# Lecture 4: Multi-armed bandits extensions and applications

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# Optimal solutions for stationary bandit problem

# MDP formulation and dynamic programming

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- Tractable approximation with Bayesian Adaptive Monte Carlo Planning (Guez, Silver, Dayan, 2012; 2014)

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  - Repeat this for every arm
  - Choose an arm with largest index in each trial: $\pi(s_t) = \text{argmax}_{a \in A} G(s_t(a))$

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Source: Lin, Zhang, Hauser (2015)

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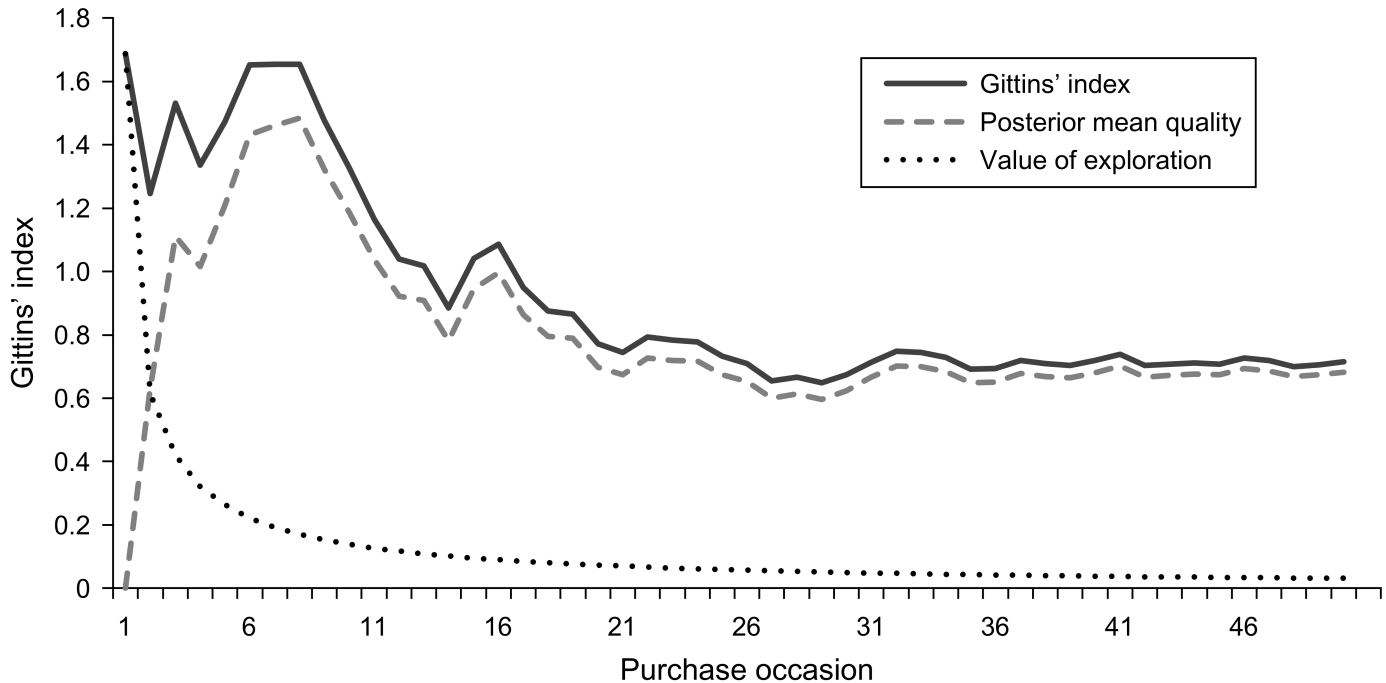


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# Restless bandits

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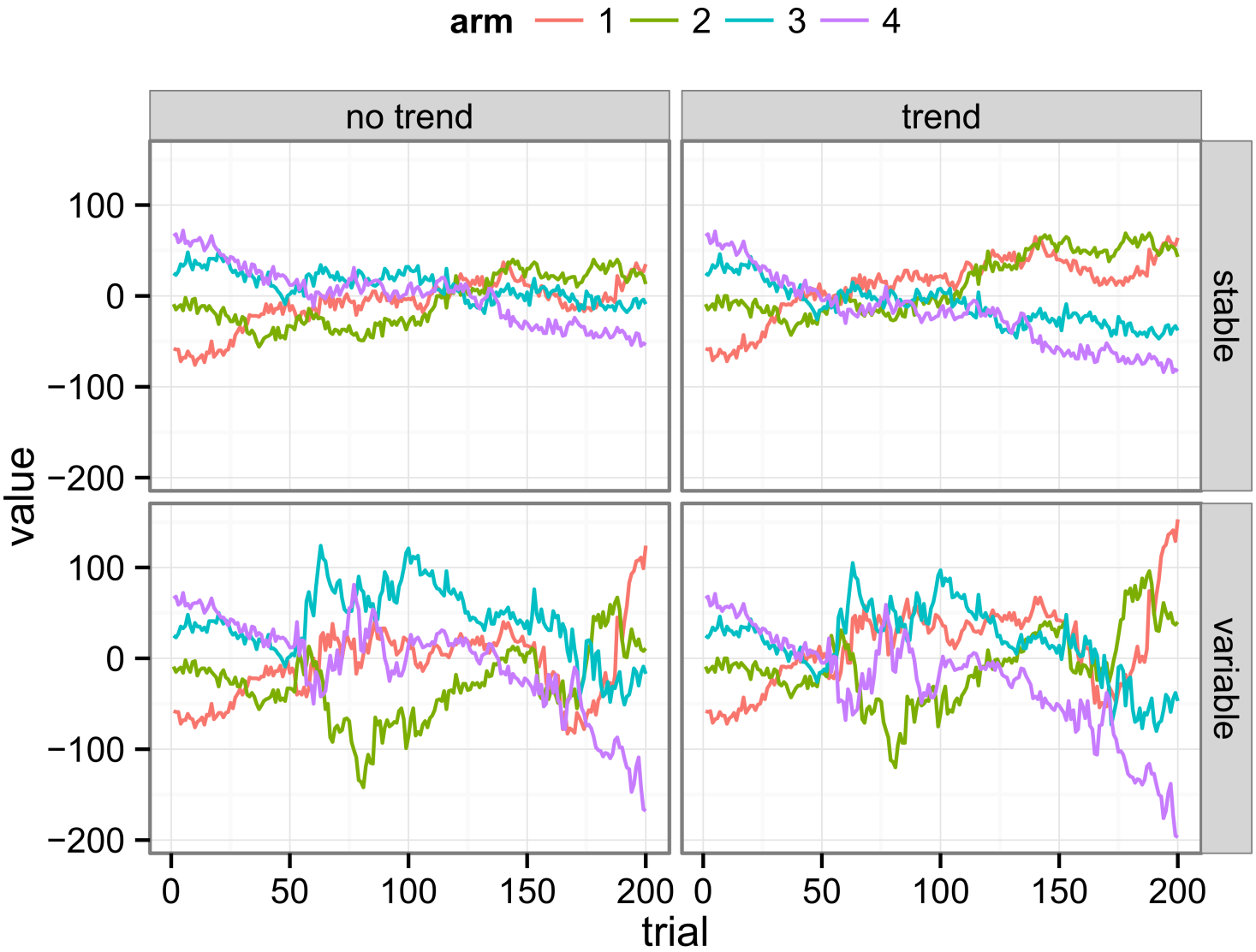
- Solutions for stationary bandits will likely not work well here

# Ornstein-Uhlenbeck process

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- Other options: sliding-window or discounting UCB (Kocsis & Szepesvári, 2006; Garivier & Moulines, 2011; Auer et al., 2019)

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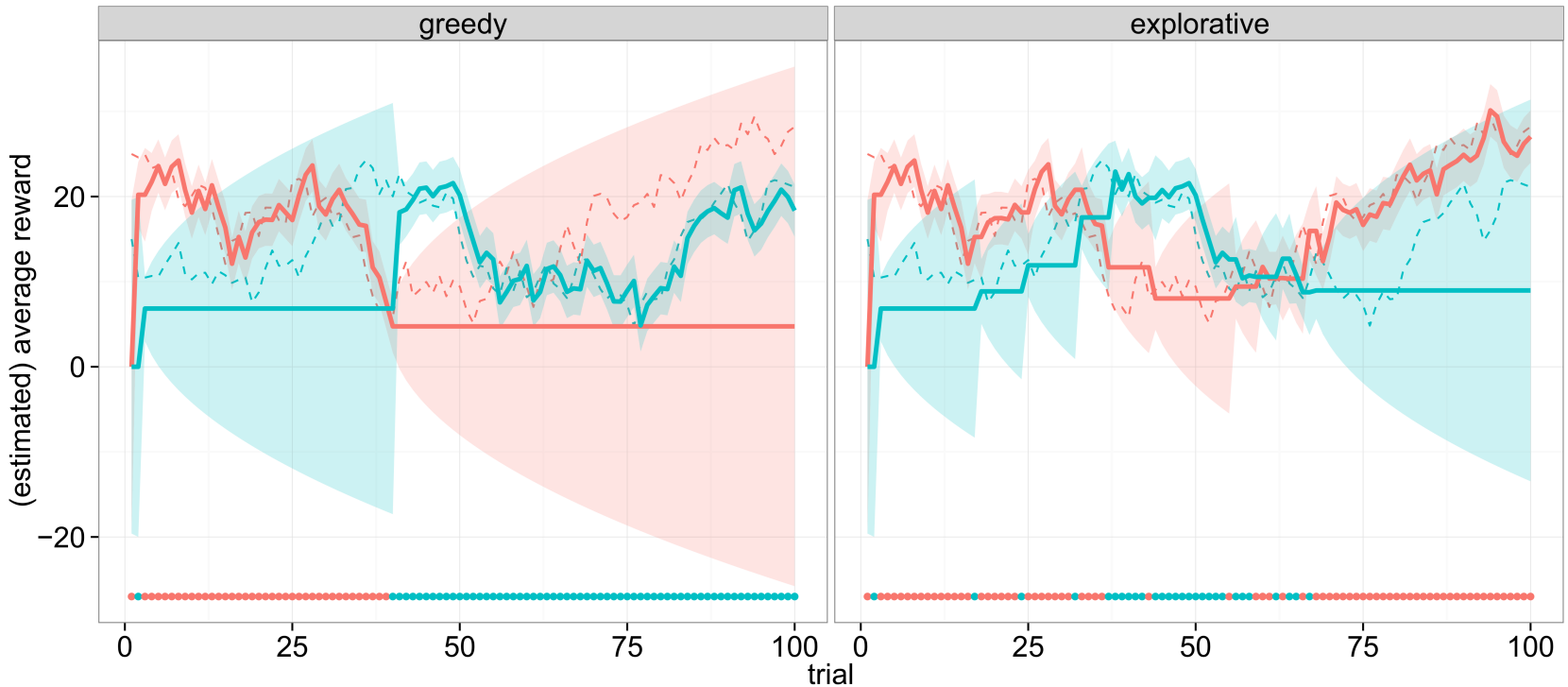


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- If you can potentially gather observations that underlie the process, you might be better off switching to a contextual bandit formulation

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- Arbitrary sequence of rewards for each arm are generated before the game begins, with a constraint that they are from a bounded real interval, $r_1, r_2, ..., r_t \in [0, 1]^A$
- Goal is to design a policy $\pi$ that keeps the regret small no matter the sequence of rewards

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  - But keeps a (stationary) idea of a single best action

- Arbitrary sequence of rewards for each arm are generated before the game begins, with a constraint that they are from a bounded real interval, $r_1, r_2, ..., r_t \in [0, 1]^A$
- Goal is to design a policy $\pi$ that keeps the regret small no matter the sequence of rewards

- Performance measure with a *weak regret*:

$$L_T(\pi) = \max_a \sum_{t=1}^{T} r_t(a) - E[\sum_{t=1}^{T} r_t(a_t)]$$

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- Upper bound on the weak regret: $L_T(\text{Exp3}) \leq (e - 1)\gamma G_{\max}(T) + \frac{K \log K}{\gamma}$

# Performance illustrations

Source: Seldin & Slivkins (2014)

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- For an example on real world trading data, see a blog post: UCB1 fares better than Exp3

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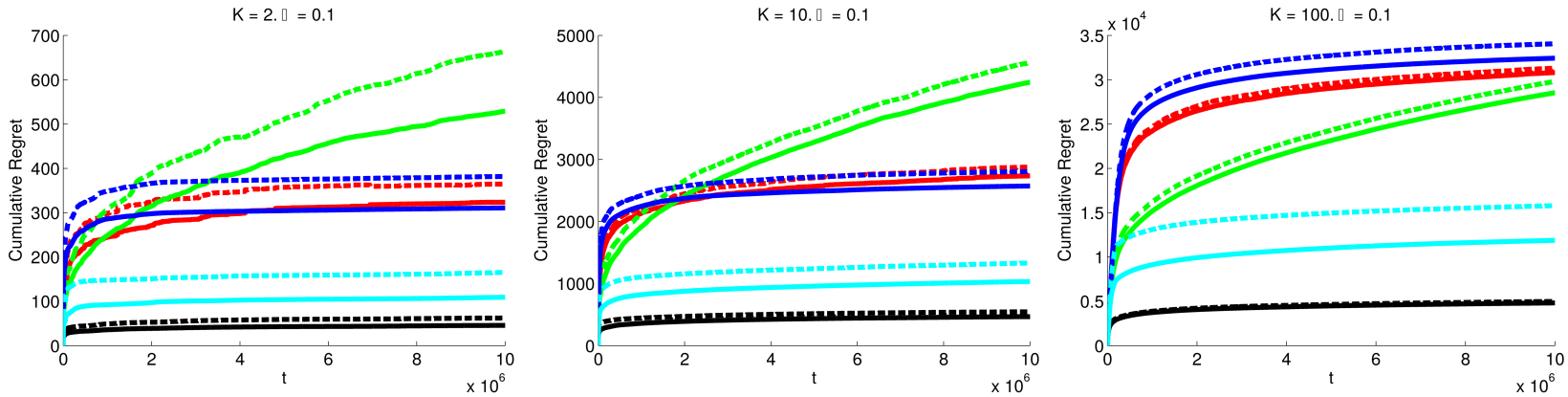
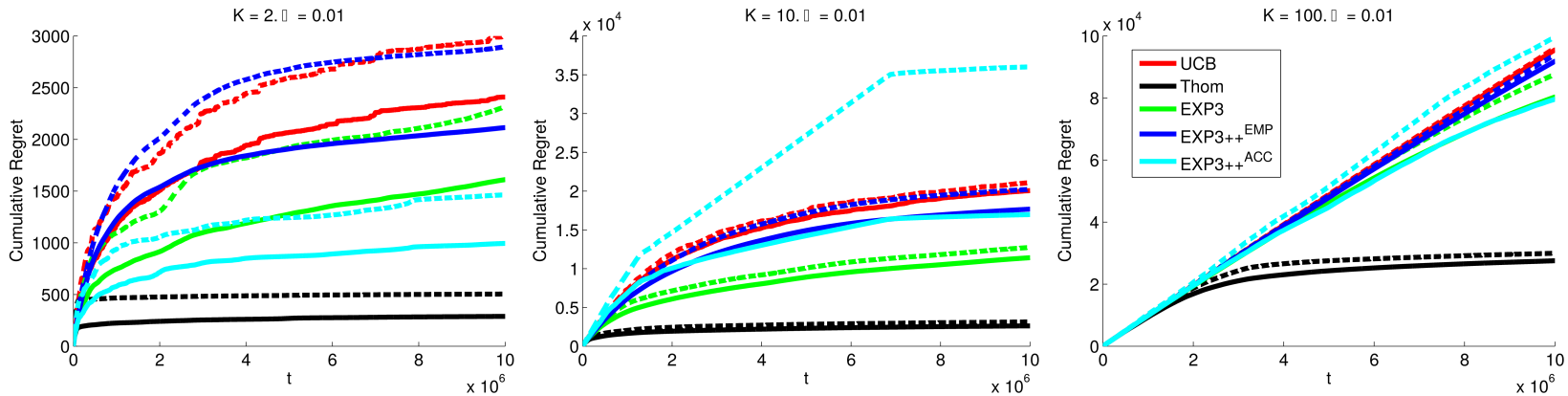



(a) $K = 2, \Delta = 0.1$ (b) $K = 10, \Delta = 0.1$ (c) $K = 100, \Delta = 0.1$

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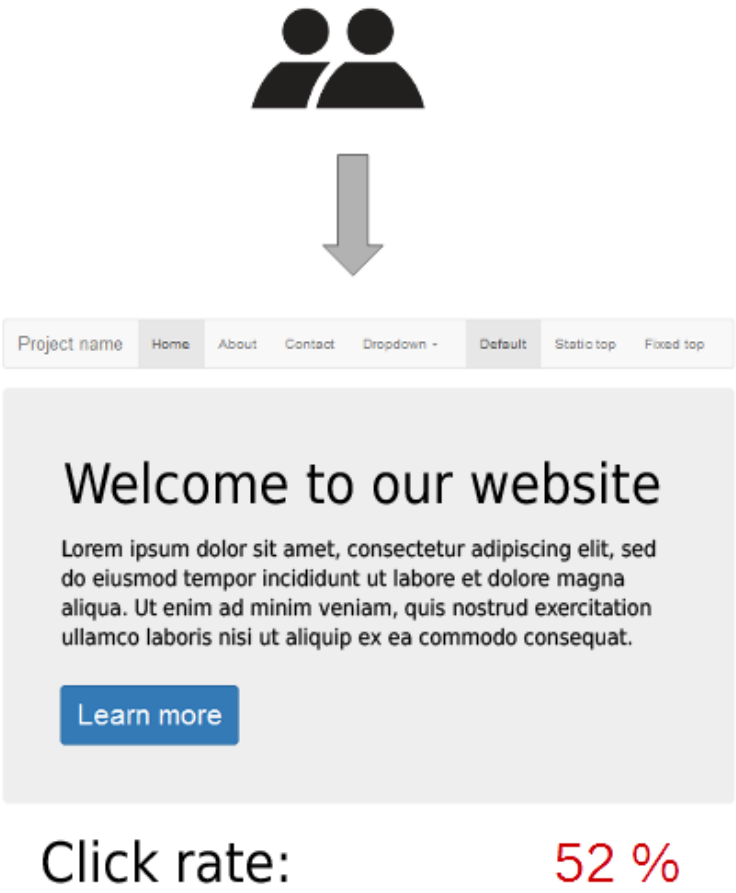
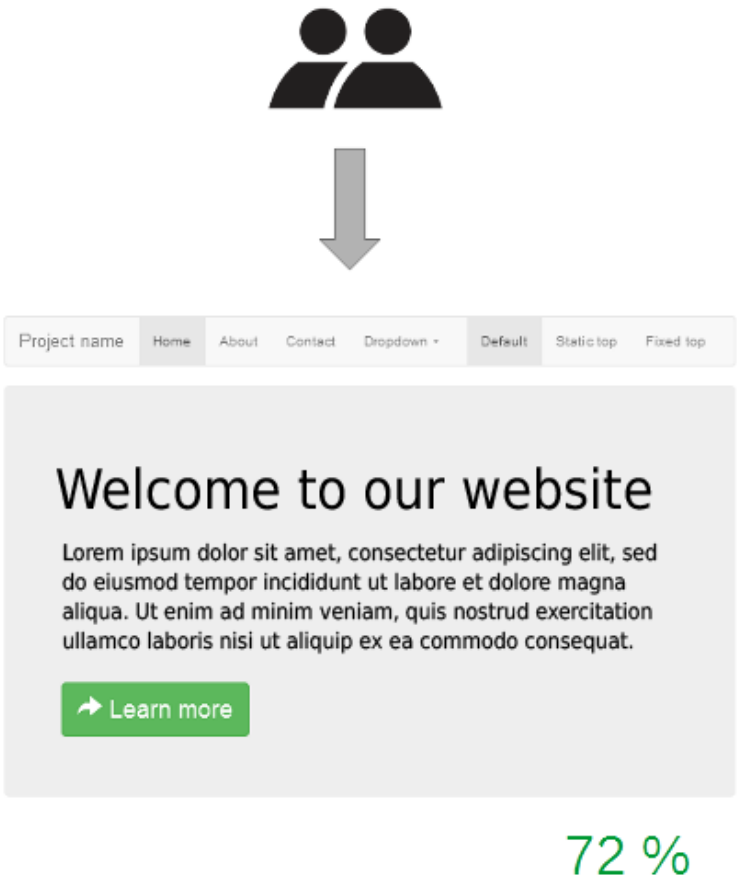
- A bit too extreme, should be called *paranoid* bandit
  - Real world payoffs are almost never entirely adversarial
  - But guarantees and performance show that we don't lose much by still using the Exp3

# Applications: A/B testing

# Version of a website that max interaction




Source: Wikipedia

# Displaying ads most likely to be clicked



Source: Me, buying a keyboard to alleviate my wrist pain :)

# Choosing frontpage stories most likely to be read


Source: theguardian.com, May 6, 2020

# Choosing a political ad that attracts the voters (and $)



Source: Optimizely Blog, retrieved May 6, 2020

# Classical hypothesis testing approach

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- Randomized controlled trial aka experiment:
  - Assign $N$ users randomly to conditions A and B (e.g. default and new website design)
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- This can have big business impact, clicks missed out due to more users in bad conditions can result in a loss of revenue

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# What can research tell us which algo to choose?

Source: Chapelle & Li (2011)

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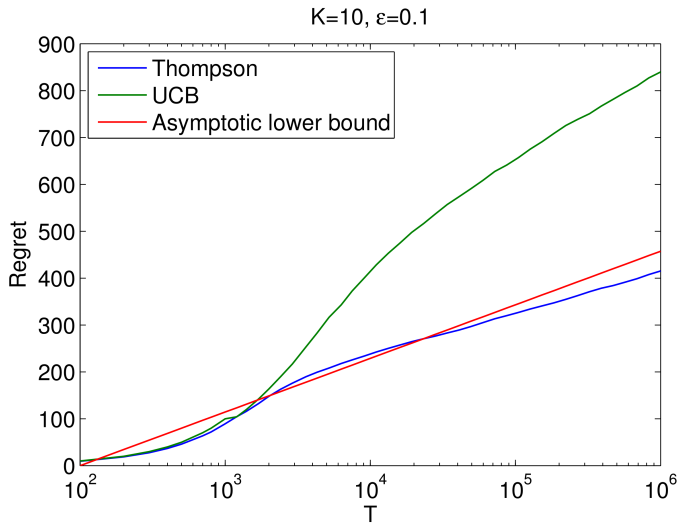


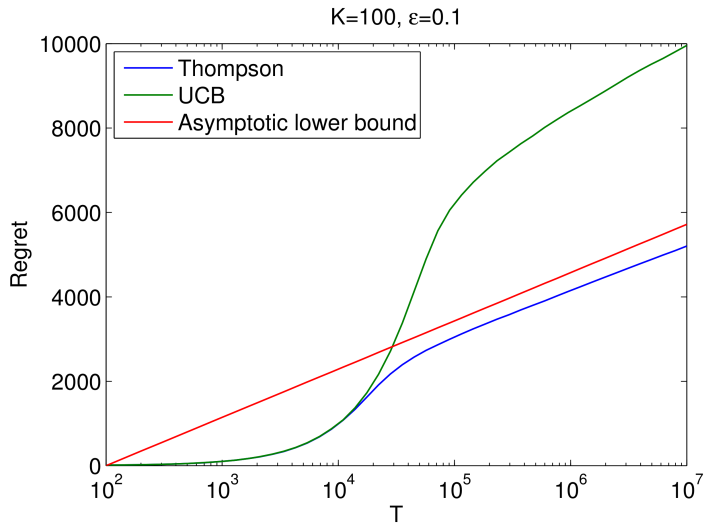


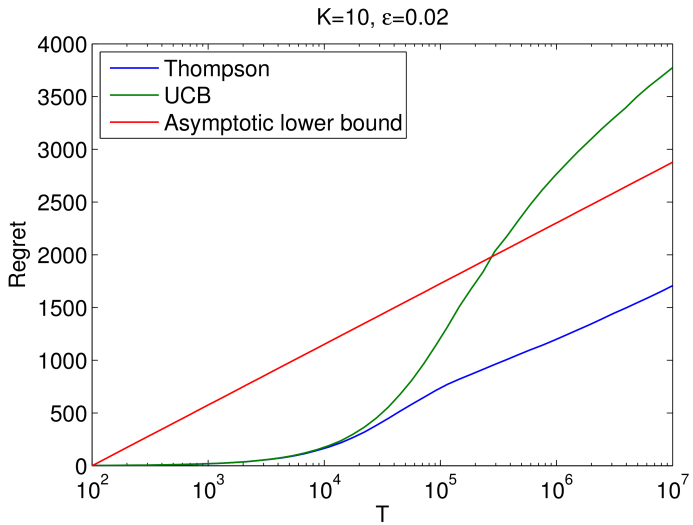


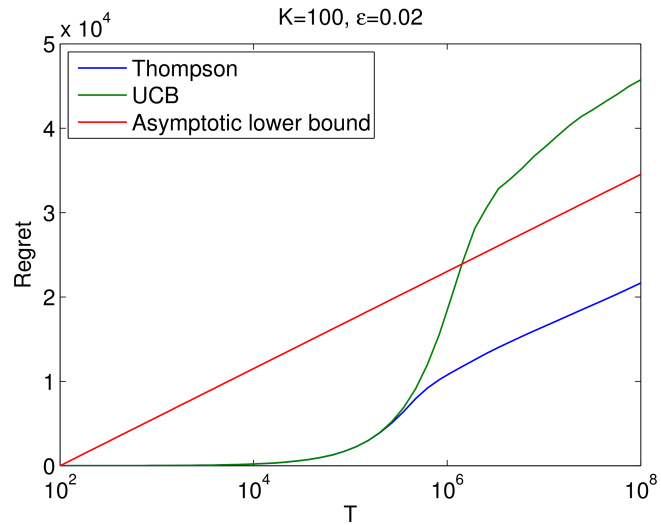


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# Some ethical dilemmas as well


Source: Dilbert