Networks assignment I

Problem I

n = number of nodes
l = number of undirected links

undirected unweighted Graph G

Proof:

a) The number of nodes with odd degree is even

We know:

$$\sum_{n \in G} \text{degrees}(n) = 2l$$

split this into nodes with even degree and odd degree

Set $N \nearrow N_{even}$, $\searrow N_{odd}$

$$\underbrace{\sum_{n \in N_{even}} d(n)}_{\text{Sum of even degrees}} + \sum_{n \in N_{odd}} d(n) = \underbrace{2m}_{\text{even}}$$

$\Rightarrow$ even

$$\Rightarrow \text{even} + \sum_{n \in N} d(n) = 2m \text{ (even)}$$

this needs to be even as well !!

b) There are always at least two nodes with the same degree!

Proof by contradiction

Let G has n nodes. For these nodes n, it is possible that we have degrees which are in the range 0 to n-1 (no neighbors, and linked to all ~~neighbors~~ nodes).

If we assume now:

$\{0, \dots, n-1\}$ i.e. all nodes have different degrees, we have a contradiction, as

0 and n-1 cannot be true at the same time.

Hence, at least two nodes need to have the same degree!!

Networks assignment I

Problem 2

G adjacency matrix of directed and unweighted network. Provide interpretation of G ~~and~~ G^T and G^TG.

Let's first consider $G*G^T$

G has entries

$$\underset{n\times n}{G} = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \\ g_{n1} & \cdots & g_{nn} \end{pmatrix}$$ where $\sum_j^n g_{ij} = d_i^{out}$ (sum of row entries $= d_i^{out}$)

$\sum_{j=1}^n g_{ji} = d_i^{in}$ sum of entries per column represent d^{in} of the respective column.

Now:

$$G*G^T = \begin{pmatrix} g_{11} & \cdots & g_{1n} \\ \vdots & & \\ g_{n1} & \cdots & g_{nn} \end{pmatrix} \begin{pmatrix} g_{11} & g_{n1} \\ \vdots & \vdots \\ g_{1n} & g_{nn} \end{pmatrix}$$

=> consider => $g_{11}^2 + g_{12}^2 + \ldots g_{1n}^2 = g_{11} + g_{12} + \ldots + g_{1n}$.

=> as we have here an unweighted graph, $g_{ij} \in \{0,1\}$.

=> So we have GG^T has diagonal elements:

=> $\cancel{GG^T} = \text{diag}(d_1^{out}, \ldots, d_n^{out})$

-> So the $G*G^T$ matrix has diagonal entries that correspond to the respective out degree of node i

If we consider now: $G^T \cdot G$

$\cancel{G^T G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ & & & \end{pmatrix}\begin{pmatrix} \\ \end{pmatrix}}$

$$G^T G = \begin{pmatrix} g_{11} & g_{21} & \cdots & g_{n1} \\ & & & \\ & & & \end{pmatrix}\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & & & \\ \vdots & & & \\ g_{n1} & & & \end{pmatrix} = \begin{pmatrix} \bullet & & \\ & & \\ & & \end{pmatrix}$$

where $\bullet = g_{11}^2 + g_{21}^2 + \dots g_{n1}^2$ as $g_{ij} \in \{0,1\}$

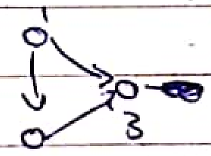
$= g_{11} + g_{21} \dots + g_{n1}$

$= \sum_{j=1}^{n} g_{j1} = \underline{\underline{a_1^{in}}}$

So $G^T G$ has diagonal elements $(a_1^{in}, \dots a_n^{in})$

Example

G:



$\Rightarrow G = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$a_1^{in} = 0$ $\quad a_1^{out} = 2$

$a_2^{in} = 1$ $\quad a_2^{out} = 1$

$a_3^{in} = 2$ $\quad a_3^{out} = 0$

$G^T G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$ $\qquad G \cdot G^T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

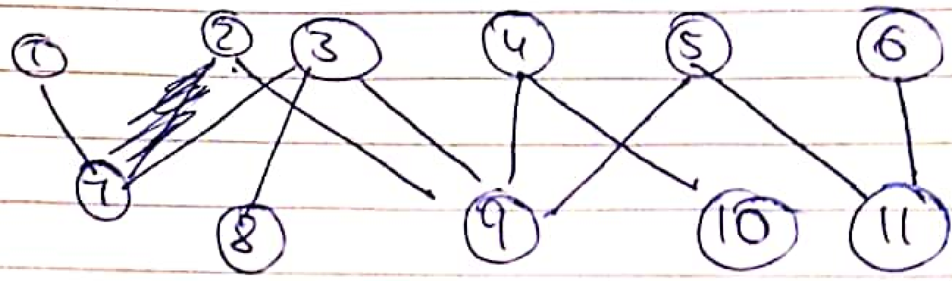
Section 2 Bipartite network

A bipartite ~~network~~ graph is a network that connects two disjoint sets U and V, such that each link connects a U-node and a V-node.

→ I can make projection of U and V

Problem 3

Bipartite network



Construct Adjacency matrix

G:

	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	1	0	0	0	0
2	0	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	1	1	1	0	0
4	0	0	0	0	0	0	0	0	1	1	0
5	0	0	0	0	0	0	0	0	1	0	1
6	0	0	0	0	0	0	0	0	0	0	1
7	1	0	1	0	0	0	0	0	0	0	0
8	0	0	1	0	0	0	0	0	0	0	0
9	0	1	1	1	1	0	0	0	0	0	0
10	0	0	0	1	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0	0

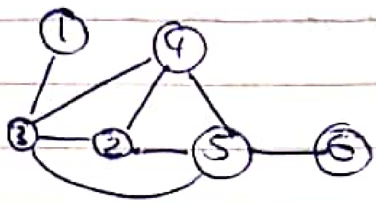
It is block diagonal matrix as it is the adjacency matrix of a bipartite graph. In that, no connections with the two parts of U. Here it is block diagonal, because nodes 1..6 are not connected, and nodes 7 to 11.

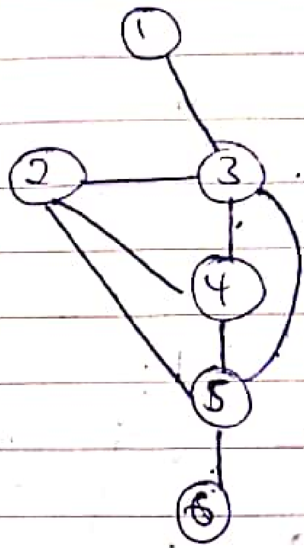
Problem 3 con't'd

Construct adjacency matrix of two projections

First connection: Connect two U nodes if they have the same V-node

First purple $N = \{1,2,3,4,5,6\}$





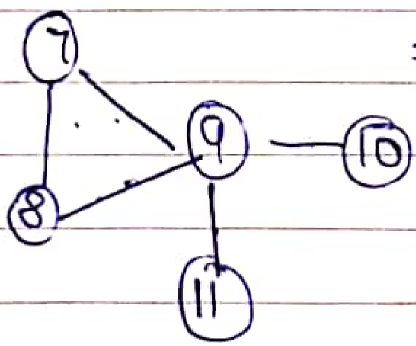
Adjacency matrix

$$G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Now green on purple Green nodes

$N = \{7,8,9,10,11\}$

$$G = \begin{array}{c|ccccc} & 7 & 8 & 9 & 10 & 11 \\ \hline 7 & 0 & 1 & 1 & 0 & 0 \\ 8 & 1 & 0 & 1 & 0 & 0 \\ 9 & 1 & 1 & 0 & 1 & 1 \\ 10 & 0 & 0 & 1 & 0 & 0 \\ 11 & 0 & 0 & 1 & 0 & 0 \end{array}$$



Network assignment I

Problem 3 cont'd

1) Calculate average degrees of the purple nodes and green nodes in the bipartite network

a) purple nodes

$d(1) = 1$
$d(2) = 1$
$d(3) = 3$
$d(4) = 2$
$d(5) = 2$
$d(6) = 1$

10

avg d. $= \frac{10}{6} = 5/3$

green nodes

average degree $\frac{10}{5} = 2$

Check!

Now:

Average degree in each of the partitions

purple on green

$d(1) = 1$
$d(2) = 3$
$d(3) = 4$
$d(4) = 3$
$d(5) = 3$
$d(6) = 1$

$15/6$

~~purple~~ green on purple

$d(7) = 2$
$d(8) = 2$
$d(9) = \cancel{3}\ 4$
$d(10) = 1$
$d(11) = 1$

10

$\rightarrow \frac{10}{5} = 2$

Problem 3 cont'd

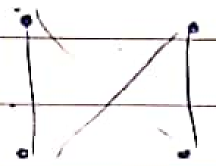
→ Second problem of the book

Bipartite networks general considerations

Bipartite network N_1 nodes N_2 nodes

- L_{max} is fully connected

N_1



N_2

4 links tops!

→ In general $N_1 \cdot N_2$ links!!

- How many links cannot occur compared to non-bipartite network?

Non bipartite graph: for $N = N_1 + N_2$

Maximum links $\frac{N(N-1)}{2}$

The difference $\frac{(N_1+N_2)(N_1+N_2-1)}{2} - \frac{N_1N_2}{2}$?

cannot occur due to the constraint

Network assignment I

Problem 3 cont'd

if $N_1 << N_2$, what can you say about network density?

$\Rightarrow$ density $= \frac{\# \text{ links}}{L_{max}}$

$L_{max} = N_1 N_2$

$\Rightarrow$ density $= \frac{\# \text{ links}}{N_1 N_2}$ if $N_1 << N_2$ then density c.p. $\uparrow$

we can see that from

~~d~~ =

$d = \frac{L_{links}}{N_1 N_2} = \frac{L_{links}}{\left(\frac{N_1}{N_2}\right) N_2^2}$ $\rightarrow$ so $d \uparrow\uparrow$

where as $\frac{N_1}{N_2} << 1$

$\Rightarrow$ close to zero

- Find an expression connecting N_1, N_2 and average degree for the two sets in the bipartite network

$\langle k_1 \rangle$ and $\langle k_2 \rangle$ $\Rightarrow$??

In general we have: $\rightarrow$ TBD

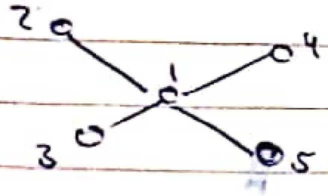
$\langle k \rangle = \frac{1}{N} \sum k_i$

$\Rightarrow$ TBD

Network assignment I

Problem 4

Architecture of a star network   exemplary network



a) Without using matrix calculation

How many paths of length $k \geq 1$ are there starting at 1 and end?

length $k=1$ => ~~paths(~~ # paths = 1

$k=2$ => 0 not possible!

$k=3$ => 4 as we can go from (1,4,1,2) for example

$k=4$ => 0 as it is not possible again

~~[illegible]~~

⇓

this pattern should continue

for even number => 0 to a power of $(k-1)/2$

odd number => $n^{\frac{k-1}{2}}$ ~~number of nodes~~ − 1 (that is $(n-1)^{k-1/2}$ not ~~the 1 node~~)

↳ $(n-1)^{\frac{k-1}{2}}$

ii) cycles for $k=1$ => 0

$k=2$ => n (here in our example it should be 4)

$k=3$ => 0 not possible

$k=4$ => $\cancel{n^2}$ $(n-1)^2$ (~~again something to the power of~~ k)

general: odd => 0

even => $n^{k/2}$

Network assignment I

Problem 4

b) Write down the adjacency matrix

For the exemplary network

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

or in general with n nodes

$$\underset{n\times n}{G} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & & & & \\ \vdots & & & & \ddots & & 0 & \\ 1 & & 0 & & & \ddots & & \end{pmatrix} = G^1$$

c) Compute few powers

$$\underset{n\times n}{G} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & & \\ \vdots & & & 0 & & \\ 1 & & & & & \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & \\ \vdots & & 0 & \\ 1 & & & \ddots \end{pmatrix}$$

$$G^2 = \begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & & 1 & 1 & 1 \\ 0 & \cdots & & & 1 \end{pmatrix}$$

$$G^3 = G^2 \cdot G$$

$$= \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$$

$$G^1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & \ddots & & 0 \\ \vdots & & 0 & \\ 1 & 0 & & \ddots \end{pmatrix}$$

$$G^3 = \begin{pmatrix} 0 & n-1 & \cdots & n-1 \\ n-1 & 0 & & \\ \vdots & & \ddots & \\ n-1 & & & 0 \end{pmatrix}$$

$$G^2 = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 1 & 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \\ 0 & 1 & & 1 \end{pmatrix}$$

$$G^3 = \begin{pmatrix} 0 & n-1 & \dots & n-1 \\ n-1 & \ddots & & \\ \vdots & & \ddots & \\ n-1 & & & 0 \end{pmatrix}$$

$$G^4 = G^3 G$$

$$= \begin{pmatrix} 0 & n-1 & \dots & n-1 \\ n-1 & \ddots & & \\ \vdots & & \ddots & \\ n-1 & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & & & \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$$

$$G^4 = \begin{pmatrix} (n-1)^2 & 0 & \dots & 0 \\ 0 & n-1 & \dots & n-1 \\ \vdots & & & \\ 0 & n-1 & \dots & n-1 \end{pmatrix}$$

In general we have the pattern

if k is even

$$G^k = \begin{pmatrix} (n-1)^{k/2} & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

where A is $(n-1) \times (n-1)$ matrix with elements $(n-1)^{k/2 - 1}$

if k is odd

$$G^k = \begin{pmatrix} 0 & (n-1)^{\frac{k-1}{2}} & \dots & (n-1)^{\frac{k-1}{2}} \\ (n-1)^{\frac{k-1}{2}} & & & \\ \vdots & & A & \\ (n-1)^{\frac{k-1}{2}} & & & \end{pmatrix}$$

where A is $(n-1) \times (n-1)$ matrix with 0 elements.

Network assignment

Problem 4

a) Compute Katz-Bonacich centrality Frage

$$\begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = \alpha \vec{1} + \beta G \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix}$$

$$\Leftrightarrow (I - \beta G) \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = \alpha \vec{1}$$

for each node.

$$\Rightarrow \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = (I - \beta G)^{-1} \alpha \vec{1}$$

where $G = \begin{pmatrix} 0 & 1 \dots & 1 \\ 1 & & \\ \vdots & A & \\ 1 & & \end{pmatrix}$ $\cdot \underset{(n-1)\times(n-1)}{A} = 0$

e) Frage:

TBD