problem set 4 (solutions)

1: Independency $\Longrightarrow$ uncorrelation

$$K_{U_1,U_2,U_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I \;,\; f_{U_1,U_2,U_3}(u_1,u_2,u_3) = f_{U_1}(u_1) f_{U_2}(u_2) f_{U_3}(u_3)$$

$$\Rightarrow f_{U_1,U_2,U_3}(u_1,u_2,u_3) = \frac{1}{(2\pi)^{3/2}} e^{\frac{-u_1^2 - u_2^2 - u_3^2}{2}}$$

a: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} \Rightarrow K_{X,Y,Z} = A K_{U_1,U_2,U_3} A^T = A I A^T = A A^T$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

b: $$\begin{cases} U_1 = X \\ U_2 = Y - X \\ U_3 = Z - Y \end{cases} \Rightarrow f_{X,Y,Z}(x,y,z) = \frac{f_{U_1,U_2,U_3}(x, y-x, z-y)}{\underbrace{|\det(A)|}_{1}}$$

$$= \frac{1}{(2\pi)^{3/2}} e^{\frac{-x^2 - (y-x)^2 - (z-y)^2}{2}}$$

ⓒ: $$f_{Y,Z/X}(y,z/x) = \frac{f_{X,Y,Z}(x,y,z)}{f_X(x)}$$

$$= \frac{\frac{1}{(2\pi)^{3/2}} e^{\frac{-x^2-(y-x)^2-(z-y)^2}{2}}}{\frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}} = \frac{1}{2\pi} e^{\frac{-(y-x)^2-(z-y)^2}{2}}$$

ⓓ: $$f_{Z/X,Y}(z/x,y) = \frac{f_{X,Y,Z}(x,y,z)}{f_{X,Y}(x,y)}$$

$$f_{X,Y}(x,y) = \int_{-\infty}^{+\infty} f_{X,Y,Z}(x,y,z)\,dz = \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{3/2}} e^{\frac{-x^2-(y-x)^2-(z-y)^2}{2}} dz =$$

$$\frac{1}{2\pi} e^{\frac{-x^2-(y-x)^2}{2}} \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-(z-y)^2}{2}} dz}_{1} = \frac{1}{2\pi} e^{\frac{-x^2-(y-x)^2}{2}}$$

$$\Rightarrow f_{Z/X,Y}(z/x,y) = \frac{\frac{1}{(2\pi)^{3/2}} e^{\frac{-x^2-(y-x)^2-(z-y)^2}{2}}}{\frac{1}{2\pi} e^{\frac{-x^2-(y-x)^2}{2}}}$$

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{-(z-y)^2}{2}}$$

(2): $f_T(t) = \alpha e^{-\alpha t}$; $t \geq 0$

$$pr\{N_i = k / T = t\} = \frac{e^{-\lambda_i t} (\lambda_i t)^k}{k!} \quad \begin{cases} i = 1, 2, 3 \\ k = 0, 1, 2, \cdots \end{cases}$$

$$\Rightarrow pr\{N_i = k\} = \int_0^{+\infty} pr\{N_i = k / T = t\} \times f_T(t)\, dt$$

$$= \int_0^{+\infty} \frac{e^{-\lambda_i t} (\lambda_i t)^k}{k!} \times \alpha e^{-\alpha t}\, dt = \frac{\lambda_i^k \cdot \alpha}{k!} \int_0^{+\infty} t^k e^{-(\alpha + \lambda_i) t}\, dt =$$

$$\frac{\lambda_i^k \alpha}{k!} \times \frac{k!}{(\alpha + \lambda_i)^{k+1}} = \underbrace{\frac{\alpha}{\alpha + \lambda_i}}_{p} \underbrace{\left(\frac{\lambda_i}{\alpha + \lambda_i}\right)^k}_{1-p} \rightarrow \text{geometric random variable}$$

$$\Rightarrow Var\{N_i\} = \frac{1-p}{p^2} = \frac{1 - \frac{\alpha}{\alpha + \lambda_i}}{\left(\frac{\alpha}{\alpha + \lambda_i}\right)^2} = \frac{\lambda_i (\alpha + \lambda_i)}{\alpha^2}$$

since N_i's are independent, we have $Cov(N_i, N_j) = 0$ for $i \neq j$

$$\Rightarrow \text{Covariance matrix} = \begin{pmatrix} \frac{\lambda_1(\alpha + \lambda_1)}{\alpha^2} & 0 & 0 \\ 0 & \frac{\lambda_2(\alpha + \lambda_2)}{\alpha^2} & 0 \\ 0 & 0 & \frac{\lambda_3(\alpha + \lambda_3)}{\alpha^2} \end{pmatrix}$$

③: $X \perp\!\!\!\perp Y \quad f_X(x) = \lambda_1 e^{-\lambda_1 x} \;;\; x \geq 0 \quad f_Y(y) = \lambda_2 e^{-\lambda_2 y} \;;\; y \geq 0$

find the PDF of $Z = \frac{X}{Y}$

first method: (Jacobian matrix)

we define the new random variable $W = X$

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 y} \;;\; x,y \geq 0$$

$$J = \begin{pmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{y} & \frac{-x}{y^2} \\ 1 & 0 \end{pmatrix} \Rightarrow \det(J) = \frac{x}{y^2}$$

$$\begin{cases} z = \frac{x}{y} \\ w = x \end{cases} \Longrightarrow \begin{cases} x = w \\ y = \frac{w}{z} \end{cases} \Rightarrow f_{Z,W}(z,w) = \frac{f_{X,Y}\left(w, \frac{w}{z}\right)}{\left|\det J\right| \begin{cases} x = w \\ y = w/z \end{cases}}$$

$$= \frac{\lambda_1 \lambda_2 e^{-\lambda_1 w} e^{-\lambda_2 \frac{w}{z}}}{\frac{w}{\left(\frac{w}{z}\right)^2}} = \lambda_1 \lambda_2 \frac{w}{z^2} e^{-\lambda_1 w} e^{-\lambda_2 \frac{w}{z}}$$

$$f_Z(z) = \int_0^{+\infty} f_{Z,W}(z,w)\, dw = \frac{\lambda_1 \lambda_2}{z^2} \int_0^{+\infty} w e^{-\left(\lambda_1 + \frac{\lambda_2}{z}\right) w} dw =$$

$$\frac{\lambda_1 \lambda_2}{z^2} \times \frac{1}{\left(\lambda_1 + \frac{\lambda_2}{z}\right)^2} = \frac{\lambda_1 \lambda_2}{\left(\lambda_2 + \lambda_1 z\right)^2}$$

second method (CDF function)

$$F_Z(z)=pr\{Z\leqslant z\}=pr\left\{\frac{X}{Y}\leqslant z\right\}=\int_0^{+\infty} pr\left\{\frac{X}{y}\leqslant z\right\} f_Y(y)\,dy$$

$$=\int_0^{+\infty} pr\{X\leqslant yz\}\lambda_2 e^{-\lambda_2 y}\,dy=\int_0^{+\infty} F_X(yz)\lambda_2 e^{-\lambda_2 y}\,dy=$$

$$F_X(x)=pr\{X\leqslant x\}=\int_0^{x} \lambda_1 e^{-\lambda_1 t}\,dt=-e^{-\lambda_1 t}\Big]_0^{x}=1-e^{-\lambda_1 x}$$

$$\Rightarrow F_Z(z)=\int_0^{+\infty}\left(1-e^{-\lambda_1 yz}\right)\lambda_2 e^{-\lambda_2 y}\,dy$$

$$=1-\int_0^{+\infty}\lambda_2 e^{-(\lambda_1 z+\lambda_2)y}\,dy=\cancel{1-\left[\frac{-1}{\lambda_1 z+\lambda_2}\right]_0^{+\infty}}=\frac{\lambda_1 z}{\lambda_1 z+\lambda_2}$$

$$\Rightarrow f_Z(z)=\frac{d}{dz}F_Z(z)=\frac{\lambda_1(\lambda_1 z+\lambda_2)-\lambda_1^2 z}{(\lambda_1 z+\lambda_2)^2}=\frac{\lambda_1\lambda_2}{(\lambda_1 z+\lambda_2)^2}$$

④: $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is a jointly Gaussian random vector.

$m_X = 0 \quad K_X = I_{3\times 3} \rightsquigarrow X_i$'s are uncorrelated

Since X is jointly Gaussian, we conclude that they are independent.

$Y \triangleq \max\{X_1, X_2, X_3\}$

$F_Y(y) = pr\{Y \leqslant y\} = pr\{\max\{X_1, X_2, X_3\} \leqslant y\} = pr\{X_1 \leqslant y, X_2 \leqslant y, X_3 \leqslant y\}$

$= pr\{X_1 \leqslant y\} \times pr\{X_2 \leqslant y\} \times pr\{X_3 \leqslant y\} = F_{X_1}(y) F_{X_2}(y) F_{X_3}(y) = F^3_{X_1}(y)$

$\Rightarrow f_Y(y) = \frac{d}{dy} F_Y(y) = 3 \times f_{X_1}(y) \times F^2_{X_1}(y)$

$\Rightarrow f_Y(y) = 3 \times \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} \times \left(\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}} dt \right)^2$

⑤ $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$ is a jointly Gaussian random vector.

$K_X = \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 1 \end{pmatrix}$, $Y = AX$

Y is a linear transformation of X. Therefore, Y_i's are independent if and only if Y_i's are uncorrelated.

$\det(K_X) = 0 \Rightarrow K_X$ is a singular matrix.

Since K_X is singular, there is not any square matrix $A_{3\times3}$ to make Y_i's uncorrelated.

Now, we want to find a non-square matrix $A_{2\times3}$ to generate a new vector with uncorrelated elements. To this end, we first write X in terms of Y as:

$X = BY$

let us assume that $k_y = I$

$$\Rightarrow C_x = B k_y B^T = B I B^T = B B^T$$

To simplify the task of finding B, let us assume that B is a lower triangular matrix. Then we need to solve the following matrix equation

$$\underbrace{\begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 1 \end{pmatrix}}_{k_x} = \underbrace{\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}}_{B} \underbrace{\begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix}}_{B^T}$$

{this decomposition is called cholesky decomposition in linear algebra.}

$$\Rightarrow 1 = a^2 \Rightarrow \boxed{a = 1} \qquad ad = \frac{-1}{2} \Rightarrow \boxed{d = \frac{-1}{2}}$$

$$ab = \frac{-1}{2} \Rightarrow \boxed{b = \frac{-1}{2}}$$

$$b^2 + c^2 = 1 \Rightarrow \frac{1}{4} + c^2 = 1 \Rightarrow \boxed{c = \frac{\sqrt{3}}{2}}$$

$$bd + ce = \frac{-1}{2} \Rightarrow \frac{1}{4} + \frac{\sqrt{3}}{2} e = \frac{-1}{2} \Rightarrow \boxed{e = \frac{-\sqrt{3}}{2}}$$

$$d^2 + e^2 + f^2 = 1 \Rightarrow \frac{1}{4} + \frac{3}{4} + f^2 = 1 \Rightarrow \boxed{f = 0}$$

$$\Rightarrow B = \begin{pmatrix} 1 & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$

since last column of B is zero, we can remove it and write the following transformation:

$$X = BY \Rightarrow \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} X_1 = Y_1 \\ X_2 = \frac{-1}{2} Y_1 + \frac{\sqrt{3}}{2} Y_2 \\ X_3 = \frac{-1}{2} Y_1 - \frac{\sqrt{3}}{2} Y_2 \end{cases} \quad (\star)$$

Let $\tilde{X}_i = X_i - EX_i$ and $\tilde{Y}_i = Y_i - EY_i$. Therefore, we have:

$$\begin{cases} \tilde{X}_1 = \tilde{Y}_1 \\ \tilde{X}_2 = \frac{-1}{2} \tilde{Y}_1 + \frac{\sqrt{3}}{2} \tilde{Y}_2 \\ \tilde{X}_3 = \frac{-1}{2} \tilde{Y}_1 - \frac{\sqrt{3}}{2} \tilde{Y}_2 \end{cases}$$

$$\Rightarrow \tilde{X}_1 + \tilde{X}_2 + \tilde{X}_3 = 0 \Rightarrow \boxed{X_1 + X_2 + X_3 = EX_1 + EX_2 + EX_3}$$

As it can be seen, X_i's are linearly dependent.
That is why K_X was a singular matrix.
Now, let us find matrix $A_{2\times3}$. Noting (✯), we can find Y_i's in terms of X_i's:

$$\begin{cases} Y_1 = X_1 \\ Y_2 = \frac{1}{\sqrt{3}}X_2 - \frac{1}{\sqrt{3}}X_3 \end{cases}$$

⇝ This is <u>not</u> unique.

$$\Rightarrow \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}_{2\times1} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix}}_{A_{2\times3}} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}_{3\times1}$$

It is not hard to see that:

$$K_Y = AK_XA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & \frac{-1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & 1 & \frac{-1}{2} \\ \frac{-1}{2} & \frac{-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & \frac{-1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow Y_1$ and Y_2 are uncorrelated.

⑥: PDF of Cauchy random variable: $f_X(x) = \dfrac{\gamma}{\pi(\gamma^2 + x^2)}$

$\phi_X(\omega) = E\{e^{j\omega X}\} = e^{-\gamma|\omega|}$ $\quad Y = \sum_{i=1}^{n} X_i$

X_i's are independent Cauchy random variables with parameter γ_i.

$\phi_Y(\omega) = E\{e^{j\omega Y}\} = E\{e^{j\omega(X_1+X_2+\cdots+X_n)}\} = E\{e^{j\omega X_1} e^{j\omega X_2} \cdots e^{j\omega X_n}\}$

$= E\{e^{j\omega X_1}\}E\{e^{j\omega X_2}\} \cdots E\{e^{j\omega X_n}\} = e^{-\gamma_1|\omega|} \times e^{-\gamma_2|\omega|} \times \cdots \times e^{-\gamma_n|\omega|}$

X_i's are independent

$= e^{-(\gamma_1+\gamma_2+\cdots+\gamma_n)|\omega|} \rightsquigarrow \phi_Y(\omega)$ is the characteristic function of a Cauchy random variable with parameter $\sum_{i=1}^{n} \gamma_i$

$$f_Y(y) = F^{-1}\{\phi_Y(\omega)\} = \frac{\sum_{i=1}^{n} \gamma_i}{\pi\left(\left(\sum_{i=1}^{n} \gamma_i\right)^2 + y^2\right)}$$

⑦

X_i	0	1
P	0.9	0.1

$(i=1,2,\dots,n)$

feedback of each person is a Bernouli random variable.

$y \triangleq \frac{1}{n}\sum_{i=1}^{n} X_i \Rightarrow 1-y$ shows the portion of participants that are unsatisfied

$$pr\{0.89 \leq 1-y \leq 0.91\} \geq 0.99 \Rightarrow pr\{0.09 \leq y \leq 0.11\} \geq 0.99$$

$$\Rightarrow pr\{|y-0.1| \leq 0.01\} \geq 0.99 \Rightarrow pr\{|y-\underbrace{0.1}_{m_y}| \geq \underbrace{0.01}_{a}\} \leq 0.01$$

$$\sigma_y^2 = \frac{1}{n^2} \times \sum_{i=1}^{n} \sigma_{X_i}^2 = \frac{1}{n^2} \times n \times (0.1-0.01) = \frac{0.09}{n}$$

~~chebyshev inequality:~~

using chebyshev inequality we have:

$$pr\{|y-0.1| \geq 0.01\} \leq \frac{\sigma_y^2}{a^2} = \frac{0.09}{n(0.01)^2}$$

To be sure that $pr\{|y-0.1| \geq 0.01\} \leq 0.01$, the following condition should be satisfied:

$$\frac{0.09}{n(0.01)^2} \leq 0.01 \Longrightarrow \boxed{n \geq 90000}$$

(8): $n=1000$

$$pr(\underbrace{y>\tfrac{1}{2}}_{a}) \leqslant \min_{s>0} e^{-\frac{1}{2}s} E\{e^{sy}\}$$

$$E\{e^{sy}\} = E\{e^{\frac{1}{n}s\sum_{i=1}^{n}x_i}\} = \left(E\{e^{\frac{1}{n}sx_1}\}\right)^n$$

x_i's are independent

$$E\{e^{\frac{1}{n}sx_1}\} = 0.9 + 0.1\times e^{\frac{s}{n}}$$

$$\Rightarrow \text{upper bound} = \min_{s>0} e^{\frac{-s}{2}} \times \left(0.9 + 0.1 e^{\frac{s}{1000}}\right)^{1000}$$

$$f(s) \triangleq e^{\frac{-s}{2}}\left(0.9 + 0.1 e^{\frac{s}{1000}}\right)^{1000}$$

$$\Rightarrow f'(s) = \frac{-1}{2} e^{\frac{-s}{2}}\left(0.9+0.1e^{\frac{s}{1000}}\right)^{1000} + e^{\frac{-s}{2}} \times \frac{1000 \times e^{\frac{s}{1000}}}{10000}\left(0.9+0.1e^{\frac{s}{1000}}\right)^{999} = 0$$

$$\Rightarrow \frac{-1}{2}\left(0.9+0.1e^{\frac{s}{1000}}\right) + \frac{1}{10} e^{\frac{s}{1000}} = 0 \Rightarrow \boxed{s^* = 1000\ln(9)}$$

$$\Rightarrow \text{upper bound} = \frac{1}{9^{500}} \times (0.9 + 0.1\times 9)^{1000} = \frac{1.8^{1000}}{9^{500}}$$