Lecture 3: Multi-armed bandits

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April 29, 2020

The roadmap

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- Multi-armed bandits (MAB) - 2 lectures
  - Naive classics: ϵ-greedy, Softmax
  - Optimism in the face of uncertainty with Upper confidence bound (UCB)
  - Bayesian bandits with Thompson sampling
  - Overview of extensions
  - Application: A/B testing
  - Problem set 1

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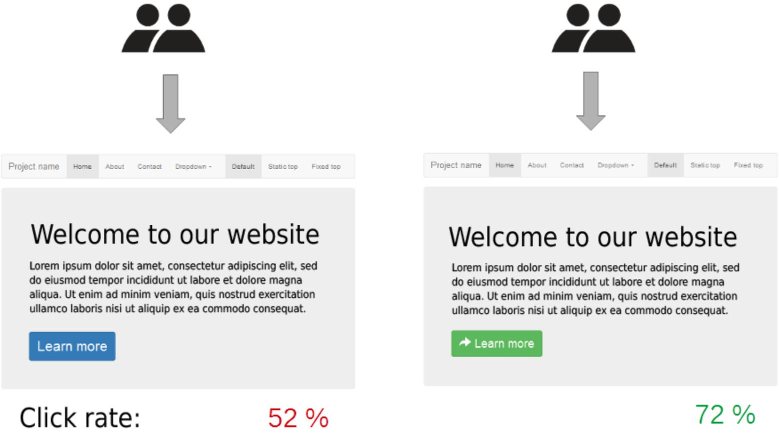
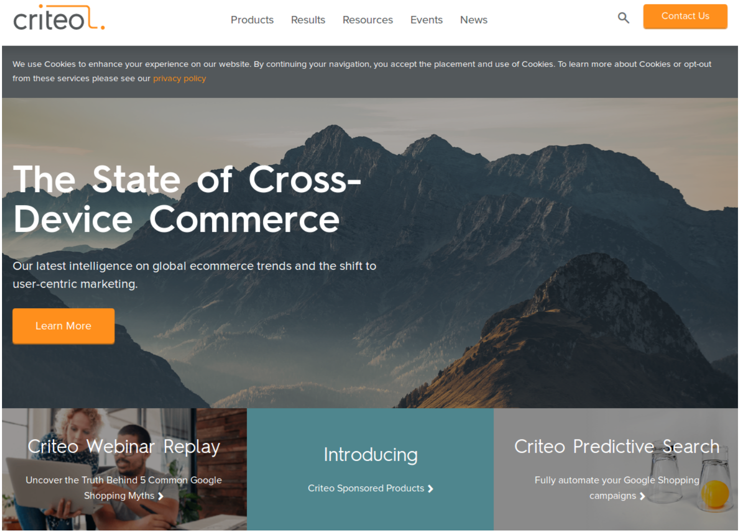
- Contextual bandits (CMAB) - 2 lectures
  - Linear models with UCB
  - Introduction to Gaussian processes (GP)
  - Bayesian optimization with GP-UCB
  - Overview of extensions
  - Application: Optimizing hyperparameters
  - Problem set 2

A/B testing



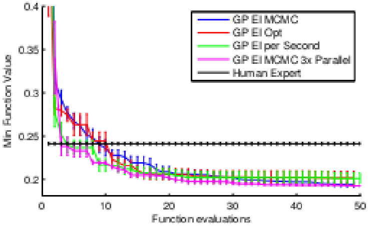
Source: Wikipedia

Recommender systems and ad placement



Source: Criteo webpage

AutoML and hyperparameter tuning



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TPE	**14.13** ±0.30 %	**44.55** ±0.44%
GP	16.70 ± 0.32%	47.08 ± 0.44%
Manual	18.63 ± 0.34%	47.39 ± 0.44%
Random	18.97 ± 0.34 %	50.52 ± 0.44%

Table 2: The test set classification error of the best model found by each search algorithm on each problem. Each search algorithm was allowed up to 200 trials. The manual searches used 82 trials for **convex** and 27 trials **MRBI**.

Source: Snoek et al 2012; Bergstra et al 2011

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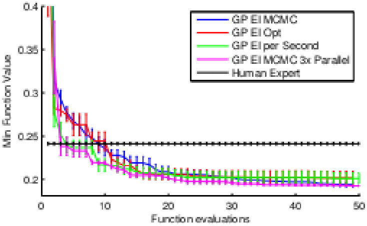

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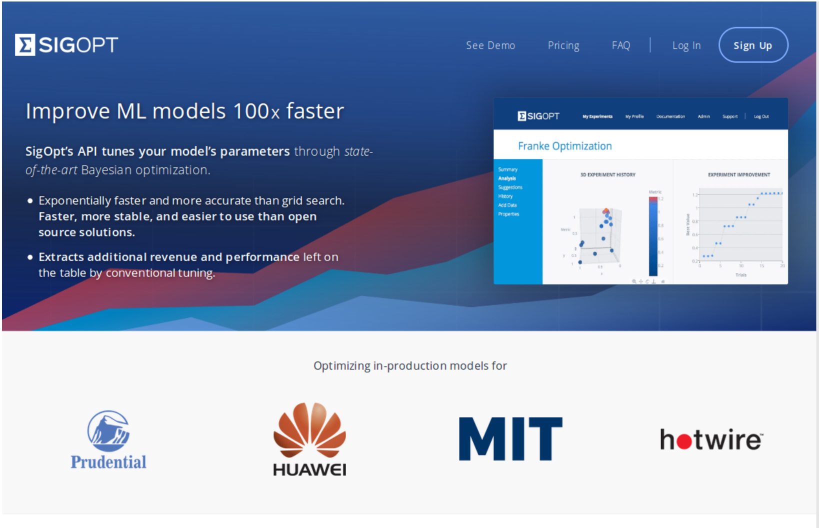
- CIFAR 10: state of the art was test error of 18%, they achieved 14.98%
- MNIST rotated background images

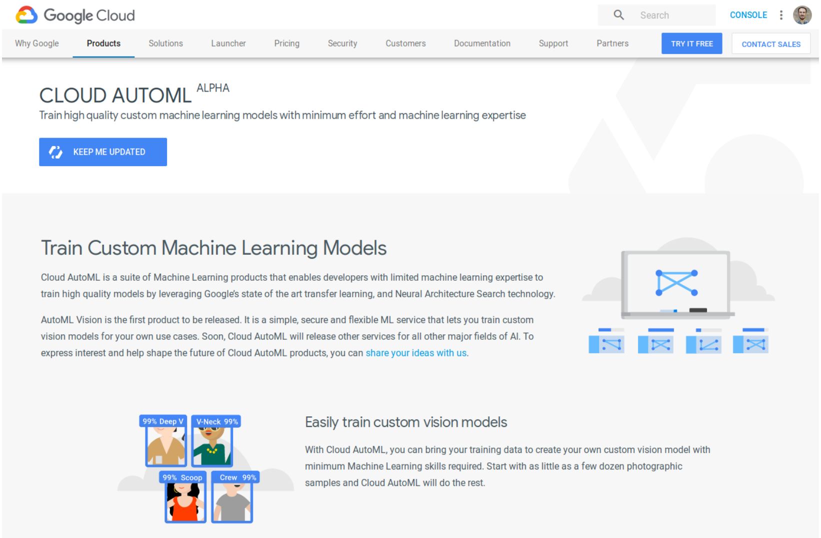
Source: Snoek et al 2012; Bergstra et al 2011

Bayesian optimization going mainstream



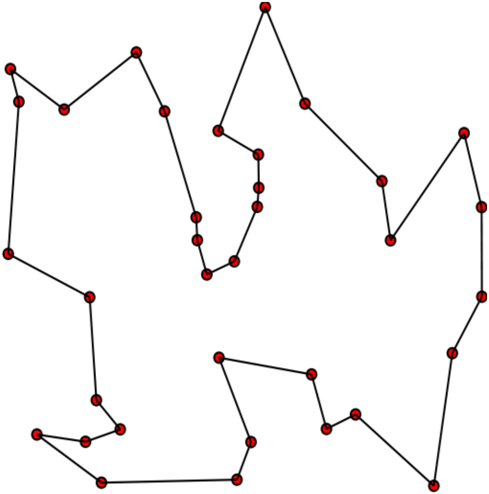
Source: SigOpt webpage

Google Cloud AutoML for computer vision



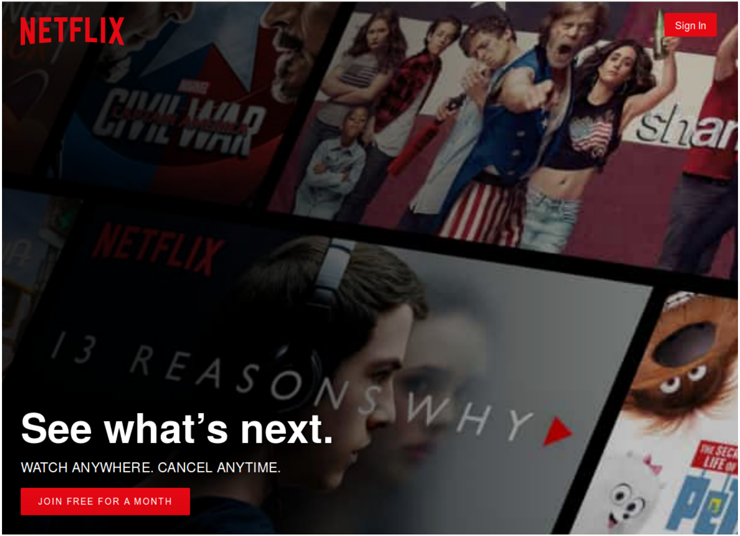
Source: Google AutoML webpage

Optimizing parameters of combinatorial optimization software



Source: Wikipedia

Preference learning and interactive user interfaces



Source: Netflix webpage

References

- Bandits

  - Sutton, R., & Barto, A. (2018). Introduction to Reinforcement Learning (book free of charge: www.incompleteideas.net/sutton/book/the-book.html)
  - Lattimore, T., & Szepesvári, C. (2020). Bandit algorithms. (book free of charge: banditalgs.com/)
  - Szepesvári, C. (2010). Algorithms for Reinforcement Learning.
  - D. Silver's lectures (videos and slides: www0.cs.ucl.ac.uk/staff/D.Silver/web/Teaching.html)

- Gaussian Processes

  - Rasmussen, C. E., & Williams, C. K. I. (2006). Gaussian processes for machine learning. MIT Press. (book free of charge: www.gaussianprocess.org/gpml/)
  - Carl Rasmussen's lectures (videos)
  - Nando De Freitas' lectures (videos and slides: www.youtube.com/user/ProfNandoDF/videos)

References

- Contextual bandits, Bayesian optimization

  - Auer, P., Cesa-Bianchi, N., & Fischer, P. (2002). Finite-time analysis of the multiarmed bandit problem. Machine learning, 47(2-3), 235-256
  - Li, L., Chu, W., Langford, J., & Schapire, R. E. (2010). A contextual-bandit approach to personalized news article recommendation. In Proceedings of the 19th international conference on World wide web (pp. 661-670).
  - Snoek, J., Larochelle, H., & Adams, R. P. (2012). Practical Bayesian Optimization of Machine Learning Algorithms. Advances in Neural Information Processing Systems, 2951-2959.
  - Shahriari, B., Swersky, K., Wang, Z., Adams, R. P., & de Freitas, N. (2016). Taking the Human Out of the Loop: A Review of Bayesian Optimization. Proceedings of the IEEE, 104(1), 148–175.

Software

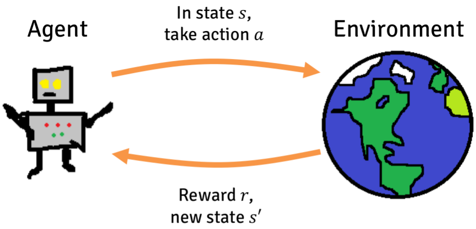
- Python libraries
  - RLlib
  - Tensorflow agents
  - scikit-learn, auto-sklearn
  - Hyperopt (Bergstra et al., 2011)
  - Spearmint (Snoek et al., 2014)
- R packages
  - GPfit, gptk, FastGP
  - rBayesianOptimization
  - DiceOptim (Roustant et al., 2012)
- Matlab
  - GPML (Rasmussen)
- C++
  - BayesOpt (Martinez-Cantin, 2014)
- Java
  - SMAC (Hutter et al., 2011)
  - AutoWEKA

Practicalities

- Contact:
  - hrvoje.stojic_youknowwhat_protonmail.com
  - Office hours by video calls
- Materials:
  - Videos and slides will be uploaded to Box
  - In addition, I will have the whole course posted on Github, where you will have access to the source code
- Evaluation:
  - No exam
  - Individual problem sets 50% and group projects 50%
  - We are still deciding on the exact form of the group projects, we will let you know the details soon!

Multi-armed bandits

A subclass of reinforcement learning problems



Learning to
- maximize reward
- in a reactive environment
- under partial feedback

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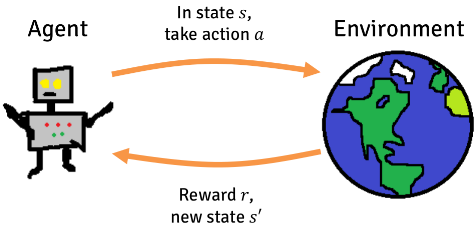

Learning to
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Two key challenges of RL:

1. Dealing with long-term effects of actions
2. Dealing with uncertainty due to partial feedback

Partial feedback -> Exploration exploitation problem

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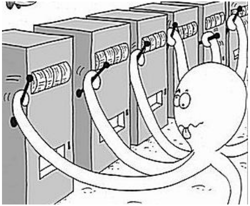
- Examples:
  - Going to a favourite restaurant (**exploitation**), or try a new restaurant (**exploration**)
  - Show the most successful ad (**exploitation**), or show a new ad (**exploration**)

Formulation



- A tuple $\langle \mathcal{A}, \mathcal{R} \rangle$
- $\mathcal{A}$ is a (stationary) set of K actions/arms
- $\mathcal{R}^a(r) = P[r|a]$ is an unknown but stationary probability distribution over rewards
- At each step t the agent selects an action $a_t \in \mathcal{A}$
- The environment generates a reward $r_t \sim \mathcal{R}^{a_t}$
- The goal is to maximise cumulative reward $\sum_{t=1}^{\tau} r_t$

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- The **cumulative regret** is the total opportunity loss $L_t = E[\sum_{\tau=1}^{t} V^* - Q(a_\tau)]$
- Regret can be expressed in terms of counts and gaps:
  - The count $N_t(a)$ is number of selections for action a
  - The gap Δ_a is the difference in value between action a and optimal action a^*, $\Delta_a = V^* - Q(a)$
  - The cumulative regret, stated differently:

$$L_t = \sum_{a \in \mathcal{A}} E[N_t(a)](V^* - Q(a)) = \sum_{a \in \mathcal{A}} E[N_t(a)]\Delta_a$$

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$$\hat{Q}_t(a) = \frac{1}{N_t(a)} \sum_{t'=1}^{t} r_{t'} \mathbf{1}(a_{t'} = a)$$

or

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   - A policy, $\pi(a)$, given estimates of action values, $\hat{Q}_t(a)$, gudies actions

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  - I.e agents would identify the optimal action value, V^*

Rewards and % optimal actions

Source: Sutton & Barto 2018

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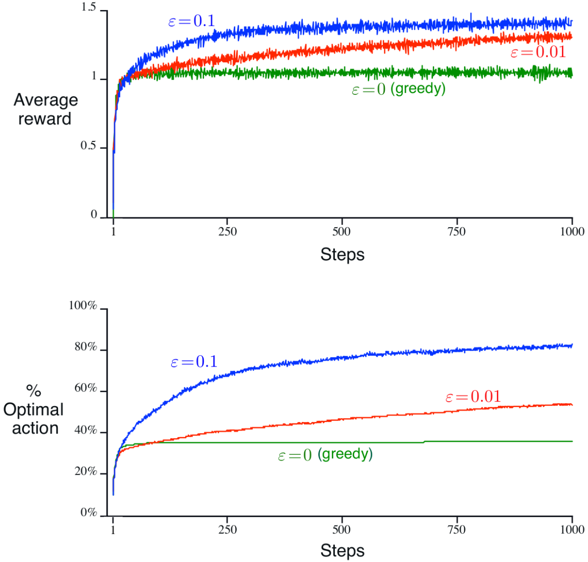

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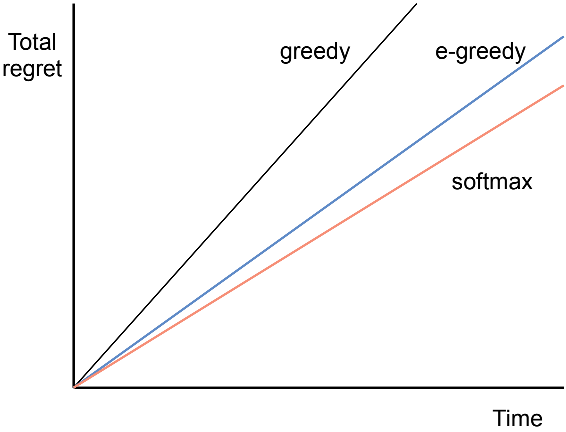

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- There are similar formulations for decaying temperature in Softmax
- These algorithms can achieve very good performance
- But these are heuristic approaches and it is usually difficult to tune the decay

Brief diversion into gradient bandit algorithms

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  - and initial preferences set to $H_1(a) = 0$

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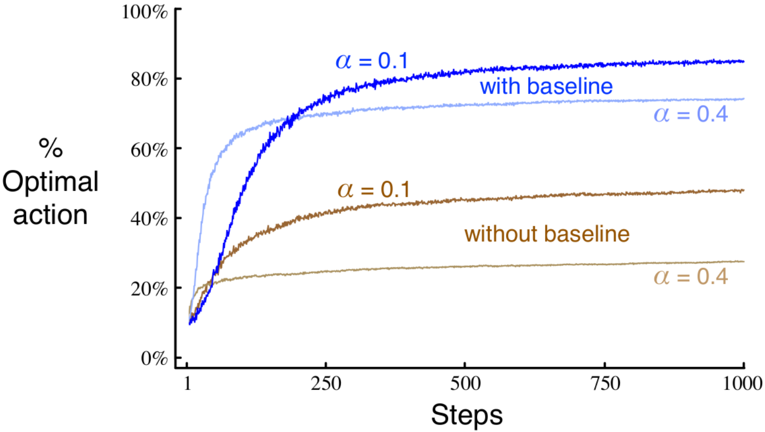

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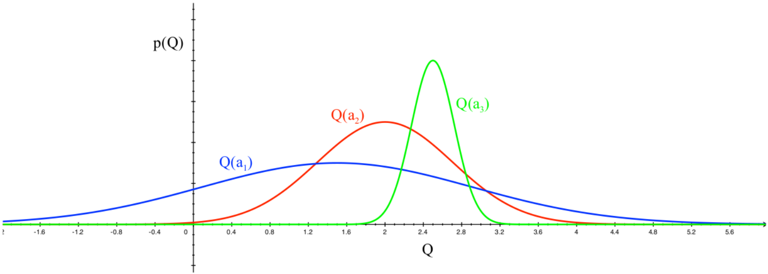

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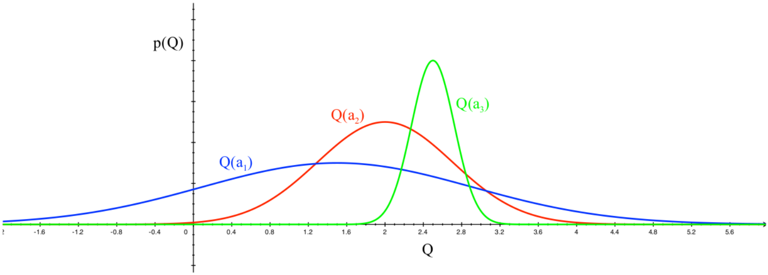

- Optimistic initialization and fixed uncertainty bonus approaches are based on the same principle

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- UCB1 algorithm (Auer et al, 2002)

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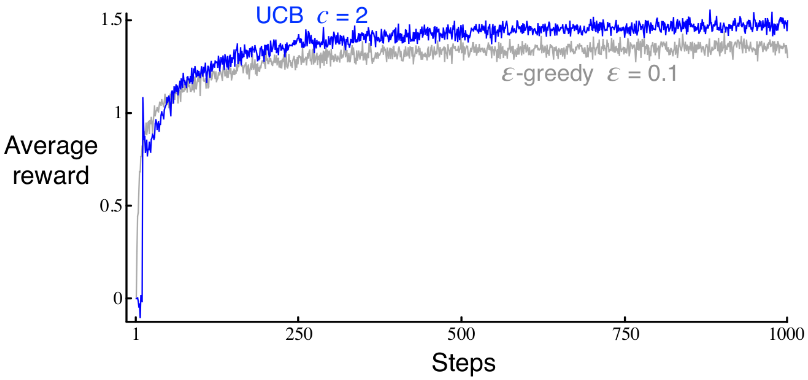
UCB performance



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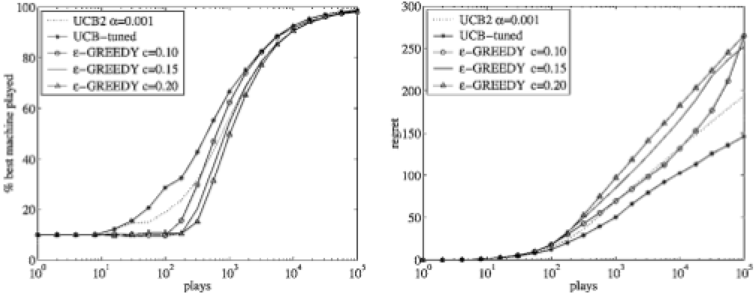

Figure 10. Comparison on distribution 12 (10 machines with parameters 0.9, 0.8, 0.8, 0.8, 0.7, 0.7, 0.7, 0.6, 0.6, 0.6).

Source: Auer, P., Cesa-Bianchi, N., & Fischer, P. (2002). Finite-time analysis of the multiarmed bandit problem. Machine Learning, 47, 235-256.

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- Wrong distribution assumption and priors might cause issues.

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- Observations are collected in $\mathcal{D}_t = \{(a_\tau, r_\tau)\}_\tau^t$ as a set of tuples, where a_τ identifies the arm and r_τ is the reward

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- Thompson sampling (Thompson, 1933; Chapelle, Li, 2010)
  - Sample $\mathbf{w}'$ from each posterior and then maximize,

$$a_{t+1} = \mathrm{argmax}_a f_{\mathbf{w}'}(a), \text{where } \mathbf{w}' \sim P[\mathbf{w}|\mathcal{D}_t]$$

  - Thompson sampling achieves Lai and Robbins lower bound!

Algorithm & Example

Algorithm 2: Thompson Sampling for Beta-Bernoulli Bandit

Require: α, β: hyperparameters of the beta prior
Initialize $n_{a,0} = n_{a,1} = i = 0$ for all a
repeat
  for $a = 1, \ldots, K$ **do**
    $\tilde{w}_a \sim \text{beta}(\alpha + n_{a,1}, \beta + n_{a,0})$
  end for
  $a_i = \arg\max_a \tilde{w}_a$
  Observe y_i by pulling arm a_i
  if $y_i = 0$ **then**
    $n_{a_i,0} = n_{a_i,0} + 1$
  else
    $n_{a_i,1} = n_{a_i,1} + 1$
  end if
  $i = i + 1$
until stopping criterion reached

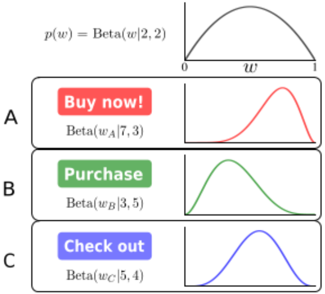

Fig. 2. *Example of the beta-Bernoulli model for A/B testing. Three different buttons are being tested with various colors and text. Each option is given two successes (click-throughs) and two failures as a prior (top). As data are observed, each option updates its posterior over w. Option A is the current best with five successes and only one observed failure.*

Source: Shahriari, B., Swersky, K., Wang, Z., Adams, R. P., & de Freitas, N. (2016).
Taking the Human Out of the Loop: A Review of Bayesian Optimization.
Proceedings of the IEEE, 104(1), 148-175.

Thompson sampling performance