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PROBLEMS

Section 7.1: Sums of Random Variables

7.1. Let $Z = X + Y + Z$, where X, Y, and Z are zero-mean, unit-variance random variables with $\text{COV}(X, Y) = 1/2$, and $\text{COV}(Y, Z) = -1/4$ and $\text{COV}(X, Z) = 1/2$.

(a) Find the mean and variance of Z.

(b) Repeat part a assuming X, Y, and Z are uncorrelated random variables.

7.2. Let $X_1, \ldots, X_n$ be random variables with the same mean and with covariance function:

$$\text{COV}(X_i, X_j) = \begin{cases} \sigma^2 & \text{if } i = j \\ \rho\sigma^2 & \text{if } |i - j| = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

7.3. Let $X_1, \ldots, X_n$ be random variables with the same mean and with covariance function

$$\text{COV}(X_i, X_j) = \sigma^2 \rho^{|i-j|},$$

where $|\rho| < 1$. Find the mean and variance of $S_n = X_1 + \cdots + X_n$.

7.4. Let X and Y be independent Cauchy random variables with parameters 1 and 4, respectively. Let $Z = X + Y$.

(a) Find the characteristic function of Z.

(b) Find the pdf of Z from the characteristic function found in part a.

7.5. Let $S_k = X_1 + \cdots + X_k$, where the X_i's are independent random variables, with X_i a chi-square random variable with n_i degrees of freedom. Show that S_k is a chi-square random variable with $n = n_1 + \cdots + n_k$ degrees of freedom.

7.6. Let $S_n = X_1^2 + \cdots + X_n^2$, where the X_i's are iid zero-mean, unit-variance Gaussian random variables.

(a) Show that S_n is a chi-square random variable with n degrees of freedom. *Hint:* See Example 4.34.

(b) Use the methods of Section 4.5 to find the pdf of

$$T_n = \sqrt{X_1^2 + \cdots + X_n^2}.$$

(c) Show that T_2 is a Rayleigh random variable.

(d) Find the pdf for T_3. The random variable T_3 is used to model the speed of molecules in a gas. T_3 is said to have the Maxwell distribution.

7.7. Let X and Y be independent exponential random variables with parameters 2 and 10, respectively. Let $Z = X + Y$.

(a) Find the characteristic function of Z.

(b) Find the pdf of Z from the characteristic function found in part a.

7.8. Let $Z = 3X - 7Y$, where X and Y are independent random variables.

(a) Find the characteristic function of Z.

(b) Find the mean and variance of Z by taking derivatives of the characteristic function found in part a.

7.9. Let M_n be the sample mean of n iid random variables X_j. Find the characteristic function of M_n in terms of the characteristic function of the X_i's.

7.10. The number X_j of raffle winners in classroom j is a binomial random variable with parameter n_j and p. Suppose that the school has K classrooms. Find the pmf of the total number of raffle winners in the school, assuming the X_i's are independent random variables.

7.11. The number of packet arrivals X_i at port i in a router is a Poisson random variable with mean α_i. Given that the router has k ports, find the pmf for the total number of packet arrivals at the router. Assume that the X_i's are independent random variables.

7.12. Let $X_1, X_2, \ldots$ be a sequence of independent integer-valued random variables, let N be an integer-valued random variable independent of the X_j, and let

$$S = \sum_{k=1}^{N} X_k.$$

(a) Find the mean and variance of S.

(b) Show that

$$G_S(z) = E(z^S) = G_N(G_X(z)),$$

where $G_X(z)$ is the generating function of each of the X_k's.

7.13. Let the number of smashed-up cars arriving at a body shop in a week be a Poisson random variable with mean L. Each job repair costs X_j dollars, the X_j's are iid random variables that are equally likely to be \$500 or \$1000.

(a) Find the mean and variance of the total revenue R arriving in a week.

(b) Find the $G_R(z) = E[z^R]$.

7.14. Let the number of widgets tested in an assembly line in 1 hour be a binomial random variable with parameters $n = 600$ and p. Suppose that the probability that a widget is faulty is a. Let S be the number of widgets that are found faulty in a 1-hour period.

(a) Find the mean and variance of S.

(b) Find $G_S(z) = E[z^S]$.

Section 7.2: The Sample Mean and the Laws of Large Numbers

7.15. Suppose that the number of particle emissions by a radioactive mass in t seconds is a Poisson random variable with mean λt. Use the Chebyshev inequality to obtain a bound for the probability that $|N(t)/t - \lambda|$ exceeds ε.

7.16. Suppose that 20% of voters are in favor of certain legislation. A large number n of voters are polled and a relative frequency estimate $f_A(n)$ for the above proportion is obtained.

Use Eq. (7.20) to determine how many voters should be polled in order that the probability is at least .95 that $f_A(n)$ differs from 0.20 by less than 0.02.

7.17. A fair die is tossed 20 times. Use Eq. (7.20) to bound the probability that the total number of dots is between 60 and 80.

7.18. Let X_i be a sequence of independent zero-mean, unit-variance Gaussian random variables. Compare the bound given by Eq. (7.20) with the exact value obtained from the Q function for $n = 16$ and $n = 81$.

7.19. Does the weak law of large numbers hold for the sample mean if the X_i's have the covariance functions given in Problem 7.2? Assume the X_i have the same mean.

7.20. Repeat Problem 7.19 if the X_i's have the covariance functions given in Problem 7.3.

7.21. (The **sample variance**) Let $X_1, \ldots, X_n$ be an iid sequence of random variables for which the mean and variance are unknown. The sample variance is defined as follows:

$$V_n^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n)^2,$$

where M_n is the sample mean.

(a) Show that

$$\sum_{j=1}^{n}(X_j - \mu)^2 = \sum_{j=1}^{n}(X_j - M_n)^2 + n(M_n - \mu)^2.$$

(b) Use the result in part a to show that

$$E\left[k\sum_{j=1}^{n}(X_j - M_n)^2\right] = k(n-1)\sigma^2.$$

(c) Use part b to show that $E[V_n^2] = \sigma^2$. Thus V_n^2 is an unbiased estimator for the variance.

(d) Find the expected value of the sample variance if $n - 1$ is replaced by n. Note that this is a biased estimator for the variance.

Section 7.3: The Central Limit Theroem

7.22. **(a)** A fair coin is tossed 100 times. Estimate the probability that the number of heads is between 40 and 60. Estimate the probability that the number is between 50 and 55.

(b) Repeat part a for $n = 1000$ and the intervals [400, 600] and [500, 550].

7.23. Repeat Problem 7.16 using the central limit theorem.

7.24. Use the central limit theorem to estimate the probability in Problem 7.17.

7.25. The lifetime of a cheap light bulb is an exponential random variable with mean 36 hours. Suppose that 16 light bulbs are tested and their lifetimes measured. Use the central limit theorem to estimate the probability that the sum of the lifetimes is less than 600 hours.

7.26. A student uses pens whose lifetime is an exponential random variable with mean 1 week. Use the central limit theorem to determine the minimum number of pens he should buy at the beginning of a 15-week semester, so that with probability .99 he does not run out of pens during the semester.

7.27. Let S be the sum of 80 iid Poisson random variables with mean 0.25. Compare the exact value of $P[S = k]$ to an approximation given by the central limit theorem as in Eq. (7.30).

7.28. The number of messages arriving at a multiplexer is a Poisson random variable with mean 15 messages/second. Use the central limit theorem to estimate the probability that more than 950 messages arrive in one minute.

7.29. A binary transmission channel introduces bit errors with probability .15. Estimate the probability that there are 20 or fewer errors in 100 bit transmissions.

7.30. The sum of a list of 64 real numbers is to be computed. Suppose that numbers are rounded off to the nearest integer so that each number has an error that is uniformly distributed in the interval $(-0.5, 0.5)$. Use the central limit theorem to estimate the probability that the total error in the sum of the 64 numbers exceeds 4.

7.31. **(a)** A fair coin is tossed 100 times. Use the Chernoff bound to estimate the probability that the number of heads is greater than 90. Compare to an estimate using the central limit theorem.

(b) Repeat part a for $n = 1000$ and the probability that the number of heads is greater than 650.

7.32. A binary transmission channel introduces bit errors with probability .01. Use the Chernoff bound to estimate the probability that there are more than 3 errors in 100 bit transmissions. Compare to an estimate using the central limit theorem.

7.33. **(a)** When you play the rock/paper/scissors game against your sister you lose with probability 3/5. Use the Chernoff bound to estimate the probability that you win more than half of 20 games played.

(b) Repeat for 100 games.

(c) Use trial and error to find the number of games n that need to be played so that the probability that your sister wins more than 1/2 the games is 90%.

7.34. Show that the Chernoff bound for X, a Poisson random variable with mean α, is $P[X \geq a] \leq e^{-a\ln(a/\alpha)+a-\alpha}$ for $a > \alpha$. *Hint:* Use $E[e^{sX}] = e^{\alpha(e^s-1)}$.

7.35. Redo Problem 7.26 using the Chernoff bound.

7.36. Show that the Chernoff bound for X, a Gaussian random variable with mean μ and variance σ^2, is $P[X \geq a] \leq e^{-(a-\mu)^2/2\sigma^2}$, $a > \mu$. *Hint:* Use $E[e^{sX}] = e^{s\mu+s^2\sigma^2/2}$.

7.37. Compare the Chernoff bound for the Gaussian random variable with the estimates provided by Eq. (4.54).

7.38. **(a)** Find the Chernoff bound for the exponential random variable with rate λ.

(b) Compare the exact probability of $P[X \geq k/\lambda]$ with the Chernoff bound.

7.39. **(a)** Generalize the approach in Problem 7.38 to find the Chernoff bound for a gamma random variable with parameters λ and α.

(b) Use the result of part a to obtain the Chernoff bound for a chi-square random variable with k degrees of freedom.

*Section7.4: Convergence of Sequences of Random Variables

7.40. Let $U_n(\zeta)$, $W_n(\zeta)$, $Y_n(\zeta)$, and $Z_n(\zeta)$ be the sequences of random variables defined in Example 7.18.

(a) Plot the sequence of functions of ζ associated with each sequence of random variables.

(b) For $\zeta = 1/4$, plot the associated sample sequence.

7.41. Let ζ be selected at random from the interval $S = [0, 1]$, and let the probability that ζ is in a subinterval of S be given by the length of the subinterval. Define the following sequences of random variables for $n \geq 1$:

$$X_n(\zeta) = \zeta^n, Y_n(\zeta) = \cos^2 2\pi\zeta, Z_n(\zeta) = \cos^n 2\pi\zeta.$$

Do the sequences converge, and if so, in what sense and to what limiting random variable?

7.42. Let b_i, $i \geq 1$, be a sequence of iid, equiprobable Bernoulli random variables, and let ζ be the number between [0, 1] determined by the binary expansion

$$\zeta = \sum_{i=1}^{\infty} b_i 2^{-i}.$$

(a) Explain why ζ is uniformly distributed in [0, 1].

(b) How would you use this definition of ζ to generate the sample sequences that occur in the urn problem of Example 7.20?

7.43. Let X_n be a sequence of iid, equiprobable Bernoulli random variables, and let

$$Y_n = 2^n X_1 X_2 \ldots X_n.$$

(a) Plot a sample sequence. Does this sequence converge almost surely, and if so, to what limit?

(b) Does this sequence converge in the mean square sense?

7.44. Let X_n be a sequence of iid random variables with mean m and variance $\sigma^2 < \infty$. Let M_n be the associated sequence of arithmetic averages,

$$M_n = \frac{1}{n} \sum_{i=0}^{n} X_i.$$

Show that M_n converges to m in the mean square sense.

7.45. Let X_n and Y_n be two (possibly dependent) sequences of random variables that converge in the mean square sense to X and Y, respectively. Does the sequence $X_n + Y_n$ converge in the mean square sense, and if so, to what limit?

7.46. Let U_n be a sequence of iid zero-mean, unit-variance Gaussian random variables. A "low-pass filter" takes the sequence U_n and produces the sequence

$$X_n = \frac{1}{2}(U_n + U_{n-1}).$$

(a) Does this sequence converge in the mean square sense?

(b) Does it converge in distribution?

7.47. Does the sequence of random variables introduced in Example 7.20 converge in the mean square sense?

7.48. Customers arrive at an automated teller machine at discrete instants of time, $n = 1, 2, \ldots$. The number of customer arrivals in a time instant is a Bernoulli random variable with parameter p, and the sequence of arrivals is iid. Assume the machine services a customer in less than one time unit. Let X_n be the total number of customers served by the machine up to time n. Suppose that the machine fails at time N, where N is a geometric random variable with mean 100, so that the customer count remains at X_N thereafter.

(a) Sketch a sample sequence for X_n.

(b) Do the sample sequences converge almost surely, and if so, to what limit?

(c) Do the sample sequences converge in the mean square sense?

7.49. Show that the sequence $Y_n(\zeta)$ defined in Example 7.18 converges in distribution.

7.50. Let X_n be a sequence of Laplacian random variables with parameter $\alpha = n$. Does this sequence converge in distribution?