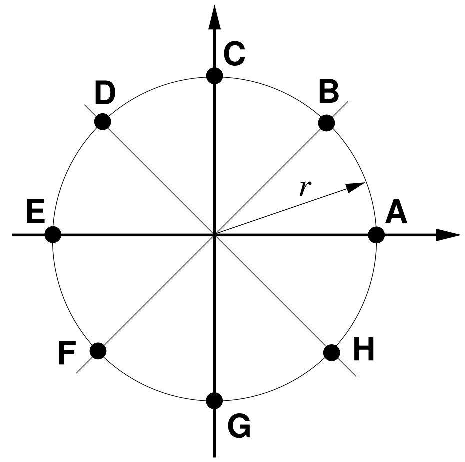

Fig. 1. Initial values of coordinates q_{01}, q_{04}

have been adopted, where $\omega = 2\pi/T$ and $T = 1$. Initial values of λ are the following $\lambda_0 = (\lambda_{0\,1}, \lambda_{0\,2}) = (-5.2, -0.08, -0.01, 0.0024, 0.005, -0.007, -0.005, 0.0004)$. The computations were stopped when the taskspace error decreased below 10^{-6}. The regularization parameter was set to $\epsilon = 10^{-6}$. Convergence ratio $\gamma = 0.2$. Selected results of computer simulations are presented in Figures 2-7. We have observed that the algorithm lacks convergence from points C and G. Additional tests accomplished from initial points close to C and G, having the coordinate $q_{0\,4} = \pm 10$, confirmed this observation. Taking into account the chained system equations (16) and the form of the control functions (27) one can conclude that in these cases the solution of the inverse kinematic problem requires setting $\lambda_{10} = 0$.

VI. ALGORITHMIC SINGULARITIES

As we have already mentioned, the inverse kinematics algorithm based on the extended Jacobian inverse (26) is well defined provided that $\det E_{q_0,T}(u(\cdot)) \neq 0$. Obviously, this determinant vanishes at singularities of the original kinematics. Other endogenous configurations zeroing $\det E_{q_0,T}(u(\cdot))$ are called algorithmic singularities of the algorithm. In this section we shall examine the algorithmic singularities. To this aim, let us rewrite the formula (25)

$$E_{q_0,T}(u(\cdot)) = \int_0^T \Phi(T,t)B(t)F_{q_0,T}(u(\cdot))(t)dt. \quad (28)$$

The matrix $F_{q_0,T}(u(\cdot))(t)$ appearing under the above integral has been defined by (24). The other matrices can be computed in accordance with (6) and (7) applied to the chained form system equations (16). Specifically, for a control-trajectory pair $(u(t), q(t))$ we obtain

$$B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ q_2(t) & 0 \\ q_3(t) & 0 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & u_1(t) & 0 & 0 \\ 0 & 0 & u_1(t) & 0 \end{bmatrix}$$

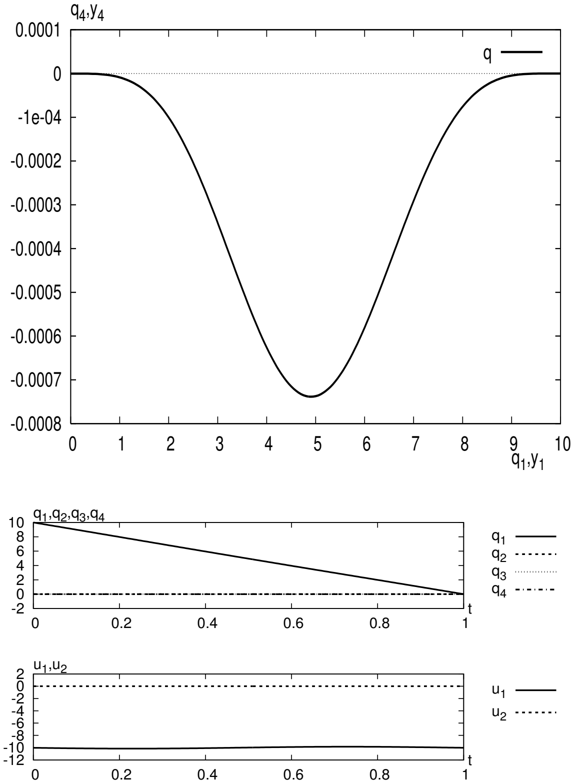

Fig. 2. Solution to the inverse kinematic problem: point A, number of iterations 71, final taskspace error $8.35041e-07$

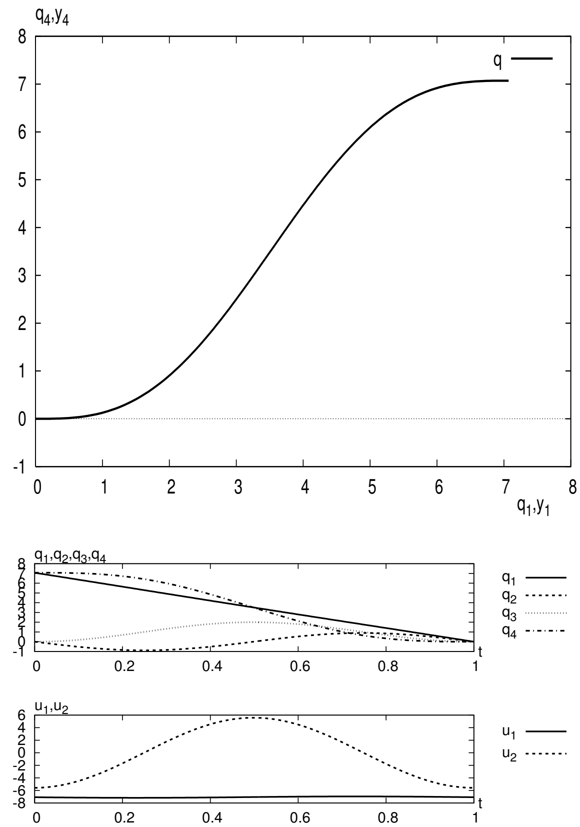

Fig. 3. Solution to the inverse kinematics problem: point B, number of iterations 95, final taskspace error $9.40894e-07$

and

$$\Phi(T,t)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & \Phi_{32}(T,t) & 1 & 0\\ 0 & \Phi_{42}(T,t) & \Phi_{43}(T,t) & 1\end{bmatrix},$$

where $\Phi_{32}(T,t)=\int_t^T u_1(s)ds=\Phi_{43}(T,t)$ and $\Phi_{42}(T,t)=\int_t^T \Phi_{32}(s,t)u_1(s)ds$. A substitution of matrices $B(t)$, $\Phi(T,t)$ and $F_{q_0,T}(u(\cdot))(t)$ into (28) results in

$$E_{q_0,T}(u(\cdot))=\begin{bmatrix}\int_0^T(\phi_0(t)+\frac{\lambda_{10}\tilde{u}_1(t)}{\lambda_{10}^2+\epsilon})dt & 0 & 0 & 0\\ 0 & \int_0^T(\phi_0(t)+\frac{\lambda_{20}\tilde{u}_2(t)}{\lambda_{20}^2+\epsilon})dt & 0 & 0\\ \int_0^T(\phi_0(t)+\frac{\lambda_{10}\tilde{u}_1(t)}{\lambda_{10}^2+\epsilon})(q_3(t)+q_2(t)\Phi_{32}(T,t))dt & \int_0^T(\phi_0(t)+\frac{\lambda_{20}\tilde{u}_2(t)}{\lambda_{20}^2+\epsilon})\Phi_{32}(T,t)dt & \int_0^T\phi_1(t)\Phi_{32}(T,t)dt & \int_0^T\phi_2(t)\Phi_{32}(T,t)dt\\ \int_0^T(\phi_0(t)+\frac{\lambda_{10}\tilde{u}_1(t)}{\lambda_{10}^2+\epsilon})(q_3(t)+q_2(t)\Phi_{32}(T,t))dt & \int_0^T(\phi_0(t)+\frac{\lambda_{20}\tilde{u}_2(t)}{\lambda_{20}^2+\epsilon})\Phi_{42}(T,t)dt & \int_0^T\phi_1(t)\Phi_{42}(T,t)dt & \int_0^T\phi_2(t)\Phi_{42}(T,t)dt\end{bmatrix}.$$

Now, using the form (17) and (18) of controls, and the properties of the orthogonal basis $\{\phi_k(t)|k\geq 0\}$ we notice that the above expression is equivalent to

$$E_{q_0,T}(u(\cdot))=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ * & * & <\phi_1,\Phi_{32}> & <\phi_2,\Phi_{32}>\\ * & * & <\phi_1,\Phi_{42}> & <\phi_2,\Phi_{42}>\end{bmatrix},$$

where asterisks stand for the entries that will not contribute to $\det E_{q_0,T}(u(\cdot))$. This being so, we conclude

$$\det E_{q_0,T}(u(\cdot))=<\phi_1,\Phi_{32}><\phi_2,\Phi_{42}>-<\phi_1,\Phi_{42}><\phi_2,\Phi_{32}>. \quad (29)$$

For the specific form of controls (27) employed in the computer simulations reported in section 5 a further computation provides us with the following result

$$\det E_{q_0,T}(u(\cdot))=\frac{\lambda_{10}}{64\pi^3}(8\lambda_{10}^2+\lambda_{12}^2-6\lambda_{10}\lambda_{12}-\lambda_{11}^2). \quad (30)$$

Now, from the fact that during the operation of the extended Jacobian inverse kinematics algorithm the augmenting kinematics map (19) remains invariant, we obtain the equality

$$\frac{\tilde{u}_1(t)}{\sqrt{\lambda_{10}^2+\epsilon}}=\frac{\tilde{u}_{0\,1}(t)}{\sqrt{\lambda_{0\,10}^2+\epsilon}}$$

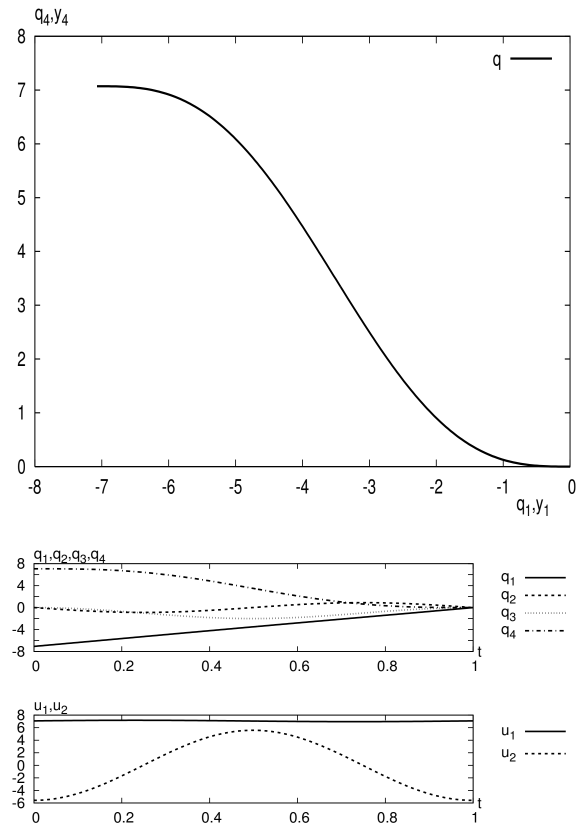

Fig. 4. Solution to the inverse kinematics problem: point D, number of iterations 99, final taskspace error $9.84385e-07$

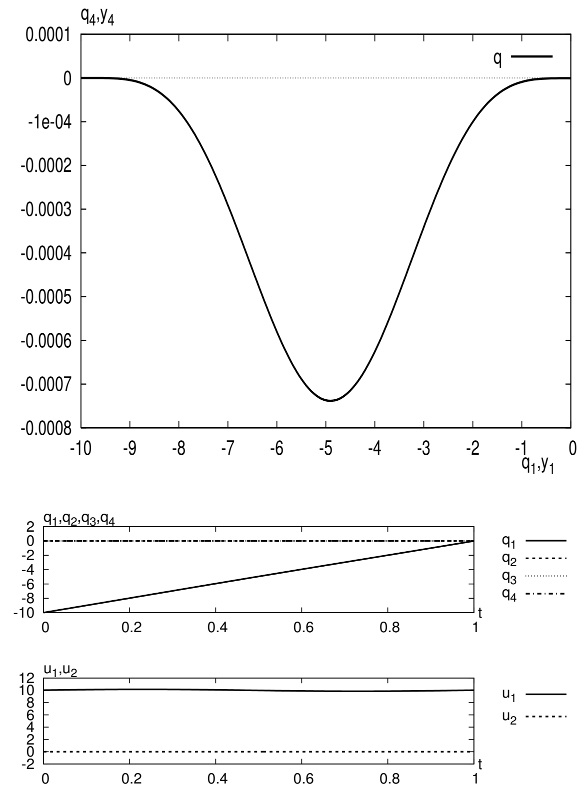

Fig. 5. Solution to the inverse kinematics problem: point E, number of iterations 82, final taskspace error $9.1763e-079.60468e-07$

and by means of (27) we deduce that

$$\lambda_{11} = \lambda_{0\,11}\sqrt{\frac{\lambda_{10}^2+\epsilon}{\lambda_{0\,10}^2+\epsilon}},\ \lambda_{12} = \lambda_{0\,12}\sqrt{\frac{\lambda_{10}^2+\epsilon}{\lambda_{0\,10}^2+\epsilon}}.$$

A substitution of the above expressions into (30) yields a formula

$$\det E_{q_0,T}(u(\cdot)) = \frac{\lambda_{10}}{64\pi^3(\lambda_{0\,01}^2+\epsilon)}((8(\lambda_{0\,10}^2+\epsilon)+\lambda_{0\,12}^2-\lambda_{0\,11}^2)\lambda_{10}^2- 6\lambda_{10}\lambda_{0\,12}\sqrt{\lambda_{10}^2+\epsilon}\sqrt{\lambda_{0\,10}^2+\epsilon}+(\lambda_{0\,12}^2-\lambda_{0\,11}^2)\epsilon) \quad (31)$$

that depends on the constant component λ_{10} of $u_1(t)$ and on the initial values $\lambda_{0\,10}$, $\lambda_{0\,11}$ and $\lambda_{0\,11}$. For the data used in the computer simulations plots of the determinant against λ_{10} are presented in figures 8 and 9. It turns out that the extended Jacobian inverse kinematics algorithm used in the simulations, in addition to the original singular configuration $\lambda_{10} = 0$, has 2 algorithmic singularities at $\lambda_{10} = -0.061e-05$ and $\lambda_{10} = 0.47e-05$. For the reason that, for a quite wide region of $|\lambda_{10}| \leq 0.05$ that includes the algorithmic singularities, the determinant function (30) takes values less than 10^{-6}, for λ_{10} from this region the algorithm is practically inapplicable. In the computer simulations such a situation has been met when the algorithm is initialized from around the point C or G.

VII. CONCLUSION

In this paper, relying on the analogy between stationary and mobile robots, we have presented a procedure of deriving extended Jacobian inverse kinematics algorithms for mobile robots. A cornerstone of this procedure is the identification of the mobile robot kinematics with the end point map of a control system representation of the kinematics. An advantage of the extended Jacobian algorithms is that they are repeatable, i.e. for fixed q_0 and T they transform closed paths of the desirable tasks into closed paths of configurations. The general definitional procedure has been exemplified with the chained form system, feedback equivalent to the kinematics of the kinematic car. This example not only serves as an illustration of the extended Jacobian algorithm, but also offers a possibility of carrying out an examination of the algorithmic singularities. The performance and convergence of the extended Jacobian inverse kinematics algorithm has been tested by computer simulations. As can be seen, the algorithm converges in a very smooth way, except from a neighbourhood of y_4 axis, including the points C and G, where the algorithm from the computational point of view behaves in a singular way.

We would like to admit that our procedure does not produce a canonical inverse kinematics algorithm, but rather delivers a collection of diverse algorithms, depending on the choice of the augmenting kinematics map. This choice

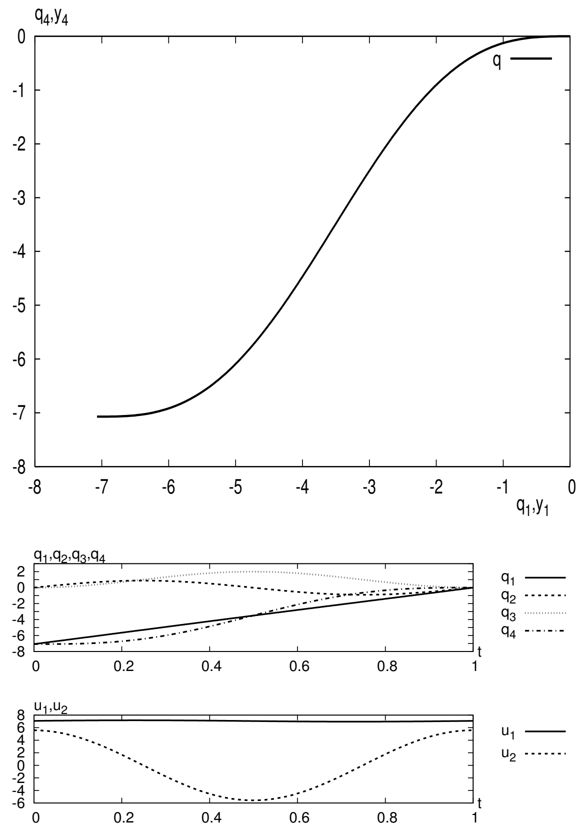

Fig. 6. Solution to the inverse kinematics problem: point F, number of iterations 100, final taskspace error $8.76824e-07$

should correspond to a secondary task accompanying the solution of the inverse kinematic problem for the mobile robot. In this context our choice of the augmenting map for the chained form system and the resulting inverse kinematics algorithm are perhaps far from being natural. The question whether for every mobile robot there exists a natural choice of the augmenting kinematics map seems to be a challenge for future research. Next, we do not expect that it would be possible to define a universal extended Jacobian inverse kinematics algorithm applicable to all mobile robots. Rather that that, the form of the algorithm will have to be tuned to the mobile robot kinematics. A similar situation takes place with the Pomet stabilizing algorithm for driftless systems [17].

Characteristic to the extended Jacobian algorithms is the problem of algorithmic singularities. From the analysis carried out in this paper we have learnt that these singularities are dependent on initial endogenous configuration. Moreover, besides formal algorithmic singularities, the algorithm may exhibit computational algorithmic singularities that affect the algorithm performance in the same as ”true” singularities. The question how to restrict the computational algorithmic singularities should be another subject of future research.

REFERENCES

[1] K. Tchoń and J. Jakubiak, Extended Jacobian inverse kinematics algorithm for mobile robots, *Int. J. Control*, vol. 79, 2006, pp. 895-909.

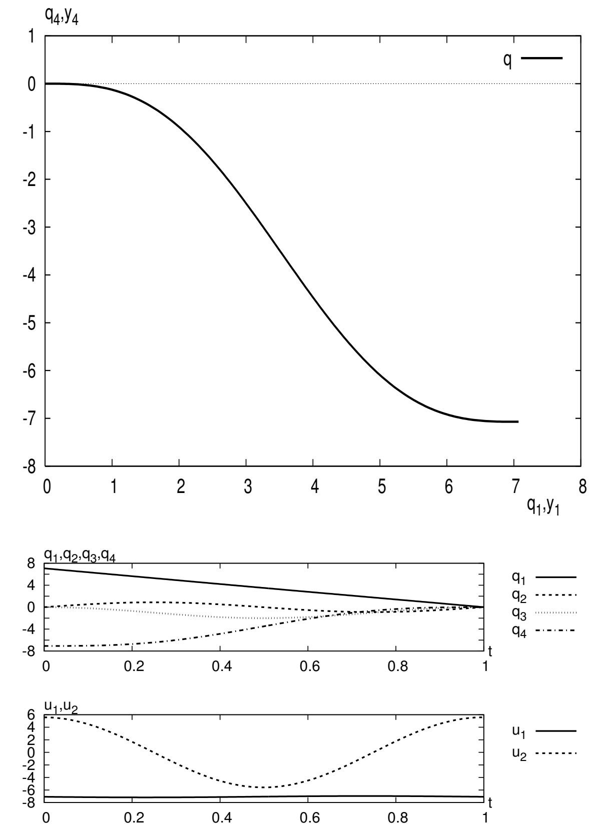

Fig. 7. Solution to the inverse kinematics problem: point H, number of iterations 95, final taskspace error $9.46389e-07$

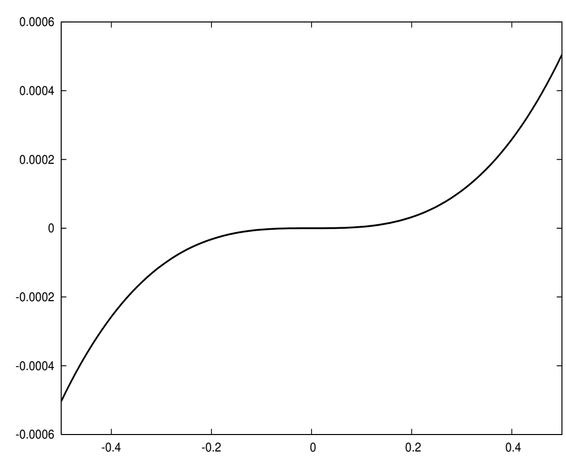

Fig. 8. Plot of $\det E_{q_0,T}(\lambda_{10})$

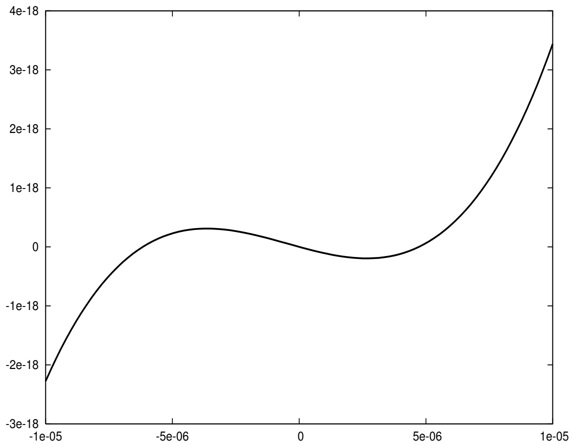

Fig. 9. Plot of $\det E_{q_0,T}(\lambda_{10})$ in a close vicinity of $\lambda_{10} = 0$

[2] K. Tchoń and R. Muszyński, Instantaneous kinematics and dexterity of mobile manipulators, *in Proc. 2000 IEEE Int. Conf. Robot. Automat*, San Francisco, CA, 2000, pp. 2493-2498.

[3] K. Tchoń and J. Jakubiak, Endogenous configuration space approach to mobile manipulators: a derivation and performance assessment of Jacobian inverse kinematics algorithms, *Int. J. Control*, vol. 76, 2003, pp. 1387-1419.

[4] Y. Chitour and H. J. Sussmann, Motion planning using the continuation method, *in Essays on Mathematical Robotics*, J. Baillieul, S. S. Sastry, and H. J. Sussmann, Eds., Springer-Verlag, New York, 1998, pp. 91-125.

[5] A. W. Divelbiss, S. Seereeram and J. T. Wen, Kinematic path planning for robots with holonomic and nonholonomic constraints, *in Essays on Mathematical Robotics*, J. Baillieul, S. S. Sastry, and H. J. Sussmann, Eds., Springer-Verlag, New York, 1998, pp. 127-150.

[6] Y. Chitour, A homotopy continuation method for trajectories generation of nonholonomic systems, *ESAIM: Control, Optim. Calc. Var.*, vol. 12, 2006, pp. 139-168.

[7] J. Baillieul, Kinematic programming alternatives for redundant manipulators, *in Proc. 1985 IEEE Int. Conf. Robot. Automat.*, St. Louis, LO, 1985, pp. 818-823.

[8] R. G. Roberts and A. A. Maciejewski, Repeatable generalized inverse control strategies for kinematically redundant manipulators, *IEEE Trans. Automat. Contr.*, vol. 38, 1993, pp. 689-699.

[9] C. A. Klein, C. Chu-Jenq and S. Ahmed, A new formulation of the extended Jacobian method and its use in mapping algorithmic singularities for kinematically redundant manipulators, *IEEE Trans. Robot. Automat.*, vol. 11, 1995, pp. 50-55.

[10] K. Tchoń and J. Jakubiak, A repeatable inverse kinematics algorithm with linear invariant subspaces for mobile manipulators, *IEEE Trans. Syst., Man, Cybernet. - Part B: Cybernetics*, vol. 35, 2005, pp. 1051-1057.

[11] K. Tchoń, Repeatable, extended Jacobian inverse kinematics algorithm for mobile manipulators, *Syst. Control Lett.*, vol. 55, 2006, pp. 87-93.

[12] K. Tchoń, Repeatability of inverse kinematics algorithms for mobile manipulators, *IEEE Trans. Autom. Control*, vol. 47, 2002, pp. 1376-1380.

[13] R. M. Murray, Z. Li and S. S. Sastry, *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, 1994.

[14] B. Bonnard and M. Chyba, *Singular Trajectories and their Role in Control*, Springer-Verlag, Berlin, 2003.

[15] H. J. Sussmann, New differential geometric methods in nonholonomic path finding, *in Systems, Models, and Feedback*, A. Isidori and T. J. Tarn, Eds., Birkhäuser, Boston, 1992, pp. 365-384.

[16] T. Ważewski, Sur l'évaluation du domaine d'existence des fonctions implicites réelles ou complexes, *Ann. Soc. Pol. Math.*, vol. 20, 1947, pp. 81-120.

[17] J. B. Pomet, Explicit design of time varying stabilization laws for a class of controllable systems without drift, *Syst. Contr. Lett.*, vol. 18, 1992, pp. 147-158.