ECE537

Random Processes

Problem Set 5

1. Consider a random variable X that has a Gaussian distribution with variance $\sigma^2 = 1$.
   Use the bounds to estimate the probability $|X| > a$, as a function of a.
   In different curves on the same plot, plot the exact value versus a, then the bounds versus a, for the Markov, Chebychev, and Chernoff bounds. In the case of the Chernoff bound use the best parameter.

2. The time that the purifier filter of a water fountain works is exponentially distributed with parameter $\lambda = 1$ (month)−1 . After that it becomes clogged, we change it with a new filter. Approximately how many filters should we buy to make sure with probability 99%, we don't need to buy any more filter in a year?

3. Let ζ be a random variable with uniform probability law in $(0,1)$. Discuss about the convergence of the following processes. Discuss about the sense of the convergence and the convergence value (random variable).

   (a) $X_n(\omega) = \omega n$
   (b) $Y_n(\omega) = \cos^2(2\pi\omega)$
   (c) $Z_n(\omega) = \cos^n(2\pi\omega)$
   (d) $P_n(\omega) = \sum_{k=0}^{n} \frac{\omega^k}{k!}$ (skip the mean square sense)

4. Consider a sequence of events defined as follows, for $n = 1,2,\cdots$

$$A_n = \begin{cases} \left(0,1+\frac{1}{n}\right) & \text{for } n \text{ odd} \\ \left(\left(-1,1-\frac{1}{n}\right)\right. & \text{for } n \text{ even} \end{cases}$$

   a) List the first 5 events in the sequence.
   b) Find lim sup A_n
   c) Find lim inf A_n
   d) Specify the event (A_n i.o).
   e) Specify the event (A_n a.a.).

5. Consider a sequence of independent random variables X_n with the probability law $P(X_n = 1) = \frac{1}{2n}$, $P(X_n = -1) = \frac{1}{2n}$, $P(X_n = 0) = 1 - \frac{1}{n}$. Use the Borel-Cantelli lemma to determine whether or not this sequence converges almost surely?

6. Let X_i's be independent Bernoulli (with $p = 0.5$) random variables. Define $Y_n = 2^n X_1 X_2 \cdots X_n$

   a) Plot a sample sequence of these random variables.
   b) Discuss the convergence types and values to which the sequence converges.

7. Let $X_n(\omega)$, and $Y_n(\omega)$ be two random sequences that converge in the mean square sense to the random variables X and Y, respectively. Does the sequence of random variables $Z_n(\omega) = X_n(\omega) + Y_n(\omega)$ converge in the mean square sense, and if so, to what limit?

8. Consider the independent sequence X_n specified as follows:

$$X_n = \begin{cases} n^3 & \text{with probability } \dfrac{1}{n^2} \\ 0 & \text{with probability } 1 - \dfrac{1}{n^2} \end{cases}$$

   Does this process converge? And if so, in what sense?