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# ECCCoS from the Black Box: Letting Models speak for Themselves

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## Abstract

Counterfactual Explanations offer an intuitive and straightforward way to explain Black Box Models but they are not unique. To address the need for plausible explanations, existing work has primarily relied on surrogate models to learn how the input data is distributed. This effectively reallocates the task of learning realistic representations of the data from the model itself to the surrogate. Consequently, the generated explanations may look plausible to humans but not necessarily conform with the behaviour of the Black Box Model. We formalise this notion of model conformity through the introduction of tailored evaluation measures and propose a novel algorithmic framework for generating **Energy-Constrained Conformal Counterfactuals** that are only as plausible as the model permits. To do so, **ECCCo** leverages recent advances in energy-based modelling and predictive uncertainty quantification through conformal inference. Through illustrative examples and extensive empirical studies, we demonstrate that ECCCoS reconcile the need for plausibility and model conformity.

## 1 Introduction

Counterfactual Explanations provide a powerful, flexible and intuitive way to not only explain Black Box Models but also enable affected individuals to challenge them through the means of Algorithmic Recourse. Instead of opening the black box, Counterfactual Explanations work under the premise of strategically perturbing model inputs to understand model behaviour [29]. Intuitively speaking, we generate explanations in this context by asking simple what-if questions of the following nature: ‘Our credit risk model currently predicts that this individual’s credit profile is too risky to offer them a loan. What if they reduced their monthly expenditures by 10%? Will our model then predict that the individual is credit-worthy?’

This is typically implemented by defining a target outcome  $\mathbf{y}^* \in \mathcal{Y}$  for some individual  $\mathbf{x} \in \mathcal{X} = \mathbb{R}^D$  described by  $D$  attributes, for which the model  $M_\theta : \mathcal{X} \mapsto \mathcal{Y}$  initially predicts a different outcome:  $M_\theta(\mathbf{x}) \neq \mathbf{y}^*$ . Counterfactuals are then searched by minimizing a loss function that compares the predicted model output to the target outcome:  $\text{yloss}(M_\theta(\mathbf{x}), \mathbf{y}^*)$ . Since Counterfactual Explanations (CE) work directly with the Black Box Model, valid counterfactuals always have full local fidelity by construction [17]. Fidelity is defined as the degree to which explanations approximate the predictions of the Black Box Model. This is arguably one of the most important evaluation metrics for model explanations, since any explanation that explains a prediction not actually made by the model is useless [16].

In situations where full fidelity is a requirement, CE therefore offers a more appropriate solution to Explainable Artificial Intelligence (XAI) than other popular approaches like LIME [22] and SHAP [12], which involve local surrogate models. But even full fidelity is not a sufficient condition for ensuring that an explanation adequately describes the behaviour of a model. That is because two

37 very distinct explanations can both lead to the same model prediction, especially when dealing with  
 38 heavily parameterized models:

39           [. . .] deep neural networks are typically very underspecified by the available  
 40           data, and [. . .] parameters [therefore] correspond to a diverse variety of compelling  
 41           explanations for the data. — Wilson [30]

42 When people talk about Black Box Models, this is usually the type of model they have in mind.

43 In the context of CE, the idea that no two explanations are the same arises almost naturally. Even  
 44 the baseline approach proposed by Wachter et al. [29] can yield a diverse set of explanations  
 45 if counterfactuals are initialised randomly. This multiplicity of explanations has not only been  
 46 acknowledged in the literature but positively embraced: since individuals seeking Algorithmic  
 47 Recourse (AR) have unique preferences, Mothilal et al. [17], for example, have prescribed *diversity*  
 48 as an explicit goal for counterfactuals. More generally, the literature on CE and AR has brought  
 49 forward a myriad of desiderata for explanations, which we will discuss in more detail in the following  
 50 section.

## 51 2 Background and Related Work

52 In this section, we provide some background on Counterfactual Explanations and our motivation for  
 53 this work. To start off, we briefly introduce the methodology underlying most state-of-the-art (SOTA)  
 54 counterfactual generators.

### 55 2.1 Gradient-Based Counterfactual Search

56 While Counterfactual Explanations can be generated for arbitrary regression models [24], existing  
 57 work has primarily focused on classification problems. Let  $\mathcal{Y} = (0, 1)^K$  denote the one-hot-encoded  
 58 output domain with  $K$  classes. Then most SOTA counterfactual generators rely on gradient descent  
 59 to optimize different flavours of the following counterfactual search objective:

$$\mathbf{Z}' = \arg \min_{\mathbf{Z}' \in \mathcal{Z}^M} \{ \text{yloss}(M_\theta(f(\mathbf{Z}')), \mathbf{y}^*) + \lambda \text{cost}(f(\mathbf{Z}')) \} \quad (1)$$

60 Here  $\text{yloss}$  denotes the primary loss function already introduced above and  $\text{cost}$  is either a single  
 61 penalty or a collection of penalties that are used to impose constraints through regularization. Fol-  
 62 lowing the convention in Altmeyer et al. [2] we use  $\mathbf{Z}' = \{\mathbf{z}_m\}_M$  to denote the  $M$ -dimensional  
 63 array of counterfactual states. This is to explicitly account for the fact that we can generate multiple  
 64 counterfactuals  $M$ , as with DiCE [17], and may choose to traverse a latent encoding  $\mathcal{Z}$  of the feature  
 65 space  $\mathcal{X}$  where we denote  $f^{-1} : \mathcal{X} \mapsto \mathcal{Z}$ . Encodings may involve simple feature transformations or  
 66 more advanced techniques involving generative models, as we will discuss further below.

67 Solutions to Equation 1 are considered valid as soon as the predicted label matches the target label. A  
 68 stripped-down counterfactual explanation is therefore little different from an adversarial example.  
 69 In Figure 1, for example, we have the baseline approach proposed in Wachter et al. [29] to MNIST  
 70 data (centre panel). This approach solves Equation 1 through gradient-descent in the feature space  
 71 with a penalty for the distance between the factual  $\mathbf{x}$  and the counterfactual  $\mathbf{x}'$ . The underlying  
 72 classifier  $M_\theta$  is a simple Multi-Layer Perceptron (MLP) with good test accuracy. For the generated  
 73 counterfactual  $\mathbf{x}'$  the model predicts the target label with high confidence (centre panel in Figure 1).  
 74 The explanation is valid by definition, even though it looks a lot like an Adversarial Example [6].  
 75 Schut et al. [23] make the connection between Adversarial Examples and Counterfactual Explanations  
 76 explicit and propose using a Jacobian-Based Saliency Map Attack (JSMA) to solve Equation 1. They  
 77 demonstrate that this approach yields realistic and sparse counterfactuals for Bayesian, adversarially  
 78 robust classifiers. Applying their approach to our simple MNIST classifier does not yield a realistic  
 79 counterfactual but this one, too, is valid (right panel in Figure 1).

### 80 2.2 From Adversarial Examples to Plausible Explanations

81 The crucial difference between Adversarial Examples (AE) and Counterfactual Explanations is one  
 82 of intent. While an AE is intended to go unnoticed, a CE should have certain desirable properties.

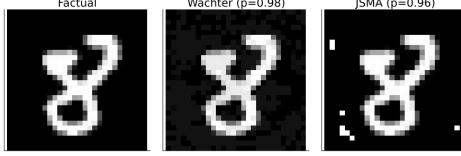


Figure 1: You may not like it, but this is what stripped-down counterfactuals look like. Counterfactuals for turning an 8 (eight) into a 3 (three): original image (left); counterfactual produced using Wachter et al. [29] (centre); and a counterfactual produced using JSMA-based approach introduced by [23].

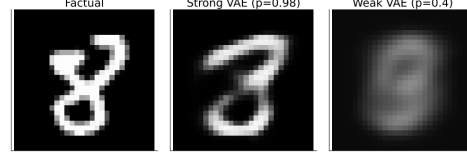


Figure 2: Using surrogates can improve plausibility, but also increases vulnerability. Counterfactuals for turning an 8 (eight) into a 3 (three): original image (left); counterfactual produced using REVISE [9] with a well-specified surrogate (centre); and a counterfactual produced using REVISE [9] with a poorly specified surrogate (right).

83 The literature has made this explicit by introducing various so-called *desiderata*. To properly serve  
 84 both AI practitioners and individuals affected by AI decision-making systems, counterfactuals should  
 85 be sparse, proximate [29], actionable [27], diverse [17], plausible [9, 21, 23], robust [26, 20, 2] and  
 86 causal [11] among other things.

87 Researchers have come up with various ways to meet these desiderata, which have been extensively  
 88 surveyed and evaluated in various studies [28, 10, 19, 4, 8]. Perhaps unsurprisingly, the different  
 89 desiderata are often positively correlated. For example, Artelt et al. [4] find that plausibility typically  
 90 also leads to improved robustness. Similarly, plausibility has also been connected to causality in the  
 91 sense that plausible counterfactuals respect causal relationships [13].

### 92 2.2.1 Plausibility through Surrogates

93 Arguably, the plausibility of counterfactuals has been among the primary concerns and some have  
 94 focused explicitly on this goal. Joshi et al. [9], for example, were among the first to suggest that  
 95 instead of searching counterfactuals in the feature space  $\mathcal{X}$ , we can instead traverse a latent embedding  
 96  $\mathcal{Z}$  that implicitly codifies the data generating process (DGP) of  $\mathbf{x} \sim \mathcal{X}$ . To learn the latent embedding,  
 97 they introduce a surrogate model. In particular, they propose to use the latent embedding of a  
 98 Variational Autoencoder (VAE) trained to generate samples  $\mathbf{x}^* \leftarrow \mathcal{G}(\mathbf{z})$  where  $\mathcal{G}$  denotes the decoder  
 99 part of the VAE. Provided the surrogate model is well-trained, their proposed approach —REVISE—  
 100 can yield compelling counterfactual explanations like the one in the centre panel of Figure 2.

101 Others have proposed similar approaches. Dombrowski et al. [5] traverse the base space of a  
 102 normalizing flow to solve Equation 1, essentially relying on a different surrogate model for the  
 103 generative task. Poyiadzi et al. [21] use density estimators ( $\hat{p} : \mathcal{X} \mapsto [0, 1]$ ) to constrain the  
 104 counterfactual paths. Karimi et al. [11] argue that counterfactuals should comply with the causal  
 105 model that generates the data. All of these different approaches share a common goal: ensuring that  
 106 the generated counterfactuals comply with the true and unobserved DGP. To summarize this broad  
 107 objective, we propose the following definition:

108 **Definition 2.1** (Plausible Counterfactuals). *Let  $\mathcal{X}|\mathbf{y}^*$  denote the true conditional distribution of*  
 109 *samples in the target class  $\mathbf{y}^*$ . Then for  $\mathbf{x}'$  to be considered a plausible counterfactual, we need:*  
 110  *$\mathbf{x}' \sim \mathcal{X}|\mathbf{y}^*$ .*

111 Note that Definition 2.1 is consistent with the notion of plausible counterfactual paths, since we can  
 112 simply apply it to each counterfactual state along the path.

113 Surrogate models offer an obvious solution to achieve this objective. Unfortunately, surrogates also  
 114 introduce a dependency: the generated explanations no longer depend exclusively on the Black Box  
 115 Model itself, but also on the surrogate model. This is not necessarily problematic if the primary  
 116 objective is not to explain the behaviour of the model but to offer recourse to individuals affected by  
 117 it. It may become problematic even in this context if the dependency turns into a vulnerability. To  
 118 illustrate this point, we have used REVISE [9] with an underfitted VAE to generate the counterfactual  
 119 in the right panel of Figure 2: in this case, the decoder step of the VAE fails to yield plausible values  
 120 ( $\{\mathbf{x}' \leftarrow \mathcal{G}(\mathbf{z})\} \not\sim \mathcal{X}|\mathbf{y}^*$ ) and hence the counterfactual search in the learned latent space is doomed.

### 2.2.2 Plausibility through Minimal Predictive Uncertainty

Schut et al. [23] show that to meet the plausibility objective we need not explicitly model the input distribution. Pointing to the undesirable engineering overhead induced by surrogate models, they propose that we rely on the implicit minimisation of predictive uncertainty instead. Their proposed methodology solves Equation 1 by greedily applying JSMA in the feature space with standard cross-entropy loss and no penalty at all. They demonstrate theoretically and empirically that their approach yields counterfactuals for which the model  $M_\theta$  predicts the target label  $\mathbf{y}^*$  with high confidence. Provided the model is well-specified, these counterfactuals are plausible. Unfortunately, this idea hinges on the assumption that the Black Box Model provides well-calibrated predictive uncertainty estimates.

## 2.3 From Fidelity to Model Conformity

Above we explained that since Counterfactual Explanations work directly with the Black Box model, the fidelity of explanations as we defined it earlier is not a concern. This may explain why research has primarily focused on other desiderata, most notably plausibility (Definition 2.1). Enquiring about the plausibility of a counterfactual essentially boils down to the following question: ‘Is this counterfactual consistent with the underlying data’? To introduce this section, we posit a related, slightly more nuanced question: ‘Is this counterfactual consistent with what the model has learned about the underlying data’? We will argue that fidelity is not a sufficient evaluation measure to answer this question and propose a novel way to assess if Counterfactual Explanations conform with model behaviour.

The word *fidelity* stems from the Latin word ‘fidelis’, which means ‘faithful, loyal, trustworthy’ [15]. As we explained in Section 2, model explanations are generally considered faithful if their corresponding predictions coincide with the predictions made by the model itself. Since this definition of faithfulness is not useful in the context of Counterfactual Explanations, we propose an adapted version:

**Definition 2.2** (Conformal Counterfactuals). *Let  $\mathcal{X}_\theta|\mathbf{y}^* = p_\theta(x|\mathbf{y}^*)$  denote the conditional distribution of  $\mathbf{x}$  in the target class  $\mathbf{y}^*$ , where  $\theta$  denotes the parameters of model  $M_\theta$ . Then for  $\mathbf{x}'$  to be considered a conformal counterfactual, we need:  $\mathbf{x}' \sim \mathcal{X}_\theta|\mathbf{y}^*$ .*

In words, conformal counterfactuals conform with what the predictive model has learned about the input data  $\mathbf{x}$ . Since this definition works with distributional properties, it explicitly accounts for the multiplicity of explanations we discussed earlier. To assess counterfactuals with respect to Definition 2.2, we need to be able to quantify the posterior conditional distribution  $p_\theta(\mathbf{x}|\mathbf{y}^*)$ . This is very much at the core of our proposed methodological framework, which reconciles the notions of plausibility and model conformity and which we will introduce next.

## 3 Methodological Framework

The primary objective of this work has been to develop a methodology for generating maximally plausible counterfactuals under minimal intervention. Our proposed framework is based on the premise that explanations should be plausible but not plausible at all costs. Energy-Constrained Conformal Counterfactuals (ECCCo) achieve this goal in two ways: firstly, they rely on the Black Box itself for the generative task; and, secondly, they involve an approach to predictive uncertainty quantification that is model-agnostic.

### 3.1 Quantifying the Model’s Generative Property

Recent work by Grathwohl et al. [7] on Energy Based Models (EBM) has pointed out that there is a ‘generative model hidden within every standard discriminative model’. The authors show that we can draw samples from the posterior conditional distribution  $p_\theta(\mathbf{x}|\mathbf{y})$  using Stochastic Gradient Langevin Dynamics (SGLD). The authors use this insight to train classifiers jointly for the discriminative task using standard cross-entropy and the generative task using SGLD. They demonstrate empirically that among other things this improves predictive uncertainty quantification for discriminative models. Our findings in this work suggest that Joint Energy Models (JEM) also tend to yield more plausible

Counterfactual Explanations. Based on the definition of plausible counterfactuals (Definition 2.1) this is not surprising.

Crucially for our purpose, one can apply their proposed sampling strategy during inference to essentially any standard discriminative model. Even models that are not explicitly trained for the joint objective learn about the distribution of inputs  $X$  by learning to make conditional predictions about the output  $y$ . We can leverage this observation to quantify the generative property of the Black Box model itself. In particular, note that if we fix  $y$  to our target value  $y^*$ , we can sample from  $p_\theta(\mathbf{x}|\mathbf{y}^*)$  using SGLD as follows,

$$\mathbf{x}_{j+1} \leftarrow \mathbf{x}_j - \frac{\epsilon^2}{2} \mathcal{E}(\mathbf{x}_j|\mathbf{y}^*) + \epsilon \mathbf{r}_j, \quad j = 1, \dots, J \quad (2)$$

where  $\mathbf{r}_j \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$  is the stochastic term and the step-size  $\epsilon$  is typically polynomially decayed. The term  $\mathcal{E}(\mathbf{x}_j|\mathbf{y}^*)$  denotes the energy function where we use  $\mathcal{E}(\mathbf{x}_j|\mathbf{y}^*) = -M_\theta(\mathbf{x}_j)[\mathbf{y}^*]$ , that is the negative logit corresponding to the target class label  $y^*$ . Generating multiple samples in this manner yields an empirical distribution  $\hat{\mathcal{X}}_\theta|\mathbf{y}^*$  that we use in our search for plausible counterfactuals, as discussed in more detail below. Appendix A provides additional implementation details for any tasks related to energy-based modelling.

### 3.2 Quantifying the Model’s Predictive Uncertainty

To quantify the model’s predictive uncertainty we use Conformal Prediction (CP), an approach that has recently gained popularity in the Machine Learning community [3, 14]. Crucially for our intended application, CP is model-agnostic and can be applied during inference without placing any restrictions on model training. Intuitively, CP works under the premise of turning heuristic notions of uncertainty into rigorous uncertainty estimates by repeatedly sifting through the training data or a dedicated calibration dataset. Conformal classifiers produce prediction sets for individual inputs that include all output labels that can be reasonably attributed to the input. These sets tend to be larger for inputs that do not conform with the training data and are therefore characterized by high predictive uncertainty.

In order to generate counterfactuals that are associated with low predictive uncertainty, we use a smooth set size penalty introduced by Stutz et al. [25] in the context of conformal training:

$$\Omega(C_\theta(\mathbf{x}; \alpha)) = \max \left( 0, \sum_{\mathbf{y} \in \mathcal{Y}} C_{\theta, \mathbf{y}}(\mathbf{x}_i; \alpha) - \kappa \right) \quad (3)$$

Here,  $\kappa \in \{0, 1\}$  is a hyper-parameter and  $C_{\theta, \mathbf{y}}(\mathbf{x}_i; \alpha)$  can be interpreted as the probability of label  $y$  being included in the prediction set.

In order to compute this penalty for any Black Box Model we merely need to perform a single calibration pass through a holdout set  $\mathcal{D}_{\text{cal}}$ . Arguably, data is typically abundant and in most applications, practitioners tend to hold out a test data set anyway. Consequently, CP removes the restriction on the family of predictive models, at the small cost of reserving a subset of the available data for calibration. This particular case of conformal prediction is referred to as Split Conformal Prediction (SCP) as it involves splitting the training data into a proper training dataset and a calibration dataset. Details concerning our implementation of Conformal Prediction can be found in Appendix B.

### 3.3 Energy-Constrained Conformal Counterfactuals (ECCCo)

Our framework for generating ECCCos combines the ideas introduced in the previous two subsections. Formally, we extend Equation 1 as follows,

$$\begin{aligned} \mathbf{Z}' = \arg \min_{\mathbf{Z}' \in \mathcal{Z}^M} \{ & \text{yloss}(M_\theta(f(\mathbf{Z}')), \mathbf{y}^*) + \lambda_1 \text{dist}(f(\mathbf{Z}'), \mathbf{x}) \\ & + \lambda_2 \text{dist}(f(\mathbf{Z}'), \hat{\mathbf{x}}_\theta) + \lambda_3 \Omega(C_\theta(f(\mathbf{Z}'); \alpha)) \} \end{aligned} \quad (4)$$

where  $\hat{\mathbf{x}}_\theta$  denotes samples generated using SGLD (Equation 2) and  $\text{dist}(\cdot)$  is a generic term for a distance metric. Our default choice for  $\text{dist}(\cdot)$  is the Manhattan Distance since it enforces sparsity.

209 The first two terms in Equation 4 correspond to the counterfactual search objective defined in Wachter  
 210 et al. [29] which merely penalises the distance of counterfactuals from their factual values. The  
 211 additional two penalties in ECCCo ensure that counterfactuals conform with the model’s generative  
 212 property and lead to minimally uncertain predictions, respectively. The hyperparameters  $\lambda_1, \dots, \lambda_3$   
 213 can be used to balance the different objectives: for example, we may choose to incur larger deviations  
 214 from the factual in favour of conformity with the model’s generative property by choosing lower  
 215 values of  $\lambda_1$  and relatively higher values of  $\lambda_2$ . Figure 3 illustrates this balancing act for an example  
 216 involving synthetic data: vector fields indicate the direction of gradients with respect to the different  
 217 components our proposed objective function (Equation 4).

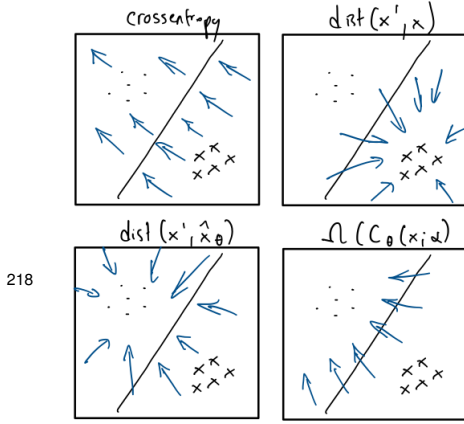


Figure 3: Vector fields indicating the direction of gradients with respect to the different components of the ECCCo objective (Equation 4).

Algorithm 1: Generating ECCCos (For more details, see Appendix C)

**Input:**  $\mathbf{x}, \mathbf{y}^*, M_\theta, f, \Lambda, \alpha, \mathcal{D}, T, \eta, n_B, N_B$   
 where  $M_\theta(\mathbf{x}) \neq \mathbf{y}^*$   
**Output:**  $\mathbf{x}'$   
 1: Initialize  $\mathbf{z}' \leftarrow f^{-1}(\mathbf{x})$   
 2: Generate buffer  $\mathcal{B}$  of  $N_B$  conditional samples  $\hat{\mathbf{x}}_\theta | \mathbf{y}^*$  using SGLD (Equation 2)  
 3: Run SCP for  $M_\theta$  using  $\mathcal{D}$   
 4: Initialize  $t \leftarrow 0$   
 5: **while** not converged or  $t < T$  **do**  
 6:    $\hat{\mathbf{x}}_{\theta,t} \leftarrow \text{rand}(\mathcal{B}, n_B)$   
 7:    $\mathbf{z}' \leftarrow \mathbf{z}' - \eta \nabla_{\mathbf{z}'} \mathcal{L}(\mathbf{z}', \mathbf{y}^*, \hat{\mathbf{x}}_{\theta,t})$   
 8:    $t \leftarrow t + 1$   
 9: **end while**  
 10:  $\mathbf{x}' \leftarrow f(\mathbf{z}')$

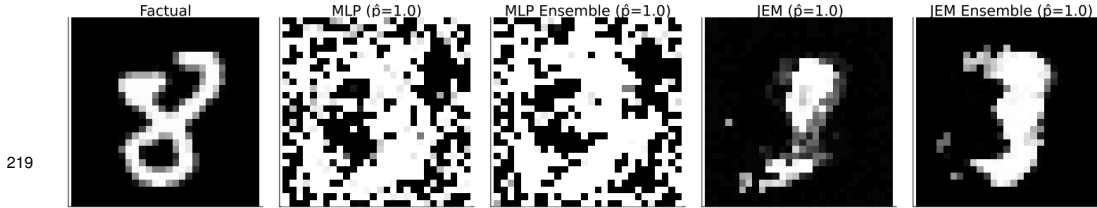


Figure 4: Original image (left) and ECCCos for turning an 8 (eight) into a 3 (three) for different Black Boxes from left to right: Multi-Layer Perceptron (MLP), Ensemble of MLPs, Joint Energy Model (JEM), Ensemble of JEMs.

220 The entire procedure for Generating ECCCos is described in Algorithm 1. For the sake of simplicity  
 221 and without loss of generality, we limit our attention to generating a single counterfactual  $\mathbf{x}' = f(\mathbf{z}')$   
 222 where in contrast to Equation 4  $\mathbf{z}'$  denotes a 1-dimensional array containing a single counterfactual  
 223 state. That state is initialized by passing the factual  $\mathbf{x}$  through the encoder  $f^{-1}$  which in our case cor-  
 224 responds to a simple feature transformer, rather than the encoder part of VAE as in REVISE [9]. Next,  
 225 we generate a buffer of  $N_B$  conditional samples  $\hat{\mathbf{x}}_\theta | \mathbf{y}^*$  using SGLD (Equation 2) and conformalise  
 226 the model  $M_\theta$  through Split Conformal Prediction on training data  $\mathcal{D}$ .

227 Finally, we search counterfactuals through gradient descent. Let  $\mathcal{L}(\mathbf{z}', \mathbf{y}^*, \hat{\mathbf{x}}_{\theta,t})$  denote our loss  
 228 function defined in Equation 4. Then in each iteration, we first randomly draw  $n_B$  samples from  
 229 the buffer  $\mathcal{B}$  before updating the counterfactual state  $\mathbf{z}'$  by moving in the negative direction of that  
 230 loss function. The search terminates once the convergence criterium is met or the maximum number  
 231 of iterations  $T$  has been exhausted. Note that the choice of convergence criterium has important  
 232 implications on the final counterfactual. For more detail on this see Appendix C).

233 Figure 4 presents ECCCos for the MNIST example from Section 2 for various Black Box models of  
 234 increasing complexity from left to right: a simple Multi-Layer Perceptron (MLP); an Ensemble of  
 235 MLPs, each of the same architecture as the single MLP; a Joint Energy Model (JEM) based on the

236 same MLP architecture; and finally, an Ensemble of these JEMs. Since Deep Ensembles have an  
 237 improved capacity for predictive uncertainty quantification and JEMs are explicitly trained to learn  
 238 plausible representations of the input data, it is intuitive to see that the plausibility of counterfactuals  
 239 visibly improves from left to right.

## 240 4 Experiments

### 241 4.1 Evaluation Measures

242 Above we have defined plausibility (2.1) and conformity (2.2) for Counterfactual Explanations.  
 243 In this subsection, we introduce evaluation measures that facilitate a quantitative evaluation of  
 244 counterfactuals for these objectives.

245 Firstly, in order to assess the plausibility of counterfactuals we adapt the implausibility metric  
 246 proposed in Guidotti [8]. The authors propose to evaluate plausibility in terms of the distance of the  
 247 counterfactual  $\mathbf{x}'$  from its nearest neighbour in the target class  $\mathbf{y}^*$ : the smaller this distance, the more  
 248 plausible the counterfactual. Instead of focusing only on the nearest neighbour of  $\mathbf{x}'$ , we suggest  
 249 computing the average over distances from multiple (possibly all) observed instances in the target  
 250 class. Formally, for a single counterfactual, we have:

$$\text{impl} = \frac{1}{|\mathbf{x} \in \mathcal{X}|\mathbf{y}^*|} \sum_{\mathbf{x} \in \mathcal{X}|\mathbf{y}^*} \text{dist}(\mathbf{x}', \mathbf{x}) \quad (5)$$

251 This measure is straightforward to compute and should be less sensitive to outliers in the target class  
 252 than the one based on the nearest neighbour. It also gives rise to a very similar evaluation measure for  
 253 conformity. We merely swap out the subsample of individuals in the target class for the empirical  
 254 distribution of generated conditional samples:

$$\text{conf} = \frac{1}{|\mathbf{x} \in \mathcal{X}_\theta|\mathbf{y}^*|} \sum_{\mathbf{x} \in \mathcal{X}_\theta|\mathbf{y}^*} \text{dist}(\mathbf{x}', \mathbf{x}) \quad (6)$$

255 As noted by Guidotti [8], these distance-based measures are simplistic and more complex alternative  
 256 measures may ultimately be more appropriate for the task. For example, we considered using statisti-  
 257 cal divergence measures instead. This would involve generating not one but many counterfactuals and  
 258 comparing the generated empirical distribution to the target distributions in Definitions 2.1 and 2.2.  
 259 While this approach is potentially more rigorous, generating enough counterfactuals is not always  
 260 practical.

## 261 5 Experiments

- 262 • BatchNorm does not seem compatible with JEM
- 263 • Coverage and temperature impacts CCE in somewhat unpredictable ways
- 264 • It seems that models that are not explicitly trained for generative task, still learn it implicitly
- 265 • Batch size seems to impact quality of generated samples (at inference, but not so much  
 266 during JEM training)
- 267 • ECCCo is sensitive to optimizer (Adam works well), learning rate and distance metric (11  
 268 currently only one that works)
- 269 • SGLD takes time
- 270 • REVISE has benefit of lower dimensional space
- 271 • For MNIST it seems that ECCCo is better at reducing pixel values than increasing them  
 272 (better at erasing than writing)

## 6 Discussion

Consistent with the findings in Schut et al. [23], we have demonstrated that predictive uncertainty estimates can be leveraged to generate plausible counterfactuals. Interestingly, Schut et al. [23] point out that this finding — as intuitive as it is — may be linked to a positive connection between the generative task and predictive uncertainty quantification. In particular, Grathwohl et al. [7] demonstrate that their proposed method for integrating the generative objective in training yields models that have improved predictive uncertainty quantification. Since neither Schut et al. [23] nor we have employed any surrogate generative models, our findings seem to indicate that the positive connection found in Grathwohl et al. [7] is bidirectional.

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## Appendices

### A JEM

While  $\mathbf{x}_J$  is only guaranteed to distribute as  $p_\theta(\mathbf{x}|\mathbf{y}^*)$  if  $\epsilon \rightarrow 0$  and  $J \rightarrow \infty$ , the bias introduced for a small finite  $\epsilon$  is negligible in practice [18, 7]. While Grathwohl et al. [7] use Equation 2 during training, we are interested in applying the conditional sampling procedure in a post hoc fashion to any standard discriminative model.

## 363 B Conformal Prediction

364 The fact that conformal classifiers produce set-valued predictions introduces a challenge: it is not  
 365 immediately obvious how to use such classifiers in the context of gradient-based counterfactual  
 366 search. Put differently, it is not clear how to use prediction sets in Equation 1. Fortunately, Stutz et al.  
 367 [25] have recently proposed a framework for Conformal Training that also hinges on differentiability.  
 368 Specifically, they show how Stochastic Gradient Descent can be used to train classifiers not only  
 369 for the discriminative task but also for additional objectives related to Conformal Prediction. One  
 370 such objective is *efficiency*: for a given target error rate  $\alpha$ , the efficiency of a conformal classifier  
 371 improves as its average prediction set size decreases. To this end, the authors introduce a smooth set  
 372 size penalty defined in Equation 3

373 Formally, it is defined as  $C_{\theta, \mathbf{y}}(\mathbf{x}_i; \alpha) := \sigma((s(\mathbf{x}_i, \mathbf{y}) - \alpha)T^{-1})$  for  $\mathbf{y} \in \mathcal{Y}$  where  $\sigma$  is the sigmoid  
 374 function and  $T$  is a hyper-parameter used for temperature scaling [25].

375 Intuitively, CP works under the premise of turning heuristic notions of uncertainty into rigorous  
 376 uncertainty estimates by repeatedly sifting through the data. It can be used to generate prediction  
 377 intervals for regression models and prediction sets for classification models [1]. Since the literature  
 378 on CE and AR is typically concerned with classification problems, we focus on the latter. A particular  
 379 variant of CP called Split Conformal Prediction (SCP) is well-suited for our purposes because it  
 380 imposes only minimal restrictions on model training.

381 Specifically, SCP involves splitting the data  $\mathcal{D}_n = \{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=1, \dots, n}$  into a proper training set  $\mathcal{D}_{\text{train}}$   
 382 and a calibration set  $\mathcal{D}_{\text{cal}}$ . The former is used to train the classifier in any conventional fashion.  
 383 The latter is then used to compute so-called nonconformity scores:  $\mathcal{S} = \{s(\mathbf{x}_i, \mathbf{y}_i)\}_{i \in \mathcal{D}_{\text{cal}}}$  where  
 384  $s : (\mathcal{X}, \mathcal{Y}) \mapsto \mathbb{R}$  is referred to as *score function*. In the context of classification, a common choice for  
 385 the score function is just  $s_i = 1 - M_{\theta}(\mathbf{x}_i)[\mathbf{y}_i]$ , that is one minus the softmax output corresponding  
 386 to the observed label  $\mathbf{y}_i$  [3].

387 Finally, classification sets are formed as follows,

$$C_{\theta}(\mathbf{x}_i; \alpha) = \{\mathbf{y} : s(\mathbf{x}_i, \mathbf{y}) \leq \hat{q}\} \quad (7)$$

388 where  $\hat{q}$  denotes the  $(1 - \alpha)$ -quantile of  $\mathcal{S}$  and  $\alpha$  is a predetermined error rate. As the size of the  
 389 calibration set increases, the probability that the classification set  $C(\mathbf{x}_{\text{test}})$  for a newly arrived sample  
 390  $\mathbf{x}_{\text{test}}$  does not cover the true test label  $\mathbf{y}_{\text{test}}$  approaches  $\alpha$  [3].

391 Observe from Equation 7 that Conformal Prediction works on an instance-level basis, much like  
 392 Counterfactual Explanations are local. The prediction set for an individual instance  $\mathbf{x}_i$  depends only  
 393 on the characteristics of that sample and the specified error rate. Intuitively, the set is more likely  
 394 to include multiple labels for samples that are difficult to classify, so the set size is indicative of  
 395 predictive uncertainty. To see why this effect is exacerbated by small choices for  $\alpha$  consider the case  
 396 of  $\alpha = 0$ , which requires that the true label is covered by the prediction set with probability equal to  
 397 one.

## 398 C Conformal Prediction

### 399 A Submission of papers to NeurIPS 2023

400 Please read the instructions below carefully and follow them faithfully.

#### 401 A Style

402 Papers to be submitted to NeurIPS 2023 must be prepared according to the instructions presented  
 403 here. Papers may only be up to **nine** pages long, including figures. Additional pages *containing only*  
 404 *acknowledgments and references* are allowed. Papers that exceed the page limit will not be reviewed,  
 405 or in any other way considered for presentation at the conference.

406 The margins in 2023 are the same as those in previous years.

407 Authors are required to use the NeurIPS L<sup>A</sup>T<sub>E</sub>X style files obtainable at the NeurIPS website as  
408 indicated below. Please make sure you use the current files and not previous versions. Tweaking the  
409 style files may be grounds for rejection.

## 410 **B Retrieval of style files**

411 The style files for NeurIPS and other conference information are available on the website at

412 <http://www.neurips.cc/>

413 The file `neurips_2023.pdf` contains these instructions and illustrates the various formatting re-  
414 quirements your NeurIPS paper must satisfy.

415 The only supported style file for NeurIPS 2023 is `neurips_2023.sty`, rewritten for L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub>.  
416 **Previous style files for L<sup>A</sup>T<sub>E</sub>X 2.09, Microsoft Word, and RTF are no longer supported!**

417 The L<sup>A</sup>T<sub>E</sub>X style file contains three optional arguments: `final`, which creates a camera-ready copy,  
418 `preprint`, which creates a preprint for submission to, e.g., arXiv, and `nonatbib`, which will not  
419 load the `natbib` package for you in case of package clash.

420 **Preprint option** If you wish to post a preprint of your work online, e.g., on arXiv, using the  
421 NeurIPS style, please use the `preprint` option. This will create a nonanonymized version of your  
422 work with the text “Preprint. Work in progress.” in the footer. This version may be distributed as you  
423 see fit, as long as you do not say which conference it was submitted to. Please **do not** use the `final`  
424 option, which should **only** be used for papers accepted to NeurIPS.

425 At submission time, please omit the `final` and `preprint` options. This will anonymize your  
426 submission and add line numbers to aid review. Please do *not* refer to these line numbers in your  
427 paper as they will be removed during generation of camera-ready copies.

428 The file `neurips_2023.tex` may be used as a “shell” for writing your paper. All you have to do is  
429 replace the author, title, abstract, and text of the paper with your own.

430 The formatting instructions contained in these style files are summarized in Sections B, C, and D  
431 below.

## 432 **B General formatting instructions**

433 The text must be confined within a rectangle 5.5 inches (33 picas) wide and 9 inches (54 picas) long.  
434 The left margin is 1.5 inch (9 picas). Use 10 point type with a vertical spacing (leading) of 11 points.  
435 Times New Roman is the preferred typeface throughout, and will be selected for you by default.  
436 Paragraphs are separated by 1/2 line space (5.5 points), with no indentation.

437 The paper title should be 17 point, initial caps/lower case, bold, centered between two horizontal  
438 rules. The top rule should be 4 points thick and the bottom rule should be 1 point thick. Allow 1/4 inch  
439 space above and below the title to rules. All pages should start at 1 inch (6 picas) from the top of the  
440 page.

441 For the final version, authors’ names are set in boldface, and each name is centered above the  
442 corresponding address. The lead author’s name is to be listed first (left-most), and the co-authors’  
443 names (if different address) are set to follow. If there is only one co-author, list both author and  
444 co-author side by side.

445 Please pay special attention to the instructions in Section D regarding figures, tables, acknowledg-  
446 ments, and references.

## 447 **C Headings: first level**

448 All headings should be lower case (except for first word and proper nouns), flush left, and bold.

449 First-level headings should be in 12-point type.

## 450 **A Headings: second level**

451 Second-level headings should be in 10-point type.

### 452 **A.1 Headings: third level**

453 Third-level headings should be in 10-point type.

454 **Paragraphs** There is also a `\paragraph` command available, which sets the heading in bold, flush  
455 left, and inline with the text, with the heading followed by 1 em of space.

## 456 **D Citations, figures, tables, references**

457 These instructions apply to everyone.

### 458 **A Citations within the text**

459 The `natbib` package will be loaded for you by default. Citations may be author/year or numeric, as  
460 long as you maintain internal consistency. As to the format of the references themselves, any style is  
461 acceptable as long as it is used consistently.

462 The documentation for `natbib` may be found at

463 `http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

464 Of note is the command `\citet`, which produces citations appropriate for use in inline text. For  
465 example,

466 `\citet{hasselmo}` investigated\dots

467 produces

468 Hasselmo, et al. (1995) investigated...

469 If you wish to load the `natbib` package with options, you may add the following before loading the  
470 `neurips_2023` package:

471 `\PassOptionsToPackage{options}{natbib}`

472 If `natbib` clashes with another package you load, you can add the optional argument `nonatbib`  
473 when loading the style file:

474 `\usepackage[nonatbib]{neurips_2023}`

475 As submission is double blind, refer to your own published work in the third person. That is, use “In  
476 the previous work of Jones et al. [4],” not “In our previous work [4].” If you cite your other papers  
477 that are not widely available (e.g., a journal paper under review), use anonymous author names in the  
478 citation, e.g., an author of the form “A. Anonymous” and include a copy of the anonymized paper in  
479 the supplementary material.

### 480 **B Footnotes**

481 Footnotes should be used sparingly. If you do require a footnote, indicate footnotes with a number<sup>1</sup>  
482 in the text. Place the footnotes at the bottom of the page on which they appear. Precede the footnote  
483 with a horizontal rule of 2 inches (12 picas).

484 Note that footnotes are properly typeset *after* punctuation marks.<sup>2</sup>



Figure 5: Sample figure caption.

Table 1: Sample table title

Part		
Name	Description	Size ( $\mu\text{m}$ )
Dendrite	Input terminal	$\sim 100$
Axon	Output terminal	$\sim 10$
Soma	Cell body	up to $10^6$

## 485 C Figures

486 All artwork must be neat, clean, and legible. Lines should be dark enough for purposes of reproduction.  
 487 The figure number and caption always appear after the figure. Place one line space before the figure  
 488 caption and one line space after the figure. The figure caption should be lower case (except for first  
 489 word and proper nouns); figures are numbered consecutively.

490 You may use color figures. However, it is best for the figure captions and the paper body to be legible  
 491 if the paper is printed in either black/white or in color.

## 492 D Tables

493 All tables must be centered, neat, clean and legible. The table number and title always appear before  
 494 the table. See Table 1.

495 Place one line space before the table title, one line space after the table title, and one line space after  
 496 the table. The table title must be lower case (except for first word and proper nouns); tables are  
 497 numbered consecutively.

498 Note that publication-quality tables *do not contain vertical rules*. We strongly suggest the use of the  
 499 booktabs package, which allows for typesetting high-quality, professional tables:

500 <https://www.ctan.org/pkg/booktabs>

501 This package was used to typeset Table 1.

## 502 E Math

503 Note that display math in bare TeX commands will not create correct line numbers for sub-  
 504 mission. Please use LaTeX (or AMSTeX) commands for unnumbered display math. (You  
 505 really shouldn't be using \$\$ anyway; see <https://tex.stackexchange.com/questions/503/why-is-preferable-to> and [https://tex.stackexchange.com/questions/40492/](https://tex.stackexchange.com/questions/40492/what-are-the-differences-between-align-equation-and-displaymath)  
 506 [what-are-the-differences-between-align-equation-and-displaymath](https://tex.stackexchange.com/questions/40492/what-are-the-differences-between-align-equation-and-displaymath) for more infor-  
 507 mation.)  
 508

---

<sup>1</sup>Sample of the first footnote.

<sup>2</sup>As in this example.

## 509 **F Final instructions**

510 Do not change any aspects of the formatting parameters in the style files. In particular, do not modify  
511 the width or length of the rectangle the text should fit into, and do not change font sizes (except  
512 perhaps in the **References** section; see below). Please note that pages should be numbered.

## 513 **E Preparing PDF files**

514 Please prepare submission files with paper size “US Letter,” and not, for example, “A4.”

515 Fonts were the main cause of problems in the past years. Your PDF file must only contain Type 1 or  
516 Embedded TrueType fonts. Here are a few instructions to achieve this.

- 517 • You should directly generate PDF files using `pdflatex`.
- 518 • You can check which fonts a PDF files uses. In Acrobat Reader, select the menu  
519 Files>Document Properties>Fonts and select Show All Fonts. You can also use the program  
520 `pdffonts` which comes with `xpdf` and is available out-of-the-box on most Linux machines.
- 521 • `xfig` “patterned” shapes are implemented with bitmap fonts. Use “solid” shapes instead.
- 522 • The `\bbold` package almost always uses bitmap fonts. You should use the equivalent AMS  
523 Fonts:

524 `\usepackage{amsfonts}`

525 followed by, e.g., `\mathbb{R}`, `\mathbb{N}`, or `\mathbb{C}` for  $\mathbb{R}$ ,  $\mathbb{N}$  or  $\mathbb{C}$ . You can also  
526 use the following workaround for reals, natural and complex:

```
527 \newcommand{\RR}{I\!\!R} %real numbers  
528 \newcommand{\Nat}{I\!\!N} %natural numbers  
529 \newcommand{\CC}{I\!\!C} %complex numbers
```

530 Note that `amsfonts` is automatically loaded by the `amssymb` package.

531 If your file contains type 3 fonts or non embedded TrueType fonts, we will ask you to fix it.

## 532 **A Margins in L<sup>A</sup>T<sub>E</sub>X**

533 Most of the margin problems come from figures positioned by hand using `\special` or other  
534 commands. We suggest using the command `\includegraphics` from the `graphicx` package.  
535 Always specify the figure width as a multiple of the line width as in the example below:

```
536 \usepackage[pdftex]{graphicx} ...  
537 \includegraphics[width=0.8\linewidth]{myfile.pdf}
```

538 See Section 4.4 in the graphics bundle documentation ([http://mirrors.ctan.org/macros/](http://mirrors.ctan.org/macros/latex/required/graphics/grfguide.pdf)  
539 [latex/required/graphics/grfguide.pdf](http://mirrors.ctan.org/macros/latex/required/graphics/grfguide.pdf))

540 A number of width problems arise when L<sup>A</sup>T<sub>E</sub>X cannot properly hyphenate a line. Please give LaTeX  
541 hyphenation hints using the `\-` command when necessary.

## 542 **F Supplementary Material**

543 Authors may wish to optionally include extra information (complete proofs, additional experiments  
544 and plots) in the appendix. All such materials should be part of the supplemental material (submitted  
545 separately) and should NOT be included in the main submission.

## 546 **References**

547 References follow the acknowledgments in the camera-ready paper. Use unnumbered first-level  
548 heading for the references. Any choice of citation style is acceptable as long as you are consistent. It

549 is permissible to reduce the font size to small (9 point) when listing the references. Note that the  
550 Reference section does not count towards the page limit.

551 [1] Alexander, J.A. & Mozer, M.C. (1995) Template-based algorithms for connectionist rule extraction. In  
552 G. Tesauro, D.S. Touretzky and T.K. Leen (eds.), *Advances in Neural Information Processing Systems 7*, pp.  
553 609–616. Cambridge, MA: MIT Press.

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555 *GENeral NEural Simulation System*. New York: TELOS/Springer-Verlag.

556 [3] Hasselmo, M.E., Schnell, E. & Barkai, E. (1995) Dynamics of learning and recall at excitatory recurrent  
557 synapses and cholinergic modulation in rat hippocampal region CA3. *Journal of Neuroscience* **15**(7):5249-5262.