ECE537
Random Processes
Midterm 1
Oct 18, 2023
E. S. Sousa

1. Consider a probably experiment consisting of flipping two coins repeatedly until they both turn up heads.

a) Define the sample space Ω. How many points are there in Ω. Enumerate them.

b) Describe the smallest sigma field that contains all singleton sets. Specify the collection of sets in the σ-field.

c) Let the total number of flips be N. Give the probability mass function for the random variable N.

d) Determine the probability that the number of flips will be greater than 5.

e) Now, define a random variable equal to the total number of heads, $X = \sum_{i=1}^{N} X_i$, where X_i is the number of heads at the i^{th} flip, and N is the number of flips (note that we must have $X_N = 2$, and $X_i < 2$ for $i < N$). Find the expected value of the random variable X.

2. Consider a random vector $\boldsymbol{X} = (X_1, X_2)$ with covariance matrix equal to $C_X = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$

a) Determine a linear transformation $\boldsymbol{Y} = \boldsymbol{AX}$ so that $\boldsymbol{Y} = (Y_1, Y_2)$ is a random vector with uncorrelated components.

b) For the linear transformation given in a) determine the covariance matrix of Y, i.e. C_Y.

3. Consider a random vector $\boldsymbol{X} = (X_1, X_2, X_3)$ with mean $\boldsymbol{m} = (2,0,-1)$ and covariance matrix equal to

$C_X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $Y = 2X_1 + X_2 + 2X_3$.

a) Use the Chebychev inequality to give an upper bound for the probability $P(Y > 5)$.

b) Assuming that X is a Gaussian random vector with the same mean and covariance matrix, give a procedure to determine the probability $P(Y > 5)$ exactly.