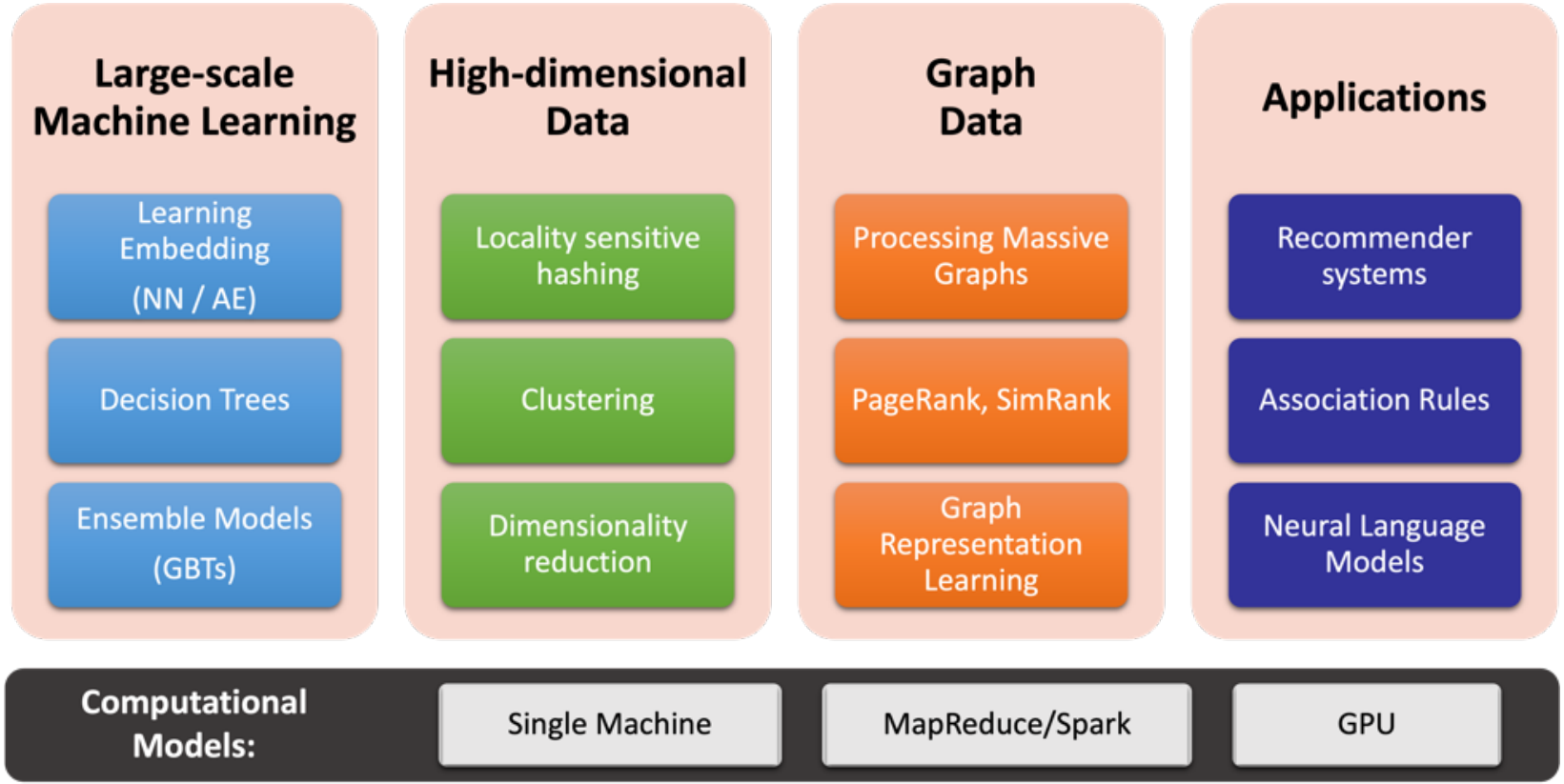
MIE524 Data Mining

PageRank

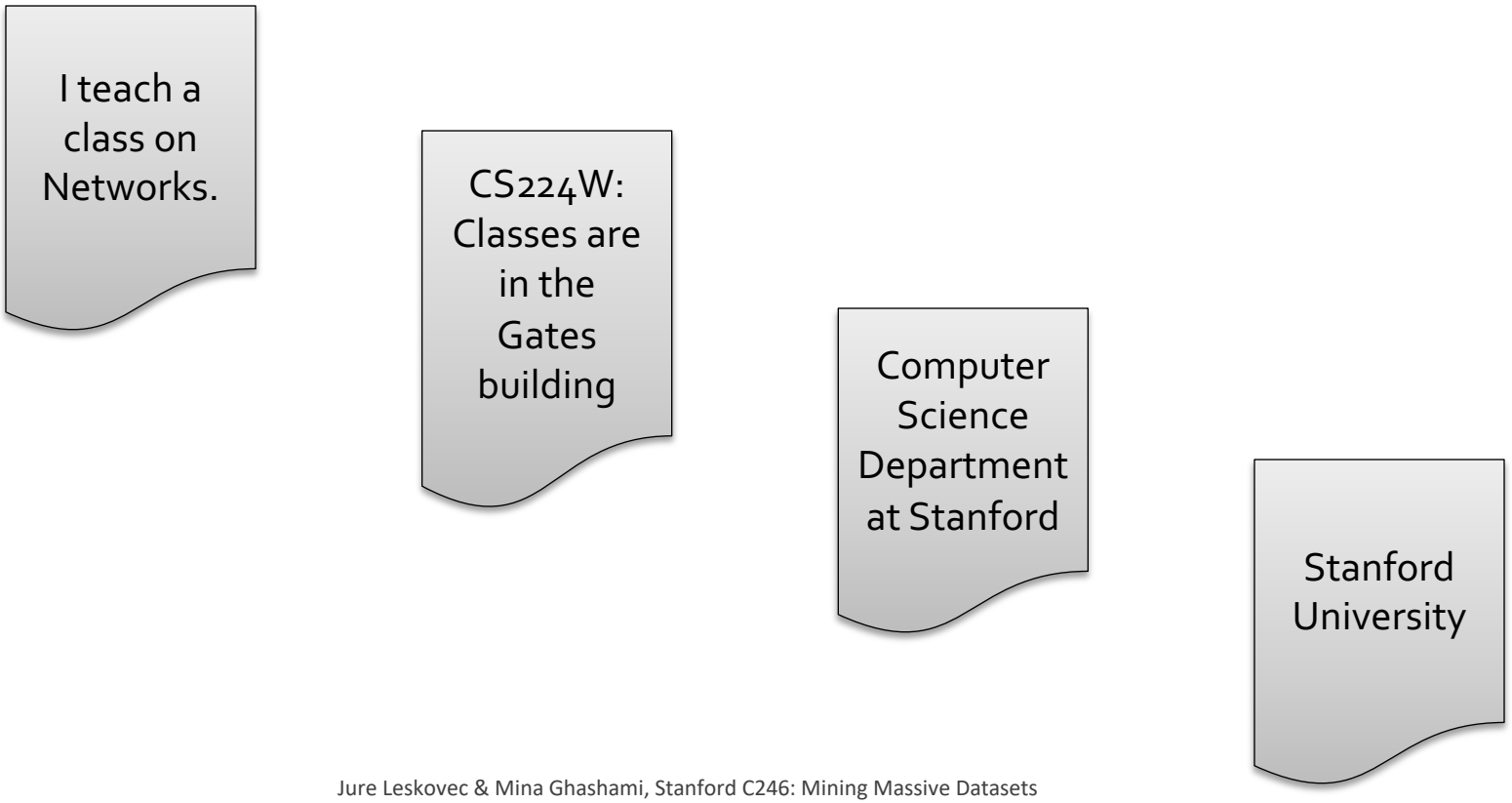
Slides Credits:
Slides from Leskovec, Rajaraman, Ullman (http://www.mmds.org), Leskovec & Ghashami

MIE524: Course Topics (Tentative)



Web as a Graph

- **Web as a directed graph:**
  - **Nodes: Webpages**
  - **Edges: Hyperlinks**



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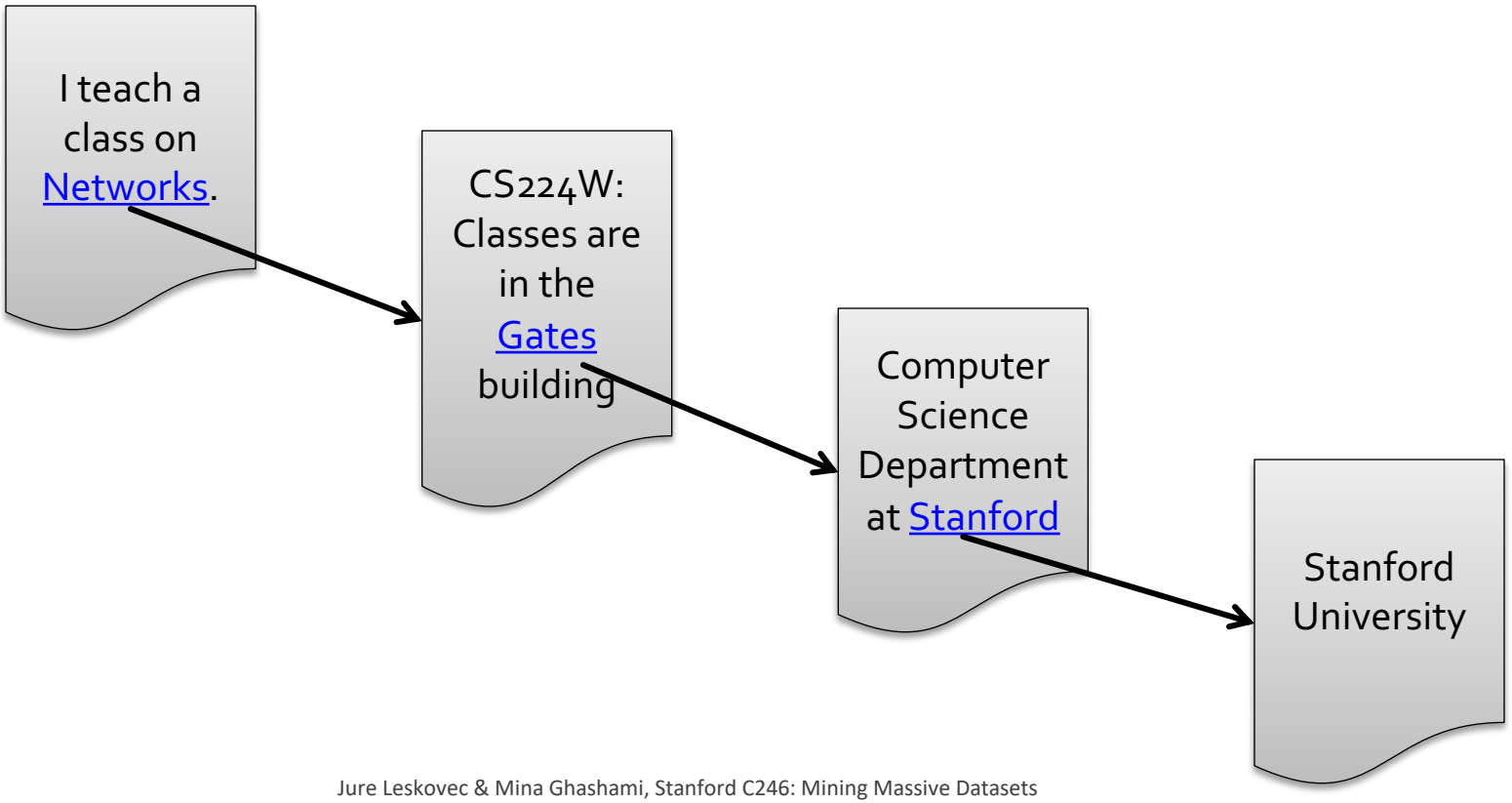
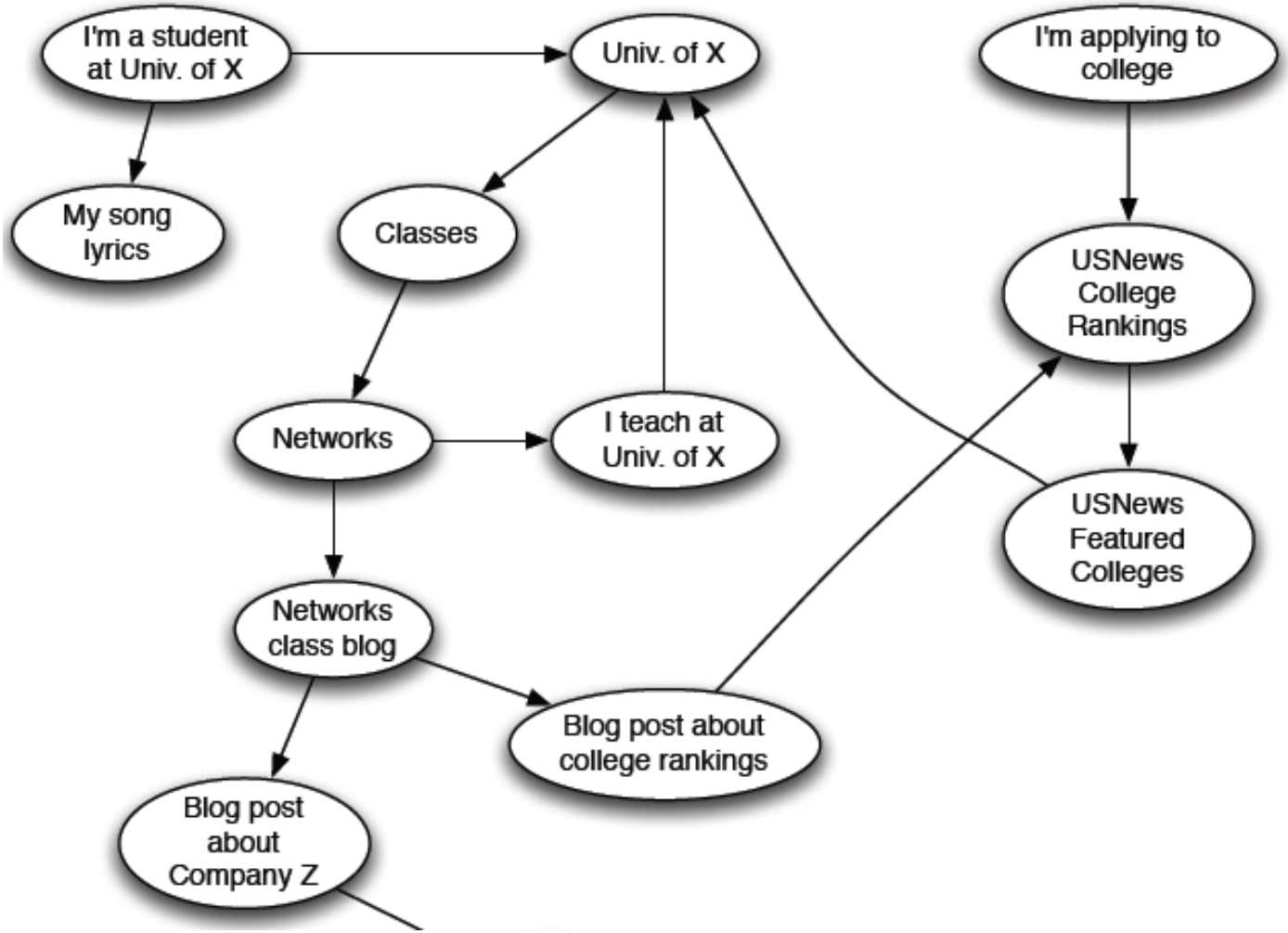


Web as a Directed Graph



Broad Question

- **How to organize the Web?**
- **First try:** Human curated
  Web directories
  - Yahoo, DMOZ, LookSmart
- **Second try: Web Search**
  - **Information Retrieval** investigates:
    Find relevant docs in a small
    and trusted set
    - Newspaper articles, Patents, etc.
  - **But:** Web is **huge**, full of untrusted documents, random things, web spam, etc.

Web Search: 2 Challenges

2 challenges of web search:

- **(1) Web contains many sources of information Who to "trust"?**
  - **Trick:** Trustworthy pages may point to each other!
- **(2) What is the "best" answer to query "newspaper"?**
  - No single right answer
  - **Trick:** Pages that actually know about newspapers might all be pointing to many newspapers

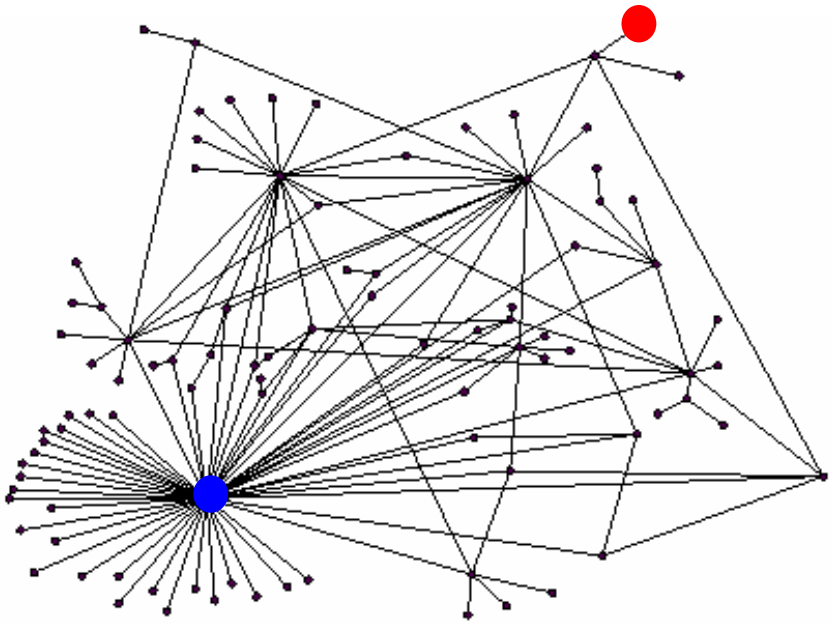
Ranking Nodes on the Graph

- **All web pages are not equally "important"**

  thispersondoesnotexist.com vs. www.stanford.edu

- There is a large diversity in the web-graph node connectivity.

  Let's rank the pages by the link structure!



PageRank: The "Flow" Formulation

Links as Votes

- **Idea: Links as votes**
  - **Page is more important if it has more links**
    - In-coming links? Out-going links?
- **Think of in-links as votes:**
  - www.stanford.edu has **millions** in-links
  - thispersondoesnotexist.com has a **few thousands** in-link
- **Are all in-links equal?**
  - **Links from important pages count more**
  - Recursive question!

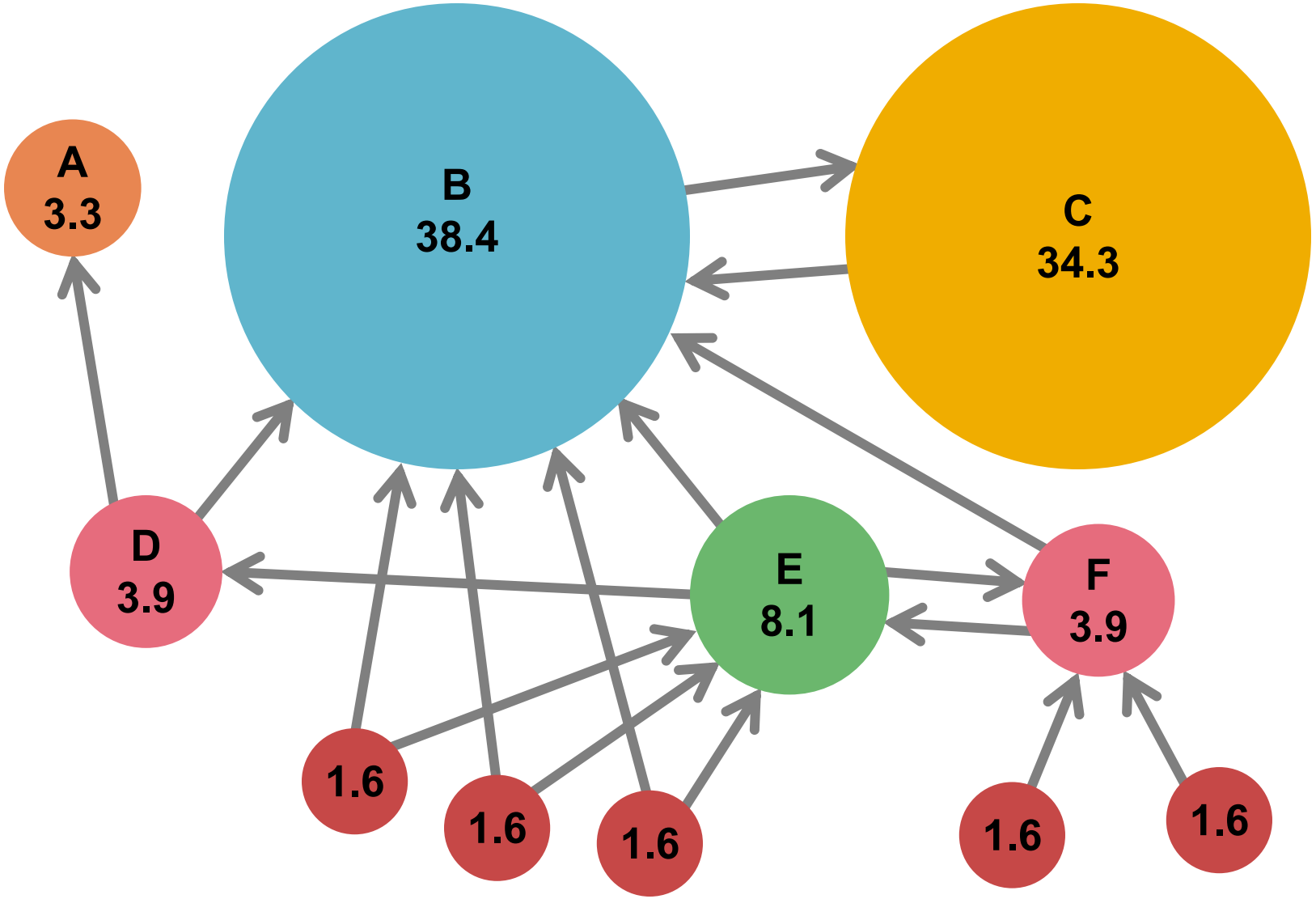
Intuition – (1)

- Web pages are important if people visit them a lot.
- But we can't watch everybody using the Web.
- A good surrogate for visiting pages is to assume people follow links randomly.
- Leads to *random surfer* model:
  - Start at a random page and follow random out-links repeatedly, from whatever page you are at.
  - *PageRank* = limiting probability of being at a page.

Intuition – (2)

- **Solve the recursive equation:** "importance of a page = its share of the importance of each of its predecessor pages"
  - Equivalent to the random-surfer definition of PageRank

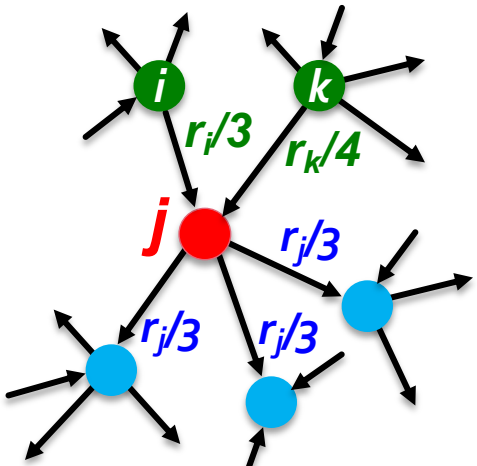
Example: PageRank Scores



Simple Recursive Formulation

- Each link's vote is proportional to the **importance** of its source page
- If page ***j*** with importance r_j has ***n*** out-links, each link gets r_j / n votes
- Page ***j***'s own importance is the sum of the votes on its in-links

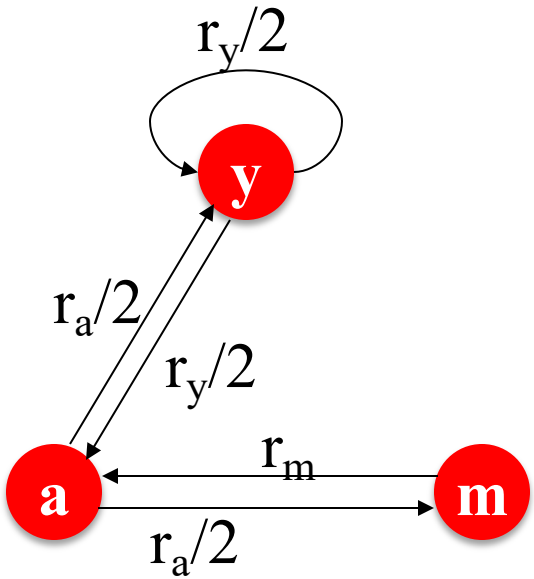
$$r_j = r_i/3 + r_k/4$$



PageRank: The "Flow" Model

- A "vote" from an important page is worth more
- A page is important if it is pointed to by other important pages
- Define a "rank" r_j for page j

The web in 1839



$$r_j = \sum_{i \to j} \frac{r_i}{d_i}$$

d_i … out-degree of node i

"Flow" equations:

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

r_j are the solutions to the "flow" equation

Solving the Flow Equations

- **3 equations, 3 unknowns, no constants**
  - No unique solution
  - All solutions equivalent modulo the scale factor
- **Additional constraint forces uniqueness:**
  - $r_y + r_a + r_m = 1$
  - **Solution:** $r_y = \frac{2}{5}$, $r_a = \frac{2}{5}$, $r_m = \frac{1}{5}$
- **Gaussian elimination method works for small examples, but we need a better method for large web-size graphs**
- **We need a new formulation!**

Flow equations:

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

PageRank: Matrix Formulation

- **Define stochastic adjacency matrix $\boldsymbol{M}$**
  - Let page i has d_i out-links
  - If $i \rightarrow j$, then $M_{ji} = \frac{1}{d_i}$ else $M_{ji} = 0$
    - $\boldsymbol{M}$ is a **column stochastic matrix**
      - Each column sums to 1
- **Define rank vector $\boldsymbol{r}$:** a vector with one entry per page; it captures importance of the page
  - r_i = importance score of page i
  - $\sum_i r_i = 1$
- **The flow equations can be written**

$$r_j = \sum_{i \rightarrow j} \frac{r_i}{\mathrm{d_i}}$$

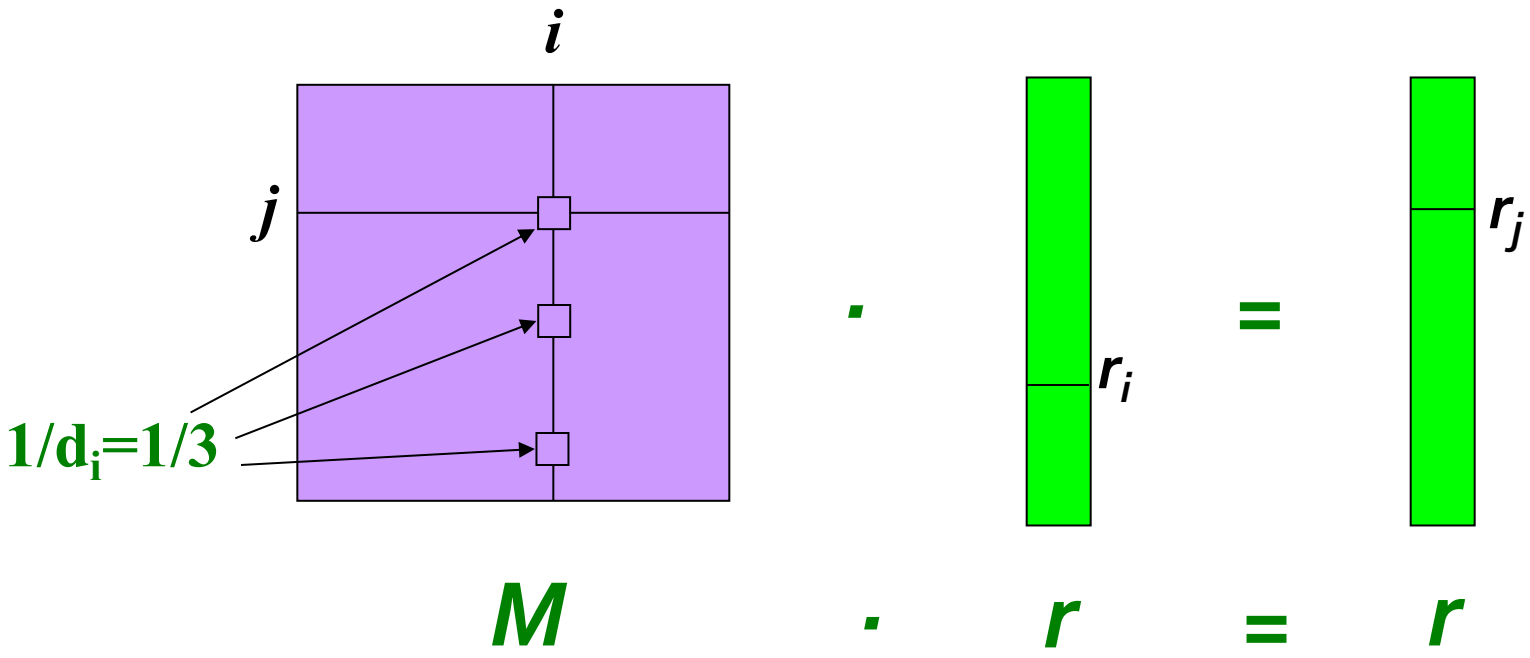
$$\boldsymbol{r} = \boldsymbol{M} \cdot \boldsymbol{r}$$

Example

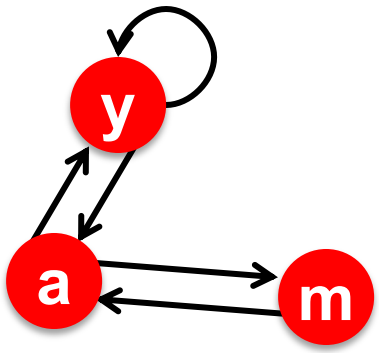
- **Remember the flow equation:** $r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d_i}}$
- **Flow equation in the matrix form**

$$M \cdot r = r$$

  - **Suppose page *i* links to 3 pages, including *j***



Example: Flow Equations & M



$$M = \begin{array}{c|ccc} & y & a & m \\ y & 1/2 & 1/2 & 0 \\ a & 1/2 & 0 & 1 \\ m & 0 & 1/2 & 0 \end{array}$$

$$r_y = r_y/2 + r_a/2$$
$$r_a = r_y/2 + r_m$$
$$r_m = r_a/2$$

$$r = M \cdot r$$

$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix}$$

Eigenvector Formulation

- **The flow equations can be written**

$$r = M \cdot r$$

- So the **rank vector** ***r*** is an **eigenvector** of the stochastic web matrix ***M***
  - In fact, its first or principal eigenvector, with corresponding eigenvalue ***1***

NOTE: ***x*** is an eigenvector with the corresponding eigenvalue **λ** if:

$$Ax = \lambda x$$

- **We can now efficiently solve for *r*!**
  The method is called Power iteration

Power Iteration Method

- **Given a web graph with *N* nodes, where the nodes are pages and edges are hyperlinks**
- **Power iteration:** a simple iterative scheme
  - Suppose there are *N* web pages
  - Initialize: $\mathbf{r}^{(0)} = [1/N,\ldots.,1/N]^T$
  - Iterate: $\mathbf{r}^{(t+1)} = \mathbf{M} \cdot \mathbf{r}^{(t)}$
  - Stop when $|\mathbf{r}^{(t+1)} - \mathbf{r}^{(t)}|_1 < \varepsilon$

    $|\mathbf{x}|_1 = \sum_{1 \le i \le N} |x_i|$ is the $\mathbf{L_1}$ norm
    So that **r** is a distribution (sums to 1)

$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

d_i …. out-degree of node i

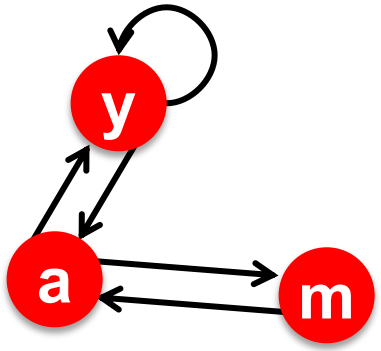
About 50 iterations is sufficient to estimate the limiting solution.

PageRank: How to solve?

- **Power Iteration:**
  - Set $r = [1/N, 1/N, 1/N]$
  - **1:** $r' = M.r$
  - **2:** $r = r'$
  - Goto **1**
- **Example:**

$$r = \begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 \\ 1/3 \\ 1/3 \end{matrix}$$

Iteration 0, 1, 2, …



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

$r_m = r_a/2$

PageRank: How to solve?

- **Power Iteration:**
  - Set $r = [1/N, 1/N, 1/N]$
  - **1:** $r' = M.r$
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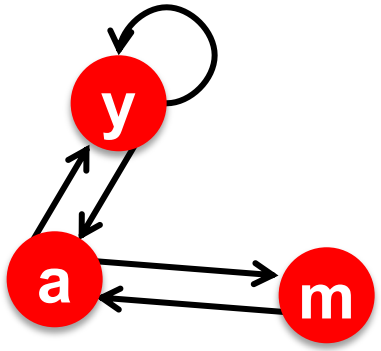
- **Example:**



	y	a	m
y	½	½	0
a	½	0	1
m	0	½	0

$r_y = r_y/2 + r_a/2$

$r_a = r_y/2 + r_m$

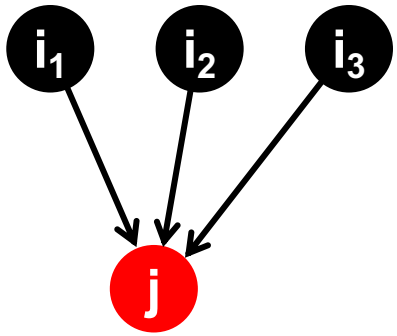
$r_m = r_a/2$

$$r = \begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 1/3 & 5/12 & 9/24 & & 6/15 \\ 1/3 & 3/6 & 1/3 & 11/24 & \ldots & 6/15 \\ 1/3 & 1/6 & 3/12 & 1/6 & & 3/15 \end{matrix}$$

Iteration 0, 1, 2, …

Random Walk Interpretation

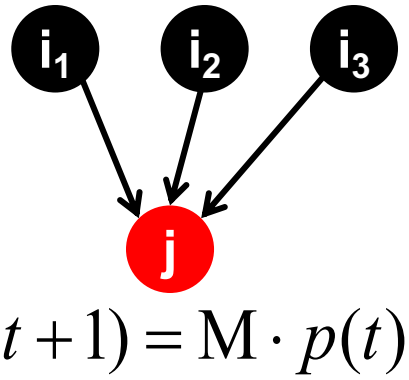
- **Imagine a random web surfer:**
  - At any time t, surfer is on some page i
  - At time $t+1$, the surfer follows an out-link from i uniformly at random
  - Ends up on some page j linked from i
  - Process repeats indefinitely



$$r_j = \sum_{i \to j} \frac{r_i}{\mathrm{d}_{\mathrm{out}}(\mathrm{i})}$$

- **Let:**
  - $p(t)$ … vector whose i^{th} coordinate is the prob. that the surfer is at page i at time t
  - So, $p(t)$ is a probability distribution over pages

The Stationary Distribution



- **Where is the surfer at time *t+1*?**
  - Follows a link uniformly at random

$$\boldsymbol{p(t+1) = M \cdot p(t)}$$

- Suppose the random walk reaches a state

$$\boldsymbol{p(t+1) = M \cdot p(t) = p(t)}$$

  then $\boldsymbol{p(t)}$ is **stationary distribution** of a random walk

- **Our original rank vector $\boldsymbol{r}$** satisfies $\boldsymbol{r = M \cdot r}$
  - **So, r is a stationary distribution for the random walk**

Existence and Uniqueness

- **A central result from the theory of random walks (a.k.a. Markov processes):**

For graphs that satisfy **certain conditions**, the **stationary distribution is unique** and eventually will be reached no matter what is the initial probability distribution at time **t = 0**

PageRank: The Google Formulation

PageRank: Three Questions

$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

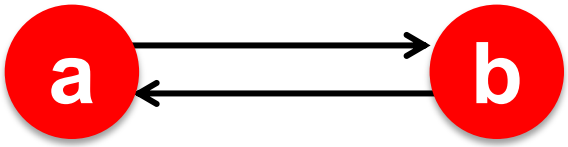
or equivalently

$$r = Mr$$

- **Does this converge?**
- **Does it converge to what we want?**
- **Are results reasonable?**

Does this converge?

$$\mathbf{M}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

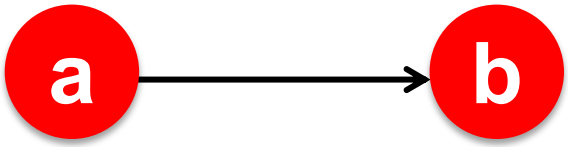
- **Example:**

$$\begin{bmatrix} r_a \\ r_b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Iteration 0, 1, 2, …

Does it converge to what we want?

$$\mathbf{M}=\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$



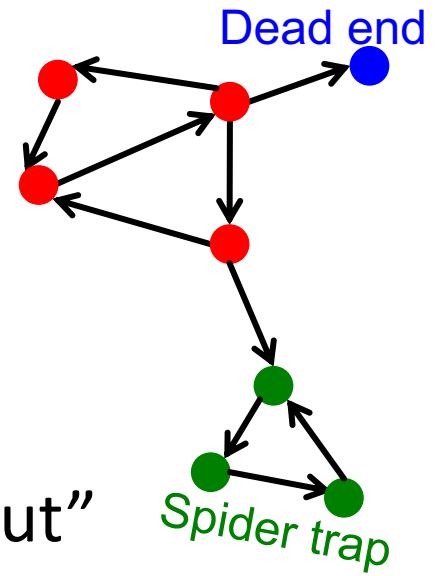
$$r_j^{(t+1)} = \sum_{i \to j} \frac{r_i^{(t)}}{\mathrm{d_i}}$$

- **Example:**

$$\begin{bmatrix} r_a \\ r_b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Iteration 0, 1, 2, …

PageRank: Problems

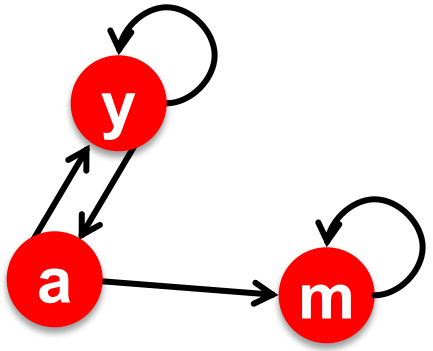


Two problems:

- **(1) Dead ends:** Some pages have no out-links
  - Random walk has "nowhere" to go to
  - Such pages cause importance to "leak out"
- **(2) Spider traps:** (all out-links are within the group)
  - Random walk gets "stuck" in a trap
  - And eventually spider traps absorb all importance

Problem: Spider Traps

- **Power Iteration:**
  - Set $r_j = 1/N$
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
    - And iterate



m is a spider trap

	y	a	m
y	½	½	0
a	½	0	0
m	0	½	1

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2$
$r_m = r_a/2 + r_m$

- **Example:**

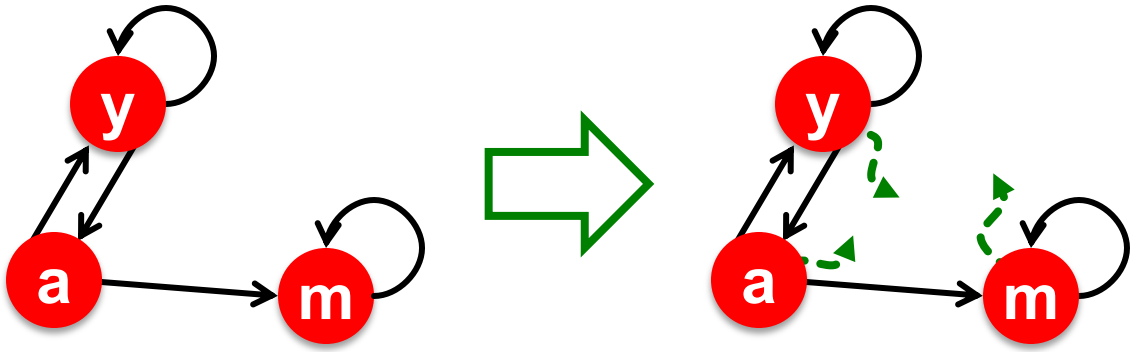
$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 2/6 & 3/12 & 5/24 & & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \dots & 0 \\ 1/3 & 3/6 & 7/12 & 16/24 & & 1 \end{matrix}$$

Iteration 0, 1, 2, …

All the PageRank score gets "trapped" in node m.

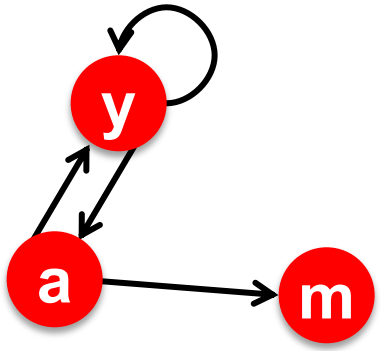
Solution: Probabilistically Teleport!

- **The Google solution for spider traps: At each time step, the random surfer has two options**
  - With prob. β, follow a link at random
  - With prob. **1-**β, jump to some random page
  - β is typically in the range 0.8 to 0.9
- **Surfer will teleport out of spider trap within a few time steps**



Problem: Dead Ends

- **Power Iteration:**
  - Set $r_j = 1/N$
  - $r_j = \sum_{i \to j} \frac{r_i}{d_i}$
    - And iterate



m is a dead end

	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

$r_y = r_y/2 + r_a/2$
$r_a = r_y/2$
$r_m = r_a/2$

- **Example:**

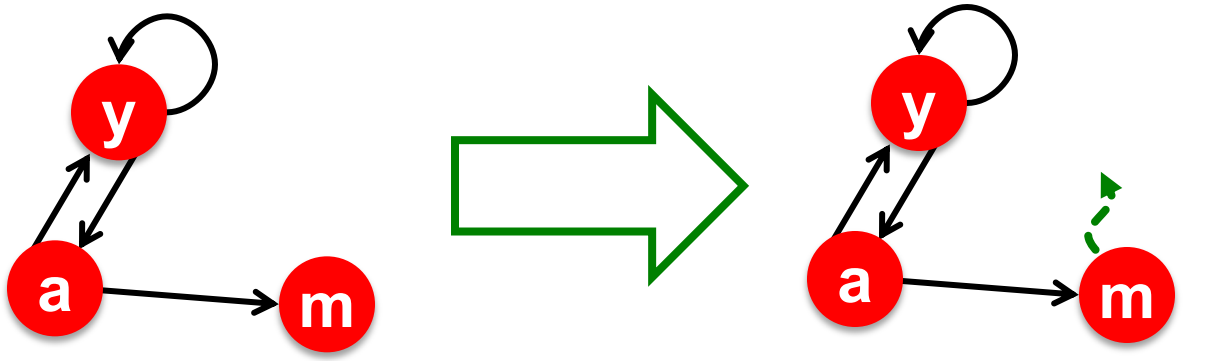
$$\begin{bmatrix} r_y \\ r_a \\ r_m \end{bmatrix} = \begin{matrix} 1/3 & 2/6 & 3/12 & 5/24 & & 0 \\ 1/3 & 1/6 & 2/12 & 3/24 & \dots & 0 \\ 1/3 & 1/6 & 1/12 & 2/24 & & 0 \end{matrix}$$

Iteration 0, 1, 2, …

Here the PageRank score "leaks" out since the matrix is not stochastic.

Solution: Always Teleport!

- **Teleports:** Follow random teleport links with probability 1.0 from dead-ends
  - Adjust matrix accordingly



	y	a	m
y	½	½	0
a	½	0	0
m	0	½	0

	y	a	m
y	½	½	**⅓**
a	½	0	**⅓**
m	0	½	**⅓**

Why Teleports Solve the Problem?

Why are dead-ends and spider traps a problem and why do teleports solve the problem?

- **Spider-traps** are not a problem, but with traps PageRank scores are **not** what we want
  - **Solution:** Never get stuck in a spider trap by teleporting out of it in a finite number of steps
- **Dead-ends** are a problem
  - The matrix is not column stochastic so our initial assumptions are not met
  - **Solution:** Make matrix column stochastic by always teleporting when there is nowhere else to go

Solution: Random Teleports

- **Google's solution that does it all:**
  At each step, random surfer has two options:
  - With probability β, follow a link at random
  - With probability $1\text{-}\beta$, jump to some random page

- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1 - \beta)\frac{1}{N}$$

d_i ... out-degree of node i

This formulation assumes that $\boldsymbol{M}$ has no dead ends. We can either preprocess matrix $\boldsymbol{M}$ to remove all dead ends or explicitly follow random teleport links with probability 1.0 from dead-ends.

The Google Matrix

- **PageRank equation** [Brin-Page, '98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1 - \beta) \frac{1}{N}$$

- **The Google Matrix *A*:**

$[1/N]_{NxN}$…N by N matrix where all entries are 1/N

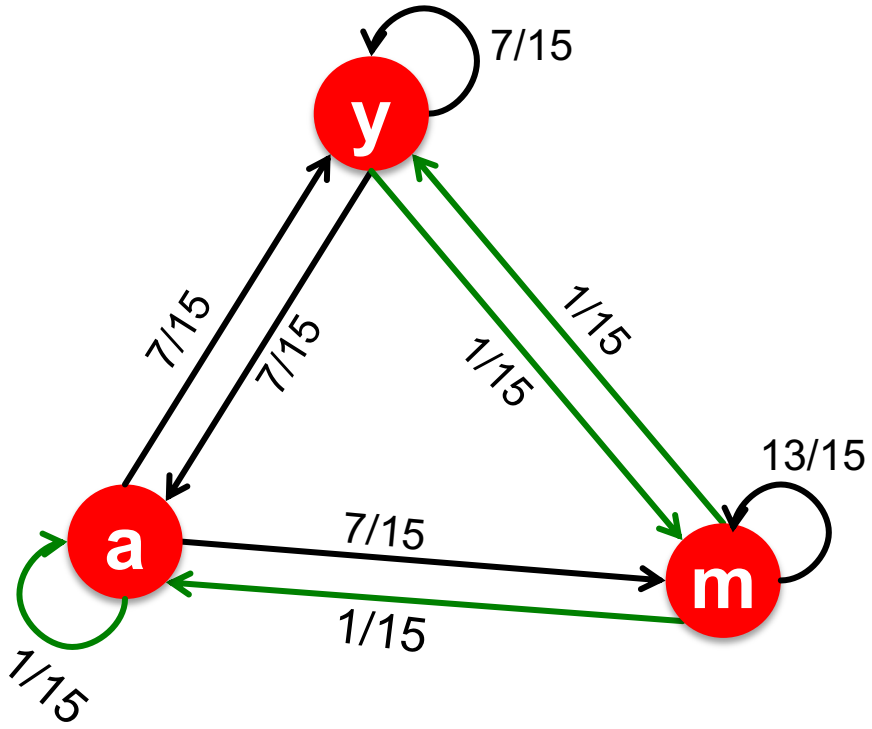
$$A = \beta\, M + (1 - \beta) \left[\frac{1}{N}\right]_{N \times N}$$

- **We have a recursive problem: $\boldsymbol{r = A \cdot r}$**

  And the Power method still works!

- **What is β?**
  - In practice $\beta = 0.8, 0.9$ (jump every 5 steps on avg.)

Random Teleports ($\beta = 0.8$)



M

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$[1/N]_{NxN}$

$$\begin{array}{c} \\ y \\ a \\ m \end{array} \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

A

$$\begin{array}{c} y \\ a \\ m \end{array} = \begin{array}{c} 1/3 \\ 1/3 \\ 1/3 \end{array} \begin{array}{c} 0.33 \\ 0.20 \\ 0.46 \end{array} \begin{array}{c} 0.24 \\ 0.20 \\ 0.52 \end{array} \begin{array}{c} 0.26 \\ 0.18 \\ 0.56 \end{array} \ldots \begin{array}{c} 7/33 \\ 5/33 \\ 21/33 \end{array}$$

How do we actually compute the PageRank?

Computing PageRank

- **Key step is matrix-vector multiplication**
  - $r^{new} = A \cdot r^{old}$
- Easy if we have enough main memory to hold **A**, $\mathbf{r}^{old}$, $\mathbf{r}^{new}$
- **Say N = 1 billion pages**

$$\mathbf{A} = \beta \cdot \mathbf{M} + (1-\beta)\,[1/N]_{NxN}$$

$$A = 0.8 \begin{bmatrix} ½ & ½ & 0 \\ ½ & 0 & 0 \\ 0 & ½ & 1 \end{bmatrix} + 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

- Matrix **A** has N^2 entries
  - 10^{18} is a large number!

$$= \begin{bmatrix} 7/15 & 7/15 & 1/15 \\ 7/15 & 1/15 & 1/15 \\ 1/15 & 7/15 & 13/15 \end{bmatrix}$$

Rearranging the Equation

- $\boldsymbol{r} = \boldsymbol{A} \cdot \boldsymbol{r}$, where $\boldsymbol{A_{ji}} = \boldsymbol{\beta}\, \boldsymbol{M_{ji}} + \frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}$
- $r_j = \sum_{i=1}^{N} A_{ji} \cdot r_i$
- $r_j = \sum_{i=1}^{N} \left[\beta\, M_{ji} + \frac{1-\beta}{N}\right] \cdot r_i$

$$= \sum_{i=1}^{N} \beta\, M_{ji} \cdot r_i + \frac{1-\beta}{N} \sum_{i=1}^{N} r_i$$

$$= \sum_{i=1}^{N} \beta\, M_{ji} \cdot r_i + \frac{1-\beta}{N}$$ since $\sum r_i = 1$

- **So we get:** $\boldsymbol{r} = \boldsymbol{\beta}\, \boldsymbol{M} \cdot \boldsymbol{r} + \left[\frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}\right]_{\boldsymbol{N}}$

Note: Here we assume **M** has no dead-ends

$[x]_N$ ... a vector of length N with all entries x

Sparse Matrix Formulation

- We just rearranged the **PageRank equation**

$$\boldsymbol{r} = \boldsymbol{\beta M} \cdot \boldsymbol{r} + \left[\frac{\boldsymbol{1-\beta}}{\boldsymbol{N}}\right]_{\boldsymbol{N}}$$

  - where $\boldsymbol{[(1-\beta)/N]_N}$ is a vector with all $\boldsymbol{N}$ entries $\boldsymbol{(1-\beta)/N}$

- $\boldsymbol{M}$ is a **sparse matrix!** (with no dead-ends)
  - 10 links per node, approx $10N$ entries
- So in each iteration, we need to:
  - Compute $\boldsymbol{r}^{new} = \beta \boldsymbol{M} \cdot \boldsymbol{r}^{old}$
  - Add a constant value **(1-β)/N** to each entry in $\boldsymbol{r}^{new}$
    - **Note if M contains dead-ends then $\sum_j r_j^{new} < 1$ and we also have to renormalize $\boldsymbol{r}^{new}$ so that it sums to 1**

PageRank: The Complete Algorithm

- **Input: Graph G and parameter β**
  - Directed graph G (can have **spider traps** and **dead ends**)
  - Parameter β
- **Output: PageRank vector r^{new}**
  - **Set:** $r_j^{old} = \frac{1}{N}$
  - **repeat until convergence:** $\sum_j \left| r_j^{new} - r_j^{old} \right| < \varepsilon$
    - $\forall j$: $r'^{new}_j = \sum_{i \to j} \beta \frac{r_i^{old}}{d_i}$

      $r'^{new}_j = 0$ if in-degree of j is 0
    - **Now re-insert the leaked PageRank:**

      $\forall j$: $r_j^{new} = r'^{new}_j + \frac{1-S}{N}$ **where:** $S = \sum_j r'^{new}_j$
    - $r^{old} = r^{new}$

If the graph has no dead-ends then the amount of leaked PageRank is **1-β**. But since we have dead-ends the amount of leaked PageRank may be larger. We have to explicitly account for it by computing **S**.

Sparse Matrix Encoding

- **Encode sparse matrix using only nonzero entries**
  - Space proportional roughly to number of links
  - Say 10N, or 4*10*1 billion = 40GB
  - **Still won't fit in memory, but will fit on disk**

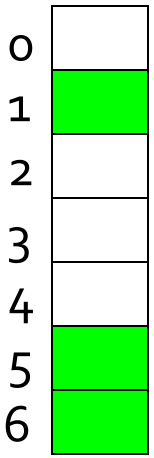
source node	degree	destination nodes
0	3	1, 5, 7
1	5	17, 64, 113, 117, 245
2	2	13, 23

Basic Algorithm: Update Step

- **Assume enough RAM to fit r^{new} into memory**
  - Store r^{old} and matrix **M** on disk
- **1 step of power-iteration is:**

Initialize all entries of $r^{new} = (1-\beta) / N$
For each page i (of out-degree d_i):
  Read into memory: i, d_i, $dest_1, \ldots, dest_{d_i}$, $r^{old}(i)$
  For $j = 1 \ldots d_i$
    $r^{new}(dest_j) \mathrel{+}= \beta\, r^{old}(i) / d_i$

Assuming no dead ends



source	degree	destination
0	3	1, 5, 6
1	4	17, 64, 113, 117
2	2	13, 23