Redefining Material Implication with Subjective Logic [1]

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Abstract—**Material implication is traditionally denoted as $(x \rightarrow y)$, where x represents the antecedent and y the consequent of the logical relationship between the propositions x and y. Material implication is a truth functional connective, meaning that it is defined by a truth table. While truth functional connectives normally have a relatively clear interpretation in normal language, this is not the case for material implication. It could for example be expressed as: "if x is true, then y is true". However, this does not say anything about the case when x is false, which is problematic for the interpretation of the corresponding entries in the truth table. In this paper we introduce a probabilistic view of material implication and show that it is not closed under binary truth values, and that it in fact produces uncertainty in the form of a vacuous opinion. When seen in this light, it becomes clear that the traditional definition of material implication is based on the over-simplistic and misleading interpretation of complete uncertainty as binary logic TRUE. We redefine material implication with subjective logic to preserve the uncertainty that it unavoidably produces in specific cases. We then compare the new definition of material implication with conditional deduction, and show that they reflect the same mathematical equation rearranged in different forms.**

I. Introduction

By definition, logical propositions in binary logic can only be evaluated to TRUE or FALSE. A logical proposition can be composed of sub-propositions that are combined with logical connectives. For example, the conjunctive connective $\wedge$ can be used to combine propositions x and y into the conjunctive proposition $(x \wedge y)$. In this case the proposition $(x \wedge y)$ is a complex proposition because it is composed of sub-propositions. The $\wedge$ connective has its natural language interpretation expressed as: "x and y are both TRUE". A logical proposition is said to be truth functional when its truth depends on the truth of its sub-propositions alone [1]. Traditionally, it is required that the complex proposition has a defined truth value for all the possible combinations of truth values of the sub-propositions, in order for the truth function to be completely defined.

As an example of a simple truth functional connective, the conjunction of two propositions is defined as the truth table shown in Table I below.

Table I
TRUTH TABLE FOR CONJUNCTION $(x \wedge y)$

x	y	$x \wedge y$
F	F	F
F	T	F
T	F	F
T	T	T

Logical AND reflects the natural intuitive understanding of the linguistic term "and", and is at the same time extremely useful for specifying computer program logic.

Material implication, also known as truth functional implication, is a conditional proposition usually denoted as $(x \rightarrow y)$, where x is the antecedent, y is the consequent. A conditional proposition is a complex proposition consisting of the two sub-propositions x and y connected with the material implication connective "$\rightarrow$". The natural language interpretation of material implication is: "if x is TRUE, then y is TRUE", or simply as "x implies y". However, it can be noted that the natural language interpretation does not say anything about the case when x is FALSE. Nevertheless, material implication is defined both in case x is TRUE and x FALSE. The natural language interpretation thus only covers half the definition of material implication, and this interpretation vacuum is the source of the confusion around material implication.

Defining material implication as truth-functional means that its truth values are determined as a function of the truth values of x and y alone, as shown in Table II.

Table II
TRADITIONAL TRUTH TABLE FOR MATERIAL IMPLICATION

	x	y	$x \rightarrow y$
Case 1:	F	F	T
Case 2:	F	T	T
Case 3:	T	F	F
Case 4:	T	T	T

The truth table of Table II happens to be equal to the truth table of $(\bar{x} \vee y)$, which is the reason why the traditional definition leads to the equivalence $(x \rightarrow y) \Leftrightarrow (\bar{x} \vee y)$.

However, treating conditionals as truth functional in this fashion leads to well known problems. The natural language interpretation assumes that there is a relevance connection between x and y which does not emerge from the truth table of Table II. The relevance property which intuitively, but mistakenly, is assumed by $(x \rightarrow y)$ can be expressed as: "the truth value of x is relevant for the truth value of y. For example, connecting a false antecedent proposition with an arbitrary consequent proposition gives a true implication according to material implication, but sounds counterintuitive when expressed in normal language such as "if 2 is odd, then 2 is even". Similarly, connecting an arbitrary antecedent with a true consequent proposition is true according to material implication although the antecedent and consequent might not have any relevance to each other. An example expressed in normal language is e.g. "if rain is wet, then sugar is sweet".

The problem is that it takes more than a truth table to determine whether a proposition x is relevant for another proposition y. In natural language, the term "relevance" assumes that when the truth value of the antecedent varies, so does that of the consequent. Correlation of truth variables between antecedent and consequent is thus a necessary element for relevance. Material implication defined in terms of Table II does not express any relevance between the propositions, and therefore does not reflect the meaning of the natural language concept of implication. Table II gives a case-by-case static view of truth values which is insufficient to derive any relevance relationships.

There is thus a difference between e.g. conjunctive propositions and conditional propositions in that conjunctive propositions are intuitively truth functional, whereas conditional propositions are not. Treating a conditional proposition as truth functional is problematic because its truth cannot be determined in binary logic terms solely as a function of the truth of its components.

In this paper we will show that the truth function of material implication is not closed under binary logic, and that its truth value can be uncertain, which is something that cannot be expressed with binary logic. We show that material implication can be redefined to preserve uncertainty, and compare this to probabilistic and subjective logic conditional deduction where uncertainty can be expressed either in the form of probability density functions or as subjective opinions. Subjective logic [2] represents a probabilistic logic of opinions where degrees of uncertainty can be explicitly expressed and is therefore suitable for expressing the truth function of material implication.

II. Probabilistic Conditional Deduction

A meaningful conditional deduction requires that the antecedent is relevant to the consequent, or in other words

that the consequent depends on the antecedent. Conditionals that are based on the dependence between consequent and antecedent are considered to be universally valid, and are called logical conditionals [3]. Deduction with logical conditionals reflect human intuitive conditional reasoning.

Both binary logic and probability calculus have mechanisms for conditional reasoning. In binary logic, Modus Ponens (MP) and Modus Tollens (MT) are the classical operators which are used in any field of logic that requires conditional deduction. In probability calculus, binomial conditional deduction is expressed as:

$$p(y\|x) = p(x)p(y|x) + p(\bar{x})p(y|\bar{x}) \qquad (1)$$

where the terms are interpreted as follows:

$p(y|x):$ probability of y given x is TRUE
$p(y|\bar{x}):$ probability of y given x is FALSE
$p(x):$ probability of the antecedent x
$p(\bar{x}):$ complement probability $(= 1 - p(x))$
$p(y\|x):$ probability of the consequent y as a function of x

We follow the convention whereby a conditional relationship is denoted on the form "consequent | antecedent", i.e. with the consequent first and the antecedent second.

The notation $y\|x$, introduced in [4], denotes that the truth or probability of proposition y has been derived as a function of the probability of the antecedent x together with the conditionals. This notation will also be used for subjective logic in Sec.VI.

III. Material Probabilistic Implication

By material probabilistic implication we mean that the probability value of the conditional $p(y|x)$ shall be determined as a function of other probability variables. This then corresponds directly with propositional logic material implication where the truth value of the conditional is determined as a function of the antecedent and the consequent truth values according to the truth table.

The difference between probabilistic conditional deduction and probabilistic material implication is a question of rearranging Eq.(1) so $p(y|x)$ is expressed as:

$$p(y|x) = \frac{p(y\|x) - p(\bar{x})p(y|\bar{x})}{p(x)} \qquad (2)$$

Below we will use Eq.(2) and Eq.(1) to determine the value of the conditional $(y|x)$.

A. Cases 1 & 2: $p(x) = 0$

The case $p(x) = 0$ in Eq.(2) immediately appears as problematic. It is therefore necessary to consider Eq.(1).

It can be seen that the term involving $p(y|x)$ disappears from Eq.(1) when $p(x) = 0$. As a result $p(y|x)$ can take any value in the range $[0, 1]$, so $p(y|x)$ must be expressed as a probability density function. Without any prior information, the density function must be considered to be uniform, which in subjective logic has a specific interpretation as will be explained below.

A realistic example could for example be when considering the propositions x:"The switch is on" and y:"The light is on". Recall that x is FALSE (i.e. "the switch is off"') in the cases under consideration here.

Let us first consider the situation corresponding to Case 1 in Table 2 where $y\|x$ is FALSE (i.e. "the light is off with the given switch position, which happens to be off"), which would be the case when $y|\bar{x}$ is FALSE. In this situation it is perfectly possible that $y|x$ is FALSE too (i.e. "the light is off whenever the switch is on"). It is for example possible that the switch in question is not connected to the lamp in question, or that the bulb is blown.

Let us now consider the situation corresponding to Case 2 in Table 2 where $y\|x$ is TRUE (i.e. "the light is on with the given switch position, which happens to be off"), which would be the case when $y|\bar{x}$ is TRUE. In this situation it is also perfectly possible that $y|x$ is FALSE (i.e. "the light is off whenever the switch is on"). It is for example possible that the electric cable connections have been inverted, so that the light is on when the switch is off, and vice versa.

These examples are in direct contradiction with Cases 1 & 2 of Table 2 which dictates that the corresponding implication $(x \rightarrow y)$ should be TRUE in both cases. The observation of this contradiction proves that the traditional definition of material implication is inconsistent with standard probability calculus.

B. Cases 3 & 4: $p(x) = 1$

Necessarily $p(\bar{x}) = 0$, so that Eq.(2) is transformed into $p(y|x) = p(y\|x)$. Thus when x is TRUE (i.e. $p(x) = 1$) then necessarily $(x \rightarrow y)$ will have the same truth value as y. This does not necessarily mean that the truth value of x is relevant to the truth value of y. In fact it could be either relevant or irrelevant. For example consider the antecedent proposition x:"It rains" combined with the consequent y:"I wear an umbrella", then it is plausibly relevant, but combined with the consequent y:"I wear glasses", then it is plausibly irrelevant. It can be assumed that x and y are TRUE in this example so that the implication is TRUE. The unclear level of relevance can also be observed in examples where the consequent y is FALSE so that the implication $(x \rightarrow y)$ becomes FALSE. The level of relevance between the antecedent and the consequent is thus independent of the truth value of the implication $(x \rightarrow y)$ alone. The criteria for relevance will be described in more detail below.

IV. RELEVANCE

A meaningful conditional relationship between x and y requires that the antecedent x is relevant to the consequent y, or in other words that the consequent depends on the antecedent, as explicitly expressed in relevance logics [5]. Conditionals that are based on the dependence between consequent and antecedent are considered to be universally valid (and not truth functional), and are called logical conditionals [3]. Deduction with logical conditionals reflects

human intuitive conditional reasoning, and do not lead to any of the paradoxes of material implication.

Material implication, which is purely truth functional, ignores any relevance connection between antecedent x and the consequent y, and defines the truth value of the conditional as a function of the truth values of the antecedent and consequent alone.

It is possible to express the relevance between the antecedent and the consequent as a function of the conditionals. For probabilistic conditional deduction, the relevance denoted as $R(x,y)$ can be defined as:

$$R(x,y) = |p(y|x) - p(y|\bar{x})| . \tag{3}$$

It can be seen that $R(x,y) \in [0,1]$, where $R(x,y) = 0$ expresses total irrelevance/independence, and $R(x,y) = 1$ expresses total relevance/dependence between x and y.

Relevance cannot be derived from the traditional truth functional definition of material implication because the truth value of $(\bar{x} \rightarrow y)$ is missing from the truth table. In order to rectify this, an augmented truth table that includes $(\bar{x} \rightarrow y)$ is given below.

Table III
TRUTH AND RELEVANCE TABLE FOR MATERIAL IMPLICATION

	x	y	$\bar{x} \rightarrow y$	$x \rightarrow y$	Relevance
Case 1a:	F	F	F	Any	Any
Case 1b:	F	F	T	n.d.	n.d.
Case 2a:	F	T	F	n.d.	n.d.
Case 2b:	F	T	T	Any	Any
Case 3a:	T	F	F	F	None
Case 3b:	T	F	T	F	Total
Case 4a:	T	T	F	T	Total
Case 4b:	T	T	T	T	None

From Table III it can be seen that the truth table is either ambiguous ("Any") or not defined ("n.d".) in the cases 1 and 2. The term "Any" is used to indicate that any truth or probability value for $x \rightarrow y$ is possible in the cases 1a and 2b, not just binary TRUE or FALSE. Only in the cases 3 and 4 is the truth table clear about the truth value of $(x \rightarrow y)$. The same applies to the relevance between x and y where any relevance value is possible in the cases 1a and 2b, and only the cases 3 and 4 define the relevance crisply as either no relevance or total relevance. Total relevance can be interpreted in the sense $x \Leftrightarrow y$ or $x \Leftrightarrow \bar{y}$, i.e. that x and y are either equivalent or inequivalent.

Our analysis shows that the natural conditional relationship between two propositions can not be meaningfylly be described with a truth table ecause other values than binary TRUE and FALSE are possible. The immediate conclusion is that material implication is not closed under a binary truth value space. Not even by allowing probability values for $(x \rightarrow y)$ in Table III can material implication be made meaningful. Below we show that subjective logic which can express uncertainty is suitable for defining material implication.

V. Expressing Material Implication with Subjective Logic

Subjective logic [2] is a type of probabilistic logic [6], [7] where connectives are defined by mathematical expressions instead of look-up truth tables. Subjective logic explicitly takes uncertainty and belief ownership into account, and is suitable for modeling and analysing situations involving uncertainty and incomplete knowledge. For example, it can be used for modeling trust networks and for analysing Bayesian networks [8].

Arguments in subjective logic are opinions about propositions. The opinion space is related to the belief function space used in Dempster-Shafer belief theory [9]. A binomial opinion applies to a single proposition x, and can be represented as a Beta distribution. A multinomial opinion applies to a collection of propositions in a frame of discernment, and can be represented as a Dirichlet distribution [10]. Through the correspondence between opinions and Beta/Dirichlet distributions, subjective logic provides an algebra for these functions.

The notation for a binomial opinion consists of an ordered tuple containing the three specific belief masses b_x: belief, d_x: disbelief, u_x: uncertainty as well as the base rate (also called relative atomicity) a_x of the proposition x.

The ordered quadruple

$$\omega_x = (b_x,\, d_x,\, u_x,\, a_x) \tag{4}$$

$$\text{where} \quad b_x,\, d_x,\, u_x,\, a_x \in [0,1] \tag{5}$$

$$\text{and} \quad b_x + d_x + u_x = 1 \tag{6}$$

is an opinion in subjective logic [2]. The default base rate is $a_x = 0.5$, but any other base rate in the range $[0,1]$ can be defined.

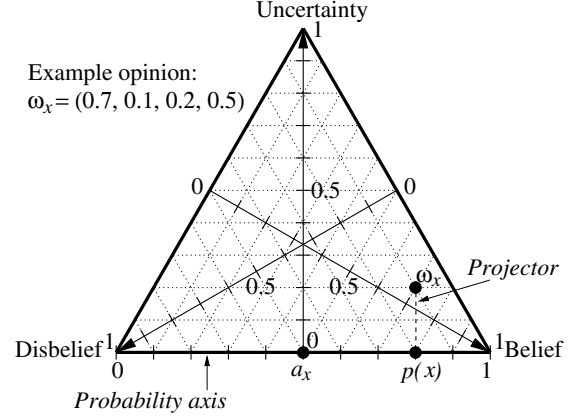The probability expectation value of a binomial opinion is:

$$p(\omega_x) = b_x + a_x u_x\,. \tag{7}$$

The expression of Eq.(7) is equivalent to the pignistic probability defined in the transferable belief model, an interpretation of the belief function theory [11]. It is based on the principle that the belief mass assigned to the whole frame is split equally among the singletons of the frame. In Eq.(7) the base rate a_x must be interpreted in the sense that the relative proportion of singletons contained in x is equal to a_x.

Binomial opinions can be mapped to a point in an equal-sided triangle. The relative distances from the left side edge to the point represent belief, from the right side edge to the point represent disbelief, and from the base line to the point represents uncertainty. For an arbitrary opinion $\omega_x = (b_x, d_x, u_x, a_x)$, the three parameters b_x, d_x and u_x thus determine the position of the opinion point in the triangle.

The base line is the probability axis, and the base rate value can be indicated as a point on the probability axis.

Fig.V illustrates an example opinion about x with the value $\omega_x = (0.7,\ 0.1,\ 0.2,\ 0.5)$ indicated by a black dot in the triangle.

The projector going through the opinion point, parallel to the line that joins the uncertainty corner and the base rate point, determines the probability expectation value $p(x) = b_x + a_x u_x$.

The parameters b_x, d_x and u_x are equivalent to the traditional $Bel(x)$ (Belief) and $Pl(x)$ (Plausibility) pair of Shaferian belief theory through the correspondence $Bel(x) = b_x$ and $Pl(x) = b_x + u_x$.

Although a binomial opinion has 4 parameters, it only has 3 degrees of freedom because the three parameters b_x, d_x and u_x are dependent through Eq.(6). The reason why a redundant parameter is kept in the opinion representation is that it allows for more compact expressions of opinion operators than otherwise would have been possible.

Various visualisations of opinions are possible to facilitate human interpretation. See for example http://folk.uio.no/josang/sl/BV.html

Binomial opinions are equivalent to Beta distributions. Beta distributions are normally denoted as $\text{Beta}(\alpha, \beta)$ where $\alpha, \beta \in R^+$ are its two parameters. Examples of Beta distributions are illustrated below. The Beta distribution of an opinion $\omega = (b, d, u, a)$ is the function $\text{Beta}(\alpha, \beta)$ where

$$\text{for } u \neq 0: \begin{cases} \alpha &= \frac{2b}{u} + 2a \\[2mm] \beta &= \frac{2d}{u} + 2(1-a) \end{cases} \tag{8}$$

$$\text{for } u = 0: \begin{cases} \alpha &= b \cdot \xi \\[2mm] \beta &= d \cdot \xi \end{cases} \text{where } \xi \mapsto \infty \tag{9}$$

It can be noted that Eq.(8) and Eq.(9) lead to the following constraint for Beta distributions derived from opinions:

$$\alpha + \beta \geq 2 \tag{10}$$

The special case of $\alpha + \beta = 2$ leads to $a = \frac{\alpha}{2}$, and in particular when $\alpha = \beta = 1$ then $a = \frac{1}{2}$.

As an example of the mapping from opinions to Beta distributions, the opinion of Fig.V corresponds to the Beta PDF of Fig.1.
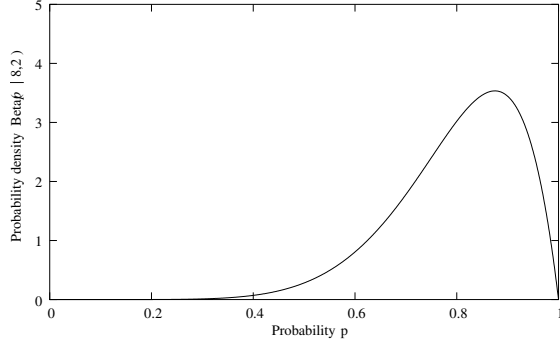

Figure 1.   Beta$(8,2)$ PDF

The uniform Beta distribution is $\mathsf{Beta}(1,1)$ which corresponds to the opinion $w_x = (0, 0, 1, \frac{1}{2})$. This particular opinion thus represents the case when nothing is known about the probability of x, i.e. its distribution is uniform, as illustrated in Fig.2.
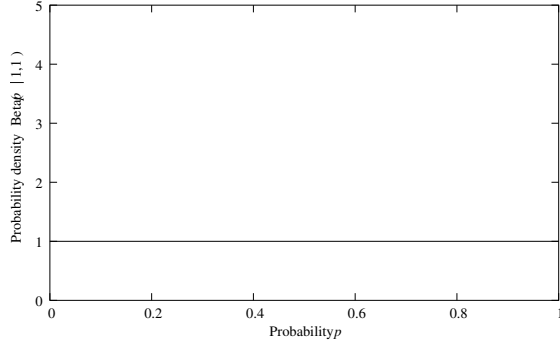

Figure 2.   Uniform Beta$(1,1)$

Using Eq.(6) and Eq.(8) the mapping from Beta distributions to opinions can be derived:

$$\text{Beta} \to \text{opinion mapping} \quad \begin{cases} b = \frac{\alpha - 2a}{\alpha + \beta} \\[2mm] d = \frac{\beta - 2(1-a)}{\alpha + \beta} \\[2mm] u = \frac{2}{\alpha + \beta} \end{cases} \quad (11)$$

The set of Beta distributions $\text{Beta}(\alpha, \beta)$ where $\alpha + \beta = 2$ correspond to the set of vacuous opinions defined by $\omega = (0, 0, 1, \frac{\alpha}{2})$. Every other Beta distributions for which Eq.(10) is satisfied correspond to an infinity of different opinions. In other words, in case $\alpha + \beta = 2$ then there is a bijective mapping between opinions and Beta distributions, and in case $\alpha + \beta > 2$ there is a surjective (onto) mapping from opinions to Beta distributions.

The discussion in Sec.III concluded that any probability value is possible in the cases 1 and 2 of the truth table of material implication. The uniform probability density function expressed by $\mathsf{Beta}(1,1)$ which is equivalent to the vacuous opinion $\omega = (0, 0, 1, \frac{1}{2})$ is therefore a meaningful representation of the term "Any" in Table III.

Given that material implication is not closed in the binary truth value space, a correct truth table can be defined that reflects the actual behaviour of $(x \to y)$ as a function of the truth values of x and y, as shown in Table IV.

Table IV
ACTUAL NON-CLOSED TRUTH TABLE FOR MATERIAL IMPLICATION

	x	y	$x \to y$
Case 1:	F	F	$\omega = (0,0,1,\frac{1}{2})$
Case 2:	F	T	$\omega = (0,0,1,\frac{1}{2})$
Case 3:	T	F	F
Case 4:	T	T	T

Table IV defines material implication as truth functional in the sense that it is determined as a function of binary truth values of x and y. Specifying vacuous opinions in the rightmost column of the truth table is a necessary consequence of the analysis in Sec.III, but this means that the truth table no longer is closed under binary truth values.

It can be argued that if values other than binary logic TRUE and FALSE are allowed for $(x \to y)$, then it would be natural to allow the same for x and y. This is indeed possible, and it can be expressed in terms of subjective logic. This will be described in the next section.

VI. DEDUCTION WITH SUBJECTIVE LOGIC

Conditional deduction over binomial opinions in subjective logic is e.g. described in [4], [8]. With the mathematical detail omitted, the notation is:

$$\omega_{y \| x} = \omega_x \circledcirc (\omega_{y|x}, \omega y | \bar{x}) \qquad (12)$$

where the terms are interpreted as follows:

$\omega_{y|x}$:   opinion about y given x is TRUE
$\omega_{y|\bar{x}}$:   opinion about y given x is FALSE
ω_x :   opinion about the antecedent x
$\omega_{y \| x}$:   opinion about the consequent y given x

An online demonstration of subjective logic operators including conditional deduction is available at http://folk.uio.no/josang/sl/, and Fig.3 shows a screen snapshot.

The input variables in the example of Fig.3 are binomial opinions which can be mapped to Beta distributions according to Eq.(8). The leftmost triangle represents the opinion of x, and the rightmost triangle that of y. The middle triangle represents the conditional opinions of $y|x$ and $y|\bar{x}$. The particular example illustrates Case 2b of Table III and Case 2 of Table IV.

Eq.(12) corresponds directly to the probabilistic version of Eq.(1). Both expressions take three input variables, the
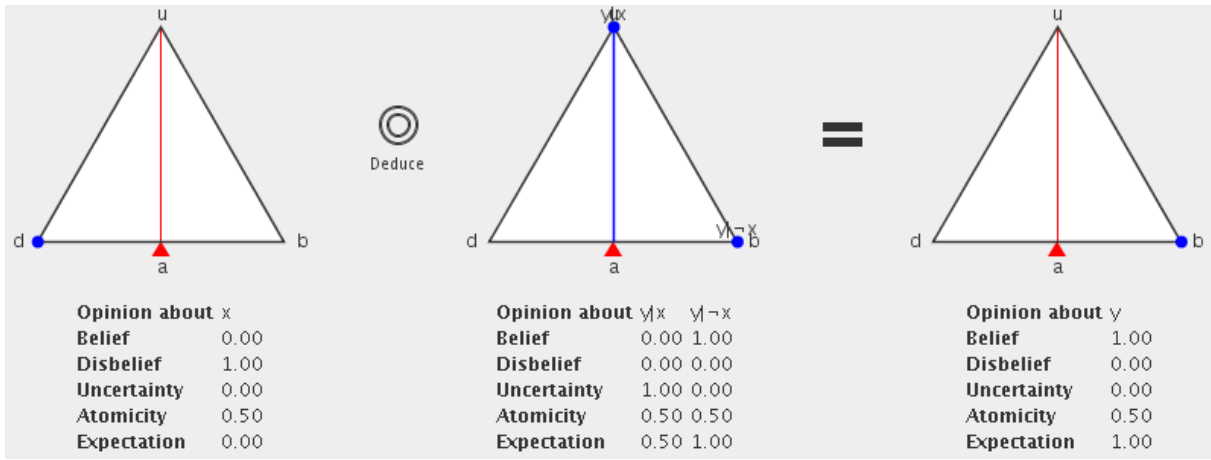
Figure 3.   Screen-shot from subjective logic demonstrator of deduction

only difference is that the input variables in Eq.(12) are scalars, where as those of Eq.(12) are 3-dimensional. Given that the base rates of $\omega_{y|x}$ and $\omega_{y|\bar{x}}$ are equal, Eq.(12) takes eight scalar input parameters.

VII. DISCUSSION AND CONCLUSION

We have shown that material implication is inconsistent with traditional probabilistic logic. This is nothing new, where e.g. Nute and Cross (2002) pointed out that "There can be little doubt that neither material implication nor any other truth function can be used by itself to provide an adequate representation of the logical and semantical properties of English conditionals" [12].

In this paper we have presented a redefinition of material implication as a probabilistic material implication. The difference between probabilistic material implication and conditional deduction is a question of rearranging equations, as in the transition from Eq.(1) to Eq.(2).

The analysis of material implication has shown that it is impractical to determine the conditional $p(y|x)$ or the corresponding implication $(x \to y)$ as a truth function because the required conditional $p(y|\bar{x})$ or the corresponding implication $(\bar{x} \to y)$ are missing. Material implication produces an uncertain conclusion precisely because it attempts to determine the conditional relationship without the necessary evidence.

Probabilistic conditional relationships are routinely determined from statistical data to be used as input to e.g. Bayesian networks. It is when the conditionals are known and expressed for example in a boolean, probabilistic or a subjective logic form that they are applicable for deriving conclusions about propositions of interest. The idea of material implication is to turn the concept of deduction on its head and try to determine the conditional from the antecedent argument and the consequent conclusion. We have shown that the cases when the antecedent is FALSE then the truth value of the material implication should be "uncertain", not TRUE.

We have extended the truth table of material implication to make it correct and consistent with conditional deduction. By doing so, material implication can no longer be considered as a connective of propositional logic because its truth table is not closed under binary logic values. A more general reasoning calculus such as subjective logic is needed to allow a consistent definition of material implication because it allows the truth table to express the required uncertainty in the form of vacuous opinions.

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