problem set 3 (solutions)

1): X is uniform over $\{-4,-3,\dots,3,4\}$, $y \triangleq \sin^2\left(\frac{\pi X}{4}\right)$

$\Rightarrow pr\{X=i\}=\frac{1}{9}$ for $-4\leqslant i\leqslant 4$

$$Ey=E\left\{\sin^2\left(\frac{\pi X}{4}\right)\right\}=\sum_{i=-4}^{+4}\sin^2\left(\frac{\pi i}{4}\right)\times pr\{X=i\}=\frac{1}{9}\sum_{i=-4}^{+4}\sin^2\left(\frac{\pi i}{4}\right)$$

$$=\frac{2}{9}\sum_{i=1}^{4}\sin^2\left(\frac{\pi i}{4}\right)=\frac{2}{9}\left(\sin^2\left(\frac{\pi}{4}\right)+\sin^2\left(\frac{2\pi}{4}\right)+\sin^2\left(\frac{3\pi}{4}\right)+\sin^2\left(\frac{4\pi}{4}\right)\right)$$

$$=\frac{2}{9}\left(\frac{1}{2}+1+\frac{1}{2}+0\right)=\frac{4}{9}$$

$$Ey^2=E\left\{\sin^4\left(\frac{\pi X}{4}\right)\right\}=\sum_{i=-4}^{+4}\sin^4\left(\frac{\pi i}{4}\right)\times pr\{X=i\}=\frac{2}{9}\sum_{i=1}^{4}\sin^4\left(\frac{\pi i}{4}\right)$$

$$=\frac{2}{9}\left(\sin^4\left(\frac{\pi}{4}\right)+\sin^4\left(\frac{2\pi}{4}\right)+\sin^4\left(\frac{3\pi}{4}\right)+\sin^4\left(\frac{4\pi}{4}\right)\right)=$$

$$=\frac{2}{9}\left(\frac{1}{4}+1+\frac{1}{4}\right)=\frac{2}{9}\times\frac{3}{2}=\frac{1}{3}$$

$$\Rightarrow Var(y)=Ey^2-(Ey)^2=\frac{1}{3}-\left(\frac{4}{9}\right)^2=\frac{11}{81}$$

② M is a geometric random variable $pr(M=k)=p^k(1-p)$; $k=0,1,2,\dots$

$$pr\{M \geq k+j / M \geq j\} = \frac{pr\{M \geq k+j , M \geq j\}}{pr\{M \geq j\}} = \frac{pr\{M \geq k+j\}}{pr\{M \geq j\}}$$

$$= \frac{\sum_{l=k+j}^{+\infty} p^l(1-p)}{\sum_{l=j}^{+\infty} p^l(1-p)} = \frac{(1-p)\times\frac{p^{k+j}}{1-p}}{(1-p)\times\frac{p^j}{1-p}} = p^k = \sum_{l=k}^{+\infty} p^l(1-p) = pr\{M \geq k\}$$

③: $f_X(x;\alpha,x_m) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$; $x \geq x_m$ and zero elsewhere.

$$\begin{cases}\alpha=1 \\ x_m=1\end{cases} \Rightarrow f_X(x;1,1) = \frac{1}{x^2} \; ; \; x \geq 1$$

$$\text{mean} = \int_1^{+\infty} x f_X(x;1,1)\,dx = \int_1^{+\infty} x \times \frac{1}{x^2}\,dx = \ln(x)\Big]_1^{+\infty} = +\infty$$

5: $f_X(x) = \frac{\alpha}{\pi(x^2+\alpha^2)}$, $Y \triangleq \frac{1}{X} \Rightarrow g(x) = \frac{1}{x}$, $g'(x) = \frac{-1}{x^2}$

$y = \frac{1}{x} \Rightarrow \boxed{x = \frac{1}{y}}$ $f_Y(y) = \frac{f_X\left(\frac{1}{y}\right)}{\left|g'(x)\right|_{x=\frac{1}{y}}}$

$$\Rightarrow f_Y(y) = \frac{\frac{\alpha}{\pi\left(\frac{1}{y^2}+\alpha^2\right)}}{\left|\frac{-1}{\left(\frac{1}{y}\right)^2}\right|} = \frac{\alpha}{\pi y^2\left(\frac{1}{y^2}+\alpha^2\right)} = \frac{\alpha}{\pi(1+\alpha^2 y^2)}$$

$$\Rightarrow f_Y(y) = \frac{\frac{1}{\alpha}}{\pi\left(y^2+\left(\frac{1}{\alpha}\right)^2\right)}$$ ⇝ Cauchy random variable with parameter $\frac{1}{\alpha}$

6: U uniform over $[0,1]$, $X \triangleq -\ln\left(\left|1-\frac{1}{U}\right|\right)$

$0 < U < 1 \Rightarrow 1-\frac{1}{U} < 0 \Rightarrow X = -\ln\left(\frac{1}{U}-1\right)$

$$F_X(x) = \Pr\{X \leqslant x\} = \Pr\left\{-\ln\left(\frac{1}{U}-1\right) \leqslant x\right\} = \Pr\left\{U \leqslant \frac{1}{1+e^{-x}}\right\} = \frac{1}{1+e^{-x}}$$

$$f_X(x) = \frac{d}{dx}F_X(x) = \frac{-(-1)e^{-x}}{(1+e^{-x})^2} = \frac{e^{-x}}{(1+e^{-x})^2}$$

$$\Rightarrow f_X(x) = \frac{e^{x}}{(1+e^{x})^2}$$

⑦: $f_x(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $y \triangleq \begin{cases} 0 ; & x \leqslant 0 \\ x ; & x > 0 \end{cases}$

$F_y(y) = pr\{y \leqslant y\}$

we consider the following two cases:

case 1 $(y < 0) \Rightarrow F_y(y) = pr\{x \leqslant y\} = 0$

case 2 $(y \geqslant 0) \Rightarrow F_y(y) = pr\{x \leqslant y\} = pr\{y=0\} + pr\{0 < y \leqslant y\}$

$= pr\{x \leqslant 0\} + pr\{0 < x \leqslant y\} = \cancel{pr\{x \leqslant 0\}} + pr\{x \leqslant y\} - \cancel{pr\{x \leqslant 0\}}$

$= pr\{x \leqslant y\} = F_x(y)$

$\Rightarrow F_y(y) = \begin{cases} 0 & ; y < 0 \\ F_x(y) ; & y \geqslant 0 \end{cases}$

$\Rightarrow f_y(y) = \frac{d}{dy} F_y(y) = F_x(0)\delta(y) + \begin{cases} 0 ; & y < 0 \\ f_x(y) ; & y \geqslant 0 \end{cases}$

⑧ let $X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$ be a random vector

$$K_X = E\left\{(X - EX)(X - EX)^T\right\}$$ — covariance matrix which is symmetric

A symmetric matrix M with real entries is positive semi-definite if the real number $Z^T M Z$ is nonnegative for every nonzero column vector Z.

Let $Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{pmatrix} \in \mathbb{R}^n$

$$\Rightarrow Z^T K_X Z = Z^T E\left\{(X-EX)(X-EX)^T\right\} Z = E\left\{Z^T(X-EX)\left(Z^T(X-EX)\right)^T\right\}$$

Defining the new random variable $Y \triangleq Z^T(X-EX)$

we have:

$$Z^T K_X Z = E\{Y^2\} \geqslant 0$$

$\Rightarrow K_X$ is positive semi-definite

9: X is a $n \times 1$ random vector.

$$\mu = EX \quad K_x = E\{(X - EX)(X - EX)^T\}$$

$$Y = AX + b$$

$$\boxed{EY = AEX + b}$$

$$K_y = E\{(Y - EY)(Y - EY)^T\} = E\{(AX + \cancel{b} - AEX - \cancel{b})(AX + \cancel{b} - AEX - \cancel{b})^T\}$$

$$= E\{A(X - EX)(X - EX)^T A^T\} = A K_x A^T$$