MINI PUPPER

FORWARD & INVERSE KINEMATICS AND JACOBIAN

COMPUTATION OF FORWARD KINEMATICS USING DENAVIT–HARTENBERG PARAMETERS

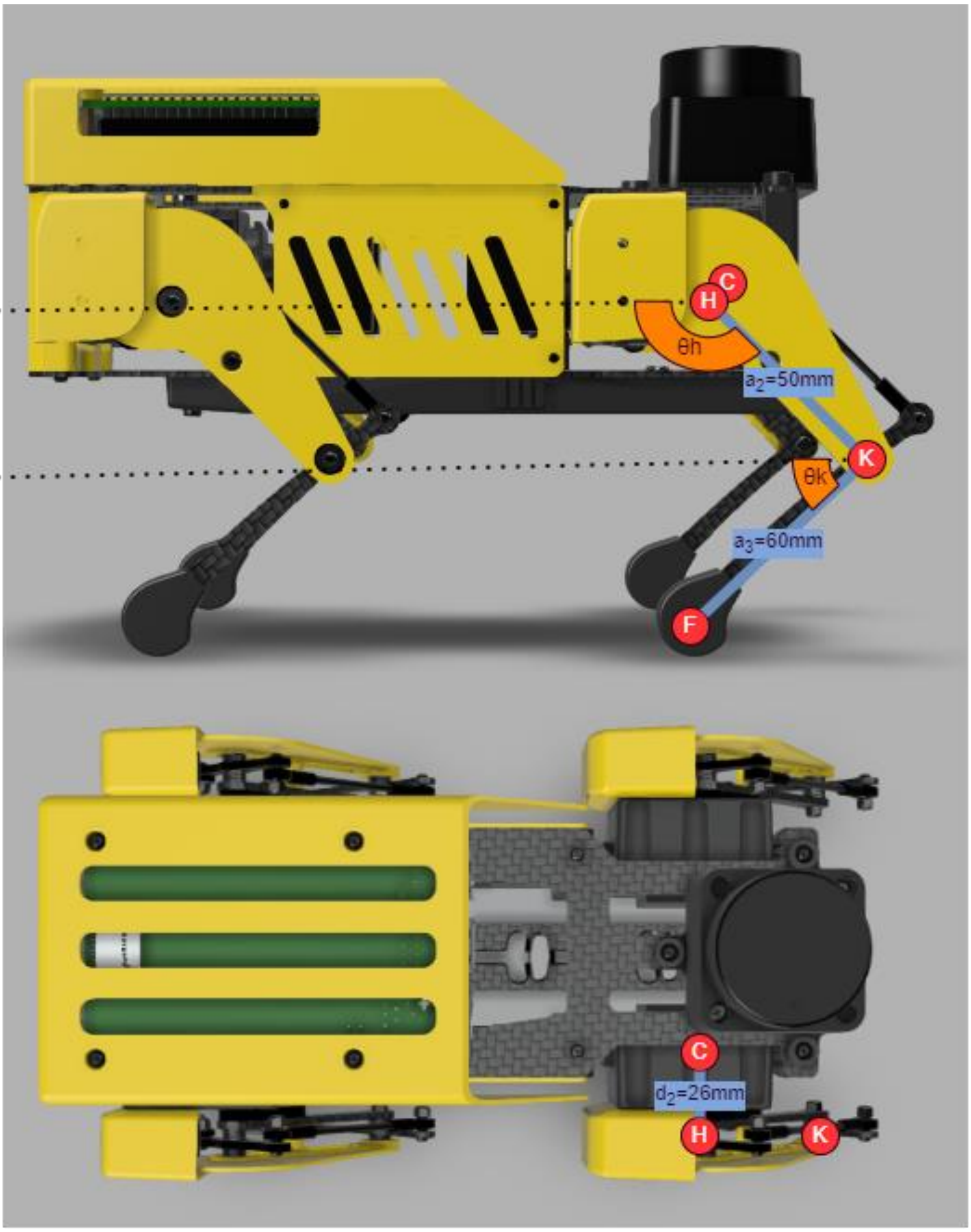
FIXED LEG REFERENCE COORDINATE FRAME #0

- Right-hand coordinate system
- Centred on coxa revolute joint
- X upward
- Y leftward
- Z backward

DH PARAMETERS

- $\boldsymbol{\theta}$ is rotation about previous Z axis, from old X to new X
- **d** is offset along previous Z axis
- **a** is offset along new X axis
- **α** is rotation about new X, from old Z to new Z

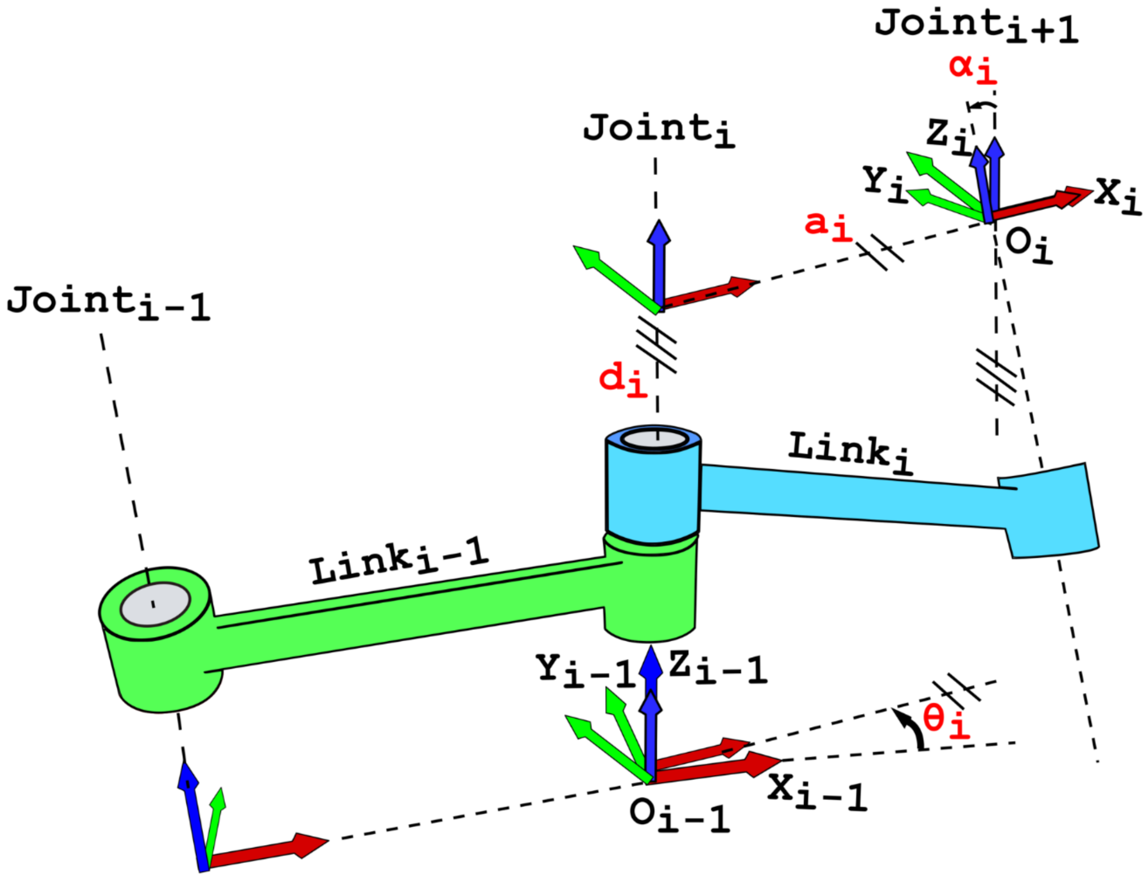
Links	θ	d	a	α
1	θc	0	0	-90°
2	$\theta h + 90$	Lc (Left Legs) $-Lc$ (Right Legs)	Lf	0
3	$\theta k - \theta h$	0	Lt	0



Variables	Description	Values
Lc	Distance from coxa revolute joint to hips revolute joint along X_1	26 mm
Lf	Distance from hips revolute joint to knee revolute joint along X_2	50 mm
Lt	Distance from knee revolute joint to foot along X_3	60 mm
θc	Coxa revolute joint angle, from X_0 axis to X_1 axis	*controlled by servo #1*
θh	Hips revolute joint angle, from body HORIZONTAL to FEMUR (X_2)	*controlled by servo #2*
θk	Knee revolute joint angle, from body HORIZONTAL to TIBIA (X_3)	*controlled by servo #3*

TRANSFORMATION MATRICES USING DH PARAMETERS

Coordinate frame of link i, noted O_i, is located at the end of the link, aligned with new X and Z axis, after all rotations and translations.



(REMINDER) DH TRANSFORMATION MATRIX

Transformation from coordinate frame j to coordinate frame j-1:

$$T_j^{j-1}(\theta_j d_j a_j \alpha_j) = \begin{bmatrix} \cos\theta_j & -\sin\theta_j\cos\alpha_j & \sin\theta_j\sin\alpha_j & a_j\cos\theta_j \\ \sin\theta_j & \cos\theta_j\cos\alpha_j & -\cos\theta_j\sin\alpha_j & a_j\sin\theta_j \\ 0 & \sin\alpha_j & \cos\alpha_j & d_j \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

COXA (J=1)

$$T_1^0 = \begin{bmatrix} c_c & 0 & -s_c & 0 \\ s_c & 0 & c_c & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

c_c	$\cos\theta_c$
s_c	$\sin\theta_c$

FEMUR (J=2)

- For left legs:

$$T_2^1 = \begin{bmatrix} c_h & -s_h & 0 & L_f c_h \\ s_h & c_h & 0 & L_f s_h \\ 0 & 0 & 1 & L_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For right legs:

$$T_2^1 = \begin{bmatrix} c_h & -s_h & 0 & L_f c_h \\ s_h & c_h & 0 & L_f s_h \\ 0 & 0 & 1 & -L_c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

c_h	$\cos\theta_h + 90$
s_h	$\sin\theta_h + 90$

TIBIA (J=3)

$$T_3^2 = \begin{bmatrix} c_{kh} & -s_{kh} & 0 & L_t c_{kh} \\ s_{kh} & c_{kh} & 0 & L_t s_{kh} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

c_{kh}	$\cos\theta k - \theta h$
s_{kh}	$\sin\theta k - \theta h$

« END-EFFECTOR » TRANSFORMATION MATRIX TO LEG REFERENCE FRAME #0

- For left legs:

$$T_3^0 = T_1^0 T_2^1 T_3^2 = \begin{bmatrix} c_c c_k & -c_c s_k & -s_c & L_t c_c c_k + L_f c_c c_h - L_c s_c \\ s_c c_k & -s_c s_k & c_c & L_t s_c c_k + L_f s_c c_h + L_c c_c \\ -s_k & -c_k & 0 & -L_t s_k - L_f s_h \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For right legs:

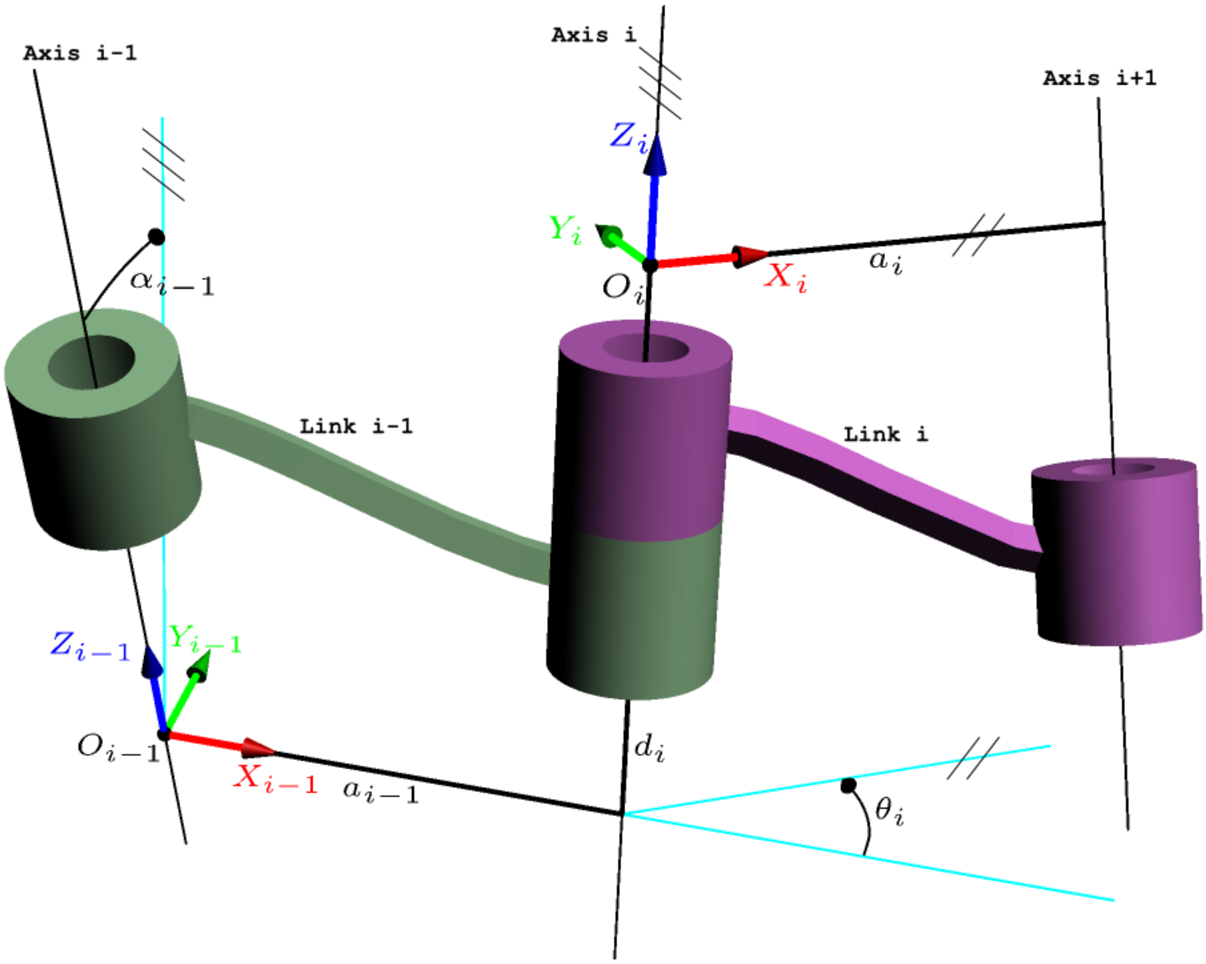
$$T_3^0 = T_1^0 T_2^1 T_3^2 = \begin{bmatrix} c_c c_k & -c_c s_k & -s_c & L_t c_c c_k + L_f c_c c_h + L_c s_c \\ s_c c_k & -s_c s_k & c_c & L_t s_c c_k + L_f s_c c_h - L_c c_c \\ -s_k & -c_k & 0 & -L_t s_k - L_f s_h \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where

c_k	$\cos\theta_k + 90$	$c_{kh} c_h - s_{kh} s_h$
s_k	$\sin\theta_k + 90$	$s_{kh} c_h + c_{kh} s_h$

TRANSFORMATION MATRICES USING MODIFIED DH PARAMETERS

Coordinate frame of link i, noted O_i, is located the beginning of the link, aligned with previous Z axis and new X axis, after rotation θ_i and translation d_i.



(REMINDER) MODIFIED DH TRANSFORMATION MATRIX

Transformation from coordinate frame j to coordinate frame j-1:

$$T_j^{j-1}(\theta_j d_j a_{j-1} \alpha_{j-1}) = \begin{bmatrix} \cos\theta_j & -\sin\theta_j & 0 & a_j \\ \sin\theta_j \cos\alpha_{j-1} & \cos\theta_j \cos\alpha_{j-1} & -\sin\alpha_{j-1} & -d_j \sin\alpha_{j-1} \\ \sin\theta_j \sin\alpha_{j-1} & \cos\theta_j \sin\alpha_{j-1} & \cos\alpha_{j-1} & d_j \cos\alpha_{j-1} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

COXA (J=1)

$$\boldsymbol{T_1^0} = \begin{bmatrix} \boldsymbol{c_c} & \boldsymbol{-s_c} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{s_c} & \boldsymbol{c_c} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

where

$\boldsymbol{c_c}$	$\boldsymbol{\cos\theta_c}$
$\boldsymbol{s_c}$	$\sin\theta_c$

FEMUR (J=2)

- For left legs:

$$\boldsymbol{T_2^1} = \begin{bmatrix} \boldsymbol{c_h} & \boldsymbol{-s_h} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{L_c} \\ \boldsymbol{-s_h} & \boldsymbol{-c_h} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

- For right legs:

$$\boldsymbol{T_2^1} = \begin{bmatrix} \boldsymbol{c_h} & \boldsymbol{-s_h} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{-L_c} \\ \boldsymbol{-s_h} & \boldsymbol{-c_h} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

where

$\boldsymbol{c_h}$	$\boldsymbol{\cos\theta_h + 90}$
$\boldsymbol{s_h}$	$\sin\theta_h + 90$

TIBIA (J=3)

$$\boldsymbol{T_3^2} = \begin{bmatrix} \boldsymbol{c_{kh}} & \boldsymbol{-s_{kh}} & \boldsymbol{0} & \boldsymbol{L_f} \\ \boldsymbol{s_{kh}} & \boldsymbol{c_{kh}} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} \end{bmatrix}$$

where

$\boldsymbol{c_{kh}}$	$\boldsymbol{\cos\theta k - \theta h}$
$\boldsymbol{s_{kh}}$	$\sin\theta k - \theta h$

FOOT (J=4)

$$T_4^3 = \begin{bmatrix} 1 & 0 & 0 & L_t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

« END-EFFECTOR » TRANSFORMATION MATRIX TO LEG REFERENCE FRAME #0

- For left legs:

$$T_4^0 = T_1^0 T_2^1 T_3^2 T_4^3 = \begin{bmatrix} c_c c_k & -c_c s_k & -s_c & L_t c_c c_k + L_f c_c c_h - L_c s_c \\ s_c c_k & -s_c s_k & c_c & L_t s_c c_k + L_f s_c c_h + L_c c_c \\ -s_k & -c_k & 0 & -L_t s_k - L_f s_h \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- For right legs:

$$\boxed{T_4^0 = T_1^0 T_2^1 T_3^2 T_4^3 = \begin{bmatrix} c_c c_k & -c_c s_k & -s_c & L_t c_c c_k + L_f c_c c_h + L_c s_c \\ s_c c_k & -s_c s_k & c_c & L_t s_c c_k + L_f s_c c_h - L_c c_c \\ -s_k & -c_k & 0 & -L_t s_k - L_f s_h \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

where

c_k	$\cos\theta_k + 90$	$c_{kh}c_h - s_{kh}s_h$
s_k	$\sin\theta_k + 90$	$s_{kh}c_h + c_{kh}s_h$

Note: both methods (DH and Modified DH) give the same result.

FORWARD KINEMATIC

Position of foot in leg reference frame #0 is given by the 3x1 submatrix of T_3^0 describing translation:

- For left legs:

$$\boxed{P^0_{\theta_c,\theta_h,\theta_k} = \begin{bmatrix} L_t c_c c_k + L_f c_c c_h - L_c s_c \\ L_t s_c c_k + L_f s_c c_h + L_c c_c \\ -L_t s_k - L_f s_h \end{bmatrix}}$$

- For right legs:

$$\boxed{P^0_{\theta_c,\theta_h,\theta_k} = \begin{bmatrix} L_t c_c c_k + L_f c_c c_h + L_c s_c \\ L_t s_c c_k + L_f s_c c_h - L_c c_c \\ -L_t s_k - L_f s_h \end{bmatrix}}$$

COMPUTATION OF JACOBIAN

Velocity of the foot in leg reference frame #0 is given by the Jacobian J :

$$\dot{P}^0 = J(\theta) \times \dot{\theta} \qquad \text{where} \qquad \dot{\theta} = \begin{matrix} \dot{\theta}_c \\ \dot{\theta}_h \\ \dot{\theta}_k \end{matrix}$$

PARTIAL DIFFERENTIATION METHOD

$$J = \begin{matrix} \frac{dPx}{d\theta_c} & \frac{dPx}{d\theta_h} & \frac{dPx}{d\theta_k} \\ \frac{dPy}{d\theta_c} & \frac{dPy}{d\theta_h} & \frac{dPy}{d\theta_k} \\ \frac{dPz}{d\theta_c} & \frac{dPz}{d\theta_h} & \frac{dPz}{d\theta_k} \end{matrix}$$

- For left legs:

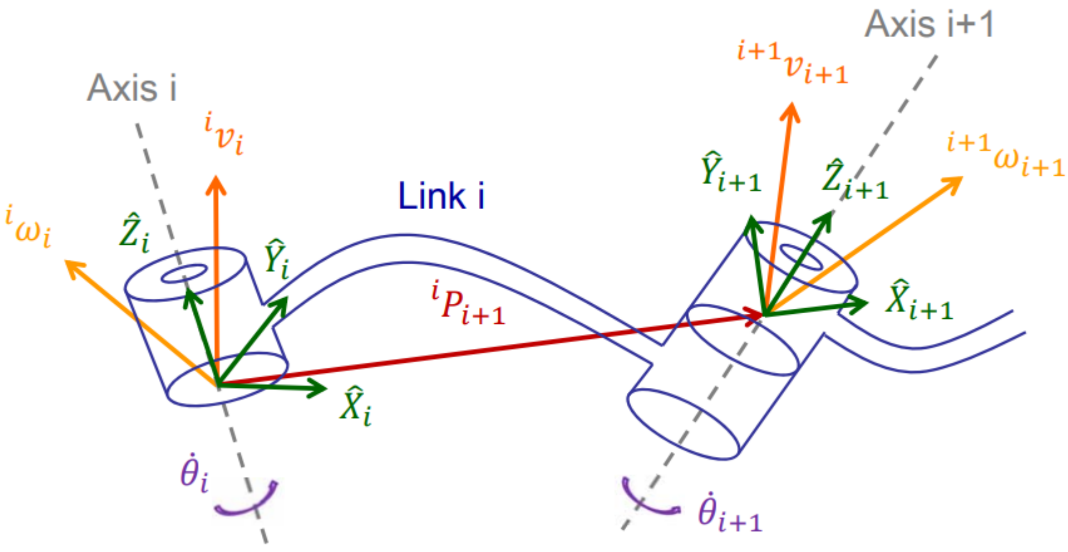
$$\boldsymbol{J} = \begin{bmatrix} -L_t s_c c_k - L_f s_c c_h - L_c c_c & -L_f c_c s_h & -L_t c_c s_k \\ L_t c_c c_k + L_f c_c c_h - L_c s_c & -L_f s_c s_h & -L_t s_c s_k \\ \mathbf{0} & -L_f c_h & -L_t c_k \end{bmatrix}$$

- For right legs:

$$\boldsymbol{J} = \begin{bmatrix} -L_t s_c c_k - L_f s_c c_h + L_c c_c & -L_f c_c s_h & -L_t c_c s_k \\ L_t c_c c_k + L_f c_c c_h + L_c s_c & -L_f s_c s_h & -L_t s_c s_k \\ \mathbf{0} & -L_f c_h & -L_t c_k \end{bmatrix}$$

VELOCITY PROPAGATION METHOD

VELOCITY PROPAGATION (THEORY)



$$\boldsymbol{{}^{i+1}\omega_{i+1} = R_i^{i+1} . \; {}^{i}\omega_i + \dot{\theta}_{i+1} \; {}^{i+1}_{i+1}\hat{Z}}$$

$$\boldsymbol{{}^{i+1}v_{i+1} = R_i^{i+1}.\left({}^{i}v_i + {}^{i}\omega_i \wedge {}^{i}p_{i+1}\right)}$$

$$\boldsymbol{{}^{0}\omega_0 = 0}$$

$$\boldsymbol{{}^{0}v_0 = 0}$$

COXA (I=1)

$${}^{1}\omega_1 = R_0^1.\ {}^{0}\omega_0 + \dot{\theta}_1\ \underset{1}{\overset{1\frown}{Z}}$$

$${}^{i1}v_1 = R_0^1.\left({}^{0}v_0 + {}^{0}\omega_0 \wedge {}^{0}p_1\right)$$

then

$$\boldsymbol{{}^{1}\omega_1 = \dot{\theta}_c\ \underset{1}{\overset{1\frown}{Z}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_c \end{bmatrix}}$$

$$\boldsymbol{{}^{i1}v_1 = 0}$$

HIPS (I=2)

$${}^{2}\omega_2 = R_1^2.\ {}^{1}\omega_1 + \dot{\theta}_2\ \underset{2}{\overset{2\frown}{Z}}$$

$${}^{2}v_2 = R_1^2.\left({}^{1}v_1 + {}^{1}\omega_1 \wedge {}^{1}p_2\right)$$

where

$$T_2^1 = \begin{bmatrix} c_h & -s_h & 0 & 0 \\ 0 & 0 & 1 & L_c \\ -s_h & -c_h & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$\boldsymbol{{}^{2}\omega_2 = \begin{bmatrix} -s_h\dot{\theta}_c \\ -c_h\dot{\theta}_c \\ \dot{\theta}_h \end{bmatrix}}$$

$$\boldsymbol{{}^{2}v_2 = \begin{bmatrix} -L_c c_h\dot{\theta}_c \\ L_c s_h\dot{\theta}_c \\ 0 \end{bmatrix}}$$

KNEE (I=3)

$${}^{3}\omega_3 = R_2^3.\ {}^{2}\omega_2 + \dot{\theta}_3\ \underset{3}{\overset{3\frown}{Z}}$$

$${}^{3}v_3 = R_2^3.\left({}^{2}v_2 + {}^{2}\omega_2 \wedge {}^{2}p_3\right)$$

where

$$T_3^2 = \begin{bmatrix} c_{kh} & -s_{kh} & 0 & L_f \\ s_{kh} & c_{kh} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$\boldsymbol{{}^3\omega_3} = \begin{bmatrix} -s_k\dot{\theta}_c \\ -c_k\dot{\theta}_c \\ \dot{\theta}_h + \dot{\theta}_{kh} \end{bmatrix} = \begin{bmatrix} -s_k\dot{\theta}_c \\ -c_k\dot{\theta}_c \\ \dot{\theta}_k \end{bmatrix}$$

$$\boldsymbol{{}^3v_3} = \begin{bmatrix} -c_kL_c\dot{\theta}_c + s_{kh}L_f\dot{\theta}_h \\ s_kL_c\dot{\theta}_c + c_{kh}L_f\dot{\theta}_h \\ L_fc_h\dot{\theta}_c \end{bmatrix}$$

TIBIA/FOOT (I=4)

$${}^4\omega_4 = R_3^4 .\ {}^3\omega_3$$

$${}^4v_4 = R_3^4 .\left({}^3v_3 + {}^3\omega_3 \wedge {}^3p_4 \right)$$

where

$$T_4^3 = \begin{bmatrix} 1 & 0 & 0 & L_t \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then

$$\boldsymbol{{}^4\omega_4} = \begin{bmatrix} -s_k\dot{\theta}_c \\ -c_k\dot{\theta}_c \\ \dot{\theta}_k \end{bmatrix}$$

$$\boldsymbol{{}^4v_4} = \begin{bmatrix} -c_kL_c\dot{\theta}_c + s_{kh}L_f\dot{\theta}_h \\ s_kL_c\dot{\theta}_c + c_{kh}L_f\dot{\theta}_h + L_t\dot{\theta}_k \\ L_fc_h\dot{\theta}_c + L_tc_k\dot{\theta}_c \end{bmatrix}$$

JACOBIAN IN LAST FRAME

Extracting $\dot{\theta}$ vector:

$${}^4v_4 = \begin{bmatrix} -c_kL_c & s_{kh}L_f & 0 \\ s_kL_c & c_{kh}L_f & L_t \\ L_fc_h + L_tc_k & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_c \\ \dot{\theta}_h \\ \dot{\theta}_k \end{bmatrix}$$

then

$$\boldsymbol{{}^4J} = \begin{bmatrix} -c_kL_c & s_{kh}L_f & 0 \\ s_kL_c & c_{kh}L_f & L_t \\ L_fc_h + L_tc_k & 0 & 0 \end{bmatrix}$$

JACOBIAN IN BASE FRAME

$$ {}^{0}J = {}^{0}_{4}R\ {}^{4}J $$

where

$$ R_4^0 = \begin{bmatrix} c_c c_k & -c_c s_k & -s_c \\ s_c c_k & -s_c s_k & c_c \\ -s_k & -c_k & 0 \end{bmatrix} $$

then

$$ \boxed{\boldsymbol{J = \begin{bmatrix} -L_t s_c c_k - L_f s_c c_h - L_c c_c & -L_f c_c s_h & -L_t c_c s_k \\ L_t c_c c_k + L_f c_c c_h - L_c s_c & -L_f s_c s_h & -L_t s_c s_k \\ 0 & -L_f c_h & -L_t c_k \end{bmatrix}}} $$

Note: both methods (partial differentiation and velocity propagation) give the same result.

COMPUTATION OF GROUND REACTION FORCE

Ground reaction force in leg reference frame #0 is given by the inverse transposed Jacobian J and the torque of each joint:

$$ GRF^0 = -F^0 = -{J^T}^{-1}\tau \qquad \text{where} \qquad \tau = \begin{matrix} \tau_c \\ \tau_h \\ \tau_k \end{matrix} $$

SINGULARITIES

Numerical approach may be used for singularities computation

INVERSE MATRIX (IF NON-ZERO DETERMINANT)

Numerical approach may be used for inverse Jacobian matrix computation

INVERSE KINEMATIC

Geometric approach.