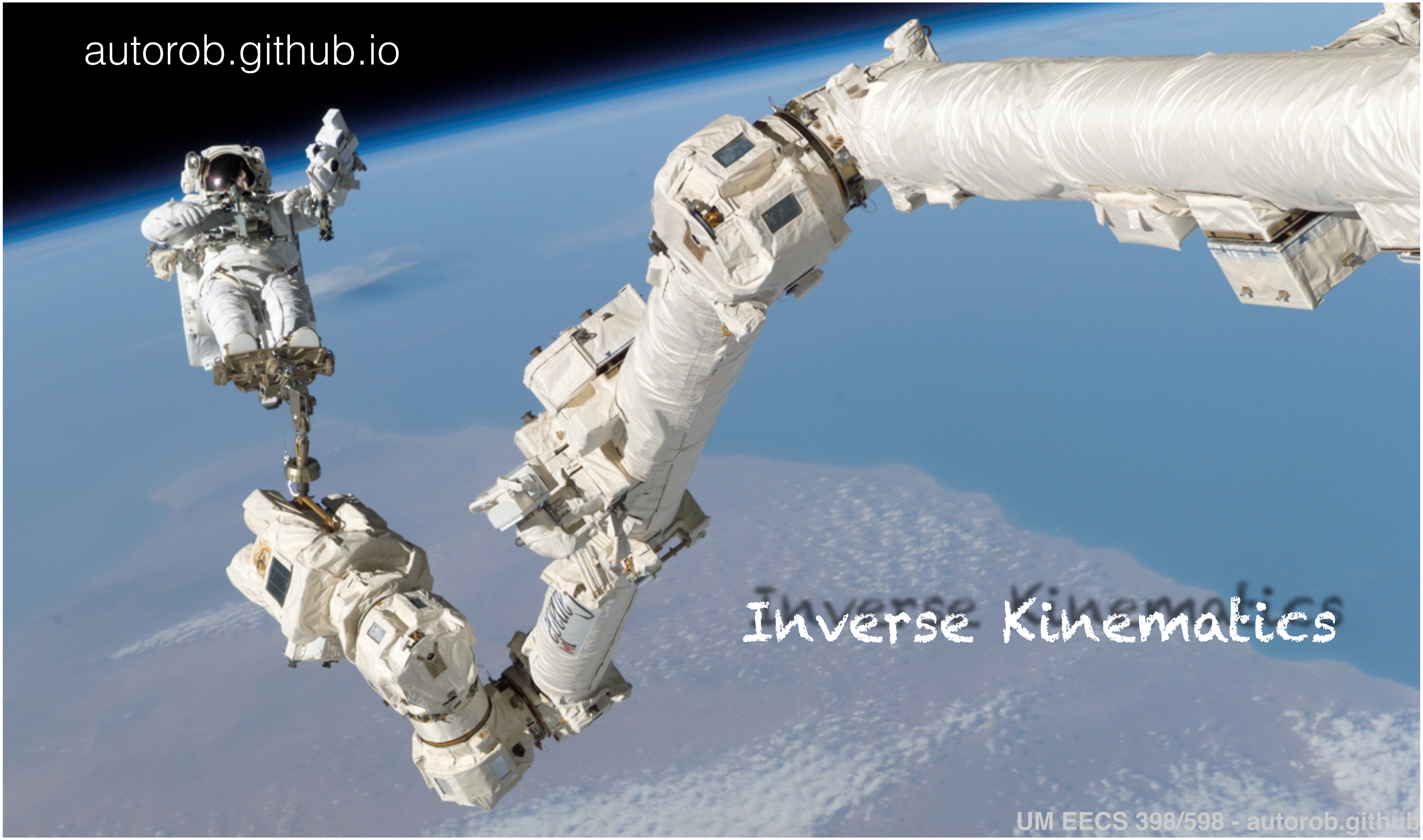

autorob.github.io
Inverse Kinematics
UM EECS 398/598 - autorob.github.io

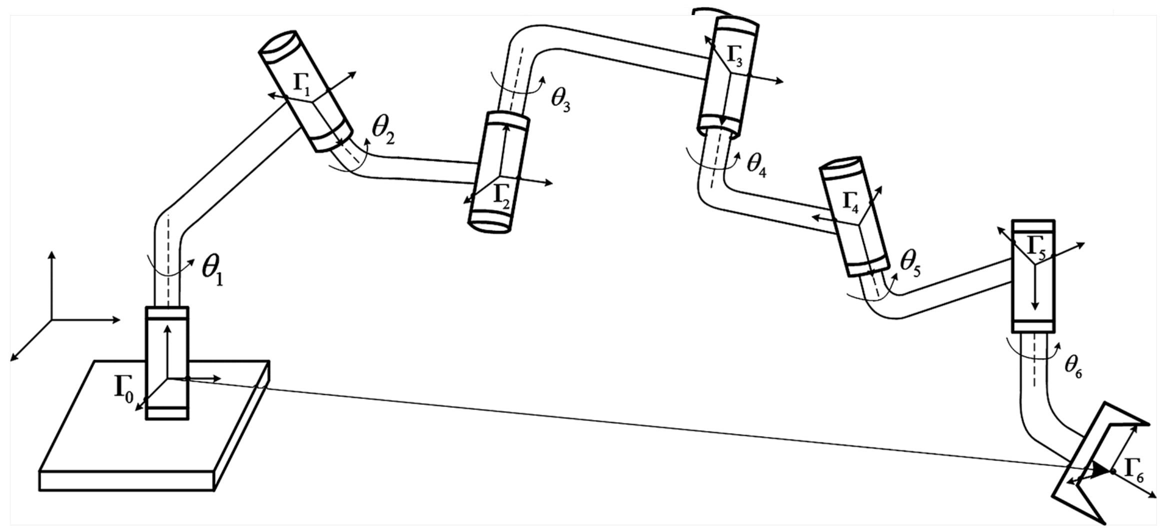
Objective (revisited)

Goal: Given the structure of a robot arm, compute

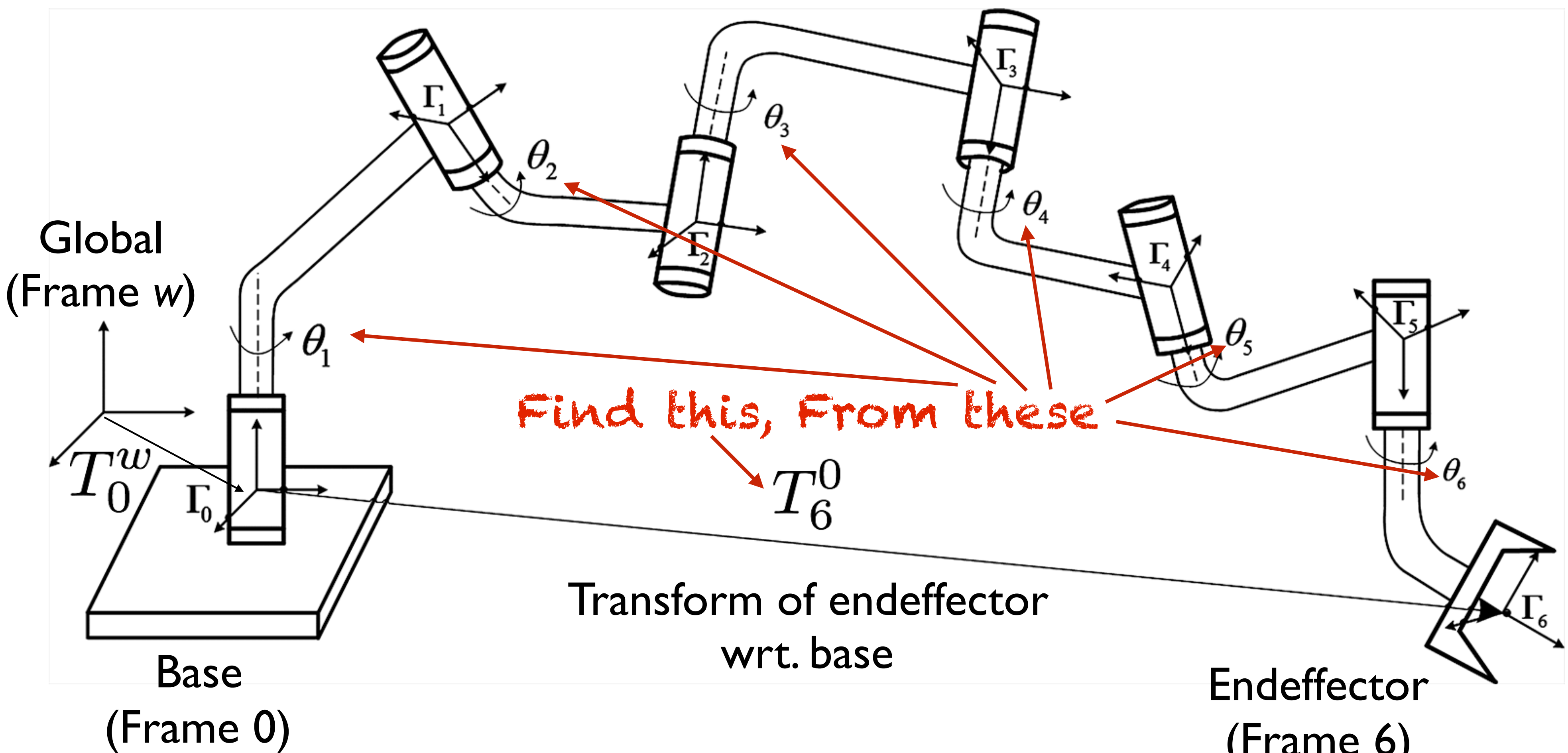
– **Forward kinematics**: predicting the pose of the end-effector, given joint positions.

– **Inverse kinematics**: inferring the joint positions necessary to reach a desired end-effector pose.

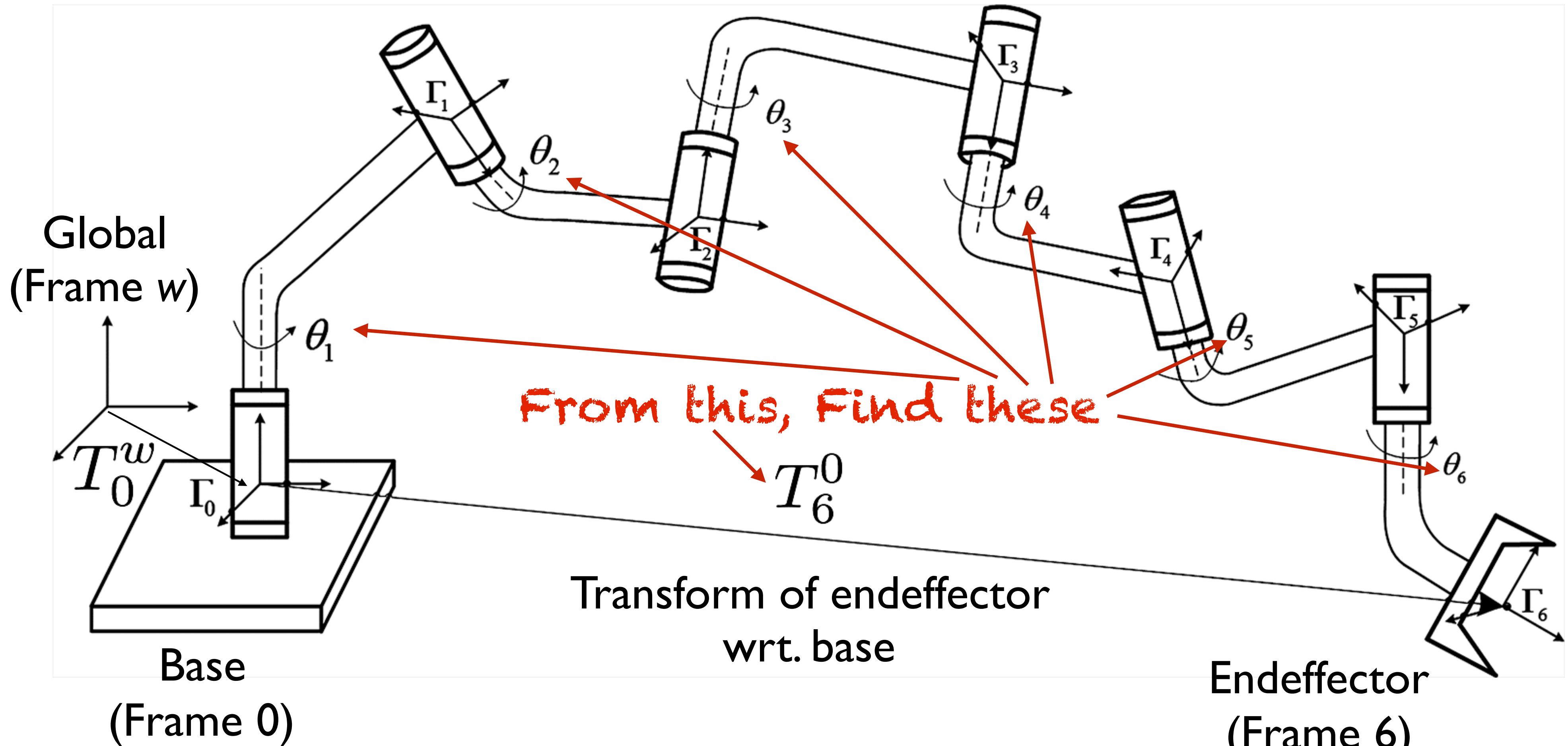
We need to solve for configuration from a transform between world and endeffector frames.



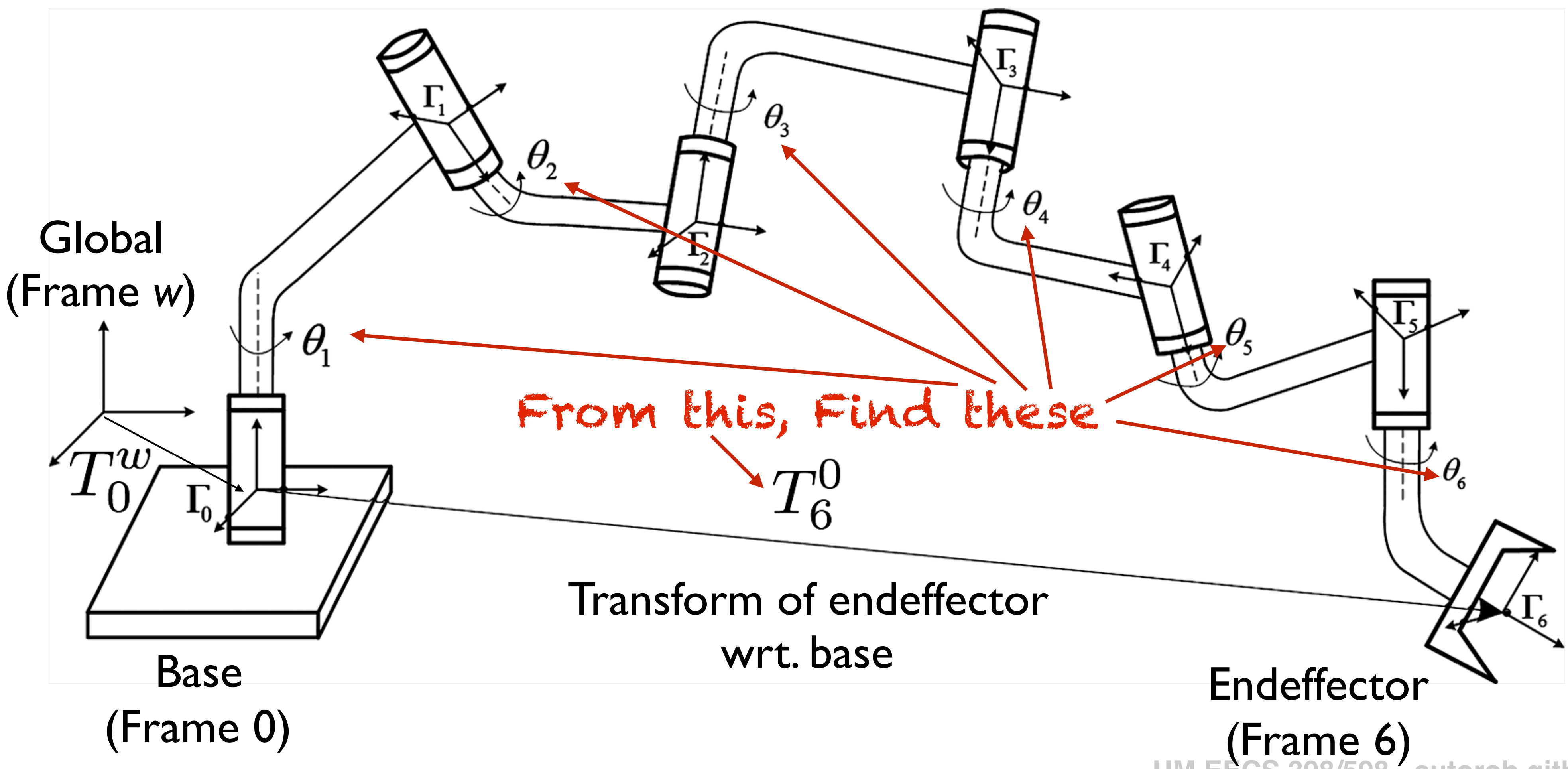
Forward kinematics: many-to-one mapping of robot configuration to reachable workspace endeffector poses



Inverse kinematics: one-to-many mapping of workspace endeffector pose to robot configuration

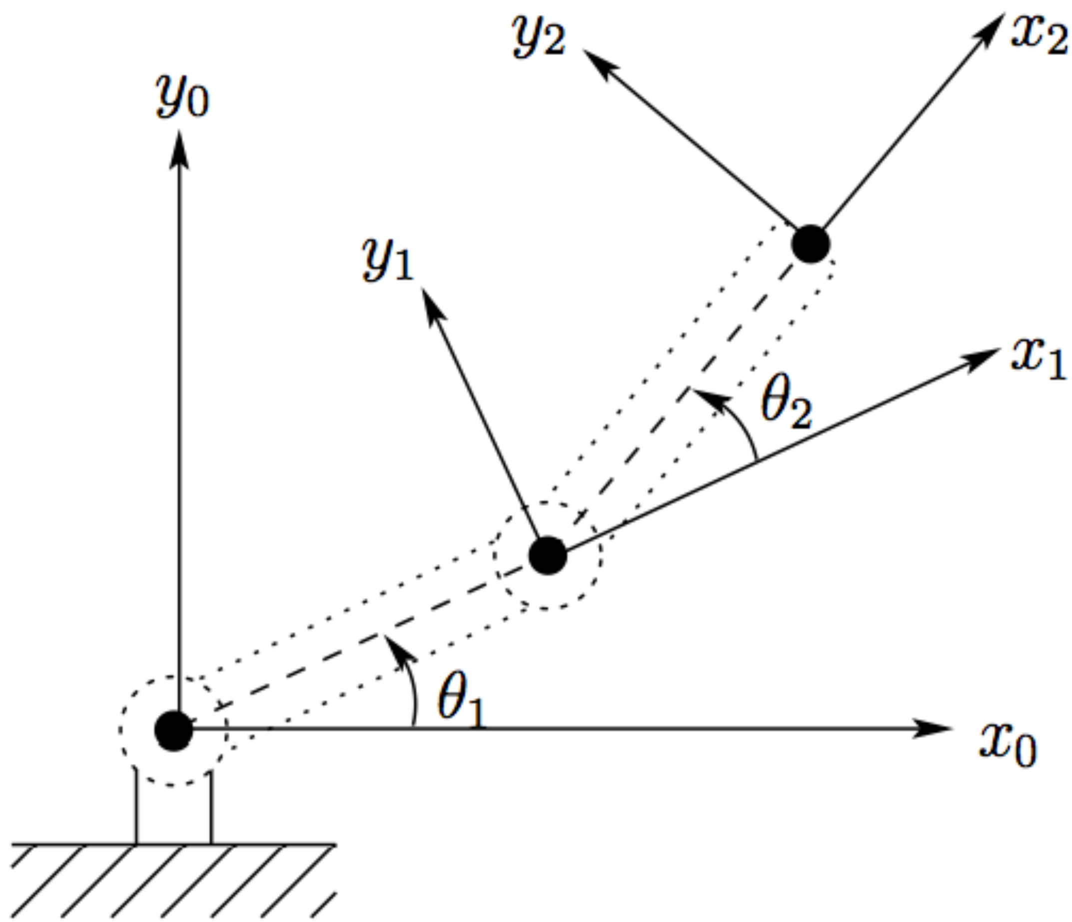


Inverse kinematics: how to solve for $q = \{\theta_1, \ldots, \theta_N\}$ from T^0_N?

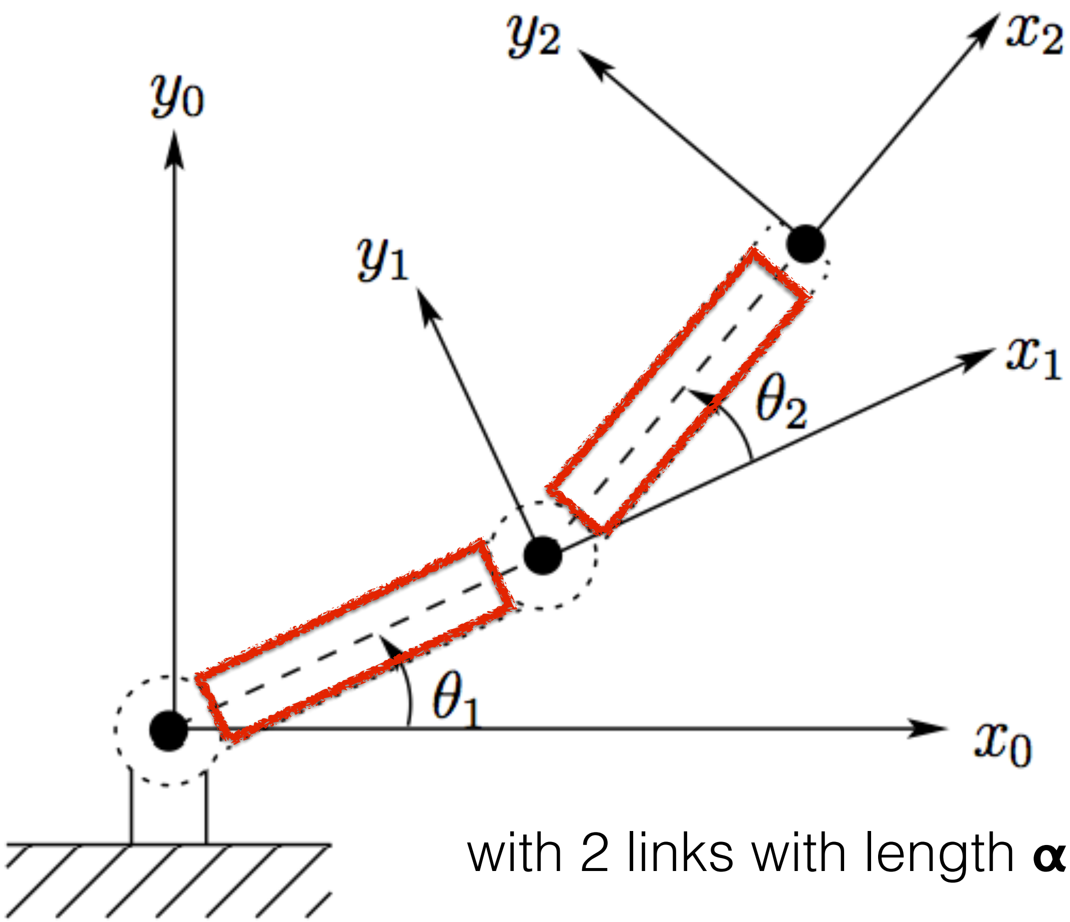


Let's define IK
starting from FK

Consider a planar 2-link arm as an example

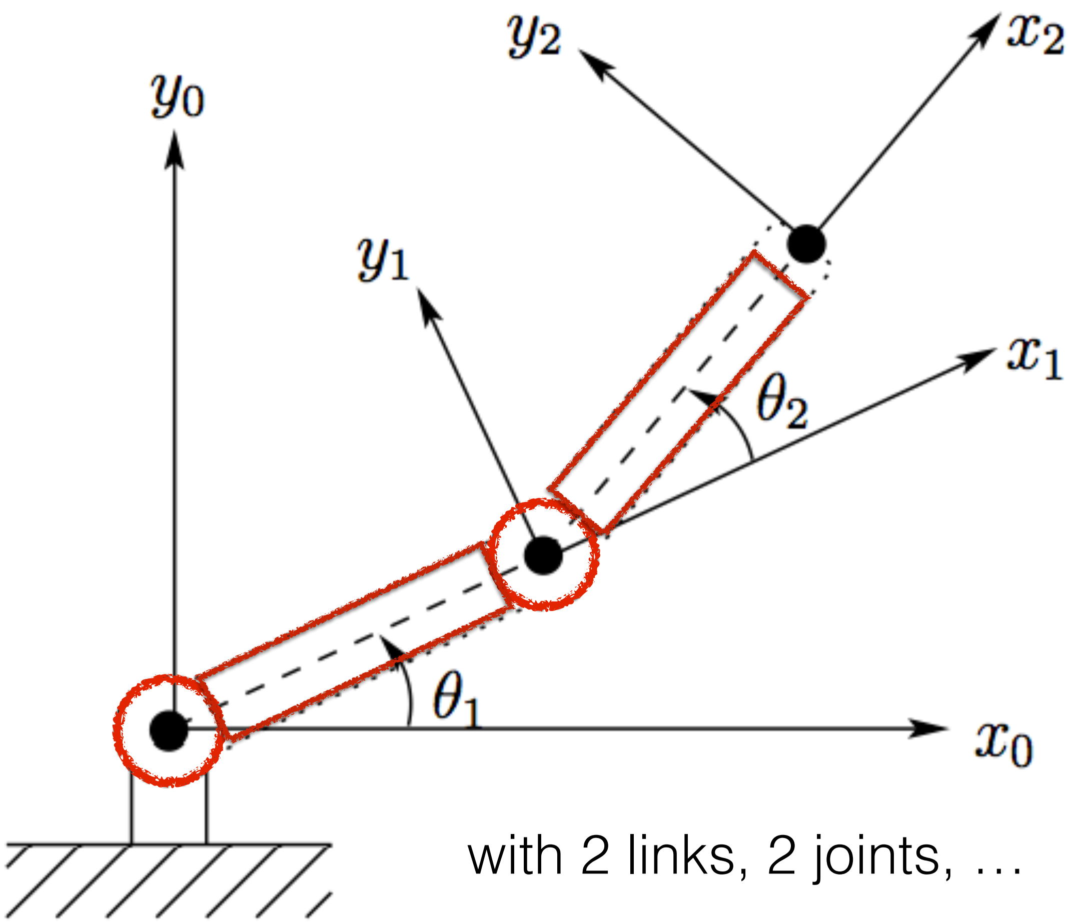


Consider a planar 2-link arm as an example



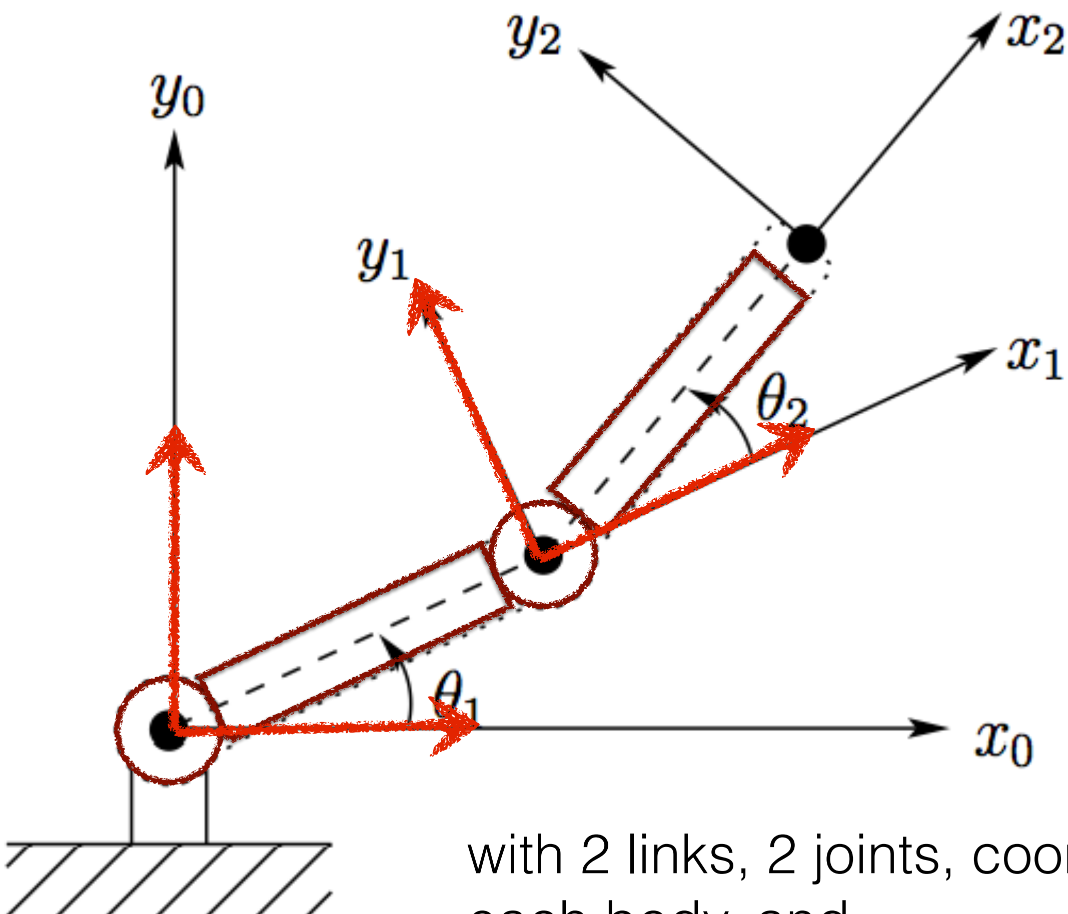
with 2 links with length $\boldsymbol{\alpha}_i$, …

Consider a planar 2-link arm as an example



with 2 links, 2 joints, …

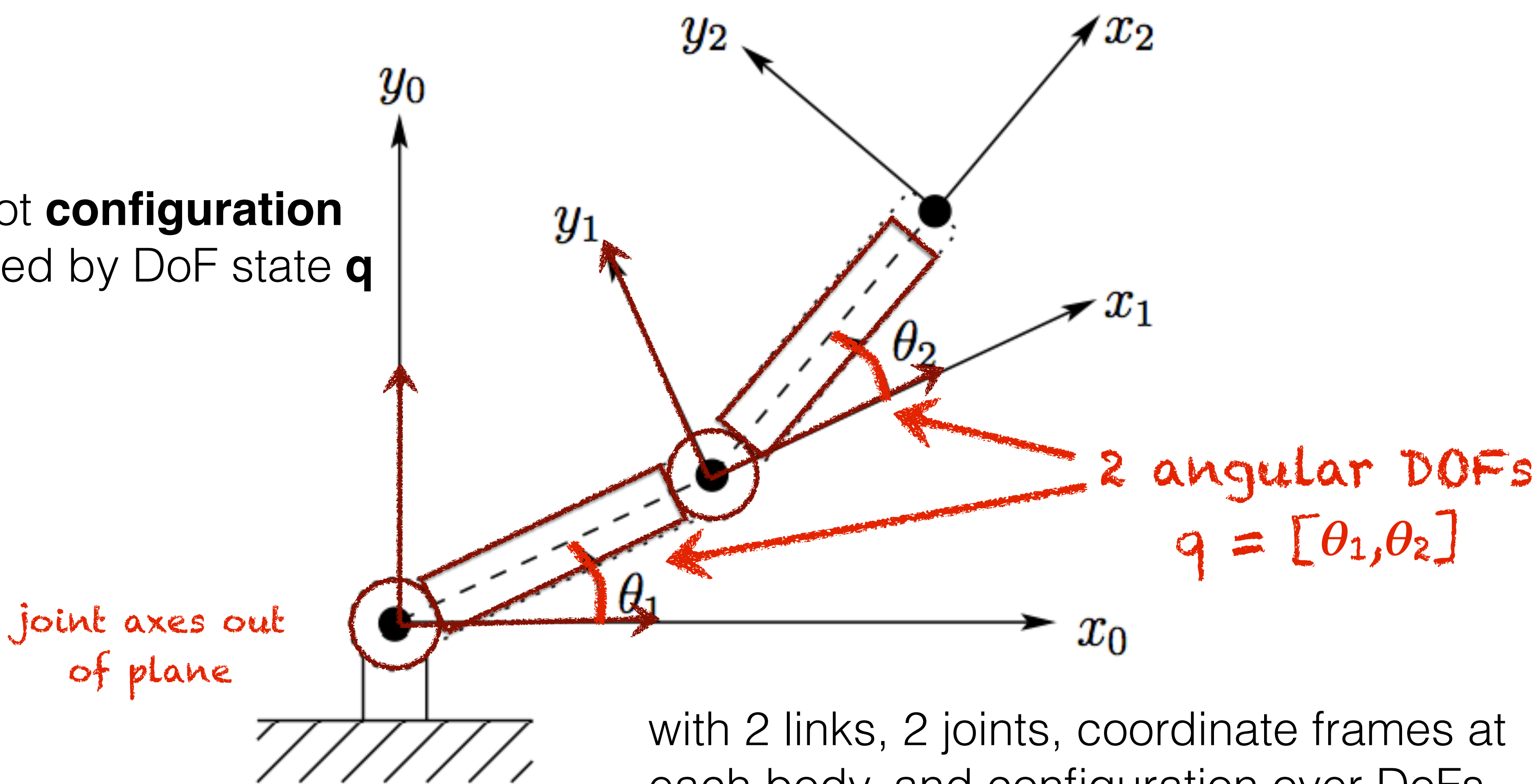
Consider a planar 2-link arm as an example



with 2 links, 2 joints, coordinate frames at each body, and …

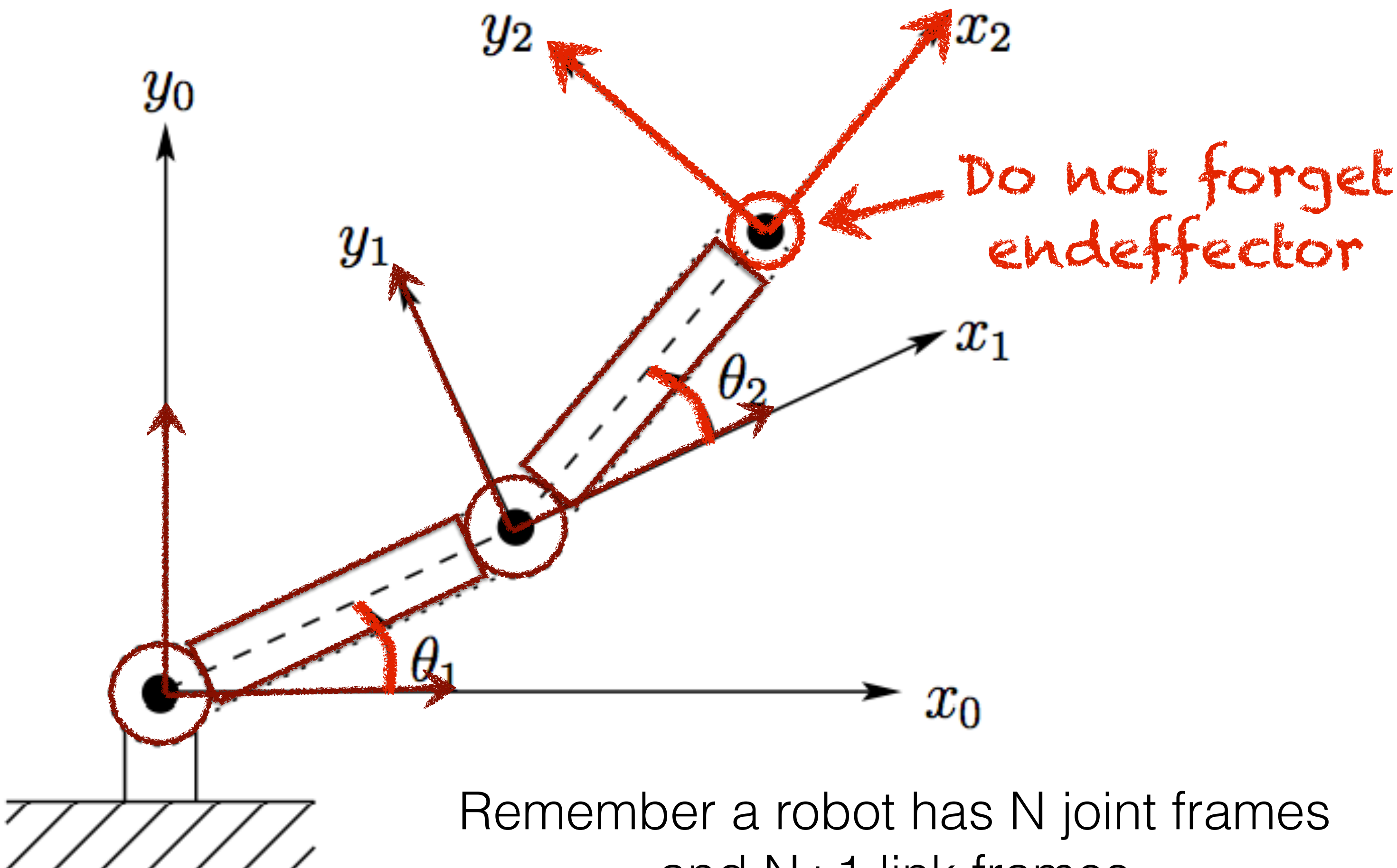
Consider a planar 2-link arm as an example

Robot **configuration** defined by DoF state **q**



with 2 links, 2 joints, coordinate frames at each body, and configuration over DoFs

Consider a planar 2-link arm as an example



Remember a robot has N joint frames
and N+1 link frames

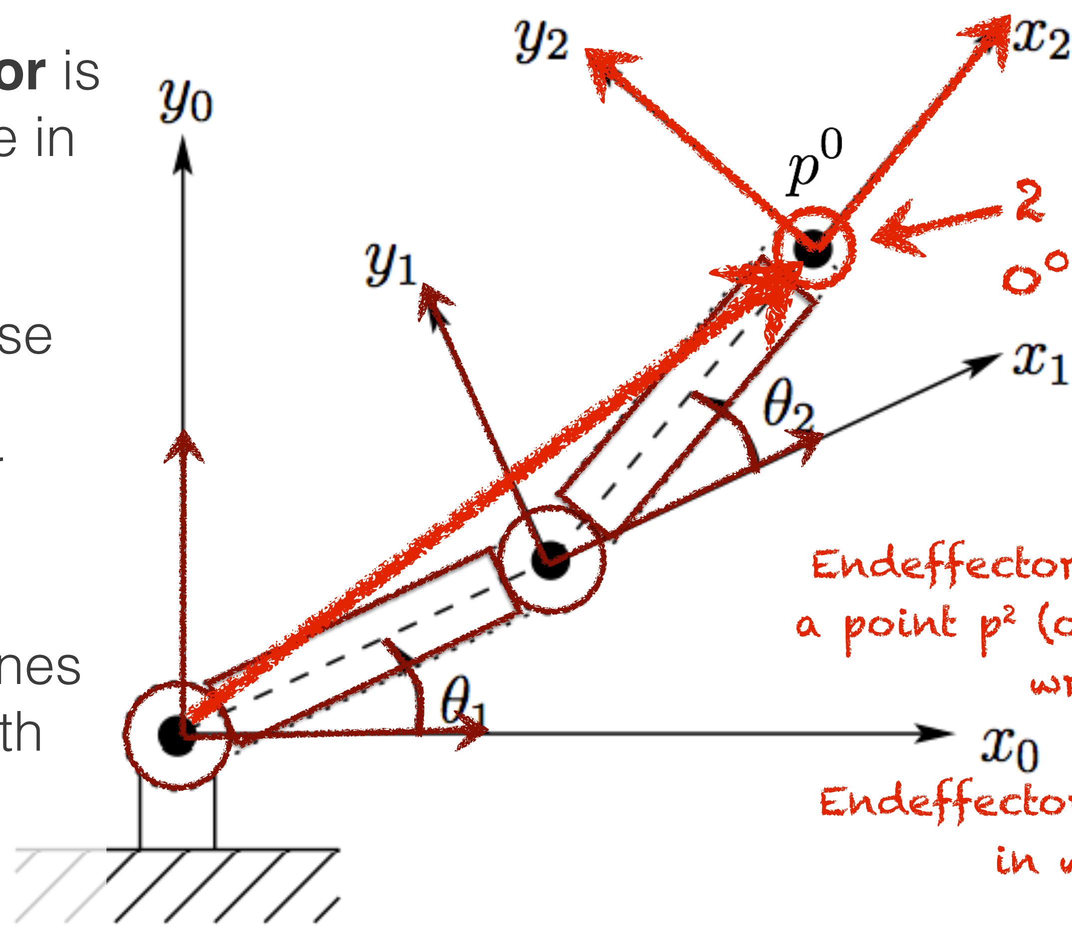
Consider a planar 2-link arm as an example

Frame 2 is the "tool frame"

Robot **endeffector** is the gripper pose in world frame

Endeffector pose has position can consider orientation

Endeffector defines "tool frame" with transform world frame



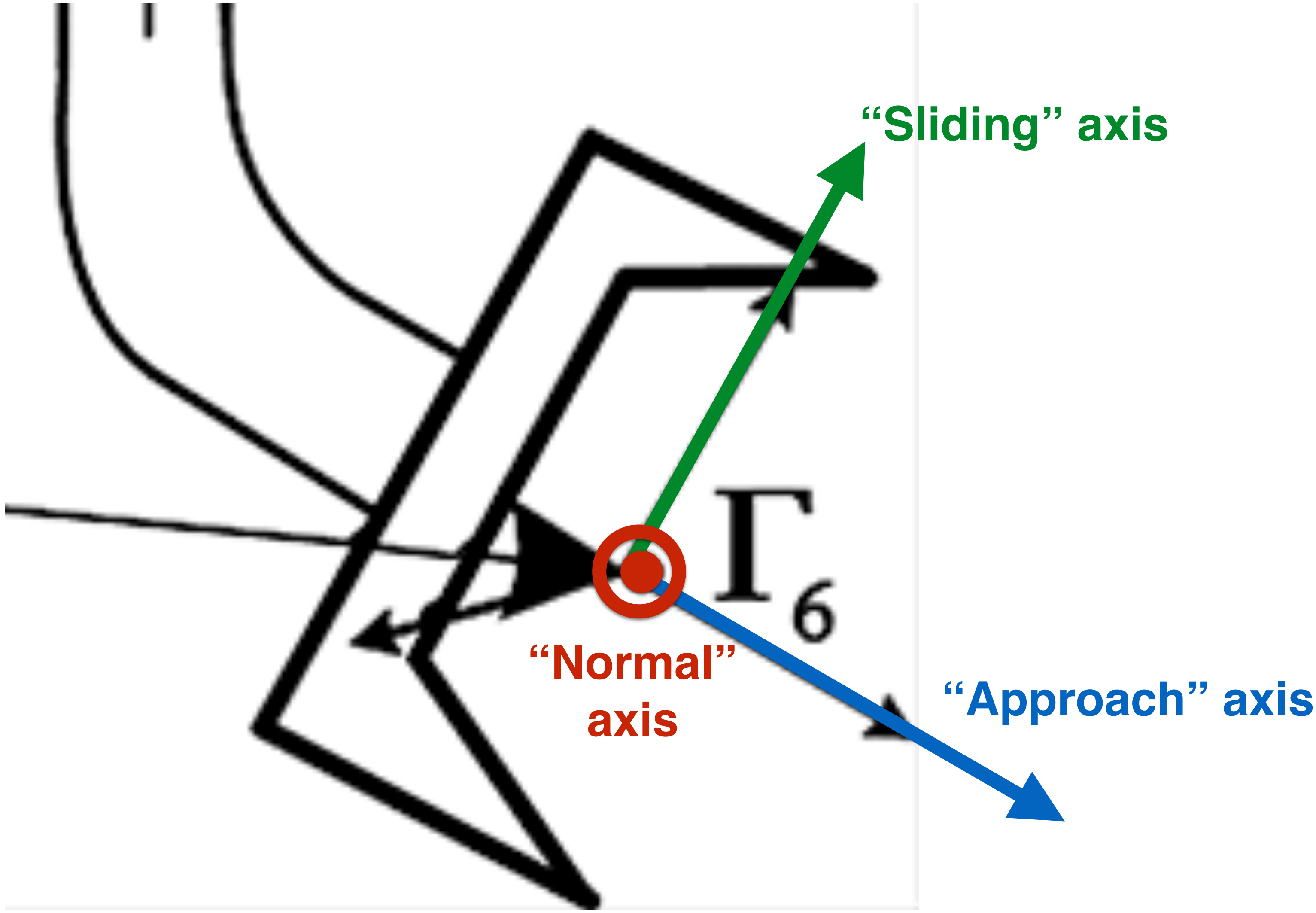
2 Cartesian DOFs

$o^0_N = p^0 = (p^{x0}, p^{y0})$

Endeffector can be specified as a point p^2 (origin of tool frame) wrt. Frame 1

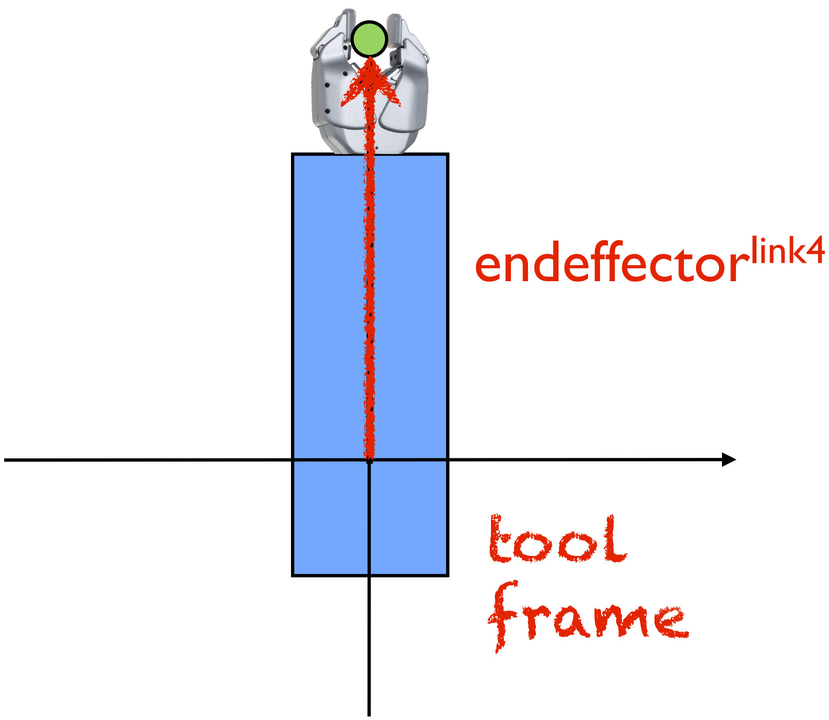
Endeffector transforms into p^0 in world frame

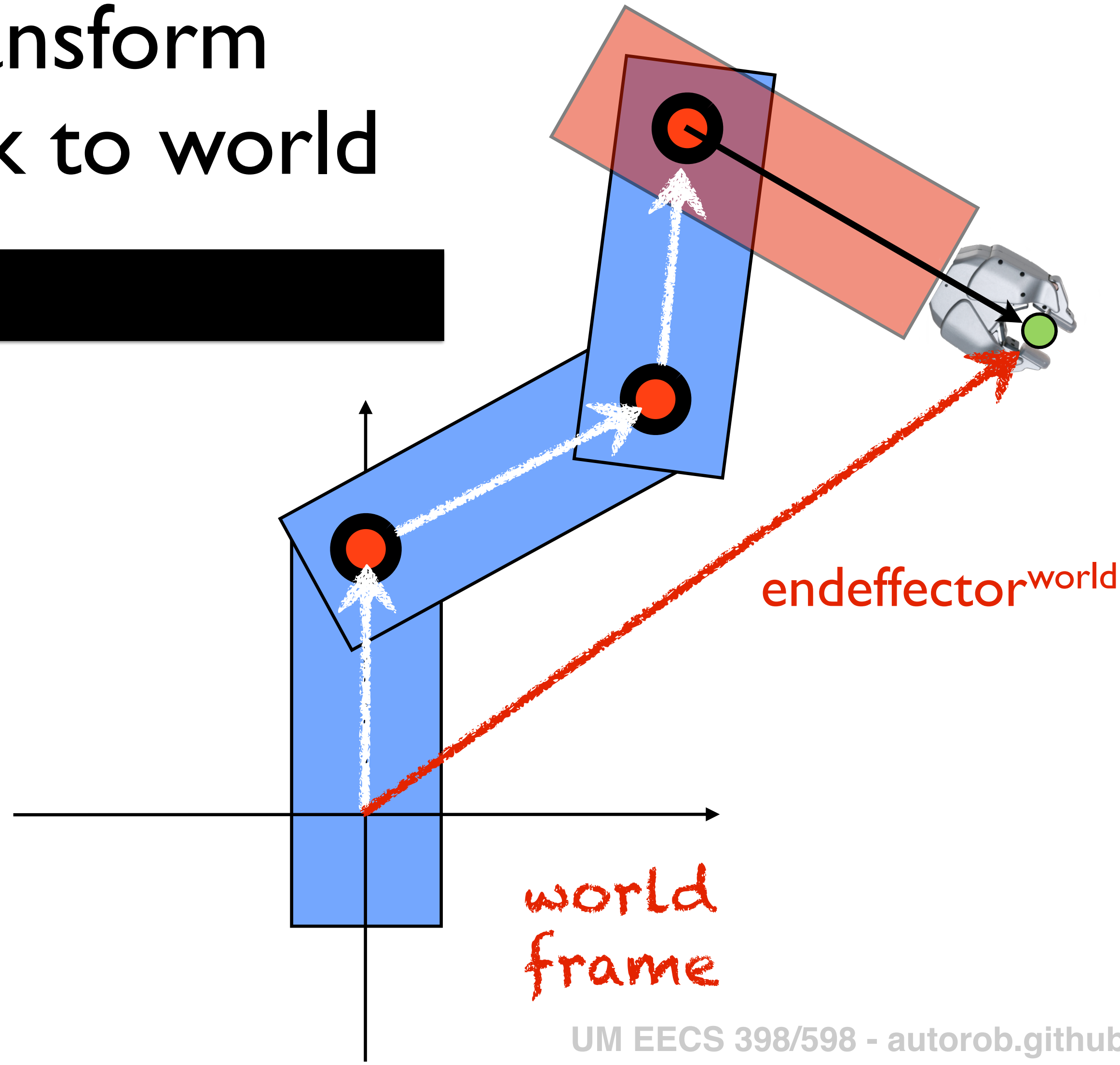
Endeffector axes



Checkpoint: Transform endeffector on link to world

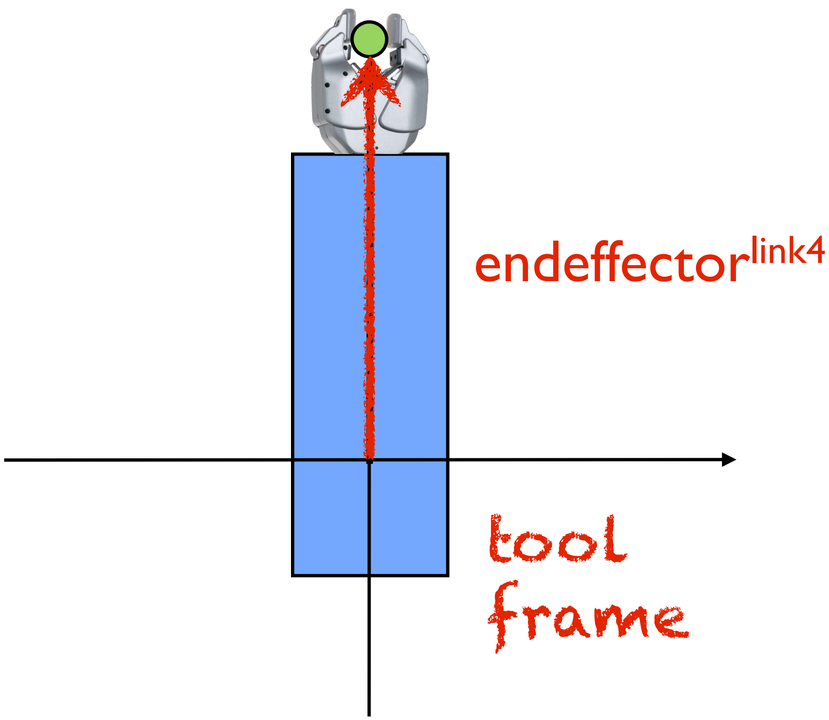
$$\text{endeffector}^{world} = \text{[redacted]}$$

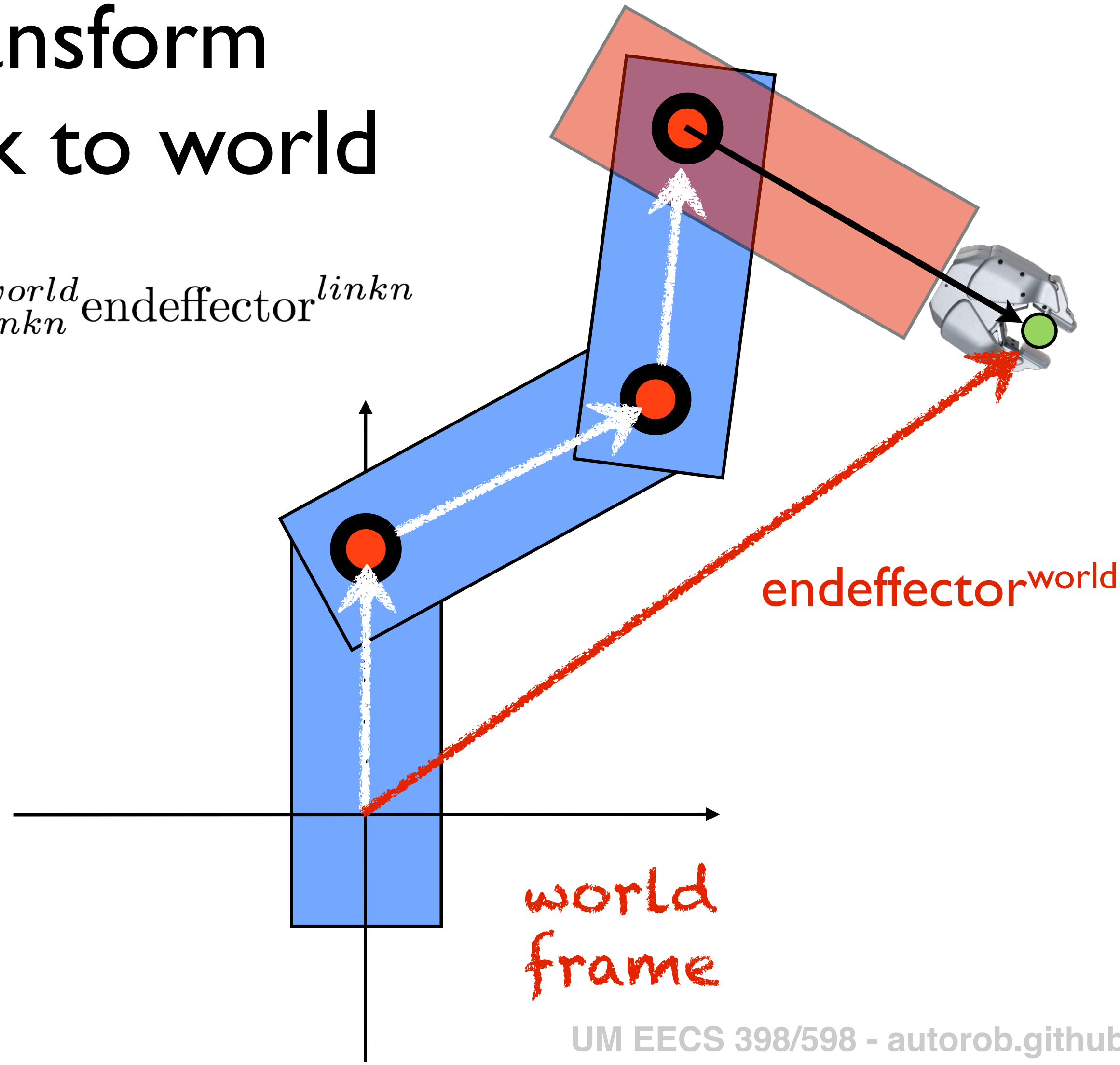




Checkpoint: Transform endeffector on link to world

$$\text{endeffector}^{world} = T_{linkn}^{world}\text{endeffector}^{linkn}$$



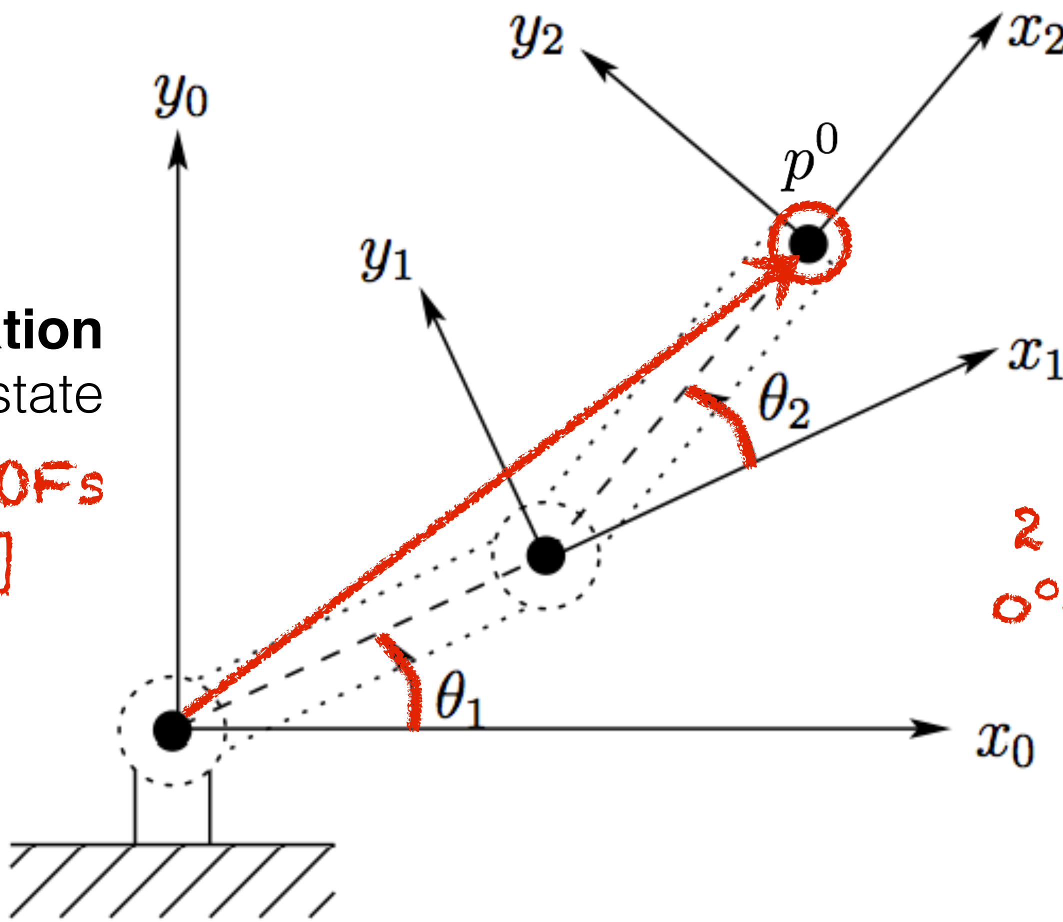


Forward kinematics: "given configuration, compute endeffector"

Robot **configuration** defined by DoF state

2 angular DOFs

$q = [\theta_1, \theta_2]$



Robot **endeffector** is the gripper pose in world frame

2 Cartesian DOFs

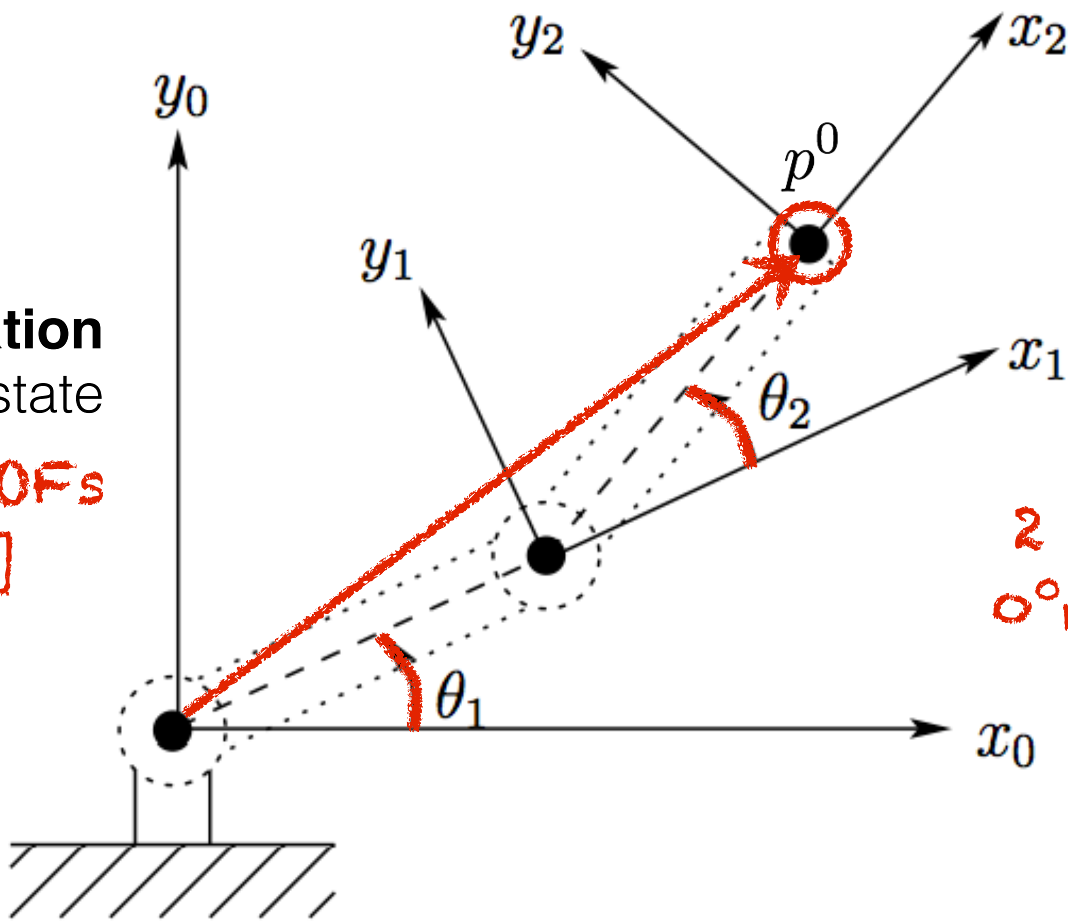
$o^0_N = p^0 = (p^{x0}, p^{y0})$

Forward kinematics: [$\mathbf{o}^0_N$, $\mathbf{R}^0_N$] = f($\mathbf{q}$)

$p^0 = f(\theta_1, \theta_2)$

Robot **configuration** defined by DoF state

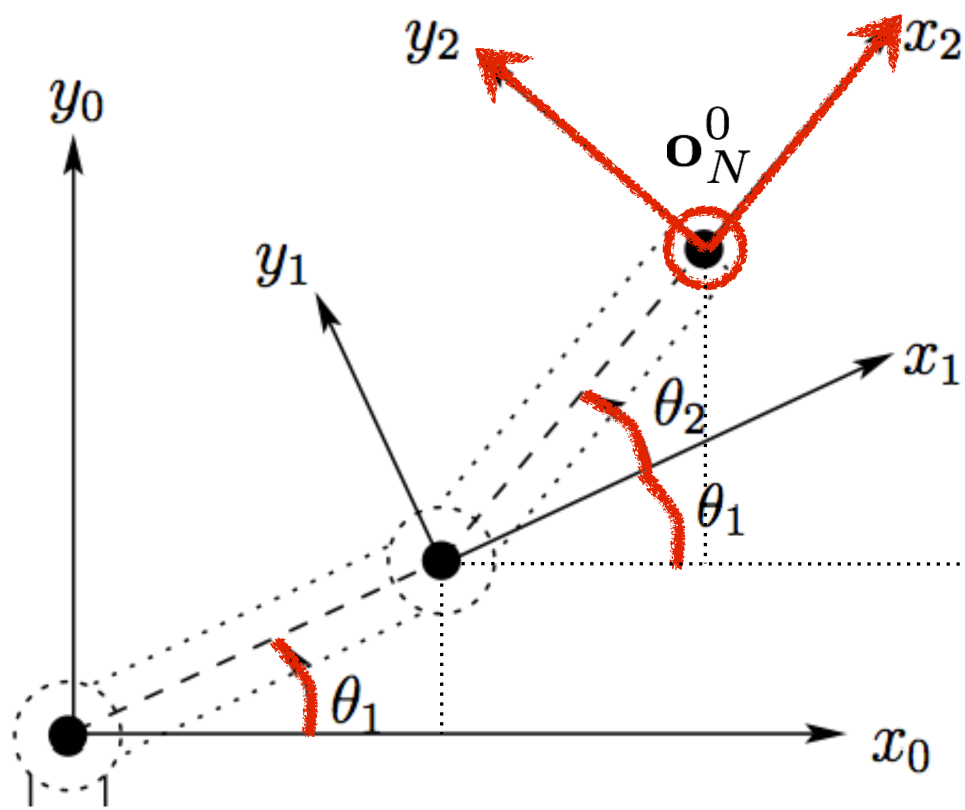
2 angular DOFs
$q = [\theta_1, \theta_2]$



Robot **endeffector** is the gripper pose in world frame

2 Cartesian DOFs
$o^0_N = p^0 = (p^{x0}, p^{y0})$

Forward kinematics: [$\mathbf{o}^0_N$, $\mathbf{R}^0_N$] = f($\mathbf{q}$)



What is the position and orientation of the tool wrt. the world?

remember:

$$p^0 = T^0_1 T^1_2 p^2$$

$$\mathbf{R}^0_N = \left[\text{What are the elements of this matrix?} \right] \qquad \mathbf{o}^0_N = \left[\text{What are the elements of this vector?} \right]$$

Forward kinematics: $[\mathbf{o}^0_N, \mathbf{R}^0_N] = f(\mathbf{q})$

What is the position and orientation of the tool wrt. the world?

remember:

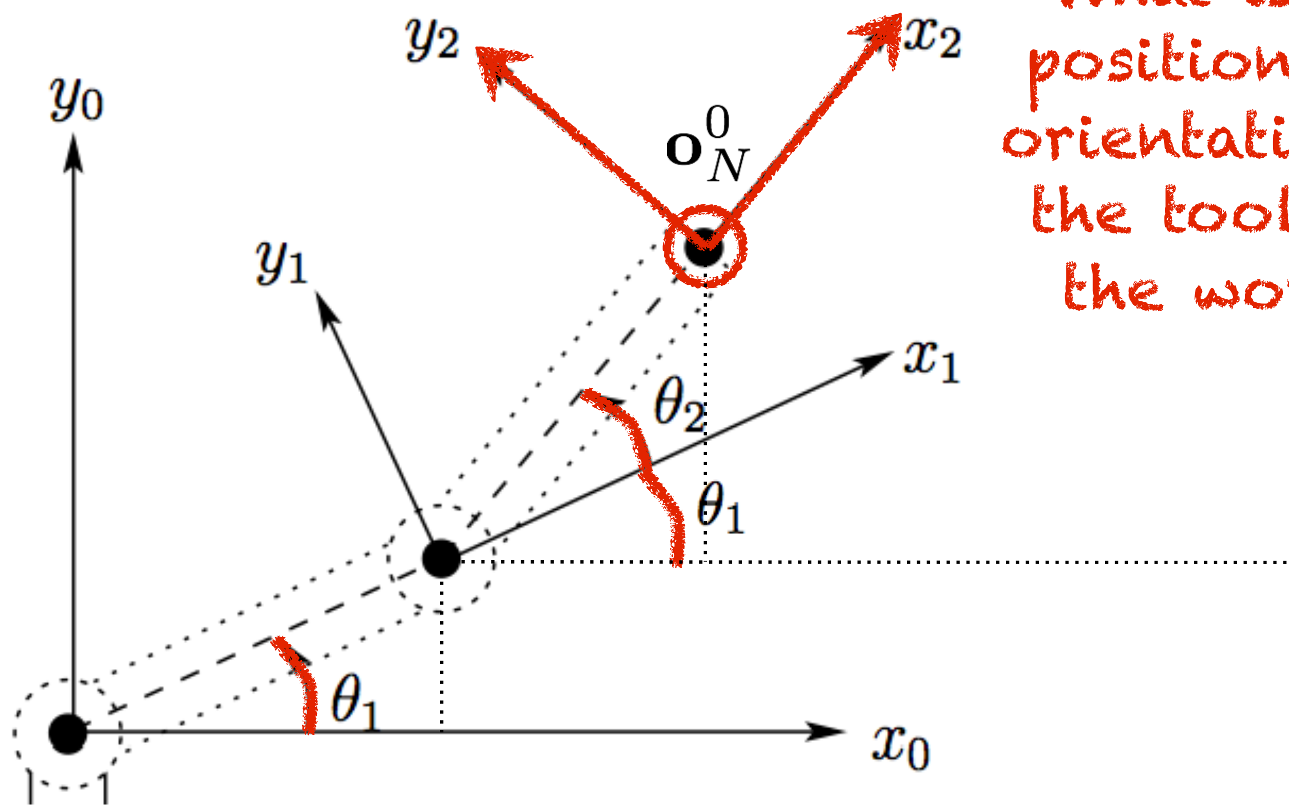
$$p^0 = T^0_1 T^1_2 p^2$$

$p^0 = R_2{}^0 p^2 + d_2{}^0$

where

$R_2{}^0 = R_1{}^0 R_2{}^1$

[redacted]



$$\mathbf{R}^0_N = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \quad \mathbf{o}^0_N = \Big[\text{What are the elements of this vector?} \Big]$$

Start with:

$$d_2^0 = R_1^0 d_2^1 + d_1^0$$

substitute in variables then perform operations:

$$\begin{bmatrix} cos\theta_2 & -sin\theta_2 \\ sin\theta_2 & cos\theta_2 \end{bmatrix} \begin{bmatrix} \alpha_2 cos\theta_2 \\ \alpha_2 sin\theta_2 \end{bmatrix} + \begin{bmatrix} \alpha_1 cos\theta_1 \\ \alpha_1 sin\theta_1 \end{bmatrix}$$

then substitute trig identities

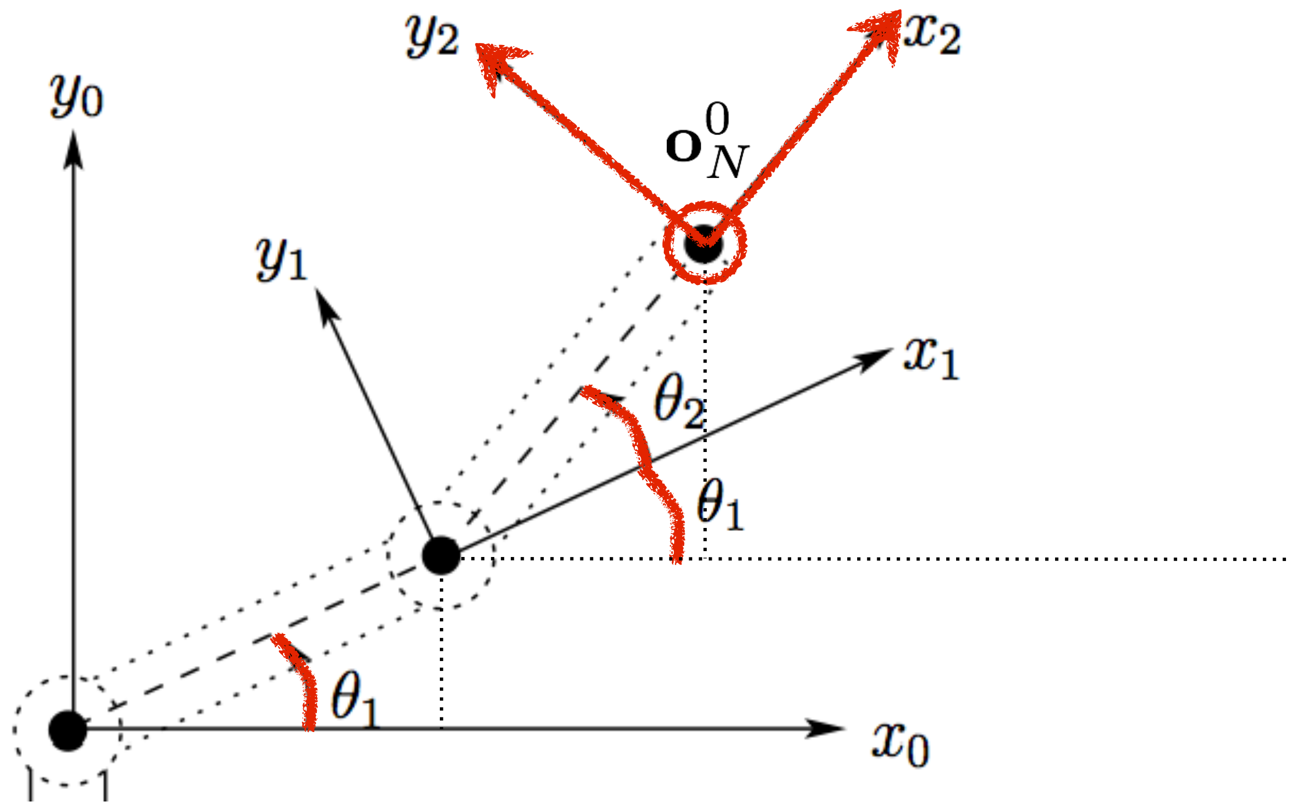
$$cos(x+y) = cos(x)cos(y) - sin(x)sin(y)$$

$$sin(x+y) = sin(x)cos(y) - cos(x)sin(y)$$

to get:

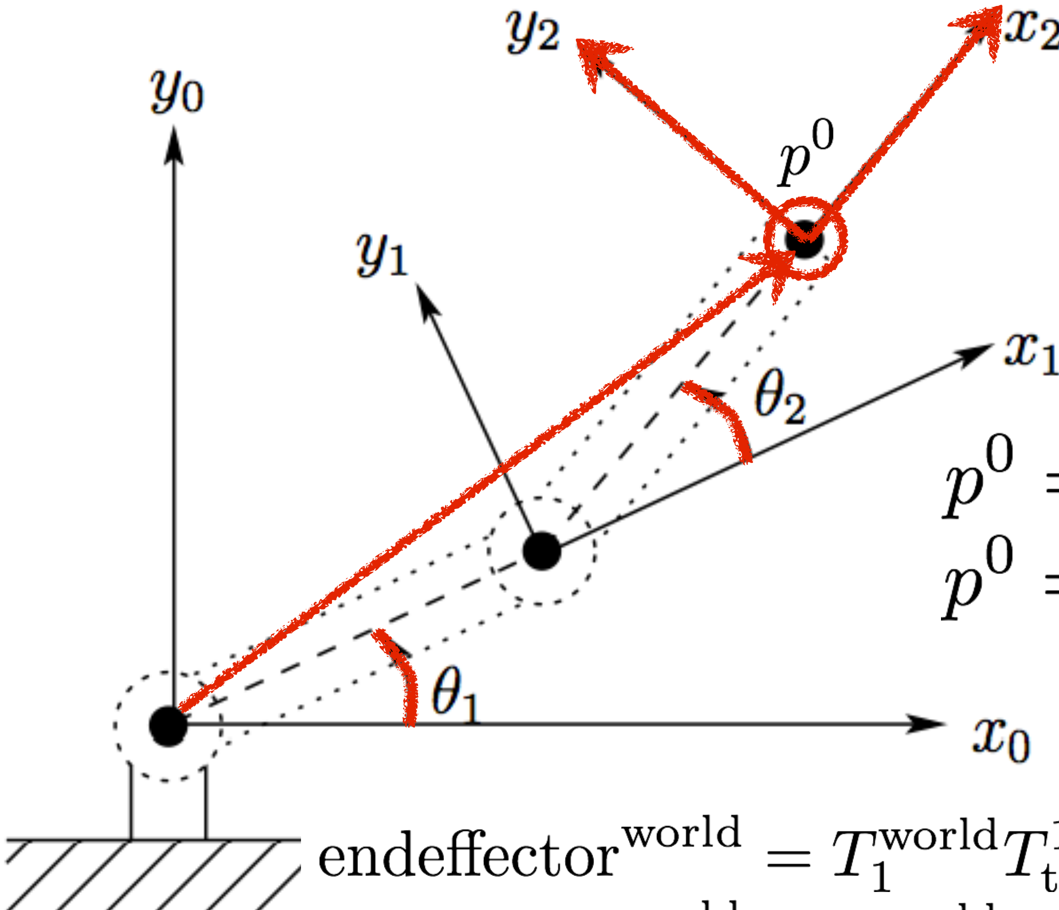
$$\mathbf{o}_N^0 = \begin{bmatrix} \text{What are the elements of this vector?} \end{bmatrix}$$

Forward kinematics: $[\mathbf{o}^0_N, \mathbf{R}^0_N] = f(\mathbf{q})$



$$\mathbf{R}^0_N = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{bmatrix} \quad \mathbf{o}^0_N = \begin{bmatrix} \alpha_1\cos\theta_1 + \alpha_2\cos(\theta_1+\theta_2) \\ \alpha_1\sin\theta_1 + \alpha_2\sin(\theta_1+\theta_2) \end{bmatrix}$$

Forward kinematics: $[\mathbf{o}^0_N, \mathbf{R}^0_N] = f(\mathbf{q})$

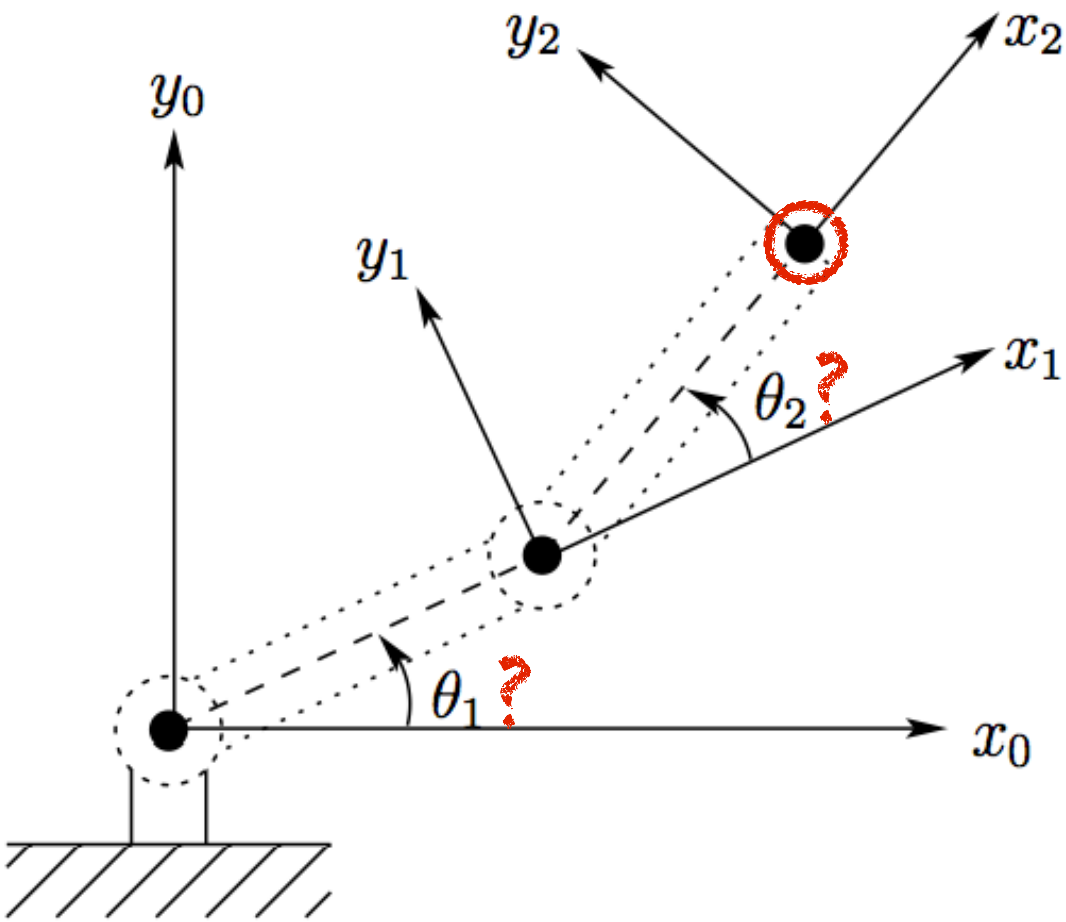


$p^\circ = f(\theta_1, \theta_2)$

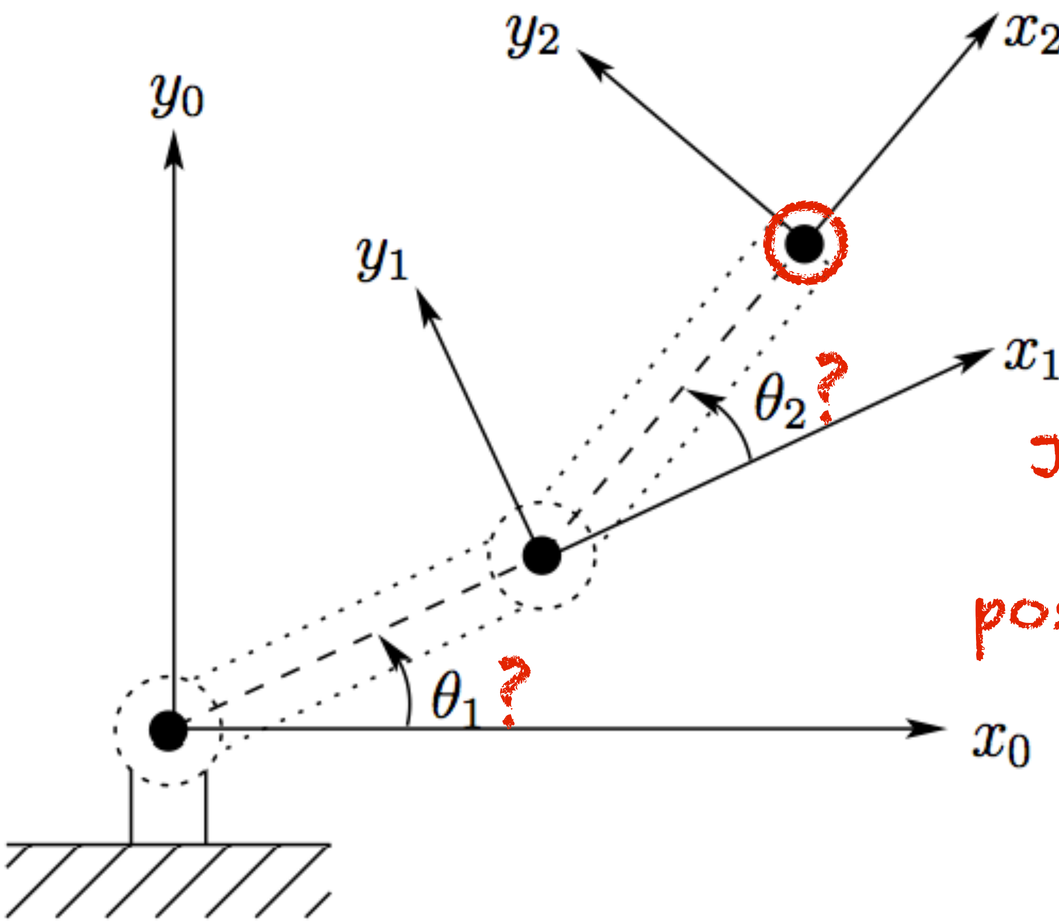
$$p^0 = T_1^0 T_2^1 p^2$$
$$p^0 = T_2^0(\theta_1, \theta_2) p^2$$

$$\text{endeffector}^{\text{world}} = T_1^{\text{world}} T_{\text{tool}}^1 \text{endeffector}^{\text{tool}}$$
$$\text{endeffector}^{\text{world}} = T_{\text{tool}}^{\text{world}}(\theta_1, \theta_2) \text{endeffector}^{\text{tool}}$$

Inverse kinematics: “given endeffector, compute configuration”



Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(p^0)$

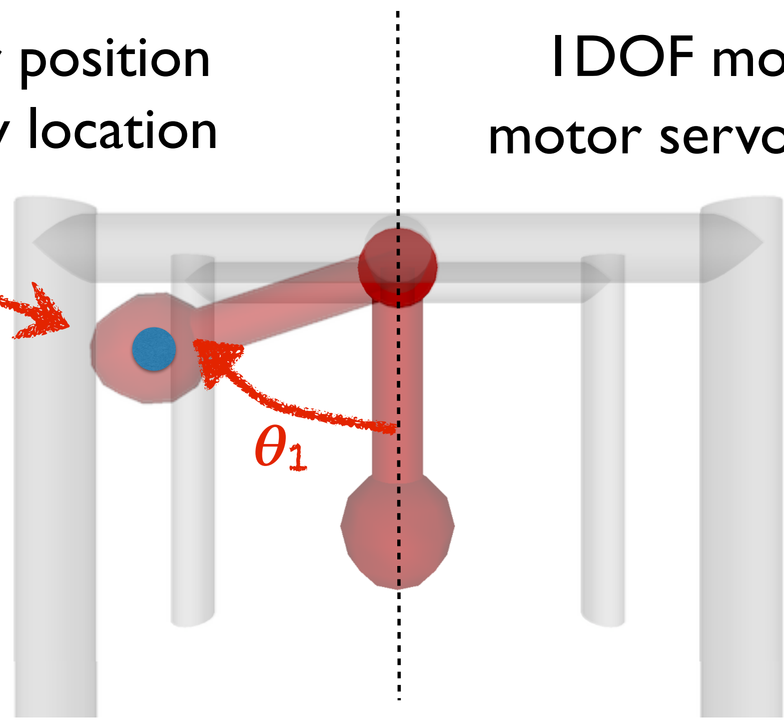


Just consider endeffector position for now

1DOF pendulum example

desired endeffector position ($\mathbf{o}^0_N$) given as an x,y location

assume:
1DOF motor at pendulum axis,
motor servo moves arm to angle θ_1



what is θ_1?

[redacted]

1DOF pendulum example

desired endeffector position ($\mathbf{o}^0_N$) given as an x,y location

assume:
1DOF motor at pendulum axis,
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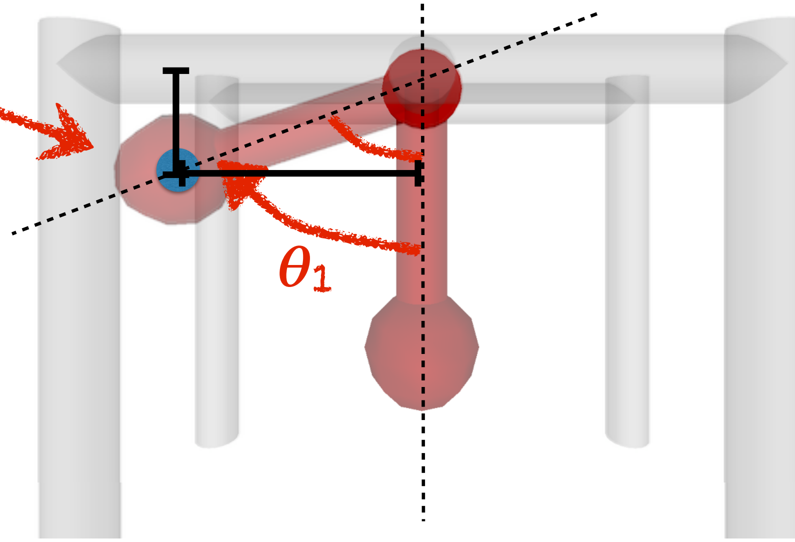
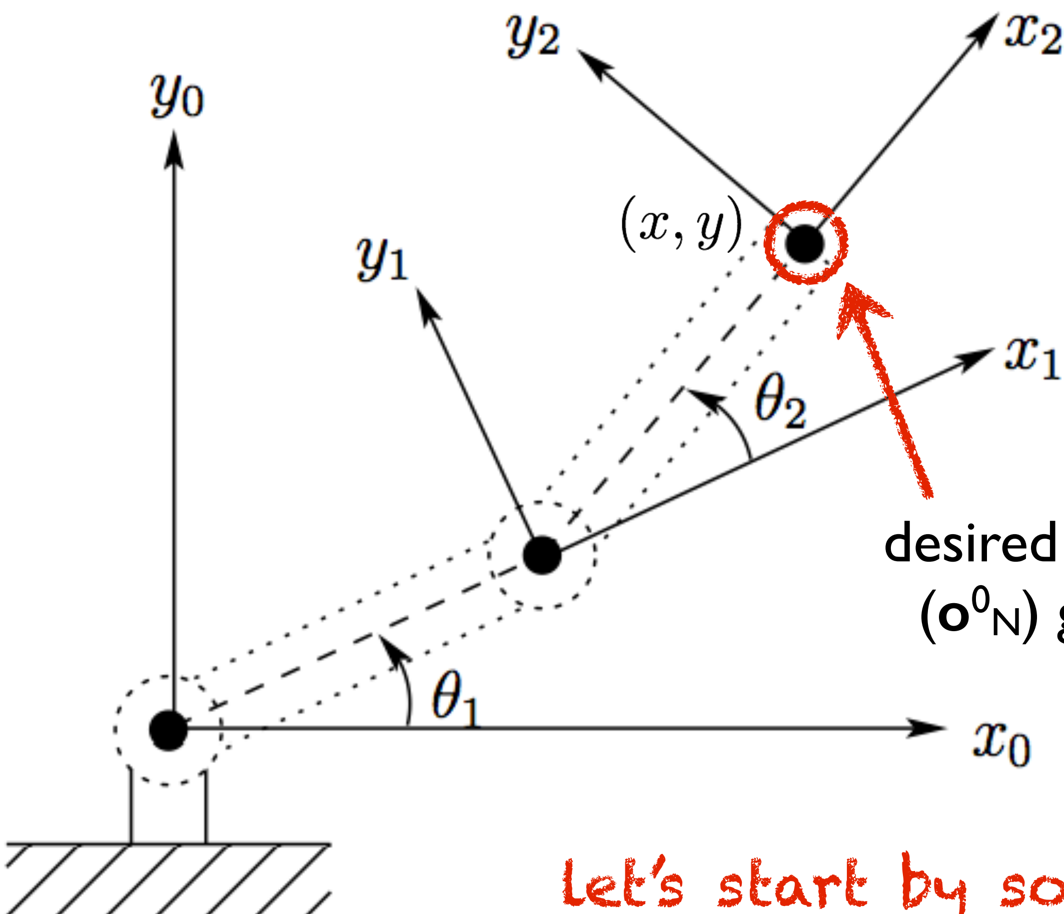


what is θ_1?

$$\theta_1 = \tan^{-1}(y/x)$$

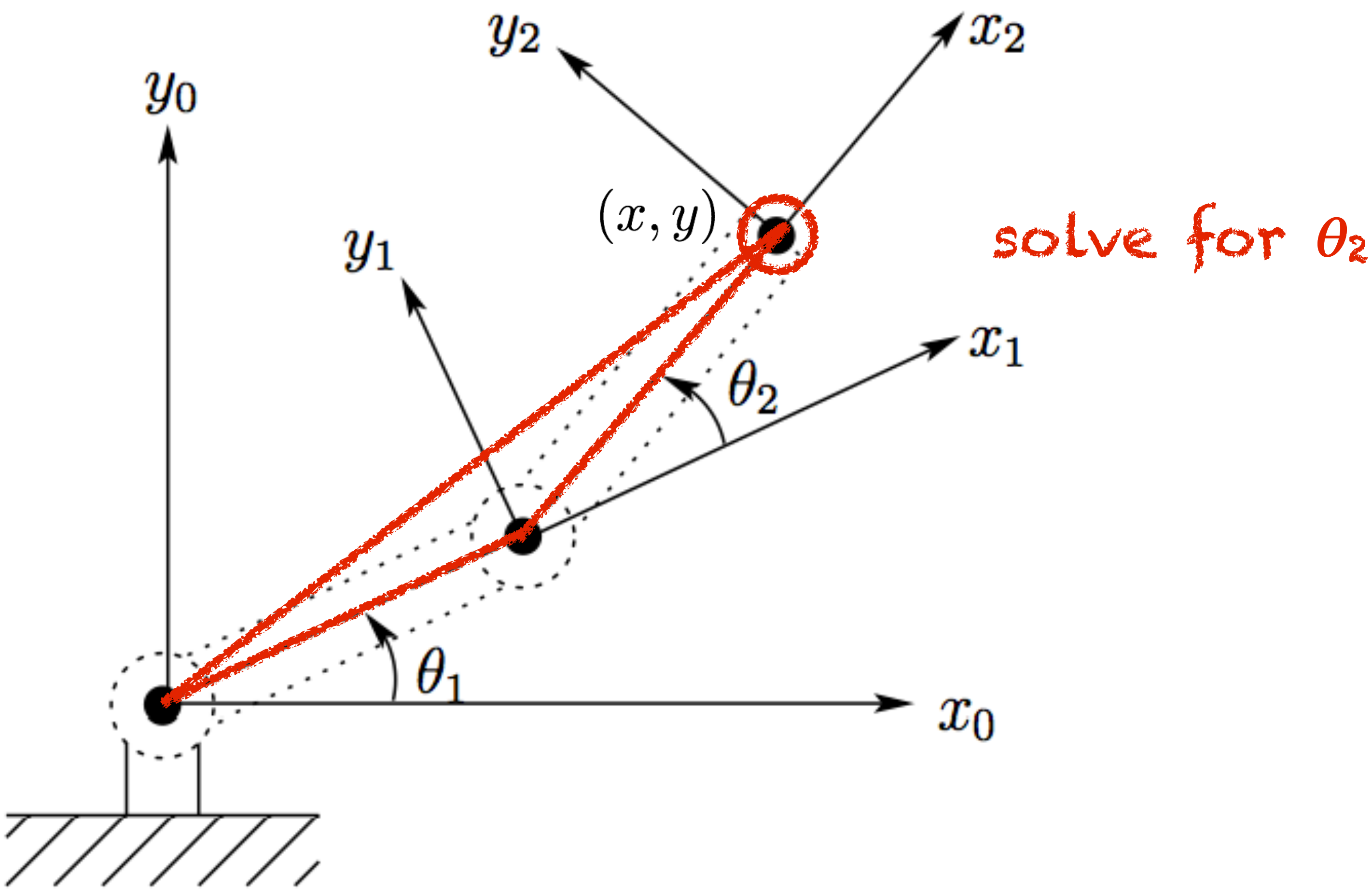
Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(x, y)$



desired endeffector position ($\mathbf{o}^0_N$) given as location *x*,*y*

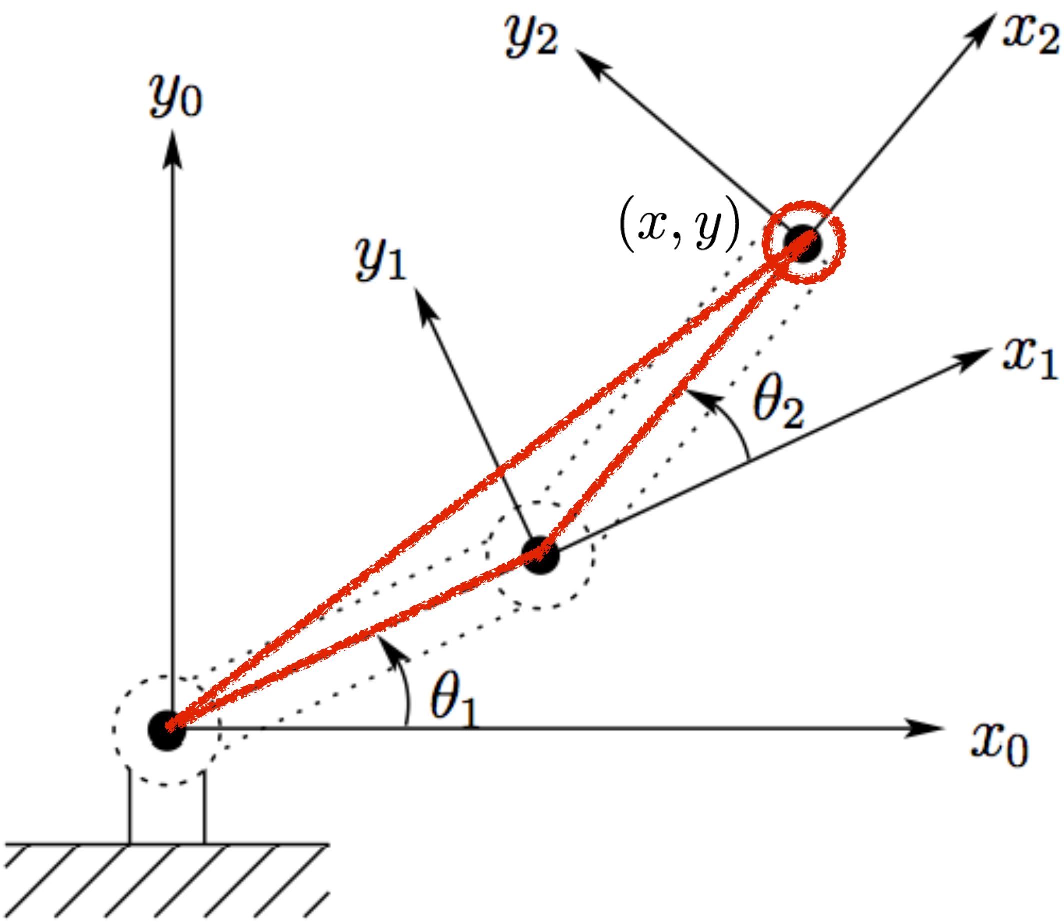
let's start by solving for θ_2

Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(x,y)$



Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$

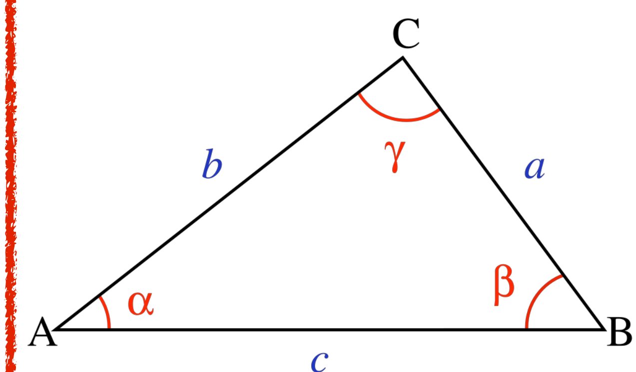
$[\theta_1, \theta_2] = f^{-1}(x, y)$



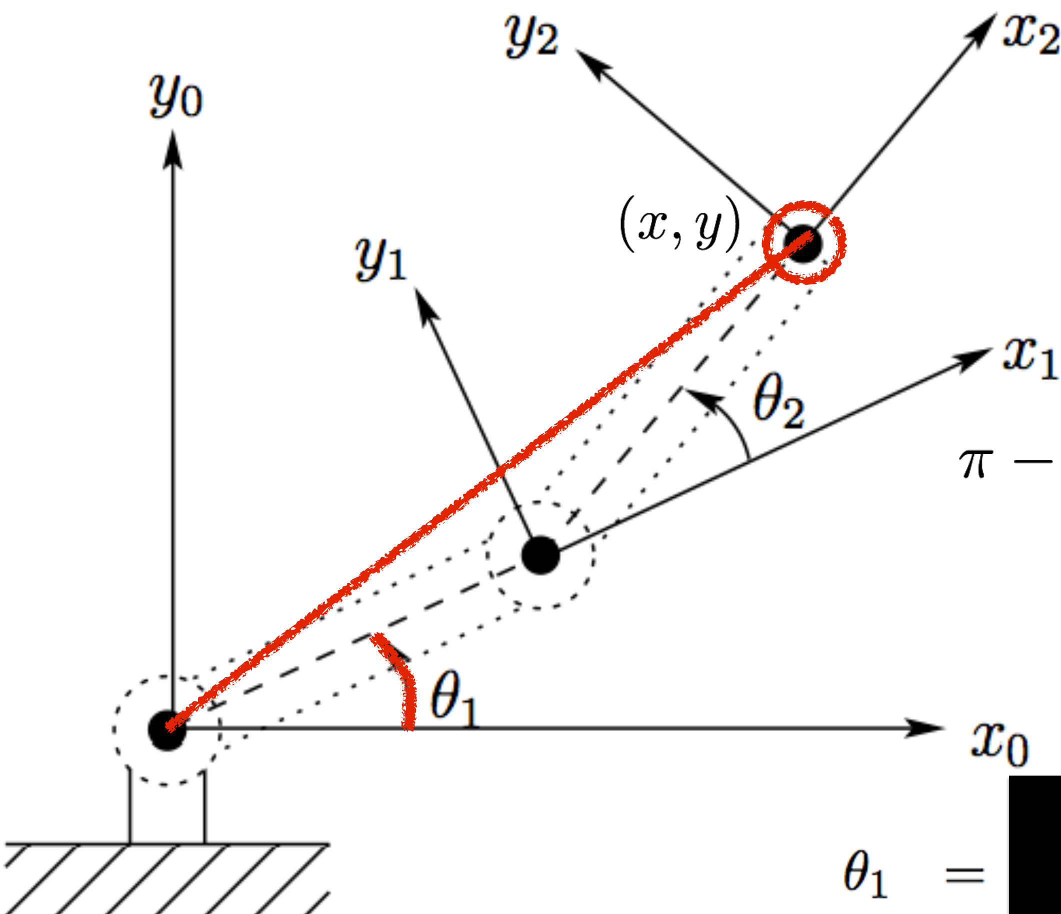
solve for θ_2

Law of Cosines

$$\gamma = \arccos\left(\frac{a^2 + b^2 - c^2}{2ab}\right)$$



Inverse kinematics: $\mathbf{q} = f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(x, y)$



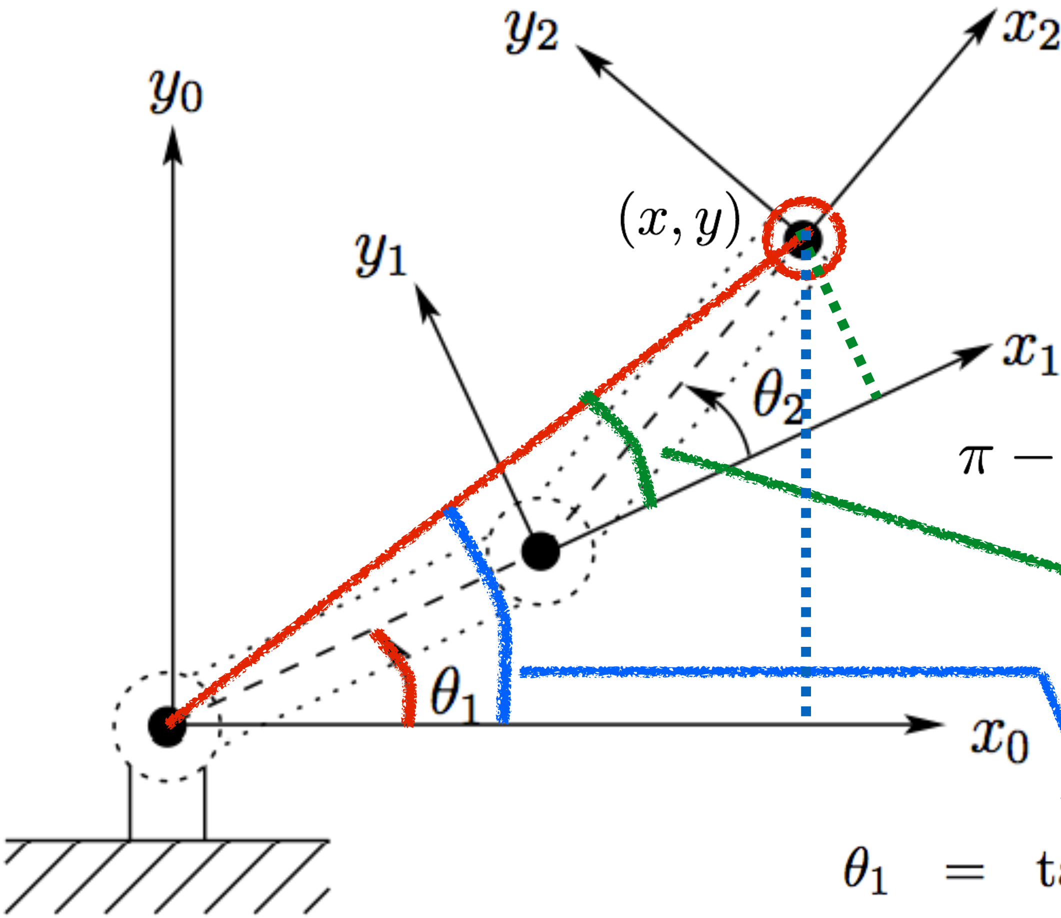
solve for θ_2

$$\pi - \theta_2 = cos^{-1}\left(\frac{\sqrt{x^2 + y^2} - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2}\right)$$

solve for θ_1

$\theta_1 =$ Consider two triangles

Inverse kinematics: **q** = $f^{-1}([\mathbf{o}^0_N, \mathbf{R}^0_N])$ $[\theta_1, \theta_2] = f^{-1}(x, y)$



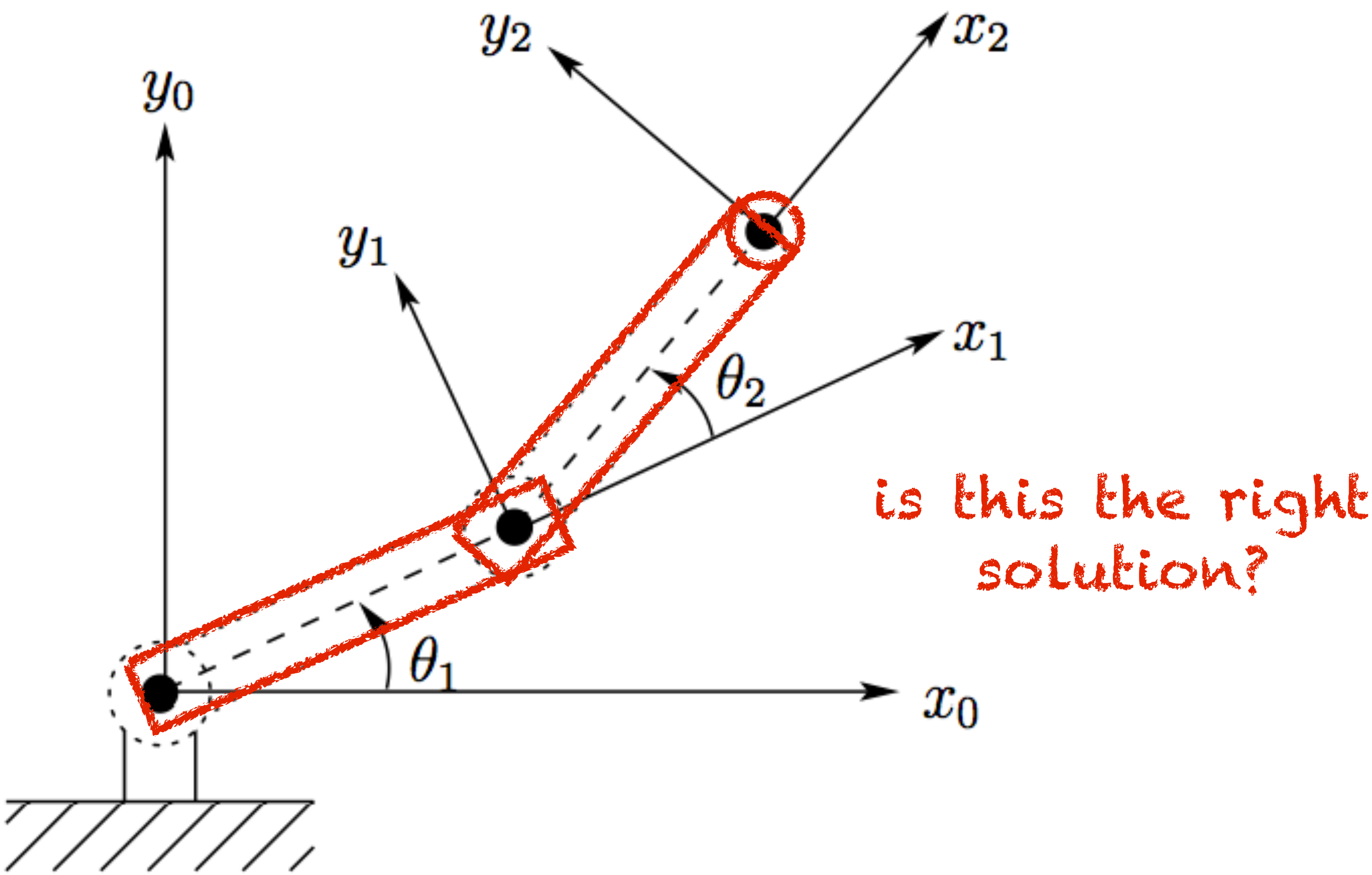
solve for θ_2

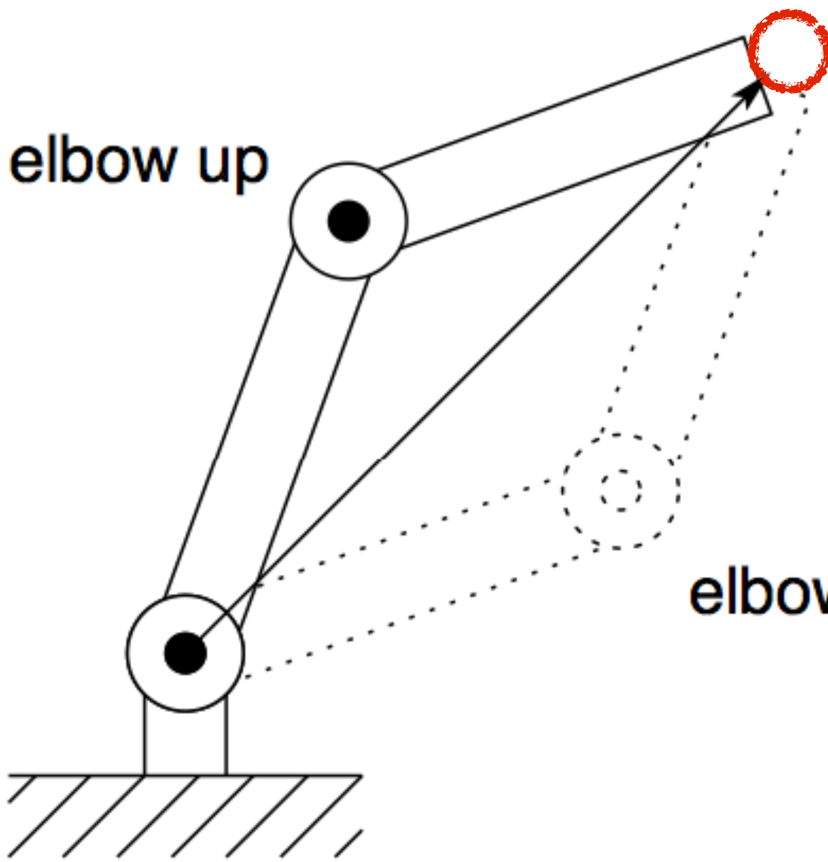
$$\pi - \theta_2 = cos^{-1}\left(\frac{\sqrt{x^2+y^2} - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2}\right)$$

solve for θ_1

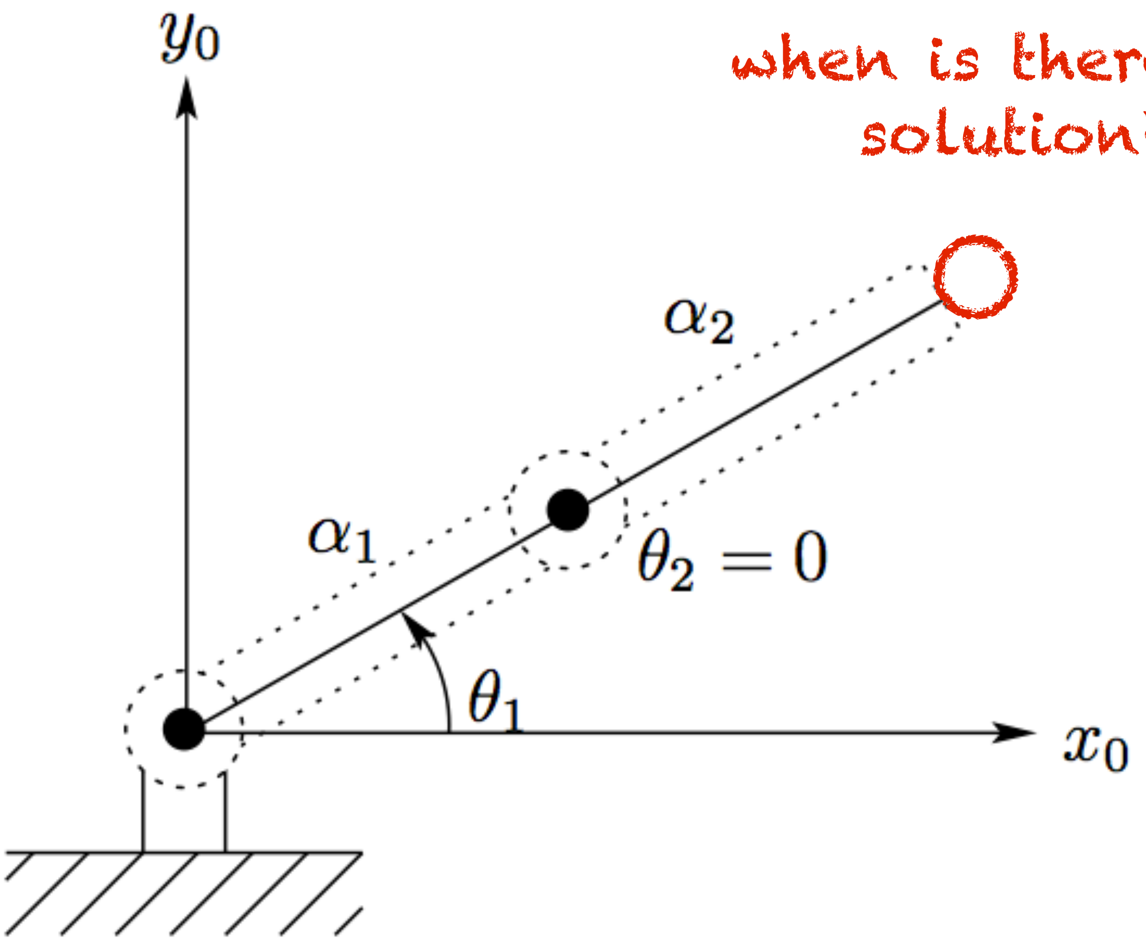
$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2 \cos\theta_2}\right)$$

inverse kinematics: $(\theta_1, \theta_2) = f^{-1}(x,y)$

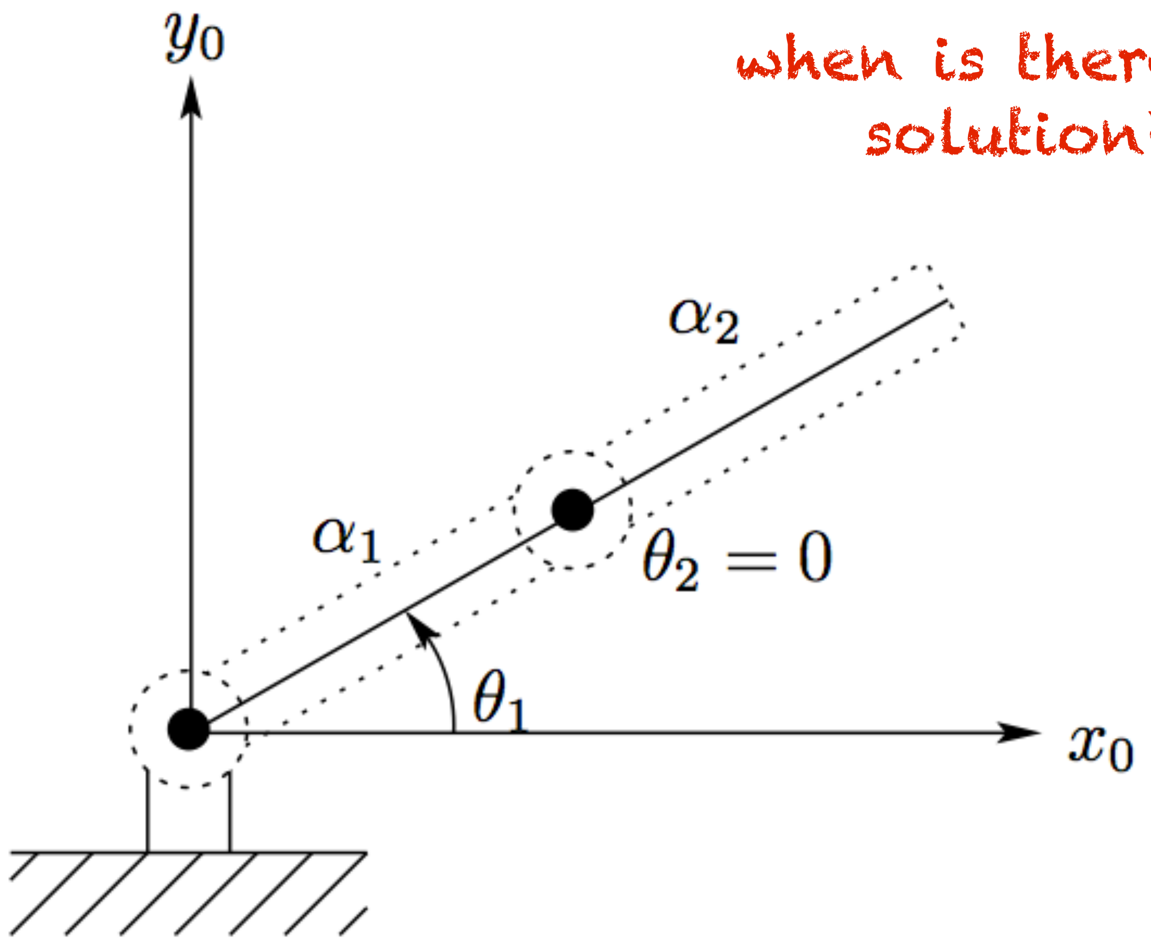




when is there one solution?



when is there no solution?

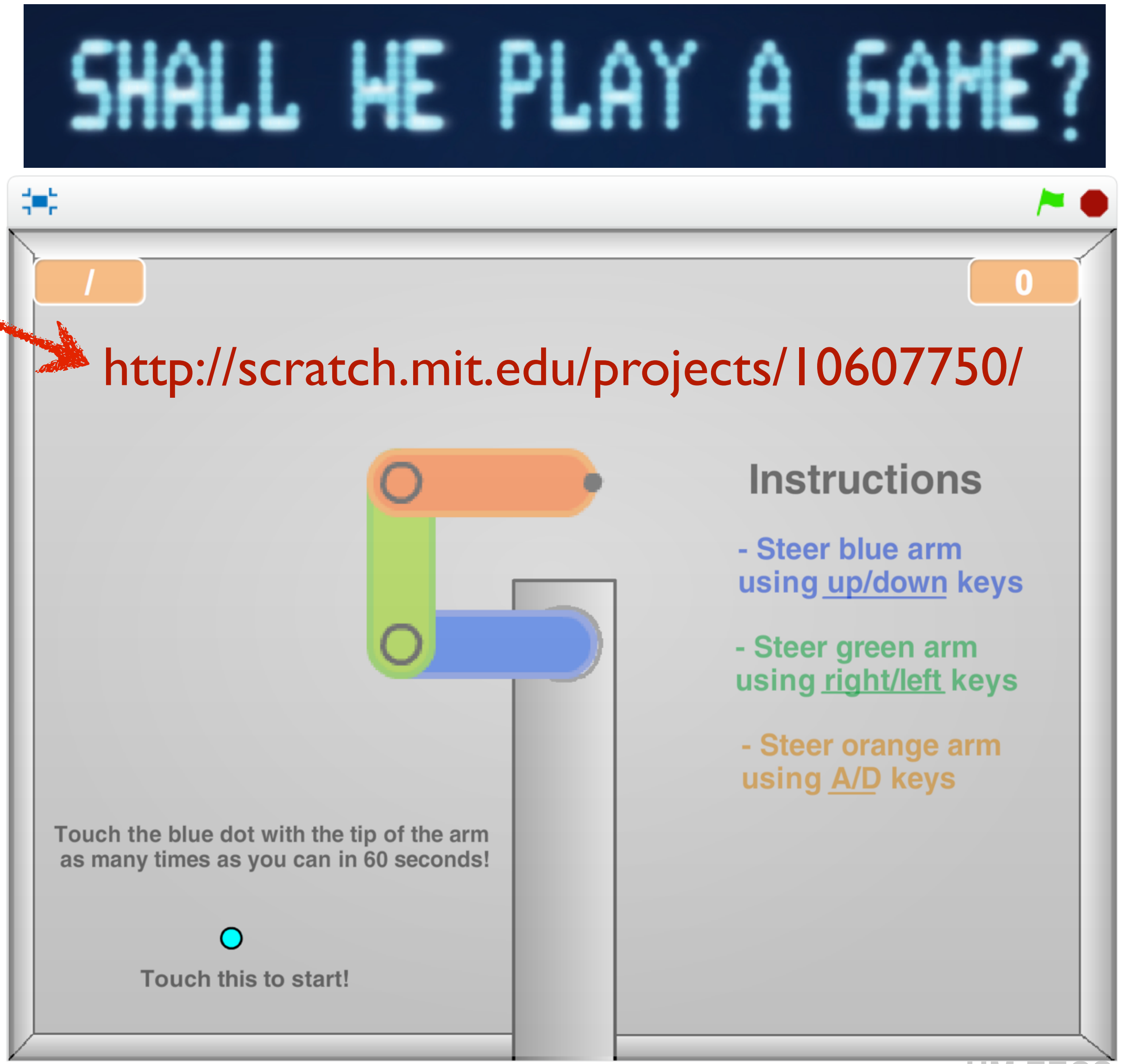


when is there no solution?

Can we do IK for 3 links?

Try this

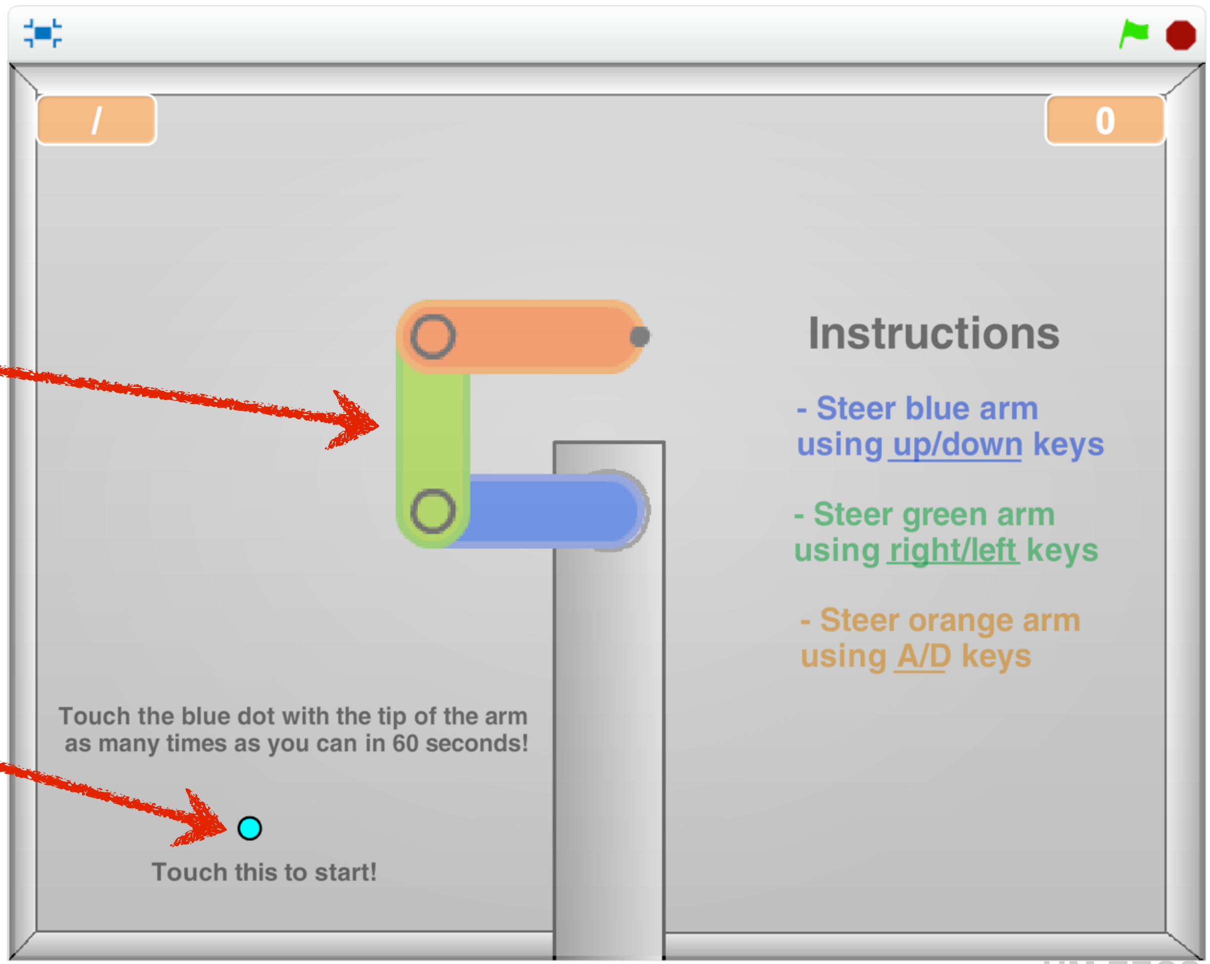
http://scratch.mit.edu/projects/10607750/



How many solutions for this arm?

3 unknowns

2 constraints



Remember: Ax=b

Inverse Kinematics: 2D

$$T_n^0(q_1, \ldots, q_n) = H$$

Inverse Kinematics: 2D

Configuration

$$T_n^0(q_1, \ldots, q_n) \quad = \quad H$$

Transform from endeffector frame to world frame

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \ldots \\ q_n \end{bmatrix}$$

$$H \quad = \quad \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

$$H \quad = \quad \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 2D

Configuration

$$T_n^0(q_1, \ldots, q_n) \quad = \quad H$$

Transform from endeffector frame to world frame

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \ldots \\ q_n \end{bmatrix}$$

$$H \quad = \quad \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

Inverse orientation

$$R_n^0(q_1, \ldots, q_n) = R$$

$$o_n^0(q_1, \ldots, q_n) = o$$

Inverse position

$$H \quad = \quad \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 2D

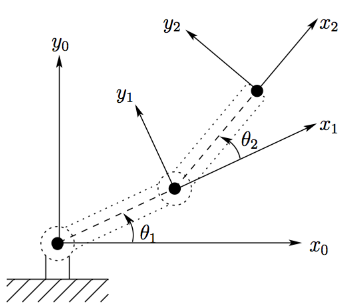
Configuration

$$T_n^0(q_1, \ldots, q_n) = H$$

Transform from endeffector

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$



Closed form solution

$$\theta_2 = cos^{-1}(\frac{x^2 + y^2 - \alpha_1^2 - \alpha_2^2}{2\alpha_1\alpha_2})$$

$$\theta_1 = \tan^{-1}(y/x) - \tan^{-1}\left(\frac{\alpha_2 \sin\theta_2}{\alpha_1 + \alpha_2 \cos\theta_2}\right)$$

$$H = \begin{bmatrix} r_{11} & r_{12} & o_x \\ r_{21} & r_{22} & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematics: 3D

Configuration

$$T_n^0(q_1, \ldots, q_n) = H$$

Transform from endeffector

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \ldots \\ q_n \end{bmatrix}$$

$$H = \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

6 DOF position and orientation of endeffector

$$\boxed{H = \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Inverse Kinematics: 3D

Configuration

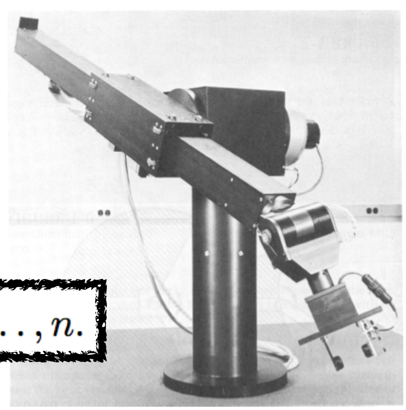
$$T_n^0(q_1, \ldots, q_n) \quad = \quad H$$

Transform from endeffector

$$q = \begin{bmatrix} q_1 \\ q_2 \\ \ldots \\ q_6 \end{bmatrix}$$

$$H \quad = \quad \begin{bmatrix} R & o \\ 0 & 1 \end{bmatrix}$$

Closed form solution?

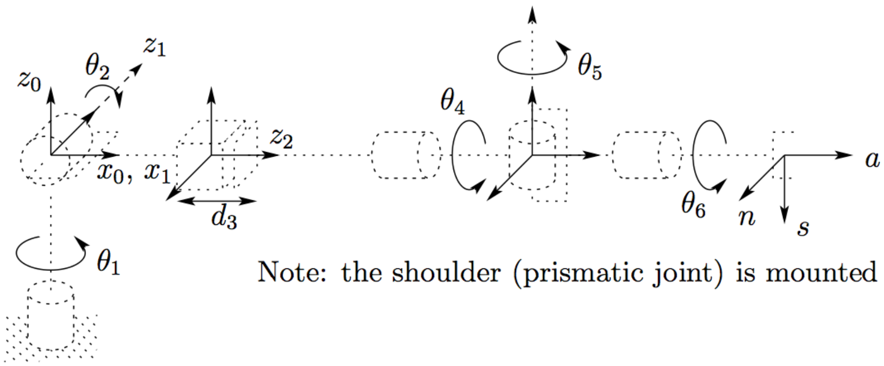


$$\boxed{q_k \quad = \quad f_k(h_{11}, \ldots, h_{34}), \qquad k = 1, \ldots, n.}$$

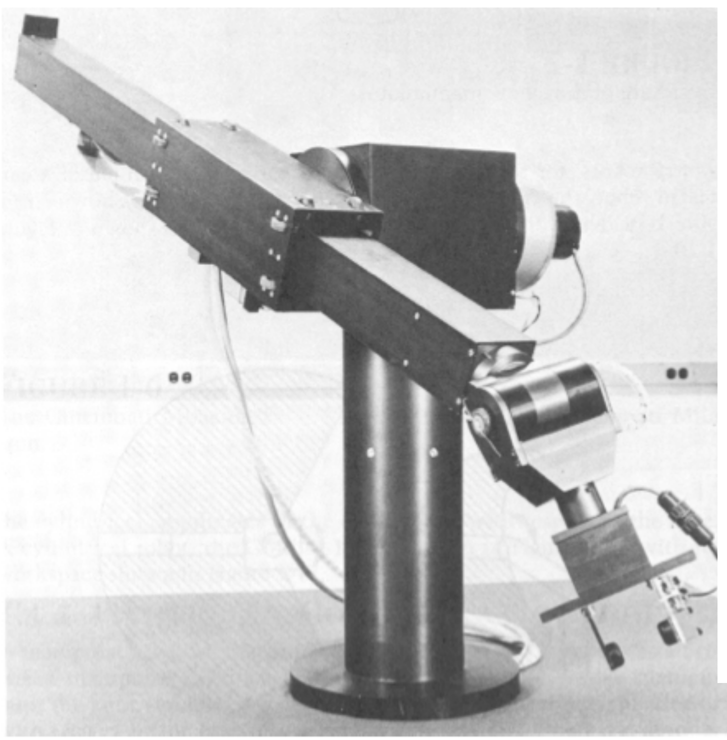
$$H \quad = \quad \begin{bmatrix} r_{11} & r_{12} & r_{13} & o_x \\ r_{21} & r_{22} & r_{23} & o_y \\ r_{31} & r_{32} & r_{33} & o_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6 DOF position and orientation of endeffector

Stanford Manipulator

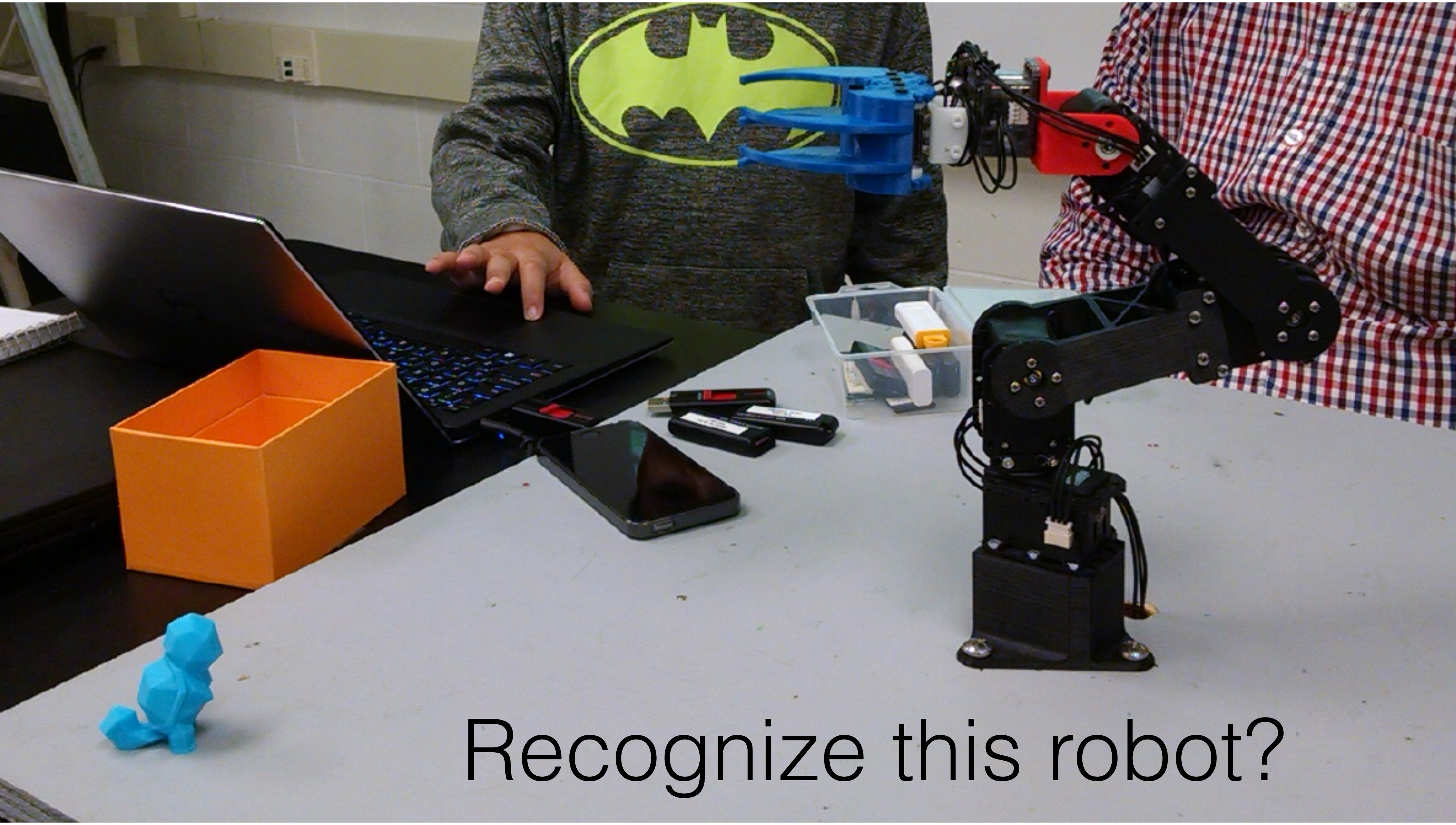


Note: the shoulder (prismatic joint) is mounted wrong.

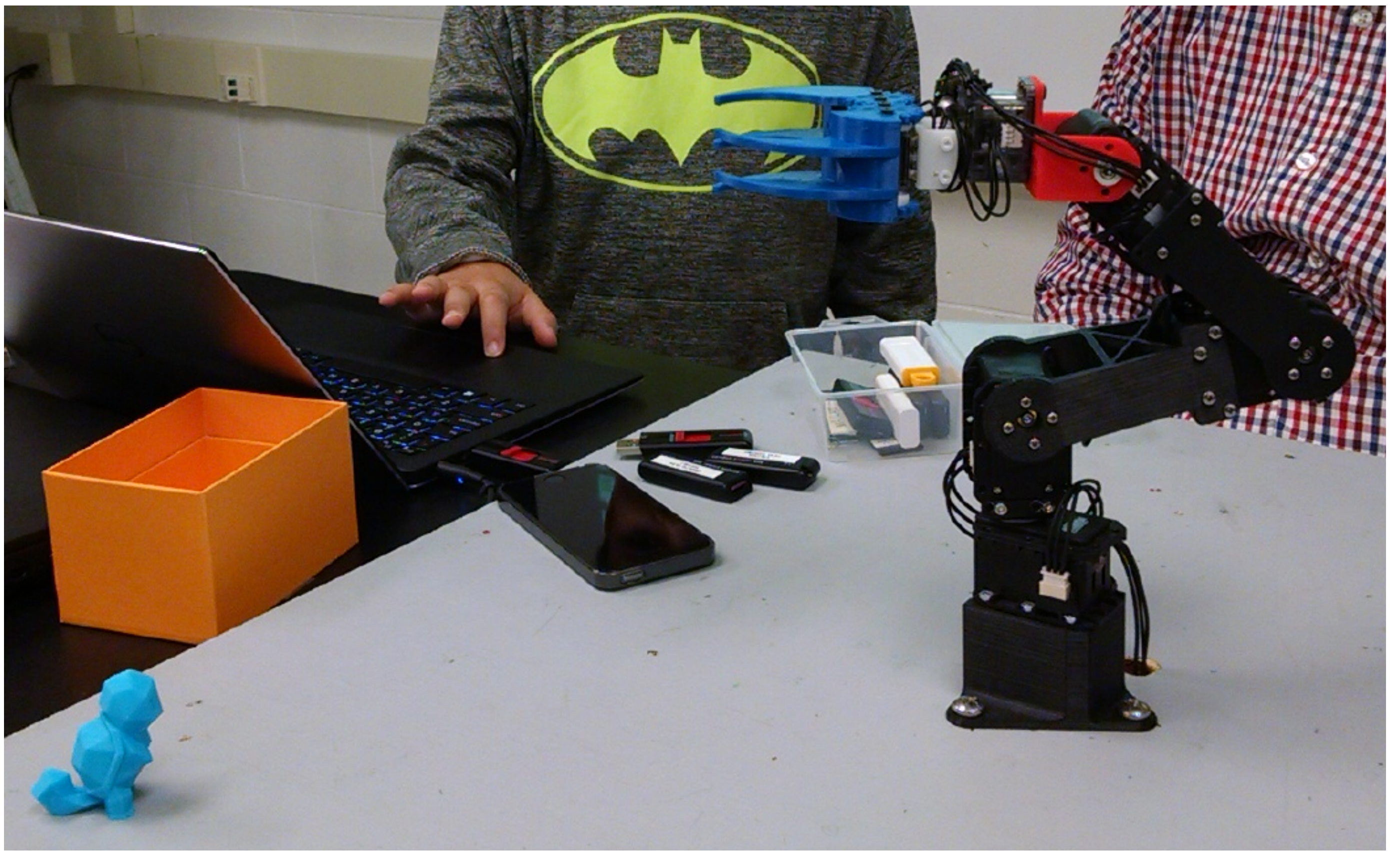


$$
\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= r_{11} \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= r_{21} \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5s_6 &= r_{31} \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= r_{12} \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= r_{22} \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= r_{32} \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= r_{13} \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= r_{23} \\
-s_2c_4s_5 + c_2c_5 &= r_{33} \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= o_x \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= o_y \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= o_z.
\end{aligned}
$$

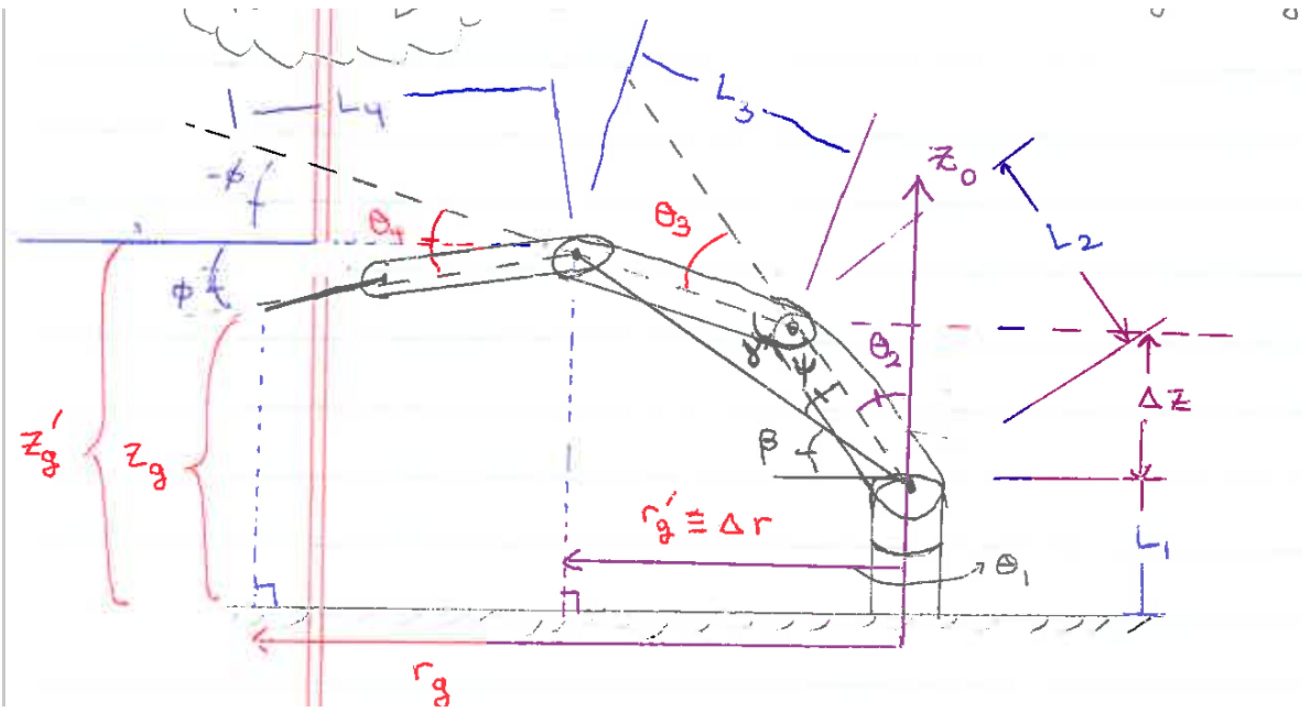
assumes D-H frames


Recognize this robot?

RexArm (EECS 467 / ROB 550)



RexArm (EECS 467 / ROB 550)



Find: configuration
$\boldsymbol{q} = [\theta_1\ \theta_2\ \theta_3\ \theta_4]$
as robot joint angles

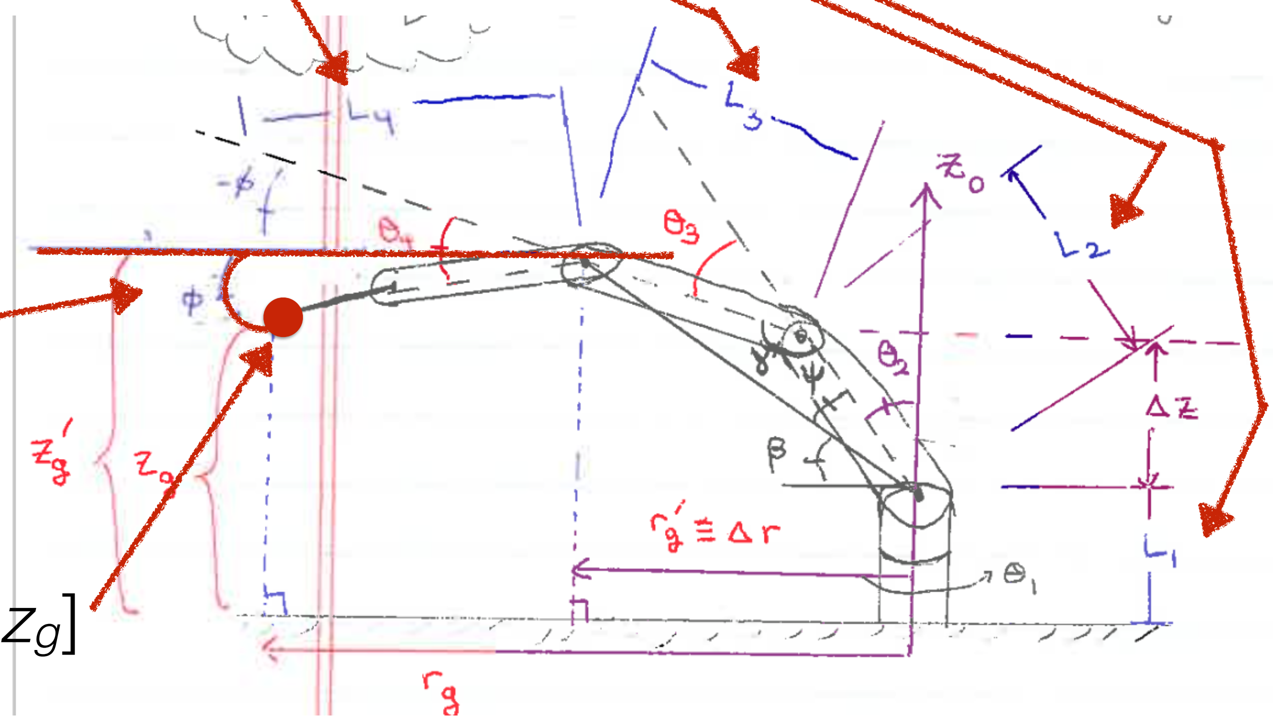
Given:

link lengths (L_4,L_3,L_2,L_1)

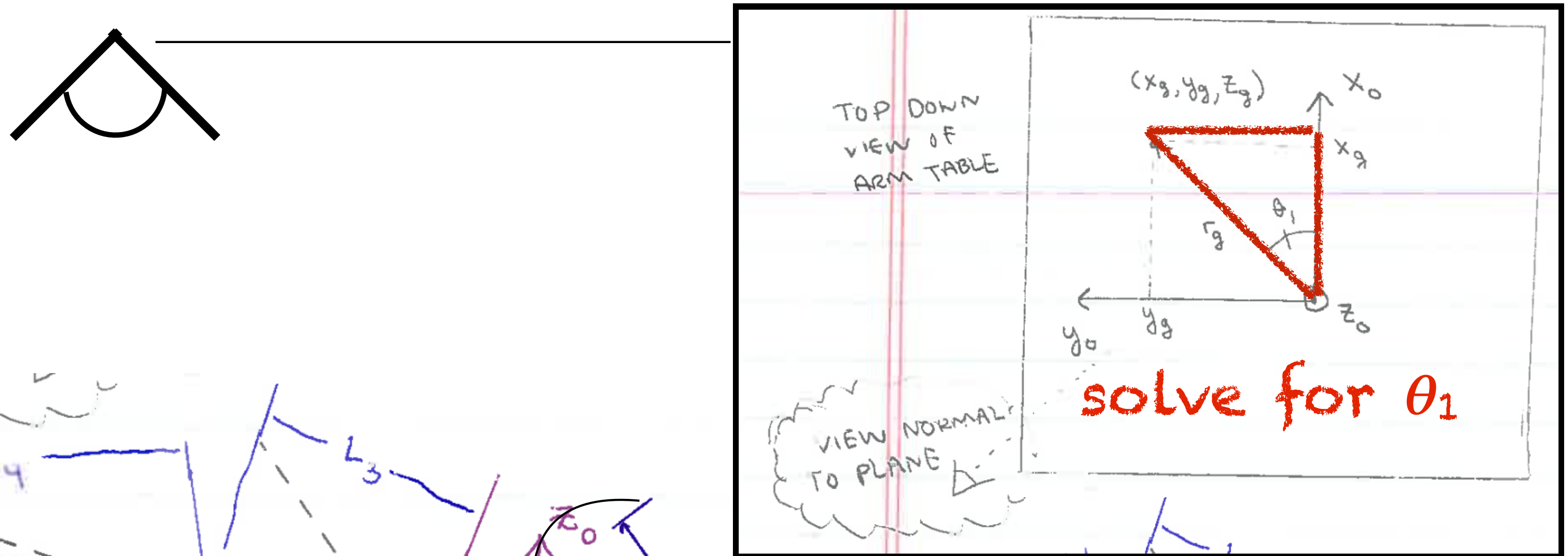
endeffector orientation ϕ as angle wrt. plane centered at $\boldsymbol{o_3}$ and parallel to ground plane

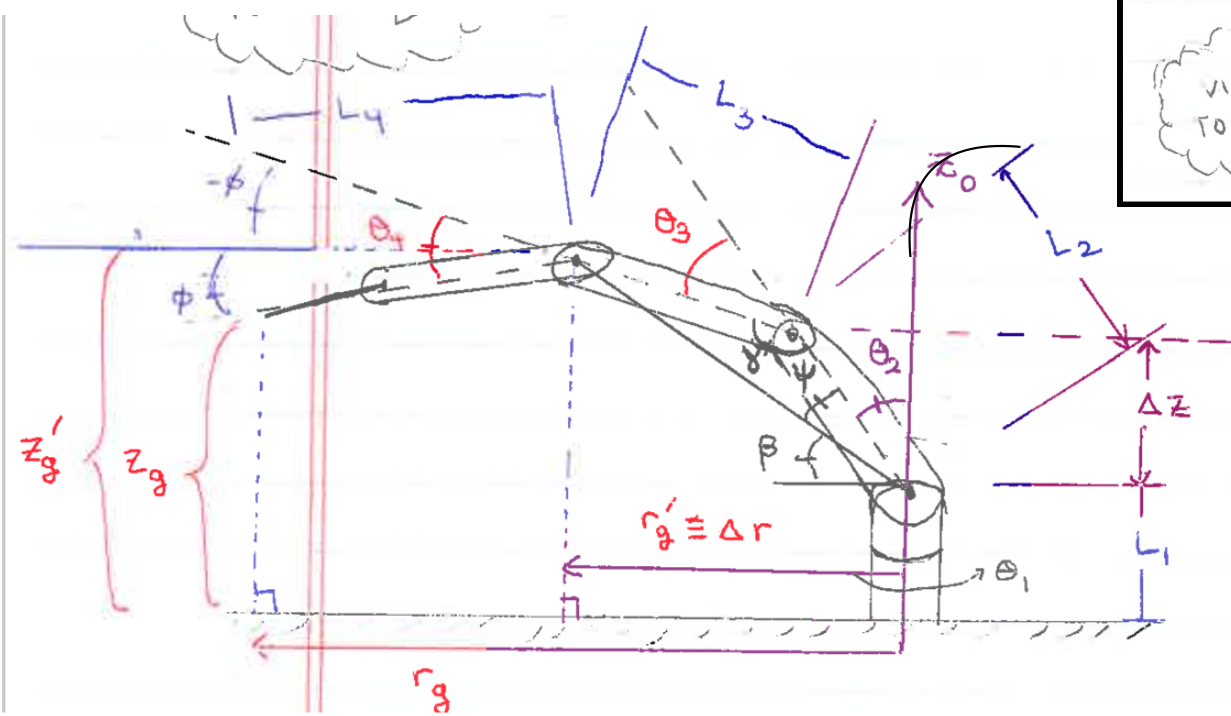
endeffector position [x_g y_g z_g] wrt. base frame

Find: configuration $\boldsymbol{q} = [\boldsymbol{\theta}_1\ \boldsymbol{\theta}_2\ \boldsymbol{\theta}_3\ \boldsymbol{\theta}_4]$ as robot joint angles



overhead view





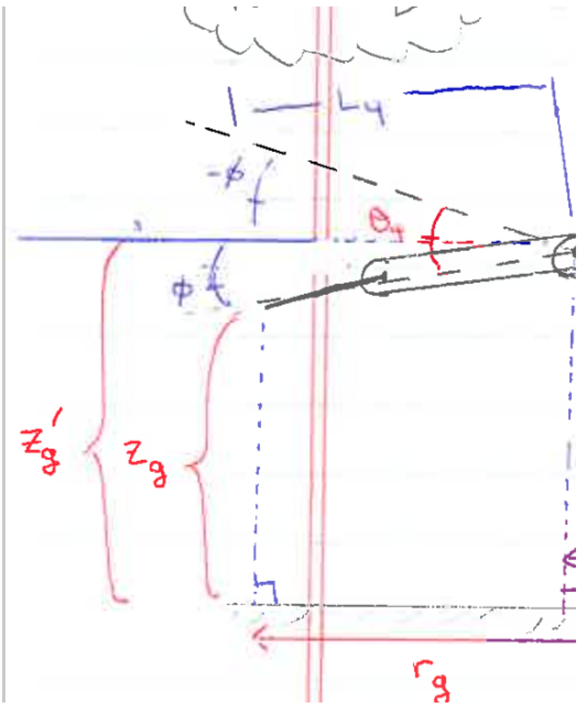
$$\boxed{\theta_1 = \text{atan2}(y_g, x_g)}$$

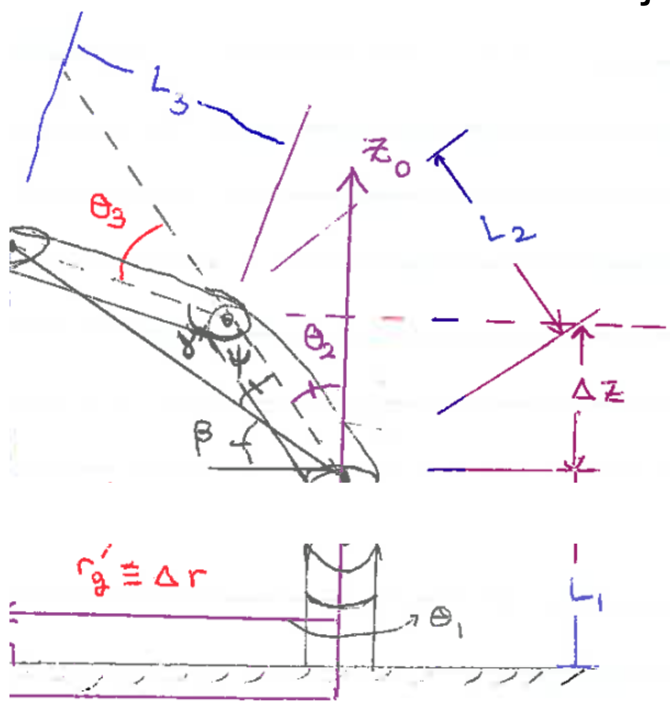
solve for θ_1

$$\boxed{\theta_1 = \text{atan2}(y_g, x_g)}$$

Decoupling:
separate endeffector from rest of the robot at last joint

solve for θ_3



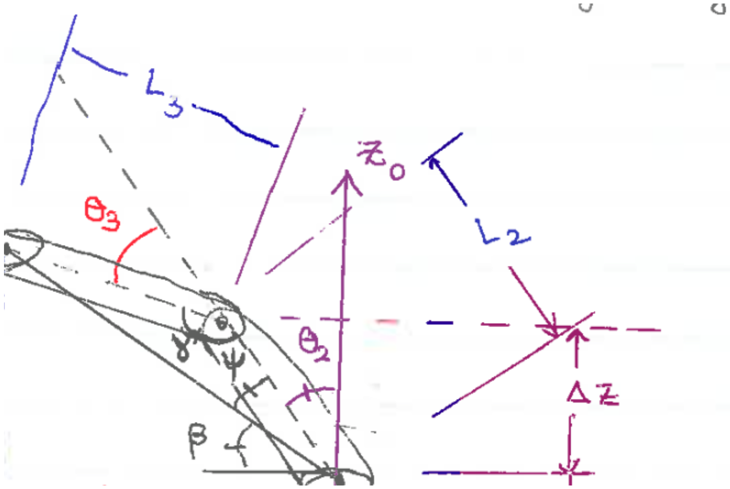


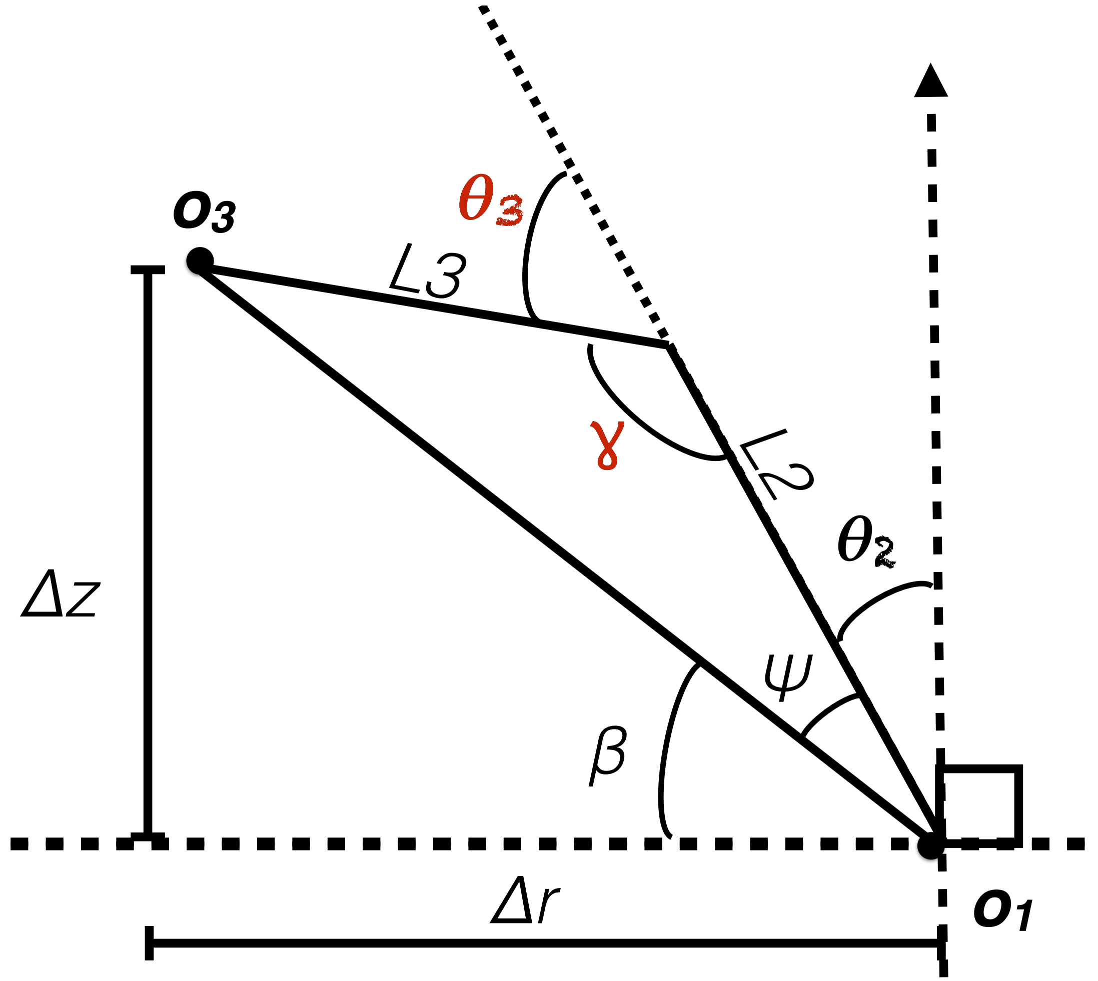
and joint 1 from rest of robot

solve for θ_1

$$\boxed{\theta_1 = \text{atan2}(y_g, x_g)}$$

solve for θ_3





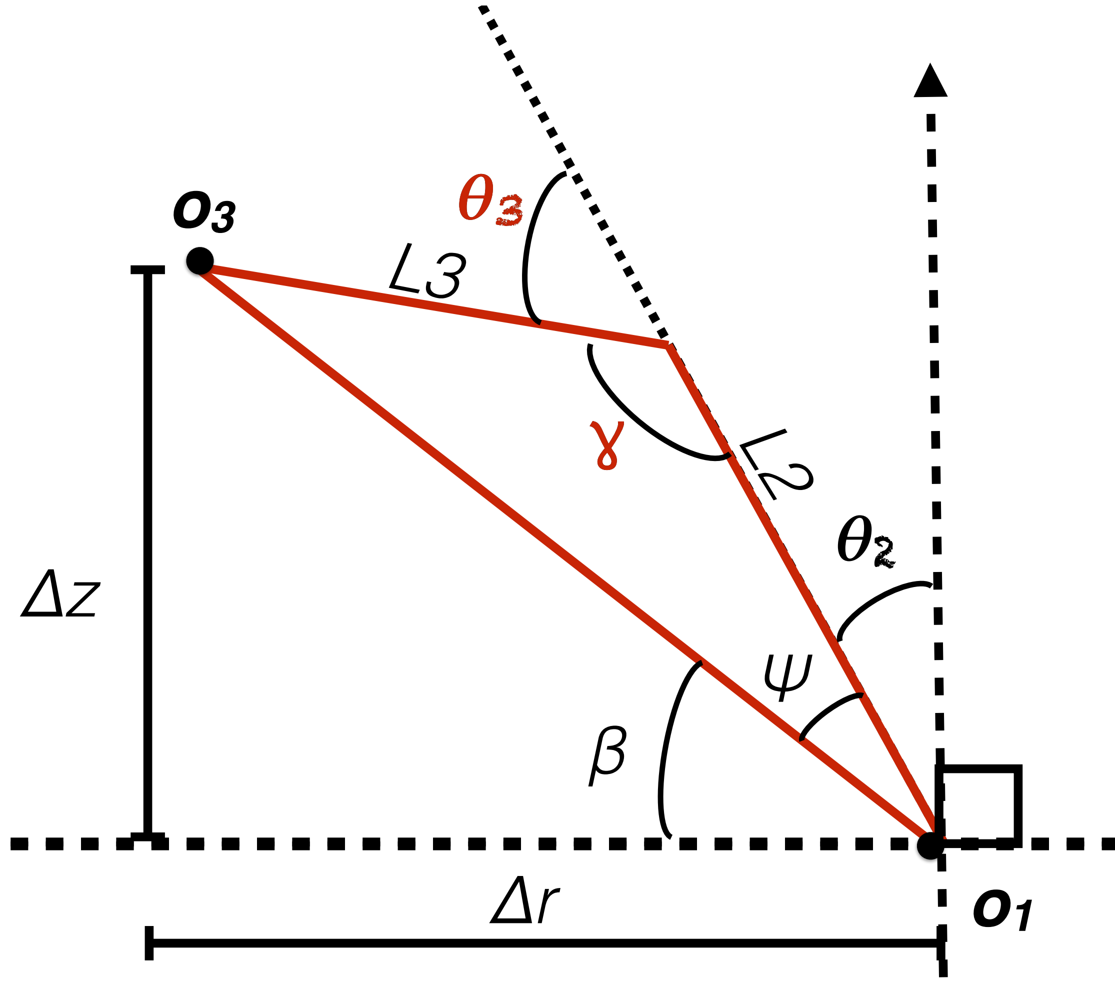
solve for θ_1

$$\theta_1 = \mathrm{atan2}(y_g, x_g)$$

solve for θ_3

(Law of cosines with supplementary angle γ)

$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$



solve for θ_1

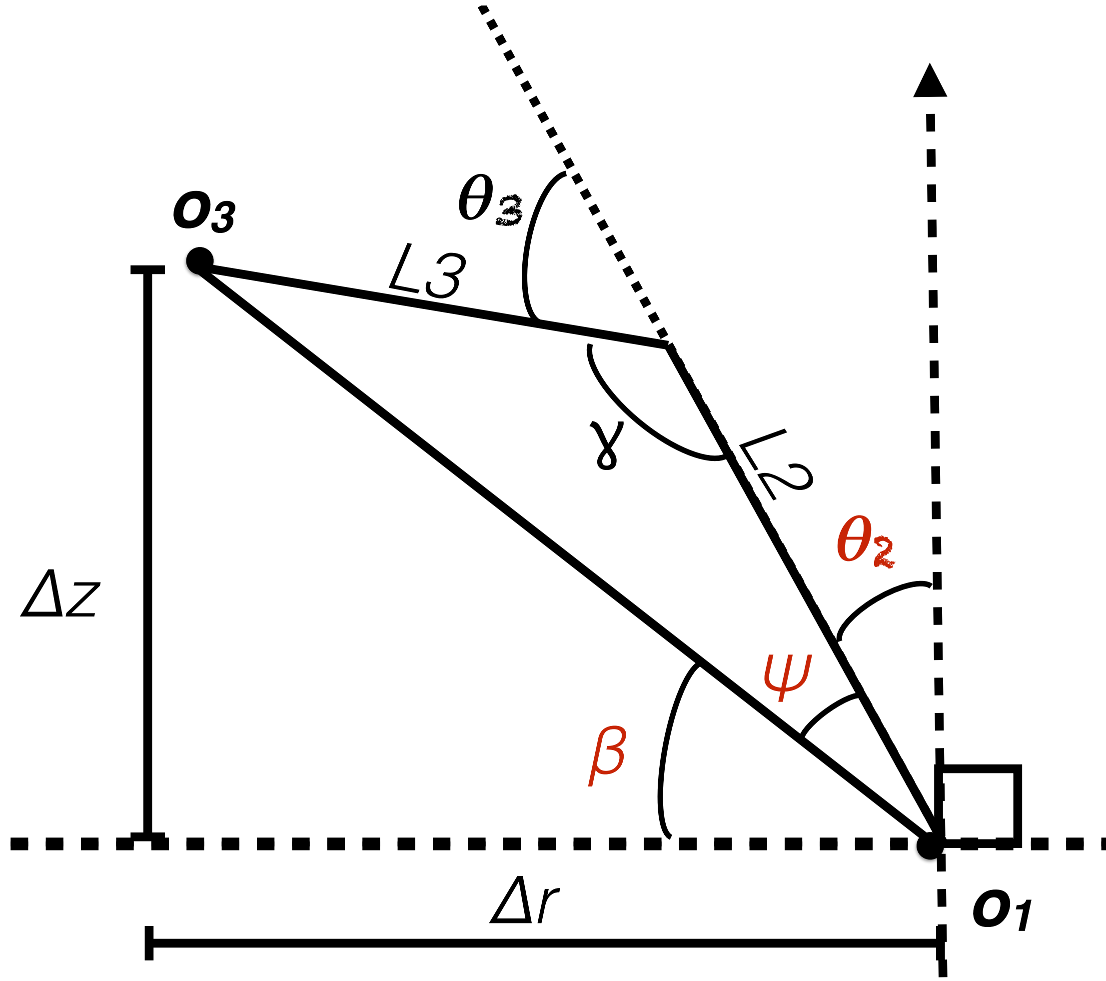
$$\theta_1 = \mathrm{atan2}(y_g, x_g)$$

solve for θ_3

$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

(Law of cosines with angle Ψ, arctan with angle β)



solve for θ_1

$$\theta_1 = \operatorname{atan2}(y_g, x_g)$$

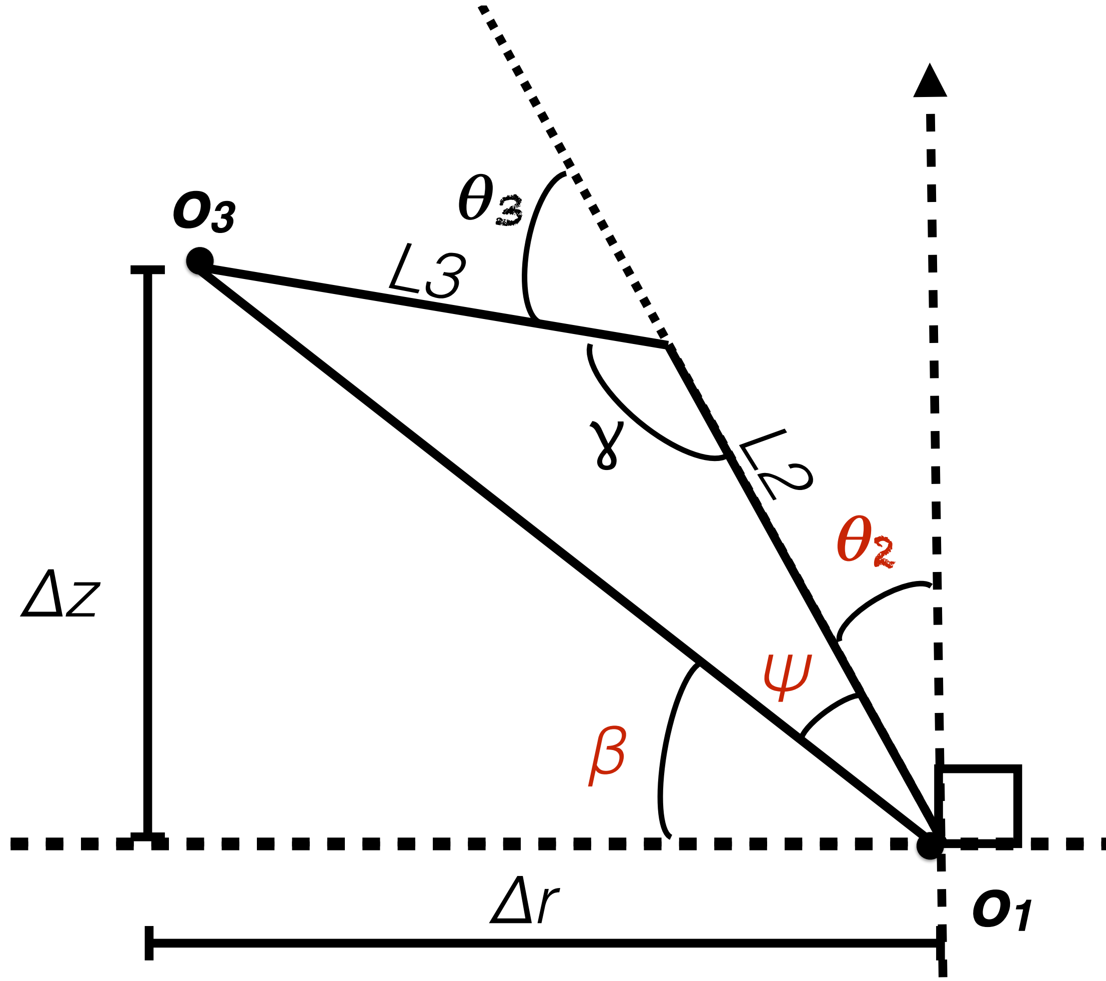
solve for θ_3

$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 > 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

two potential solutions depending on elbow angle



solve for θ_1

$$\theta_1 = \operatorname{atan2}(y_g, x_g)$$

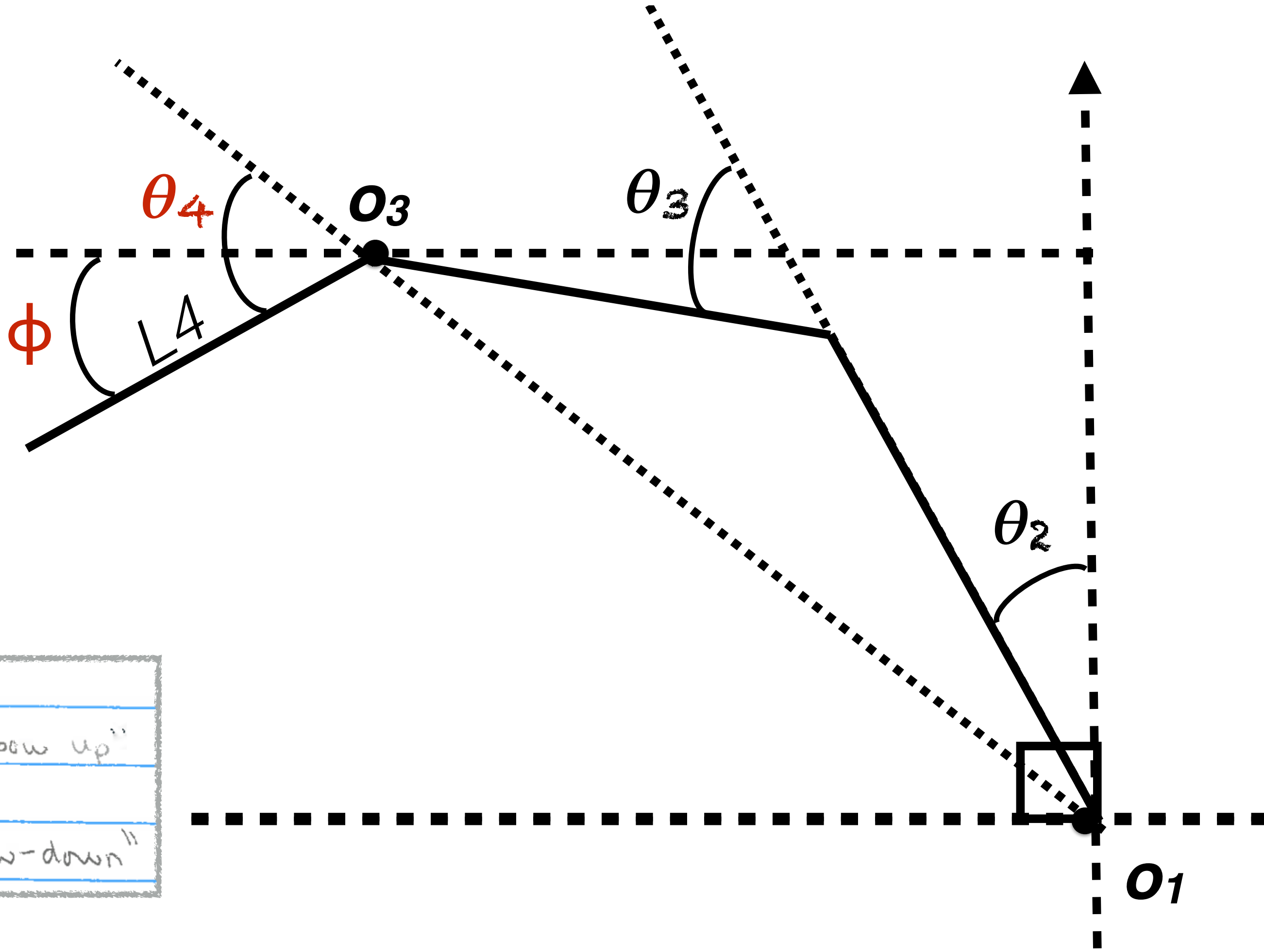
solve for θ_3

$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 > 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4



solve for θ_1

$$\theta_1 = \mathrm{atan2}(y_g, x_g)$$

solve for θ_3

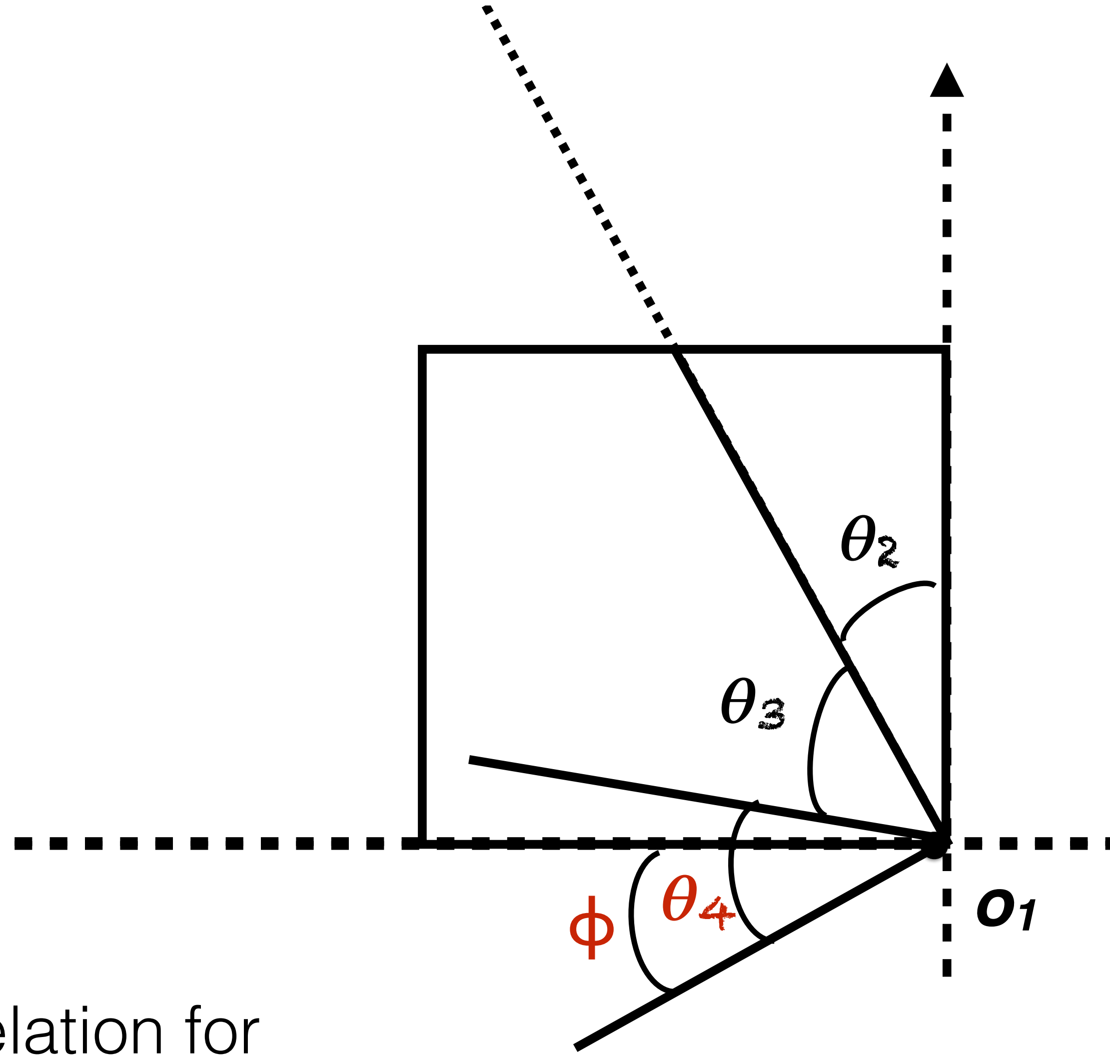
$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 > 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4

(Equilvalence relation for adding angles from $\boldsymbol{z_0}$)



solve for θ_1

$$\theta_1 = \operatorname{atan2}(y_g, x_g)$$

solve for θ_3

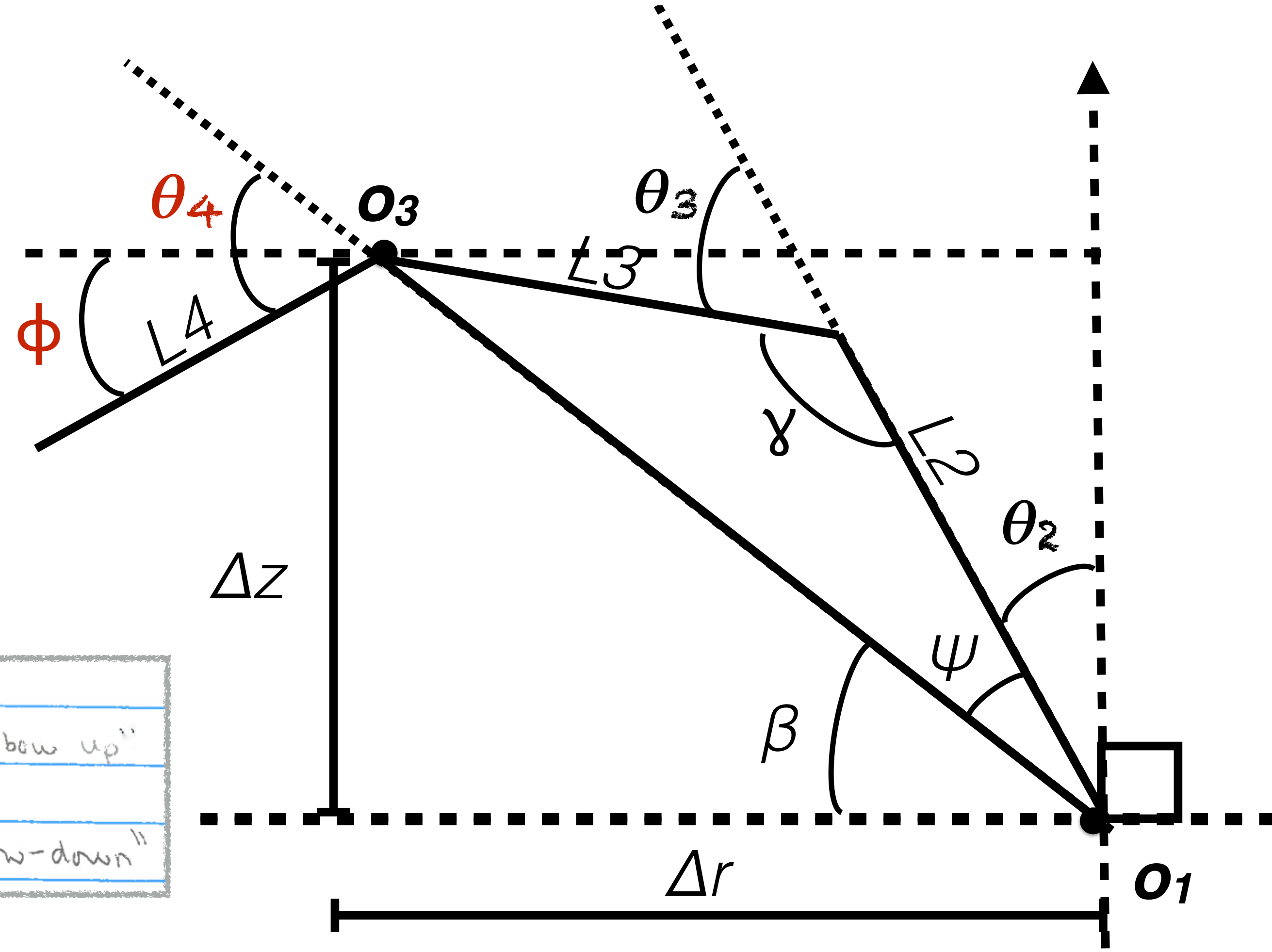
$$\cos\theta_3 = \frac{\Delta z^2 + \Delta r^2 - L_2^2 - L_3^2}{2 L_2 L_3}$$

solve for θ_2

$$\theta_2 = \begin{cases} \frac{\pi}{2} - \beta - \psi & \text{if } \theta_3 \geq 0 \quad \text{"Elbow up"} \\ \frac{\pi}{2} - \beta + \psi & \text{if } \theta_3 < 0 \quad \text{"Elbow-down"} \end{cases}$$

solve for θ_4

$$\theta_4 = \phi - \theta_2 - \theta_3 + \frac{\pi}{2}$$



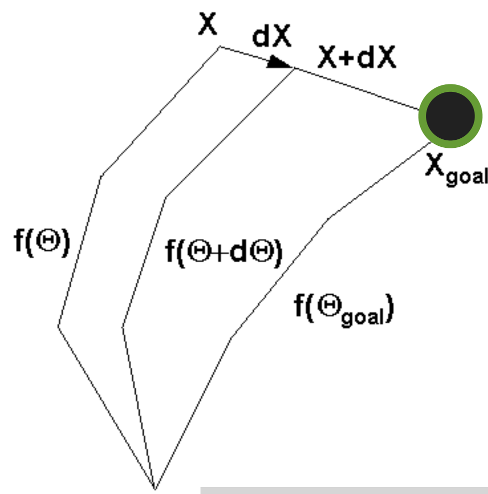
(Addition of angles in arm plane starting from ***z*₀**)

Why Closed Form?

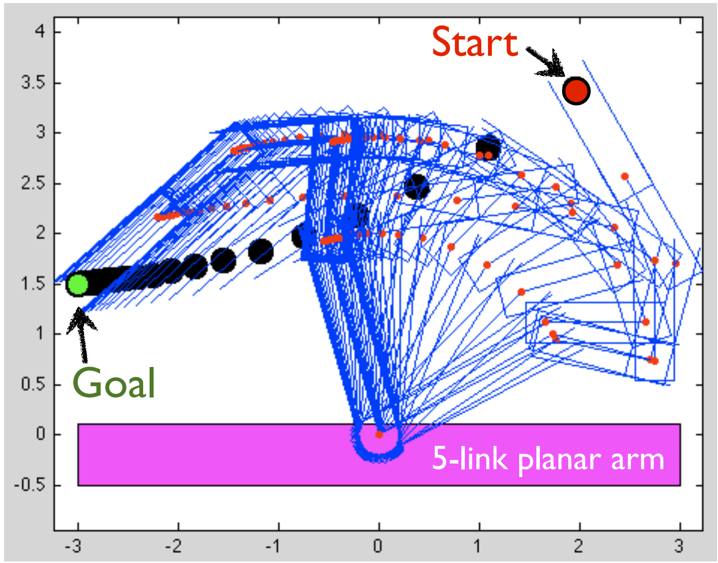
- Advantages
  - Speed: IK solution computed in constant time
  - Predictability: consistency in selecting satisfying IK solution

- Disadvantage
  - Generality: general form for arbitrary kinematics difficult to express

Iterative Solutions to IK



- Minimize error between current endeffector and its desired position
- Transform desired endeffector velocity into configuration space
- Repeatedly step to convergence at desired endeffector position



Next Class

- IK as an optimization problem
  - Gradient descent optimization
  - Manipulator Jacobian as the derivative of configuration
- Advanced: IK by Cyclic Coordinate Descent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Inverse Kinematics:
Manipulator Jacobian
GM
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