ECE537

Problem Set 12

1. An urn initially contains 5 black balls and 5 white balls. The following experiment is repeated indefinitely: A ball is drawn from the urn at random: if the ball is white it is put back in the urn, otherwise it is left out. Let X_n be the number of black balls remaining in the urn after n draws.

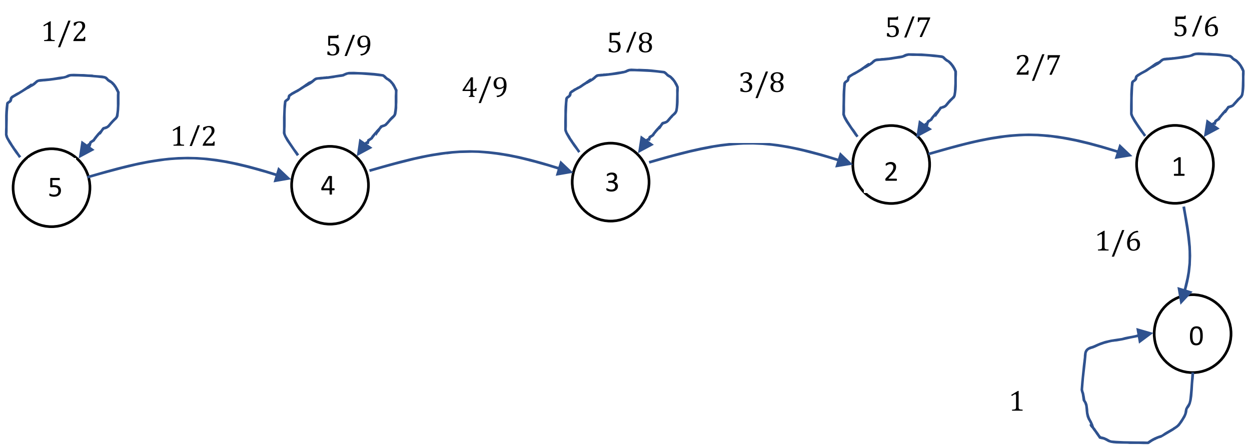
a) Is X_n a Markov Chain? Why? If so list the states.

This is a Markov chain because the number of black balls at time n+1 is the same as the number at time n, or it is greater than the number at time n by one. It does not depend on the number of black balls at the times $n-1, n-2, \cdots$

There are 6 states: The states are 0,1,2,3,4,5

b) Draw the state diagram and find the transition probabilities.

The state diagram and the transition probabilities are shown below



c) Do the transition probabilities depend on n?

No they don't. This is a homogeneous Markov Chain. The decision as to what is done does not depend on the time index n

d) Give the transition probability matrix.

The transition probability matrix is

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/6 & 5/6 & 0 & 0 & 0 & 0 \\ 0 & 2/7 & 5/7 & 0 & 0 & 0 \\ 0 & 0 & 3/8 & 5/8 & 0 & 0 \\ 0 & 0 & 0 & 4/9 & 5/9 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

e) List the communicating classes of the Markov Chain

No two states communicate, but a state can communicate with itself. Hence there are 6 classes. Each class has only one state.

f) Is the Markov Chain Irreducible?

No, It is not irreducible because it has more than one class.

g) List the Transient classes.

A class is a set of communicating states.

Classes {1}, {2}, {3}, {4}, {5} are transient classes.

h) List the recurrent classes.

There is only one recurrent class, i.e. {0}

i) Find the steady state probability vector $\boldsymbol{p}_\infty$.

The stead state probability vector is $\boldsymbol{p}_\infty = (1,0,0,0,0,0)$

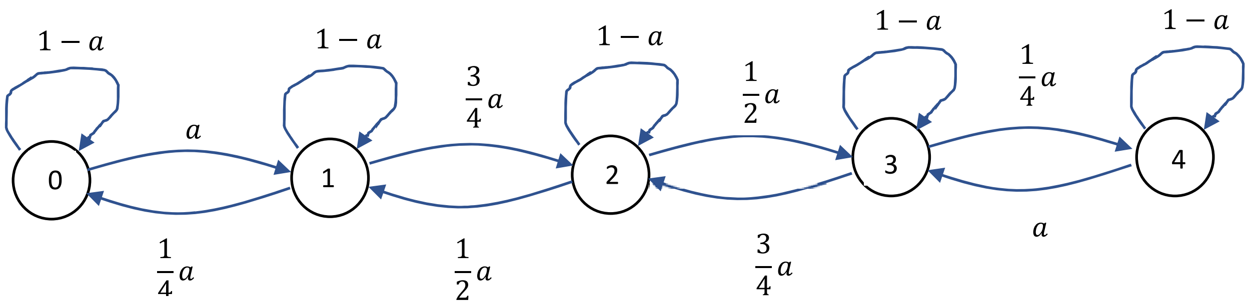
2. Consider an urn with 2 black balls and 2 white balls. The following experiment is repeated indefinitely: We draw a ball at random and with probability a we change the colour of the ball and put it back in the urn, otherwise we put the ball back without change. Let X_n be the number of black balls in the urn at time n.

a) Is X_n a Markov Chain? Why? If so list the states.

Yes, because the number of black balls at time n only depends on the number of black balls at time $n-1$ and not on the number of black balls at earlier times.

b) Draw the state diagram

The state diagram is shown below. Note that if $a = 0$, or $a = 1$, we would remove some of the edges from the graph, because we don't include edges with transition probability equal to 0.



c) Find the state transition probabilities

They are shown in the graph. If we are at state 0, i.e. 0 block balls, then with probability $1-a$ *we will remain in state 0, but with probability* a *we will change the colour. Since all balls are white in state 0 we must be changing one ball to black, hence we transition to state 1.*

If we are in state 1 then with probability $1-a$ *we will remain in state 1, but with probability* a *we will change state. If we draw a black ball then we will change it to white, i.e. decrease the number of black and go to state 0. This occurs with probability* $\frac{1}{4}$*. Hence the probability of a transition to state 0 is* $\frac{1}{4}a$*. But if we draw a white ball (the probability is ¾) then the transition probability to state 2 is* $\frac{3}{4}a$*. In the same way we compute the transition probabilities for all states.*
For each state we remain in the state with probability $1-a$*. We decrement the state if we draw a black ball, and increment the state if we draw a white ball.*

d) Do the transition probabilities depend on n?

No they don't because the decision depends only on the current state.

e) Give the transition probability matrix.

$$P = \begin{bmatrix} 1-a & a & 0 & 0 & 0 \\ \frac{1}{4}a & 1-a & \frac{3}{4}a & 0 & 0 \\ & \frac{1}{2}a & 1-a & \frac{1}{2}a & 0 \\ 0 & 0 & \frac{3}{4}a & 1-a & \frac{1}{4}a \\ 0 & 0 & 0 & 0 & 1-a \end{bmatrix}$$

f) List the communicating classes of the Markov Chain

There is a single communicating Class consisting of all the states

g) Is the Markov Chain irreducible?

Yes, it is. Because there is a single communicating class

h) List the Transient classes.

There are no transient states

i) List the recurrent classes.

There is only one communicating class $\{0,1,2,3,4\}$

j) Assuming $a = \frac{1}{2}$, find the state probability vector at the times $n = 1,2,3,10,100,1000$. Use the computer. Repeat for $a = \frac{3}{4}$.

First we do $a = 1/2$

$$P = \begin{pmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.125 & 0.5 & 0.375 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0 & 0.375 & 0.5 & 0.125 \\ 0 & 0 & 0 & 0.5 & 0.5 \end{pmatrix}$$

We are given that $\boldsymbol{p_0} = (0,0,1,0,0)$

Hence $\boldsymbol{p_1} = \boldsymbol{p_0 P}, \cdots, \boldsymbol{p_n} = \boldsymbol{p_0 P^n}$

We show the results below obtained with the computer. We can see the convergence

$$p0 \cdot P = (0 \quad 0.25 \quad 0.5 \quad 0.25 \quad 0)$$

$$p0 \cdot P^2 = (0.031 \quad 0.25 \quad 0.438 \quad 0.25 \quad 0.031)$$

$$p0 \cdot P^3 = (0.047 \quad 0.25 \quad 0.406 \quad 0.25 \quad 0.047)$$

$$p0 \cdot P^{10} = (0.062 \quad 0.25 \quad 0.375 \quad 0.25 \quad 0.062)$$

$$p0 \cdot P^{100} = (0.063 \quad 0.25 \quad 0.375 \quad 0.25 \quad 0.063)$$

Now for $a = 3/4$

$$P = \begin{pmatrix} 0.25 & 0.75 & 0 & 0 & 0 \\ 0.188 & 0.25 & 0.563 & 0 & 0 \\ 0 & 0.375 & 0.25 & 0.375 & 0 \\ 0 & 0 & 0.563 & 0.25 & 0.188 \\ 0 & 0 & 0 & 0.75 & 0.25 \end{pmatrix}$$

$$p0 \cdot P = (0 \quad 0.375 \quad 0.25 \quad 0.375 \quad 0)$$

$$p0 \cdot P^2 = (0.07 \quad 0.188 \quad 0.484 \quad 0.188 \quad 0.07)$$

$$p0 \cdot P^3 = (0.053 \quad 0.281 \quad 0.332 \quad 0.281 \quad 0.053)$$

$$p0 \cdot P^{10} = (0.063 \quad 0.25 \quad 0.375 \quad 0.25 \quad 0.063)$$

$$p0 \cdot P^{100} = (0.063 \quad 0.25 \quad 0.375 \quad 0.25 \quad 0.063)$$

We can see that the sequence of probability vectors is different depending on a but they both converge to the same value.

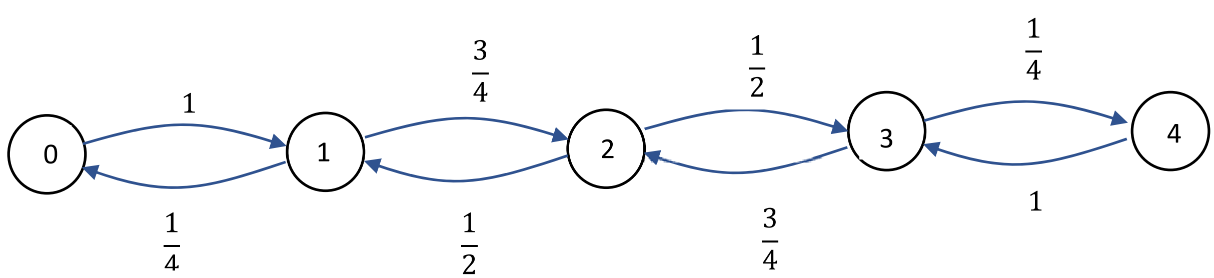
k) Assume that $a = \frac{1}{2}$, is the Markov chain periodic? Show your work.

We should compute the period of the states. For a class of communicating states all the states have the same period. So we need to check the period of only one state.

Consider state 0. Then there are possible returns at $n = 1,2,3,\cdots$
Since the $gcd(1,2,3,\cdots) = 1$. The Chain has period 1. It is therefor not periodic, i.e. aperiodic. In fact if a Chain has a "loop" in one of the states then it is aperiodic.

l) Assuming that $a = 1$, modify the state diagram. Is the Markov Chain periodic?

The simplified Chain is shown below



Let us check the period of state 0.

Returns to state 0 may occur at $n = 2,4,6,\cdots$
Hence the period (equal to the gcd) is $d_0 = 2$. The chain is periodic with period equal to 2.

m) Find the steady state probability vector $\boldsymbol{p}_\infty$ in the case that $a = 1/2$ using manual computation.

We need to solve the equation

$$\boldsymbol{p} = \boldsymbol{p}\boldsymbol{P}$$

Along with the equation $\boldsymbol{p}(1,1,1,1,1)^T = 1$*, i.e. the vector* p *is a probability vector.*

The above matrix equation can be solved as a system of 5 equations in 5 unknows using Gaussian elimination and then the normalizing condition. We obtain the following

$$\boldsymbol{p} = \left(\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}\right)$$