ECE 537 problem set 5 (solutions)

① Let us assume that $X \sim N(0,1)$.

$\Rightarrow$ chebyshev inequality: $p(|x| \geq a) \leq \frac{\sigma_x^2}{a^2} = \frac{1}{a^2}$

Markov inequality cannot be used because X is not non negative.

Now, let us find the chernoff bound.

If $X \sim N(\mu, \sigma^2) \Rightarrow M_X(s) = E\{e^{sX}\} = e^{\mu s + \frac{1}{2}\sigma^2 s^2}$

$\hookrightarrow$ moment generating function

chernoff bound: $p(X \geq a) \leq \min_{s>0} e^{-as} M_X(s)$

$$\Rightarrow p(X \geq a) \leq \min_{s>0} e^{-as} \times e^{0s + \frac{1}{2} \times 1 \times s^2} = \min_{s>0} e^{\frac{s^2}{2} - as}$$

To find the optimal value of s, we set the derivative of $\frac{s^2}{2} - as$ to 0.

$\frac{1}{2}\times 2S - a = 0 \Rightarrow \boxed{S^* = a}$

$$\Rightarrow \boxed{P(X \geq a) \leq e^{-\frac{1}{2}a^2}}$$

As it can be observed, chernoff bound exponentially approaches to zero as a grows. Therefore, for large values of a, chernoff bound gives a better upper bound compared to the chebyshev bound.

2: Let $T_i \sim \lambda e^{-\lambda t} u(t)$ be the life time of i'th filter for $i = 1, 2, \dots, n$.

$\Rightarrow T = \sum_{i=1}^{n} T_i$ is the total life time.

since T_i's are independent, we can use the central limit theorem and approximate T by the normal distribution $N(ET, \sigma_T^2)$

$ET = \sum_{i=1}^{n} E(T_i) = n \quad \sigma_T^2 = \sum_{i=1}^{n} \sigma_{T_i}^2 = n$

$\Rightarrow T \sim N(n, n)$

Now, we need to solve the following inequality:

$$P(T \geq 12) \geq 0.99 \Rightarrow P\Big(\underbrace{\frac{T-n}{\sqrt{n}}}_{\sim N(0,1)} \geq \frac{12-n}{\sqrt{n}}\Big) \geq 0.99$$

$$\Rightarrow Q\left(\frac{12-n}{\sqrt{n}}\right) \geq 0.99 \Rightarrow \frac{12-n}{\sqrt{n}} \leq \underbrace{Q^{-1}(0.99)}_{-2.326}$$

$$\Rightarrow 12-n \leq -2.326\sqrt{n} \Rightarrow 0 \leq n - 2.326\sqrt{n} - 12$$

$$\Rightarrow n \geq 23.23 \Rightarrow \boxed{n_{min} = 24 \text{ filters}}$$

We need at least 24 filters.

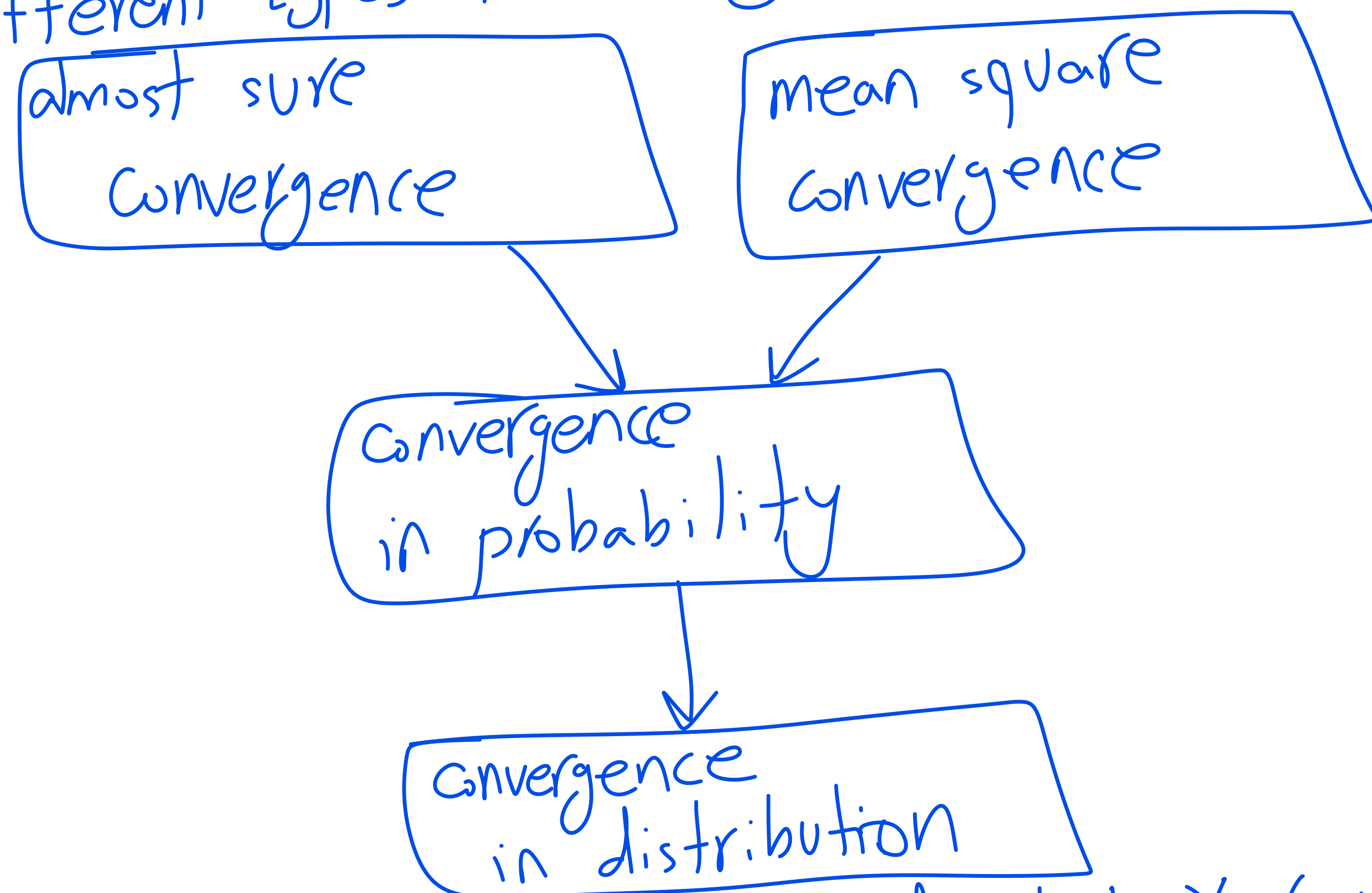
3a $X_n(w) = w^n \qquad w \sim U(0,1)$

$$\forall w \in (0,1): \lim_{n \to +\infty} X_n(w) = \lim_{n \to +\infty} w^n = 0$$

Since $X_n(w)$ converges to 0 for all values of w, We have convergence in almost sure sense.

$$P\left(\lim_{n \to +\infty} X_n(w) = 0\right) = 1$$

The following diagram shows the relation between different types of convergence.



Noting this diagram, we conclude that $X_n(w)$ is convergent to zero in probability and distribution senses as well.

Now, let us check the mean square sense.

$$\lim_{n\to+\infty} E\left\{\left(X_n(w)-0\right)^2\right\} = \lim_{n\to+\infty} E\{w^{2n}\} = \lim_{n\to+\infty} \int_0^1 w^{2n}\,dw$$

$$= \lim_{n\to+\infty} \frac{w^{2n+1}}{2n+1}\Bigg]_0^1 = \lim_{n\to+\infty} \frac{1}{2n+1} = 0 \quad \checkmark$$

$\Rightarrow X_n(w)$ converges to 0 in mean square sense.

(b): $Y_n(w) = \cos^2(2\pi w)$

$\forall w \in (0,1)$: $\lim_{n\to+\infty} Y_n(w) = \lim_{n\to+\infty} \cos^2(2\pi w) = \cos^2(2\pi w)$

$$\Rightarrow P\left(\lim_{n\to+\infty} Y_n(w) = \cos^2(2\pi w)\right) = 1$$

$\Rightarrow Y_n(w)$ converges to $Y(w) = \cos^2(2\pi w)$ in almost surely sense. Noting the previous diagram, we also have convergence in probability and distribution. Now, let us check mean square convergence.

$$\lim_{n\to+\infty} E\left\{\left(Y_n(w) - Y(w)\right)^2\right\} = \lim_{n\to+\infty} E\left\{\left(\cos^2(2\pi w) - \cos^2(2\pi w)\right)^2\right\}$$

$= 0$ ✓ $Y_n(w)$ converges in mean square sense.

(C) $Z_n(w) = \cos^n(2\pi w)$

If $w \neq \frac{1}{2}$: $\lim_{n\to+\infty} Z_n(w) = \lim_{n\to+\infty} (\underbrace{\cos(2\pi w)}_{<1})^n = 0$

If $w = \frac{1}{2}$: $Z_n(w) = \cos^n(\pi) = (-1)^n$ ⇝ will <u>not</u> converge

But $P(w = \frac{1}{2}) = 0$. In other words, the set $\{\frac{1}{2}\}$ has measure 0. Therefore, we conclude that $Z_n(w)$ converges to 0 almost surely.

Noting the diagram, we also have convergence in probability and distribution.

Now, we check mean square convergence:

$$\lim_{n\to+\infty} E\left\{\left(Z_n(\omega)-0\right)^2\right\} = \lim_{n\to+\infty} E\left\{\cos^{2n}(2\pi\omega)\right\}$$

$$= \lim_{n\to+\infty}\int_0^1 \cos^{2n}(2\pi\omega)\,d\omega \underbrace{=}_{\text{change the order of integral and limit.}} \int_0^1 \lim_{n\to+\infty}\cos^{2n}(2\pi\omega)\,d\omega = \int_0^1 (0)\,d\omega = 0$$

$\Rightarrow Z_n(\omega)$ converges to 0 in mean square sense.

d) $P_n(\omega) = \sum_{k=0}^{n} \frac{\omega^k}{k!}$

$$\forall \omega\in(0,1): \lim_{n\to+\infty} P_n(\omega) = \lim_{n\to+\infty}\sum_{k=0}^{n}\frac{\omega^k}{k!} = \sum_{k=0}^{+\infty}\frac{\omega^k}{k!} = e^{\omega}$$

$$\Rightarrow P\left(\lim_{n\to+\infty} P_n(\omega) = e^{\omega}\right) = 1$$

Therefore, $P_n(\omega)$ approaches to $P(\omega)=e^{\omega}$ in almost sure, probability, and distribution senses.

(4) $A_n = \begin{cases} (0, 1+\frac{1}{n}) & ; \ n \text{ odd} \\ (-1, 1-\frac{1}{n}) & ; \ n \text{ even} \end{cases}$

for $n = 1, 2, 3, \ldots$

(a): $A_1 = (0, 2)$ $A_2 = (-1, \frac{1}{2})$ $A_3 = (0, \frac{4}{3})$

$A_4 = (-1, \frac{3}{4})$ $A_5 = (0, \frac{6}{5})$

(b) $\limsup A_n = \bigcap_{m=1}^{+\infty} \bigcup_{n=m}^{+\infty} A_n$

If $m=1$: $\bigcup_{n=1}^{+\infty} A_n = \left(\bigcup_{n=1}^{+\infty} A_{2n}\right) \cup \left(\bigcup_{n=1}^{+\infty} A_{2n-1}\right)$

$= (-1, 1) \cup A_1 = (-1, 1) \cup (0, 2) = (-1, 2)$

If $m=2$: $\bigcup_{n=2}^{+\infty} A_n = (-1, 1) \cup A_3 = (-1, 1) \cup (0, \frac{4}{3}) = (-1, \frac{4}{3})$

If $m=3$: $\bigcup_{n=3}^{+\infty} A_n = (-1,1)\cup A_3 = \left(-1,\frac{4}{3}\right)$

If $m=4$: $\bigcup_{n=4}^{+\infty} A_n = (-1,1)\cup A_5 = (-1,1)\cup\left(0,\frac{6}{5}\right) = \left(-1,\frac{6}{5}\right)$

$\vdots$

$$\Longrightarrow \bigcap_{m=1}^{+\infty}\bigcup_{n=m}^{+\infty} A_n = (-1,2)\cap\left(-1,\frac{4}{3}\right)\cap\left(-1,\frac{6}{5}\right)\cap\cdots = (-1,1)$$

$$\boxed{\limsup A_n = (-1,1)}$$

Ⓒ: $\liminf A_n = \bigcup_{m=1}^{+\infty}\bigcap_{n=m}^{+\infty} A_n$

If $m=1$: $\bigcap_{n=1}^{+\infty} A_n = \left(\bigcap_{n=1}^{+\infty} A_{2n}\right)\cap\left(\bigcap_{n=1}^{+\infty} A_{2n-1}\right)$

$= A_2\cap(0,1) = \left(-1,\frac{1}{2}\right)\cap(0,1) = \left(0,\frac{1}{2}\right)$

If $m=2$: $\bigcap_{n=2}^{+\infty} A_n = A_2\cap(0,1) = \left(0,\frac{1}{2}\right)$

If $m=3$: $\bigcap_{n=3}^{+\infty} A_n = A_4\cap(0,1) = \left(-1,\frac{3}{4}\right)\cap(0,1) = \left(0,\frac{3}{4}\right)$

$\vdots$

$$\bigcup_{m=1}^{+\infty} \bigcap_{n=m}^{+\infty} A_n = (0,\tfrac{1}{2}) \cup (0,\tfrac{3}{4}) \cup \cdots = (0,1)$$

$$\Rightarrow \boxed{\liminf A_n = (0,1)}$$

since $\limsup A_n \neq \liminf A_n$, $\lim A_n$ does <u>not</u> exist.

(d): $(A_n \ i.o) = \limsup A_n = (-1,1)$

(e) $(A_n \ a.a) = \liminf A_n = (0,1)$

(5): Borel-Cantelli lemma has different equivalent variations. we use the following one for solving the question:

Let $\{Z_n\}_{n=1}^{+\infty}$ be a sequence of random variables.

If for each $\varepsilon > 0$, we have $\sum_{n=1}^{+\infty} P(|Z_n - Z| > \varepsilon) < +\infty$

Then Z_n converges to Z almost surely.

X_n	-1	0	1
$P(X_n)$	$\frac{1}{2n}$	$1-\frac{1}{n}$	$\frac{1}{2n}$

As it can be seen, $P(X_n = 0) = 1 - \frac{1}{n}$ will approach to 1 as n increases. So, we guess that X_n approaches to $X=0$. Now, we prove it using Borel-Cantelli Lemma:

If $0 < \varepsilon < 1$: $\sum_{n=1}^{+\infty} P(|X_n - 0| > \varepsilon) = \sum_{n=1}^{+\infty}\left(P(X_n=1) + P(X_n=-1)\right)$

$= \sum_{n=1}^{+\infty}\left(\frac{1}{2n} + \frac{1}{2n}\right) = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$

$\Rightarrow X_n$ does <u>not</u> converge to 0 almost surely.

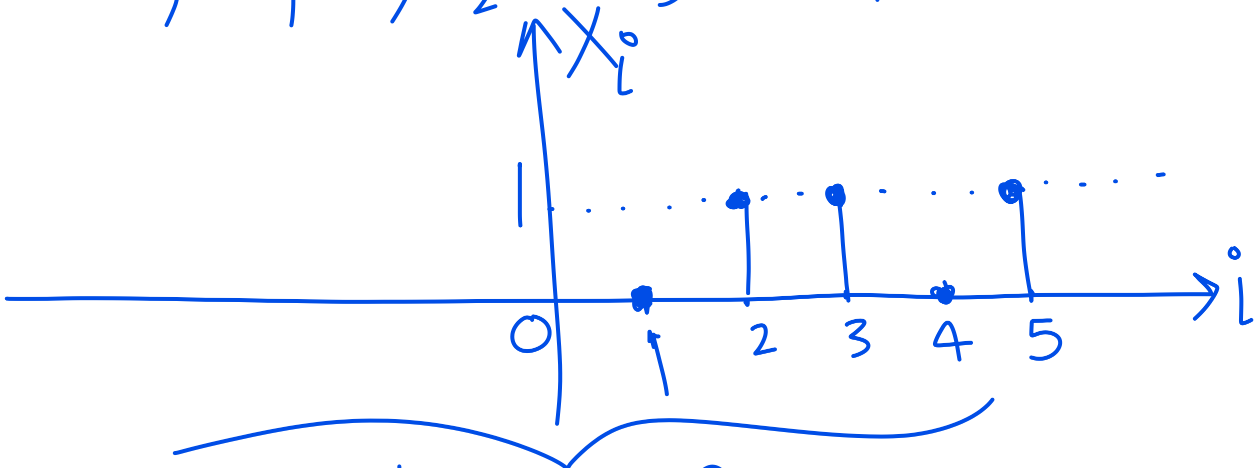
⑥: X_i's are independent Bernoulli random variables.

X_i	0	1
$P(X_i)$	$\frac{1}{2}$	$\frac{1}{2}$

for $i = 1, 2, \cdots, n$

$$Y = 2^n X_1 X_2 \cdots X_n$$

ⓐ Let $n = 5$, $X_1 = 0, X_2 = 1, X_3 = 1, X_4 = 0, X_5 = 1$



A sample sequence

ⓑ: Y_n only takes two value 0 and 2^n.

$$P(Y_n = 2^n) = P(X_1 = 1, X_2 = 1, \cdots, X_n = 1) = (P(X_1 = 1))^n = \frac{1}{2^n}$$

$\hookrightarrow$ X_i's are independent

$$P(Y_n = 0) = 1 - P(Y_n = 2^n) = 1 - \frac{1}{2^n}$$

$P(Y_n=0)$ will approach to 1 as n increases.

So, we guess that Y_n approaches to $Y=0$.

Borel-cantelli :

Let $\varepsilon>0$. $\sum_{n=1}^{+\infty} P(|Y_n-0|>\varepsilon)=\sum_{n=1}^{+\infty} P(2^n>\varepsilon)$

If $0<\varepsilon<2$: $\sum_{n=1}^{+\infty} P(2^n>\varepsilon)=\sum_{n=1}^{+\infty}\frac{1}{2^n}=\frac{\frac{1}{2}}{1-\frac{1}{2}}=1<+\infty$ ✓

If $2\leqslant\varepsilon$: $\sum_{n=1}^{+\infty} P(2^n>\varepsilon)=\sum_{n:n>\log_2\varepsilon}\frac{1}{2^n}<\sum_{n=1}^{+\infty}\frac{1}{2^n}=1<+\infty$

$\Rightarrow Y_n$ converges to 0 almost surely.

Y_n also converges to 0 in probability and distribution senses.

Finally, we check the convergence in mean square sense:

$$\lim_{n\to+\infty} E\left\{(Y_n - 0)^2\right\} = \lim_{n\to+\infty} E\left\{2^{2n} X_1^2 X_2^2 \cdots X_n^2\right\}$$

$$= \lim_{n\to+\infty} 2^{2n} E\left\{X_1^2\right\} E\left\{X_2^2\right\} \times \cdots \times E\left\{X_n^2\right\}$$

$$E\left\{X_i^2\right\} = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \lim_{n\to+\infty} 2^{2n} \times \underbrace{\frac{1}{2} \times \frac{1}{2} \times \cdots \times \frac{1}{2}}_{n \text{ times}} = \lim_{n\to+\infty} 2^{2n} \times \frac{1}{2^n} = \lim_{n\to+\infty} 2^n = +\infty$$

$\Rightarrow Y_n$ does <u>not</u> converge in mean square sense.

⑦: $$\lim_{n\to+\infty} E\left\{(X_n(\omega) - X(\omega))^2\right\} = 0$$

$$\lim_{n\to+\infty} E\left\{(Y_n(\omega) - Y(\omega))^2\right\} = 0$$

We guess that $Z_n(\omega)=X_n(\omega)+Y_n(\omega)$ converges to $X(\omega)+Y(\omega)$ in mean square sense. We need to prove it.

$$\lim_{n\to+\infty} E\left\{\left(Z_n(\omega)-X(\omega)-Y(\omega)\right)^2\right\}$$

$$=\lim_{n\to+\infty} E\left\{\left(\left(X_n(\omega)-X(\omega)\right)+\left(Y_n(\omega)-Y(\omega)\right)\right)^2\right\}$$

$$=\underbrace{\lim_{n\to+\infty} E\left(X_n(\omega)-X(\omega)\right)^2}_{0}+\underbrace{\lim_{n\to+\infty} E\left(Y_n(\omega)-Y(\omega)\right)^2}_{0}+2\lim_{n\to+\infty} E\left(X_n(\omega)-X(\omega)\right)\left(Y_n(\omega)-Y(\omega)\right)$$

Noting cauchy-shwartz inequality, we have:

$$\left|E\left(X_n(\omega)-X(\omega)\right)\left(Y_n(\omega)-Y(\omega)\right)\right|\leqslant\sqrt{E\left(X_n(\omega)-X(\omega)\right)^2}\sqrt{E\left(Y_n(\omega)-Y(\omega)\right)^2}$$

$$\Rightarrow\left|\lim_{n\to+\infty} E\left(X_n(\omega)-X(\omega)\right)\left(Y_n(\omega)-Y(\omega)\right)\right|\leqslant\lim_{n\to+\infty}\underbrace{\sqrt{E\left(X_n(\omega)-X(\omega)\right)^2}}_{0}\underbrace{\sqrt{E\left(Y_n(\omega)-Y(\omega)\right)^2}}_{0}$$

$$\Rightarrow\lim_{n\to+\infty} E\left(X_n(\omega)-X(\omega)\right)\left(Y_n(\omega)-Y(\omega)\right)=0$$

So, $Z_n(w)$ converges to $X(w)+Y(w)$ in the mean square sense.

8:

X_n	0	n^3
$P(X_n)$	$1-\frac{1}{n^2}$	$\frac{1}{n^2}$

X_n's are independent.

$$\lim_{n\to+\infty} P(X_n=0) = \lim_{n\to+\infty} 1-\frac{1}{n^2} = 1 \Longrightarrow$$

we guess that X_n approaches 0 as n increases.

Let $0<\varepsilon<1$: $\sum_{n=1}^{+\infty} P(|X_n-0|>\varepsilon) = \sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < +\infty$

If $\varepsilon \geqslant 1$, $\sum_{n=1}^{+\infty} P(|X_n|>\varepsilon) = \sum_{n:n>\sqrt[3]{\varepsilon}} \frac{1}{n^2} \leqslant \sum_{n=1}^{+\infty} \frac{1}{n^2} < +\infty$

$\Rightarrow X_n$ converges to $X=0$ almost surely, in probability, and in distribution.

Now, we check mean square convergence:

$$\lim_{n \to +\infty} E\left\{(X_n - 0)^2\right\} = \lim_{n \to +\infty} E(X_n)^2$$

$$= \lim_{n \to +\infty} \left(0^2 \times \left(1 - \frac{1}{n^2}\right) + n^6 \times \frac{1}{n^2}\right) = \lim_{n \to +\infty} n^4 = +\infty$$

$\Rightarrow X_n$ does <u>not</u> converge to $X = 0$ in mean square sense.