Inversion

The algorithm we will perform is the powerful *MergeSort*, but in each step of the merging phase we will count the inversions. Taking advantage of the fact that the two arrays $\mathbf{l}$ and $\mathbf{r}$ that we are comparing at each step k are already sorted in non-decreasing order, we have that if $\mathbf{l}_i > \mathbf{r}_j$ then this implies that $\mathbf{l}_i > \mathbf{r}_m$ for all $m > j$.

Pseudocode

The pseudo-code of this algorithm will consists of two functions, `count_inversions` and `merge_inversion`. The former is the main, recursive function that uses the `merge_inversion` as a sub-method. To understand the pseudo-code let us consider a simple example of a list: `A=[3,1,2,4]`, which has only one inversion. The `count_inversions` function will recursively split `A` into halves until it reaches the K-th step when A_K is of length 1. At that point it exits the recursion and simply returns the list A_K itself and a pseudo-inversion count of 0. At all other steps it returns the output from `merge_inversion(left,right,inv_left,inv_right)` - a list sorted in non-decreasing order and the cumulative count of inversions.

```
def count_inversions(A):
    if len(A) = 1
        # In case we are already at final divisions:
        return A, 0
    else:
        # For all other steps k<K:
        middle = len(A)/2
        intialize left, right
        for x in A up to middle:
            add x to left
        for x in A after middle:
            add x to right
        left, inv_left = count_inversions(left)
        right, inv_right = count_inversions(right)
        return merge_inversion(left,right,inv_left,inv_right)
```

The `merge_inversion` function implements the k-th step logic: *MergeSort* and *counting inversions*. The latter works in the same way it always does, it compares two elements `left[0]` and `right[0]` and appends the output list `combine` by the smaller value.

How inversions are counted should be straight-forward to see as well: when comparing `left[0]` and `right[0]` suppose we have `left[0] > right[0]`, so we have encountered an inversion. Then this implies that since `left` is already sorted in non-decreasing order the `inversion` counter should be incremented by the length of `left`.

```
def merge_inversion(left,right,inv_left,inv_right):
     # Initialize:
    combine = []
    inversion = 0
    # Merge and count inversions:
    while len(left) != 0 and len(right)!=0:
        if left[0] > right[0]:
            add right[0] to combine
            add len(left) to inversion
            remove right[0] from right
        else:
            add left[0] to combine
            remove left[0] from left
    endwhile
    # Take care of remaining elements:
    while len(left) != 0:
        add left[0] to combine
        remove left[0] from left
    endwhile
    while len(right) != 0:
        add right[0] to combine
        remove right[0] from right
    endwhile
    # Add them all up:
    total_inversion = inversion + inv_left + inv_right
    return combine, total_inversion
```

Complexity

Let $T(n)$ the time complexity of the algorithm. Given that *MergeSort* involves splitting the list in then recursively *MergeSort* twice at each recursion we have

$$T(n) = 2T\left(\frac{n}{2}\right) + \mathcal{O}(n)$$

where $\mathcal{O}(n)$ corresponds to the complexity of merging the two sub-arrays again. Then we can derive the complexity as follows:

$$
\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + \mathcal{O}(n) \\
&= 2\left[2T\left(\frac{n}{4}\right) + \mathcal{O}(\frac{n}{2})\right] + \mathcal{O}(n) \\
&= 2^K T\left(\frac{n}{2^K}\right) + \sum_{k=0}^{K-1} 2^k \mathcal{O}(\frac{n}{2^k}) \\
&= nT(1) + \sum_{k=0}^{K-1} n\mathcal{O}(1)
\end{aligned}
$$

Note that $n\mathcal{O}(1) = \mathcal{O}(n)$ and hence

$$
T(n) = k\mathcal{O}(n) = \mathcal{O}(n \log n)
$$

Proof of correctness

Case where $|A| \leq 1$:

In this case it is trivial that the algorithm is corrects since we will jump straight to

```
if len(A) = 1
    # In case we are already at final divisions:
    return A, 0
```

so the algorithm correctly returns 0.

General case

We will prove by induction. Suppose we are at the k-th step of the algorithm and assume that in all previous steps the number of inversions have been computed correctly and list A_k is sorted correctly. At the step k we then split A_k into `left` and `right`. Then we have already demonstrated above how the `merge_inversion` will correctly merge `left` and `right` and increment the `inversion` counter. Applying the principle of mathematical induction we conclude that the algorithm is correct.

Example

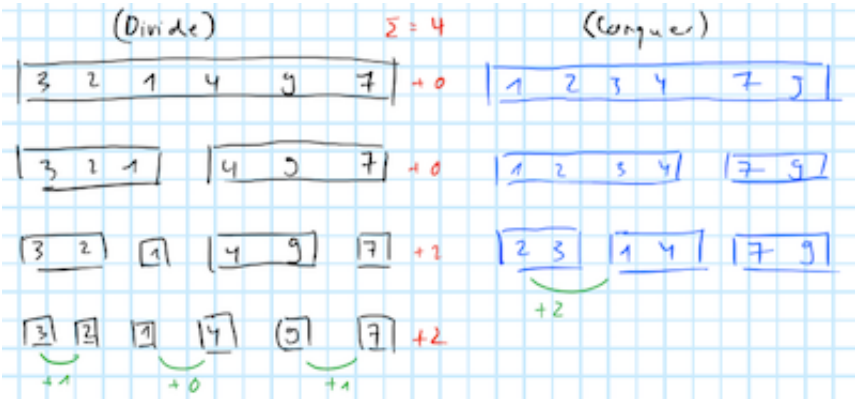
Below we have sketched a schematic example of how the algorithm works. On the left in black we show the *divide* phase. Individual rows show the K steps we need to divide (and then again to conquer). In blue we have sketched the *conquer* phase. Green annotations show where inversions occure and in red have added a simple iterator that counts the inversions at each step k and the sum of all of them above.

Note that the bottom row of the *divide* phase is also our first point of action of the *conquer*

phase. We compare adjacent lists two each other, all of which have just one elements. Sorting them and counting inversions is very simple. We find two inversions - $3 > 2$ and $9 > 7$ - and for each of them we increment the iterator by $+1$.

At the next step we have three lists so we just merge the first two. Here we first compare 2 - at position $i = 0$ - to 1 at position $m = 2$. We find that $2 > 1$ so we have encountered an inversion. But this time we increment the iterator by $m - i = +2$ since 3 is also greater than 1. Once again we also sort the elements we look at.

It should be straight-forward to see that in the remainder of the divide phase we find no more inversions and that we therefore end up with 4 inversions in total.



String distance

Pseudocode

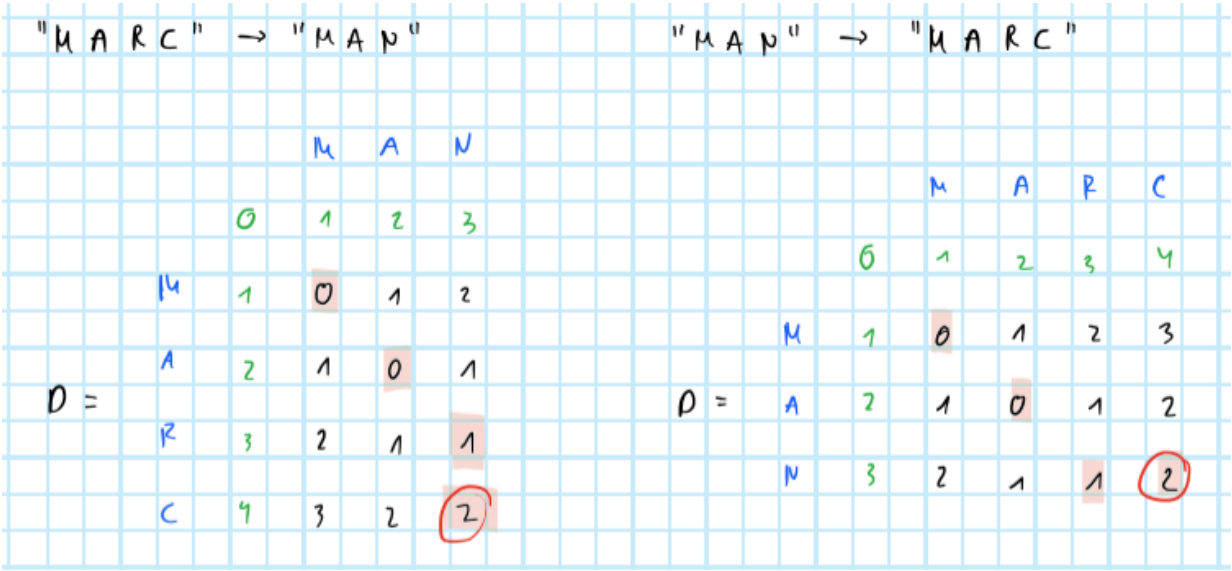
(a) Simple

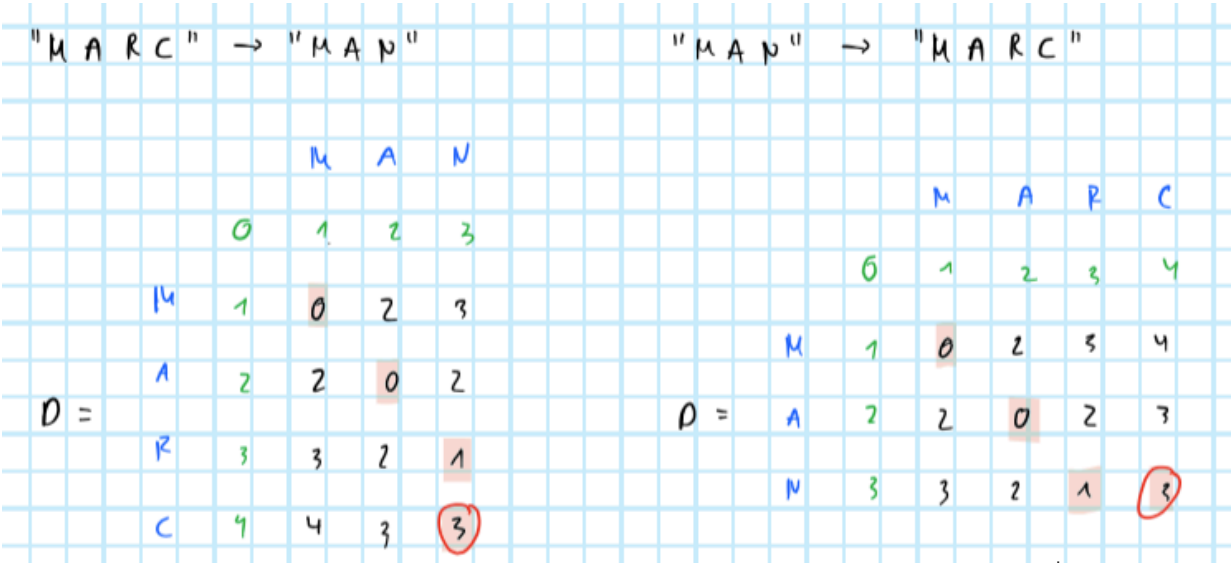
```
def edit_distance(str_1, str_2):
    # Initialize a distance matrix:
    m = len(str_1)
    n = len(str_2)
    D = matrix (m x n)
    set each element in D to zero
    optimal_sequence =
    # Set 0-th row...
    for i from 1 to m:
        D[i,0] = i
    for j from 1 to n:
        D[0,j] = j
    # Recursion:
    for j from 1 to n:
        for i from 1 to m:
            if s[i] == t[j]:
                substitution_cost = 0
            else:
                deletion_cost = 1
                insertion_cost = 1
                substitution_cost = 1
            edit_costs = [
                D[i-1,j] + deletion_cost,           # delete
                D[i,j-1] + insertion_cost,          # insert
                D[i-1,j-1] + substitution_cost      # substitute
            ]
            D[i,j] = minimum(edit_choices)
            if minimum(edit_costs) == D[i-1,j-1] + substitution_cost:
                if substitution_cost == 0:
                    append optimal_sequence by None
                else:
                    append optimal_sequence by "S"
            else if minimum(edit_costs) == D[i-1,j] + deletion_cost:
                append optimal_sequence by "D"
            else:
                append optimal_sequence by "I"
    return D[m,n], optimal_sequence
```

(b) Increased penalty for delete/insert

```
def edit_distance(str_1, str_2):
    # Initialize:
    m = len(str_1)
    n = len(str_2)
    D = matrix (m x n)
    set each element in D to zero
    # Set 0-th row...
    for i from 1 to m:
        D[i,0] = i
    for j from 1 to n:
        D[0,j] = j
    # Recursion:
    for j from 1 to n:
        for i from 1 to m:
            if s[i] == t[j]:
                substitution_cost = 0
            else:
                deletion_cost = 2
                insertion_cost = 2
                substitution_cost = 1
            edit_costs = [
                D[i-1,j] + deletion_cost,           # delete
                D[i,j-1] + insertion_cost,          # insert
                D[i-1,j-1] + substitution_cost      # substitute
            ]
            D[i,j] = minimum(edit_choices)
            if minimum(edit_costs) == D[i-1,j-1] + substitution_cost:
                if substitution_cost == 0:
                    append optimal_sequence by None
                else:
                    append optimal_sequence by "S"
            else if minimum(edit_costs) == D[i-1,j] + deletion_cost:
                append optimal_sequence by "D"
            else:
                append optimal_sequence by "I"
    return D[m,n], optimal_sequence
```

Example





Complexity

Note that here we are looking at nested loop where the j-loop requires n operations - one for each column - and the i-loop requires m operations - one for each row. Hence the total complexity is of order $\mathcal{O}(nm)$.

Proof of correctness