Networks assignment 1

Problem 1

n = number of nodes — undirected

l = number of undirected links — unweighted Graph G

Proof:

a) The number of nodes with odd degree is even

We know:

$$\sum_{n \in G} \text{degree}(n) = 2l$$

split this into nodes with even degree and odd degree

Set N → N_{even}, N_{odd}

$$\sum_{n \in N_{even}} d(n) + \sum_{n \in N_{odd}} d(n) = 2m$$

$\sum_{n \in N_{even}} d(n)$ ↓ Sum of even degrees → even

$2m$ → even

$$\Rightarrow \text{even} + \sum_{n \in N} d(n) = 2m \text{ (even)}$$

↑ this needs to be even as well!!

b) There are always at least two nodes with the same degree!

Proof by contradiction

Let G has n nodes. For these nodes n, it is possible that we have degrees which are in the range 0 to n-1 (no negative, and limited to all nodes).

If we assume now:

$\{0, \ldots, n-1\}$ i.e. all nodes have different degrees, we have a contradiction, as

① and n-1 cannot be true at the same time.
at least

Hence, two nodes need to have the same degree!! ∎

Networks assignment I

Problem 2

G adjacency matrix of directed and unweighted network. Provide interpretation of G ~~and~~ G^T and G^TG.

Let's first consider G^*G^T

G has entries

$G_{n \times n} = \begin{pmatrix} g_{11} \cdots g_{1n} \\ \vdots \\ g_{n1} \cdots g_{nn} \end{pmatrix}$ where $\sum_{j=1}^{n} g_{ij} = d_i^{out}$ (sum of row entries $= d_i^{out}$)

$\sum_{i=1}^{n} g_{ij} = d_i^{in}$ sum of entries per column represent d^{in} of the respective column.

Now:

$G^*G^T = \begin{pmatrix} g_{11} \cdots g_{1n} \\ \vdots \\ g_{n1} \cdots g_{nn} \end{pmatrix} \begin{pmatrix} g_{11} & g_{n1} \\ \vdots & \vdots \\ g_{1n} & g_{nn} \end{pmatrix}$

$\Rightarrow$ consider $\Rightarrow g_{11}^2 + g_{12}^2 + \cdots + g_{1n}^2 = g_{11} + g_{12} + \cdots + g_{1n}$

$\Rightarrow$ As we have here an unweighted graph, $g_{ij} \in \{0, 1\}$.

$\Rightarrow$ So we have GG^T has diagonal elements:

$\Rightarrow$ ~~G^T~~ $= \text{diag}(d_1^{out}, \ldots, d_n^{out})$

-> So the G^*G^T matrix has diagonal entries that correspond to the respective out degree of node i

If we consider now: $G^T \cdot G$

$G^T G = \begin{pmatrix} g_{11} & g_{21} & \dots & g_{n1} \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} & \dots & g_{1n} \\ g_{21} & & & \\ \vdots & & & \\ g_{n1} & & & \end{pmatrix} = \begin{pmatrix} \cdot & & \\ & \ddots & \\ & & \end{pmatrix}$

where $\cdot = g_{11}^2 + g_{21}^2 + \dots g_{n1}^2$ as $g_{ij} \in \{0,1\}$

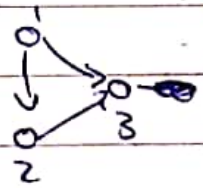
$= g_{11} + g_{21} \dots + g_{n1}$

$= \sum_{j=1}^{n} g_{j1} = a_1^{in}$

So $G^T G$ has diagonal elements $(a_1^{in} \dots a_n^{in})$

Example

G:



$\Rightarrow G = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$a_1^{in} = 0 \quad a_1^{out} = 2$

$a_2^{in} = 1 \quad a_2^{out} = 1$

$a_3^{in} = 2 \quad a_3^{out} = 0$

$G^T G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad G \cdot G^T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

Section 2 Bipartite network

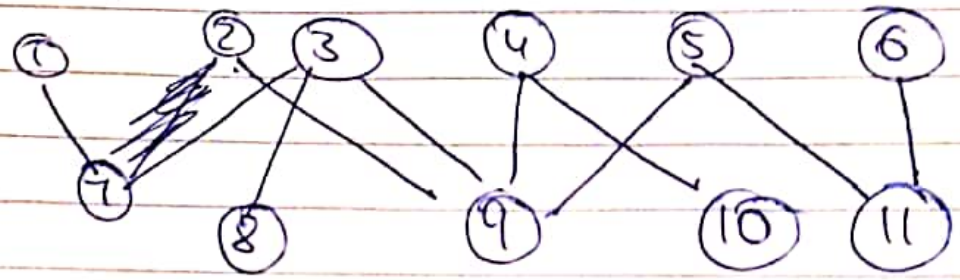
graph

A bipartite ~~network~~ is a network that connects two disjoint sets U and V, such that each link connects a U-node and a V-node.

→ I can make projection of U and V

Problem 3

Bipartite network



Construct Adjacency matrix

G:

	1	2	3	4	5	6	7	8	9	10	11
1	0	0	0	0	0	0	1	0	0	0	0
2	0	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	1	1	1	0	0
4	0	0	0	0	0	0	0	0	1	1	0
5	0	0	0	0	0	0	0	0	1	0	1
6	0	0	0	0	0	0	0	0	0	0	1
7	1	0	1	0	0	0	0	0	0	0	0
8	0	0	1	0	0	0	0	0	0	0	0
9	0	1	1	1	1	0	0	0	0	0	0
10	0	0	0	1	0	0	0	0	0	0	0
11	0	0	0	0	0	1	0	0	0	0	0

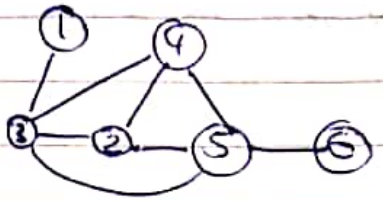
It is block diagonal matrix so it is the adjacency matrix of a bipartite graph. It block, no connections with the two parts of U. Here it is block diagonal, because nodes 1.. 6 are not connected, and nodes 7 to 11.

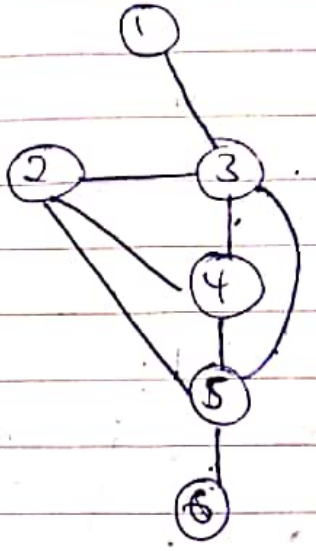
Problem 3 call'd

Construct adjacency matrix of two projections

First connection: Connect two le nodes if they have le same V-node

First purple $V = \{1, 2, 3, 4, 5, 6\}$



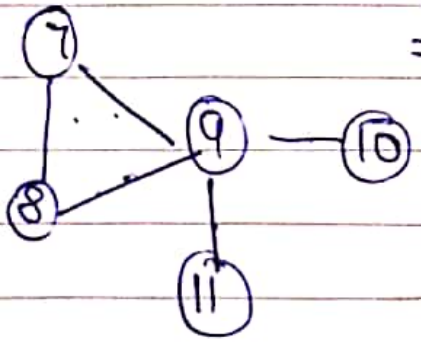


Adjacency matrix

$$G = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Also green or purple Green nodes

$V = \{7, 8, 9, 10, 11\}$



$$G = \begin{array}{c} \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{ccccc} 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Network assignment I

Problem 3 cont'd

1) Calculate average degrees of the purple nodes and green nodes in the bipartite network

a) purple nodes

$d(1) = 1$
$d(2) = 1$
$d(3) = 3$
$d(4) = 2$
$d(5) = 2$
$d(6) = 1$

10

avg d. $= \frac{10}{6} = 5/3$ //

green nodes

average degree $\frac{10}{5} = 2$

Check!

Now:

Average degree in each of the partitions

purple or green

$d(1) = 1$
$d(2) = 3$
$d(3) = 4$
$d(4) = 3$
$d(5) = 3$
$d(6) = 1$

$15/6$ //

~~purple~~ green or purple

$d(7) = 2$
$d(8) = 2$
$d(9) = 4$
$d(10) = 1$
$d(11) = 1$

10

$\rightarrow \frac{10}{5} = 2$ //

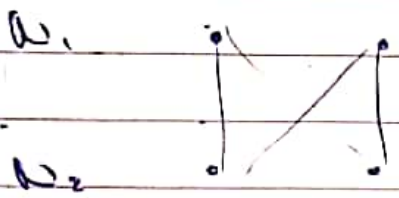
Problem 3 cal'd

⇒ Second problem of the book

Bipartite networks general considerations

Bipartite network N_1 nodes N_2 nodes

- K_{max} is fully connected



4 links ops!

⇒ In general $N_1 \cdot N_2$ links!!

- How many links cannot occur compared to non-bipartite network?

Non bipartite graph: for $N = N_1 + N_2$

Maximum links $\frac{N(N-1)}{2}$

The difference $\frac{(N_1+N_2)(N_1+N_2-1)}{2} - \frac{N_1 N_2 2}{2}$

cannot occur due to the constraint

Network assignment I

Problem 3 cal'd

if $N_1 << N_2$, what can you say about network density?

$\Rightarrow$ density = $\frac{\#\ links}{L_{max}}$

$L_{max} = N_1 N_2$

$\Rightarrow$ density = $\frac{\#\ links}{N_1 N_2}$ if $N_1 << N_2$ then density $\downarrow$

we can see that from

$Cl = \frac{L_{links}}{N_1 N_2} = \frac{L_{links}}{\left(\frac{N_1}{N_2}\right) N_2^2}$ $\rightarrow$ so Cl $\uparrow\uparrow$

where $\alpha \frac{N_1}{N_2} << 1$

$\rightarrow$ close to zero

- Find an expression correctly N_1, N_2 and average degree for the two sets in the bipartite network.

$\langle k_1 \rangle$ and $\langle k_2 \rangle$ $\Rightarrow$??

In general we have: $\rightarrow$ TBD

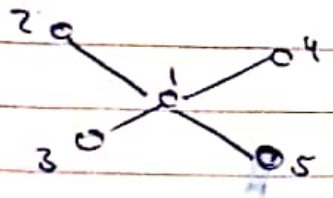
$\langle k \rangle = \frac{1}{N} \sum k_i$

$\Rightarrow$ TBD

Network assignment 1

Problem 4

Architecture of a star network exemplary network



a) Without using matrix calculation

How many paths of length $k \geq 1$ are there starting at 1 and end 2

length $k=1$ ⇒ ~~path~~ # paths = 1

$k=2$ ⇒ 0 not possible!

$k=3$ ⇒ 4 as we can go from (1,4,1,2) for example

$k=4$ = 0 as it is not possible again

~~[illegible]~~

⇓

this pattern should continue

for even number ⇒ 0 to a power of $(k-1)/2$

odd number ⇒ ~~# number of nodes / 1~~ (that is $(n-1)^{\frac{k-1}{2}}$ $(n-1)^{k-1/2}$ not ~~the # nodes~~)

ii) cycles for $k=1$ ⇒ 0

$k=2$ ⇒ n (not in as example it should be 4)

$k=3$ ⇒ 0 not possible

$k=4$ ⇒ ~~n^2~~ $(n-1)^2$ (~~again something to the power of~~ k)

general: odd ⇒ 0
even ⇒ $n^{k/2}$

Network assignment I

Problem 4

b) Write down the adjacency matrix

For the exemplary network

$$G = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$ or in general with n nodes

$$\underset{n \times n}{G} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & & & & & & \\ \vdots & & \ddots & & & & \\ 1 & & & & & & \end{pmatrix} = G^1$$

c) Compute few powers

$$\underset{n \times n}{G} = \begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & & \\ \vdots & & & 0 & & \\ 1 & & & & & \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & \ddots & \\ \vdots & & & 0 \\ 1 & & & \end{pmatrix}$$

$$G^2 = \begin{pmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & & 1 & \ddots & 1 \\ 0 & \cdots & & & 1 \end{pmatrix} \qquad G^3 = G^2 \cdot G$$

$$= \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$$

$$G^1 = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & \ddots & 0 \\ \vdots & & 0 & \\ 1 & 0 & & \end{pmatrix} \qquad G^3 = \begin{pmatrix} 0 & n-1 & \cdots & n-1 \\ n-1 & 0 & \ddots & \\ \vdots & \ddots & \ddots & \\ n-1 & & & 0 \end{pmatrix}$$

$$G^2 = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \\ 0 & 1 & & \ddots & 1 \end{pmatrix}$$

$$G^3 = \begin{pmatrix} 0 & n-1 & \cdots & n-1 \\ n-1 & & & \\ \vdots & & \ddots & \\ n-1 & & & 0 \end{pmatrix}$$

$$G^4 = G^3 G$$

$$= \begin{pmatrix} 0 & n-1 & \cdots & n-1 \\ n-1 & & & \\ \vdots & & \ddots & \\ n-1 & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & 0 & \\ 1 & & & \end{pmatrix}$$

$$G^4 = \begin{pmatrix} (n-1)^2 & 0 & \cdots & 0 \\ 0 & n-1 & \cdots & n-1 \\ \vdots & & & \\ 0 & n-1 & \cdots & n-1 \end{pmatrix}$$

In general we have the pattern

if k is even

$$G^k = \begin{pmatrix} (n-1)^{k/2} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}$$

where A is $(n-1)\times(n-1)$ matrix with elements $(n-1)^{k/2-1}$

if k is odd

$$G^k = \begin{pmatrix} 0 & (n-1)^{\frac{k-1}{2}} & \cdots & (n-1)^{\frac{k-1}{2}} \\ (n-1)^{\frac{k-1}{2}} & & & \\ \vdots & & A & \\ (n-1)^{\frac{k-1}{2}} & & & \end{pmatrix}$$

where A is $(n-1)\times(n-1)$ matrix with 0 elements.

Debrah assignment

Problem 4

a) Compute Katz-Bonacich centrality for:

$$\begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = \alpha \vec{1} + \beta G \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix}$$

$$\Leftrightarrow (I - \beta G) \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = \alpha \vec{1}$$

for each node.

$$\Rightarrow \begin{pmatrix} \hat{c}_1 \\ \vdots \\ \hat{c}_n \end{pmatrix} = (I - \beta G)^{-1} \alpha \vec{1}$$

where $G = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & A & \\ 1 & & & \end{pmatrix}$ A $_{(n-1)\times(n-1)}$ $= 0$

e) Fragen: TBD

Networks assignment I

Problem 6 Diameter

Diameter = the (longest) shortest path in a graph, or the distance between the two furthest nodes.

- maximum distance between two nodes in a network

→ The greatest shortest path
= distance

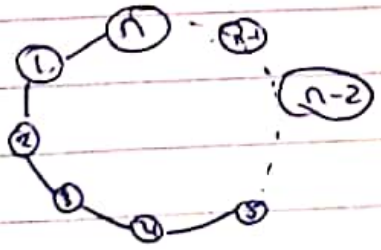
→ connected graph !!

not complete → two nodes are not adjacent!!

→ greatest → length of the shortest path !!

Problem 6

a) Circle network

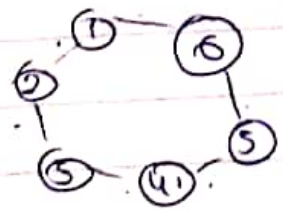


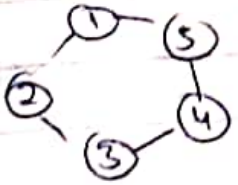
node 1 is connected to node n and 2.
node 2 to 1 and 3

	1	2	3
n	1	2	
		3	

What is the diameter of this?

$d = 3$ $\quad n/2$





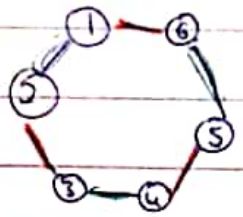
→ here diameter is (2)

$\frac{n}{2}$ if n is even

$n/2$ else $\frac{(n-1)}{2}$

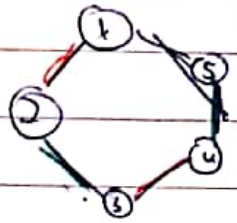
Diameter: length of greatest stated distance
or distance between nodes furthest away

n = even



$diam(G) = n/2 = 3$ //

n odd



$diam(G) = \frac{(n-1)}{2} = \frac{(5-1)}{2} = 2$ //

So if n is even → $diam(G) = n/2$

n is odd ⇒ $diam(G) = \frac{(n-1)}{2}$

The network is only bipartite if n is even! If n is odd it is not bipartite!

b) Find a network with largest possible diameter

① → ② – ... ⓝ (all nodes in a row)

diameter is n

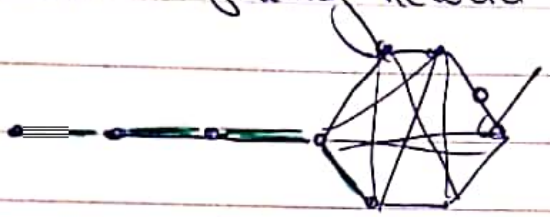
c) smallest possible network!

All complete graph has a diameter of 1 // (all nodes are connected!)

Problem 6 Network assignment I

a) Find a network that has a diameter 4 times larger than average distance

Consider the following network with $n+3$ nodes



Complete graph of n nodes !!

Diameter is 4.

As $n \to \infty$, the average distance should go to 1, as the complete graph has a distance of 1 (connecting only its nodes within). The properties of distance 1 grows as $n \to \infty$, so eventually the average distance $= \frac{1}{4}$ diameter (where diameter is 4).