ECE537

Random Processes

Midterm 2

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1. Consider the probability sample space $\Omega = [0,1]$, and the set of events $A_n = \left(0, \frac{2+(-1)^n}{4} + \frac{(-1)^n}{2^{n+2}}\right)$, for $n = 1,2,3,\cdots$

   a) Determine $A_1, A_2, \cdots, A_5$. Hint: Keep it in the form $(0, x + y)$ and find the pattern.
   b) Find the event $A = \cup_{i=1}^{\infty} A_n$
   c) Find the event $B = \cap_{n=1}^{\infty} A_n$
   d) Find the event $C = A_n$ i.o. (infinitely often). Hint: $C = \limsup A_n = \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} A_n$
   e) Find the event $D = A_n$ a.a. (almost always). Hint: $D = \liminf A_n = \cup_{m=1}^{\infty} \cap_{n=m}^{\infty} A_n$
   f) For a uniform distribution on $[0,1]$, determine the probabilities for each of the above 4 events A, B, C, D.

2. Consider a Bernoulli random process X_n, where for $n \geq 0$, X_n takes values in $\{0,1\}$ with $P(X_n = 1) = p$, and $P(X_n = 0) = 1 - p$, and for $n < 0$, $X_n = 0$. Note that the X_n are independent random variables. Now form the process $Z_n = X_n + \alpha X_{n-1}$ where $\alpha = \frac{1}{2}$.

   a) Find the mean of the process Z_n.
   b) Find the auto-correlation function for the process Z_n.
   c) Find the auto-covariance function for the process Z_n.
   d) Find the probability mass function (PMF) for the process Z_n at time n.
   e) Find the joint PMF for the process Z_n at the points n and $n + 1$.

3. A Poisson random variable has probability mass function $P(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, where $\lambda > 0$, is a constant. Note that the mean is λ and the second moment is $\lambda^2 + \lambda$. Now consider a Poisson process $N(t)$ with expected number of arrivals per unit time equal to 4.

   a) Determine the variance of $N(t)$ at time $t = 8$.
   b) Determine the correlation function value $R_N(2, 8)$.

Midterm 2 (solutions)

① $\Omega = [0,1]$

$$A_n = \left(0, \frac{2+(-1)^n}{4} + \frac{(-1)^n}{2^{n+2}}\right) \text{ for } n=1,2,\dots$$

ⓐ $A_1 = \left(0, \frac{1}{4} - \frac{1}{8}\right)$ $\quad A_4 = \left(0, \frac{3}{4} + \frac{1}{64}\right)$

$A_2 = \left(0, \frac{3}{4} + \frac{1}{16}\right)$ $\quad A_5 = \left(0, \frac{1}{4} - \frac{1}{128}\right)$

$A_3 = \left(0, \frac{1}{4} - \frac{1}{32}\right)$

ⓑ $A = \bigcup_{n=1}^{+\infty} A_n = \left(0, \frac{3}{4} + \frac{1}{16}\right) = A_2$

ⓒ $B = \bigcap_{n=1}^{+\infty} A_n = \left(0, \frac{1}{4} - \frac{1}{8}\right) = A_1$

ⓓ $\bigcup_{n=1}^{+\infty} A_n = A_2 \quad \bigcup_{n=2}^{+\infty} A_n = A_2$

$\bigcup_{n=3}^{+\infty} A_n = A_4 \quad \bigcup_{n=4}^{+\infty} A_n = A_4$

$\vdots \quad \vdots$

$$\bigcap_{m=1}^{+\infty}\bigcup_{n=m}^{+\infty}A_n=\bigcap_{m=1}^{+\infty}A_{2m}=\underbrace{\left(0,\frac{3}{4}\right]}_{C}$$

(e) $$\bigcap_{n=1}^{+\infty}A_n=A_1 \qquad \bigcap_{n=2}^{+\infty}A_n=A_3$$

$$\bigcap_{n=3}^{+\infty}A_n=A_3 \qquad \bigcap_{n=4}^{+\infty}A_n=A_5$$

$$\vdots \qquad \vdots$$

$$\Rightarrow \bigcup_{m=1}^{+\infty}\bigcap_{n=m}^{+\infty}A_n=\bigcup_{m=1}^{+\infty}A_{2m-1}=\underbrace{\left(0,\frac{1}{4}\right)}_{D}$$

(f) $$P(A)=P\left(\left(0,\frac{13}{16}\right)\right)=\frac{\frac{13}{16}-0}{1-0}=\frac{13}{16}$$

$$P(B)=P\left(\left(0,\frac{1}{8}\right)\right)=\frac{1}{8}$$

$$P(C)=P\left(\left(0,\frac{3}{4}\right]\right)=\frac{3}{4}$$

$$P(D)=P\left(\left(0,\frac{1}{4}\right)\right)=\frac{1}{4}$$

2a) If $n<0$, $EX_n=0$

If $n\geq 0$, $EX_n = 1\times P\{X_n=1\}+0\times P\{X_n=0\} = 1\times p+0\times(1-p)=p$

$\Rightarrow \boxed{EX_n = pU[n]}$ $U[n]$ is the discrete step function.

$EZ_n = EX_n + \frac{1}{2}EX_{n-1} = pU[n]+\frac{p}{2}U[n-1]$

2b) $R_X[n_1,n_2]=E\{X_{n_1}X_{n_2}\}$

If $n_1<0$ or $n_2<0$, $R_X[n_1,n_2]=0$

If n_1 and n_2 are non-negative, we consider the following two cases:

case I: $n_1=n_2 \Rightarrow R_X[n_1,n_2]=E\{X_{n_1}^2\}$

$=1^2\times p+0^2\times(1-p)=p$

Case II: If $n_1\neq n_2 \Rightarrow R_X[n_1,n_2]=\underbrace{EX_{n_1}EX_{n_2}}_{\text{(independency)}}=p^2$

$\Rightarrow R_X[n_1,n_2]=\Big(p\delta[n_2-n_1]+p^2(1-\delta[n_2-n_1])\Big)U[n_1]U[n_2]$

$=p^2U[n_1]U[n_2]+(p-p^2)\delta[n_2-n_1]U[n_1]U[n_2]$

Noting the sifting property of delta function, we have:

$$\delta[n_2-n_1]u[n_1]u[n_2]=\delta[n_2-n_1]u[n_1]u[n_1]=\delta[n_2-n_1]u[n_1]$$

$$\Rightarrow \boxed{R_x[n_1,n_2]=p^2u[n_1]u[n_2]+p(1-p)\delta[n_2-n_1]u[n_1]}$$

Now, let us find $R_z[n_1,n_2]$ in terms of $R_x[n_1,n_2]$:

$$R_z[n_1,n_2]=E\{Z_{n_1}Z_{n_2}\}=E\left\{\left(X_{n_1}+\frac{1}{2}X_{n_1-1}\right)\left(X_{n_2}+\frac{1}{2}X_{n_2-1}\right)\right\}$$

$$=E\{X_{n_1}X_{n_2}\}+\frac{1}{2}E\{X_{n_1}X_{n_2-1}\}+\frac{1}{2}E\{X_{n_1-1}X_{n_2}\}+\frac{1}{4}E\{X_{n_1-1}X_{n_2-1}\}$$

$$=R_x[n_1,n_2]+\frac{1}{2}R_x[n_1,n_2-1]+\frac{1}{2}R_x[n_1-1,n_2]+\frac{1}{4}R_x[n_1-1,n_2-1]$$

$$=p^2u[n_1]u[n_2]+p(1-p)\delta[n_2-n_1]u[n_1]+\frac{p^2}{2}u[n_1]u[n_2-1]+\frac{p(1-p)}{2}\delta[n_2-n_1-1]u[n_1]$$

$$+\frac{p^2}{2}u[n_1-1]u[n_2]+\frac{p(1-p)}{2}\delta[n_2-n_1+1]u[n_1-1]$$

$$+\frac{p^2}{4}u[n_1-1]u[n_2-1]+\frac{p(1-p)}{4}\delta[n_2-n_1]u[n_1-1]$$

2c $C_z[n_1,n_2]=R_z[n_1,n_2]-EZ_{n_1}EZ_{n_2}$

$$=R_z[n_1,n_2]-\left(pu[n_1]+\frac{p}{2}u[n_1-1]\right)\left(pu[n_2]+\frac{p}{2}u[n_2-1]\right)$$

$$= p(1-p)\delta[n_2-n_1]u[n_1] + \frac{p(1-p)}{2}\delta[n_2-n_1-1]u[n_1]$$

$$+ \frac{p(1-p)}{2}\delta[n_2-n_1+1]u[n_1-1] + \frac{p(1-p)}{4}\delta[n_2-n_1]u[n_1-1]$$

2d If $n<0$, $P\{z_n=0\}=1$.

If $n=0$, $P\{z_0=1\}=p$ and $P\{z_0=0\}=1-p$.

If $n\geq 1$:

$$P\{z_n=0\}=P\{x_n=0\}\times P\{x_{n-1}=0\}=(1-p)^2$$

$$P\{z_n=\tfrac{1}{2}\}=P\{x_n=0\}\times P\{x_{n-1}=1\}=p(1-p)$$

$$P\{z_n=1\}=P\{x_n=1\}\times P\{x_{n-1}=0\}=p(1-p)$$

$$P\{z_n=\tfrac{3}{2}\}=P\{x_n=1\}\times P\{x_{n-1}=1\}=p^2$$

2e: If $n\leq -2$, $P(z_n=0, z_{n+1}=0)=1$

If $n=-1$, $P(z_{-1}=0, z_0=0)=1-p$ $\quad P(z_{-1}=0, z_0=1)=p$

If $n=0$, $\begin{cases} Z_0 = X_0 \\ Z_1 = X_1 + \frac{1}{2}X_0 \end{cases}$

X_0	X_1	(Z_0, Z_1)	probability
0	0	$(0,0)$	$(1-p)^2$
0	1	$(0,1)$	$p(1-p)$
1	0	$(1,\frac{1}{2})$	$p(1-p)$
1	1	$(1,\frac{3}{2})$	p^2

If $n \geqslant 1$, $\begin{cases} Z_n = X_n + \frac{1}{2}X_{n-1} \\ Z_{n+1} = X_{n+1} + \frac{1}{2}X_n \end{cases}$

X_{n-1}	X_n	X_{n+1}	(Z_n, Z_{n+1})	probability
0	0	0	$(0,0)$	$(1-p)^3$
0	0	1	$(0,1)$	$(1-p)^2p$
0	1	0	$(1,\frac{1}{2})$	$(1-p)^2p$
0	1	1	$(1,\frac{3}{2})$	$(1-p)p^2$
1	0	0	$(\frac{1}{2},0)$	$(1-p)^2p$
1	0	1	$(\frac{1}{2},1)$	$(1-p)p^2$
1	1	0	$(\frac{3}{2},\frac{1}{2})$	$(1-p)p^2$
1	1	1	$(\frac{3}{2},\frac{3}{2})$	p^3

(3) $N(t)$ is a poisson random variable with parameter λt.

(a): $$\mathrm{Var}\{N(8)\} = E\{N^2(8)\} - (EN(8))^2$$

$$= (4\times 8) + (4\times 8)^2 - (4\times 8)^2 = 32$$

(b): $$R_N(2,8) = E\{N(2)N(8)\}$$

$$= E\{N(2)((N(8)-N(2)) + N(2))\}$$

$$= E\{N(2)(N(8)-N(2))\} + E\{N^2(2)\}$$

$$\overset{\text{independent}}{\underset{\text{increment}}{=}} EN(2)(EN(8)-EN(2)) + E\{N^2(2)\}$$

$$= EN(2)EN(8) - (EN(2))^2 + E\{N^2(2)\}$$

$$= (4\times 2)\times(4\times 8) - \cancel{(4\times 2)^2} + (4\times 2) + \cancel{(4\times 2)^2}$$

$$= 8\times 33 = 264$$