Georgia Tech Spring 2023 Heinrich Matzinger

Solution for Test 0 for math3670

Name: GT-number: Major:

0a) Prove that $P(A \cup B) = P(a) + P(B) - P(A \cap B)$

0b) Prove the Law of large numbers. That is assume that $X_1, X_2, \ldots,$ are i.i.d. Show then that the average converges to the expectation

$$\frac{X_1 + X_2 + X_3 + \ldots + X_n}{n} \to E[X_1],$$

as n goes to infinity. You can assume that the variables X_i can take only values $1, 2, 3, 4$.

1) The probability that you are infected with a given illness is 10%. Let I designate the event that you are infected. (Only a blood test can reveal if you are infected, there are no visible symptoms). If you are infected and take a test, then the test has a 98% probability to detect that you are infected. If you are not infected the test always works correctly. Before doing a test, you have a 10% probability of being infected. Now, you do the test and the test tells you that you are not infected. After that test (given the result of the test), what is the probability that you are infected. Let N be the event that the test gave a negative (not-infected) result. **Answer:** We have

$$P(I|N) = \frac{P(N|I)P(I)}{P(N|I)P(I) + P(N|I^c)P(I^c)} = \frac{0.02 \cdot 0.1}{0.02 \cdot 0.1 + 1 \cdot 0.9} = 0.0022.$$

2) When you do not learn for an exam with Matzinger, the probability to get an f is 0.8. If you learn, the probability to get an f is 0.2. In Matzinger's exams on the long run, there are 40% of students getting an f. Let F designate the event that a student (randomly chosen in Matzinger's class) gets the grade f. Let NL be the event that this student did not learn for the test. Assuming we know that that student did get an f. What is the conditional probability, that he/she did not learn for that test? **Answer:** We want to determine $P(NL|F)$. We can use Bayes rule:

$$P(NL|F) = \frac{P(F|NL) \cdot P(NL)}{P(F)}$$

where $P(F|NL) = 0.8$ and $P(F) = 0.4$. To determine $P(NL)$ note that by total probability, we have

$$P(F) = P(F|NL)P(NL) + P(F|NL^c)P(NL^c).$$

Recall that $P(NL^c) = 1 - P(NL)$. So we obtain

$$P(F) = P(F|NL)P(NL) + P(F|NL^c)(1 - P(NL))$$

and hence

$$0.4 = 0.8P(NL) + 0.2(1 - P(NL)).$$

The last equality above can be solved for the unknown $P(NL)$. We find $P(NL) = 1/3$. Hence,

$$P(NL|F) = \frac{P(F|NL) \cdot P(NL)}{P(F)} = \frac{0.8 \cdot (1/3)}{0.4} = \frac{2}{3} = 0.\bar{6}$$

3) Assume given an electrical network with 4 points and six segments between those points: the segment a goes from x to y, b goes from x to z, c goes from y to z and d goes from y to v whilst f goes from z to v. Finally, the line g goes from x to v. Each of the segments works independently form the others. Let A be the event that segment a works, let B be the event that segment b works,... Let p_a be the probability that a works, hence $p_a = P(A)$. Let p_b be equal to $P(B)$,... We assume that p_a, p_b, p_c, p_d, p_f, p_g are all given to us. Let E be the event that electricity can flow from x to v. Calculate $P(E)$ in terms of p_a, p_b, p_c, p_d,p_f and p_g. (HINT: use law of total probability. Consider for example the case when c is known to work and on the other hand when c is known to not work....) **Answer:** When G works, we are sure that E holds. So, we get by Law of total probablility:

$$P(E) = P(G) + P(E|G^c)P(G^c) = p_g + P(E|G^c) \cdot (1 - p_g) \tag{1}$$

Now, $P(E|G^c)$ is just the probability when we delete the segment g from graph. So, the probability for that simplified graph obtained by deleting g will be denoted by $P_{G^c}(.)$. Use law of total probability:

$$P_{G^c}(E) = P_{G^c}(E|C)P(C) + P_{G^c}(E|C^c)P(C^c) = P_{G^c}(E|C)p_c + P_{G^c}(E|C^c)(1 - p_c).$$

Now,

$$P_{G^c}(E|C^c) = P((A \cap D) \cup (B \cap F)) = p_a p_d + p_b p_f - p_a p_d p_b p_f$$

and

$$P_{G^c}(E|C) = P((A \cup B) \cap (D \cup F)) = (p_a + p_b - p_a p_b)(p_d + p_f - p_d p_f).$$

Now to get the answer, simply plug the values found for $P_{G^c}(E|C)$ and $P_{G^c}(E|C^c)$ into

$$P_{G^c}(E) = P_{G^c}(E|C)p_c + P_{G^c}(E|C^c)(1 - p_c)$$

and then back to equation 1, you find:

$$P(E) = p_g + (1 - p_g)\left[(p_a + p_b - p_a p_b)(p_d + p_f - p_d p_f)p_c + (p_a p_d + p_b p_f - p_a p_d p_b p_f)(1 - p_c)\right].$$

4) Same setting as in the previous problem. What is the (conditional) probability of the event that segment a works , if we know that E holds ? Hence $P(A|E) =$?)(HINT: use Bayse rule....)
Answer: By Bayse we find:

$$P(A|E) = P(E|A) \cdot \frac{P(A)}{P(E)},$$

now $P(E)$ has been calculated in the previous problem and $P(A)$ nis given. So, we are only left with calculating the probability $P(E|A)$. When we assume that A works, then the points y and x become "equivlaent, so we get conditional on A, a netwwork with one less point. Thath is y dissappears and all lines which were out of y are now out of x. In other words in the new network we have c is from x to z and d is from x to v, and line b, g and f are left as before meaning b is from x to z and f is from z to v and g is from x to v So, we need d to work or g or (f and (c or d)). Hence,

$$P(E|A) = P(D \cup G \cup (F \cap (B \cup C))) = P(D \cup G) + P(F \cap (B \cup C)) - P(D \cup G) \cdot P(F \cap (B \cup C)) \quad (2)$$

Now,

$$P(D \cup G) = P(D) + P(G) - P(D \cap G) = P(D) + P(G) - P(D) \cdot P(G) \quad (3)$$

And

$$P(F \cap (B \cup C))) = P(F) \cdot P(B \cup C) = P(F) \cdot (P(B) + P(C) - P(B) \cdot P(C))$$

which you can plug into equation 2 also using 3 to find the formula for $P(E|A)$.

5) A cell phone company offers cell phone insurance for one year. If the cell phone breaks down completely, the company replaces it and this costs the company 500 dollars. This event has a probability of 5 percent, that is the cell phone will break down completely within a year with five percent probability. With ten percent probability the cell phone will need to be repaired and this will cost the company 100 dollars on average. Except these two cases, the cell phone will work and not cost the cell phone company anything. How much does the company need to charge for the one year cell phone insurance, excluding administrative and handling costs and excluding profit per year? **Answer:** $500 \cdot 0.05 + 100 \cdot 0.1 = 35$

6) You are given a four sided die with probabilities given as follows :

x	1	2	3	4
$P(X = x)$	0.1	0.2	0.2	0.5

You throw the die many times independently. So, X_i is the result of the i-th throw.
a)Find the probability

$$P((X_2 = X_3 = X_5) \cap (X_5 > X_6))$$

Answer: Let A be the event

$$A = \{X_2 = X_3 = X_4\} \cap \{X_5 > X_6\}$$

then

$$P(A) = P(A|X_6 = 1)P(X_6 = 1) + P(A|X_6 = 2)P(X_6 = 2) + P(A|X_6 = 2)P(X_6 = 2) = \quad (4)$$

$$= P(A|X_6 = 1)0.1 + P(A|X_6 = 2)0.2 + P(A|X_6 = 2)0.2 \quad (5)$$

Now, when $X_6 = 1$, and the event A is to hold then X_5 can be 2, 3 or 4. Hence

$$P(A|X_6 = 1) = P(X_2 = X_3 = X_5 = 2) + P(X_2 = X_3 = X_5 = 3) + P(X_2 = X_3 = X_5 = 4) = 0.2^3 + 0.2^3 + 0.5^3$$

Similarly when $X_6 = 2$ and the event A holds, then X_5 must be equal to 3 or 4 and hence

$$P(A|X_6 = 2) = P(X_2 = X_3 = X_5 = 3) + P(X_2 = X_3 = X_5 = 4) = 0.2^3 + 0.5^3.$$

Finally, when $X_6 = 3$, then X_5 must be equal to 4 and hence

$$P(A|X_6 = 3) = P(X_2 = X_3 = X_5 = 4) = 0.5^3$$

We put all this back into equation 4 to find

$$P(A) = 0.1 \cdot (0.2^3 + 0.2^3 + 0.5^3) + 0.2 \cdot (0.2^3 + 0.5^3) + 0.2 \cdot 0.5^3 = 0.77$$

b) What is the probability

$$P(X_1 \cdot X_2 > 4)$$

equal to?**Answer:** You can have 2 times 3 to get a number strictly larger than 4 or any two numbers above 2 and 3:

$$\begin{aligned}
&P(X_1 \cdot X_2 > 4) = \\
&= P(X_1 = 3, x_2 = 3) + P(X_1 = 4, X_2 = 4) + \\
&P(X_1 = 2, X_2 = 3) + P(X_1 = 3, X_2 = 2) + P(X_1 = 2, X_2 = 4) + P(X_1 = 4, X_2 = 2) + \\
&+ P(X_1 = 4, X_2 = 3) + P(X_1 = 4, X_2 = 2) = \\
&0.2^2 + 0.5^2 + 2 \cdot (0.2 \cdot 0.2 + 0.2 \cdot 0.5 + 0.2 \cdot 0.5) = \\
&= 0.0657
\end{aligned}$$

8) You play Russian Roullette three times in a row independently. Each time there is a probability of $1/6$ that a bullet is fired. What is the probability that no bullet is fired after you played three times? **Answer:** the probability that no bullet is fired when you shoot once, is $5/6$. So, when you shoot three, the probability that no bullet is fired is equal to:

$$P(\text{no bullet fired after three rounds}) = \left(\frac{5}{6}\right)^3$$

9) You throw the previous die from Problem 6) many times independently. When you get a face value of x, you earn an amount of money equal to $x^2 + x + 3$. So, for example when you get side 1, you obtain 5 dollars. If you throw the die and get a 2, than you earn $4 + 2 + 3 = 9$ dollars...How much do you make on average per round after playing many times? **Answer:** Let $Z = X^2 + X + 3$. You have

z	5	9	15	23
x	1	2	3	4
$P(X = x)$	0.1	0.2	0.2	0.5

So,

$$E[Z] = 5 \cdot P(Z = 5) + 9P(Z = 9) + 15 \cdot P(Z = 15) + 23 \cdot P(Z = 23) = 5 \cdot 0.1 + 9 \cdot 0.2 + 15 \cdot 0.2 + 23 \cdot 0.5 = 16.8$$