problem set 10 solutions:

1

$R_{xx}[k] = a^{|k|}$; $|a|<1$ $\quad EX[n]=0 \quad X[n]$ is WSS

$X[n]$ is WSS iff $\lim\limits_{N\to+\infty} \frac{1}{2N+1} \sum\limits_{k=-2N}^{2N} \left(1-\frac{|k|}{2N+1}\right) C_{xx}[k] = 0$

$C_{xx}[k] = R_{xx}[k] - (\underbrace{EX[n]}_{0})^2 = R_{xx}[k]$

$$\frac{1}{2N+1} \sum_{k=-2N}^{2N} \left(1-\frac{|k|}{2N+1}\right) a^{|k|} \leqslant \left|\frac{1}{2N+1} \sum_{k=-2N}^{2N} \left(1-\frac{|k|}{2N+1}\right) a^{|k|}\right|$$

$$\leqslant \frac{1}{2N+1} \sum_{k=-2N}^{2N} \left|1-\frac{|k|}{2N+1}\right| |a|^{|k|} \leqslant \frac{1}{2N+1} \sum_{k=-2N}^{2N} |a|^{|k|}$$

triangle inequality

$$\lim_{N\to+\infty} \frac{1}{2N+1} \sum_{k=-2N}^{2N} |a|^{|k|} = \lim_{N\to+\infty} \frac{1}{2N+1} \left(1+2\sum_{k=1}^{2N} |a|^{k}\right)$$

$$= \lim_{N\to+\infty} \frac{1}{2N+1} \left(1+2\times \frac{|a|\left(1-|a|^{2N}\right)}{1-|a|}\right) = 0$$

$$\Rightarrow \lim_{N\to+\infty} \frac{1}{2N+1} \sum_{k=-2N}^{2N} \left(1-\frac{|k|}{2N+1}\right) a^{|k|} = 0$$

The process is mean-ergodic.

②: $\langle X(t)\rangle \overset{ms}{=} EX$, $X(t)$ WSS, $EX(t)\neq 0$

$Y(t)=CX(t)$, $C \perp\!\!\!\perp X(t)$, $EC\neq 0$

$\langle Y(t)\rangle = C\langle X(t)\rangle$, $EY(t)=E\{CX(t)\} \underbrace{=}_{C \perp\!\!\!\perp X(t)} EC\,EX(t)$

$$\Rightarrow E\big(\langle Y(t)\rangle - EY(t)\big)^2 = E\big(C\langle X(t)\rangle - EC\,EX(t)\big)^2$$

$$= E\big(C\langle X(t)\rangle - CEX(t) + CEX(t) - EC\,EX(t)\big)^2$$

$$= E\Big(C\big(\langle X(t)\rangle - EX(t)\big) + \big(C-EC\big)EX(t)\Big)^2$$

$$= EC^2 \times \underbrace{\cancel{E\big(\langle X(t)\rangle - EX(t)\big)^2}}_{0} + E\big(C-EC\big)^2 \times \big(EX(t)\big)^2$$

$X(t)$ is mean ergodic

$$+\,2E\{C(C-EC)\}\,EX(t)\,\underbrace{\cancel{E\big(\langle X(t)\rangle - EX(t)\big)}}_{0} = \sigma_C^2\,\big(EX(t)\big)^2 \neq 0$$

$\Rightarrow$ The process is not ergodic in mean.

③ : $X(t) = b\cos(2\pi\psi t + \Theta)$ $\Theta \sim U[0, 2\pi)$ $\Phi \perp\!\!\!\perp \Theta$

$\Phi \sim f_\psi(\psi)$

$EX(t) = E\{b\cos(2\pi\psi t)\cos\Theta - b\sin(2\pi\psi t)\sin\Theta\}$

$= bE(\cos(2\pi\psi t))\underbrace{E(\cos\Theta)}_{0} - bE(\sin(2\pi\psi t))\underbrace{E(\sin\Theta)}_{0} = 0$

$E(X(t+\tau)X(t)) = E(b^2\cos(2\pi\psi t + \Theta)\cos(2\pi\psi(t+\tau)+\Theta))$

$= \frac{1}{2}b^2 E(\cos(2\pi\psi\tau)) + \underbrace{\frac{1}{2}b^2 E(\cos(2\pi\psi(2t+\tau)+2\Theta))}_{0}$

$= \frac{b^2}{2}E(\cos(2\pi\psi\tau))$

characteristic function.

$= \frac{b^2}{4}\left(E(e^{j2\pi\psi\tau}) + E(e^{-j2\pi\psi\tau})\right) = \frac{b^2}{4}\left(\phi_\psi(2\pi\tau) + \phi_\psi(-2\pi\tau)\right)$

$\Rightarrow X(t)$ is WSS.

$S_X(f)$
PSD $= F\{R_X(\tau)\} = \frac{b^2}{4}F\{\phi_\psi(2\pi\tau) + \phi_\psi(-2\pi\tau)\} =$

$\frac{b^2}{4}\left(f_\psi(f) + f_\psi(-f)\right)$

We know that $S_{X,X}(f) = F\{R_{X,X}(\tau)\}$ is a real, non-negative, and even function of f when $X(t)$ is a real process.

Therefore, we can construct the PDF $f_\psi(f)$ as:

$$f_\psi(f) = \frac{S_{XX}(f)U(f)}{\frac{b^2}{4}} \quad ; \text{where } \frac{b^2}{4} = \int_0^{+\infty} S_{XX}(f)\,df.$$

④: $x(t) = \sum_{n=1}^{N} a_n \cos(\omega_n t + \theta_n)$ θ_n's are i.i.d $\sim [0, 2\pi]$

ⓐ: $E\{x(t+\tau)x(t)\} = E\left\{\sum_{n=1}^{N} a_n \cos(\omega_n(t+\tau)+\theta_n) \times \sum_{m=1}^{N} a_m \cos(\omega_m t + \theta_m)\right\}$

$= \sum_{n=1}^{N}\sum_{m=1}^{N} a_n a_m E\left\{\cos(\omega_n(t+\tau)+\theta_n)\cos(\omega_m t + \theta_m)\right\}$

$$E\left\{\cos(\omega_n(t+\tau)+\theta_n)\cos(\omega_m t+\theta_m)\right\} = \begin{cases} 0 ; & n \neq m \\ \frac{1}{2}\cos(\omega_n \tau) ; & n = m \end{cases}$$

$\Longrightarrow E(x(t+\tau)x(t)) = \frac{1}{2}\sum_{n=1}^{N} a_n^2 \cos(\omega_n \tau)$ $\Downarrow$

$EX(t) = \sum_{n=1}^{N} a_n E(\cancel{\cos(\omega_n t + \theta_n)}) = 0 \Rightarrow x(t)$ is WSS

ⓑ: $S_x(f) = F\{R_x(\tau)\} = F\left\{\frac{1}{2}\sum_{n=1}^{N} a_n^2 \cos(\omega_n \tau)\right\}$

$= \frac{1}{4}\sum_{n=1}^{N} a_n^2 \left(\delta\left(f - \frac{\omega_n}{2\pi}\right) + \delta\left(f + \frac{\omega_n}{2\pi}\right)\right)$

(5): Let $\Omega \triangleq \frac{\omega_0 v}{c}$, $\omega_0 = 2\pi f_0$. $y(t) = b\cos((\omega_0+\Omega)t+\theta)$

$\Omega \sim U\left[\frac{-\omega_0 v_0}{c}, \frac{\omega_0 v_0}{c}\right]$, $\theta \sim [0, 2\pi)$ $\quad \theta \perp\!\!\!\perp \Omega$

$$Ey(t) = bE(\cos(\omega_0+\Omega)t)\cancel{\times}\underbrace{\cancel{E(\cos\theta)}}_{0} - bE(\sin(\omega_0+\Omega)t)\underbrace{\cancel{E(\sin\theta)}}_{0} = 0$$

$$E(y(t+\tau)y(t)) = b^2 E\Big(\cos((\omega_0+\Omega)(t+\tau)+\theta)\cos((\omega_0+\Omega)t+\theta)\Big)$$

$$= \frac{b^2}{2}E\Big(\cos((\omega_0+\Omega)\tau)\Big) + \frac{b^2}{2}E\Big(\cos((\omega_0+\Omega)(2t+\tau)+2\theta)\Big)$$

$$\frac{b^2}{2}E\Big(\cos((\omega_0+\Omega)(2t+\tau))\Big)\underbrace{\cancel{E(\cos 2\theta)}}_{0}$$

$$-\frac{b^2}{2}E\Big(\sin((\omega_0+\Omega)(2t+\tau))\Big)\underbrace{\cancel{E(\sin(2\theta))}}_{0}$$

$$= \frac{b^2}{2}E\Big(\cos((\omega_0+\Omega)\tau)\Big)$$

$$= \frac{b^2}{2} \times \frac{c}{2\omega_0 v_0}\int_{-\frac{\omega_0 v_0}{c}}^{\frac{\omega_0 v_0}{c}} \cos((\omega_0+\omega)\tau)\,d\omega =$$

$$\frac{cb^2}{4\omega_0 v_0 \tau} \times \left\{\sin\left(\left(\omega_0 + \frac{\omega_0 v_0}{c}\right)\tau\right) - \sin\left(\left(\omega_0 - \frac{\omega_0 v_0}{c}\right)\tau\right)\right\}$$

$$= \frac{cb^2}{2\omega_0 v_0 \tau} \times \sin\left(\frac{\omega_0 v_0 \tau}{c}\right)\cos(\omega_0\tau) = \frac{b^2}{2}\frac{\sin\left(\frac{\omega_0 v_0 \tau}{c}\right)}{\frac{\omega_0 v_0 \tau}{c}}\cos(\omega_0\tau)$$

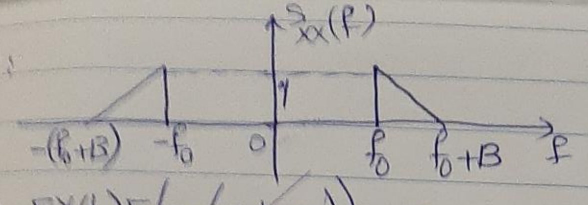
$$= \frac{b^2}{2}\operatorname{sinc}\left(\frac{2f_0 v_0 \tau}{c}\right)\cos(2\pi f_0\tau) \qquad S_y(f) = F\{R_y(\tau)\}$$

$$\Rightarrow S_y(f) = \frac{b^2}{2} \times \frac{1}{2} \times \{\delta(f-f_0) + \delta(f+f_0)\} * \frac{c}{2f_0 v_0}\operatorname{rect}\left(\frac{c}{2f_0 v_0}f\right)$$

$$= \frac{b^2 c}{8 f_0 v_0}\left(\operatorname{rect}\left(\frac{c(f-f_0)}{2f_0 v_0}\right) + \operatorname{rect}\left(\frac{c(f+f_0)}{2f_0 v_0}\right)\right)$$

The power is spread over a f_0 and $-f_0$.

6 :



, $Y(t)=X(t)\cos(\omega_0 t+\theta)$

$\theta \sim U[0,2\pi)$

$\theta \perp\!\!\!\perp X(t)$

$$EY(t)=EX(t)\underbrace{E(\cos(\omega_0 t+\theta))}_{0}=0$$

$$E(Y(t+\tau)Y(t))=E\left(X(t+\tau)\cos(\omega_0(t+\tau)+\theta)X(t)\cos(\omega_0 t+\theta)\right)$$

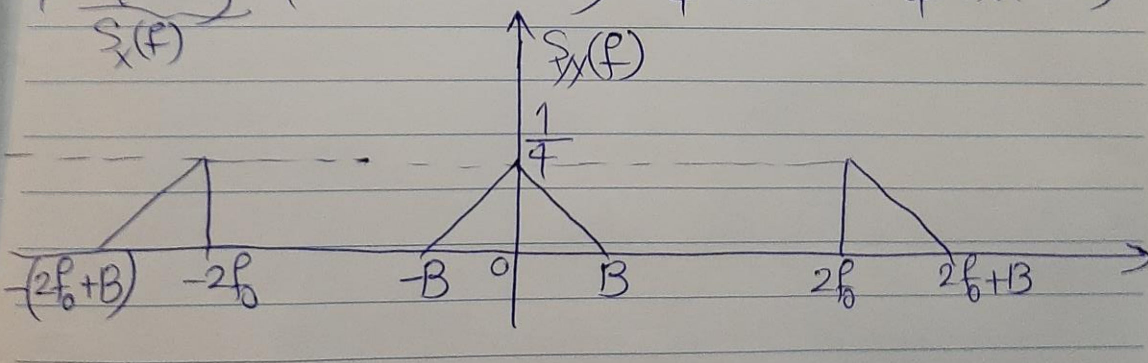
$$=E(X(t+\tau)X(t))E\left(\cos(\omega_0(t+\tau)+\theta)\cos(\omega_0 t+\theta)\right)$$

$$R_X(\tau)\times\left\{\frac{1}{2}E(\cos(\omega_0\tau))+\frac{1}{2}\underbrace{E\left(\cos(\omega_0(2t+\tau))+2\theta\right)}_{0}\right\}=\frac{1}{2}R_X(\tau)\cos(\omega_0\tau)$$

$\Longrightarrow$ Y(t) is WSS.

$$S_Y(f)=F\{R_Y(\tau)\}=F\left\{\frac{1}{2}R_X(\tau)\cos(2\pi f_0\tau)\right\}=$$

$$\frac{1}{4}\underbrace{F\{R_X(\tau)\}}_{S_X(f)}*\left(\delta(f-f_0)+\delta(f+f_0)\right)=\frac{1}{4}S_{XX}(f-f_0)+\frac{1}{4}S_{XX}(f+f_0)$$



(7): $x(t) = \cos\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)$ for $nT \leq t < (n+1)T$

$B[n]$ is a discrete time, Bernoulli random process $B_n \in \{\pm 1\}$

(a): $EX(t) = \cos(2\pi f_0 t) E\left(\underbrace{\cos\left(B[n]\frac{\pi}{2}\right)}_{0}\right) - \sin(2\pi f_0 t) E\left(\underbrace{\sin\left(B[n]\frac{\pi}{2}\right)}_{0}\right)$

$= 0$ $\quad E(X(t+\tau)X(t)) = E\left(\cos\left(2\pi f_0(t+\tau) + B[n]\frac{\pi}{2}\right)\cos\left(2\pi f_0 t + B[m]\frac{\pi}{2}\right)\right)$

where $nT \leq t+\tau < (n+1)T$ and $mT \leq t < (m+1)T$

If $n \neq m$: $E(X(t+\tau)X(t)) = \underbrace{E\left\{\cos\left(2\pi f_0(t+\tau) + B[n]\frac{\pi}{2}\right)\right\}}_{0} \times \underbrace{E\left\{\cos\left(2\pi f_0 t + B[m]\frac{\pi}{2}\right)\right\}}_{0} = 0$

If $n = m$: $E(X(t+\tau)X(t)) = E\left\{\cos\left(2\pi f_0(t+\tau) + B[n]\frac{\pi}{2}\right)\cos\left(2\pi f_0 t + B[n]\frac{\pi}{2}\right)\right\}$

$= \frac{1}{2}\cos(2\pi f_0 \tau) + \frac{1}{2} E\left\{\cos\left(2\pi f_0(2t+\tau) + B[n]\pi\right)\right\}$

$= \frac{1}{2}\cos(2\pi f_0 \tau) - \frac{1}{2}\cos\left(2\pi f_0(2t+\tau)\right) \Rightarrow$ It is not WSS.

(b): Since the process is not WSS, we need to find time average over t.

$R_{X,X}(\tau) = \langle R_{X,X}(t, t+\tau)\rangle_t = (1-\rho(\tau))\langle 0 \rangle + \rho(\tau)\left\langle \frac{1}{2}\cos(2\pi f_0 \tau) - \frac{1}{2}\cos(2\pi f_0(2t+\tau))\right\rangle$

where $\rho(\tau)$ is the fraction of the values of t that lead to t and $t+\tau$ being in the same interval. This fraction is given by $\rho(\tau) = \begin{cases} 0 ; & |\tau| > T \\ 1 - \frac{|\tau|}{T} ; & |\tau| < T \end{cases} = \Lambda\left(\frac{\tau}{T}\right)$

$\Rightarrow R_{X,X}(\tau) = \frac{1}{2}\Lambda\left(\frac{\tau}{T}\right)\cos(2\pi f_0 \tau) \Longrightarrow$

$$S_x(f) = F\{R_{x,x}(\tau)\} = \frac{1}{2} F\left\{\Lambda\left(\frac{\tau}{T}\right)\right\} * F\{\cos(2\pi f_0 \tau)\}$$

$$= \frac{1}{2} \times T \times \left(\operatorname{sinc}(Tf)\right)^2 * \left(\frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0)\right)$$

$$= \frac{T}{4}\left(\operatorname{sinc}^2\left(T(f-f_0)\right) + \operatorname{sinc}^2\left(T(f+f_0)\right)\right)$$

(8) : First of all, lets review some important lemmas.

Lemma 1 :

If X_1, X_2 are two jointly gaussian random variables with variances σ_1^2 and σ_2^2, respectively and with correlation coefficient

$$\rho = \frac{Cov(X_1, X_2)}{\sigma_1 \sigma_2} = \frac{E(X_1 X_2) - EX_1 EX_2}{\sigma_1 \sigma_2}$$, then X_1/X_2 (X_1 conditioned on X_2)

is a Gaussian random variable.

proof:

$$f_{X_1/X_2}(x_1/x_2) = \frac{f_{X_1,X_2}(x_1, x_2)}{f_{X_2}(x_2)} \qquad K = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \rightarrow \text{covariance matrix}$$

$$|K| = \sigma_1^2\sigma_2^2 - \rho^2\sigma_1^2\sigma_2^2 = (1-\rho^2)\sigma_1^2\sigma_2^2$$

$$\Rightarrow K^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2} \begin{pmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}$$

$$\Rightarrow f_{X_1,X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2} \exp\left(\frac{-1}{2}\begin{pmatrix}\tilde{X}_1 & \tilde{X}_2\end{pmatrix} K^{-1} \begin{pmatrix}\tilde{X}_1 \\ \tilde{X}_2\end{pmatrix}\right)$$

where $\tilde{X}_1 \triangleq X_1 - EX_1$ and $\tilde{X}_2 = X_2 - EX_2$

$$\Rightarrow f_{X_1/X_2}(x_1/x_2) = \frac{\frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_1\sigma_2} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{\tilde{X}_1^2}{\sigma_1^2} + \frac{\tilde{X}_2^2}{\sigma_2^2} - 2\rho\frac{\tilde{X}_1}{\sigma_1}\frac{\tilde{X}_2}{\sigma_2}\right)\right)}{\frac{1}{\sqrt{2\pi}\,\sigma_2} \exp\left(\frac{-1}{2\sigma_2^2}\tilde{X}_2^2\right)} =$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_1^2}} \exp\left(\frac{-1}{2(1-\rho^2)}\left(\frac{\tilde{X}_1}{\sigma_1} - \frac{\rho\tilde{X}_2}{\sigma_2}\right)^2\right)$$

$$= \frac{1}{\sqrt{2\pi(1-\rho^2)\sigma_1^2}} \exp\left(\frac{-1}{2(1-\rho^2)\sigma_1^2}\left(x_1 - EX_1 - \rho\frac{\sigma_1}{\sigma_2}(x_2 - EX_2)\right)^2\right)$$

As it can be seen, $f_{X_1/X_2}(x_1/X_2 = x_2)$ is in the form of a Gaussian random variable with mean $EX_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - EX_2)$ and variance $(1-\rho^2)\sigma_1^2$

In summary: $\begin{cases} E(X_1/X_2) = EX_1 + \rho\frac{\sigma_1}{\sigma_2}(X_2 - EX_2) \\ \sigma^2_{X_1/X_2} = (1-\rho^2)\sigma_1^2 \end{cases}$

Lemma 2: Let X be a zero-mean Gaussian random variable with variance σ^2. Then, we have:

$$E(X^4) = 3\sigma^4$$

proof: $\phi_X(\omega) = \int_{-\infty}^{+\infty} f_X(x) e^{j\omega x} dx = \exp\left(\frac{-1}{2}\sigma^2\omega^2\right)$

$$\Rightarrow \phi_X^{(4)}(\omega) = \int_{-\infty}^{+\infty} f_X(x) x^4 e^{j\omega x} dx \Rightarrow$$

$$E(X^4) = \int_{-\infty}^{+\infty} f_X(x) x^4 dx = \phi_X^{(4)}(0)$$

$$\phi_X'(\omega) = -\sigma^2\omega\,\phi_X(\omega) \Rightarrow \phi_X''(\omega) = -\sigma^2\phi_X(\omega) - \sigma^2\omega\,\phi_X'(\omega)$$

$$= -\sigma^2\phi_X(\omega) + \sigma^4\omega^2\phi_X(\omega) = \left(-\sigma^2 + \sigma^4\omega^2\right)\phi_X(\omega)$$

$$\Rightarrow \phi_X^{(3)}(\omega) = 2\sigma^4\omega\,\phi_X(\omega) + \left(-\sigma^2+\sigma^4\omega^2\right)\phi_X'(\omega)$$

$$= 2\sigma^4\omega\,\phi_X(\omega) - \sigma^2\omega\left(-\sigma^2+\sigma^4\omega^2\right)\phi_X(\omega) = \left(3\sigma^4\omega - \sigma^6\omega^3\right)\phi_X(\omega)$$

$$\Rightarrow \phi_X^{(4)}(\omega) = \left(3\sigma^4 - 3\sigma^6\omega^2\right)\phi_X(\omega) - \sigma^2\omega\left(3\sigma^4\omega - \sigma^6\omega^3\right)\phi_X(\omega)$$

$$\Rightarrow \phi_X^{(4)}(0) = 3\sigma^4\phi_X(0) = 3\sigma^4$$

Now, lets solve the main problem.
$X(t)$ is stationary, zero-mean Gaussian process.
with psD $S_{XX}(f)$ and auto-correlation function $R_{X,X}(\tau)$

$$Y(t)=X^2(t) \rightarrow R_{Y,Y}(t_1,t_2)=E(Y(t_1)Y(t_2))=E(X^2(t_1)X^2(t_2))$$

Let $X_1 \triangleq X(t_1)$ and $X_2 = X(t_2)$

$\Rightarrow R_{Y,Y}(t_1,t_2)=E(X_1^2X_2^2)$ X_1 and X_2 are jointly Gaussian.

$$=\int_{-\infty}^{+\infty} E(X_1^2X_2^2/X_2=x_2)f_{X_2}(x_2)dx_2$$

$$=\int_{-\infty}^{+\infty} E(X_1^2x_2^2/X_2=x_2)f_{X_2}(x_2)dx_2=\int_{-\infty}^{+\infty}x_2^2E(X_1^2/X_2=x_2)f_{X_2}(x_2)dx_2$$

$$=\int_{-\infty}^{+\infty}x_2^2\left(\sigma^2_{X_1/X_2}+(EX_1/X_2)^2\right)f_{X_2}(x_2)dx_2=?$$

$$\left\{\begin{array}{l} E(X_1/X_2=x_2)=EX_1+\frac{\sigma_1}{\sigma_2}\rho(x_2-EX_2)=0+x_2\times\rho\frac{\sigma_1}{\sigma_2} \\ \sigma^2_{X_1/X_2=x_2}=(1-\rho^2)\sigma_1^2 \end{array}\right\}$$

$$=\int_{-\infty}^{+\infty}x_2^2\left((1-\rho^2)\sigma_1^2+\rho^2\frac{\sigma_1^2}{\sigma_2^2}x_2^2\right)f_{X_2}(x_2)dx_2=$$

$$(1-\rho^2)\sigma_1^2\int_{-\infty}^{+\infty}x_2^2f_{X_2}(x_2)dx_2+\rho^2\frac{\sigma_1^2}{\sigma_2^2}\int_{-\infty}^{+\infty}x_2^4f_{X_2}(x_2)dx_2$$

$$= (1-\rho^2)\sigma_1^2\sigma_2^2 + \rho^2 \frac{\sigma_1^2}{\sigma_2^2} \times 3\sigma_2^4$$

$$= (1+2\rho^2)\sigma_1^2\sigma_2^2$$

$$\sigma_1^2 = E(X_1^2) = E(X^2(t_1)) = R_{X,X}(0)$$

$$\sigma_2^2 = E(X_2^2) = E(X^2(t_2)) = R_{X,X}(0)$$

$$\text{cov}(X_1, X_2) = E(X_1 X_2) = E(X(t_1)X(t_2)) = R_{X,X}(t_1, t_2)$$

$$\Rightarrow R_{Y,Y}(t_1,t_2) = \left(1 + 2 \times \frac{R_{X,X}^2(t_1,t_2)}{R_{X,X}^2(0)}\right) R_{X,X}^2(0)$$

$$= R_{X,X}^2(0) + 2 R_{X,X}^2(t_1,t_2)$$

let $t_2 - t_1 = \tau$

$$\Rightarrow R_Y(\tau) = R_X^2(0) + 2 R_X^2(\tau)$$

$\Rightarrow Y(t)$ is WSS.

$$S_Y(f) = F\{R_Y(\tau)\} = R_X^2(0)\delta(f) + 2 F\{R_X^2(\tau)\}$$

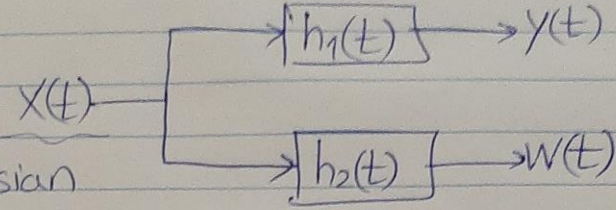
$$= \underbrace{R_X^2(0)\delta(f) + 2 S_X(f) * S_X(f)}_{\text{PSD of } Y(t) \text{ in terms of PSD of } X(t)}$$

If $S_X(f) = \text{rect}\left(\frac{f}{2B}\right) \Rightarrow R_X(0) = P_{av} = \int_{-\infty}^{+\infty} S_X(f)\,df = 2B$

$$S_X(f) * S_X(f) = \text{rect}\left(\frac{f}{2B}\right) * \text{rect}\left(\frac{f}{2B}\right) = 2B\,\Lambda\left(\frac{f}{2B}\right)$$

$$\Rightarrow S_Y(f) = 4B^2\delta(f) + 4B\,\Lambda\left(\frac{f}{2B}\right)$$

10.31



WSS Gaussian random process

$$Y(t)=\int_{-\infty}^{+\infty} X(\tau)h_1(t-\tau)d\tau \Rightarrow Y(t_1)=\int_{-\infty}^{+\infty} X(\tau)h_1(t_1-\tau)d\tau$$

$$W(t)=\int_{-\infty}^{+\infty} X(s)h_2(t-s)ds \Rightarrow W(t_2)=\int_{-\infty}^{+\infty} X(s)h_2(t_2-s)ds$$

First, we show that $Y(t_1)$ and $W(t_2)$ are jointly Gaussian random variables.

$$\alpha Y(t_1)+\beta W(t_2)=\int_{-\infty}^{+\infty}\underbrace{\left(\alpha h_1(t_1-\tau)+\beta h_2(t_2-\tau)\right)}_{\triangleq f(\tau)}X(\tau)d\tau=\int_{-\infty}^{+\infty} f(\tau)X(\tau)d\tau$$

Since $X(t)$ is Gaussian process, the result of above integral is a Gaussian random variable.

$$EY(t_1)=\int_{-\infty}^{+\infty} EX(\tau)h_1(t_1-\tau)d\tau=m_X\int_{-\infty}^{+\infty} h_1(t_1-\tau)d\tau=m_X\int_{-\infty}^{+\infty} h_1(\tau)d\tau=m_X H_1(0)$$

$$EW(t_2)=\int_{-\infty}^{+\infty} EX(\tau)h_2(t_2-\tau)d\tau=m_X\int_{-\infty}^{+\infty} h_2(t_2-\tau)d\tau=m_X\int_{-\infty}^{+\infty} h_2(\tau)d\tau=m_X H_2(0)$$

$$S_Y(f)=|H_1(f)|^2 S_X(f)\Rightarrow R_Y(0)=E(Y^2(t_1))=\int_{-\infty}^{+\infty} S_Y(f)df=\int_{-\infty}^{+\infty} S_X(f)|H_1(f)|^2df$$

$$S_W(f)=|H_2(f)|^2 S_X(f)\Rightarrow R_W(0)=E(W^2(t_2))=\int_{-\infty}^{+\infty} S_W(f)df=\int_{-\infty}^{+\infty} S_X(f)|H_2(f)|^2df$$

$$\Rightarrow Var(Y(t_1))=\int_{-\infty}^{+\infty} S_X(f)|H_1(f)|^2df-m_X^2H_1^2(0)$$

$$Var(W(t_2))=\int_{-\infty}^{+\infty} S_X(f)|H_2(f)|^2df-m_X^2H_2^2(0)$$

$$E(y(t_1)W(t_2)) = E\left\{\int_{-\infty}^{+\infty} h_1(\tau)X(t_1-\tau)d\tau \times \int_{-\infty}^{+\infty} h_2(S)X(t_2-S)dS\right\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\tau)h_2(S)E\{X(t_1-\tau)X(t_2-S)\}d\tau dS$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h_1(\tau)h_2(S)R_X(t_1-t_2+S-\tau)d\tau dS$$

$$\Rightarrow C = \begin{pmatrix} Var(Y(t_1)) & E(Y(t_1)W(t_2)) - EY(t_1)EW(t_2) \\ E(Y(t_1)W(t_2)) - EY(t_1)EW(t_2) & Var(W(t_2)) \end{pmatrix}$$

matrix C can be determind by the previous derived ralations.

$\Rightarrow$ let $\begin{cases} X_1 = Y(t_1) \\ X_2 = W(t_2) \end{cases}$, the joint PDF is:

$$\frac{1}{2\pi |C|^{1/2}} \exp\left(\frac{-1}{2}(X_1 - EX_1)^T C^{-1}\begin{pmatrix} X_1 - EX_1 \\ X_2 - EX_2 \end{pmatrix}\right)$$

(b): $X(t)$ is White Gaussian noise $\Rightarrow R_X(t_1-t_2) = 6^2\delta(t_1-t_2)$

$EX(t) = 0$ $\qquad S_X(f) = 6^2$

$\Rightarrow EY(t_1) = m_x H_1(0) = 0 \qquad EW(t_2) = m_x H_2(0) = 0$

$Var(y(t_1)) = 6^2\int_{-\infty}^{+\infty} |H_1(f)|^2 df = 6^2 \mathcal{E}_{h_1}$

$Var(w(t_2)) = 6^2\int_{-\infty}^{+\infty} |H_2(f)|^2 df = 6^2 \mathcal{E}_{h_2}$

$Cov(Y(t_1), W(t_2)) = E(Y(t_1)W(t_2)) = 6^2\int_{-\infty}^{+\infty} h_1(\tau)\int_{-\infty}^{+\infty} h_2(S)\delta(t_1-t_2+S-\tau)dS d\tau$

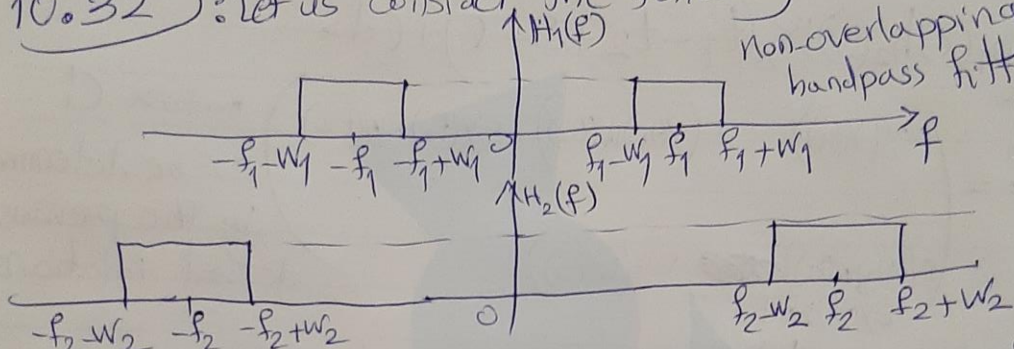
$= 6^2\int_{-\infty}^{+\infty} h_1(\tau)\int_{-\infty}^{+\infty} h_2(\tau-t_1+t_2)\delta(S-\tau+t_1-t_2)dS d\tau$

$$= \sigma^2 \int_{-\infty}^{+\infty} h_1(\tau) h_2(\tau - t_1 + t_2)\, d\tau =$$

$$= \sigma^2 h_1(t) * h_2(-t) \Big|_{at\ t = t_1 - t_2}$$

10.32 : Let us consider the following filters:

non-overlapping bandpass filters:



$H_1(f) \times H_2(f) = 0$

Noting problem 10.31, $Y(t)$ and $W(t)$ are jointly Gaussian random processes.

$$Cov(Y(t_1), W(t_2)) = \sigma^2 \int_{-\infty}^{+\infty} h_1(\tau) h_2(\tau - (t_1 - t_2))\, d\tau$$ parseval's theorem

$$= \sigma^2 \int_{-\infty}^{+\infty} H_1(f) \times \left(H_2(f) \exp(-j2\pi f(t_1 - t_2))\right)^* df$$

$$= \sigma^2 \int_{-\infty}^{+\infty} H_1(f) H_2^*(f) \exp(j2\pi f(t_1 - t_2))\, df$$

H_1 and H_2 are non-overlapping

$= 0$ $\Rightarrow Y(t_1)$ and $W(t_2)$ are uncorrelated.

since $Y(t)$ and $W(t)$ are jointly Gaussian, uncorrelation implies independency $\Rightarrow Y(t)$ and $W(t)$ are independent random processes.