Week 5 solutions

1. (a) $\mu_X(t) = \frac{1}{3} \cdot 1 + \frac{1}{3}(-3) + \frac{1}{3} \cdot \sin(2\pi t) = -\frac{2}{3} + \frac{1}{3}\sin(2\pi t)$

   (b)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \frac{1}{3} \cdot 1 + \frac{1}{3}(-3)(-3) + \frac{1}{3} \cdot \sin(2\pi t_1)\sin(2\pi t_2) \\ &= -\frac{8}{3} + \frac{1}{3}\sin(2\pi t_1)\sin(2\pi t_2) \end{aligned}$$

2. (a)

$$\begin{aligned} \mu_X(t) &= \frac{1}{5}[-2\cos(t) - 2\sin(t) + 2(\cos(t) + \sin(t)) \\ &\quad + (\cos(t) - \sin(t)) - (\cos(t) - \sin(t))] \\ &= 0 \end{aligned}$$

   (b)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[X(t_1)X(t_2)] \\ &= \frac{1}{5}[4\cos(t_1)\cos(t_2) + 4\sin(t_1)\sin(t_2) + 4(\cos(t_1) + \sin(t_1))(\cos(t_2) + \sin(t_2))] \\ &\quad + 2(\cos(t_1) - \sin(t_1))(\cos(t_2) - \sin(t_2))] \\ &= \frac{1}{5}[10\cos(t_1)\cos(t_2) + 10\sin(t_1)\sin(t_2) + 2\cos(t_1)\sin(t_2) + 2\sin(t_1)\cos(t_2) \\ &= 2\cos(t_2 - t_1) + \frac{2}{5}\sin(t_1 + t_2) \end{aligned}$$

3. (a) $\mu_X[n] = E[X[n]] = \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) = 3.5$

   (b) $R_{X,X}[k_1, k_2] = E[X[k_1]X[k_2]]$
   If $k_1 \neq k_2$:
   $R_{X,X}[k_1, k_2] = E[X[k_1]]E[X[k_2]] = (3.5)^2$
   If $k_1 = k_2$:
   $R_{X,X}[k_1, k_2] = E\left[X^2[k_1]\right] = \frac{1}{6}\left(1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2\right) = \frac{91}{6}$
   $R_{X,X}[k_1, k_2] = \begin{cases} \frac{49}{4} & k_1 \neq k_2 \\ \frac{91}{6} & k_1 = k_2 \end{cases}$

4. Case 1: n_1 even, n_2 even
   $R_{Y,Y}[n_1, n_2] = E\left[X\left[\frac{n_1}{2}\right]X\left[\frac{n_2}{2}\right]\right] = \begin{cases} 1 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$
   Case 2: n_1 odd, n_2 odd
   $R_{Y,Y}[n_1, n_2] = E[X[n_1 + 1]X[n_2 + 1]] = \begin{cases} 1 & n_1 = n_2 \\ 0 & n_1 \neq n_2 \end{cases}$
   Case 3: n_1 odd, n_2 even

$R_{Y,Y}[n_1, n_2] = E\left[X[n_1+1]\,X\left[\frac{n_2}{2}\right]\right] = \begin{cases} 1 & n_2 = 2(n_1+1) \\ 0 & n_2 \neq 2(n_1+1) \end{cases}$

Case 4: n_1 even, n_2 odd

$R_{Y,Y}[n_1, n_2] = E\left[X\left[\frac{n_1}{2}\right]X[n_2+1]\right] = \begin{cases} 1 & n_1 = 2(n_2+1) \\ 0 & n_1 \neq 2(n_2+1) \end{cases}$

In summary,

$$R_{Y,Y}[n_1, n_2] = \begin{cases} 1 & n_1 = n_2 \\ 1 & n_2 = 2(n_1+1) \quad (n_1 \text{ odd}) \\ 1 & n_1 = 2(n_2+1) \quad (n_2 \text{ odd}) \\ 0 & \text{otherwise} \end{cases}$$

5. (a) $\mu_X(t) = \mu_A \cos(\omega t) + \mu_B \sin(\omega t) = 0$

   (b)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E\left[A^2\right]\cos(\omega t_1)\cos(\omega t_2) + E\left[B^2\right]\sin(\omega t_1)\sin(\omega t_2) \\ &\quad + E[AB]\cos(\omega t_1)\sin(\omega t_2) + E[AB]\sin(\omega t_1)\cos(\omega t_2) \\ &= \frac{E\left[A^2\right] + E\left[B^2\right]}{2}\cos(\omega(t_2 - t_1)) + \frac{E\left[A^2\right] - E\left[B^2\right]}{2}\cos(\omega(t_1 + t_2)) \end{aligned}$$

6. (a) Since T is uniformly distributed over one period of M, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Since $s(t)$ is 1 half of the time and -1 half of the time, we get
   $\Pr(X(t) = 1) = \Pr(X(t) = -1) = \frac{1}{2}$

   (b) $E[X(t)] = 1 \cdot \Pr(X(t) = 1) + (-1) \cdot \Pr(X(t) = -1) = 0$

   (c)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E\left[s(t_1 - T)\,s(t_2 - T)\right] \\ &= \int_0^1 s(t_1 - u)\,s(t_2 - u)\,du \\ &= \int_0^1 s(v)s(v + t_2 - t_1)\,dv \\ &= s(t) * s(-t)\big|_{t = t_1 - t_2} \end{aligned}$$

7. (a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Given the linear functional form of $s(t)$, $X(t)$ will be uniform over (-1, 1).
   $f_X(x;t) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$

   (b) $E[X(t)] = 0$ since the PDF above is symmetric about zero.

(c)

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\begin{aligned}
R_{X,X}(t_1,t_2) &= E\left[s(t_1-T)\,s(t_2-T)\right] \\
&= \int_0^1 s(t_1-u)\,s(t_2-u)\,du \\
&= \int_0^1 s(v)s(v+t_2-t_1)\,dv \\
&= s(t) * s(-t)|_{t=\cancel{t_2-t_1}}\ t_1-t_2
\end{aligned}
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8. (a) $E[X(t)] = \Pr(X(t)=+1) - \Pr(X(t)=-1) = \frac{1}{2} - \frac{1}{2} = 0$

(b) $X(t_1)\,X(t_2) = \begin{cases} 1 \text{ if no pulse transitions in } (t_1,t_2) \\ \pm 1 \text{ with equal prob. if 1 or more pulse transitions in } (t_1,t_2)\,. \end{cases}$

$E[X(t)] = 0 \cdot \Pr(\geq 1 \text{ transitions in } (t_1,t_2)) + 1 \cdot \Pr(\text{no transitions in } (t_1,t_2))$

For $|t_2-t_1| < \Delta$,

$$\Pr(\text{no transitions in } (t_1,t_2)) = 1 - \frac{|t_2-t_1|}{\Delta}$$

Therefore,

$$R_{X,X}(t_1,t_2) = \begin{cases} 1 - \frac{|t_2-t_1|}{\Delta} & |t_2-t_1| < \Delta \\ 0 & |t_2-t_1| > \Delta \end{cases}$$

9. (a)

$$f_X(x;t) = \left.\frac{f_A(a)}{\left|\frac{dX}{dA}\right|}\right|_{A=-\frac{1}{t}\ln(x)} = \frac{f_A\left(-\frac{1}{t}\ln(x)\right)}{tx}$$

(b)

$$E[X(t)] = E\left[e^{-At}\right] = \int_0^\infty e^{-at}e^{-a}da = \frac{1}{1+t}$$

$$R_{X,X}(t_1,t_2) = E\left[X(t_1)\,X(t_2)\right] = E\left[e^{-A(t_1+t_2)}\right] = \frac{1}{1+t_1+t_2}$$

10.

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\begin{aligned}
R_{Z,Z}[k] &= R_{X,X}[k] + R_{Y,Y}[k] + R_{X,Y}[k] + R_{Y,X}[k] \\
R_{X,Y}[k] &= E[X[n]Y[n+k] = \mu_X[n]\mu_Y[n+k] = 0 \\
\Rightarrow R_{Z,Z}[k] &= R_{X,X}[k] + R_{Y,Y}[k] = \left(\frac{1}{2}\right)^{|k|} + \left(\frac{1}{3}\right)^{|k|}
\end{aligned}
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