Problem set 9 (solutions)

1- a. $R_Y[n_1,n_2] = E\{Y[n_1]Y[n_2]\} = E\{(X[n_1]+c)(X[n_2]+c)\}$

$= R_{XX}[n_1,n_2] + 2c\mu_X + c^2$,

Since $X[n]$ is WSS, $E\{X[n]\}$ is constant, thus $E\{X\} = \mu_X$.

b. $E\{X[n_1]Y[n_2]\} = E\{X[n_1](X[n_2]+c)\}$

$= R_{XX}[n_2-n_1] + c\mu_X$

$E\{X[n_1]\}E\{Y[n_2]\} = \mu_X^2 + c\mu_X$

- $R_{XY} \neq 0 \rightarrow$ not orthogonal.
- $R_{XY} \neq \mu_X \mu_Y \rightarrow$ not uncorrelated.
- $Y[n] = X[n]+c \rightarrow$ not independent.

2. $E\{(X[n+m]-X[n-m])^2\} = E\{X^2[n+m] + X^2[n-m] - 2X[n+m]X[n-m]\}$

$= R_X[0] + R_X[0] - 2R_X[2m] = 2(R_X[0] - R_X[2m])$.

3. $E\{X(t)\} = E\{X(t)\} = 0 \rightarrow$ mean is constant for both.

$R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{A(t_1)A(t_2)\}\cos t_1 \cos t_2$

$= R_A(t_2-t_1)\cos t_1 \cos t_2 \rightarrow$ not a function of just t_2-t_1 thus $X(t)$ is not WSS.

$$R_Y(t_1,t_2) = E\{Y(t_1)Y(t_2)\} = E\{B(t_1)B(t_2)\}\sin(t_1)\sin(t_2)$$

$$= R_B(t_2-t_1)\sin t_1 \sin t_2 \rightarrow$$ not a t_2-t_1 function

thus not WSS.

$E\{Z(t)\} = 0.$

$Z(t) = X(t) + Y(t).$, $R_A(t_2-t_1) = R_B(t_2-t_1) \triangleq R(t_2-t_1)$

$$R_Z(t_1,t_2) = E\{(X(t_1)+Y(t_1))(X(t_2)+Y(t_2))\}$$

$$= R(t_2-t_1)(\cos t_1 \cos t_2 + \sin t_1 \sin t_2)$$

$$= R(t_2-t_1)\cos(t_2-t_1)$$ → thus it is WSS.

4. $X(t) = A(t)\cos(W_0 t + \theta)$

$$E\{X(t)\} = E\{A(t)\}\,E\{\cos W_0 t + \theta\} = 0$$

$$R_{XX}(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{A(t_1)A(t_2)\}\,E\{\cos(W_0 t_1+\theta)\cdot\cos(W_0 t_2+\theta)\}$$

$$= \frac{1}{2}R_{AA}(t_2-t_1)\left[E\{\cos W_0(t_2-t_1)\} + E\{\cos(W_0(t_1+t_2)+2\theta)\}\right]$$

$$= \frac{1}{2}R_{AA}(t_2-t_1)\cos W_0(t_2-t_1)$$ → $X(t)$ is WSS

Similarly for $Y(t)$ → $R_{YY}(t_1,t_2) = \frac{1}{2}R_{AA}(t_2-t_1)\cos((W_0+W_1)(t_2-t_1))$, → WSS.

$Z(t) = X(t) + Y(t)$ →

→ $E\{Z(t)\} = E\{X(t) + Y(t)\}$

$= 0$ → constant.

$R_{ZZ}(t_1,t_2) = R_{XX}(t_1,t_2) + R_{YY}(t_1,t_2) + R_{XY}(t_1,t_2)$
$+ R_{YX}(t_1,t_2)$

$R_{XX}(t_1,t_2)$ is a function of $t_2 - t_1$,

$R_{YY}(t_1,t_2)$ is a function of $t_2 - t_1$.

now we consider $R_{XY}(t_2,t_1)$. If it is a function of (t_2-t_1) then $Z(t)$ is also WSS, but in the following we prove that $R_{XY}(t_1,t_2)$ is not a function of $t_2 - t_1$.

$R_{XY}(t_2,t_1) = E\{X(t_1)Y(t_2)\}$

$= E\{A(t_1)A(t_2)\} \cos\{(w_0 t_1 + \theta) \cos((w_0 + w_1) t_2 + \theta)\}$

$= \frac{1}{2} R_{AA}(t_2 - t_1) \left[\cos\left(w_0(t_2 - t_1) + w_1 t_2\right)\right]$

↳ Thus not WSS.

5. $E\{X^2(t_1) Y^2(t_2)\}$

$$= E\{X^2(t_1)\} E\{Y^2(t_2)\} + 2\left(E\{X(t_1)Y(t_2)\}\right)^2$$

$$= R_{XX}(0) R_{YY}(0) + 2R^2_{XY}(t_2 - t_1)$$

- Mean square continuity: The random process $X(t)$ is continuous at the point $\underline{t_0}$ in the mean square sense if:

$$E\{(X(t)-X(t_0))^2\} \to 0 \quad \text{as } t \to t_0$$

- Mean square derivative: The random process $X(t)$ has mean square derivative $X'(t)$ at $\underline{t}$ defined by

$$\lim_{\varepsilon \to 0} \frac{X(t+\varepsilon)-X(t)}{\varepsilon},$$

provided that the mean square limit exists, that is:

$$\lim_{\varepsilon \to 0} E\left\{\left(\frac{X(t+\varepsilon)-X(t)}{\varepsilon} - X'(t)\right)^2\right\} = 0.$$

- Mean square Integrals: The mean square ~~k~~integral of $X(t)$ exists if the following double integral exists:

$$\int_{t_0}^{t}\int_{t_0}^{t} R_X(u,v)\,du\,dv. \quad , \quad Y(t) = \int_{t_0}^{t} X(t')\,dt'$$

$$\to E\{Y(t)\} = E\left\{\int_{t_0}^{t} X(t')\,dt'\right\} = \int_{t_0}^{t} m_X(t')\,dt'$$

$$R_Y(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u,v)\,du\,dv.$$

- If $R_X(t_1, t_2)$ is continuous on both t_1, t_2 at the point (t_0, t_0), then $X(t)$ is mean squrare continuous at the point t_0.

$$R_X(t_0+\varepsilon_1, t_0+\varepsilon_2) - R_X(t_0, t_0) \to 0, \text{ as } \varepsilon_1, \varepsilon_2 \to 0.$$

9.82) $X(t) = U(t-S)$, $\underbrace{S \sim \lambda e^{-\lambda x} u(x)}_{\text{exponential}}$

a) $R_X(t_1,t_2) = E\{X(t_1)X(t_2)\} = E\{U(t_1-S)U(t_2-S)\}$

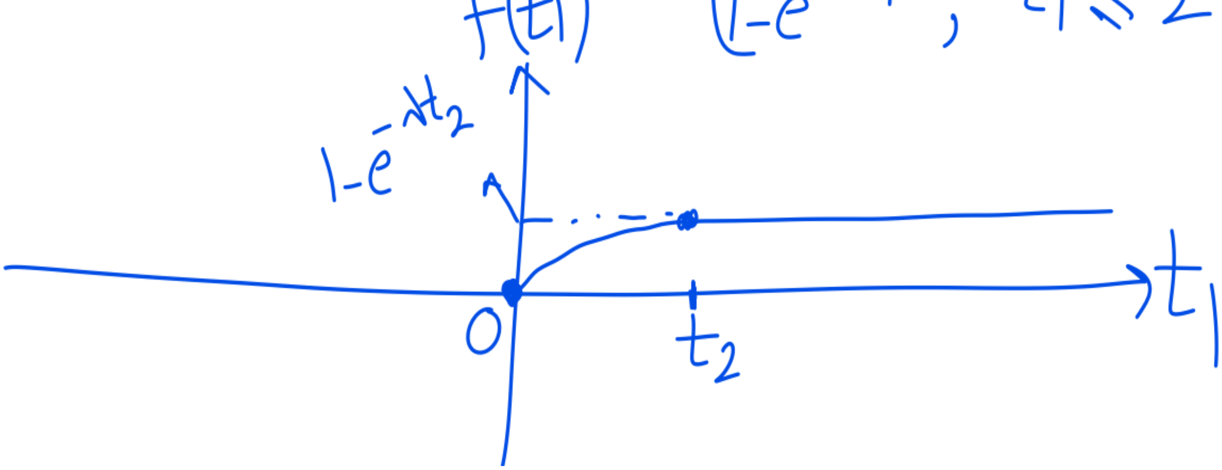
Assuming that $t_1, t_2 > 0$:

$$= \int_0^{\min\{t_1,t_2\}} \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big]_0^{\min\{t_1,t_2\}}$$

$$= 1 - e^{-\lambda \min\{t_1,t_2\}}$$

b): Let t_2 be fixed $\Longrightarrow$

$$R_X(t_1,t_2) = 1 - e^{-\lambda \min\{t_1,t_2\}} = \begin{cases} 1-e^{-\lambda t_2} & t_1 \geq t_2 \\ 1-e^{-\lambda t_1}; & t_1 \leq t_2 \end{cases} = f(t_1)$$



As you can see, $f(t_1)$ is a continuous function of $t_1 \Rightarrow \lim_{t_1 \to t_0} R_X(t_1,t_2) = R_X(t_0,t_2)$ ✓

In a similar way, we can show that:

$$\lim_{t_2 \to t_0} R_X(t_1,t_2) = R_X(t_1,t_0)$$

$\Longrightarrow X(t)$ is mean square continuous.

(C): since $X(t) = U(t-S)$ only takes constant values 0 and 1, we guess that

$X'(t) = 0$.

$$\lim_{\varepsilon \to 0} \left(\frac{X(t+\varepsilon) - X(t)}{\varepsilon} - \underbrace{X'(t)}_{0} \right)^2 = \lim_{\varepsilon \to 0} \frac{E\left(X(t+\varepsilon) - X(t)\right)^2}{\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} \frac{EX^2(t+\varepsilon) + EX^2(t) - 2E\left(X(t+\varepsilon)X(t)\right)}{\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0} \frac{1 - e^{-\lambda(t+\varepsilon)} + 1 - e^{-\lambda t} - 2\left(1 - e^{-\lambda \min\{t+\varepsilon, t\}}\right)}{\varepsilon^2}$$

Let us check the right limit. $(\varepsilon > 0)$

$$\lim_{\varepsilon \to 0^+} \frac{1 - e^{-\lambda(t+\varepsilon)} + 1 - e^{-\lambda t} - 2\left(1 - e^{-\lambda t}\right)}{\varepsilon^2}$$

$$= \lim_{\varepsilon \to 0^+} \frac{e^{-\lambda t}\left(1 - e^{-\lambda \varepsilon}\right)}{\varepsilon^2} =$$ Hopital rule

$$= \lim_{\varepsilon \to 0^+} \frac{e^{-\lambda t} \times \lambda e^{-\lambda \varepsilon}}{2\varepsilon} = +\infty \Rightarrow$$

Mean square derivative does <u>not</u> exist.

d: In this part, we want to check the existanse of Integral.

$$I=\int_0^t\int_0^t R_X(u,v)\,du\,dv=\int_0^t\int_0^t\left(1-e^{-\lambda\min\{u,v\}}\right)du\,dv$$

$$=\int_0^t dv\int_0^t\left(1-e^{-\lambda\min\{u,v\}}\right)du$$

$$=\int_0^t dv\int_0^v\left(1-e^{-\lambda u}\right)du+\int_0^t dv\int_v^t\left(1-e^{-\lambda v}\right)du=$$

$$\int_0^t dv\left[u+\frac{1}{\lambda}e^{-\lambda u}\right]_0^v+\int_0^t dv\left[\left(1-e^{-\lambda v}\right)u\right]_v^t$$

$$=\int_0^t\left(v+\frac{1}{\lambda}e^{-\lambda v}-\frac{1}{\lambda}\right)dv+\int_0^t\left(1-e^{-\lambda v}\right)(t-v)\,dv$$

$$\left[\frac{v^2}{2} - \frac{1}{\lambda^2} e^{-\lambda v} - \frac{1}{\lambda} v\right]_0^t + \left[tv - \frac{v^2}{2} + \frac{1}{\lambda} e^{-\lambda v} t - e^{-\lambda v}\left(\frac{v}{\lambda} + \frac{1}{\lambda^2}\right)\right]_0^t$$

$$\cancel{\frac{t^2}{2}} - \frac{1}{\lambda^2} e^{-\lambda t} - \frac{t}{\lambda} + \frac{1}{\lambda^2} + t^2 - \cancel{\frac{t^2}{2}} + \frac{t}{\lambda} e^{-\lambda t} - e^{-\lambda t}\left(\frac{t}{\lambda} + \frac{1}{\lambda^2}\right)$$

$$-\frac{t}{\lambda} + \frac{1}{\lambda^2} = t^2 - \frac{2t}{\lambda} + \frac{2}{\lambda^2} - 2e^{-\lambda t}\frac{1}{\lambda^2}$$

As you can see, integral exists ✓

$\Rightarrow X(t)$ is mean square integrable.

Let $Y(t) = \int_0^t X(u)\,du \Rightarrow E(Y(t)) = \int_0^t EX(\tau)\,d\tau$

$$EX(\tau) = E(U(\tau - S)) = \int_0^\tau 1 \times \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big]_0^\tau$$

$$= 1 - e^{-\lambda \tau}$$

$$\rightsquigarrow EY(t) = \int_0^t \left(1 - e^{-\lambda \tau}\right) d\tau = \tau + \frac{1}{\lambda} e^{-\lambda \tau}\Big]_0^t$$

$$= t + \frac{1}{\lambda} e^{-\lambda t} - \frac{1}{\lambda} = t + \frac{1}{\lambda}\left(e^{-\lambda t} - 1\right)$$

$$R_Y(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} R_X(u, v)\, du\, dv$$

$$= \int_0^{t_1} \int_0^{t_2} \left(1 - e^{-\lambda \min\{u, v\}}\right) du\, dv$$

$$\vdots$$

we can simplify it in a similar way

9.33 · $\lim_{t_2 \to t_1} E\{(X(t_2) - X(t_1))^2\} \to 0$

a- $X(t)$ can be either $+1$ or -1. Thus $X(t_2) - X(t_1)$ can take the following values:

$X(t_2) - X(t_1) =$

1	1	= 0
-1	-1	= 0
1	-1	= 2
-1	1	= -2

Therefore: $E\{(X(t_2) - X(t_1))^2\} =$

$= 0 \times P(X(t_2)=1, X(t_1)=1)$

$+ 0 \times P(X(t_2)=-1, X(t_1)=-1)$

$+ 2 \times P(X(t_2)=1, X(t_1)=-1)$

$+ -2 \times P(X(t_2)=-1, X(t_1)=1)$.

• $P(X(t_2)=-1, X(t_1)=+1) = P(X(t_1)=1)\, P(X(t_2)=-1 \mid X(t_1)=1)$

$= \frac{1}{2} \cdot P(X(t_2)=-1 \mid X(t_1)=1) = \frac{1}{2} \cdot \frac{1}{2}\left(1 - e^{-2\alpha|t_2 - t_1|}\right)$

• $P(X(t_2)=+1, X(t_1)=-1) = \frac{1}{4}\left(1 - e^{-2\alpha|t_2 - t_1|}\right)$

$E\{(X(t_2) - X(t_1))^2\} = \frac{4}{4}\left(1 - e^{-2\alpha|t_2 - t_1|}\right) + \frac{4}{4}\left(1 - e^{-2\alpha|t_2 - t_1|}\right) = 0$

$= 2\left(1 - e^{-2\alpha|t_2 - t_1|}\right)$

$\rightarrow \lim_{t_2 \to t_1} E\{(X(t_2) - X(t_1))^2\} = \lim_{t_2 \to t_1} 2\left(1 - e^{-2\alpha|t_2 - t_1|}\right) \to 0$

Thus, it is m.s. continuous.

9.33.b.

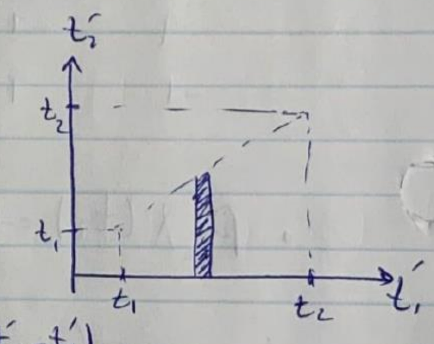
$$\lim_{\epsilon\to 0} E\left\{\frac{(X(t+\epsilon)-X(t))^2}{\epsilon^2} - X'(t)\right\} = 0, \quad X'(t)=0$$

$$\to \lim_{\epsilon\to 0}\frac{1}{\epsilon^2} E\left\{(X(t+\epsilon)-X(t))^2\right\} = \lim_{\epsilon\to 0}\frac{1}{\epsilon^2}\cdot 2\left(1-e^{-2\alpha\epsilon}\right) \to \infty$$

Thus, m.s. derivative does not exist.

9.33.C. Supposing $t_2 - t_1 > 0$, we get:

$$\int_{t_1}^{t_2}\int_{t_1}^{t_2} e^{-2\alpha|t_2'-t_1'|}\, dt_1'\, dt_2'$$



$$= \int_{t_1}^{t_2} dt_1' \int_{t_1}^{t_1'} e^{2\alpha(t_2'-t_1')}\, dt_2' + \int_{t_1}^{t_2} dt_2' \int_{t_1}^{t_2'} e^{-2\alpha(t_2'-t_1')}\, dt_1'$$

$$= 2\int_{t_1}^{t_2} \left[\frac{1}{2\alpha} e^{2\alpha(t_2'-t_1')}\right]_{t_1}^{t_1'} dt_1' = \frac{1}{\alpha}\int_{t_1}^{t_2}\left(1-e^{2\alpha(t_1-t_1')}\right) dt_1'$$

$$= \frac{1}{\alpha}\left((t_2-t_1) + \frac{1}{2\alpha}\left(e^{2\alpha(t_1-t_2)} - 1\right)\right), \quad t_2 - t_1 = t$$

Therefore, $X(t)$ has m.s. integral.

$$= \frac{1}{\alpha}\left(t + \frac{1}{2\alpha}\left(e^{-2\alpha t} - 1\right)\right)$$

, setting $t_1 = 0$ would make ~~the~~ for this specific case.

$$m_Y(t) = \int_{t_0}^{t} m_X(t')\,dt' \,,\quad m_X(t') = 0 \qquad , \; Y(t) = \int_{0}^{t} X(t')\,dt'$$

$$\Rightarrow m_Y(t) = 0$$

$$\int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u,v)\,du\,dv = \int_{t_0}^{t_1}\int_{t_0}^{t_2} e^{-2\alpha|t_2'-t_1'|}\,dt_1'\,dt_2'$$

Which is similar to the previous calculations.

9.84:

If WSS $\rightarrow$ $E\{(X(t_0+\tau)-X(t_0))^2\} = 2(R_X(0)-R_X(\tau))$

If $R_X(\tau)$ is continuous at $\tau=0$, then the WSS random process $X(t)$ is m.s. continuous at t_0.

a- The function $R_X(\tau)=\sigma^2 e^{-a\tau^2}$ is continuous at $\tau=0$, thus m.s. continuous.

b- The m.s. derivative of a WSS random process $X(t)$ exists for all t, if $R_X(\tau)$ has derivatives up to order two at $\tau=0$.

$$\frac{d^2}{d\tau^2}R_X(\tau)\Big|_{\tau=0} .$$

The function $R_X(t)=\sigma^2 e^{-at^2}$ is double-differentiable, thus it has m.s. derivative. ~~[illegible]~~

9.85 → $E\{(N(t)-N(t_0))^2\}$

Since the poisson distribution is memoryless, thus the increments are independent. Therefore, the difference in the # of occurences between t, and t_0, is equal to the # of occurences in $t-t_0$ period of time.

so → $E\{(N(t)-N(t_0))^2\} = E\{(N(t-t_0))^2\}$

$$= \lambda(t-t_0) + \lambda^2(t-t_0)^2$$

$$\lim_{t \to t_0} \left(\lambda(t-t_0) + \lambda^2(t-t_0)^2\right) \to 0,$$

thus m.s. continuous.