2.2 The outcome of this experiment consists of a pair of numbers (x, y) where x = number of dots in first toss and y = number of dots in second toss. Therefore, S = set of ordered pairs (x, y) where $x, y \epsilon \{1, 2, 3, 4, 5, 6\}$ which are listed in the table below:

a)

$x \backslash y$	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

checkmarks indicate elements of events

b)

$x \backslash y$	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6	✓	✓	✓	✓	✓	✓

$A = \{N_1 < N_2\}^C = \{N_1 \geq N_2\}$

c)

$x \backslash y$	1	2	3	4	5	6
1						
2						
3						
4						
5						
6	✓	✓	✓	✓	✓	✓

$B = \{N_1 = 6\}$

d) B is a subset of A so when B occurs then A also occurs, thus B implies A

e) $A \cap B^C = \{N_2 \leq N_1 < 6\}$

f) $A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$

2.3 ⓐ $A = \{0, 1, 2, 3, 4, 5\}$

ⓑ $A = \{3\}$

ⓒ $\{0\} = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$

$\{1\} = \{(1,2), (2,3), (3,4), (4,5), (5,6), (2,1), (3,2), (4,3), (5,4), (6,5)\}$

$\{2\} = \{(1,3), (2,4), (3,5), (4,6), (3,1), (4,2), (5,3), (6,4)\}$

$\{3\} = \{(1,4), (2,5), (3,6), (4,1), (5,2), (6,3)\}$

$\{4\} = \{(1,5), (2,6), (5,1), (6,2)\}$

$\{5\} = \{(1,6), (6,1)\}$

2.4)

a)

X \ Y	−2	−1	0	1	2
+2	—	—	(2,0)	(2,1)	(2,2)
−2	(−2,−2)	(−2,−1)	(−2,0)	—	—

b) "X definitely +2" (based on observed Y): $\{(2,1), (2,2)\}$

c) $\{Y=0\} = \{(2,0), (-2,0)\}$

"observed output is zero"
cannot determine input

x	1	2	3	4	5	6
1	✓					
2	✓	✓				
3	✓	✓	✓			
4	✓	✓	✓	✓		
5	✓	✓	✓	✓	✓	
6						

f) C = "number of dots differ by 2"

	1	2	3	4	5	6
1			✓			
2				✓		
3	✓				✓	
4		✓				✓
5			✓			
6				✓		

Comparing the tables for A and C we see

$A \cap C = \{(3,1), (4,2), (5,3), (6,4)\}$

2.6 (a) $S = \{abc, cab, bca, acb, bac, cba\}$

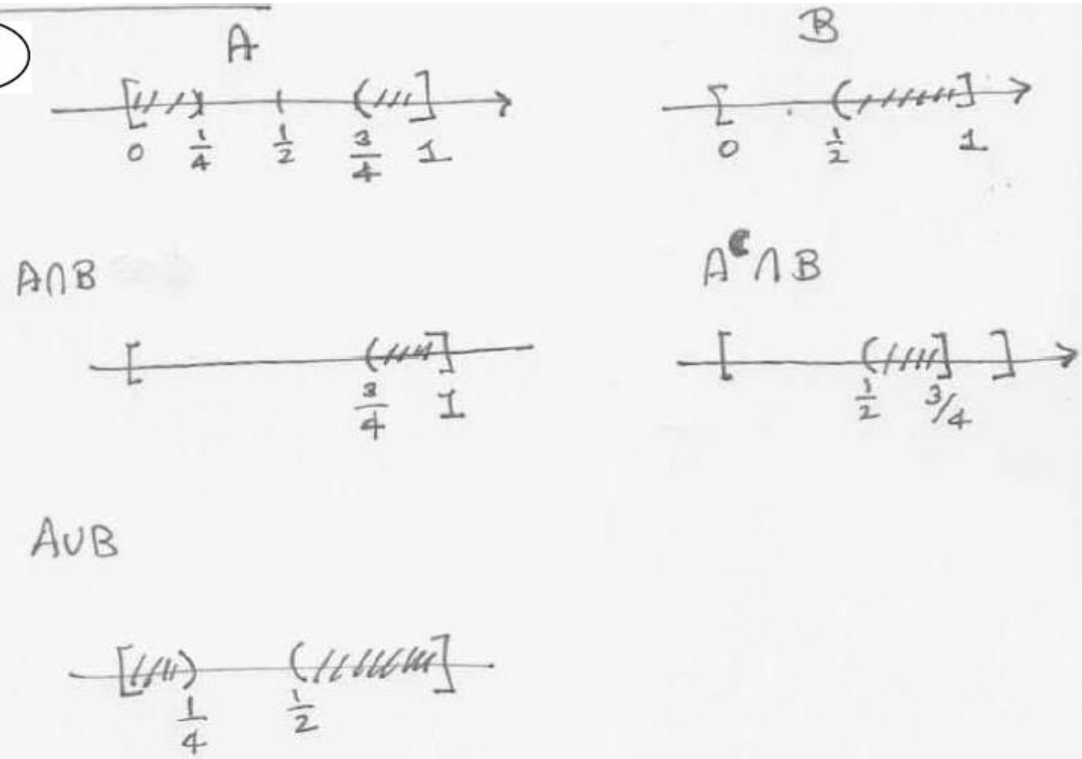
(b) $A = \{abc, acb\}$ $\quad B = \{abc, cba\}$ $\quad C = \{abc, bac\}$

(c) $(A \cup B \cup C)^c = \{abc, acb, cba, bac\}^c = \{cab, bca\}$

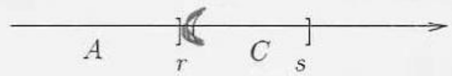
(d) $A \cap B \cap C = \{abc\}$

(e) $A \cup B \cup C = \{abc, acb, cba, bac\}$.

2.8



2.9 If we sketch the events A and B we see that $B = A \cup C$. We also see that the intervals corresponding to A and C have no points in common so $A \cap C = \emptyset$



We also see that $(r, s] = (r, \infty) \cap (-\infty, s] = (-\infty, r]^C \cap (-\infty, s]$
that is $C = A^C \cap B$

2.16 (a) "System j is up" $= A_{j1} \cap A_{j2}$

"System is up" $= (A_{11} \cap A_{12}) \cup (A_{21} \cap A_{22}) \cup (A_{31} \cap A_{32})$

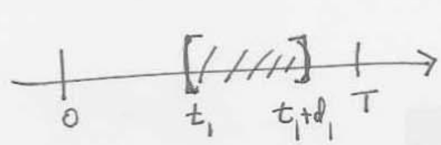
(b) "jth level connection active" if $A_{j1} \cap A_{j2}$

"connection active" if any of 3 connections is active

(2.18) (a) $S = \{(t_1, t_2) : 0 < t_1 < T,\ 0 < t_2 < T\}$

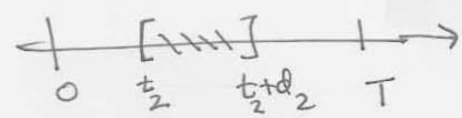
(b) A = train in crossing

$= \{t_1 < t < t_1 + d_1\}$



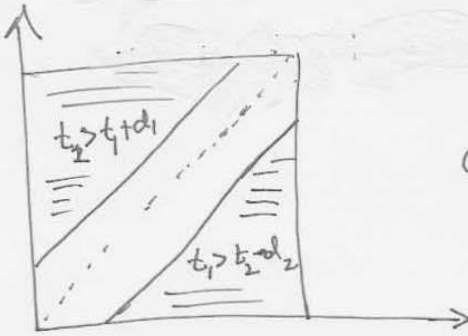
B = car in crossing

$= \{t_2 < t < t_2 + d_2\}$

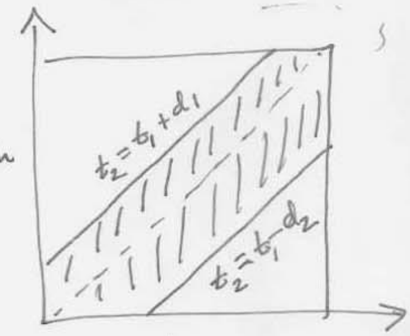


No collision occurs if $A \cap B$ is empty $\iff$ $t_1 > t_2 + d_2$ or $t_2 > t_1 + d_1$

No collision

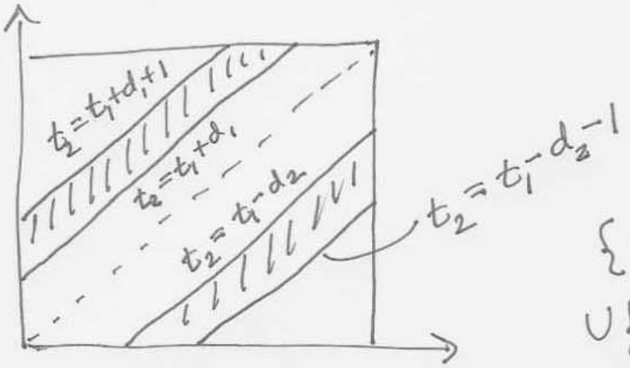


Collision



(c) Collision missed by 1 second or less.

$= \{\text{No Collision}\} \cap \{\text{Within 1 second of collision}\}$



$\{t_1 + d_1 < t_2 < t_1 + d_1 + 1\}$

$\cup \{t_2 + d_2 < t_1 < t_2 + d_2 + 1\}$

(2.21) The sample space in tossing a die is $S = \{1, 2, 3, 4, 5, 6\}$. Let $p_i = P[\{i\}] = p$ since all faces are equally likely. By Axiom 1

$$
\begin{aligned}
1 &= P[S] \\
&= P[\{1\} \cup \{2\} \cup \{3\} \cup \{4\} \cup \{5\} \cup \{6\}]
\end{aligned}
$$

The elementary events $\{i\}$ are mutually exclusive so by Corollary 4:

(a)

$$1 = p_1 + p_2 + \ldots + p_6 = 6p \Rightarrow p_i = p = \tfrac{1}{6} \text{ for } i = 1, \ldots, 6$$

(b) $P[A] = P[> 3 \text{ dots}] = P[\{4,5,6\}] = P[\{4\}] + P[\{5\}] + P[\{6\}] = \frac{3}{6}$

$P[B] = P[\text{odd } \#] = P[\{1,3,5\}] = P[\{1\}] + P[\{3\}] + P[\{5\}] = \frac{3}{6}$

(c) $P[A \cup B] = P[\{1, 3, 4, 5, 6\}] = \frac{5}{6}$

$P[A \cap B] = P[\{5\}] = \frac{1}{6}$

$P[A^c] = 1 - P[A] = \frac{3}{6}$

(2.23)

$P[\{a, d\}] = p_a + p_d = \frac{3}{8}$ by expressing each event in terms of elementary events

$P[\{b, c\}] = p_b + p_c = \frac{6}{8}$

$P[\{d\}] = [illegible] \frac{1}{8}$

~~Solving this set of linear~~

~~[illegible]~~

$1 = P[S] = p_a + p_b + p_c + p_d = 1$

Solving this set of linear equations gives

$p_a = \frac{1}{8} \quad p_b = \frac{4}{8} \quad p_c = \frac{2}{8} \quad p_d = \frac{1}{8}$

2.24

(a) $P[A \cap B^c] = P[A] - P[A \cap B]$

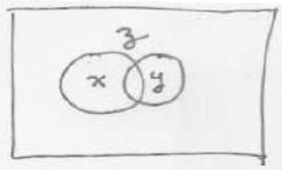
$P[A^c \cap B] = P[B] - P[A \cap B]$

(b) $P[(A \cap B^c) \cup (A^c \cap B)] = P[A] + P[B] - 2P[A \cap B]$

(c) $P[(A \cup B)^c] = 1 - P[A \cup B] = 1 - P[A] - P[B] + P[A \cap B]$

2.25

$z = P[A \cup B] = P[A] + P[B] - P[A \cap B] = x + y - P[A \cap B]$



$P[A \cap B] = x + y - z$

$P[A^c \cap B^c] = 1 - P[(A^c \cap B^c)^c] = 1 - P[A \cup B]$

$= 1 - z$

$P[A^c \cup B^c] = 1 - P[(A^c \cup B^c)^c] = 1 - P[A \cap B] = 1 - x - y + z$

$P[A \cap B^c] = P[A] - P[A \cap B] = x - (x + y - z) = z - y$

$P[A^c \cup B] = 1 - P[A \cap B^c] = 1 - z + y$

2.26 Identities of this type are shown by application of the axioms. We begin by treating $(A \cup B)$ as a single event, then

$P[A \cup B \cup C]$	$= P[(A \cup B) \cup C]$	
	$= P[A \cup B] + P[C] - P[(A \cup B) \cap C]$	by Cor. 5
	$= P[A] + P[B] - P[A \cap B] + P[C]$	by Cor. 5 on $A \cup B$
	$- P[(A \cap C) \cup (B \cap C)]$	and by distributive property
	$= P[A] + P[B] + P[C] - P[A \cap B]$	
	$- P[A \cap C] - P[B \cap C]$	by Cor. 5 on
	$+ P[(A \cap B) \cap (B \cap C)]$	$(A \cap C) \cup (B \cap C)$
	$= P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C]$	since
	$- P[B \cap C] + P[A \cap B \cap C]$.	$(A \cap B) \cap (B \cap C) = A \cap B \cap C$

2.33 $S=\{1,2,\ldots,59,60\}$

(a) $P[k]=\frac{1}{60}$ $\quad k\in S$

(b) $p_2=\frac{1}{2}p_1$ $\quad p_3=\frac{1}{3}p_1$ $\quad\ldots\quad p_{60}=\frac{1}{60}p_1$

$$1=p_1+p_2+\cdots+p_{60}=p_1\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{60}\right)=4.68\,p_1$$

$p_1=0.2137$

(c) $p_2=\frac{1}{2}p_1$ $\quad p_3=\frac{1}{4}p_1$ $\quad p_3=\frac{1}{8}p_1$ $\quad\ldots\quad p_{60}=\left(\frac{1}{2}\right)^{59}p_1$

$$1=p_1\left(1+\frac{1}{2}+\frac{1}{4}+\cdots+\left(\frac{1}{2}\right)^{59}\right)\approx 2p_1$$

$p_1=\frac{1}{2}$

(d) for a: $p[60]=\frac{1}{60}$ $\quad$ b: $p[60]=\frac{0.2137}{60}=0.00356$ $\quad$ c: $p[60]=0.86\times10^{-18}$

2.34) $\Omega=[-1,2]$ $\quad A=[-1,0)$ $\quad B=(0,1)$ $\quad C=(0.75,2]$

(a): $P(A)=\frac{0-(-1)}{2-(-1)}=\frac{1}{3}$ $\quad P(B)=\frac{1-0}{2-(-1)}=\frac{1}{3}$ $\quad P(C)=\frac{2-0.75}{2-(-1)}=\frac{1.25}{3}=\frac{5}{12}$

$P(A\cap B)=P(\emptyset)=0$ $\quad P(A\cap C)=P(\emptyset)=0$

Direct method

(b): $P(A\cup B)=P([-1,1))=\frac{1-(-1)}{2-(-1)}=\frac{2}{3}$

$$P(A\cup C)=P([-1,0)\cup(0.75,2])=\frac{0-(-1)+2-0.75}{2-(-1)}=\frac{2.25}{3}=\frac{9}{12}=\frac{3}{4}$$

$P(A\cup B\cup C)=P([-1,2])=1$

second method: Let us use axioms and Corollaries.

$$P(A\cup B)=P(A)+P(B)-P(A\cap B)=\frac{1}{3}+\frac{1}{3}-0=\frac{2}{3}$$

$$P(A\cup C)=P(A)+P(C)-P(A\cap C)=\frac{1}{3}+\frac{5}{12}-0=\frac{4+5}{12}=\frac{9}{12}=\frac{3}{4}$$

$$P(A\cup B\cup C)=P(A)+P(B)+P(C)-\underbrace{P(A\cap B)}_{0}-\underbrace{P(A\cap C)}_{0}-\underbrace{P(B\cap C)}_{\frac{1}{12}}+\underbrace{P(A\cap B\cap C)}_{0}$$

$$=\frac{1}{3}+\frac{1}{3}+\frac{5}{12}-\frac{1}{12}=1$$

2.34 Assume that the probability of any subinterval I of $[-1,2]$ is proportional to its length, then

$$P[I] = k \text{ length } (I).$$

If we let $I = [-1,2]$ then we must have that

$$1 = P[S] = P[[-1,2]] = k \text{ length } ([-1,2]) = 3k \Rightarrow k = \frac{1}{3}.$$

a) $P[A] = \frac{1}{3} \text{ length } ([-1,0)) = \frac{1}{3}(1) = \frac{1}{3}$

$P[B] = \frac{1}{3} \text{ length } ((-0.5,1)) = \frac{1}{3}\frac{3}{2} = \cancel{\frac{3}{4}}\ \frac{1}{2}$

$P[C] = \frac{1}{3} \text{ length } ((0.75,2)) = \cancel{\frac{1}{3}\frac{1}{4}}\ \frac{1}{3}\ \frac{5}{4} = \frac{5}{12}$

$P[A \cap B] = \frac{1}{3} \text{ length } ((-0.5,0)) = \frac{1}{3}\frac{1}{2} = \frac{1}{6}$

$P[A \cap C] = P[\emptyset] = 0$

b) $P[A \cup B] = \cancel{P[S]}\ \frac{1}{3} \text{ length } ([-1,1)) = \frac{2}{3}$

$P[A \cup C] = \frac{1}{3} \text{ length } (A \cup C)$

$= \frac{1}{3}\left(1 + \frac{5}{4}\right) = \frac{3}{4}$

$P[A \cup B \cup C] = P[S] = 1$

Now use axioms and corollaries:

$$P[A \cup B] = P[A] + P[B] - P[A \cap B] \quad \text{by Cor. 5}$$
$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3} \quad \checkmark$$

$$P[A \cup C] = P[A] + P[C] - P[\underbrace{A \cap C}_{0}] = \frac{1}{3} + \frac{5}{12} = \frac{3}{4} \quad \checkmark \quad \text{by Cor. 5}$$

$$P[A \cup B \cup C] = P[A] + P[B] + P[C]$$
$$-P[A \cap B] - P[A \cap C] - P[B \cap C]$$
$$+P[A \cap B \cap C] \quad \text{by Eq. (2.7)}$$
$$= \frac{1}{3} + \frac{1}{2} + \frac{5}{12} - \frac{1}{6} - 0 - \frac{1}{12} + 0$$
$$= 1 \quad \checkmark$$

2.37 a) Since $(-\infty, r] \subset (-\infty, s]$ when $r < s$

$$P[(-\infty, r]] \leq P[(-\infty, s]] \text{ by Corollary 7.}$$

b)

$$\begin{aligned} P[(-\infty, s]] &= P[(-\infty, r] \cup (r, s]] \\ &= P[(-\infty, r]] + P[(r, s]] \\ \Rightarrow P[(r, s]] &= P[(-\infty, s]] - P[(-\infty, r]] \end{aligned}$$

c) $(s, +\infty) = (-\infty, s]^c \Rightarrow P((s, +\infty)) = 1 - P((-\infty, s])$

2.62 $A = \{N_1 \geq N_2\}$ $B = \{N_1 = 6\}$

from problem 2.2 we have that $A \supset B$, therefore

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = \frac{P[B]}{P[B]} = 1$$

and

$$P[B|A] = \frac{P[A \cap B]}{P[A]} = \frac{P[B]}{P[A]} = \frac{6/36}{21/36} = \frac{2}{7}$$

2.69): $P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{P(\emptyset)}{P(B)} = 0$

$$P(B/C) = \frac{P(B \cap C)}{P(C)} = \frac{P((0.75, 1))}{P((0.75, 2]))} = \frac{1 - 0.75}{2 - 0.75} = \frac{0.25}{1.25} = \frac{1}{5}$$

$$P(A/C^c) = \frac{P(A \cap C^c)}{P(C^c)} = \frac{P([-1, 0))}{P([-1, 0.75])} = \frac{0 - (-1)}{0.75 + 1} = \frac{1}{1.75} = \frac{4}{7}$$

$$P(B/C^c) = \frac{P(B \cap C^c)}{P(C^c)} = \frac{P((0, 0.75]))}{P([-1, 0.75]))} = \frac{0.75 - 0}{0.75 - (-1)} = \frac{0.75}{1.75} = \frac{3}{7}$$

2.69 Proceeding as in Problem 2.84

$$P[A|B] = \frac{P[A\cap B]}{P[B]} = \frac{P[(-0.5,0)]}{P[(-0.5,1)]} = \frac{1/6}{1/2} = \frac{1}{3}$$

$$P[B|C] = \frac{P[B\cap C]}{P[C]} = \frac{P[(0.75,1)]}{P[(0.75,2)]} = \frac{1/12}{5/12} = \frac{1}{5}$$

$$P[A|C^c] = \frac{P[A\cap C^c]}{P[C^c]} = \frac{P[(-1,0)]}{P[[-1,0.75]]} = \frac{1/3}{7/12} = \frac{4}{7}$$

$$P[B|C^c] = \frac{P[B\cap C^c]}{P[C^c]} = \frac{P[(-0.5,0.75)]}{P[[-1,0.75]]} = \frac{5/12}{7/12} = \frac{5}{7}.$$

2.71 P[2 or more students have some birthday]

$= 1 - P$[all students have different birthdays]

P[all students have different birthdays]

$$= \frac{365}{365}\frac{364}{365}\frac{363}{365}\cdots\frac{346}{365} = 0.588$$

P[2 or more have same birthday] $= 0.412$

P[2 or more have same birthday in class of 23] $= 0.507$

For a general class of size N, we have:

$$P = 1 - \frac{\prod_{n=1}^{N}(366-n)}{365^N}$$

N	20	21	22	23	24	25
P	0.412	0.443	0.475	0.507	0.538	0.568

$\underbrace{0.507 \quad 0.538 \quad 0.568}_{N \geq 23}$

2.73 **a)** The results follow directly from the definition of conditional probability. $P[A|B] = \dfrac{P[A \cap B]}{P[B]}$.

$$\text{If} \quad A \cap B = \emptyset \quad \text{then } P[A \cap B] = 0 \text{ by Corollary 3 and thus} \quad P[A|B] = 0 \; .$$

$$\text{If} \quad A \subset B \quad \text{then } A \cap B = A \text{ and} \quad P[A|B] = \frac{P[A]}{P[B]} \; .$$

$$\text{If} \quad A \supset B \Rightarrow \quad A \cap B = B \text{ and} \quad P[A|B] = \frac{P[B]}{P[B]} = 1.$$

b) If $P[A|B] = \dfrac{P[A \cap B]}{P[B]} > P[A]$ then multiplying both sides by $P[B]$ we have: $P[A \cap B] > P[A]P[B]$

We then also have that $P[B|A] = \dfrac{P[A \cap B]}{P[A]} > \dfrac{P[A]P[B]}{P[A]} = P[B]$.

~~We conclude that if $P[A|B] > P[A]$ then B and A tend to occur jointly.~~

2.74

$$P[A|B] = \frac{P[A\cap B]}{P[B]} \quad \text{for } P[B] > 0.$$

(i) $P[A\cap B] \geq 0 \Rightarrow P[A|B] \geq 0.$ ✓

$A\cap B \subset B \Rightarrow P[A\cap B] \leq P[B] \Rightarrow P[A|B] \leq 1.$ ✓

(ii) $P[S|B] = \frac{P[B\cap S]}{P[B]} = \frac{P[B]}{P[B]} = 1$ ✓

(iii) If $A\cap C = \phi$ then

$$P[A\cup C|B] = \frac{P[(A\cup C)\cap B]}{P[B]} = \frac{P[(A\cap B)\cup(C\cap B)]}{P[B]}$$

$$= \frac{P[A\cap B] + P[C\cap B]}{P[B]} \quad \text{since } (A\cap B)\cap(C\cap B) = A\cap B\cap C = \phi$$

$$= P[A|B] + P[C|B]$$ ✓

2.75

$$\begin{aligned} P[A\cap B\cap C] &= P[A|B\cap C]P[B\cap C] \\ &= P[A|B\cap C]P[B|C]P[C] \end{aligned}$$

2.77 Let X denote the input and Y the output

(a) $P[Y=0] = P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=1]$

$= (1-\varepsilon_1)q + \varepsilon_2 p.$ $(q = 1-p)$

Similarly

$P[Y=1] = (1-\varepsilon_2)p + \varepsilon_1 q$

(b) $P[X=0|Y=1] = \dfrac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \dfrac{\varepsilon_1 q}{(1-\varepsilon_2)p + \varepsilon_1 q}$

$P[X=1|Y=1] = \dfrac{(1-\varepsilon_2)p}{(1-\varepsilon_2)p + \varepsilon_1 q}$

$P[X=1|Y=1] > P[X=0|Y=1]$

$\Leftrightarrow \quad (1-\varepsilon_2)p > \varepsilon_1 q = \varepsilon_1(1-p)$

$\Leftrightarrow \quad p > \dfrac{\varepsilon_1}{1-\varepsilon_2+\varepsilon_1}$

2.81 Let X denote the input and Y the output.

a)
$$\begin{aligned}P[Y=0] &= P[Y=0|X=0]P[X=0] + P[Y=0|X=1]P[X=0] \\ &\quad + P[Y=0|X=2]P[X=2] \\ &= \cancel{(1-\varepsilon)\tfrac{1}{2} + 1\cdot\tfrac{1}{4} + \varepsilon\tfrac{1}{4}} = (1-\varepsilon)\tfrac{1}{3} + \varepsilon\tfrac{1}{3} = \tfrac{1}{3} \\ &\cancel{= \tfrac{1}{2} + \tfrac{\varepsilon}{4}}\end{aligned}$$

Similarly

$$\begin{aligned}P[Y=1] &= \cancel{\varepsilon\frac{1}{2} + (1-\varepsilon)\frac{1}{4} + 0\frac{1}{4} = \frac{1}{4} + \frac{\varepsilon}{4}}\ \frac{1}{3} \\ P[Y=2] &= \cancel{0\cdot\frac{1}{2} + \varepsilon\cdot\frac{1}{4} + (1-\varepsilon)\frac{1}{4} = \frac{1}{4}}\ \frac{1}{3}\end{aligned}$$

b) Using Bayes' Rule

$$\begin{aligned}P[X=0|Y=1] &= \frac{P[Y=1|X=0]P[X=0]}{P[Y=1]} = \frac{\frac{1}{3}\varepsilon}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}}\ \frac{1}{3}} = \cancel{\frac{2\varepsilon}{1+\varepsilon}}\ \varepsilon \\ P[X=1|Y=1] &= \frac{P[Y=1|X=1]P[X=1]}{P[Y=1]} = \frac{(1-\varepsilon)\frac{1}{3}}{\cancel{\frac{1}{4}+\frac{\varepsilon}{4}}\ \frac{1}{3}} = \cancel{\frac{1-\varepsilon}{1+\varepsilon}}\ 1-\varepsilon \\ P[X=2|Y=1] &= 0\end{aligned}$$