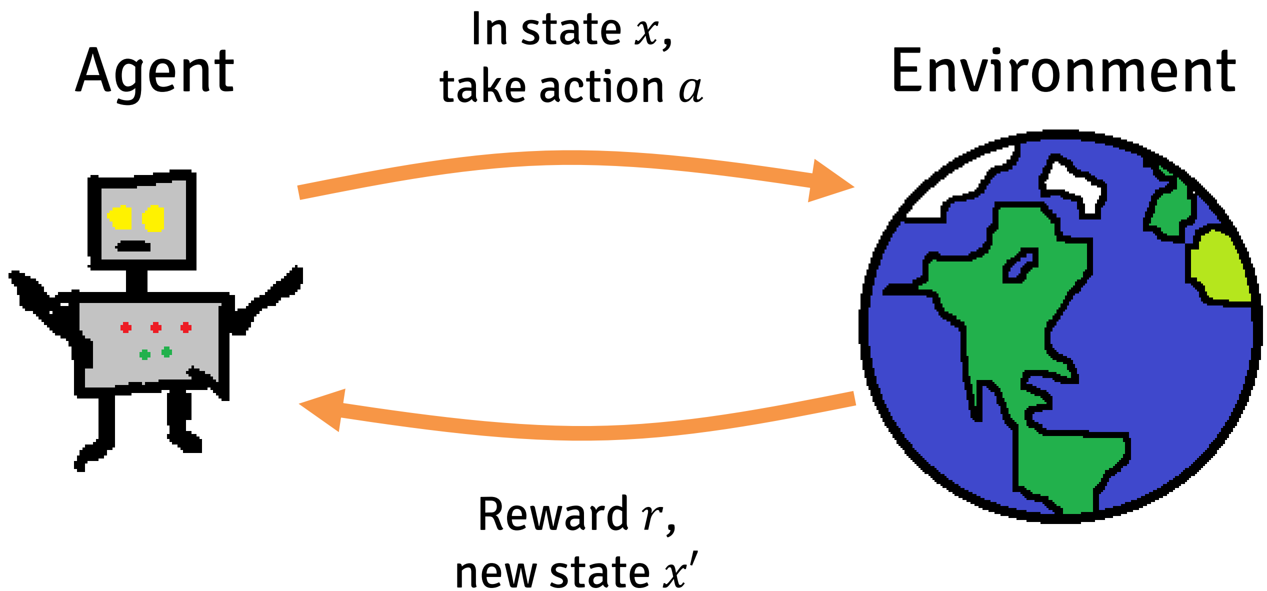
REINFORCEMENT LEARNING

Gergely Neu

upf. Universitat Pompeu Fabra Barcelona

Lecture 2: Dynamic Programming

WHAT IS REINFORCEMENT LEARNING?



Learning to
- maximize reward
- in a reactive environment
- under partial feedback

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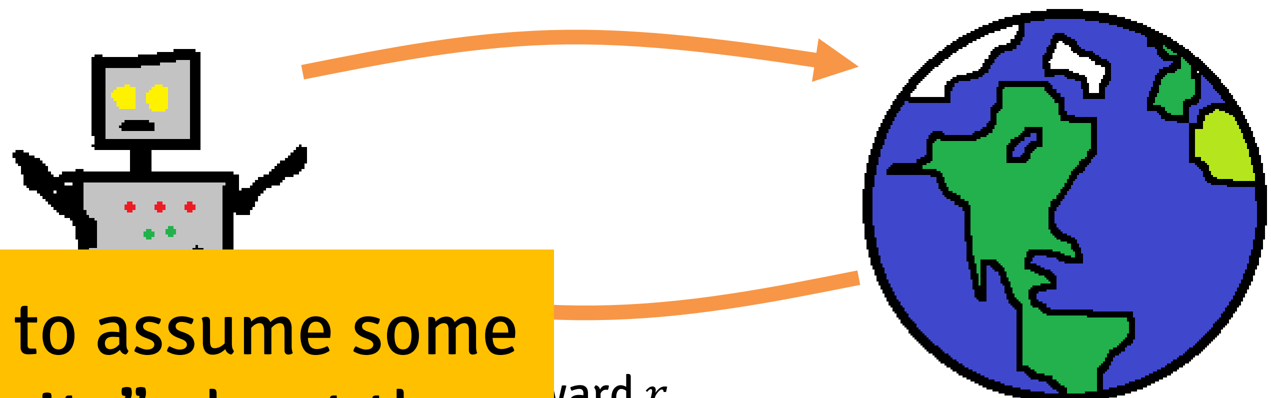

We need to assume some "regularity" about the environment so that the agent can predict the evolution of the states

aximize reward
a reactive environment
der partial feedback

EVOLUTION OF THE STATES

The states generally evolve according to the probability distribution

$$x_t \sim P(\cdot \mid x_{t-1}, a_{t-1}, x_{t-2}, a_{t-2}, \dots, x_0, a_0)$$

Problems:

- Long-term planning is intractable: too many paths!
- Learning from experience is very hard: the agent will never get to sample from the same distribution twice!

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Solution: Markov assumption

THE MARKOV ASSUMPTION

Assumption:
For any t, the states evolve according to
$$x_t \sim P(\cdot\,|x_{t-1}, a_{t-1})$$

Andrey Markov
(1856-1922)

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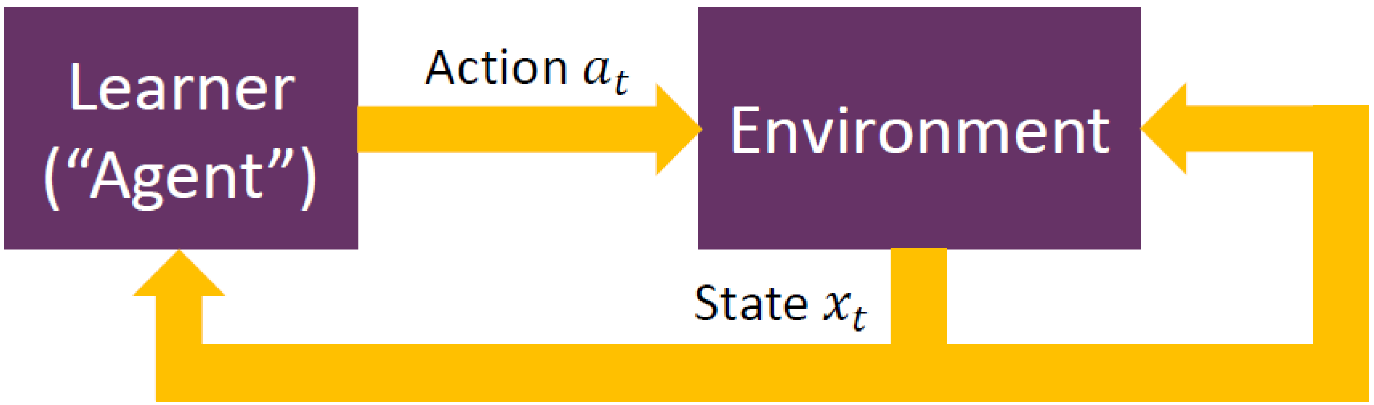
MARKOV DECISION PROCESSES (MDPs)



A Markov Decision Process (MDP) is characterized by

- X: a set of states
- A: a set of actions, possibly different in each state
- P: $X \times A \times X \rightarrow [0,1]$: a transition function with $P(\cdot\,|x,a)$ being the distribution of the next state given previous state x and action a:
$$\mathbf{P}[x_{t+1} = x' | x_t = x, a_t = a] = P(x'|x,a)$$
- $r: X \times A \rightarrow [0,1]$: a reward function

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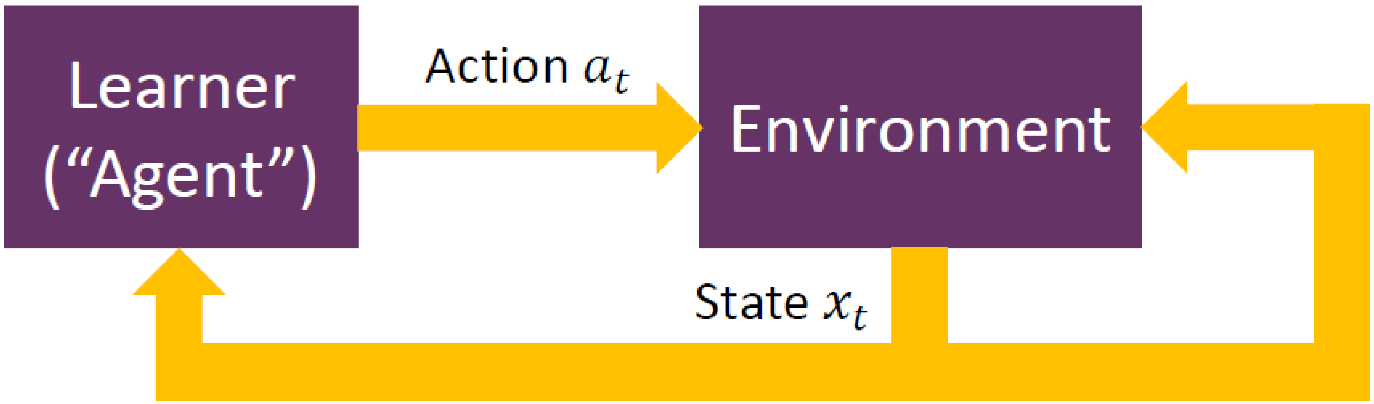

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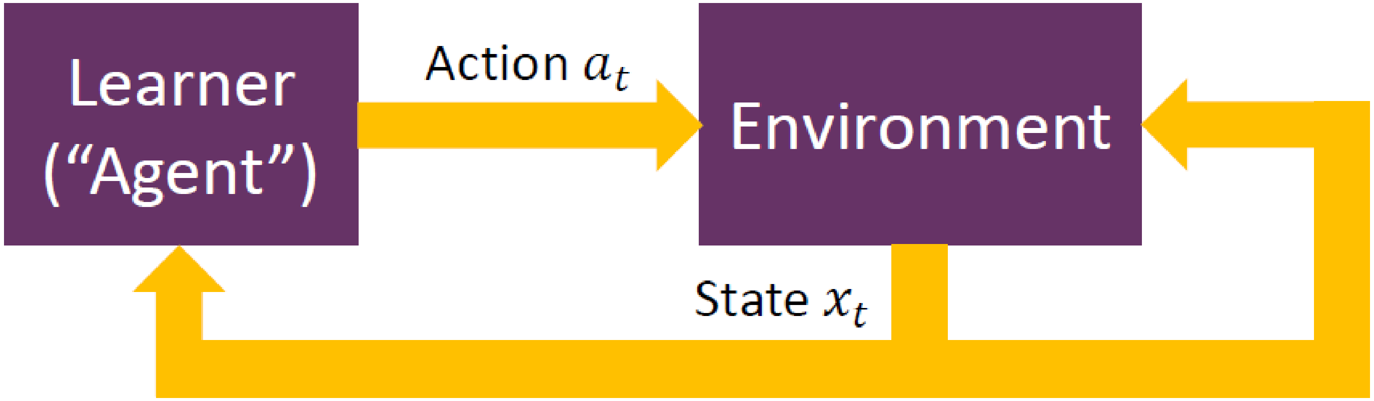

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Interaction in an MDP: in each round $t = 1,2, \ldots$

- Agent observes state x_t and selects action a_t
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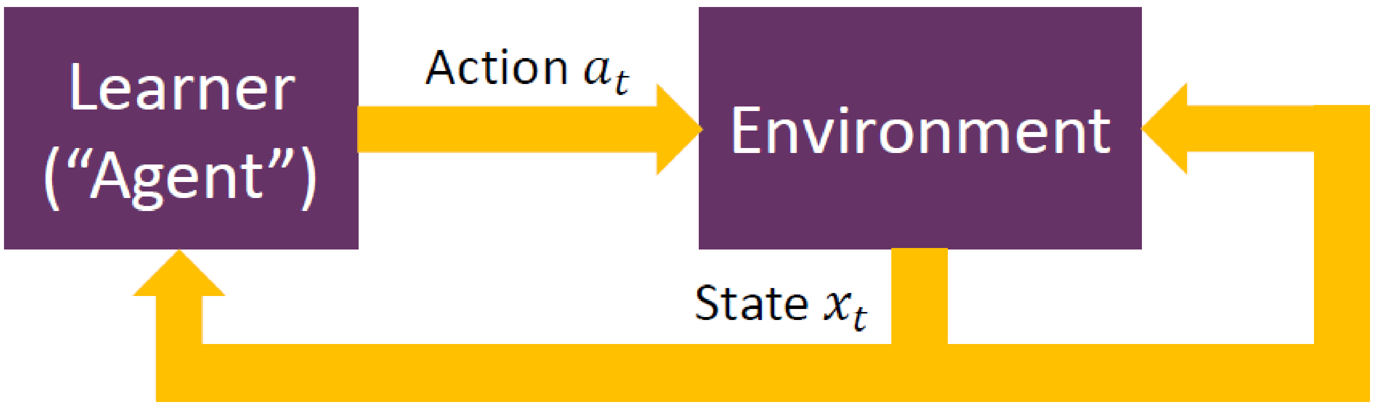

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GOAL:
maximize "total rewards"!

NOTIONS OF "TOTAL REWARD"

Episodic MDPs:

- There is a terminal state x^*
- GOAL: maximize total reward until final round T when x^* is reached:

$$R^* = \mathbf{E}[\sum_{t=0}^{T} r_t]$$

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Discounted MDPs:

- No terminal state
- Discount factor $\gamma \in (0,1)$
- GOAL: maximize total discounted reward (a.k.a. "return")

$$R_\gamma = \mathbf{E}[\sum_{t=0}^{\infty} \gamma^t r_t]$$

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+ other notions:
- long-term average reward
- total reward up to fixed horizon
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WHY DISCOUNT?

- “Earlier rewards matter more”
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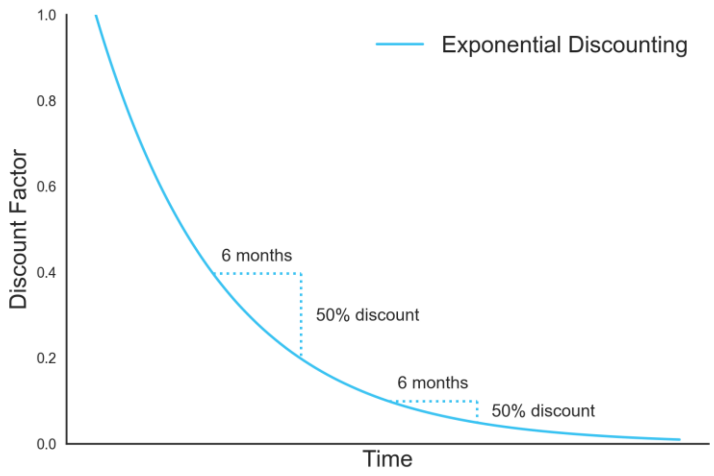

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- Mathematically convenient: if $r_t \in [0, R]$, then

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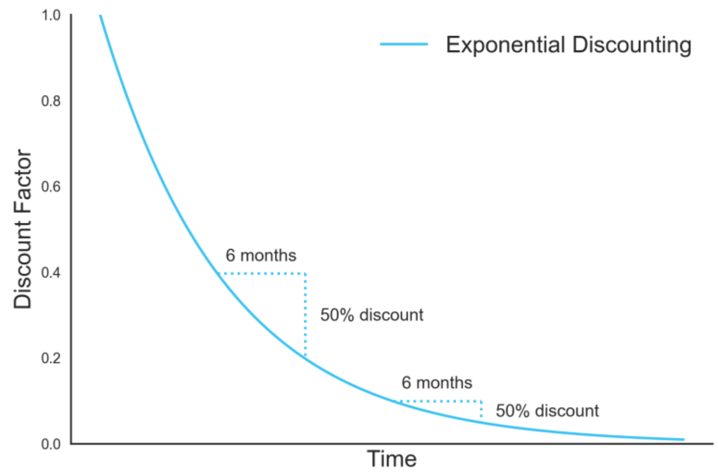

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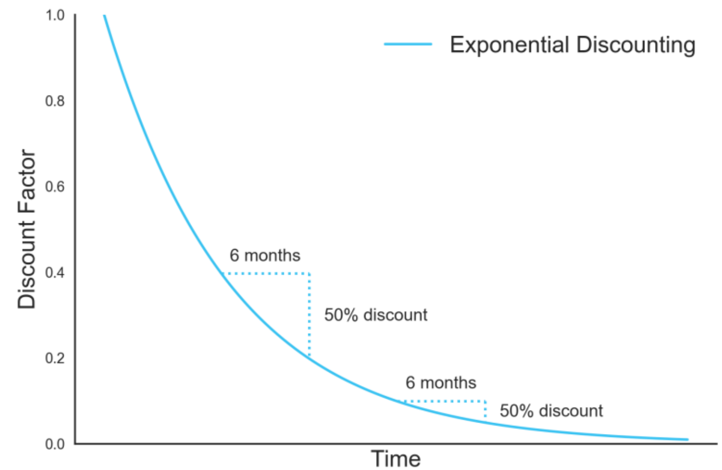

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- Discounted return blows up as $\gamma \to 1$ and becomes harder to optimize
- The factor $\frac{1}{1-\gamma}$ is sometimes called an "effective time horizon" as rewards after these many steps "don't matter too much"

Dynamic Programming for discounted rewards

Dynamic Programming for discounted rewards

1. basic definitions
2. value functions and optimal policies
3. the Bellman equations
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POLICIES AND TRAJECTORY DISTRIBUTIONS

Policy: mapping from histories to actions

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Expectation under this distribution: $\mathbf{E}_\pi[\cdot]$

DEFINING OPTIMALITY

Optimal policy π^*: a policy that maximizes

$$\mathbf{E}_\pi[R_\gamma] = \mathbf{E}_\pi\left[\sum_{t=0}^{\infty} \gamma^t r_t\right]$$

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"Markov property"

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"Optimal policy $\pi^* = \arg\max_{\pi} V^{\pi}(x_0)$"

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The optimal value function:

$$V^* = V^{\pi^*}$$

WHY IS THIS IMPORTANT?

- **Previous result** only establishes that for any initial state, there exists an optimal stationary policy maximizing $\mathbf{E}_\pi\left[R_\gamma\right]$

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An optimal policy π^* does not "make compromises"

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THE BELLMAN EQUATIONS

Richard E. Bellman
(1920-1984)

THE BELLMAN EQUATIONS

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The value function of a stationary policy π satisfies the system of equations $(\forall x \in X)$

$$V^{\pi}(x) = r\big(x, \pi(x)\big) + \gamma \sum_y P(y|x, \pi(x))\, V^{\pi}(y)$$

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= greedy with respect to Q^*

SHORT SUMMARY SO FAR

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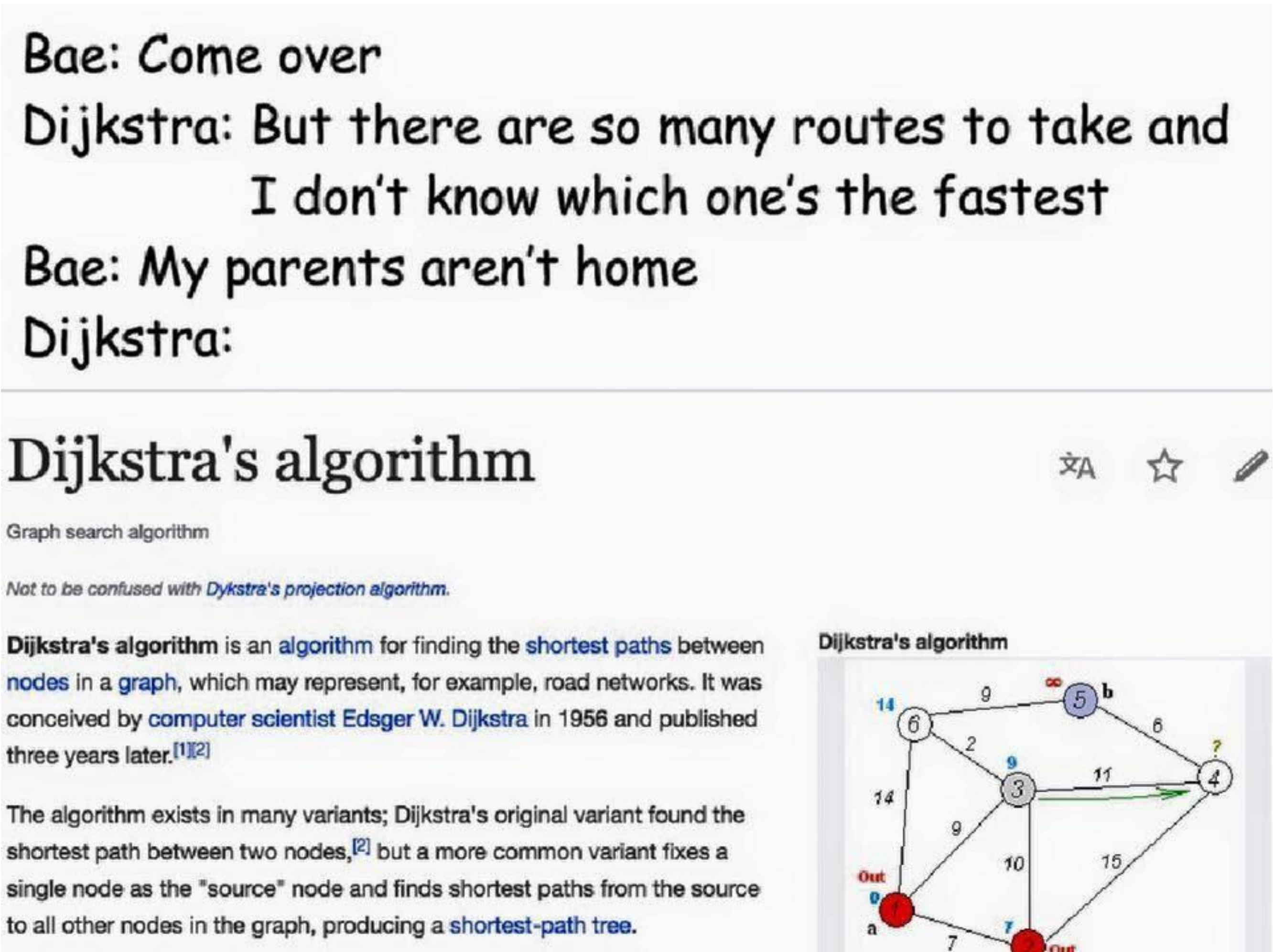
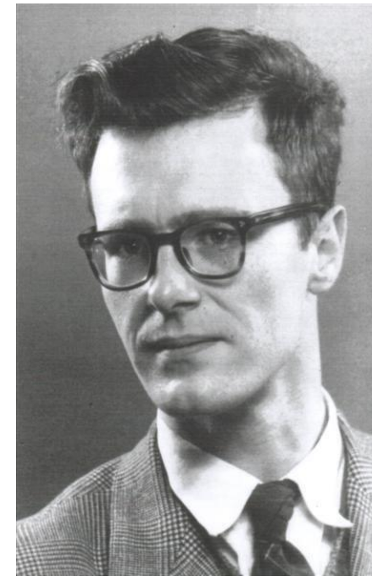
BUT HOW DO WE FIND THE OPTIMAL VALUE FUNCTION??

EASY ANSWER FOR FINITE-HORIZON PROBLEMS





Edsger Dijkstra
(1920-2002)

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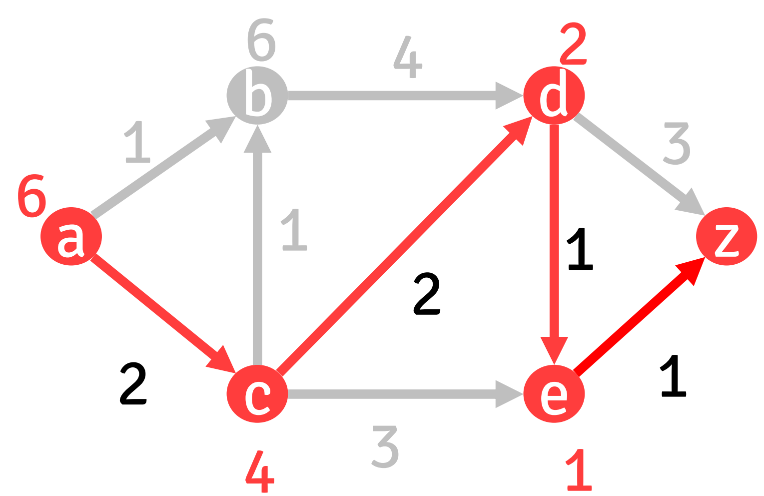
Dijkstra:

Cost-to-go = immediate cost
+ future cost-to-go

Bellman:

Value = immediate reward
+ expected future value



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DYNAMIC PROGRAMMING

Dynamic programming

=

computing value functions through repeated use of the "Bellman operators"

THE BELLMAN OPERATOR

Bellman operator T^{π}:
maps a function $f \in \mathbb{R}^X$ to another function $g = T^{\pi} f \in \mathbb{R}^X$:

$$g(x) = (T^{\pi} f)(x) = r\big(x, \pi(x)\big) + \gamma \sum_y P(y|x, \pi(x)) f(y)$$

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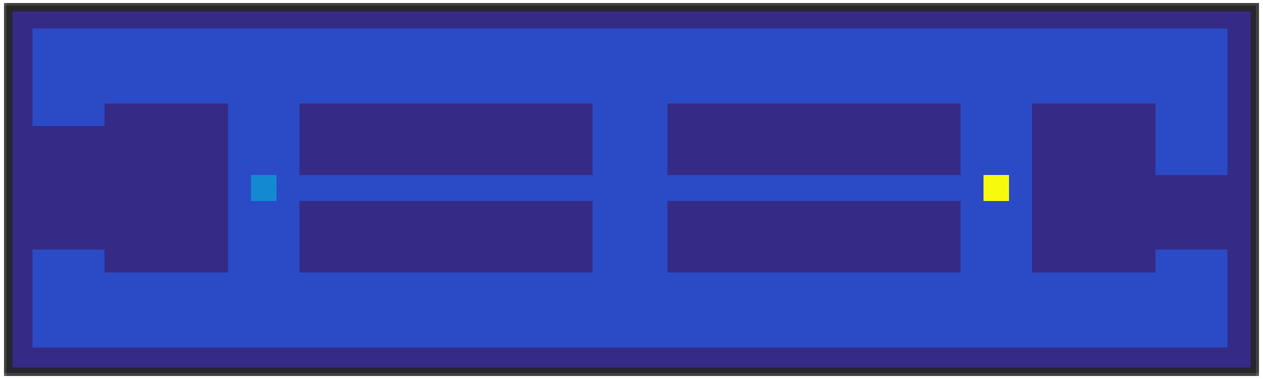
- The value function V^π satisfies

$$V^\pi = r + \gamma P^\pi V^\pi \Leftrightarrow V^\pi = (I - \gamma P^\pi)^{-1} r$$

POWER ITERATION IN ACTION

Gridworld MDP



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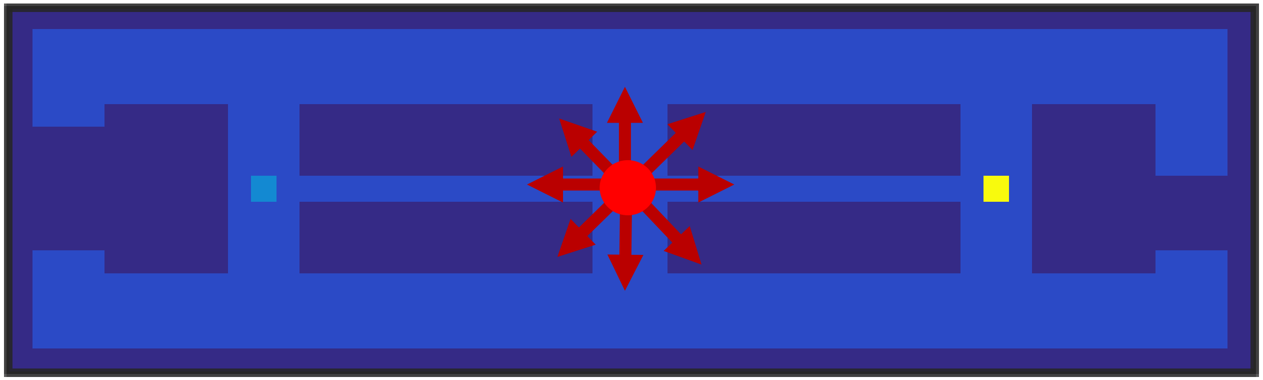

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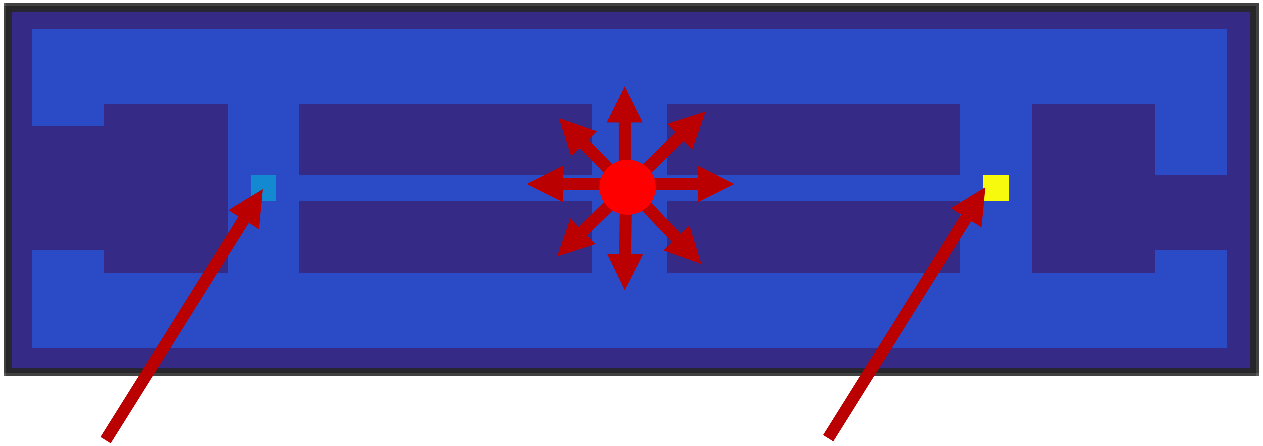

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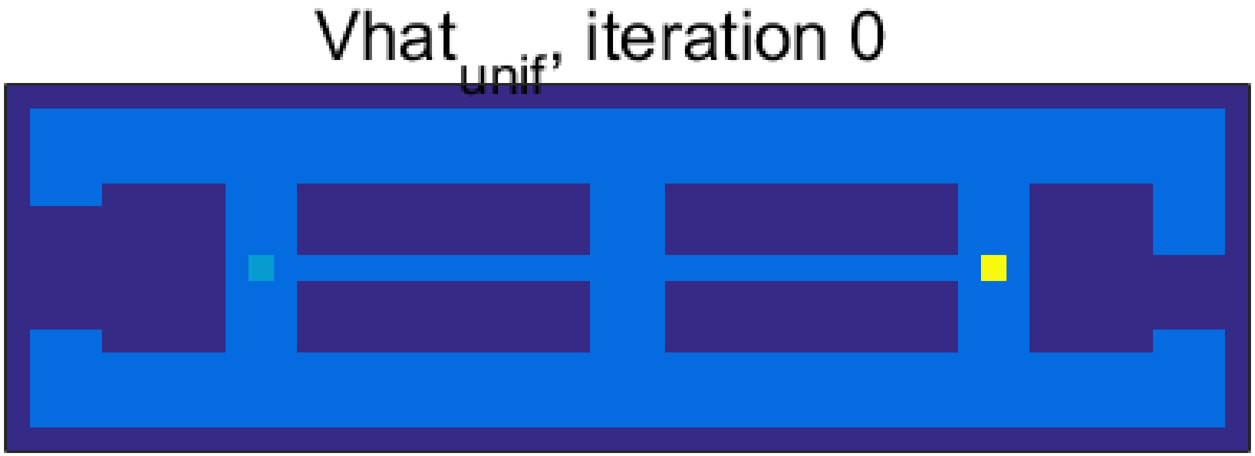

Uniform policy:

$$\pi(a|x) = \frac{1}{9}$$

for all actions $a \in \{1,2,\dots,9\}$

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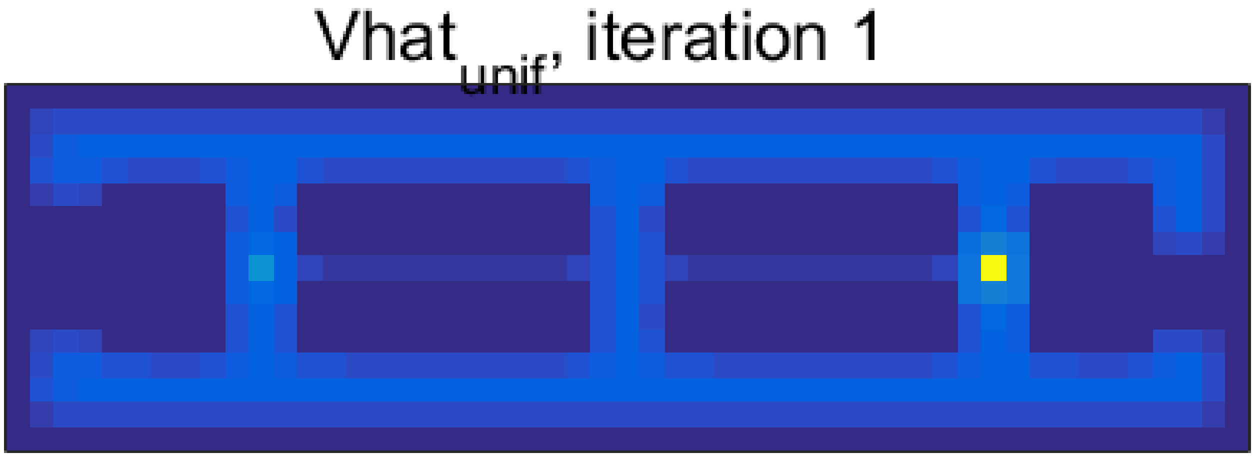

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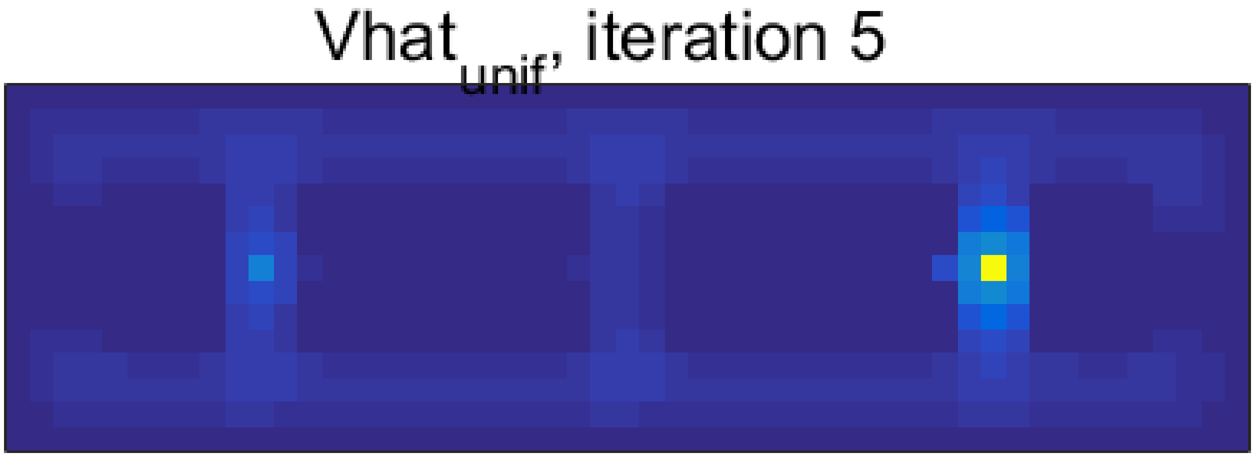

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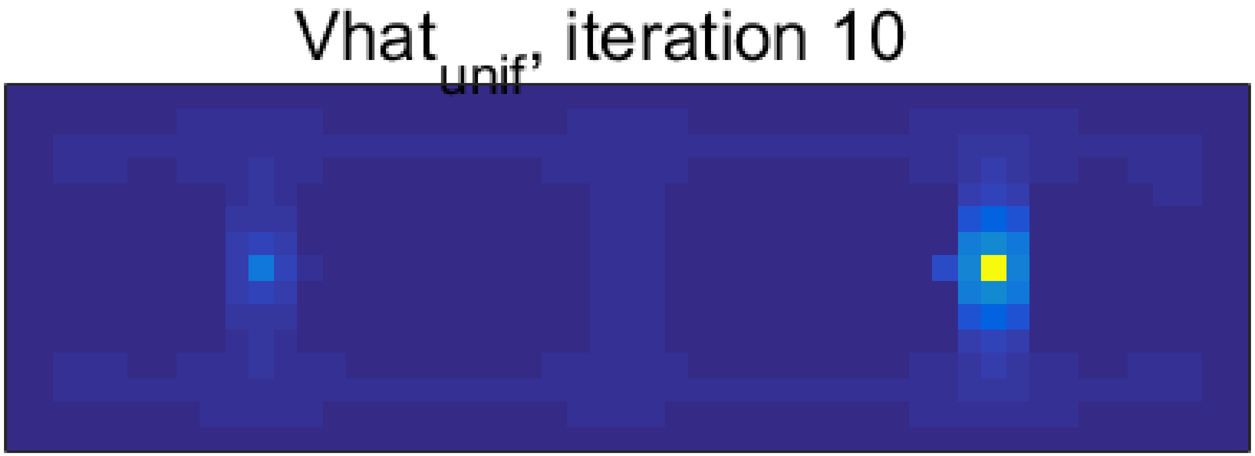

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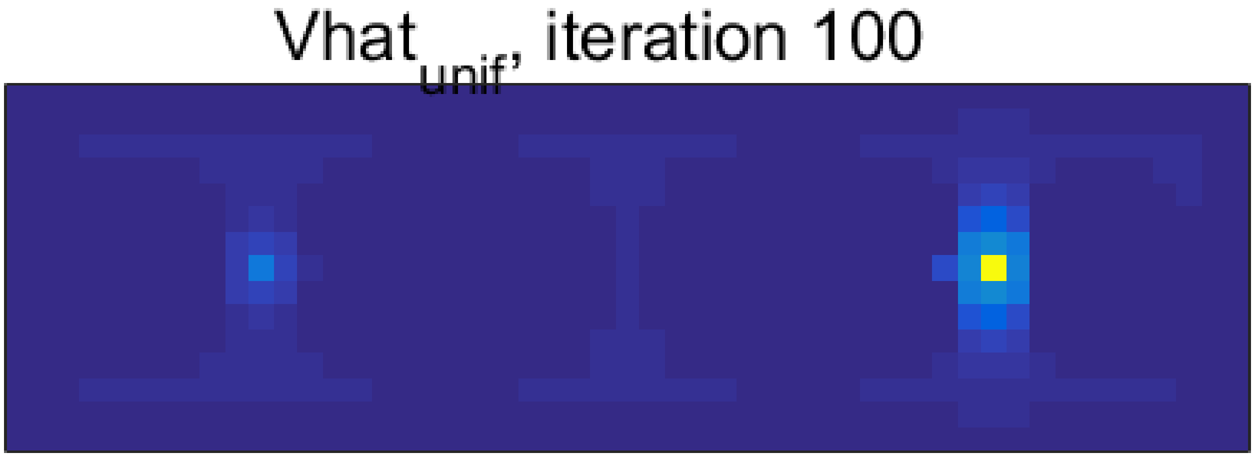

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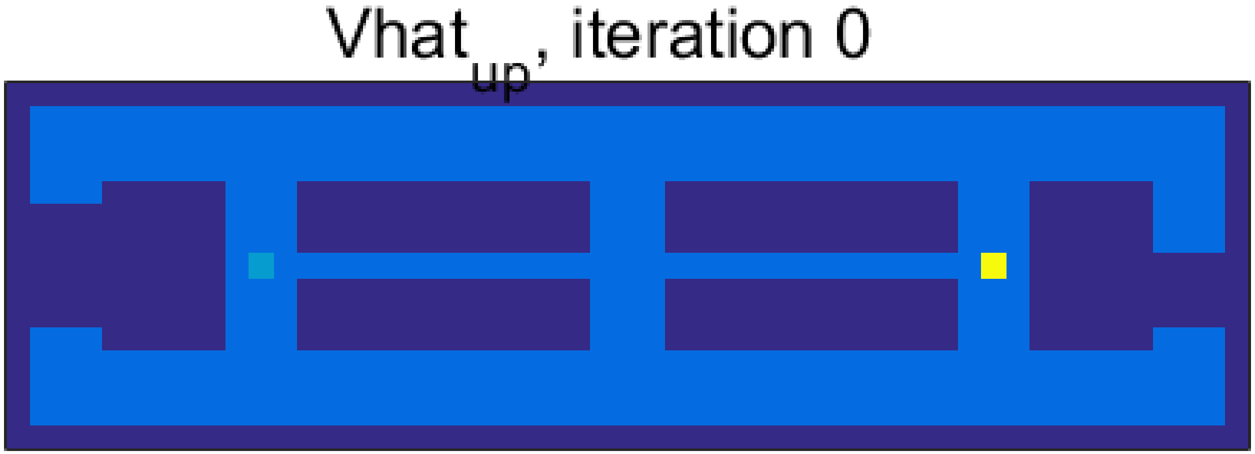

"Upwards" policy:
$\pi(\text{up}|x) = 1$

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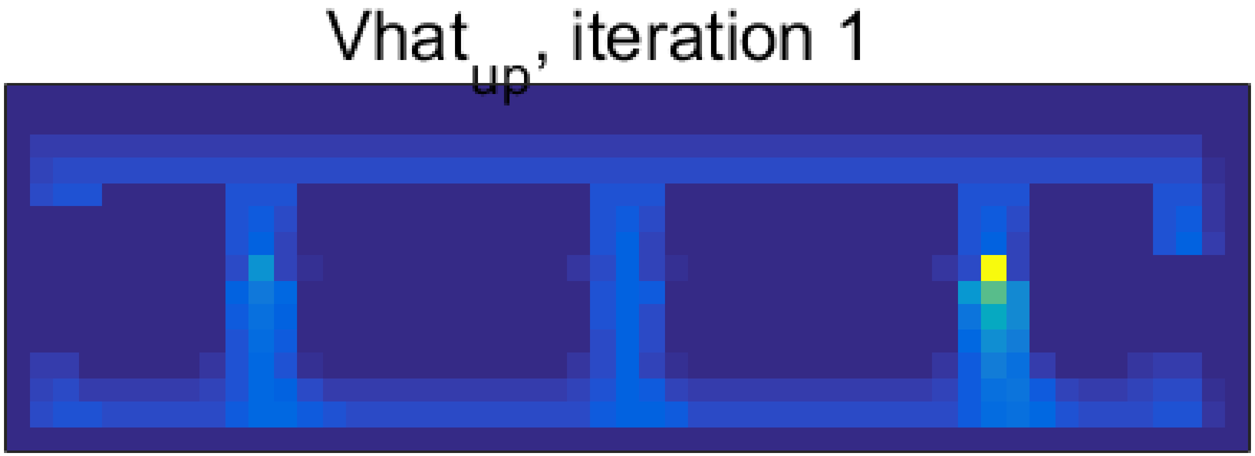

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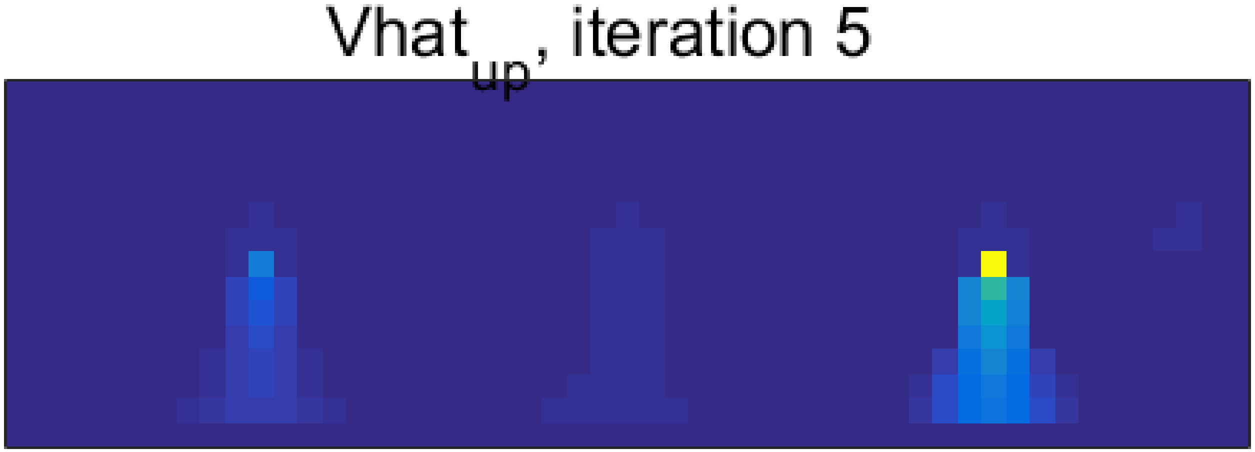

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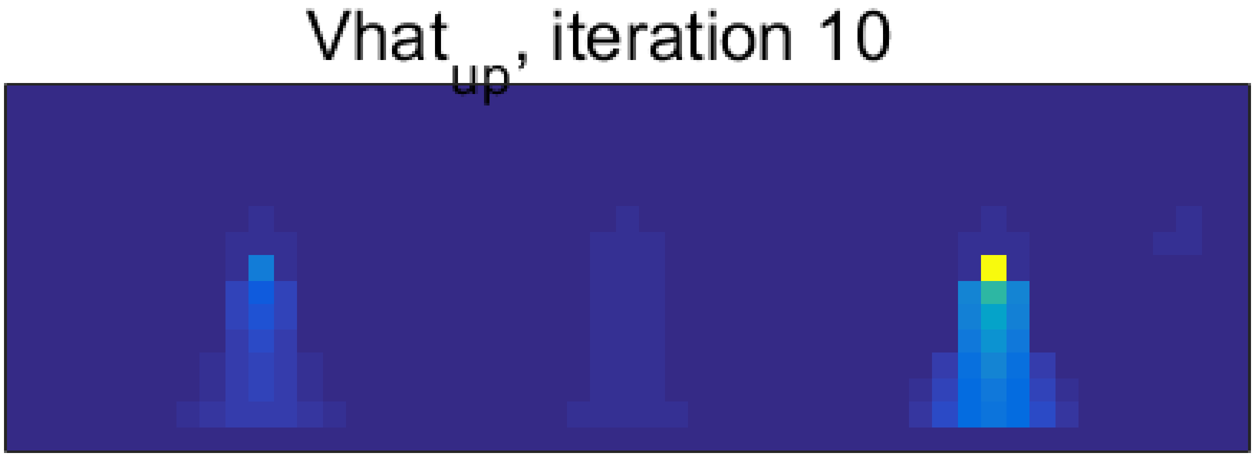

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Bellman optimality operator T^*:
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Key idea: T^* is a **contraction**

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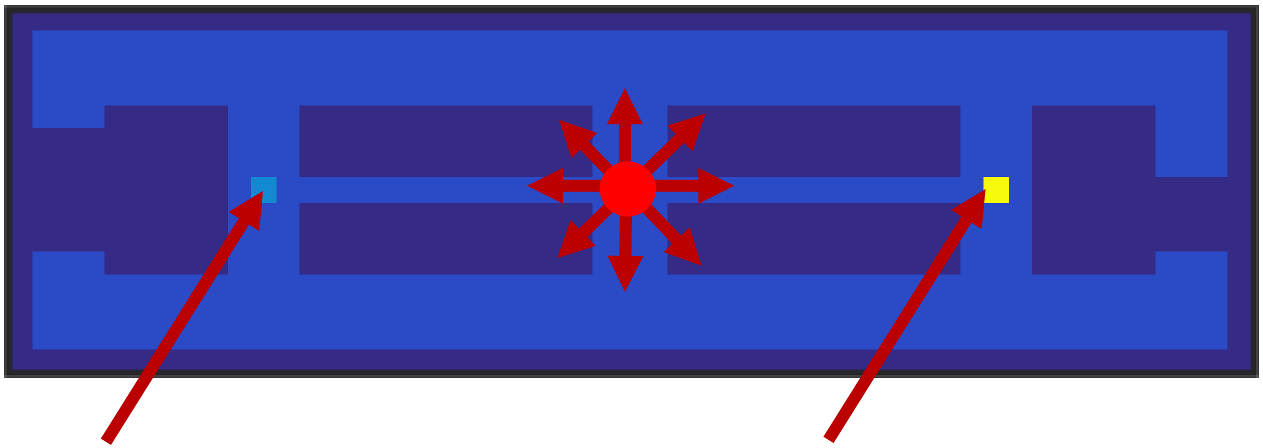

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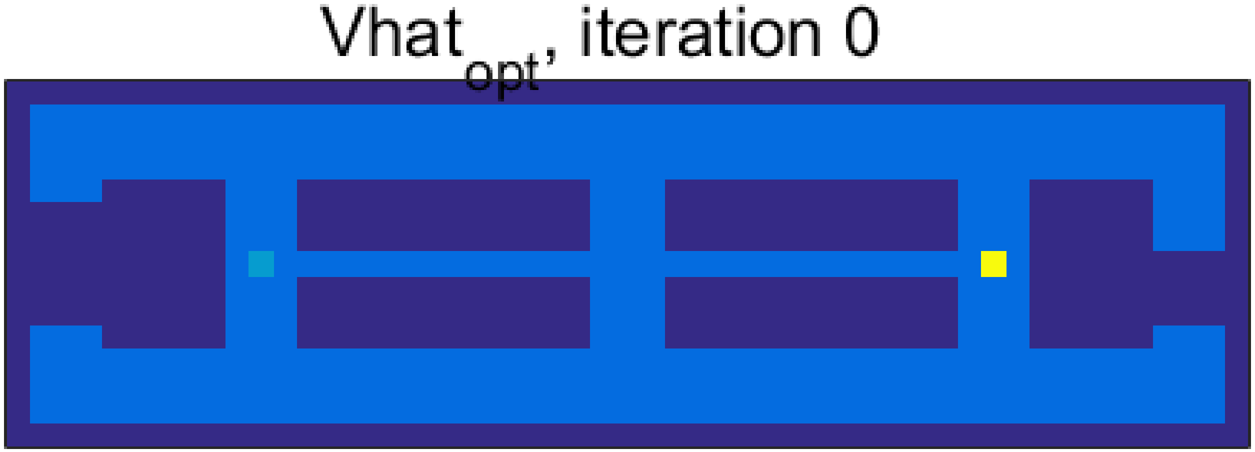

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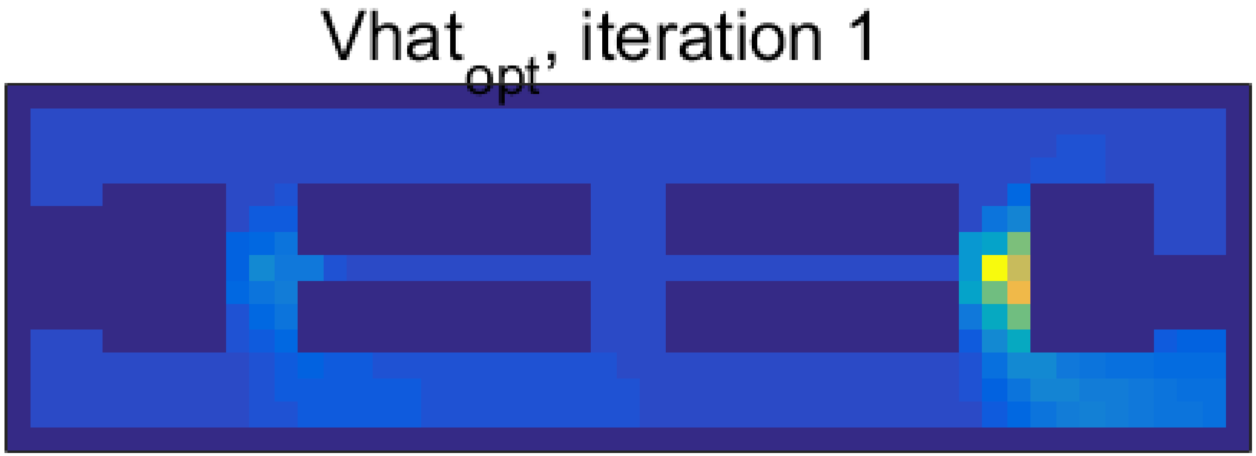

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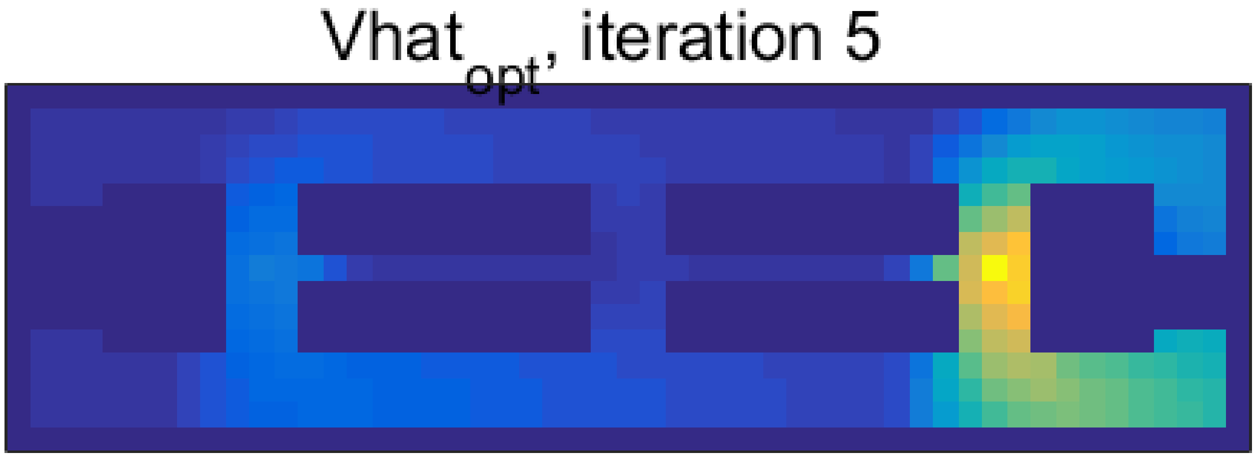

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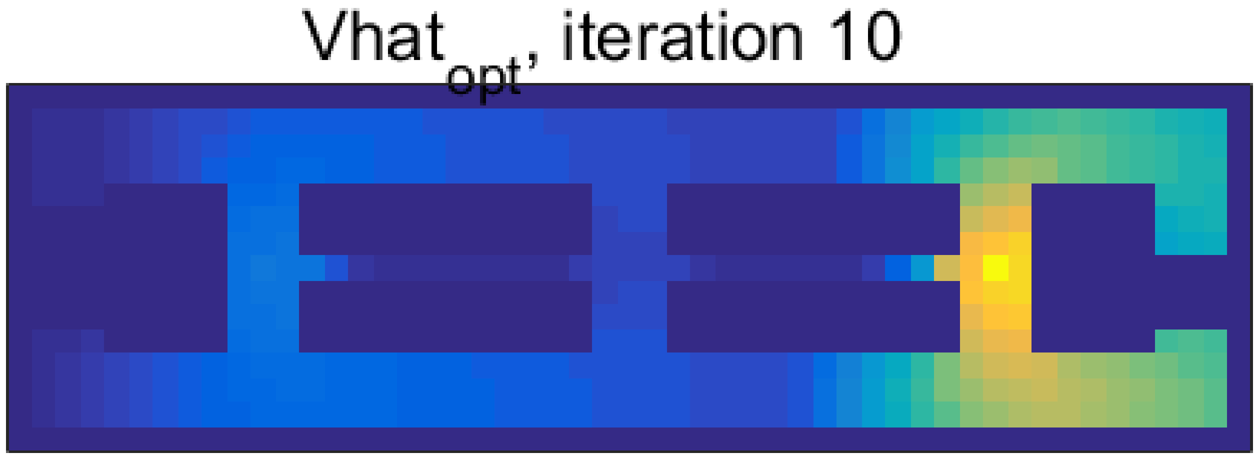

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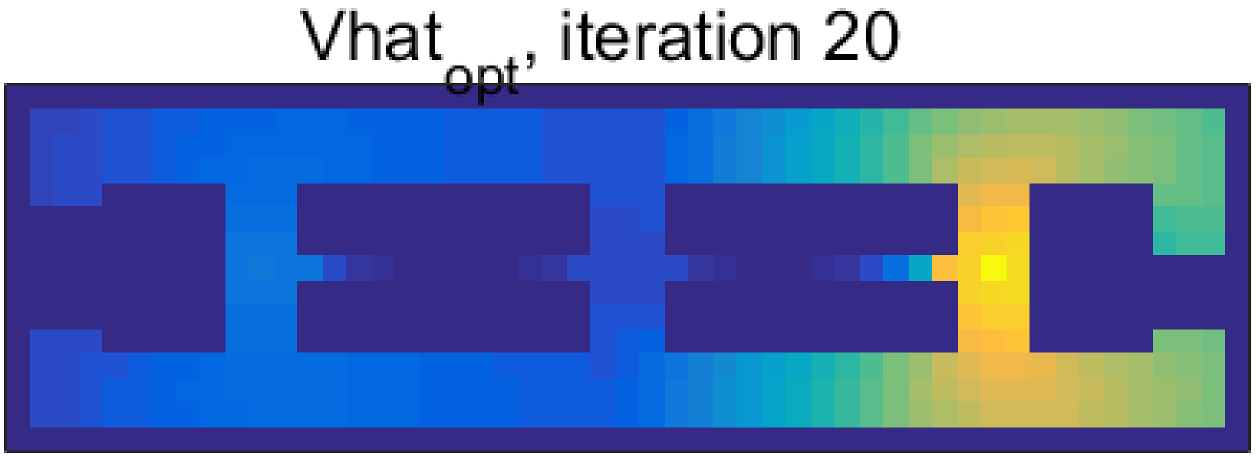

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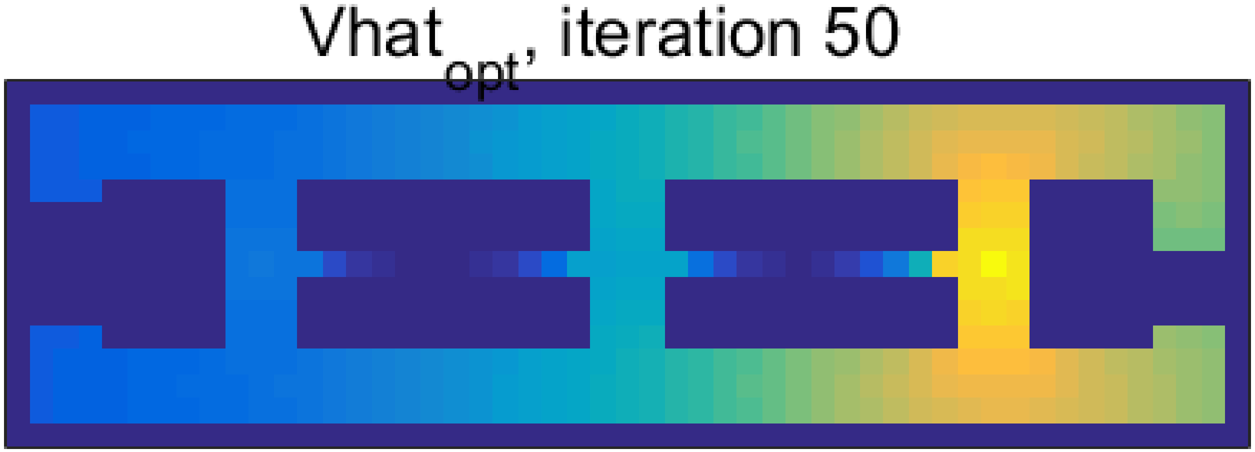

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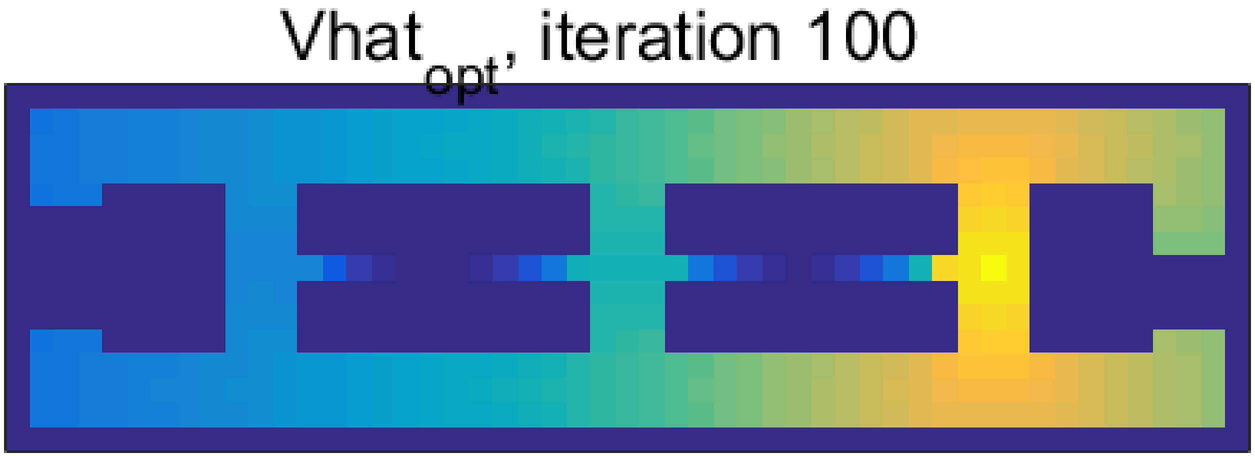

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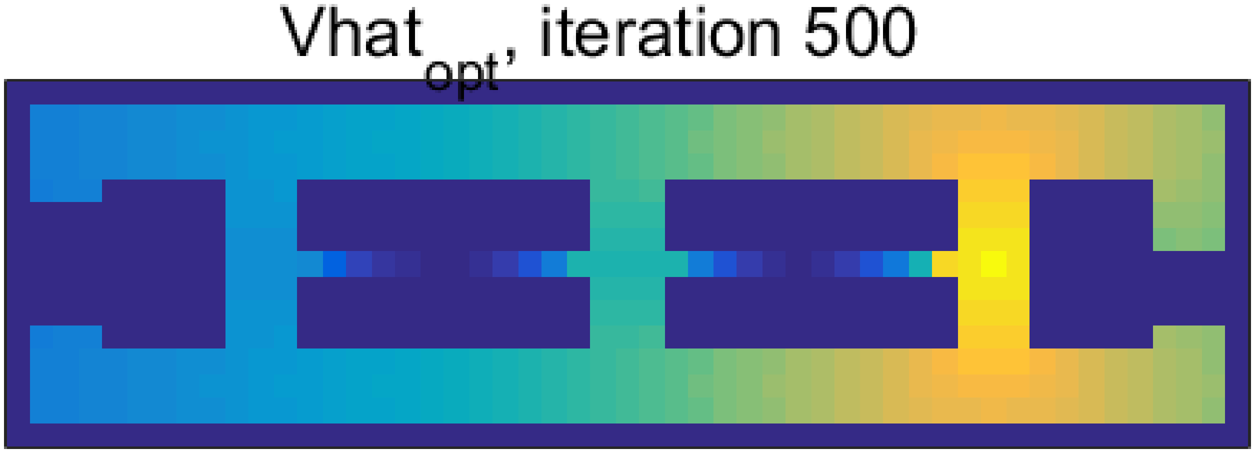

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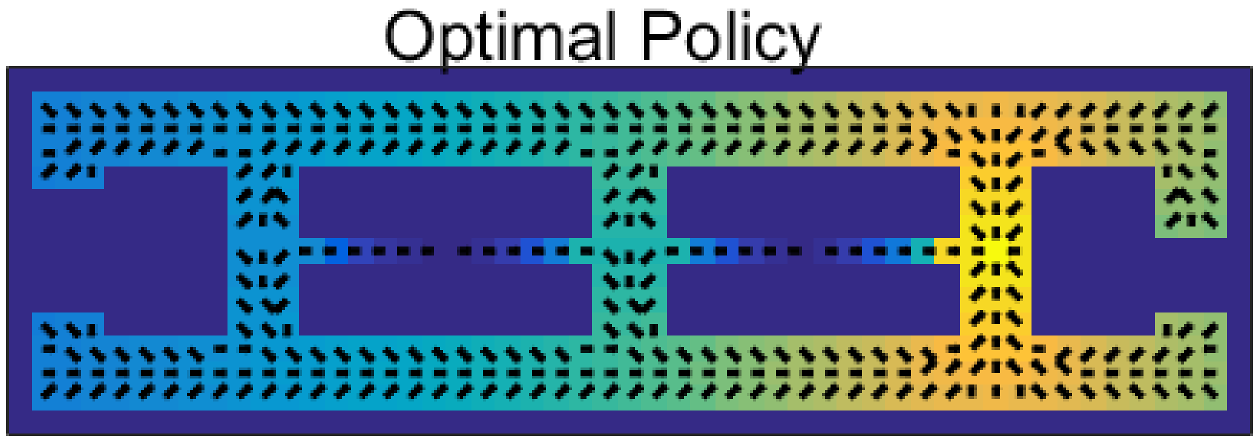

POLICY ITERATION

Greedy policy with respect to V:

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THE CONVERGENCE OF ~~VALUE~~ policy ITERATION: PROOF SKETCH

Just replace T^* with the operator
$B^*: f \mapsto (T^{\pi_f})^\infty$

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EPILOGUE

from
Dynamic Programming
to
Reinforcement Learning

FROM DYNAMIC PROGRAMMING TO VALUE-BASED REINFORCEMENT LEARNING

Policy iteration:

$$V_k$$

FROM DYNAMIC PROGRAMMING TO VALUE-BASED REINFORCEMENT LEARNING

Policy iteration:

improve policy

$$\pi_k = \pi_{V_k}$$

$$V_k \qquad \pi_k$$

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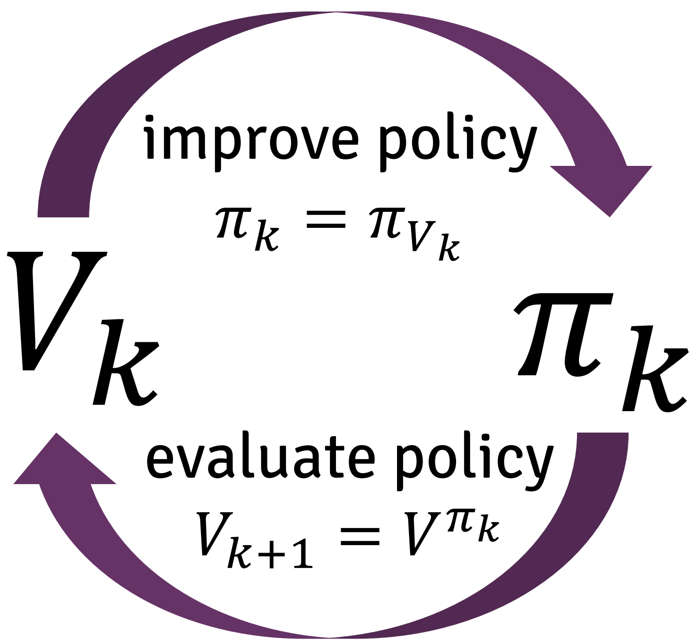

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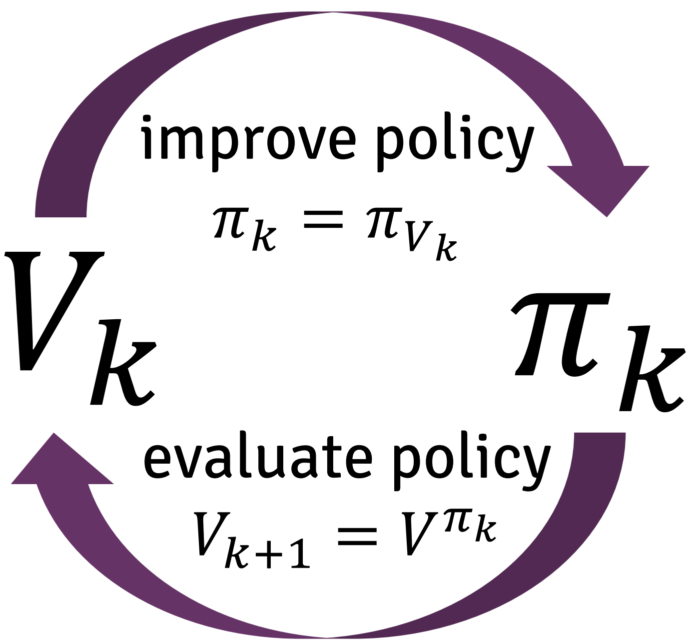

Approximate policy iteration: