problem set 6 (solutions)

7.4 : $X \perp\!\!\!\perp Y$ $\quad$ X and Y are cauchy random variables with parameters 1 and 4, respectively.

$$Z = X + Y \Rightarrow f_X(x) = \frac{1}{\pi(1+x^2)}, \quad f_Y(y) = \frac{4}{\pi(16+y^2)}$$

$$\phi_X(\omega) = e^{-|\omega|}, \quad \phi_Y(\omega) = e^{-4|\omega|}$$

ⓐ: $\phi_Z(\omega) = E\{e^{j\omega z}\} = E\{e^{j\omega x} e^{j\omega y}\} \underset{X \perp\!\!\!\perp Y}{=} E\{e^{j\omega x}\} E\{e^{j\omega y}\}$

$$= e^{-|\omega|} \times e^{-4|\omega|} = e^{-5|\omega|}$$

ⓑ: $f_Z(z) = F^{-1}\{\phi_Z(-\omega)\} = F^{-1}\{e^{-5|\omega|}\} = \dfrac{5}{\pi(25+z^2)}$

7.7 : $X \perp\!\!\!\perp Y$ $\quad f_X(x) = 2e^{-2x}; x>0 \quad f_Y(y) = 10e^{-10y}; y>0$

we first find the characteristic function of a general exponential random variable with parameter λ.

Let W be exponential with parameter λ. then we have:

$$f_W(w) = \lambda e^{-\lambda w}; w>0$$

$$\phi_W(\omega) = E\{e^{j\omega W}\} = \int_0^{+\infty} e^{j\omega w} \lambda e^{-\lambda w} dw = \lambda \int_0^{+\infty} e^{(-\lambda + j\omega)w} dw =$$

$$\frac{\lambda}{-\lambda + j\omega} e^{(-\lambda + j\omega)w}\Big]_0^{+\infty} = \frac{\lambda}{\lambda - j\omega} \Rightarrow \begin{cases} \phi_X(\omega) = \dfrac{2}{2-j\omega} \\ \phi_Y(\omega) = \dfrac{10}{10-j\omega} \end{cases}$$

ⓐ: $\phi_Z(\omega) = E\{e^{j\omega x} e^{j\omega y}\} = \phi_X(\omega)\phi_Y(\omega) = \dfrac{20}{(2-j\omega)(10-j\omega)}$

ⓑ $\Rightarrow f_Z(z) = F^{-1}\{\phi_Z(-\omega)\} = F^{-1}\left\{\dfrac{20}{(2+j\omega)(10+j\omega)}\right\}$

$$= F^{-1}\left\{\frac{5/2}{2+j\omega} - \frac{5/2}{10+j\omega}\right\} = \frac{5}{2}\left(e^{-2z} - e^{-10z}\right)u(z)$$

7.8 $Z = 3X - 7Y$, $X \perp\!\!\!\perp Y$

a: $\phi_Z(\omega) = E\{e^{j\omega Z}\} = E\{e^{j\omega 3X} e^{-j\omega 7Y}\} = E\{e^{j\omega 3X}\}E\{e^{-j\omega 7Y}\}$

$= \phi_X(3\omega)\phi_Y(-7\omega)$

b: We first find the mean and variance of a random variable in terms of derivatives of characteristic function

$$\phi_X(\omega) = E\{e^{j\omega X}\} = \int_{-\infty}^{+\infty} f_X(x) e^{j\omega x} dx$$

$$\Rightarrow \phi'_X(\omega) = \int_{-\infty}^{+\infty} jx f_X(x) e^{j\omega x} dx \Rightarrow \phi'_X(0) = j\int_{-\infty}^{+\infty} x f_X(x) dx = E\{X\} \times j$$

$$\phi''_X(\omega) = \int_{-\infty}^{+\infty} -x^2 f_X(x) e^{j\omega x} dx \Rightarrow \phi''_X(0) = -\int_{-\infty}^{+\infty} x^2 f_X(x) dx = -E\{X^2\}$$

$$\Rightarrow \begin{cases} EX = -j\phi'_X(0) \\ \sigma_X^2 = E\{X^2\} - (EX)^2 = -\phi''_X(0) + (\phi'_X(0))^2 = \{\phi'_X(0)\}^2 - \phi''_X(0) \end{cases}$$

We also point out that: $\phi_X(0) = \int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$\phi'_Z(\omega) = 3\phi'_X(3\omega)\phi_Y(-7\omega) - 7\phi'_Y(-7\omega)\phi_X(3\omega)$$

$$\Rightarrow \phi'_Z(0) = 3\phi'_X(0)\underbrace{\cancel{\phi_Y(0)}}_{1} - 7\phi'_Y(0)\underbrace{\cancel{\phi_X(0)}}_{1}$$

$$\Rightarrow E\{Z\} = -j\phi'_Z(0) = -3j\phi'_X(0) + 7j\phi'_Y(0) = 3E(X) - 7E(Y)$$

$$\phi_Z''(\omega)=9\phi_X''(3\omega)\phi_Y(-7\omega)-21\phi_X'(3\omega)\phi_Y'(-7\omega)+49\phi_Y''(-7\omega)\phi_X(3\omega)-21\phi_Y'(-7\omega)\phi_X'(3\omega)$$

$$\Rightarrow \phi_Z''(0)=9\phi_X''(0)-21\phi_X'(0)\phi_Y'(0)+49\phi_Y''(0)-21\phi_Y'(0)\phi_X'(0)$$

$$\Rightarrow \sigma_Z^2=\{\phi_Z'(0)\}^2-\phi_Z''(0)=\left(3\phi_X'(0)-7\phi_Y'(0)\right)^2-9\phi_X''(0)-49\phi_Y''(0)+42\phi_X'(0)\phi_Y'(0)$$

$$=9\left(\{\phi_X'(0)\}^2-\phi_X''(0)\right)+49\left(\{\phi_Y'(0)\}^2-\phi_Y''(0)\right)=9\sigma_X^2+49\sigma_Y^2$$

7.11: $pr\{X_i=l\}=\frac{e^{-\alpha_i}\alpha_i^l}{l!}$; $1\leq i\leq k$, $l=0,1,2,\dots$

$$M_{X_i}(s)=E\{e^{sX_i}\}=\sum_{l=0}^{+\infty}\frac{e^{-\alpha_i}\alpha_i^l}{l!}\times e^{sl}=\sum_{l=0}^{+\infty}\frac{e^{-\alpha_i}(\alpha_i e^s)^l}{l!}=e^{-\alpha_i}\times e^{\alpha_i e^s}=e^{\alpha_i(e^s-1)}$$

$$M_{\sum_{i=1}^k X_i}(s)=E\{e^{s\sum_{i=1}^k X_i}\}=\prod_{i=1}^k E\{e^{sX_i}\}=\prod_{i=1}^k e^{\alpha_i(e^s-1)}=e^{\sum_{i=1}^k\alpha_i(e^s-1)}$$

$\Rightarrow \sum_{i=1}^k X_i$ is a poisson random variable with parameter $\sum_{i=1}^k \alpha_i$

pmf of $\sum_{i=1}^k X_i$: $pr\left\{\sum_{i=1}^k X_i=l\right\}=\frac{e^{-\sum_{i=1}^k\alpha_i}\left(\sum_{i=1}^k\alpha_i\right)^l}{l!}$; $l=0,1,2,\dots$

7.14: (a): $E(S)=\sum_{k=0}^{n}E\left(S/_{\#\text{tested widgets}=k}\right)\times pr\{\#\text{tested widgets}=k\}$

$$=\sum_{k=0}^{n}k\alpha\times\binom{n}{k}p^k(1-p)^{n-k}=\alpha\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k}=\alpha np$$

I will find the variance of S using the function

$G_S(z)=E\{z^S\}$

(b): $G_s(z)=E\{z^S\}=\sum_{k=0}^{n} E\{z^S / \#\text{tested widgets}=k\}\times pr(\#\text{tested widgets}=k)$

$\Rightarrow E\{z^S/\#\text{tested widgets}=k\}=E\{z^{\sum_{i=1}^{k}X_i}\}=\prod_{i=1}^{k}E\{z^{X_i}\}=\prod_{i=1}^{k}(z\alpha+(1-\alpha))$

is a Bernoulli r.v.

$=(z\alpha+(1-\alpha))^k$

$\Rightarrow G_S(z)=E\{z^S\}=\sum_{k=0}^{n}(z\alpha+(1-\alpha))^k\times\binom{n}{k}p^k(1-p)^{n-k}=$

$=\sum_{k=0}^{n}\binom{n}{k}(pz\alpha+p(1-\alpha))^k(1-p)^{n-k}=(1-p+pz\alpha+p(1-\alpha))^n$

$=(1-p\alpha+pz\alpha)^n$

$G'_S(1)=E\{S\}=n\alpha p$

$G''_S(1)=E\{S^2\}-E\{S\}=(n^2-n)p^2\alpha^2-n\alpha p$

$\Rightarrow Var\{S\}=G''_S(1)+G'_S(1)-(G'_S(1))^2$

$=(n^2-n)p^2\alpha^2-n^2\alpha^2p^2+n\alpha p$

$=n\alpha p(1-\alpha p)$

(7.17): A fair die is tossed 20 times.

Eq. (7.20): $P(|M_n - \mu| < \varepsilon) \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$

$M_n \triangleq \frac{1}{20}\sum_{i=1}^{20} X_i$

$\mu = \frac{1}{6}(1+2+\dots+6) = \frac{1}{6}\times\frac{6\times 7}{2} = \frac{7}{2}$

$\sigma^2 = \frac{1}{6}(1^2+2^2+\dots+6^2) - \left(\frac{7}{2}\right)^2 = \frac{1}{\cancel{6}}\times\frac{\cancel{6}\times 7\times 13}{6} - \frac{49}{4} = \frac{35}{12}$

$60 < \sum_{i=1}^{20} X_i < 80 \Rightarrow 3 < M_n = \frac{1}{20}\sum_{i=1}^{20} X_i < 4 \Rightarrow$

$\underbrace{\frac{-1}{2}}_{-\varepsilon} < M_n - \underbrace{\frac{7}{2}}_{\mu} < \underbrace{\frac{1}{2}}_{\varepsilon} \Rightarrow |M_n - \mu| < \varepsilon$

$pr\left(\left|M_n - \frac{7}{2}\right| < \frac{1}{2}\right) \geq 1 - \frac{\left(\frac{35}{12}\right)}{20\times\left(\frac{1}{2}\right)^2} = 0.4167$

(7.21): $X_1, X_2, \dots, X_n$ iid sequence of random variables.

$V_n^2 \triangleq \frac{1}{n-1}\sum_{j=1}^{n}(X_j - M_n)^2, \quad M_n \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i$

(a): $\sum_{j=1}^{n}(X_j-\mu)^2 = \sum_{j=1}^{n}\left((X_j - M_n) + (M_n - \mu)\right)^2 =$

$\sum_{j=1}^{n}(X_j - M_n)^2 + \sum_{j=1}^{n}(M_n-\mu)^2 + 2\sum_{j=1}^{n}(X_j - M_n)(M_n - \mu)$

$= \sum_{j=1}^{n}(X_j - M_n)^2 + n(M_n-\mu)^2 + 2(M_n-\mu)\underbrace{\left\{\cancel{\sum_{j=1}^{n} X_j - nM_n}\right\}}_{0}$

$= \sum_{j=1}^{n}(X_j - M_n)^2 + n(M_n - \mu)^2$

(b): $E\left\{\sum_{j=1}^{n}(X_j-\mu)^2\right\}=\sum_{j=1}^{n}E\left\{\underbrace{(X_j-\mu)^2}_{\sigma^2}\right\}=n\sigma^2=E\left(\sum_{j=1}^{n}(X_j-M_n)^2\right)$

$+\cancel{n}\times E\left(\underbrace{(M_n-\mu)^2}_{\sigma^2}\right)=E\left(\sum_{j=1}^{n}(X_j-M_n)^2\right)+\sigma^2$

$\Rightarrow E\left(\sum_{j=1}^{n}(X_j-M_n)^2\right)=(n-1)\sigma^2$

$\Rightarrow E\left(k\sum_{j=1}^{n}(X_j-M_n)^2\right)=k(n-1)\sigma^2$

(c): Let $k=\frac{1}{n-1}$

$\Rightarrow E\left(\frac{1}{n-1}\sum_{j=1}^{n}(X_j-M_n)^2\right)=E(V_n^2)=\sigma^2$

(d): $V_n^2\triangleq\frac{1}{n}\sum_{j=1}^{n}(X_j-M_n)^2$

let $k=\frac{1}{n}$

$\Rightarrow E\left(\frac{1}{n}\sum_{j=1}^{n}(X_j-M_n)^2\right)=E(V_n^2)=\frac{n-1}{n}\sigma^2=\sigma^2-\frac{\sigma^2}{n}$

7.22: A fair coin is tossed 100 times.

X_i	head(1)	tail(0)
P	$\frac{1}{2}$	$\frac{1}{2}$

bernoulli $S_{100}=\sum_{i=1}^{100}X_i \rightsquigarrow$ Binomial r.v.

$ES_{100}=np=100\times\frac{1}{2}=50$

$Var\{S_{100}\}=np(1-p)=100\times\frac{1}{2}\times\frac{1}{2}=25$

$Z\triangleq\frac{S_{100}-ES_{100}}{\sqrt{Var(S_{100})}}\sim N(0,1)\Rightarrow Z=\frac{S_{100}-50}{5}\sim N(0,1)$

$$pr\{50 \leq S_{100} \leq 55\} = pr\{0 \leq \frac{S_{100}-50}{5} \leq 1\} = pr\{0 \leq Z \leq 1\}$$

$$= Q(0) - Q(1) = \frac{1}{2} - Q(1) = 0.3413$$

(b): $n = 1000$

$E(S_{1000}) = 1000 \times \frac{1}{2} = 500$

$Var(S_{1000}) = 1000 \times \frac{1}{2} \times \frac{1}{2} = 250$

$\Rightarrow Z = \frac{S_{1000}-500}{\sqrt{250}}$

$$pr\{400 \leq S_{1000} \leq 600\} = pr\left\{\frac{-100}{\sqrt{250}} \leq Z = \frac{S_{1000}-500}{\sqrt{250}} \leq \frac{100}{\sqrt{250}}\right\}$$

$$= Q\left(\frac{-100}{\sqrt{250}}\right) - Q\left(\frac{100}{\sqrt{250}}\right) = 1 - 2Q\left(\frac{100}{\sqrt{250}}\right) \simeq 1$$

$$pr\{500 \leq S_{1000} \leq 550\} = pr\left\{0 \leq Z = \frac{S_{1000}-500}{\sqrt{250}} \leq \frac{50}{\sqrt{250}}\right\} =$$

$$\frac{1}{2} - Q\left(\frac{50}{\sqrt{250}}\right) = 0.4992$$

7.27: S is the sum of 80 iid poisson random variables with mean 0.25.

Noting question 7.11, we know that S is a poisson random variable with mean $80 \times 0.25 = 20$

exact value: $pr\{S = k\} = \frac{e^{-20} \times 20^k}{k!}$; $k = 0, 1, 2, \dots$

approximation: $pr\{S = k\} = \frac{1}{\sqrt{2\pi \times 20}} \int_{k-\frac{1}{2}}^{k+\frac{1}{2}} e^{\frac{-(x-20)^2}{2 \times 20}} dx$

$$\approx \frac{1}{\sqrt{40\pi}} \times e^{\frac{-(k-20)^2}{40}}$$

7.29: $n=100$, $p=0.15$ S_{100} is a binomial r.v.

$E\{S_{100}\}=np=100\times0.15=15$

$var\{S_{100}\}=np(1-p)=100\times0.15\times0.85=12.75$

$pr\{S_{100}\leq 20\}=pr\{z=\frac{S_{100}-15}{\sqrt{12.75}}\leq\frac{5}{\sqrt{12.75}}\}=1-Q\left(\frac{5}{\sqrt{12.75}}\right)$

$=0.9193$

7.32:

X_i	error(1)	not error(0)
p	0.01	0.99

$T\triangleq\sum_{i=1}^{100}X_i$ $\quad pr(T\geq \underset{a}{3})\leq\min_{s>0}e^{-sa}E\{e^{sT}\}$

$E\{e^{sX_i}\}=e^{s}\times0.01+0.99$

$E\{e^{sT}\}=\prod_{i=1}^{100}E\{e^{sX_i}\}=(0.99+0.01e^{s})^{100}$

$f(s)\triangleq e^{-3s}(0.99+0.01e^{s})^{100}$

$f'(s)=-3e^{-3s}(0.99+0.01e^{s})^{100}+100\times e^{-3s}\times0.01e^{s}(0.99+0.01e^{s})^{99}=0$

$\Rightarrow -3e^{-3s}(0.99+0.01e^{s})+e^{-2s}=0\Rightarrow e^{s}=0.03e^{s}+3\times0.99$

$\Rightarrow 0.97e^{s}=3\times0.99\Rightarrow \boxed{e^{s^*}=\frac{3\times0.99}{0.97}}=3.0619$

$f(s^*)=0.2682\rightsquigarrow$ chernoff bound.

$ET=100\times0.01=1$

$var(T)=100\times0.01\times0.99=0.99$

$pr\{T>3\}=pr\{z=\frac{T-1}{\sqrt{0.99}}>\frac{2}{\sqrt{0.99}}\}=Q\left(\frac{2}{\sqrt{0.99}}\right)=0.0222$

(7.46): $X_n = \frac{1}{2}(U_n + U_{n-1})$

We use the Cauchy criterion: (a)

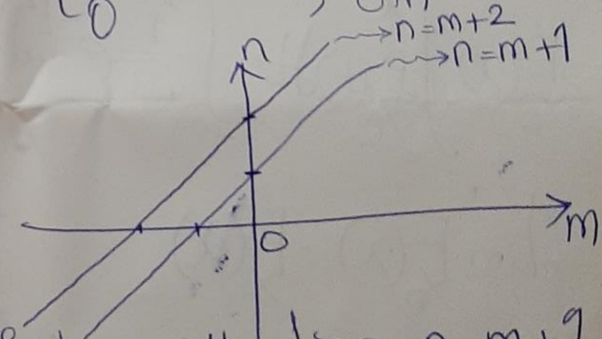
$$E\{(X_n - X_m)^2\} = E\{X_n^2\} + E\{X_m^2\} - 2E\{X_n X_m\}$$

$$E(X_n^2) = \frac{1}{4}E((U_n + U_{n-1})^2) = \frac{1}{4}\underbrace{EU_n^2}_{1} + \frac{1}{4}\underbrace{EU_{n-1}^2}_{1} + \frac{1}{2}\underbrace{EU_n EU_{n-1}}_{0} = \frac{1}{2}$$

$$E(X_m^2) = \cdots = \frac{1}{2}$$

$$E(X_n X_m) = \frac{1}{4}E((U_n + U_{n-1})(U_m + U_{m-1})) = \begin{cases} \frac{1}{2} & ; n = m \\ \frac{1}{4} & ; n = m-1 \\ \frac{1}{4} & ; n = m+1 \\ 0 & ; \text{oth} \end{cases}$$

$$\Rightarrow E\{(X_n - X_m)^2\} = \begin{cases} 0 ; & n = m \\ \frac{1}{2} ; & n = m \pm 1 \\ 1 ; & \text{oth} \end{cases}$$



If n and m approach to infinity on the line $n = m+1$ the limit will become $\frac{1}{2}$. Therefor, this limit does not exist.

$\Rightarrow$ This sequence does not converge in the mean square sense

(7.43): This question is same as question 6 in problem set 5. please see the solution of this question in problem set 5 solutions.

(b): $X_n = \frac{1}{2}(U_n + U_{n-1})$ let define $Y_n \triangleq \frac{1}{2}(U_n - U_{n-1})$

$$\Rightarrow \begin{pmatrix} X_n \\ Y_n \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix}}_{A} \begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} \Rightarrow \begin{cases} U_n = X_n + Y_n \\ U_{n-1} = X_n - Y_n \end{cases}$$

$$f_{X_n, Y_n}(x,y) = \frac{f_{U_n, U_{n-1}}(x+y, x-y)}{|\det(A)|} = \frac{\frac{1}{2\pi} e^{\frac{-(x+y)^2 - (x-y)^2}{2}}}{\frac{1}{2}}$$

$$= \frac{1}{\pi} e^{-x^2} e^{-y^2}$$

$$\Rightarrow f_{X_n}(x) = \int_{-\infty}^{+\infty} f_{X_n, Y_n}(x,y)\, dy = \frac{1}{\sqrt{\pi}} e^{-x^2} \cancel{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \frac{1}{2}}} e^{\frac{-y^2}{2 \times \frac{1}{2}}} dy}^{\,1} = \frac{1}{\sqrt{\pi}} e^{-x^2}$$

$$\Rightarrow F_{X_n}(x) = \int_{-\infty}^{x} f_{X_n}(t)\, dt = \int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-t^2} dt = F_X(x)$$

X is a Gaussian random variable with mean zero and variance $\frac{1}{2}$.

$$\Rightarrow \lim_{n \to +\infty} F_{X_n}(x) = F_X(x)$$

converges in distribution.

<u>second method</u>: (characteristic function)

$U_n \perp\!\!\!\perp U_{n-1}$ $\quad \phi_{U_n}(\omega) = \phi_{U_{n-1}}(\omega) = e^{j\omega 0 - \frac{1}{2} \times 1 \times \omega^2} = e^{\frac{-1}{2}\omega^2}$

$$\phi_{X_n}(\omega) = E\left(e^{j\omega X_n}\right) \underset{U_n \perp\!\!\!\perp U_{n-1}}{=} E\left(e^{j\omega \frac{1}{2} U_n}\right) E\left(e^{j\omega \frac{1}{2} U_{n-1}}\right) = \phi_{U_n}\left(\tfrac{1}{2}\omega\right) \phi_{U_{n-1}}\left(\tfrac{1}{2}\omega\right) = e^{\frac{-1}{4}\omega^2}$$

$$\Rightarrow f_{X_n}(x) = F^{-1}\left(\phi_{X_n}(-\omega)\right) = \frac{1}{\sqrt{\pi}} e^{-x^2} \Rightarrow F_{X_n}(x) = \int_{-\infty}^{x} f_{X_n}(t)\, dt = F_X(x)$$

$$\Rightarrow \lim_{n \to +\infty} F_{X_n}(x) = F_X(x)$$ converges