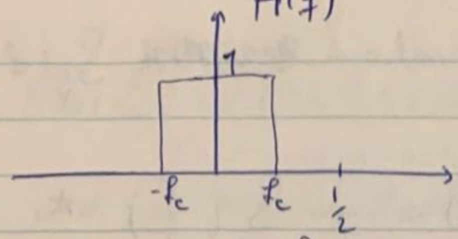
10.46. a)



$H(f) = rect\left(\frac{f}{2f_c}\right)$

Inverse Fourier transform $\rightarrow$ $h[n] = \int_{-f_c}^{f_c} e^{j2\pi f n} \, df$

$$= \int_{-f_c}^{f_c} (\cos 2\pi f n + j \sin 2\pi f n) \, df$$

$$= \frac{\sin 2\pi f_c n}{\pi n}$$

b) $S_X(f) = A \, tri\left(\frac{f}{W}\right)$. $S_{\tilde{Y}}(f) = |H(f)|^2 S_{\tilde{X}}(f)$

X is the analog signal with PSD of $S_X(f)$

$\tilde{X}$ is the sampled version of X, which has the PSD of $S_{\tilde{X}}(f)$.

what is the relation btw $S_X(f)$ and $S_{\tilde{X}}(f)$?

$$S_X(f) = F\{R_X(\tau)\} = F\{E\{X(t)\,X(t+\tau)\}\}$$

$$S_{\tilde{X}}(f) = F\{R_{\tilde{X}}[m]\} = F\{E\{\tilde{X}[n]\,\tilde{X}[n+m]\}\}$$

$$= F\{E\{X(nT)\,X((n+m)T)\}\} = F\{R_X(mT)\}$$

Therefor the relation btw $S_X(f)$ and $S_{\tilde{X}}(f)$ is the relation btw the Fourier transform of a bandlimited signal and the DTFT of its input Samples.

In just one period of ~~samples~~ $S_{\tilde{X}}(f)$ we have:

$$S_{\tilde{X}}(f) = \frac{1}{T} S_X\left(\frac{f}{T}\right) \quad \star$$

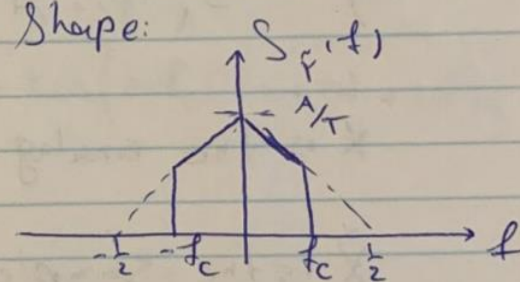
$$= \frac{A}{T} \text{tri}\left(\frac{f}{TW}\right) = \frac{A}{T} \text{tri}(2f)$$

where $\frac{1}{T} = 2W$, so we have

$$S_{\tilde{Y}}(f) = |H(f)|^2 S_{\tilde{X}}(f)$$

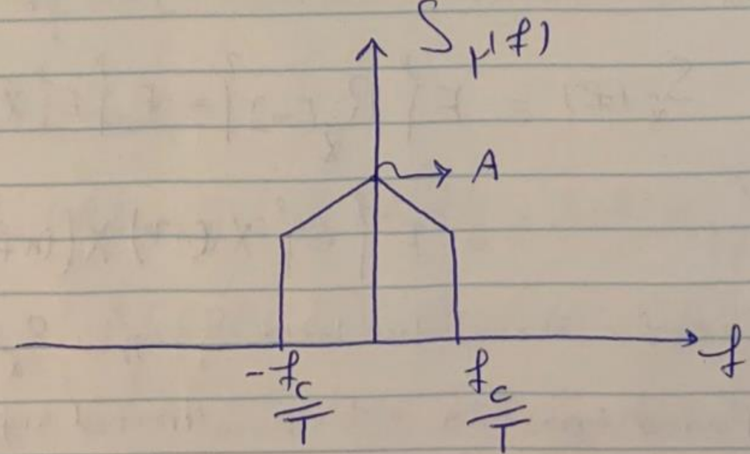
$$= H(f) \frac{A}{T} \text{tri}(2f)$$

Thus $S_{\tilde{Y}}(f)$ has the following shape:



c) The relation btw $S_{\tilde{Y}}(f)$ and $S_Y(f)$ is the same as $(\star)$

so → $S_Y(f) = T S_{\tilde{Y}}(Tf)$ and it has the shape:



10.51. $X(t) = \underbrace{2\cos(2\pi f_1 t + \phi)}_{A(t)} \cos(2\pi f_c t + \theta)$

Where ϕ and θ are uniformly distributed over $(-\pi, \pi)$ and are independent. The mean of X is zero.

$$R_X(\tau) = E\{X(t)X(t+\tau)\}$$

$$= 4E\{\cos(2\pi f_1 t + \phi)\cos(2\pi f_1(t+\tau)+\phi)\cos(2\pi f_c t + \theta) \cdot \cos(2\pi f_c(t+\tau)+\theta)\}$$

$$= R_A(\tau)\, E\{\cos(2\pi f_c t + \theta)\cos(2\pi f_c(t+\tau)+\theta)\}$$

$$= R_A(\tau) \cdot \frac{1}{2} E\{\cos(2\pi f_c \tau) + \cos(2\pi f_c(2t+\tau) + 2\theta)\}$$

$$= \frac{1}{2} R_A(\tau)\cos(2\pi f_c \tau)$$

$$S_X(f) = \frac{1}{4}\left(S_A(f-f_c) + S_A(f+f_c)\right)$$

$$R_A(\tau) = 4E\{\cos(2\pi f_1 t + \phi) \cdot \cos(2\pi f_1(t+\tau)+\phi)\}$$

$$= 4\cos 2\pi f_1 \tau$$

$$\rightarrow S_A(f) = 2\left(\delta(f-f_1) + \delta(f+f_1)\right)$$

$$\rightarrow S_X(f) = \frac{1}{2}\begin{pmatrix} \delta(f-f_1-f_c) + \delta(f-f_c+f_1) \\ +\delta(f+f_1+f_c) + \delta(f+f_c-f_1) \end{pmatrix}.$$

10.53. $X(t) = A(t)\cos(2\pi f_c t + \theta) + B(t)\sin(2\pi f_c t + \theta)$

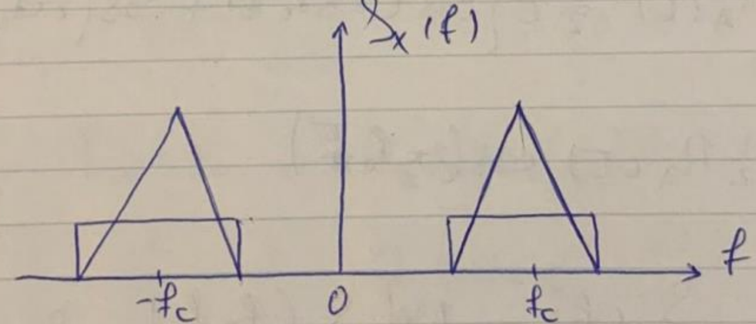
$R_X(\tau) = E\{X(t)\,X(t+\tau)\}$, $A(t), B(t)$ are independent.

$$= E\{A(t)\,A(t+\tau)\cos(2\pi f_c t + \theta)\cos(2\pi f_c (t+\tau) + \theta)\}$$

$$+ E\{B(t)\,B(t+\tau)\sin(2\pi f_c t + \theta)\sin(2\pi f_c (t+\tau) + \theta)\}$$

$$= \frac{1}{2}\cos(2\pi f_c \tau)\,R_A(\tau) + \frac{1}{2}\cos(2\pi f_c \tau)\,R_B(\tau)$$

$$\rightarrow S_X(f) = \frac{1}{4}\{S_A(f-f_c) + S_A(f+f_c) + S_B(f-f_c) + S_B(f+f_c)\}.$$



10.57) a,b) The optimum filter must satisfy the orthogonality condition which states the error e_t must be orthogonal to all observations X_α, that is.

$$0 = E\{e_t X_\alpha\}, \text{ for } \alpha \in I$$

$$= E\{(Z_t - Y_t) X_\alpha\} = 0 \quad , Y_t = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}$$

Thus, the optimum linear filter must satisfy the set of $a+b+1$ linear equations given by

$$R_{Z,X}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta), \quad -b \le m \le a.$$

$$X_\alpha = Z_\alpha + N_\alpha, \quad R_Z(k) = \sigma_Z^2 (r_1)^{|k|} \quad , |r_1| < 1$$

$$R_N(k) = \sigma_N^2 r_2^{|k|} \quad , |r_2| < 1$$

- We must solve $R_{Z,X}(m) = \sum_{\beta=0}^{p} h_\beta R_X(m-\beta)$ for $0 \le m \le p$.

$$R_{Z,X}(m) = E\{Z(n) X(n-m)\} = E\{Z(n)(Z(n-m) + N(n-m))\} = R_Z(m)$$

$$R_X(m-\beta) = R_Z(m-\beta) + R_N(m-\beta)$$

$$\rightarrow R_Z(m) = \sum_{\beta=0}^{p} h_\beta \left(R_Z(m-\beta) + R_N(m-\beta)\right), \quad 0 \le m \le p$$

if $\Gamma = \frac{\sigma_N^2}{\sigma_z^2}$ then we have:

$$r_1^{|m|} = \sum_{\beta=0}^{p} h_\beta \left(r_1^{|m-\beta|} + \Gamma r_2^{|m-\beta|} \right), \quad 0 \le m \le p$$

$$\begin{bmatrix} 1 \\ r_1 \\ r_1^2 \\ \vdots \\ r_1^p \end{bmatrix} = \begin{bmatrix} 1+\Gamma & r_1+\Gamma r_2 & r_1^2+\Gamma r_2^2 & \cdots & r_1^p+\Gamma r_2^p \\ r_1+\Gamma r_2 & 1+\Gamma & r_1+\Gamma r_2 & \cdots & r_1^{p-1}+\Gamma r_2^{p-1} \\ r_1^2+\Gamma r_2^2 & r_1+\Gamma r_2 & 1+\Gamma & \cdots & r_1^{p-2}+\Gamma r_2^{p-2} \\ \vdots & & & & \\ r_1^p+\Gamma r_2^p & r_1^{p-1}+\Gamma r_2^{p-1} & r_1^{p-2}+\Gamma r_2^{p-2} & \cdots & 1+\Gamma \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix}$$

c) $p=2$, $\sigma_z^2 = 9$, $r_1 = 2/3$, $\sigma_N^2 = 1$, $r_2 = \frac{1}{3}$

$$\begin{bmatrix} 1 \\ \frac{2}{3} \\ \frac{4}{9} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} & \frac{19}{27} & \frac{37}{81} \\ \frac{19}{27} & \frac{10}{9} & \frac{19}{27} \\ \frac{37}{81} & \frac{19}{27} & \frac{10}{9} \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} \rightarrow \begin{pmatrix} h_0 \\ h_1 \\ h_2 \end{pmatrix} \simeq \begin{pmatrix} 0.868 \\ 0.038 \\ 0.019 \end{pmatrix}$$

d) $E\{e_t^2\} = E\{e_t(z_t - Y_t)\} = E\{e_t z_t\}$

↳ for optimum filter the error (e_t) and the estimate Y_t are arthogonal.

$$= E\{(z_t - Y_t) z_t\} = E\{z_t z_t\} - E\{Y_t z_t\}$$

$$= R_z(0) - E\left\{ z_t \sum_{\beta=-b}^{a} h_\beta X_{t-\beta} \right\}$$

$$= R_z(0) - \sum_{\beta=-b}^{a} h_\beta R_{zx}(\beta).$$

Contd. →

10.57. d → cont'd)

$$E\{e_t^2\} = R_z(0) - \sum_{\beta=0}^{p} h_\beta R_{z,x}(\beta)$$

$$= R_z(0) - \sum_{\beta=0}^{2} h_\beta R_z(\beta)$$

$$= \sigma_z^2 - \sigma_z^2 \sum_{\beta=0}^{2} h_\beta r_1^{|\beta|}$$

$$= 9\left(1-h_0-h_1\left(\tfrac{2}{3}\right)-h_2\left(\tfrac{4}{9}\right)\right) = 0.884$$

10.58. a) $R_z(k) = \begin{cases} (1+\alpha^2)\sigma^2 \triangleq \sigma_z^2 & k=0 \\ \alpha\sigma^2 & k=\pm 1 \\ 0 & \text{o.w.} \end{cases}$, $R_N(m-\beta) = \sigma_N^2 \delta(m-\beta)$

$$R_z(m) = \sum_{\beta=0}^{p} h_\beta \left(R_z(m-\beta) + R_N(m-\beta) \right), \quad 0 \le m \le p.$$

$$\rightarrow R_z(m) = \sum_{\beta=0}^{p} h_\beta \left(R_z(m-\beta) + \sigma_N^2 (m-\beta) \right), \quad 0 \le m \le p$$

$$\begin{bmatrix} \sigma_z^2+\sigma_N^2 & \alpha\sigma^2 & 0 & 0 & \cdots & 0 \\ \alpha\sigma^2 & \sigma_z^2+\sigma_N^2 & \alpha\sigma^2 & 0 & \cdots & 0 \\ 0 & \alpha\sigma^2 & \sigma_z^2+\sigma_N^2 & \cdots & & 0 \\ \vdots & & \cdots & & & \\ 0 & & & & 0 & \sigma_z^2+\sigma_N^2 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_p \end{bmatrix} = \begin{bmatrix} \sigma_z^2 \\ \alpha\sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

b). $p=2$, $P = \sigma_N^2/\sigma^2$

$$\begin{bmatrix} 1+\alpha^2+P & \alpha & 0 \\ \alpha & 1+\alpha^2+P & \alpha \\ 0 & \alpha & 1+\alpha^2+P \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} 1+\alpha^2 \\ \alpha \\ 0 \end{bmatrix}$$

$$h_0 = \frac{\dfrac{(1+\alpha^2+\Gamma)^2(1+\alpha^2) - \alpha^2(1+\alpha^2)}{1+\alpha^2+\Gamma} - \alpha^2}{(1+\alpha^2+\Gamma)^2 - 2\alpha^2}$$

$$h_1 = \frac{\alpha\Gamma}{(1+\alpha^2+\Gamma)^2 - 2\alpha^2}$$

$$h_2 = \frac{-\dfrac{\alpha^2\Gamma}{1+\alpha^2+\Gamma}}{(1+\alpha^2+\Gamma)^2 - 2\alpha^2}$$

c. $E\{e_t^2\} = R_z(0) - \sum_{\beta=0}^{2} h_\beta R_z(\beta)$

$$= R_z(0) - h_0 R_z(0) - h_1 R_z(1) - h_2 \underbrace{R_z(\cancel{\beta}\,2)}_{=0}$$

$$= \sigma^2 \{(1+\alpha^2)(1-h_0) - \alpha h_1\}$$

Check: If $\sigma_N^2 = 0$, i.e. no noise, $h_0 = 1$, $h_1 = 0$, $h_2 = 0$, i.e. no filtering

and $E\{e_t^2\} = 0$, i.e. no error.

10.59. a) $X_\alpha = Z_\alpha + N_\alpha$

a) $R_{ZX}(m) = \sum_{\beta=-p}^{p} h_\beta R_X(m-\beta) \quad , \quad |m| \le p$

$$R_{ZX}(m) = E\{Z_n X_{n-m}\} = E\{Z_n(Z_{n-m} + N_{n-m})\}$$

$$= R_Z(m)$$

Noise and signal are indep. and noise is zero-mean.

$$R_X(m-\beta) = R_Z(m-\beta) + R_N(m-\beta)$$

the opt. filter must satisfy:

$$R_Z(m) = \sum_{\beta=-p}^{p} h_\beta \{R_Z(m-\beta) + R_N(m-\beta)\}, \quad |m| \le p$$

b) $$\begin{bmatrix} R_Z(0)+R_N(0) & R_Z(1)+R_N(1) & \cdots & R_Z(2p)+R_N(2p) \\ R_Z(1)+R_N(1) & R_Z(0)+R_N(0) & \cdots & R_Z(2p-1)+R_N(2p-1) \\ \vdots & & & \\ R_Z(2p)+R_N(2p) & \cdots & - & R_Z(0)+R_N(0) \end{bmatrix} \begin{bmatrix} h_{-p} \\ h_{-p+1} \\ \vdots \\ h_0 \\ \vdots \\ h_p \end{bmatrix}$$

$$= \begin{bmatrix} R_Z(-p) \\ \vdots \\ R_Z(0) \\ \vdots \\ R_Z(p) \end{bmatrix}$$

$$\begin{pmatrix} 5 & 3 & \frac{9}{4} \\ 3 & 5 & 3 \\ \frac{9}{4} & 3 & 5 \end{pmatrix} \begin{pmatrix} h_{-1} \\ h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} h_{-1} \\ h_0 \\ h_1 \end{pmatrix} = \begin{pmatrix} 0.164 \\ 0.603 \\ 0.164 \end{pmatrix}$$

$$E\{e_t^2\} = R_Z(0) - \sum_{\beta=-1}^{1} h_\beta R_Z(\beta)$$

$$= 4 - 0.492 - 2.412 - 0.492 = 0.604.$$

10.65 $\quad S_Z(f) = \dfrac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}, \quad S_N(f) = \dfrac{N_0}{2}$

$$H(f) = \frac{S_Z(f)}{S_Z(f) + S_N(f)} = \frac{4\alpha}{(4\alpha + 2N_0\alpha^2) + \frac{N_0}{2} 4\pi^2 f^2}$$

For the random telegraph signal, we have

$$R_Z(\tau) = E\{Z(t) Z(t+\tau)\} = Pr\left\{\begin{array}{l} Z(t) \,\&\, Z(t+\tau) \\ \text{have the same signs} \end{array}\right\} - P\left\{\begin{array}{l} Z(t) \,\&\, Z(t+\tau) \\ \text{have the opp. signs.} \end{array}\right\}$$

$$= \frac{1+e^{-2\alpha\tau}}{2} - \frac{1-e^{-2\alpha\tau}}{2} = e^{-2\alpha\tau}$$

In this derivation we have assumed $\tau > 0$, so $R_Z(\tau) = e^{-2\alpha|\tau|}$

$$\rightarrow S_Z(f) = \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$$

$$e^{-\alpha|t|} \leftrightarrow \frac{2\alpha}{\alpha^2 + 4\pi^2 f^2}$$

$$\alpha = \tfrac{1}{2} \rightarrow H(f) = \frac{2}{2 + \frac{N_0}{2}\left(1 + 4\pi^2 f^2\right)}$$

$$\rightarrow h(t) = \frac{e^{-\sqrt{1+2/\sigma_N^2}\,|t|}}{\sigma_N^2 \sqrt{1 + 2/\sigma_N^2}}$$

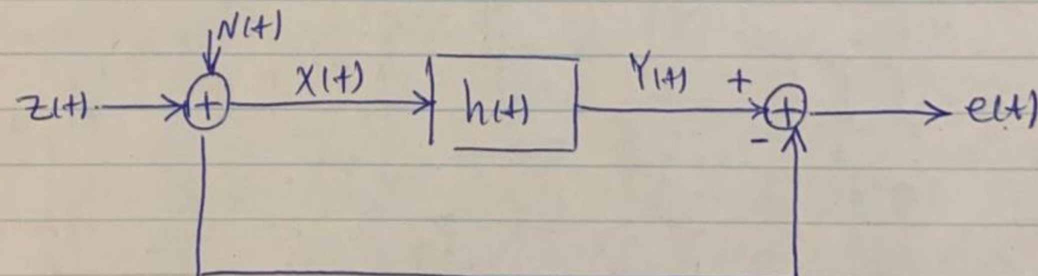
$$E\{e_t^2\} = R_Z(0) - \int_{-\infty}^{+\infty} h(u)\, R_{ZX}(u)\, du$$

$$= 1 - \int_{-\infty}^{+\infty} h(u)\, e^{-|u|}\, du$$

$$= 1 - \int_{-\infty}^{+\infty} \frac{e^{-\sqrt{1+\frac{2}{\sigma_N^2}}\,|u|}\; e^{-|u|}}{\sigma_N^2 \sqrt{1+\frac{2}{\sigma_N^2}}}\, du$$

$$\Rightarrow E\{e_t^2\} = 1 + \frac{2}{2 + \sigma_N^2 + \sigma_N^2\sqrt{1 + 2/\sigma_N^2}} \; .$$

10.70.



$$R_e(\tau) = E\{e(t+\tau)\, e(t)\} = E\{(Y(t+\tau) - Z(t+\tau))(Y(t) - Z(t))\}$$

$$= R_Y(\tau) + R_Z(\tau) - R_{YZ}(\tau) - R_{ZY}(\tau)$$

$$R_{ZY}(\tau) = E\{Z(t+\tau)\, Y(t)\} = E\{Z(t+\tau) \int_{-\infty}^{+\infty} h(\lambda)\, X(t-\lambda)\, d\lambda\}$$

$$= \int_{-\infty}^{+\infty} h(\lambda)\, R_{ZX}(\tau+\lambda)\, d\lambda \,, \quad R_{ZX}(\tau) = E\{Z(t+\tau)(Z(t)+N(t))\}$$
$$= E\{Z(t+\tau) Z(t)\} = R_Z(\tau)$$

$$= \int_{-\infty}^{+\infty} h(\lambda)\, R_Z(\tau+\lambda)\, d\lambda = h(-\tau)\, R_Z(\tau)$$

$$S_e(f) = S_r(f) + S_z(f) - S_{rz}(f) - S_{zr}(f)$$

$$= |H(f)|^2 \left(S_z(f) + S_N(f)\right) + S_z(f) - H(f)\, S_Z(f) - H^*(f)\, S_Z(f)$$

$$\rightarrow S_e(f) = |1 - H(f)|^2 S_Z(f) + |H(f)|^2 S_N(f).$$

In ~~our~~ this problem $S_N(f) = 1$

and $H(f) = \dfrac{c}{\sqrt{3} + j2\pi f}$ where $c = \dfrac{2}{1+\sqrt{3}}$

$$\rightarrow 1 - H(f) = \frac{1 + j2\pi f}{\sqrt{3} + j2\pi f}$$

$$\rightarrow S_e(f) = \left|\frac{1 + j2\pi f}{\sqrt{3} + j2\pi f}\right|^2 \left|\frac{\sqrt{2}}{1 + j2\pi f}\right|^2 + \left|\frac{c}{\sqrt{3} + j2\pi f}\right|^2$$

$$= \frac{2 + c^2}{|\sqrt{3} + j2\pi f|^2}$$

$$\rightarrow R_e(\tau) = \frac{1}{2\sqrt{3}} (2 + c^2)\, e^{-\sqrt{3}\,\tau}$$

$$\rightarrow R_e(0) = \frac{2 + c^2}{2\sqrt{3}} = 0.732$$

This is a larger error than smoothing filter which uses the entire observation of $Z(\alpha)$.