Problem set 2: Solutions

1: D: diamonds (red) H: hearts (red)

S: spades C: clubs

a: $\Omega = \begin{Bmatrix} D2, D3, \ldots, D10, DA, DJ, DQ, DK \\ H2, H3, \ldots, H10, HA, HJ, HQ, HK \\ S2, S3, \ldots, S10, SA, SJ, SQ, SK \\ C2, C3, \ldots, C10, CA, CJ, CQ, CK \end{Bmatrix}$

$|\Omega| = 52$

c: $P(\text{diamonds}) = \frac{13}{52} = \frac{1}{4}$

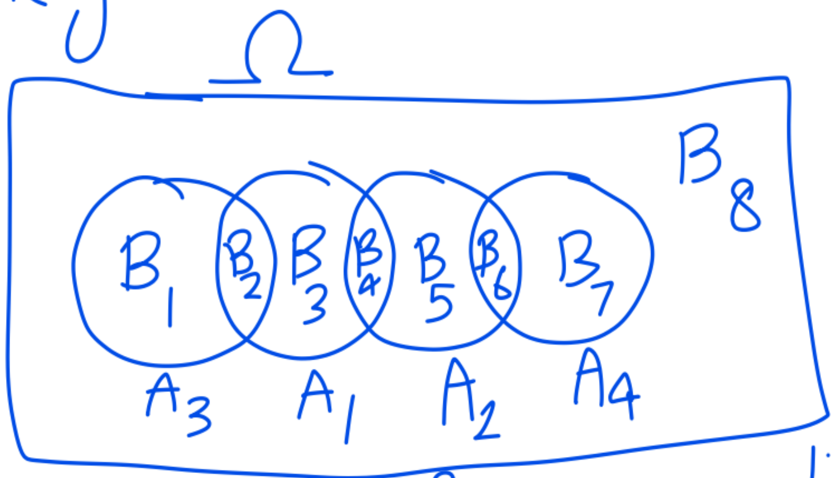
$P(\text{queen}) = \frac{4}{52} = \frac{1}{13}$

$P(\text{red } 6) = P(\text{Hearts } 6) + P(\text{diamonds } 6) = \frac{2}{52} = \frac{1}{26}$

$P(\text{clubs}) = \frac{13}{52} = \frac{1}{4}$

(d) $S = \{\underbrace{\text{diamonds}}_{A_1}, \underbrace{\text{a queen}}_{A_2}, \underbrace{\text{a red } 6}_{A_3}, \underbrace{\text{clubs}}_{A_4}\}$

Let us show the given events in the following venn diagram:



It is not hard to see that $\{B_i\}_{i=1}^{8}$ is a partition for the sample space Ω.

$B_i \cap B_j = \emptyset$ for $i \neq j$

$$\bigcup_{i=1}^{8} B_i = \Omega$$

Moreover, each B_i can be written in terms of A_i's as follows:

$$B_1 = A_3 \cap A_1^c \cap A_2^c \cap A_4^c$$

$B_2 = A_1 \cap A_3 \cap A_2^c \cap A_4^c$

$B_3 = A_1 \cap A_2^c \cap A_3^c \cap A_4^c$

$B_4 = A_1 \cap A_2 \cap A_3^c \cap A_4^c$

$B_5 = A_2 \cap A_1^c \cap A_3^c \cap A_4^c$

$B_6 = A_2 \cap A_4 \cap A_1^c \cap A_3^c$

$B_7 = A_4 \cap A_1^c \cap A_2^c \cap A_3^c$

$B_8 = A_1^c \cap A_2^c \cap A_3^c \cap A_4^c$

B_i's are called elementary events.

we can form $\sigma(S)$ as the set of all possible unions of B_i's. In forming a union, we have two choices for each elementary event B_i; either include it in the union, or exclude it.

As an example, we show some of these unions:

$$\sigma(S)=\{B_1, B_2, \dots, B_8, B_1 \cup B_2, B_1 \cup B_3, \dots, B_7 \cup B_8, B_1 \cup B_2 \cup B_3, \dots, B_6 \cup B_7 \cup B_8, \dots\}$$

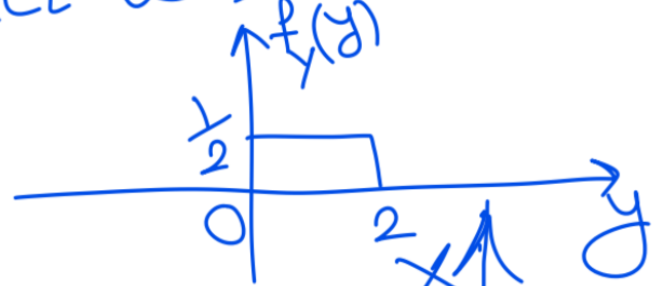
$$|\sigma(S)| = \underbrace{2 \times 2 \times \cdots \times 2}_{8 \text{ times}} = 2^8 = 256$$

e: largest σ-field = power set of Ω

number of events $= 2^{|\Omega|} = 2^{52}$

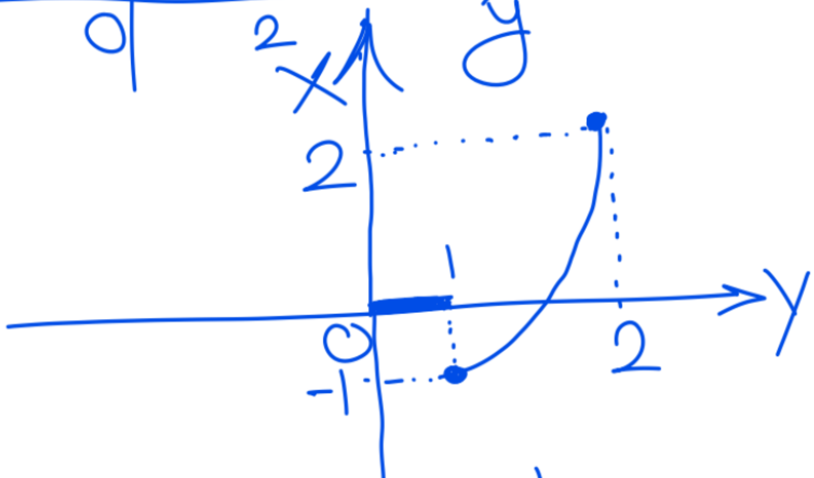
② : $\Omega = [0, 2]$ $\quad \omega \sim U[0, 2]$

Let us show the random variable ω with Y.



$$X = (Y^2 - 2)\underbrace{U}_{\text{step function}}(Y - 1)$$

$\Rightarrow$

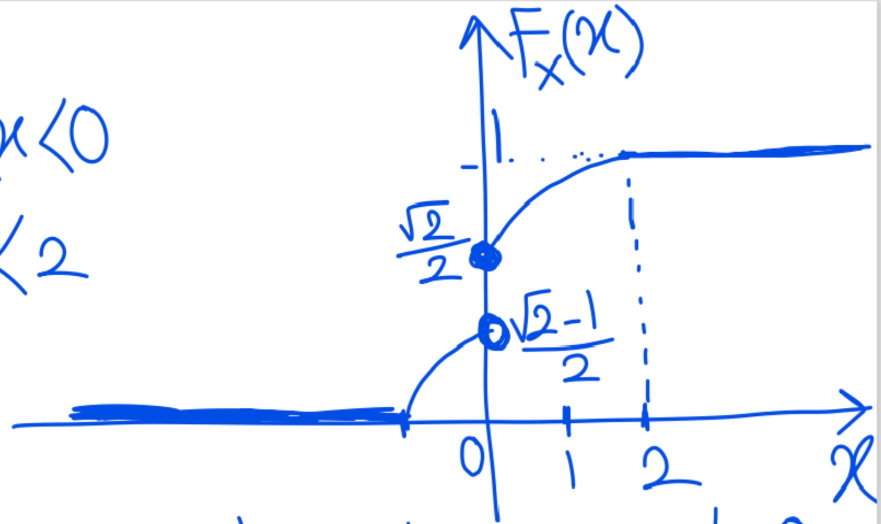


(i)

$$F_X(x) = P(X \leq x) = \begin{cases} 0; & x < -1 \\ P(-1 \leq Y^2 - 2 \leq x); & -1 \leq x < 0 \\ P(-1 \leq Y^2 - 2 \leq x) + P(0 \leq Y \leq 1); & 0 \leq x < 2 \\ 1; & x \geq 2 \end{cases}$$

$$= \begin{cases} 0; & x < -1 \\ P(1 \leq Y \leq \sqrt{x+2}); & -1 \leq x < 0 \\ P(1 \leq Y \leq \sqrt{x+2}) + \frac{1}{2}; & 0 \leq x < 2 \\ 1; & x \geq 2 \end{cases}$$

$$= \begin{cases} 0 \ ; & x < -1 \\ \frac{1}{2}(\sqrt{x+2} - 1) \ ; & -1 \leqslant x < 0 \\ \frac{1}{2}\sqrt{x+2} \ ; & 0 \leqslant x < 2 \\ 1 \ ; & x \geqslant 2 \end{cases}$$



As you can see, $F_X(x)$ is not continuous at 0.

$$\Rightarrow f_X(x) = \frac{dF_X(x)}{dx} = \begin{cases} 0 \ ; & x < -1 \\ \frac{1}{4\sqrt{x+2}} \ ; & -1 < x < 2 \ + \frac{1}{2}\delta(x) \\ 0 \ ; & x > 2 \end{cases}$$

(ii) $$EX = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \underbrace{\int_{-1}^{2} \frac{x}{4\sqrt{x+2}}\,dx}_{\text{Let } \sqrt{x+2} = Z} + \underbrace{\int_{-1}^{2} \frac{x}{2}\delta(x)\,dx}_{0}$$

$$= \int_{1}^{2} \frac{Z^2-2}{4Z} \cdot 2Z\,dZ = \int_{1}^{2} \frac{Z^2-2}{2}\,dZ = \underbrace{\frac{1}{6}}$$

$$EX^2 = \underbrace{\int_{-1}^{2} \frac{x^2}{4\sqrt{x+2}}\,dx}_{\text{Let } \sqrt{x+2}=z} + \underbrace{\int_{-1}^{2} \frac{x^2}{2}\delta(x)\,dx}_{0}$$

$$= \int_{1}^{2} \frac{(z^2-2)^2}{4z}\cdot 2z\,dz = \int_{1}^{2} \frac{(z^2-2)^2}{2}\,dz = \frac{13}{30}$$

$$\Rightarrow Var(X) = E(X^2) - (EX)^2 = \frac{13}{30} - \left(\frac{1}{6}\right)^2 = \frac{13}{30} - \frac{1}{36} = \frac{73}{180}$$

3 : We use the following Lemmas for solving the problem.

Lemma 1 : the number of nonnegative integer solutions of equation

$x_1+x_2+\cdots+x_r=n$ is $\binom{n+r-1}{r-1}$.

Lemma 2 : $\sum_{n=0}^{N}\binom{n+k}{k}=\binom{N+k+1}{k+1}$.

Now, let us solve the problem:

$\text{outcome}=0.X_1X_2X_3$

$X \triangleq X_1+X_2+X_3 \quad 0\leq X\leq 27$

Let the number of solutions of equation $\begin{cases} x_1+x_2+x_3=\ell \\ 0\leq x_1,x_2,x_3\leq 9 \end{cases}$

be N_ℓ for $\ell=0,1,\dots,27$.

Note that x_i's should be less than or equal to 9.

Let us define the following sets:

$A_1=\{\text{possible solutions when } x_1\geq 10\}$

$A_2=\{\text{possible solutions when } x_2\geq 10\}$

$A_3=\{\text{possible solutions when } x_3\geq 10\}$

Then, we have:

$A_1^c=\{\text{possible solutions when } 0\leq x_1\leq 9\}$

$A_2^c=\{\text{possible solutions when } 0\leq x_2\leq 9\}$

$A_3^c=\{\text{possible solutions when } 0\leq x_3\leq 9\}$

$\Rightarrow N_\ell=|A_1^c\cap A_2^c\cap A_3^c|=|(A_1\cup A_2\cup A_3)^c|$

From Lemma 1, we know that the total number of solutions of $x_1+x_2+x_3=\ell$ (without condition $0\leq x_1,x_2,x_3\leq 9$) is equal to $\binom{\ell+3-1}{3-1}=\binom{\ell+2}{2}$

$$\Rightarrow N_\ell = \left|(A_1 \cup A_2 \cup A_3)^c\right| = \binom{\ell+2}{2} - |A_1 \cup A_2 \cup A_3| \quad (1)$$

Noting principle of inclusion and exclusion, we have:

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \quad (2)$$

For finding $|A_1|$, we should solve the following equation:

$$\begin{cases} x_1 + x_2 + x_3 = \ell \\ x_1 \geqslant 10 \\ x_2, x_3 \geqslant 0 \end{cases} \Rightarrow \begin{cases} (x_1 - 10) + x_2 + x_3 = \ell - 10 \\ x_1 - 10 \geqslant 0 \\ x_2, x_3 \geqslant 0 \end{cases} \xRightarrow{\text{Let } x_1 - 10 = y_1} \begin{cases} y_1 + x_2 + x_3 = \ell - 10 \\ y_1, x_2, x_3 \geqslant 0 \end{cases}$$

Note that the number of solutions of last equation is 0 when $0 \leqslant \ell \leqslant 9$.

For $10 \leqslant \ell \leqslant 27$, noting lemma 1, the number of solutions is $\binom{\ell-10+3-1}{3-1} = \binom{\ell-8}{2}$

$$\Rightarrow |A_1| = \begin{cases} 0 \,;\ 0 \leqslant \ell \leqslant 9 \\ \binom{\ell-8}{2} \,;\ 10 \leqslant \ell \leqslant 27 \end{cases}$$

In a similar way, we can show that $|A_2| = |A_3| = \begin{cases} 0 \,;\ 0 \leqslant \ell \leqslant 9 \\ \binom{\ell-8}{2} \,;\ 10 \leqslant \ell \leqslant 27 \end{cases}$

In fact, $|A_1| = |A_2| = |A_3|$

Now, let us find $|A_1 \cap A_2|$.

$$\begin{cases} x_1+x_2+x_3=l \\ x_1, x_2 \geqslant 10 \\ x_3 \geqslant 0 \end{cases} \Rightarrow \begin{cases} (x_1-10)+(x_2-10)+x_3=l-20 \\ x_1-10 \geqslant 0 \\ x_2-10 \geqslant 0 \\ x_3 \geqslant 0 \end{cases} \xrightarrow[\substack{x_1-10=y_1 \\ x_2-10=y_2}]{\text{Let}} \begin{cases} y_1+y_2+x_3=l-20 \\ y_1, y_2, x_3 \geqslant 0 \end{cases}$$

The number of solutions is 0 when $0 \leqslant l \leqslant 19$.

For $20 \leqslant l \leqslant 27$, noting lemma 1, the number of solutions is $\binom{l-20+3-1}{3-1} = \binom{l-18}{2}$

$$\Rightarrow |A_1 \cap A_2| = \begin{cases} 0 ; & 0 \leqslant l \leqslant 19 \\ \binom{l-18}{2} ; & 20 \leqslant l \leqslant 27 \end{cases} = |A_2 \cap A_3| = |A_1 \cap A_3|$$

For finding $|A_1 \cap A_2 \cap A_3|$, we should solve the following equation:

$$\begin{cases} x_1+x_2+x_3=l \quad (l=0,1,\dots,27) \\ x_1, x_2, x_3 \geqslant 10 \end{cases}$$

It is obvious that this equation does not have any solution. $\Rightarrow |A_1 \cap A_2 \cap A_3| = 0$

If we substitute the obtained values in (1) and (2), we will get:

$$N_l = \binom{l+2}{2} - 3 \times \begin{cases} 0 ; & 0 \leqslant l \leqslant 9 \\ \binom{l-8}{2} ; & 10 \leqslant l \leqslant 27 \end{cases} + 3 \times \begin{cases} 0 ; & 0 \leqslant l \leqslant 19 \\ \binom{l-18}{2} ; & 20 \leqslant l \leqslant 27 \end{cases} - 0$$

$$\Rightarrow N_\ell = \begin{cases} \binom{\ell+2}{2}; & 0 \leqslant \ell \leqslant 9 \\ \binom{\ell+2}{2} - 3\binom{\ell-8}{2}; & 10 \leqslant \ell \leqslant 19 \\ \binom{\ell+2}{2} - 3\binom{\ell-8}{2} + 3\binom{\ell-18}{2}; & 20 \leqslant \ell \leqslant 27 \end{cases}$$

Let N_T be the total number of cases. Then, we have

$$N_T = \sum_{\ell=0}^{27} N_\ell = \sum_{\ell=0}^{27}\binom{\ell+2}{2} - 3\underbrace{\sum_{\ell=10}^{27}\binom{\ell-8}{2}}_{\text{let } \ell-10=\ell_1} + 3\underbrace{\sum_{\ell=20}^{27}\binom{\ell-18}{2}}_{\text{let } \ell-20=\ell_2}$$

$$\Rightarrow N_T = \sum_{\ell=0}^{27}\binom{\ell+2}{2} - 3\sum_{\ell_1=0}^{17}\binom{\ell_1+2}{2} + 3\sum_{\ell_2=0}^{7}\binom{\ell_2+2}{2} \xrightarrow{\text{Lemma 2}}$$

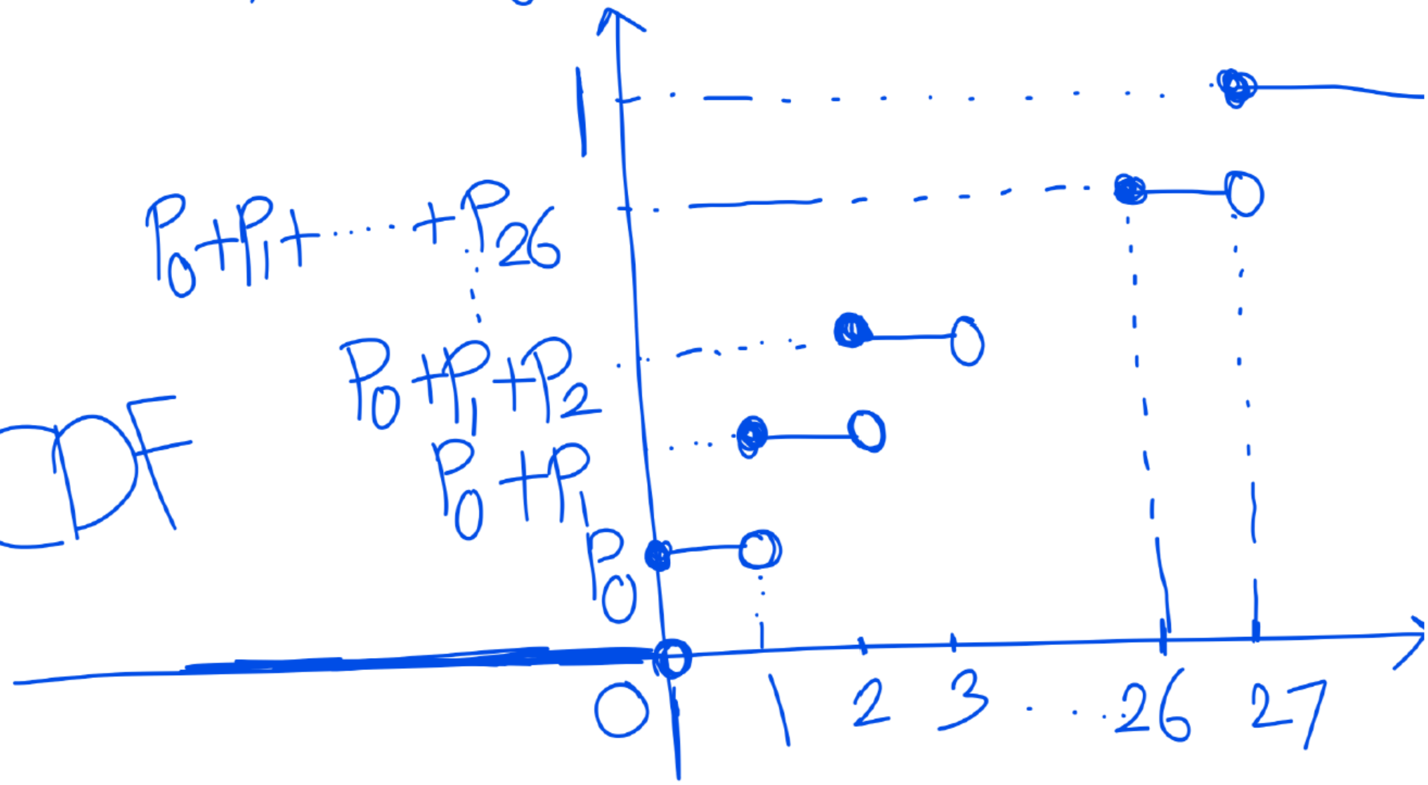
$$= \binom{27+3}{3} - 3\binom{17+3}{3} + 3\binom{7+3}{3} = \binom{30}{3} - 3\binom{20}{3} + 3\binom{10}{3} = 1000$$

$$\Rightarrow P(X=\ell) = \frac{N_\ell}{N_T} = \begin{cases} \dfrac{\binom{\ell+2}{2}}{1000}; & 0 \leqslant \ell \leqslant 9 \\ \dfrac{\binom{\ell+2}{2} - 3\binom{\ell-8}{2}}{1000}; & 10 \leqslant \ell \leqslant 19 \\ \dfrac{\binom{\ell+2}{2} - 3\binom{\ell-8}{2} + 3\binom{\ell-18}{2}}{1000}; & 20 \leqslant \ell \leqslant 27 \end{cases}$$

$\ell = 0, 1, 2, \ldots, 27$

$$\Rightarrow \text{pdf} = f_X(x) = \sum_{l=0}^{27} P_l\, \delta(x-l)$$

CDF



④ : $\Omega = \{(H,\dots,H),(H,\dots,T),\dots,(T,\dots,T)\}$ (i)

(ii) the largest σ-field is the power set of Ω, which has $2^{|\Omega|}$ members.

$|\Omega| = 2^{10} \Rightarrow |F| = 2^{2^{10}}$.

(iii) Let us define the following event:

A = the first three flips are the same or the last 3 flips are the same.

A^c = the first three flips are not the same and the last 3 flips are not the same.

$$P(A^c) = \frac{|A^c|}{|\Omega|} \qquad \underbrace{\square\square\square}_{2^3-2} \times \underbrace{\square\square\square\square}_{2^4} \times \underbrace{\square\square\square}_{2^3-2}$$

$$\Rightarrow P(A^c) = \frac{6\times 6\times 2^4}{2^{10}} = \frac{36}{2^6} = \frac{9}{16}$$

$$P(A) = 1 - P(A^c) = 1 - \frac{9}{16} = \frac{7}{16}$$

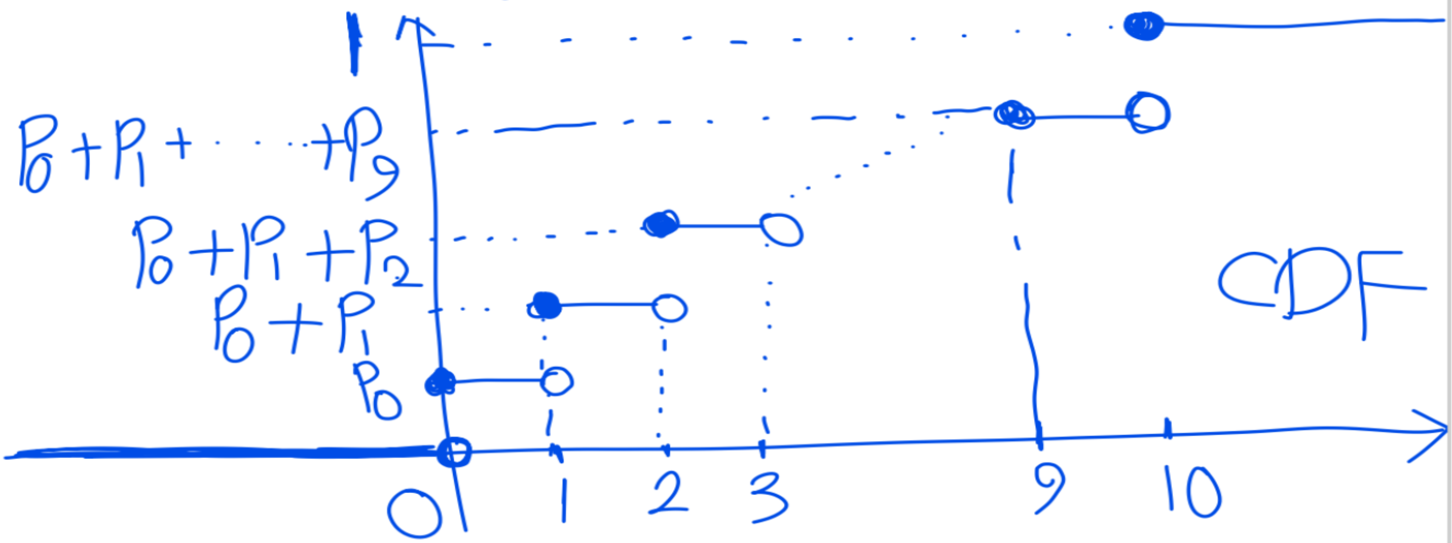
(iv): X = number of Heads

X is a binomial random variable.

$$\Pr\{X=k\} = \binom{10}{k} \times \left(\frac{1}{2}\right)^{k} \times \left(\frac{1}{2}\right)^{10-k} = \frac{\binom{10}{k}}{2^{10}} = P_k$$

for $k = 0, 1, 2, \dots, 10$

$$\Rightarrow \text{pdf} = f_X(x) = \sum_{k=0}^{10} P_k \,\delta(x-k) = \sum_{k=0}^{10} \frac{\binom{10}{k}}{2^{10}} \delta(x-k)$$



v) since it is a fair coin, we have:

$$P(\text{Head}) = p = \frac{1}{2} \quad P(\text{tail}) = q = \frac{1}{2}$$

$$E(X) = np = 10 \times \frac{1}{2} = 5$$

$$\text{Var}(X) = npq = 10 \times \frac{1}{2} \times \frac{1}{2} = 2.5$$

5) We first review the concept of continuity of measure.

a) continuity from below:

If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$ is an increasing sequence of measurable sets, then the measure of the union of these sets is the limit of the measures of the sets.

Formally:
$$\mu\left(\bigcup_{n=1}^{+\infty} A_n\right) = \lim_{n \to +\infty} \mu(A_n)$$

This means that as the sets grow, the measure converges to the measure of their union.

b) continuity from above:

If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$ is a decreasing sequence of measurable sets, and the measure of A_1 is finite, then the measure of the limit of the intersection of these sets is the limit of the measures of the sets.

Formally:
$$\mu\left(\bigcap_{n=1}^{+\infty} A_n\right) = \lim_{n \to +\infty} \mu(A_n)$$

You can find the proof of continuity of measure in the real analysis textbooks. Finally, we point out that the discussed continuity also holds for the probability because probability is indeed a measure.

Now, let us start solving the last question.

(i) $F_X(x) = P\{X \leq x\}$

noting axioms of probability, we have:

$$0 \leq P\{X \leq x\} \leq 1 \Rightarrow 0 \leq F_X(x) \leq 1.$$

(ii) For any $x_0 \in \mathbb{R}$, we need to prove that

$$\lim_{x \to x_0^+} F_X(x) = F_X(x_0)$$

Let us consider the sequence $\left\{x_n = x_0 + \frac{1}{n}\right\}_{n=1}^{+\infty}$.

It is not hard to see that x_n is decreasing and converges to x_0 from right. Formally:

$x_n > x_{n+1}$ for $n = 1, 2, \dots$

$$\lim_{n \to +\infty} x_n = x_0.$$

Let us define the following events:

$A_n = \{X \leq x_n\}$ for $n = 1, 2, \dots$

$B = \{X \leq x_0\}$

It is not hard to see that: $\begin{cases} A_1 \supset A_2 \supset A_3 \supset \cdots \\ \bigcap_{n=1}^{+\infty} A_n = B \end{cases}$

Now, we can write:

$$\lim_{x \to x_0^+} F_X(x) = \lim_{x \to x_0^+} P\{X \leq x\} = \lim_{n \to +\infty} P\{X \leq x_n\} = \lim_{n \to +\infty} P(A_n)$$

$$= \underbrace{P\left(\bigcap_{n=1}^{+\infty} A_n\right)}_{\text{(continuity of measure)}} = P(B) = P(X \leq x_0) = F_X(x_0)$$

(iii) Let us consider the events $A_n = \{X \leq -n\}_{n=1}^{+\infty}$

It is not hard to see that $A_1 \supset A_2 \supset A_3 \supset \cdots$

Now, we have:

$$\lim_{x \to -\infty} F_X(x) = \lim_{x \to -\infty} P\{X \leq x\} = \lim_{n \to +\infty} P\{X \leq -n\} = \lim_{n \to +\infty} P(A_n)$$

$$= P\left(\bigcap_{n=1}^{+\infty} A_n\right) = P(\varnothing) = 0$$

↳ (continuity of measure)

(iv) We consider the events $B_n = \{X \leq n\}_{n=1}^{+\infty}$.

It is notable that $B_1 \subset B_2 \subset B_3 \subset \cdots$

Now, we can write

$$\lim_{x \to +\infty} F_X(x) = \lim_{x \to +\infty} P\{X \leq x\} = \lim_{n \to +\infty} P\{X \leq n\} = \lim_{n \to +\infty} P(B_n)$$

$$= P\left(\bigcup_{n=1}^{+\infty} B_n\right) = P(-\infty < X < +\infty) = P(S) = 1$$

↳ (continuity of measure)

(v) Let us define the following events:

$A = \{X \leq x_1\}$ $\xrightarrow{\text{since } x_1 \leq x_2}$ $A \subseteq B$

$B = \{X \leq x_2\}$

$\Longrightarrow P(A) \leq P(B) \Rightarrow P\{X \leq x_1\} \leq P\{X \leq x_2\}$

$\Rightarrow F_X(x_1) \leq F_X(x_2)$