PS 8

1. Let $X(t)$ be a modified version of the random telegraph process. The process switches between the two states $X(t) = 1$ and $X(t) = -1$, with the time between switches following exponential distributions $f_T(s) = \lambda \exp(-\lambda) u(s)$. Also, the starting state is determined by flipping a biased coin so that $\Pr(X(0) = 1) = p$ and $\Pr(X(0) = -1) = 1 - p$.

   (a) Find $\Pr(X(t) = 1)$ and $\Pr(X(t) = -1)$

   (b) Find the mean function, $\mu_X(t)$

   (c) Find the autocorrelation function, $R_{X,X}(t_1, t_2)$

2. Let W_n be an IID sequence of zero-mean Gaussian random variables with variance σ_W^2. Define a discrete time random process $X[n] = pX[n-1] + W_n$, $n = 1, 2, 3, \cdots$, where $X[0] = W_0$ and p is a constant.

   (a) Find the mean function, $\mu_X(n)$

   (b) Find the autocorrelation function, $R_{X,X}(n_1, n_2)$

3. Prove that the family of differential equations

$$\frac{d}{dt} P_X(0;t) + \lambda P_X(0;t) = 0$$
$$\frac{d}{dt} P_X(i;t) + \lambda P_X(i;t) = \lambda P_X(i-1;t), \quad i = 1, 2, 3, \ldots$$

   leads to the Poisson distribution

$$P_X(i;t) = \frac{(\lambda t)^i}{i!} e^{-\lambda t}$$

4. Consider a Poisson counting process with arrival rate λ.

   (a) Suppose it is observed that there is exactly one arrival in the time interval $[0, t_0)$. Find the PDF of that arrival time.

   (b) Now suppose there were exactly two arrivals in the time interval $[0, t_0)$. Find the joint PDF of those two arrival times.

5. Let $X(t)$ be a Poisson counting process with arrival rate λ. Find $\Pr(N(t) = k | N(t+\tau) = m)$, where $\tau > 0$ and $m \geq k$.

6. Let $X_i(t), i = 1, 2, \cdots, n$ be a sequence of independent Poisson counting processes with arrival rates λ_i. Show that the sum of all of these Poisson processes,

$$X(t) = \sum_{i=1}^{n} X_i(t)$$

   is itself a Poisson process. What is the arrival rate of the sum process?

7. A workstation is used until it fails and is then sent out for repair. The time between failures, or the length of time the workstation functions until it needs repair, is a random variable T. Assume the times between failures, $T_1, T_2, \cdots T_n$, of the workstations available are independent random variables that are identically distributed. For $t > 0$, let the number of workstations that have failed be $N(t)$.

   (a) If the time between failures of each workstation has an exponential PDF, then what type of process is $N(t)$?

   (b) Assume that you have just purchased 10 new workstations and that each has a 90-day warranty. If the mean time between failures (MTBF) is 250 days, what is the probability that at least one workstation will fail before the end of the warranty period?

8. Suppose the arrival of calls at a switchboard is modeled as a Poisson process with the rate of calls per minute being $\lambda_a = 0.1$.

   (a) What is the probability that the number of calls arriving in a 10-minute interval is less than 10?

   (b) What is the probability that the number of calls arriving in a 10-minute interval is less than 10 if $\lambda_a = 10$?

   (c) Assuming $\lambda_a = 0.1$, what is the probability that one call arrives during the first 10-minute interval and two calls arrive during the second 10-minute interval?

9. Model lightning strikes to a power line during a thunderstorm as a Poisson impulse process. Suppose the number of lightning strikes in time interval t has a mean rate of arrival given by s, which is one strike per 3 minutes.

   (a) What is the expected number of lightning strikes in 1 minute? in 10 minutes?

   (b) What is the average time between lightning strikes?

10. Suppose the power line in the previous problem has an impulse response that may be approximated by $h(t) = te^{-at}u(t)$, where $a = 10$ sec^{-1}.

    (a) Find the mean function of the shot noise process.

    (b) Find the auto-correlation function of the shot noise process.