problem set 8 (solutions)

①: $X(t)$ is a modified version of random telegraph process.

time between switches $f_T(s) = \lambda e^{-\lambda s} u(s)$

$pr(X(0)=1) = p$, $pr(X(0)=-1) = 1-p$

ⓐ: $pr(X(t)=1) \underset{\text{total probability rule}}{=} pr(X(t)=1/X(0)=1) \times pr(X(0)=1) + pr(X(t)=1/X(0)=-1) \times pr(X(0)=-1)$

$pr(X(t)=1/X(0)=1) = pr\{\text{even number of switches in } [0,t]\}$

$= \sum_{i=0}^{+\infty} \frac{e^{-\lambda t}(\lambda t)^{(2i)}}{(2i)!} = \cosh(\lambda t) \times e^{-\lambda t}$

$pr(X(t)=1/X(0)=-1) = pr\{\text{odd number of switches in } [0,t]\}$

$= \sum_{i=0}^{+\infty} \frac{e^{-\lambda t}(\lambda t)^{2i+1}}{(2i+1)!} = \sinh(\lambda t) \times e^{-\lambda t}$

$\Rightarrow pr(X(t)=1) = pe^{-\lambda t}\cosh(\lambda t) + (1-p)e^{-\lambda t}\sinh(\lambda t)$

$pr(X(t)=-1) = 1 - pe^{-\lambda t}\cosh(\lambda t) - (1-p)e^{-\lambda t}\sinh(\lambda t)$

ⓑ: $\mu_X(t) = 1 \times pr(X(t)=1) + (-1) \times pr(X(t)=-1)$

$= pe^{-\lambda t}\cosh(\lambda t) + (1-p)e^{-\lambda t}\sinh(\lambda t) - 1 + pe^{-\lambda t}\cosh(\lambda t) + (1-p)e^{-\lambda t}\sinh(\lambda t)$

$= 2pe^{-\lambda t}\cosh(\lambda t) + 2(1-p)e^{-\lambda t}\sinh(\lambda t) - 1$

$= pe^{-\lambda t}(e^{\lambda t} + e^{-\lambda t}) + (1-p)e^{-\lambda t}(e^{\lambda t} - e^{-\lambda t}) - 1$

$= p(1 + e^{-2\lambda t}) + (1-p)(1 - e^{-2\lambda t}) - 1$

$= \cancel{p} + pe^{-2\lambda t} \cancel{+1} - e^{-2\lambda t} \cancel{-p} + pe^{-2\lambda t} \cancel{-1} = (2p-1)e^{-2\lambda t}$

ⓒ: $R_{X,X}(t_1,t_2)=E\{X(t_1)X(t_2)\}$

Let us assume that $t_1 \leqslant t_2$. $X(t_1)X(t_2)$ can take only the values ± 1.

$$\Rightarrow E\{X(t_1)X(t_2)\} = pr(X(t_1)X(t_2)=1)\times 1 + pr(X(t_1)X(t_2)=-1)\times(-1)$$

$$= pr\{\text{Even number of switches in } [t_1,t_2]\} - pr\{\text{odd number of switches in } [t_1,t_2]\}$$

$$=\cosh(\lambda(t_2-t_1))e^{-\lambda(t_2-t_1)} - \sinh(\lambda(t_2-t_1))e^{-\lambda(t_2-t_1)} = e^{-2\lambda(t_2-t_1)}$$

similarly, if $t_1>t_2$ we can conclude that: $E\{X(t_2)X(t_1)\}=e^{-2\lambda(t_1-t_2)}$

$$\Rightarrow R_{X,X}(t_1,t_2)=e^{-2\lambda|t_2-t_1|}$$

②: W_n is an i.i.d sequence of zero-mean Gaussian random variables with variance σ_W^2. $X[n]\triangleq \rho X[n-1]+W_n$ for $n=1,2,3,\dots$

$X[0]=W_0$ and ρ is constant.

ⓐ: $\mu_{X[n]} = E\{X[n]\} = \rho E\{X[n-1]\} + \underbrace{EW_n}_{0} = \rho E\{X[n-1]\} = \rho^n E\{X[0]\} = \rho^n E\underbrace{W_0}_{0} = 0$

ⓑ: $R_{X,X}(n_1,n_2)= ?$

Firstly, we apply induction to derive expression for $X[n]$ in terms of W_n's:

$X[0]=W_0$ $\quad X[2]=\rho X[1]+W_2=\rho^2 W_0+\rho W_1+W_2$

$X[1]=\rho W_0+W_1$ $\quad X[3]=\rho X[2]+W_3=\rho^3 W_0+\rho^2 W_1+\rho W_2+W_3$

$\Rightarrow$ It seems that: $X[n]=\sum_{l=0}^{n}\rho^{n-l}W_l$

Induction hypothesis: $X[k] = \sum_{l=0}^{k} p^{k-l} W_l$

$$X[k+1] = pX[k] + W_{k+1} = p \times \sum_{l=0}^{k} p^{k-l} W_l + W_{k+1} = \sum_{l=0}^{k} p^{k+1-l} W_l + W_{k+1}$$

$$= \sum_{l=0}^{k+1} p^{k+1-l} W_l \checkmark$$

Now, we have:

$$R_{X,X}(n_1,n_2) = E\{X[n_1]X[n_2]\} = E\left\{\sum_{l_1=0}^{n_1} p^{n_1-l_1} W_{l_1} \times \sum_{l_2=0}^{n_2} p^{n_2-l_2} W_{l_2}\right\}$$

$$= \sum_{l_1=0}^{n_1}\sum_{l_2=0}^{n_2} p^{n_1+n_2-l_1-l_2} \quad E(W_{l_1}W_{l_2})$$

Noting the fact that $E(W_{l_1}W_{l_2}) = \begin{cases} \sigma_W^2 \,; & l_1 = l_2 \\ 0 \,; & \text{oth} \end{cases}$, we have:

$$= \sum_{l_1=0}^{\min\{n_1,n_2\}} p^{n_1+n_2-2l_1} \times \sigma_W^2 = \sigma_W^2 p^{n_1+n_2} \sum_{l_1=0}^{\min\{n_1,n_2\}} p^{-2l_1}$$

$$= \sigma_W^2 \times p^{n_1+n_2} \times \frac{1-p^{-2\min\{n_1,n_2\}}}{1-p^{-2}}$$

3: $\begin{cases} \frac{d}{dt} P_X(0;t) + \lambda P_X(0;t) = 0 \\ \frac{d}{dt} P_X(i;t) + \lambda P_X(i;t) = \lambda P_X(i-1;t) \,; \quad i = 1,2,3,\dots \end{cases}$

we start with the first equation:

$$\int \frac{dP_X(0;t)}{P_X(0;t)} = \int -\lambda dt \Rightarrow \boxed{P_X(0;t) = e^{-\lambda t}}$$

The integrating factor of second equation is $\mu = e^{\int \lambda dt} = e^{\lambda t}$

$$\Rightarrow \frac{d}{dt} P_X(i;t) \times e^{\lambda t} + \lambda e^{\lambda t} P_X(i;t) = \lambda P_X(i-1;t) \times e^{\lambda t} \Longrightarrow$$

$$\left(e^{\lambda t}\times P_x(i;t)\right)' = \lambda e^{\lambda t} P_x(i-1;t) \Longrightarrow P_x(i;t)\times e^{\lambda t} = \lambda\int e^{\lambda t} P_x(i-1;t)\,dt \Longrightarrow$$

$$P_x(i;t) = \lambda e^{-\lambda t}\int P_x(i-1;t)\,e^{\lambda t}\,dt$$

Now, let us find $P_x(i;t)$ for some values of i:

$$P_x(1;t) = \lambda e^{-\lambda t}\int P_x(0;t)e^{\lambda t}dt = \lambda e^{-\lambda t}\int e^{-\lambda t}e^{\lambda t}dt = \lambda t e^{-\lambda t}$$

$$P_x(2;t) = \lambda e^{-\lambda t}\int P_x(1;t)e^{\lambda t}dt = \lambda e^{-\lambda t}\int \lambda t e^{-\lambda t}e^{\lambda t}dt = e^{-\lambda t}\frac{(\lambda t)^2}{2!}$$

It seems that $P_x(i;t) = e^{-\lambda t}\frac{(\lambda t)^i}{i!}$, we prove it by induction

Induction hypothesis: $P_x(k;t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$

$$P_x(k+1;t) = \lambda e^{-\lambda t}\int P_x(k;t)e^{\lambda t}dt = \lambda e^{-\lambda t}\int \cancel{e^{-\lambda t}}\frac{(\lambda t)^k}{k!}\times\cancel{e^{\lambda t}}dt =$$

$$\lambda e^{-\lambda t}\times\frac{\lambda^k}{k!}\int t^k dt = e^{-\lambda t}\frac{\lambda^{k+1}}{k!}\times\frac{t^{k+1}}{k+1} = e^{-\lambda t}\times\frac{(\lambda t)^{k+1}}{(k+1)!}$$

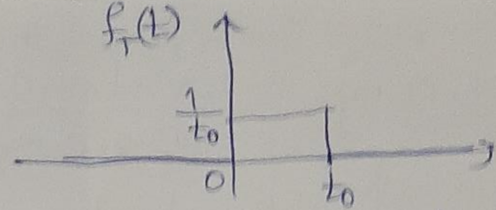
$$\Rightarrow P_x(i;t) = e^{-\lambda t}\frac{(\lambda t)^i}{i!}$$

4: a: $F_T(t) = pr\{T \leqslant t / \text{one arrival in } [0,t_0)\}$

$$= \begin{cases} 1; & t \geqslant t_0 \\ \dfrac{pr\{\text{one arrival in } [0,t]\}\times pr\{\text{no arrival in } (t,t_0)\}}{pr\{\text{one arrival in } [0,t_0)\}}; & t > t_0 \end{cases}$$

$$= \begin{cases} 1; & t \geqslant t_0 \\ \dfrac{e^{-\lambda t}\lambda t\times e^{-\lambda(t_0-t)}}{e^{-\lambda t_0}\times\lambda t_0}; & t > t_0 \end{cases} = \begin{cases} 1; & t \geqslant t_0 \\ \dfrac{t}{t_0}; & 0 \leqslant t < t_0 \end{cases}$$

$$= \begin{cases} 0; & t \geq t_0 \\ \frac{1}{t_0}; & 0 \leq t < t_0 \end{cases} = f_T(t)$$



uniform distribution

(b): Assume that T_1 is the first arrival and T_2 is the second arrival.

$$F_{T_1,T_2}(t_1,t_2) = pr\{T_1 \leq t_1, T_2 \leq t_2 / \text{two arrivals in } [0,t_0)\}$$

Let us assume that $0 < t_1 < t_2 < t_0$.

$$pr\{T_1 \leq t_1, T_2 \leq t_2 / 2 \text{ arrival in } [0,t_0)\}$$

$$= \frac{e^{-\lambda t_1}\lambda t_1 \times e^{-\lambda(t_2-t_1)}\lambda(t_2-t_1) \times e^{-\lambda(t_0-t_2)} + e^{-\lambda t_1}\frac{(\lambda t_1)^2}{2} \times e^{-\lambda(t_0-t_1)}}{e^{-\lambda t_0} \times \frac{(\lambda t_0)^2}{2}}$$

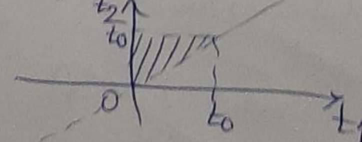
In the numinator, we have two cases: both occur in $[0,t_1]$ or first one occurs in $[0,t_1]$ and second one occurs in $[t_1,t_2]$

$$= \frac{\cancel{\frac{e^{-\lambda t_0} \times \lambda^2}{2}}\left[2t_1(t_2-t_1)+t_1^2\right]}{\cancel{\frac{e^{-\lambda t_0} \times \lambda^2}{2}} \times t_0^2} = \frac{2t_1t_2 - t_1^2}{t_0^2}$$

$$\Rightarrow f_{T_1,T_2}(t_1,t_2) = \frac{\partial F_{T_1,T_2}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{2}{t_0^2}$$

we can see that PDF is zero for the other cases such as $0 < t_1 < t_0,\ t_2 > t_0$. Therefore, the joint PDF is:

$$f_{T_1,T_2}(t_1,t_2) = \begin{cases} \frac{2}{t_0^2}; & 0 < t_1 < t_2 < t_0 \\ 0; & \text{otherwise} \end{cases}$$



Again, it is a uniform distribution in 2-dimensional space.

⑤: $X(t)$ is a poisson counting process with parameter λ.

$$pr\{N(t)=k / N(t+\tau)=m\} = \frac{pr\{N(t)=k \text{ and } N(t+\tau)=m\}}{pr\{N(t+\tau)=m\}}$$

$$= \frac{pr\{k \text{ arrivals in } [0,t]\} \times pr\{m-k \text{ arrivals in } [t,t+\tau]\}}{pr\{m \text{ arrivals in } [0,t+\tau]\}}$$

$$= \frac{e^{-\lambda t}\frac{\lambda^k t^k}{k!} \times e^{-\lambda\tau}\frac{\lambda^{m-k}\tau^{m-k}}{(m-k)!}}{e^{-\lambda(t+\tau)} \times \frac{\lambda^m (t+\tau)^m}{m!}} = \frac{t^k \times \tau^{m-k} \times m!}{k!(m-k)! \times (t+\tau)^m} = \binom{m}{k}\frac{\tau^{m-k}\, t^k}{(t+\tau)^m}$$

⑥: $X_i(t)$; $i=1,2,\dots,n$ a sequence of independent poisson counting processes. with arrival rates λ_i. $X(t) \triangleq \sum_{i=1}^{n} X_i(t)$

Let $N(t)$ be the total number of occurrences of $X(t)$ in the interval $[0,t]$.

Moreover, Let $N_i(t)$ be the number of occurrences of $X_i(t)$ in the interval $[0,t]$.

Therefore, we conclude that: $N(t) = \sum_{i=1}^{n} N_i(t)$

① $N(0) = \sum_{i=1}^{n} N_i(0) = 0$

② $N_i(t)$ is a poisson random variable with parameter $\lambda_i t$. we showed that summation of some independent poisson random variables is itsel a poisson random variable with a new parameter, which is the summation of old parameters. (see homework 6)

$\Rightarrow N(t)$ is a ~~random~~ poisson random variable with parameter $\sum_{i=1}^{n}\lambda_i \times t$.

③: proving that $X(t)$ has independent increments.

Let $t_1 < t_2 < t_3$:

$$X(t_2) - X(t_1) = \sum_{i=1}^{n} X_i(t_2) - \sum_{i=1}^{n} X_i(t_1) = \sum_{i=1}^{n}\left(X_i(t_2) - X_i(t_1)\right)$$

$$X(t_3) - X(t_2) = \sum_{i=1}^{n} X_i(t_3) - \sum_{i=1}^{n} X_i(t_2) = \sum_{i=1}^{n}\left(X_i(t_3) - X_i(t_2)\right)$$

since $X_i(t_2)-X_i(t_1)$ is independent of $X_i(t_3)-X_i(t_2)$ for $i=1,2,\dots,n$, we conclude that summation of $X_i(t_2)-X_i(t_1)$'s and $X_i(t_3)-X_i(t_2)$'s are also independent. (Note that $X_i(t)$'s are also independent)

$\Longrightarrow X(t)$ has independent increments.

In summary, $X(t)$ has all properties of poisson counting process. As a result $X(t)$ is a poisson counting process with parameter $\sum_{i=1}^{n} \lambda_i$

(7):

(a): It is a poisson counting process.

$N(0)=0$ ✓

$N(t)$ has independent increments. Let us assume that $t_1 < t_2 < t_3$.
the number of failures in $[t_2,t_3]$ is $N(t_3)-N(t_2)$, which is independent from the number of failures in $[t_1,t_2]$ $\left(N(t_2)-N(t_1)\right)$.

Now, we should prove that $N(t)$ is a poisson random variable with parameter λt. Let us define $g_k(t) \triangleq pr\{N(t)\leq k\}$

Then, we have:

$$g_k(t)=pr\{N(t)\leq k\}=1-pr\{N(t)>k\}=1-pr\{\underbrace{E_{k+1}}_{k+1\text{'th arrival}}\leq t\}=1-F_{E_{k+1}}(t)$$

E_{k+1} is an Erlang random variable with PDF: $f_{E_{k+1}}(x)=\dfrac{(\lambda x)^k \lambda e^{-\lambda x}}{k!}$

$$F_{E_{k+1}}(t)=\int_0^t f_{E_{k+1}}(x)\,dx=\frac{\lambda^{k+1}}{k!}\int_0^t x^k e^{-\lambda x}\,dx=1-\sum_{m=0}^{k} e^{-\lambda t}\frac{(\lambda t)^m}{m!}$$

Therefore, $g_k(t)=1-F_{E_{k+1}}(t)=\sum_{m=0}^{k} e^{-\lambda t}\dfrac{(\lambda t)^m}{m!}$

Now, we have:

$$pr\{N(t)=k\}=pr\{N(t)\leq k\}-pr\{N(t)\leq k-1\}$$

$$=\sum_{m=0}^{k} \frac{e^{-\lambda t}(\lambda t)^m}{m!}-\sum_{m=0}^{k-1}\frac{e^{-\lambda t}(\lambda t)^m}{m!}=\frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

As it can be seen, $N(t)$ is a poisson random variable with parameter λt.

b: mean time between failures is the mean of exponential random variable with parameter λ which is $\frac{1}{\lambda}$

$$\Rightarrow \frac{1}{\lambda}=250 \Rightarrow \boxed{\lambda=\frac{1}{250}}$$

$$pr(N(90)\geq 1)=1-pr(N(90)=0)=1-e^{-\lambda\times 90}=1-e^{-\frac{90}{250}}=0.3023$$

(8): $\lambda_a = 0.1$ calls per minute.

(a): $pr\{N(10) < 10\} = \sum_{i=0}^{9} \frac{e^{-\lambda_a \times 10} (\lambda_a \times 10)^i}{i!} = \frac{1}{e} \sum_{i=0}^{9} \frac{1}{i!}$

(b): $pr\left\{\begin{matrix} N(10) < 10 \\ \text{if } \lambda_a = 10 \end{matrix}\right\} = \sum_{i=0}^{9} \frac{e^{-100} \times 100^i}{i!} = \frac{1}{e^{100}} \sum_{i=0}^{9} \frac{100^i}{i!}$

(c): $pr\{N(10) - N(0) = 1 \text{ and } N(20) - N(10) = 2\} =$

since poisson process has independent increments:

$= pr\{N(10) = 1\} \times pr\{N(20) - N(10) = 2\}$

$= \frac{e^{-\lambda_a \times 10} (\lambda_a \times 10)^1}{1!} \times \frac{e^{-\lambda_a \times 10} (\lambda_a \times 10)^2}{2!} = e^{-1} \times \frac{e^{-1}}{2} = \frac{1}{2e^2}$

(9):

$\lambda = \frac{1}{3}$ (per minute) $\Rightarrow \lambda t = t/3$

(a): expected # of strikes in one minute $= \frac{1}{3}$

" " " in 10 minutes $= 10/3$

(b): The interarrival times are exponentially distributed with parameter λ $f_T(s) = \lambda e^{-\lambda s} u(s)$

average of this exponential distribution is $\frac{1}{\lambda} = \frac{1}{1/3} = 3$

(10): impulse response $= h(t) = t e^{-at} u(t)$; $a = 10\, sec^{-1}$

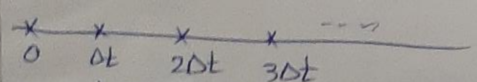
Let us assume that arrival times are $\{E_i's\}_{i=1}^{+\infty}$

Then, the input signal to system is $x(t) = \sum_{i=1}^{+\infty} \delta(t - E_i)$

Therefore, the shot noise process (output $y(t)$) will be:

$y(t) = x(t) * h(t) = \sum_{i=1}^{+\infty} \delta(t - E_i) * h(t) = \sum_{i=1}^{+\infty} h(t - E_i)$

Now, we use the Binomial approximation



In each interval $[n\Delta t, (n+1)\Delta t]$, we have a Bernouli random variable S_n.

$$\begin{cases} pr(S_n = 1) = \lambda \Delta t \\ pr(S_n = 0) = 1 - \lambda \Delta t \end{cases}$$

$\Rightarrow y(t) \approx \sum_{n=0}^{+\infty} S_n h(t - n\Delta t) \Rightarrow E y(t) \approx \sum_{n=0}^{+\infty} E(S_n) h(t - n\Delta t)$

$= \lambda \sum_{n=0}^{+\infty} h(t - n\Delta t)\Delta t$. If $\Delta t \to 0$, we have $E y(t) = \lambda \int_0^{+\infty} h(t - \tau) d\tau = \lambda \int_{-\infty}^{t} h(x) dx$

If we assume that h is causal ($h(t)=0$, for $t<0$), we will have:

$$\mu_X(t)=\lambda\int_0^t h(x)dx = \lambda\int_0^t xe^{-ax}dx=\lambda\left[\frac{-x}{a}e^{-ax}-\frac{1}{a^2}e^{-ax}\right]_0^t$$

$$=\lambda\left[\frac{-t}{a}e^{-at}-\frac{1}{a^2}e^{-at}+\frac{1}{a^2}\right]$$

ⓑ: $R_{X,X}(t+\tau,t)=E\{y(t+\tau)y(t)\}\simeq\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}E(S_nS_m)h(t+\tau-n\Delta t)h(t-m\Delta t)$

$$=\sum_{n=0}^{+\infty}E(S_n^2)h(t+\tau-n\Delta t)h(t-n\Delta t)+\sum_{n=0}^{+\infty}E(S_n)h(t+\tau-n\Delta t)\sum_{m\neq n}^{+\infty}E(S_m)h(t-m\Delta t)$$

$$=\lambda\sum_{n=0}^{+\infty}h(t+\tau-n\Delta t)h(t-n\Delta t)\Delta t+\lambda^2\sum_{n=0}^{+\infty}h(t+\tau-n\Delta t)\Delta t\sum_{m\neq n}^{+\infty}h(t-m\Delta t)\Delta t$$

$$=\sum_{n=0}^{+\infty}h(t+\tau-n\Delta t)h(t-n\Delta t)\left(\lambda\Delta t-(\lambda\Delta t)^2\right)$$ → we can ignore this term when $\Delta t \longrightarrow 0$

$$+\lambda^2\sum_{n=0}^{+\infty}h(t+\tau-n\Delta t)\Delta t\sum_{m=0}^{+\infty}h(t-m\Delta t)\Delta t$$

⇒ Noting that $\Delta t\longrightarrow 0$, we have:

$$R_{X,X}(t+\tau,t)=\lambda\int_0^{+\infty}h(t-u)h(t+\tau-u)du$$

$$+\lambda^2\int_0^{+\infty}h(t+\tau-u)du\int_0^{+\infty}h(t-u)du$$

Let $t-u=v$, then we have:

$$R_{X,X}(t+\tau,t)=\lambda\int_{-\infty}^{t}h(v)h(v+\tau)dv+\lambda^2\int_{-\infty}^{t}h(v)dv\times\int_{-\infty}^{t}h(\tau+v)dv$$

If $h(t)$ is causal:

$$R_{X,X}(t+\tau,t)=\lambda\int_0^t h(v)h(v+\tau)dv+\mu_X(t)\mu_X(t+\tau)$$