ECE537
Random Processes
Midterm 1
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1. Consider a Gaussian random vector $X = (X_1, X_2, X_3)$ with independent components, X_1, X_3 have zero mean, X_2 has mean -1, and $\mathrm{var}(X_i) = i$, $i = 1,2,3$. Now, consider the random vector $Y = AX + b$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and } b = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

   a) What is the covariance matrix for X?
   b) Determine $\mathcal{E}(Y)$.
   c) Determine the covariance matrix for Y.
   d) Give an expression for the PDF for X.

2. Consider two independent random variables X, Y. The PDFs are for X, uniform distribution on the interval $[-2,2]$, and for Y uniform distribution on $[0,6]$. Let $Z = \frac{1}{2}X + Y$.

   a) Give the PDF for Z, and plot it.
   b) Give the characteristic functions for X, Y, and Z.
   c) Determine the probability that $|Z - 3| > 5$
   d) Use the Chebychev inequality to estimate an upper bound for $P(|Z - 3| > 5)$.

ECE537 midterm 1 (solutions)

1a) X_i's are independent $\Rightarrow$ X_i's are uncorrelated

$$\Rightarrow \mathrm{Cov}(X) = \begin{pmatrix} \sigma_{X_1}^2 & 0 & 0 \\ 0 & \sigma_{X_2}^2 & 0 \\ 0 & 0 & \sigma_{X_3}^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

1b) $$EY = A\,EX + b = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

1c) $$\mathrm{Cov}(Y) = A\,\mathrm{Cov}(X)\,A^T = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 1 & 4 & 3 \\ 2 & 3 & 5 \end{pmatrix}$$

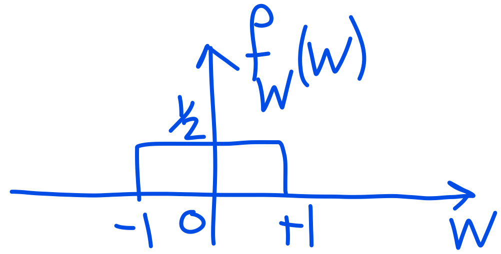
1d) $$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{X_1}(x_1)\,f_{X_2}(x_2)\,f_{X_3}(x_3)$$

$\hookrightarrow$ X_i's are independent

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x_1^2}{2}} \times \frac{1}{\sqrt{2\pi \times 2}} e^{-\frac{(x_2+1)^2}{2\times 2}} \times \frac{1}{\sqrt{2\pi\times 3}} e^{-\frac{x_3^2}{2\times 3}}$$

② $X \sim U[-2,2]$ $Y \sim U[0,6]$ $Z = \frac{1}{2}X + Y$

X and Y are independent.



②a Let $W = \frac{1}{2}X$. $f_W(w) = \dfrac{f_X(2w)}{\left|\frac{1}{2}\right|} = 2f_X(2w)$

$$\phi_Z(\omega) = E\{e^{j\omega z}\} = E\{e^{j\omega w} \times e^{j\omega y}\} = E\{e^{j\omega w}\} \times E\{e^{j\omega y}\} = \phi_W(\omega)\phi_Y(\omega)$$

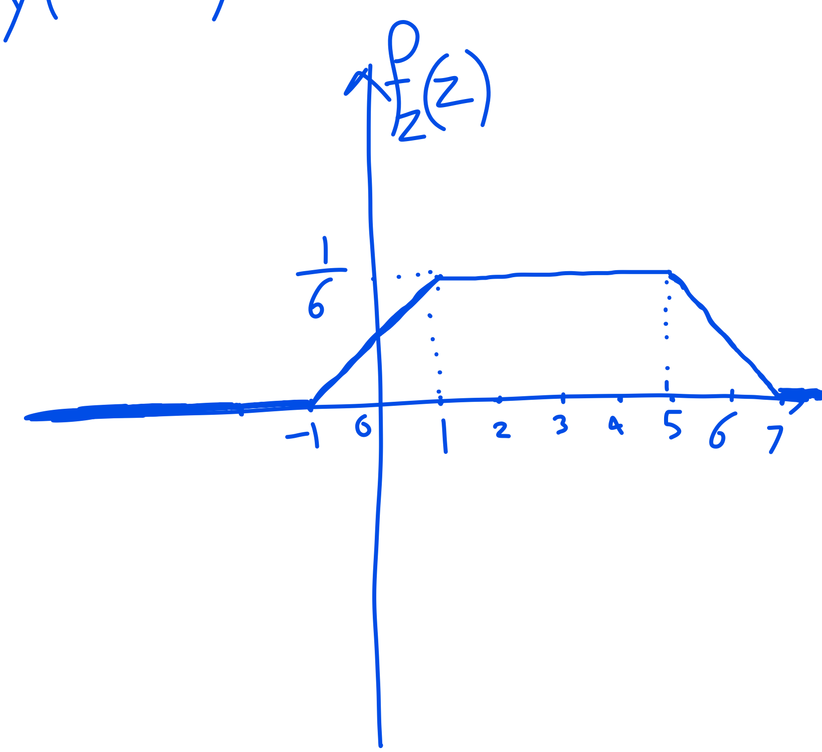
↳ w and y are independent

$$\Rightarrow \phi_Z(-\omega) = \phi_W(-\omega)\phi_Y(-\omega)$$

$$\Rightarrow F^{-1}\{\phi_Z(-\omega)\} = F^{-1}\{\phi_W(-\omega)\phi_Y(-\omega)\} \Rightarrow \boxed{f_Z(z) = f_W(z) * f_Y(z)}$$

$$\Rightarrow f_Z(z) = \int_{-\infty}^{+\infty} f_W(t) f_Y(z-t)\,dt = \frac{1}{2}\int_{-1}^{1} f_Y(-(t-z))\,dt$$

$$= \begin{cases} 0 & ; z \leq -1 \\ \dfrac{z+1}{12} & ; -1 \leq z \leq 1 \\ \dfrac{1}{6} & ; 1 \leq z \leq 5 \\ \dfrac{7-z}{12} & ; 5 \leq z \leq 7 \\ 0 & ; z \geq 7 \end{cases}$$



2b $\phi_X(w) = E\{e^{jwX}\} = \int_{-2}^{2} e^{jwx} \times \frac{1}{4}\,dx = \frac{1}{4} \times \frac{1}{jw} e^{jwx}\Big]_{-2}^{+2}$

$= \frac{1}{2w}\left(\frac{e^{j2w} - e^{-j2w}}{2j}\right) = \frac{\sin(2w)}{2w}$

$\phi_Y(w) = E\{e^{jwY}\} = \int_{0}^{6} e^{jwy} \times \frac{1}{6}\,dy = \frac{1}{6} \times \frac{1}{jw} e^{jwy}\Big]_{0}^{6} = \frac{e^{j6w} - 1}{6jw}$

$\phi_Z(w) = E\left\{e^{jw\left(\frac{1}{2}X + Y\right)}\right\} = E\left\{e^{j\left(\frac{w}{2}\right)X}\right\} E\left\{e^{jwY}\right\} = \phi_X\left(\frac{w}{2}\right)\phi_Y(w)$

$= \frac{\sin w}{w} \times \frac{e^{j6w} - 1}{6jw} = \frac{(e^{j6w} - 1)\sin(w)}{6jw^2}$

2c $P(|Z-3|>5) = P(Z-3>5 \text{ or } Z-3<-5)$

$= P(Z>8 \text{ or } Z<-2) = P(Z>8) + P(Z<-2) = 0 + 0 = 0$

2d $EX^2 = \int_{-2}^{2} \frac{1}{4} x^2\,dx = \frac{1}{12} x^3\Big]_{-2}^{2} = \frac{16}{12} = \frac{4}{3}$

$EY^2 = \int_{0}^{6} \frac{1}{6} y^2\,dy = \frac{1}{18} y^3\Big]_{0}^{6} = \frac{6 \times 36}{18} = 12$

$\Rightarrow \sigma_X^2 = E(X^2) - (EX)^2 = \frac{4}{3} - 0^2 = \frac{4}{3}$

$$\sigma_y^2 = E(Y^2) - (EY)^2 = 12 - 3^2 = 12 - 9 = 3$$

$$\sigma_Z^2 = \underbrace{\left(\tfrac{1}{2}\right)^2 \times \sigma_X^2 + \sigma_Y^2}_{X \text{ and } Y \text{ are independent}} = \frac{1}{4} \times \frac{4}{3} + 3 = \frac{10}{3}$$

$$EZ = \frac{1}{2}EX + EY = \frac{1}{2} \times 0 + 3 = 3$$

$$\Rightarrow P(|Z-3| > 5) \leq \frac{\sigma_Z^2}{5^2} = \frac{10}{25 \times 3} = \frac{2}{15}$$