Homework 1 - On Social Multipliers

A linear-in-sums model

$$V_i(l_i, l_{-i}) = \alpha l_i - \frac{1}{2} l_i^2 + \beta \sum_{j=1}^{n} g_{ij} l_i l_j \qquad i = 1, \dots, n$$

$\alpha > 0$, $\beta > 0$

a) $g_{ij} = 0 \;\forall\, i, j$. Then:

$$V_i(l_i, l_{-i}) = \alpha l_i - \frac{1}{2} l_i^2$$

Hence, each individual will maximize his utility function w.r.t l_i. The F.O.C is:

$$\alpha - \frac{1}{2} \cdot 2 \cdot l_i^* = 0 \; ; \quad \boxed{l_i^* = \alpha}$$

(note that we can do that because preferences are convex, i.e. the utility function is concave)

Therefore, if $\alpha_1 = \alpha_0 + \Delta$, the new individual effort in equilibrium is: $\boxed{l_i^* = \alpha_0 + \Delta}$. If we define the aggregate effort as $\sum_{j=1}^{n} l_j^*$, then we have:

$$\boxed{\sum_{j=1}^{n} l_j^* = \sum_{j=1}^{n} (\alpha_0 + \Delta) = n(\alpha_0 + \Delta) = \sum_{j=1}^{n} l_{j0}^* + n\Delta.}$$

where $\sum_{j=1}^{n} l_{j0}^*$ is the aggregate effort before the change in α.

b) $g_{ij} = 1 \ \forall \ i \neq j$. If this is the case, we can express the utility function as:

$$V_i(l_i, l_{-i}) = \alpha l_i - \frac{1}{2} l_i^2 + \beta \sum_{i \neq j} l_i l_j$$

Hence, the F.O.C. can be written as:

$$\alpha - l_i^* + \beta \sum_{i \neq j} l_j^* = 0$$

$$\boxed{l_i^* = \alpha + \beta \sum_{i \neq j} l_j^* \qquad i = 1, \dots, n}$$

↑ this is the Nash Equilibrium of this game

~~To find the solution of this game, let's work in vector form.~~

$$\cancel{\vec{l}^* = \vec{\alpha} + \beta \vec{l}_{-i}^*}$$

But note that, given that this will be true for all i: $l_i^o = \alpha + \beta \sum_{i \neq j} l_i^*$, given that $l_i^* = l_j^o \ \forall i \neq j$.

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$$e_i^* = \alpha + \beta e_i^* (n-1)$$

$$\Rightarrow \boxed{e_i^* = \frac{\alpha}{1-\beta(n-1)}}$$

Notice that I am focusing in the symmetric equilibria, given that $\alpha_i = \alpha \ \forall \ i$). However, we would need to show whether there are asymmetric equilibria.

First of all notice that we need $\beta < \frac{1}{n-1}$, given that otherwise there is a trivial equilibrium where $e_i^* = 0 \ \forall \ i$. If α increases by Δ, we have that, individually:

if we denote α_1 as the new value for α and α_0 as the old value, so that $\alpha_1 = \alpha_0 + \Delta$, then:

$$e_i^* = \frac{\alpha_0}{1-\beta(n-1)} + \frac{\Delta}{1-\beta(n-1)}$$

$$\Rightarrow \boxed{e_i^* = e_{i0}^* + \frac{\Delta}{1-\beta(n-1)}}$$

where we denote e_{i0}^* for the value of the effort before the change of α.

Note that now the increase in individual effort is higher than when the network was empty. The reason is the following: before nobody was internalizing others' positive effect of this increase in α, while now, when α increases for others, I internalize this effect through β - my marginal cost of exerting more effort decreases as others exert more effort. Actually, the higher the β the more I internalize this decrease in my marginal cost and the higher ~~was~~ my effort will be.

With regards to the change in the aggregate effort:

$$\boxed{\sum_{i=1}^{n} e_i^{*} = \sum_{i=1}^{n} e_{i0}^{*} + \frac{n}{1-\beta(n-1)} \cdot \Delta}$$

Notice that for $\beta = 0$, we are back to the case where the network is empty: even though $g_{ij} = 1$, if I do not internalize others' marginal utility for effort, I will not increase my effort in more than Δ.

Escaneado con CamScanner

c) $g_{1j} = g_{j1} = 1 \ \forall \ j = 2, \dots, n$ and $g_{ij} = 0$ for all other pairs. To look for this equilibrium we have to be more careful:

$$V_1(l_1, l_{-1}) = \alpha l_1 - \frac{1}{2} l_1^2 + \beta \sum_{j=1}^{n} l_1 l_j$$

$$V_i(l_i, l_{-i}) = \alpha l_i - \frac{1}{2} l_i^2 + \beta l_i \cdot l_1 \quad \forall \ i = 2, \dots, n$$

The hub of the network will face the following maximization problem:

$$\max_{l_1} \ \alpha l_1 - \frac{1}{2} l_1^2 + \beta \sum_{j=1}^{n} l_1 l_j$$

Whose F.O.C. is:

$$\boxed{\alpha - l_1^* + \beta \sum_{j=1}^{n} l_j^* = 0}$$

The spokes, on the contrary, will have the following F.O.C:

$$\boxed{\alpha - l_i^* + \beta l_1^* = 0 \quad \forall \quad i = 2, \dots, n}$$

Here:

$$l_1^* = \alpha + \beta \sum_{j=1}^{n} l_j^*$$

$$l_0^* = \alpha + \beta l_1^*$$

To solve this, we will make use of two arguments:

(1) the equilibrium will be symmetric among the spokes

(2) the hub will solve his maximization problem assuming that (1) is satisfied.

Then; $l_j^* = l_i^* \quad \forall \quad i \neq j$ and for $i, j = 2, \dots, n$

$$\Rightarrow l_1^* = \alpha + \beta l_1^* + \beta (n-1) \cdot l_i^*$$

$e_j^* = \alpha + \beta e_1^* \quad \forall \quad j = 2, \ldots, n$. Then:

$$e_1^* = \alpha + \beta e_1^* + \beta(n-1)(\alpha + \beta e_1^*)$$

$$\Rightarrow \left(1 - \beta - \beta^2(n-1)\right) e_1^* = \alpha + \alpha\beta(n-1)$$

$$\boxed{e_1^* = \frac{\alpha(1+\beta(n-1))}{1-\beta-\beta^2(n-1)}}$$

$$\boxed{e_j^* = \alpha + \frac{\alpha\beta(1+\beta(n-1))}{1-\beta-\beta^2(n-1)} \quad \text{for } j = 2, \ldots, n}$$

Here, if now we increase α by Δ, the individual effort changes by:

$$\boxed{e_1^{*} = e_{10}^{*} + \frac{\Delta(1+\beta(n-1))}{1-\beta-\beta^2(n-1)}}$$

$$e_i^{*} = e_{i0}^{*} + \Delta + \Delta \cdot \beta \cdot \frac{(1+\beta(n-1))}{1-\beta-\beta^2(n-1)}$$

$$= \boxed{e_{i0}^{*} + \Delta\left(1+\beta \cdot \frac{(1+\beta(n-1))}{(1-\beta-\beta^2(n-1))}\right)}$$

The hub internalizes the first-order effect with all the spokes and the second-order effect of all the spokes through the hub. The spokes