Inversion

The algorithm we will perform is the powerful `mergesort`, but in each step of the merging phase we will count the inversions. Taking advantage of the fact that the two arrays $\mathbf{l}$ and $\mathbf{r}$ that we are comparing at each step k are already sorted in non-decreasing order, we have that if $\mathbf{l}_i > \mathbf{r}_j$ then this implies that $\mathbf{l}_i > \mathbf{r}_m$ for all $m > j$.

Pseudocode

The pseudo-code of this algorithm will consists of two functions, `count_inversions` and `merge_inversion`. The former is the main, recursive function that uses the `merge_inversion` as a sub-method. To understand the pseudo-code let us consider a simple example of a list: `A=[3,1,2,4]`, which has only one inversion. The `count_inversions` function will recursively split `A` into halves until it reaches the K-th step when A_K is of length 1. At that point it exits the recursion and simply returns the list A_K itself and a pseudo-inversion count of 0. At all other steps it returns the output from `merge_inversion(left,right,inv_left,inv_right)` - a list sorted in non-decreasing order and the cumulative count of inversions.

```
def count_inversions(A):
    if len(A) = 1
        # In case we are already at final divisions:
        return A, 0
    else:
        # For all other steps k<K:
        middle = len(A)/2
        intialize left, right
        for x in A up to middle:
            add x to left
        for x in A after middle:
            add x to right
        left, inv_left = count_inversions(left)
        right, inv_right = count_inversions(right)
        return merge_inversion(left,right,inv_left,inv_right)
```

The `merge_inversion` function implements the k-th step logic: *mergesort* and *counting inversions*. The latter works in the same way it always does, it compares two elements `left[0]` and `right[0]` and appends the output list `combine` by the smaller value. How inversions are counted should be straight-forward to see as well: when comparing `left[0]` and `right[0]` suppose we have `left[0] > right[0]`, so we have encountered an inversion. Then this implies that since `left` is already sorted in non-decreasing order the `inversion` counter should be incremented by the length of `left`.

```
def merge_inversion(left,right,inv_left,inv_right):
     # Initialize:
    combine = []
    inversion = 0
    # Merge and count inversions:
    while len(left) != 0 and len(right)!=0:
        if left[0] > right[0]:
            add right[0] to combine
            add len(left) to inversion
            remove right[0] from right
        else:
            add left[0] to combine
            remove left[0] from left
    endwhile
    # Take care of remaining elements:
    while len(left) != 0:
        add left[0] to combine
        remove left[0] from left
    endwhile
    while len(right) != 0:
        add right[0] to combine
        remove right[0] from right
    endwhile
    # Add them all up:
    total_inversion = inversion + inv_left + inv_right
    return combine, total_inversion
```

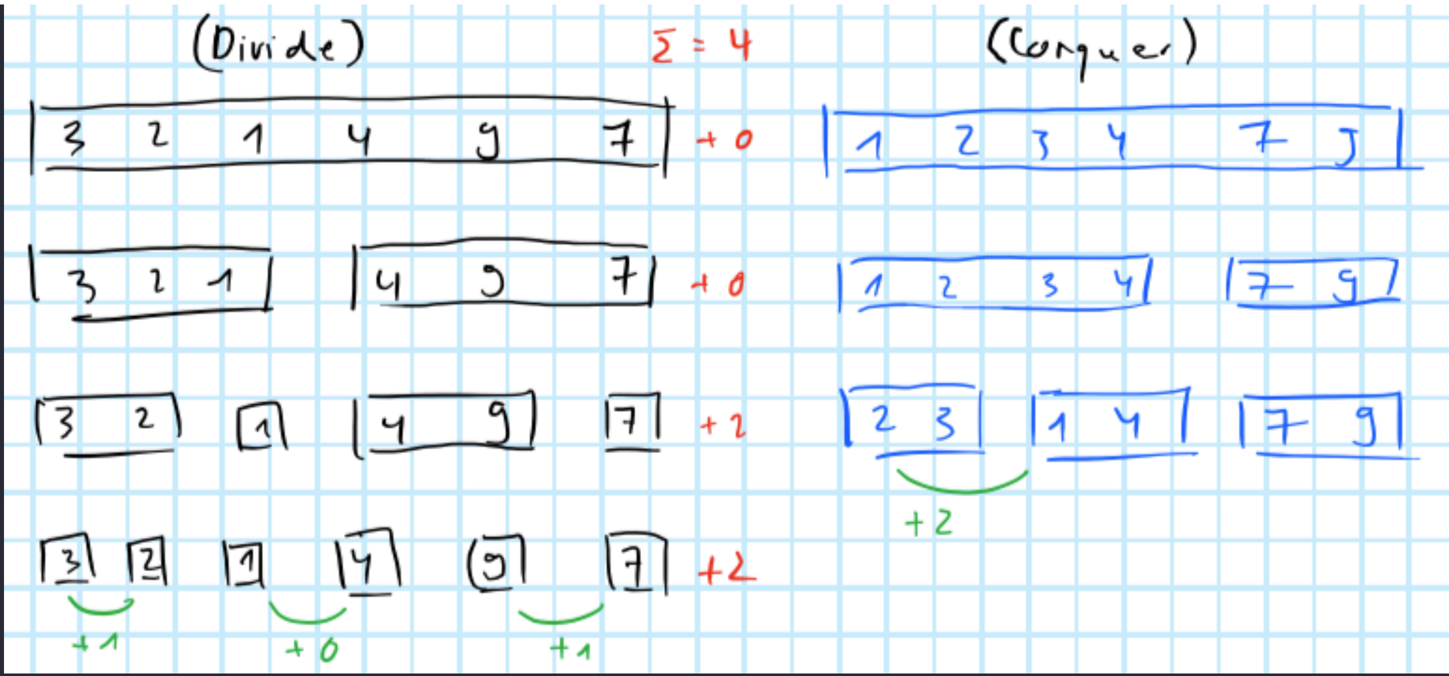
Example

Below we have sketched a schematic example of how the algorithm works. On the left in black we show the *divide* phase. Individual rows show the K steps we need to divide (and then again to conquer). In blue we have sketched the *conquer* phase. Green annotations show where inversions occure and in red have added a simple iterator that counts the inversions at each step k and the sum of all of them above.

Note that the bottom row of the *divide* phase is also our first point of action of the *conquer* phase. We compare adjacent lists two each other, all of which have just one elements. Sorting them and counting inversions is very simple. We find two inversions - $3 > 2$ and $9 > 7$ - and for each of them we increment the iterator by $+1$.

At the next step we have three lists so we just merge the first two. Here we first compare 2 - at position $i = 0$ - to 1 at position $m = 2$. We find that $2 > 1$ so we have encountered an inversion. But this time we increment the iterator by $m - i = +2$ since 3 is also greater than 1. Once again we also sort the elements we look at.

It should be straight-forward to see that in the remainder of the divide phase we find no more inversions and that we therefore end up with 4 inversions in total.



Complexity

Proof of correctness

Case where $|A| \leq 1$:

In this case it is trivial that the algorithm is corrects since we will jump straight to

```
if len(A) = 1
    # In case we are already at final divisions:
    return A, 0
```

so the algorithm correctly returns 0.

General case

We will prove by induction. Suppose we are at the k-th step of the algorithm and assume that in all previous steps the number of inversions have been computed correctly and list A_k is sorted correctly. At the step k we then split A_k into `left` and `right`. Then we have already demonstrated above how the `merge_inversion` will correctly merge `left` and `right` and increment the `inversion` counter. Applying the principle of mathematical induction we conclude that the algorithm is correct.

Strings