ECE537

Random Processes

Problem Set 2

Sept 17, 2024

Elvino S. Sousa

1. Consider an experiment where we draw a card at random from a deck of 52 cards.
   a) Specify the sample space Ω.
   b) Consider the following set of events $\mathcal{S} = \{\text{diamonds, a queen, a red 6, clubs}\}$.
   c) Find the probability of each of these events.
   d) Find the smallest σ-field of events that contain $\mathcal{S}$, i.e. find $\sigma(\mathcal{S})$. How many events are there?
   e) Specify the largest σ-field that contains $\mathcal{S}$. How many events are there?

2. Consider an experiment where $\Omega = [0,2]$. The probability law is the uniform probability law. A random variable is defined as follows: $X = (\omega^2 - 2)u(\omega - 1)$, where $u(\cdot)$ is the step function.

   i) Determine the CDF and PDF for the random variable X?

   ii) Determine the expected value of X.

   iii) Determine the variance of X.

3. Consider an experiment where we draw a number at random from the interval $[0, 1]$ with uniform probability law. Express the number as a decimal expansion, i.e. 0.345. Define a random variable X as the sum of the first 3 digits after the decimal.

   i) Find the CDF for X?
   ii) Find the PDF for X?

4. An experiments consists in the flipping of a fair coin 10 times.

   i) Specify the sample space Ω.
   ii) Specify the largest σ-field that we can define in this space. How many events are there?
   iii) What is the probability that the first 3 flips or the last 3 flips result in the same value (i.e. Heads or Tails)?
   iv) Let X be a random variable that is equal to the number of heads. Determine the CDF and PDF (using delta functions) for X.
   v) Determine the mean and variance of X.

5. Prove rigorously the following properties of the CDF
   i) $0 \leq F_X(x) \leq 1$
   ii) $F_X(x)$ is right continuous and monotone increasing.
   iii) $\lim_{x \to -\infty} F_X(x) = 0$
   iv) $\lim_{x \to \infty} F_x(x) = 1$
   v) If $x_1 \leq x_2$ then $F_X(x_1) \leq F_X(x_2)$ (definition of "monotone increasing").