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PROBLEMS

Section 2.1: Specifying Random Experiments

2.1. The (loose) minute hand in a clock is spun hard and the hour at which the hand comes to rest is noted.

(a) What is the sample space?

(b) Find the sets corresponding to the events: A = "hand is in first 4 hours"; B = "hand is between 2nd and 8th hours inclusive"; and D = "hand is in an odd hour."

(c) Find the events: $A \cap B \cap D$, $A^c \cap B$, $A \cup (B \cap D^c)$, $(A \cup B) \cap D^c$.

2.2. A die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

(a) Find the sample space.

(b) Find the set A corresponding to the event "number of dots in first toss is not less than number of dots in second toss."

(c) Find the set B corresponding to the event "number of dots in first toss is 6."

(d) Does A imply B or does B imply A?

(e) Find $A \cap B^c$ and describe this event in words.

(f) Let C correspond to the event "number of dots in dice differs by 2." Find $A \cap C$.

2.3. Two dice are tossed and the magnitude of the difference in the number of dots facing up in the two dice is noted.

(a) Find the sample space.

(b) Find the set A corresponding to the event "magnitude of difference is 3."

(c) Express each of the elementary events in this experiment as the union of elementary events from Problem 2.2.

2.4. A binary communication system transmits a signal X that is either a $+2$ voltage signal or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.

(a) Find the sample space.

(b) Find the set of outcomes corresponding to the event "transmitted signal was definitely $+2$."

(c) Describe in words the event corresponding to the outcome $Y = 0$.

2.5. A desk drawer contains six pens, four of which are dry.

(a) The pens are selected at random one by one until a good pen is found. The sequence of test results is noted. What is the sample space?

(b) Suppose that only the number, and not the sequence, of pens tested in part a is noted. Specify the sample space.

(c) Suppose that the pens are selected one by one and tested until both good pens have been identified, and the sequence of test results is noted. What is the sample space?

(d) Specify the sample space in part c if only the number of pens tested is noted.

2.6. Three friends (Al, Bob, and Chris) put their names in a hat and each draws a name from the hat. (Assume Al picks first, then Bob, then Chris.)

(a) Find the sample space.

(b) Find the sets A, B, and C that correspond to the events "Al draws his name," "Bob draws his name," and "Chris draws his name."

(c) Find the set corresponding to the event, "no one draws his own name."

(d) Find the set corresponding to the event, "everyone draws his own name."

(e) Find the set corresponding to the event, "one or more draws his own name."

2.7. Let M be the number of message transmissions in Experiment E_6.

(a) What is the set A corresponding to the event "M is even"?

(b) What is the set B corresponding to the event "M is a multiple of 3"?

(c) What is the set C corresponding to the event "6 or fewer transmissions are required"?

(d) Find the sets $A \cap B$, $A - B$, $A \cap B \cap C$ and describe the corresponding events in words.

2.8. A number U is selected at random from the unit interval. Let the events A and B be: A = "U differs from 1/2 by more than 1/4" and B = "$1 - U$ is less than 1/2." Find the events $A \cap B$, $A^c \cap B$, $A \cup B$.

2.9. The sample space of an experiment is the real line. Let the events A and B correspond to the following subsets of the real line: $A = (-\infty, r]$ and $B = (-\infty, s]$, where $r \le s$. Find an expression for the event $C = (r, s]$ in terms of A and B. Show that $B = A \cup C$ and $A \cap C = \varnothing$.

2.10. Use Venn diagrams to verify the set identities given in Eqs. (2.2) and (2.3). You will need to use different colors or different shadings to denote the various regions clearly.

2.11. Show that:

(a) If event A implies B, and B implies C, then A implies C.

(b) If event A implies B, then B^c implies A^c.

2.12. Show that if $A \cup B = A$ and $A \cap B = A$ then $A = B$.

2.13. Let A and B be events. Find an expression for the event "exactly one of the events A and B occurs." Draw a Venn diagram for this event.

2.14. Let A, B, and C be events. Find expressions for the following events:

(a) Exactly one of the three events occurs.

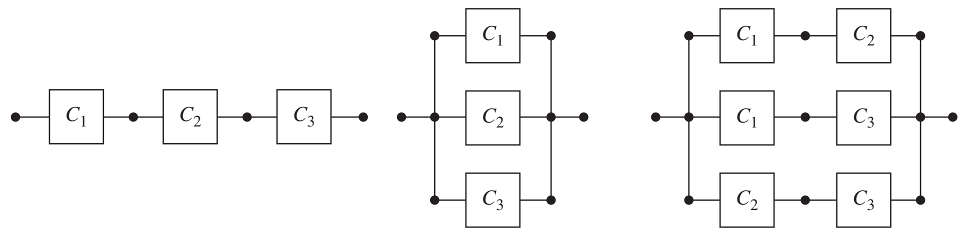
(b) Exactly two of the events occur.

(c) One or more of the events occur.

(d) Two or more of the events occur.

(e) None of the events occur.

2.15. Figure P2.1 shows three systems of three components, C_1, C_2, and C_3. Figure P2.1(a) is a "series" system in which the system is functioning only if all three components are functioning. Figure 2.1(b) is a "parallel" system in which the system is functioning as long as at least one of the three components is functioning. Figure 2.1(c) is a "two-out-of-three" system in which the system is functioning as long as at least two components are functioning. Let A_k be the event "component k is functioning." For each of the three system configurations, express the event "system is functioning" in terms of the events A_k.



(a) Series system (b) Parallel system (c) Two-out-of-three system

FIGURE P2.1

2.16. A system has two key subsystems. The system is "up" if both of its subsystems are functioning. Triple redundant systems are configured to provide high reliability. The overall system is operational as long as one of three systems is "up." Let A_{jk} correspond to the event "unit k in system j is functioning," for $j = 1, 2, 3$ and $k = 1, 2$.

(a) Write an expression for the event "overall system is up."

(b) Explain why the above problem is equivalent to the problem of having a connection in the network of switches shown in Fig. P2.2.

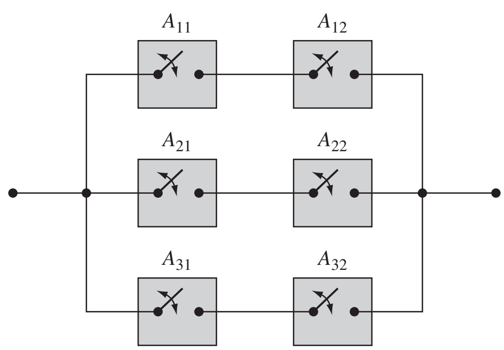


FIGURE P2.2

2.17. In a specified 6-AM-to-6-AM 24-hour period, a student wakes up at time t_1 and goes to sleep at some later time t_2.

(a) Find the sample space and sketch it on the x-y plane if the outcome of this experiment consists of the pair (t_1, t_2).

(b) Specify the set A and sketch the region on the plane corresponding to the event "student is asleep at noon."

(c) Specify the set B and sketch the region on the plane corresponding to the event "student sleeps through breakfast (7–9 AM)."

(d) Sketch the region corresponding to $A \cap B$ and describe the corresponding event in words.

2.18. A road crosses a railroad track at the top of a steep hill. The train cannot stop for oncoming cars and cars, cannot see the train until it is too late. Suppose a train begins crossing the road at time t_1 and that the car begins crossing the track at time t_2, where $0 < t_1 < T$ and $0 < t_2 < T$.

(a) Find the sample space of this experiment.

(b) Suppose that it takes the train d_1 seconds to cross the road and it takes the car d_2 seconds to cross the track. Find the set that corresponds to a collision taking place.

(c) Find the set that corresponds to a collision is missed by 1 second or less.

2.19. A random experiment has sample space $S = \{-1, 0, +1\}$.

(a) Find all the subsets of S.

(b) The outcome of a random experiment consists of pairs of outcomes from S where the elements of the pair cannot be equal. Find the sample space S' of this experiment. How many subsets does S' have?

2.20. **(a)** A coin is tossed twice and the sequence of heads and tails is noted. Let S be the sample space of this experiment. Find all subsets of S.

(b) A coin is tossed twice and the number of heads is noted. Let S? be the sample space of this experiment. Find all subsets of S'.

(c) Consider parts a and b if the coin is tossed 10 times. How many subsets do S and S' have? How many bits are needed to assign a binary number to each possible subset?

Section 2.2: The Axioms of Probability

2.21. A die is tossed and the number of dots facing up is noted.

(a) Find the probability of the elementary events under the assumption that all faces of the die are equally likely to be facing up after a toss.

(b) Find the probability of the events: $A = \{\text{more than 3 dots}\}$; $B = \{\text{odd number of dots}\}$.

(c) Find the probability of $A \cup B$, $A \cap B$, A^c.

2.22. In Problem 2.2, a die is tossed twice and the number of dots facing up in each toss is counted and noted in the order of occurrence.

(a) Find the probabilities of the elementary events.

(b) Find the probabilities of events A, B, C, $A \cap B^c$, and $A \cap C$ defined in Problem 2.2.

2.23. A random experiment has sample space $S = \{a, b, c, d\}$. Suppose that $P[\{c, d\}] = 3/8$, $P[\{b, c\}] = 6/8$, and $P[\{d\}] = 1/8$, $P[\{c, d\}] = 3/8$. Use the axioms of probability to find the probabilities of the elementary events.

2.24. Find the probabilities of the following events in terms of $P[A]$, $P[B]$, and $P[A \cap B]$:

(a) A occurs and B does not occur; B occurs and A does not occur.

(b) Exactly one of A or B occurs.

(c) Neither A nor B occur.

2.25. Let the events A and B have $P[A] = x$, $P[B] = y$, and $P[A \cup B] = z$. Use Venn diagrams to find $P[A \cap B]$, $P[A^c \cap B^c]$, $P[A^c \cup B^c]$, $P[A \cap B^c]$, $P[A^c \cup B]$.

2.26. Show that

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] - P[A \cap B] - P[A \cap C] - P[B \cap C] + P[A \cap B \cap C].$$

2.27. Use the argument from Problem 2.26 to prove Corollary 6 by induction.

2.28. A hexadecimal character consists of a group of three bits. Let A_i be the event "ith bit in a character is a 1."

(a) Find the probabilities for the following events: $A_1, A_1 \cap A_3, A_1 \cap A_2 \cap A_3$ and $A_1 \cup A_2 \cup A_3$. Assume that the values of bits are determined by tosses of a fair coin.

(b) Repeat part a if the coin is biased.

2.29. Let M be the number of message transmissions in Problem 2.7. Find the probabilities of the events $A, B, C, C^c, A \cap B, A - B, A \cap B \cap C$. Assume the probability of successful transmission is 1/2.

2.30. Use Corollary 7 to prove the following:

(a) $P[A \cup B \cup C] \le P[A] + P[B] + P[C]$.

(b) $P\left[\bigcup_{k=1}^{n} A_k\right] \le \sum_{k=1}^{n} P[A_k]$.

(c) $P\left[\bigcap_{k=1}^{n} A_k\right] \ge 1 - \sum_{k=1}^{n} P[A_k^c]$.

The second expression is called the **union bound**.

2.31. Let p be the probability that a single character appears incorrectly in this book. Use the union bound for the probability of there being any errors in a page with n characters.

2.32. A die is tossed and the number of dots facing up is noted.

(a) Find the probability of the elementary events if faces with an even number of dots are twice as likely to come up as faces with an odd number.

(b) Repeat parts b and c of Problem 2.21.

2.33. Consider Problem 2.1 where the minute hand in a clock is spun. Suppose that we now note the *minute* at which the hand comes to rest.

(a) Suppose that the minute hand is very loose so the hand is equally likely to come to rest anywhere in the clock. What are the probabilities of the elementary events?

(b) Now suppose that the minute hand is somewhat sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/3 as likely to land in the third minute as in the first, and so on. What are the probabilities of the elementary events?

(c) Now suppose that the minute hand is very sticky and so the hand is 1/2 as likely to land in the second minute than in the first, 1/2 as likely to land in the third minute as in the second, and so on. What are the probabilities of the elementary events?

(d) Compare the probabilities that the hand lands in the last minute in parts a, b, and c.

2.34. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$.

(a) Find the probabilities of A, B, $A \cap B$, and $A \cap C$.

(b) Find the probabilities of $A \cup B$, $A \cup C$, and $A \cup B \cup C$, first, by directly evaluating the sets and then their probabilities, and second, by using the appropriate axioms or corollaries.

2.35. A number x is selected at random in the interval $[-1, 2]$. Numbers from the subinterval $[0, 2]$ occur half as frequently as those from $[-1, 0)$.

(a) Find the probability assignment for an interval completely within $[-1, 0)$; completely within $[0, 2]$; and partly in each of the above intervals.

(b) Repeat Problem 2.34 with this probability assignment.

2.36. The lifetime of a device behaves according to the probability law $P[(t, \infty)] = 1/t$ for $t > 1$. Let A be the event "lifetime is greater than 4," and B the event "lifetime is greater than 8."

(a) Find the probability of $A \cap B$, and $A \cup B$.

(b) Find the probability of the event "lifetime is greater than 6 but less than or equal to 12."

2.37. Consider an experiment for which the sample space is the real line. A probability law assigns probabilities to subsets of the form $(-\infty, r]$.

(a) Show that we must have $P[(-\infty, r]] \le P[(-\infty, s]]$ when $r < s$.

(b) Find an expression for $P[(r, s]]$ in terms of $P[(-\infty, r]]$ and $P[(-\infty, s]]$

(c) Find an expression for $P[(s, \infty)]$.

2.38. Two numbers (x, y) are selected at random from the interval $[0, 1]$.

(a) Find the probability that the pair of numbers are inside the unit circle.

(b) Find the probability that $y > 2x$.

*Section 2.3: Computing Probabilities Using Counting Methods

2.39. The combination to a lock is given by three numbers from the set $\{0, 1, \ldots, 59\}$. Find the number of combinations possible.

2.40. How many seven-digit telephone numbers are possible if the first number is not allowed to be 0 or 1?

2.41. A pair of dice is tossed, a coin is flipped twice, and a card is selected at random from a deck of 52 distinct cards. Find the number of possible outcomes.

2.42. A lock has two buttons: a "0" button and a "1" button. To open a door you need to push the buttons according to a preset 8-bit sequence. How many sequences are there? Suppose you press an arbitrary 8-bit sequence; what is the probability that the door opens? If the first try does not succeed in opening the door, you try another number; what is the probability of success?

2.43. A Web site requires that users create a password with the following specifications:

- Length of 8 to 10 characters
- Includes at least one special character $\{!, @, \#, \$, \%, \hat{}, \&, *, (,), +, =, \{, \}, |, <, >, \backslash, \_, \text{-}, [,], /, ?\}$
- No spaces
- May contain numbers (0–9), lower and upper case letters (a–z, A–Z)
- Is case-sensitive.

How many passwords are there? How long would it take to try all passwords if a password can be tested in 1 microsecond?

2.44. A multiple choice test has 10 questions with 3 choices each. How many ways are there to answer the test? What is the probability that two papers have the same answers?

2.45. A student has five different t-shirts and three pairs of jeans ("brand new," "broken in," and "perfect").

(a) How many days can the student dress without repeating the combination of jeans and t-shirt?

(b) How many days can the student dress without repeating the combination of jeans and t-shirt and without wearing the same t-shirt on two consecutive days?

2.46. Ordering a "deluxe" pizza means you have four choices from 15 available toppings. How many combinations are possible if toppings can be repeated? If they cannot be repeated? Assume that the order in which the toppings are selected does not matter.

2.47. A lecture room has 60 seats. In how many ways can 45 students occupy the seats in the room?

2.48. List all possible permutations of two distinct objects; three distinct objects; four distinct objects. Verify that the number is $n!$.

2.49. A toddler pulls three volumes of an encyclopedia from a bookshelf and, after being scolded, places them back in random order. What is the probability that the books are in the correct order?

2.50. Five balls are placed at random in five buckets. What is the probability that each bucket has a ball?

2.51. List all possible combinations of two objects from two distinct objects; three distinct objects; four distinct objects. Verify that the number is given by the binomial coefficient.

2.52. A dinner party is attended by four men and four women. How many unique ways can the eight people sit around the table? How many unique ways can the people sit around the table with men and women alternating seats?

2.53. A hot dog vendor provides onions, relish, mustard, ketchup, Dijon ketchup, and hot peppers for your hot dog. How many variations of hot dogs are possible using one condiment? Two condiments? None, some, or all of the condiments?

2.54. A lot of 100 items contains k defective items. M items are chosen at random and tested.

(a) What is the probability that m are found defective? This is called the *hypergeometric distribution*.

(b) A lot is accepted if 1 or fewer of the M items are defective. What is the probability that the lot is accepted?

2.55. A park has N raccoons of which eight were previously captured and tagged. Suppose that 20 raccoons are captured. Find the probability that four of these are found to be tagged. Denote this probability, which depends on N, by $p(N)$. Find the value of N that maximizes this probability. *Hint:* Compare the ratio $p(N)/p(N-1)$ to unity.

2.56. A lot of 50 items has 40 good items and 10 bad items.

(a) Suppose we test five samples from the lot, with replacement. Let X be the number of defective items in the sample. Find $P[X = k]$.

(b) Suppose we test five samples from the lot, without replacement. Let Y be the number of defective items in the sample. Find $P[Y = k]$.

2.57. How many distinct permutations are there of four red balls, two white balls, and three black balls?

2.58. A hockey team has 6 forwards, 4 defensemen, and 2 goalies. At any time, 3 forwards, 2 defensemen, and 1 goalie can be on the ice. How many combinations of players can a coach put on the ice?

2.59. Find the probability that in a class of 28 students exactly four were born in each of the seven days of the week.

2.60. Show that

$$\binom{n}{k} = \binom{n}{n-k}$$

2.61. In this problem we derive the multinomial coefficient. Suppose we partition a set of n distinct objects into J subsets $B_1, B_2, \dots, B_J$ of size $k_1, \dots, k_J$, respectively, where $k_i \ge 0$, and $k_1 + k_2 + \dots + k_J = n$.

(a) Let N_i denote the number of possible outcomes when the ith subset is selected. Show that

$$N_1 = \binom{n}{k_1}, N_2 = \binom{n-k_1}{k_2}, \dots, N_{J-1} = \binom{n-k_1-\cdots-k_{J-2}}{k_{J-1}}.$$

(b) Show that the number of partitions is then:

$$N_1 N_2 \ldots N_{J-1} = \frac{n!}{k_1!\, k_2! \ldots k_J!}.$$

Section 2.4: Conditional Probability

2.62. A die is tossed twice and the number of dots facing up is counted and noted in the order of occurrence. Let A be the event "number of dots in first toss is not less than number of dots in second toss," and let B be the event "number of dots in first toss is 6." Find $P[A|B]$ and $P[B|A]$.

2.63. Use conditional probabilities and tree diagrams to find the probabilities for the elementary events in the random experiments defined in parts a to d of Problem 2.5.

2.64. In Problem 2.6 (name in hat), find $P[B \cap C|A]$ and $P[C|A \cap B]$.

2.65. In Problem 2.29 (message transmissions), find $P[B|A]$ and $P[A|B]$.

2.66. In Problem 2.8 (unit interval), find $P[B|A]$ and $P[A|B]$.

2.67. In Problem 2.36 (device lifetime), find $P[B|A]$ and $P[A|B]$.

2.68. In Problem 2.33, let $A = \{\text{hand rests in last 10 minutes}\}$ and $B = \{\text{hand rests in last 5 minutes}\}$. Find $P[B|A]$ for parts a, b, and c.

2.69. A number x is selected at random in the interval $[-1, 2]$. Let the events $A = \{x < 0\}$, $B = \{|x - 0.5| < 0.5\}$, and $C = \{x > 0.75\}$. Find $P[A|B]$, $P[B|C]$, $P[A|C^c]$, $P[B|C^c]$.

2.70. In Problem 2.36, let A be the event "lifetime is greater than t," and B the event "lifetime is greater than $2t$." Find $P[B|A]$. Does the answer depend on t? Comment.

2.71. Find the probability that two or more students in a class of 20 students have the same birthday. *Hint:* Use Corollary 1. How big should the class be so that the probability that two or more students have the same birthday is 1/2?

2.72. A cryptographic hash takes a message as input and produces a fixed-length string as output, called the digital fingerprint. A brute force attack involves computing the hash for a large number of messages until a pair of distinct messages with the same hash is found. Find the number of attempts required so that the probability of obtaining a match is 1/2. How many attempts are required to find a matching pair if the digital fingerprint is 64 bits long? 128 bits long?

2.73. **(a)** Find $P[A|B]$ if $A \cap B = \varnothing$; if $A \subset B$; if $A \supset B$.

(b) Show that if $P[A|B] > P[A]$, then $P[B|A] > P[B]$.

2.74. Show that $P[A|B]$ satisfies the axioms of probability.

(i) $0 \le P[A|B] \le 1$

(ii) $P[S|B] = 1$

(iii) If $A \cap C = \varnothing$, then $P[A \cup C|B] = P[A|B] + P[C|B]$.

2.75. Show that $P[A \cap B \cap C] = P[A|B \cap C]P[B|C]P[C]$.

2.76. In each lot of 100 items, two items are tested, and the lot is rejected if either of the tested items is found defective.

(a) Find the probability that a lot with k defective items is accepted.

(b) Suppose that when the production process malfunctions, 50 out of 100 items are defective. In order to identify when the process is malfunctioning, how many items should be tested so that the probability that one or more items are found defective is at least 99%?

2.77. A nonsymmetric binary communications channel is shown in Fig. P2.3. Assume the input is "0" with probability p and "1" with probability $1 - p$.

(a) Find the probability that the output is 0.

(b) Find the probability that the input was 0 given that the output is 1. Find the probability that the input is 1 given that the output is 1. Which input is more probable?

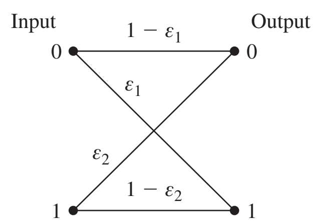


FIGURE P2.3

2.78. The transmitter in Problem 2.4 is equally likely to send $X = +2$ as $X = -2$. The malicious channel counts the number of heads in two tosses of a fair coin to decide by how much to reduce the magnitude of the input to produce the output Y.

(a) Use a tree diagram to find the set of possible input-output pairs.

(b) Find the probabilities of the input-output pairs.

(c) Find the probabilities of the output values.

(d) Find the probability that the input was $X = +2$ given that $Y = k$.

2.79. One of two coins is selected at random and tossed three times. The first coin comes up heads with probability p_1 and the second coin with probability $p_2 = 2/3 > p_1 = 1/3$.

(a) What is the probability that the number of heads is k?

(b) Find the probability that coin 1 was tossed given that k heads were observed, for $k = 0, 1, 2, 3$.

(c) In part b, which coin is more probable when k heads have been observed?

(d) Generalize the solution in part b to the case where the selected coin is tossed m times. In particular, find a threshold value T such that when $k > T$ heads are observed, coin 1 is more probable, and when $k < T$ are observed, coin 2 is more probable.

(e) Suppose that $p_2 = 1$ (that is, coin 2 is two-headed) and $0 < p_1 < 1$. What is the probability that we do not determine with certainty whether the coin is 1 or 2?

2.80. A computer manufacturer uses chips from three sources. Chips from sources A, B, and C are defective with probabilities .005, .001, and .010, respectively. If a randomly selected chip is found to be defective, find the probability that the manufacturer was A; that the manufacturer was C. Assume that the proportions of chips from A, B, and C are 0.5, 0.1, and 0.4, respectively.

2.81. A ternary communication system is shown in Fig. P2.4. Suppose that input symbols 0, 1, and 2 occur with probability 1/3 respectively.

(a) Find the probabilities of the output symbols.

(b) Suppose that a 1 is observed at the output. What is the probability that the input was 0? 1? 2?

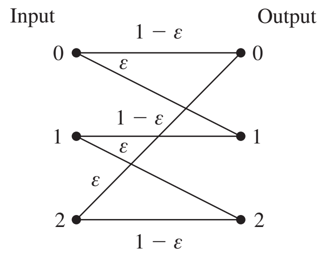


FIGURE P2.4

End of P1

Section 2.5: Independence of Events

2.82. Let $S = \{1, 2, 3, 4\}$ and $A = \{1, 2\}$, $B = \{1, 3\}$, $C = \{1, 4\}$. Assume the outcomes are equiprobable. Are A, B, and C independent events?

2.83. Let U be selected at random from the unit interval. Let $A = \{0 < U < 1/2\}$, $B = \{1/4 < U < 3/4\}$, and $C = \{1/2 < U < 1\}$. Are any of these events independent?

2.84. Alice and Mary practice free throws at the basketball court after school. Alice makes free throws with probability p_a and Mary makes them with probability p_m. Find the probability of the following outcomes when Alice and Mary each take one shot: Alice scores a basket; Either Alice or Mary scores a basket; both score; both miss.

2.85. Show that if A and B are independent events, then the pairs A and B^c, A^c and B, and A^c and B^c are also independent.

2.86. Show that events A and B are independent if $P[A|B] = P[A|B^c]$.

2.87. Let A, B, and C be events with probabilities $P[A]$, $P[B]$, and $P[C]$.

(a) Find $P[A \cup B]$ if A and B are independent.

(b) Find $P[A \cup B]$ if A and B are mutually exclusive.

(c) Find $P[A \cup B \cup C]$ if A, B, and C are independent.

(d) Find $P[A \cup B \cup C]$ if A, B, and C are pairwise mutually exclusive.

2.88. An experiment consists of picking one of two urns at random and then selecting a ball from the urn and noting its color (black or white). Let A be the event "urn 1 is selected" and B the event "a black ball is observed." Under what conditions are A and B independent?

2.89. Find the probabilities in Problem 2.14 assuming that events A, B, and C are independent.

2.90. Find the probabilities that the three types of systems are "up" in Problem 2.15. Assume that all units in the system fail independently and that a type k unit fails with probability p_k.

2.91. Find the probabilities that the system is "up" in Problem 2.16. Assume that all units in the system fail independently and that a type k unit fails with probability p_k.

2.92. A random experiment is repeated a large number of times and the occurrence of events A and B is noted. How would you test whether events A and B are independent?

2.93. Consider a very long sequence of hexadecimal characters. How would you test whether the relative frequencies of the four bits in the hex characters are consistent with independent tosses of coin?

2.94. Compute the probability of the system in Example 2.35 being "up" when a second controller is added to the system.

2.95. In the binary communication system in Example 2.26, find the value of ε for which the input of the channel is independent of the output of the channel. Can such a channel be used to transmit information?

2.96. In the ternary communication system in Problem 2.81, is there a choice of ε for which the input of the channel is independent of the output of the channel?

Section 2.6: Sequential Experiments

2.97. A block of 100 bits is transmitted over a binary communication channel with probability of bit error $p = 10^{-2}$.

(a) If the block has 1 or fewer errors then the receiver accepts the block. Find the probability that the block is accepted.

(b) If the block has more than 1 error, then the block is retransmitted. Find the probability that M retransmissions are required.

2.98. A fraction p of items from a certain production line is defective.

(a) What is the probability that there is more than one defective item in a batch of n items?

(b) During normal production $p = 10^{-3}$ but when production malfunctions $p = 10^{-1}$. Find the size of a batch that should be tested so that if any items are found defective we are 99% sure that there is a production malfunction.

2.99. A student needs eight chips of a certain type to build a circuit. It is known that 5% of these chips are defective. How many chips should he buy for there to be a greater than 90% probability of having enough chips for the circuit?

2.100. Each of n terminals broadcasts a message in a given time slot with probability p.

(a) Find the probability that exactly one terminal transmits so the message is received by all terminals without collision.

(b) Find the value of p that maximizes the probability of successful transmission in part a.

(c) Find the asymptotic value of the probability of successful transmission as n becomes large.

2.101. A system contains eight chips. The lifetime of each chip has a Weibull probability law: with parameters λ and $k = 2$: $P[(t, \infty)] = e^{-(\lambda t)^k}$ for $t \geq 0$. Find the probability that at least two chips are functioning after $2/\lambda$ seconds.

2.102. A machine makes errors in a certain operation with probability p. There are two types of errors. The fraction of errors that are type 1 is α, and type 2 is $1 - \alpha$.

(a) What is the probability of k errors in n operations?

(b) What is the probability of k_1 type 1 errors in n operations?

(c) What is the probability of k_2 type 2 errors in n operations?

(d) What is the joint probability of k_1 and k_2 type 1 and 2 errors, respectively, in n operations?

2.103. Three types of packets arrive at a router port. Ten percent of the packets are "expedited forwarding (EF)," 30 percent are "assured forwarding (AF)," and 60 percent are "best effort (BE)."

(a) Find the probability that k of N packets are not expedited forwarding.

(b) Suppose that packets arrive one at a time. Find the probability that k packets are received before an expedited forwarding packet arrives.

(c) Find the probability that out of 20 packets, 4 are EF packets, 6 are AF packets, and 10 are BE.

2.104. A run-length coder segments a binary information sequence into strings that consist of either a "run" of k "zeros" punctuated by a "one", for $k = 0, \dots, m - 1$, or a string of m "zeros." The $m = 3$ case is:

String	**Run-length k**
1	0
01	1
001	2
000	3

Suppose that the information is produced by a sequence of Bernoulli trials with $P[\text{"one"}] = P[\text{success}] = p$.

(a) Find the probability of run-length k in the $m = 3$ case.

(b) Find the probability of run-length k for general m.

2.105. The amount of time cars are parked in a parking lot follows a geometric probability law with $p = 1/2$. The charge for parking in the lot is \$1 for each half-hour or less.

(a) Find the probability that a car pays k dollars.

(b) Suppose that there is a maximum charge of \$6. Find the probability that a car pays k dollars.

2.106. A biased coin is tossed repeatedly until heads has come up three times. Find the probability that k tosses are required. *Hint:* Show that $\{\text{"}k \text{ tosses are required"}\} = A \cap B$, where $A = \{\text{"}k\text{th toss is heads"}\}$ and $B = \{\text{"2 heads occurs in } k - 1 \text{ tosses"}\}$.

2.107. An urn initially contains two black balls and two white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn; if the color of the ball is the same as the majority of balls remaining in the urn, then the ball is put back in the urn. Otherwise the ball is left out.

(a) Draw the trellis diagram for this experiment and label the branches by the transition probabilities.

(b) Find the probabilities for all sequences of outcomes of length 2 and length 3.

(c) Find the probability that the urn contains no black balls after three draws; no white balls after three draws.

(d) Find the probability that the urn contains two black balls after n trials; two white balls after n trials.

2.108. In Example 2.45, let $p_0(n)$ and $p_1(n)$ be the probabilities that urn 0 or urn 1 is used in the nth subexperiment.

(a) Find $p_0(1)$ and $p_1(1)$.

(b) Express $p_0(n + 1)$ and $p_1(n + 1)$ in terms of $p_0(n)$ and $p_1(n)$.

(c) Evaluate $p_0(n)$ and $p_1(n)$ for $n = 2, 3, 4$.

(d) Find the solution to the recursion in part b with the initial conditions given in part a.

(e) What are the urn probabilities as n approaches infinity?

*Section 2.7: Synthesizing Randomness: Number Generators

2.109. An urn experiment is to be used to simulate a random experiment with sample space $S = \{1, 2, 3, 4, 5\}$ and probabilities $p_1 = 1/3$, $p_2 = 1/5$, $p_3 = 1/4$, $p_4 = 1/7$, and $p_5 = 1 - (p_1 + p_2 + p_3 + p_4)$. How many balls should the urn contain? Generalize the result to show that an urn experiment can be used to simulate any random experiment with finite sample space and with probabilities given by rational numbers.

2.110. Suppose we are interested in using tosses of a fair coin to simulate a random experiment in which there are six equally likely outcomes, where $S = \{0, 1, 2, 3, 4, 5\}$. The following version of the "rejection method" is proposed:

1. Toss a fair coin three times and obtain a binary number by identifying heads with zero and tails with one.
2. If the outcome of the coin tosses in step 1 is the binary representation for a number in S, output the number. Otherwise, return to step 1.

(a) Find the probability that a number is produced in step 2.

(b) Show that the numbers that are produced in step 2 are equiprobable.

(c) Generalize the above algorithm to show how coin tossing can be used to simulate any random urn experiment.

2.111. Use the `rand` function in Octave to generate 1000 pairs of numbers in the unit square. Plot an x-y scattergram to confirm that the resulting points are uniformly distributed in the unit square.

2.112. Apply the rejection method introduced above to generate points that are uniformly distributed in the $x > y$ portion of the unit square. Use the `rand` function to generate a pair of numbers in the unit square. If $x > y$, accept the number. If not, select another pair. Plot an x-y scattergram for the pair of accepted numbers and confirm that the resulting points are uniformly distributed in the $x > y$ region of the unit square.

2.113. The *sample mean-squared value* of the numerical outcomes $X(1), X(2), \ldots X(n)$ of a series of n repetitions of an experiment is defined by

$$\langle X^2 \rangle_n = \frac{1}{n} \sum_{j=1}^{n} X^2(j).$$

(a) What would you expect this expression to converge to as the number of repetitions n becomes very large?

(b) Find a recursion formula for $\langle X^2 \rangle_n$ similar to the one found in Problem 1.9.

2.114. The *sample variance* is defined as the mean-squared value of the variation of the samples about the sample mean

$$\langle V^2 \rangle_n = \frac{1}{n} \sum_{j=1}^{n} \{X(j) - \langle X \rangle_n\}^2.$$

Note that the $\langle X \rangle_n$ also depends on the sample values. (It is customary to replace the n in the denominator with $n - 1$ for technical reasons that will be discussed in Chapter 8. For now we will use the above definition.)

(a) Show that the sample variance satisfies the following expression:

$$\langle V^2 \rangle_n = \langle X^2 \rangle_n - \langle X \rangle_n^2.$$

(b) Show that the sample variance satisfies the following recursion formula:

$$\langle V^2 \rangle_n = \left(1 - \frac{1}{n}\right) \langle V^2 \rangle_{n-1} + \frac{1}{n}\left(1 - \frac{1}{n}\right)(X(n) - \langle X \rangle_{n-1})^2,$$

with $\langle V^2 \rangle_0 = 0$.

2.115. Suppose you have a program to generate a sequence of numbers U_n that is uniformly distributed in $[0, 1]$. Let $Y_n = \alpha U_n + \beta$.

(a) Find α and β so that Y_n is uniformly distributed in the interval $[a, b]$.

(b) Let $a = -5$ and $b = 15$. Use Octave to generate Y_n and to compute the sample mean and sample variance in 1000 repetitions. Compare the sample mean and sample variance to $(a + b)/2$ and $(b - a)^2/12$, respectively.

2.116. Use Octave to simulate 100 repetitions of the random experiment where a coin is tossed 16 times and the number of heads is counted.

(a) Confirm that your results are similar to those in Figure 2.18.

(b) Rerun the experiment with $p = 0.25$ and $p = 0.75$. Are the results as expected?

*Section 2.8: Fine Points: Event Classes

2.117. In Example 2.49, Homer maps the outcomes from Lisa's sample space $S_L = \{\text{r}, \text{g}, \text{t}\}$ into a smaller sample space $S_H = \{\text{R}, \text{G}\} : f(\text{r}) = \text{R}, f(\text{g}) = \text{G}$, and $f(\text{t}) = \text{G}$.

Define the inverse image events as follows:

$$f^{-1}(\{\text{R}\}) = A_1 = \{r\} \quad \text{and} \quad f^{-1}(\{\text{G}\}) = A_2 = \{\text{g}, \text{t}\}.$$

Let A and B be events in Homer's sample space.

(a) Show that $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(b) Show that $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(c) Show that $f^{-1}(A^c) = f^{-1}(A)^c$.

(d) Show that the results in parts a, b, and c hold for a general mapping f from a sample space S to a set S'.

2.118. Let f be a mapping from a sample space S to a finite set $S' = \{y_1, y_2, \ldots, y_n\}$.

(a) Show that the set of inverse images $A_k = f^{-1}(\{y_k\})$ forms a partition of S.

(b) Show that any event B of S' can be related to a union of A_k's.

2.119. Let A be any subset of S. Show that the class of sets $\{\varnothing, A, A^c, S\}$ is a field.

*Section 2.9: Fine Points: Probabilities of Sequences of Events

2.120. Find the countable union of the following sequences of events:

(a) $A_n = [a + 1/n, b - 1/n]$.

(b) $B_n = (-n, b - 1/n]$.

(c) $C_n = [a + 1/n, b)$.

2.121. Find the countable intersection of the following sequences of events:

(a) $A_n = (a - 1/n, b + 1/n)$.

(b) $B_n = [a, b + 1/n)$.

(c) $C_n = (a - 1/n, b]$.

2.122. (a) Show that the Borel field can be generated from the complements and countable intersections and unions of open sets (a, b).

(b) Suggest other classes of sets that can generate the Borel field.

2.123. Find expressions for the probabilities of the events in Problem 2.120.

2.124. Find expressions for the probabilities of the events in Problem 2.121.

Problems Requiring Cumulative Knowledge

2.125. Compare the binomial probability law and the hypergeometric law introduced in Problem 2.54 as follows.

(a) Suppose a lot has 20 items of which five are defective. A batch of ten items is tested without replacement. Find the probability that k are found defective for $k = 0, \ldots, 10$. Compare this to the binomial probabilities with $n = 10$ and $p = 5/20 = .25$.

(b) Repeat but with a lot of 1000 items of which 250 are defective. A batch of ten items is tested without replacement. Find the probability that k are found defective for $k = 0, \ldots, 10$. Compare this to the binomial probabilities with $n = 10$ and $p = 5/20 = .25$.

2.126. Suppose that in Example 2.43, computer A sends each message to computer B simultaneously over two unreliable radio links. Computer B can detect when errors have occurred in either link. Let the probability of message transmission error in link 1 and link 2 be q_1 and q_2 respectively. Computer B requests retransmissions until it receives an error-free message on either link.

(a) Find the probability that more than k transmissions are required.

(b) Find the probability that in the last transmission, the message on link 2 is received free of errors.

2.127. In order for a circuit board to work, seven identical chips must be in working order. To improve reliability, an additional chip is included in the board, and the design allows it to replace any of the seven other chips when they fail.

(a) Find the probability p_b that the board is working in terms of the probability p that an individual chip is working.

(b) Suppose that n circuit boards are operated in parallel, and that we require a 99.9% probability that at least one board is working. How many boards are needed?

2.128. Consider a well-shuffled deck of cards consisting of 52 distinct cards, of which four are aces and four are kings.

(a) Find the probability of obtaining an ace in the first draw.

(b) Draw a card from the deck and look at it. What is the probability of obtaining an ace in the second draw? Does the answer change if you had not observed the first draw?

(c) Suppose we draw seven cards from the deck. What is the probability that the seven cards include three aces? What is the probability that the seven cards include two kings? What is the probability that the seven cards include three aces and/or two kings?

(d) Suppose that the entire deck of cards is distributed equally among four players. What is the probability that each player gets an ace?