Using a Python ODE solver to Determine the Oscillatory Behavior of a Satellite Mass About Lagrange Point 4 in the Earth Moon System

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INTRODUCTION

In the vastness of space, it is often difficult to maintain a precise knowledge of a body's position. Forces such as gravitational potential and inertia direct smaller, less massive bodies in varying paths at fluctuating speeds around bodies of substantially greater mass. Lagrange Points are named for mathematician and physicist Joseph-Louis Lagrange. He first proposed these mathematical phenomena in his 1772 paper, *Essai sur le Problème des Trois Corps*, as positions in space where the gravitational attraction of two large masses is equal to the centripetal acceleration of a smaller body causing the smaller body to move with the masses (Cornish 1998). In other terms, the gravitational force of each massive body creates points in space where there is little to no potential and is suitable for satellite bodies to stay at those locations and travel in orbit a constant distance from the two larger masses. A common analogy is the consideration of a marble in a bowl, which will travel in an oscillatory path as it settles to the bottom. This idea remained unobserved until 1906 when German astronomist Max Wolf discovered the Trojan asteroids at the L4 location in the Jupiter-Sun system (Steigerwald 2021).

Lagrange points are crucial to understand in the disciplines of satellite and astro dynamics. These locations enable engineers and researchers to maintain the location of an artificial satellite with some certainty while burning less fuel compared to other locations between the Earth and Moon. Smaller quantities of fuel on board the satellite would lower the launch mass and therefore the cost, which is a major consideration in aerospace applications. Naturally occurring satellites have settled in Lagrange points around other planets, such as the Trojan asteroids mentioned above. Understanding where these celestial bodies exist presents the opportunity for future exploration and a greater understanding of the solar system and its creation.

RELEVANT INFORMATION

For the sake of this analysis, the orbit of the Moon about the Earth was assumed to be circular when it is an ellipse. The three-body problem was subjected to a plane, eliminating additional equations governing the movement of the body in the Z direction. Each body was understood to be perfectly spherical. To define the scope, the motion of the satellite mass about Lagrange Point 4 is the only point of consideration between the Earth and the Moon. The units of measure were kept as metric, in kilograms, meters, and meters per second.

A crucial aspect of this analysis is the identification of a rotating coordinate system (Fitzpatrick 2021). The origin is placed at the center of mass of the system, C of Figure 1, and rotates with angular velocity ω. A numeric value for this term will be derived from the masses and radius of the system later in the report and for this portion of the analysis can be kept symbolic. By fixing the coordinate axes to the Earth-Moon system, the movement defined by the system of ODEs of the satellite mass around the Lagrange point will be easier to identify. The masses of the Earth, Moon, and satellite mass can be seen in Figure 2 represent by $m_1,\ m_2, and\ m_3$ respectively.

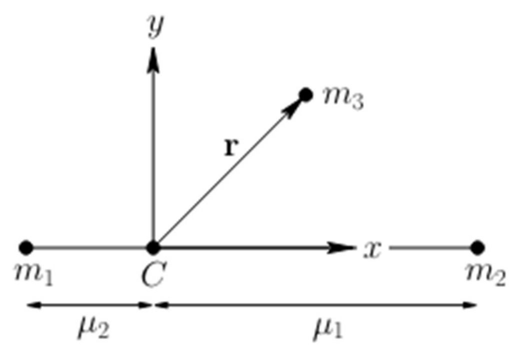

Figure 1. Coordinate system at the center of the mass for m_1 and m_2 which rotates with angular velocity, ω.

The value of μ_2 is determined by the division of the Moon's mass by the sum of the Earth and Moon's masses, shown below in Equation 1 (Drang 2016).

$$\mu_2 = \frac{m_m}{m_m + m_e} \tag{1}$$

This value determines how far over the coordinate system is shifted toward the center of the Earth-Moon system. In Figure 2, L4 is shown at the apex of an equilateral triangle in the Earth-Moon system.

To obtain a graph that adequately showed the behavior of the satellite mass, a duration of ten times the orbital period was used.

Newton's laws of planetary motion provide the potential that acts on a smaller body from the two more massive bodies in the three-body system.

$$U = \frac{GMm_s}{R}\left[-\frac{(1-\mu_2)}{\rho_s} - \frac{\mu_2}{\rho_\rho} - \frac{1}{2}\rho^2\right] \tag{2}$$

Which can be scaled to Equation 3 when looking to determine the potential of a system that has the ratio between the large body masses of $m_m = \mu_2 m_e$.

$$u = -\frac{(1-\mu_2)}{\rho_s} - \frac{\mu_2}{\rho_\rho} - \frac{1}{2}\rho^2 \tag{3}$$

This becomes particularly useful when graphing the contour plot of Figure 5 discussed in Results to evaluate where the Lagrange Points of the two large masses exist and where regions of low potential overlap in the system.

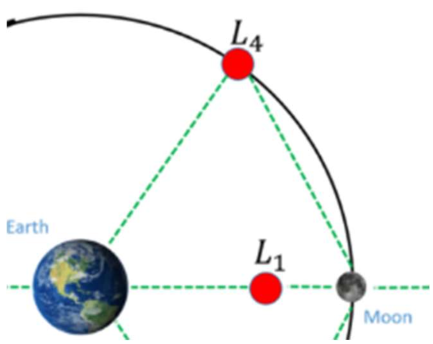

Figure 2. Lagrange Point L4 in the Earth-Moon system. The location of L1 is included.

Thus, L4 is always at a position of $\left(\frac{1}{2}R, \frac{\sqrt{3}}{2}R\right)$ relative to the larger masses, which is where the satellite mass is initialized in the Python code. After determining the tangential speed of the moon as an estimate, the x and y components are determined by multiplying the speed by each respective component.

The orbital period is important to this analysis and is shown below in Equation 2.

$$T = \frac{2\pi}{\omega} \tag{4}$$

CONSTANTS AND INITIAL CONDITIONS

The constants that govern Newton's Laws of Planetary motion and variables found for the Earth-Moon three-body system are present below. The initial conditions of the oscillating body were chosen based on the tangential velocity of moon, shown below in meters per second and multiplied by the

$$G = 6.674 \text{ x } 10^{-1} \ \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2} \tag{5}$$

$$R = 384{,}400{,}000 \text{ m} \tag{6}$$

$$\mu = 0.012 \tag{7}$$

$$m_e = 5.972 \text{ x } 10^{24} \text{ kg} \tag{8}$$

$$m_m = 7.347 \text{ x } 10^{22} \text{ kg} \tag{9}$$

$$m_s = 10000 \text{ kg} \tag{10}$$

$$v_{moon} = 1023 \text{ m/s} \tag{11}$$

$$x_1 = 0 \tag{12}$$

$$y_1 = 0 \tag{13}$$

$$v_{1x} = 1023(\frac{1}{2}R) \tag{14}$$

$$v_{1y} = 1023(\frac{\sqrt{3}}{2}R) \tag{15}$$

MATHEMATICAL MODELS

The two coupled, second order ordinary differential equations (ODEs) below describe the oscillation, or "libration" of satellite masses as a function of time about the Lagrange Points of L4 and L5 as shown above on Figure 1.

$$\frac{d^2x}{dt^2} = 2\omega\frac{dy}{dt} + \frac{3}{4}\omega^2 x + \gamma\omega^2 y \tag{16}$$

$$\frac{d^2y}{dt^2} = -2\omega\frac{dx}{dt} + \frac{9}{4}\omega^2 y + \gamma\omega^2 x \tag{17}$$

These equations are correct for any L4 and L5 in a three-body system and are solved as an initial value problem. To be used in the way that the selected numerical solver is intended to integrate, must be broken down into four first order ordinary differential equations as seen in Equations 18 to 21 below.

$$\frac{dx}{dt} = v_x \tag{18}$$

$$\frac{dv_x}{dt} = 2\omega\frac{dy}{dt} + \frac{3}{4}\omega^2 x + \gamma\omega^2 y \tag{19}$$

$$\frac{dy}{dt} = v_y \tag{20}$$

$$\frac{dvx}{dt} = 2\omega\frac{dy}{dt} + \frac{3}{4}\omega^2 x + \gamma\omega^2 y \tag{21}$$

Where ω and γ are coefficients dependent on the masses of the system, radius between the two large bodies, and Newton's gravitational constant involved in the system defined by Equations 22 and 23.

$$\omega = \sqrt{\frac{GM}{R^3}} \tag{22}$$

$$\gamma = \frac{3\sqrt{3}(m_e - m_m)}{4(m_e + m_m)} \tag{23}$$

NUMERICAL METHOD

To integrate the system of first order ordinary differential equations, odeint() was imported from the SciPy computing library in Python. Odeint was adopted from the LSODA ODEPACK originally developed for FORTRAN by researchers at the Lawrence Livermore National Laboratory. LSODA uses the Adams-Bashforth Method and Backward Differentiation Formula adaptively to solve initial value problems that are stiff or non-stiff depending on the parameters specified in the application.

The equations of libration exhibit non-stiff behavior, as there are no sudden changes in oscillation or magnitude. Therefore, the Adams-Bashforth Method is used by odeint() to solve the system of ordinary differential equations as an initial value problem. The first order Adams-Bashforth equation is shown below in Equation 12 ("Adams Method…" 2019). It is worthwhile noting that this the same as Euler's method for estimating a first order ordinary differential equation.

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{24}$$

With the Adams-Bashforth (AB) method, the LSODA solver can vary the order of the equation used for integration to achieve varying degrees of accuracy. The order of AB changes by estimating the slope at the current point based on the previous points. For instance, a second order Adams-Bashforth uses the previous two points to estimate the next point (SciPy 2024). In doing so, this solver lessens the error. However, one disadvantage to this method is its sensitivity to unstable (stiff) ordinary differential equations. Seeing how the first order differential equations in the system were determined to be non-stiff, there was no concern regarding stability inaccuracies to this analysis.

RESULTS

The graphs of this section were generated in Python to show the behavior of the satellite mass about the Lagrange point L4. Figure 3 gives the x and y positions of the small mass as it travels in an oscillating path around L4 located at the origin. The initial position, as discussed in the Relevant Information section above, is seen to be leaving the origin, which was centered at the coordinates of L4. This graph confirms the expected behavior of the mass about the point in space. To compare this to a more tangible example, this graph also shows the path you'd likely expect to see a marble take as it drops down the side of a bowl and begins to roll until finally settling in the middle.

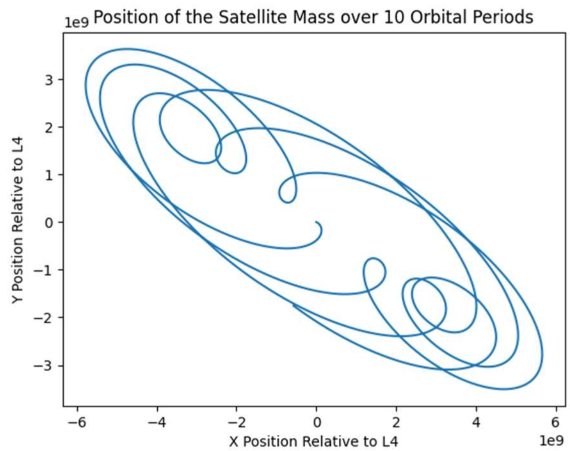

Figure 3. Path of satellite mass m_s about Lagrange Point 4 over 10 orbital periods of the Moon about the Earth.

Figure 4 describes the speed of the satellite mass during its oscillation. The graph takes on a quasiperiodic shape, consistent with information presented in literature on the subject. The peaks correspond to the movement of the satellite mass as it moves closer to the center of the Lagrange point, moving from a position of higher potential to lower potential around the point. It is important to note that as the mass oscillates about the point, it is not gaining potential in the Earth-Moon system but rather in relation to the Lagrange points. The troughs of this graph correspond to the moments in time when the mass is further from the center of the Lagrange point and is being pushed back to L4 by the gravity of the Earth and moon. Like in L3, the timescale ten times that of one lunar orbit about the Earth.

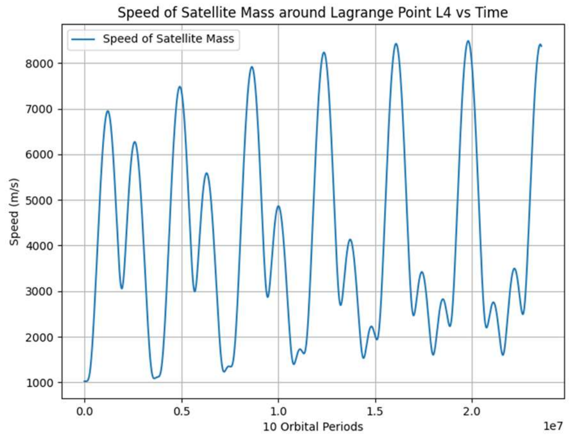

Figure 4. Speed of the satellite mass m_s about Lagrange Point 4 over 10 orbital periods of the Moon about the Earth.

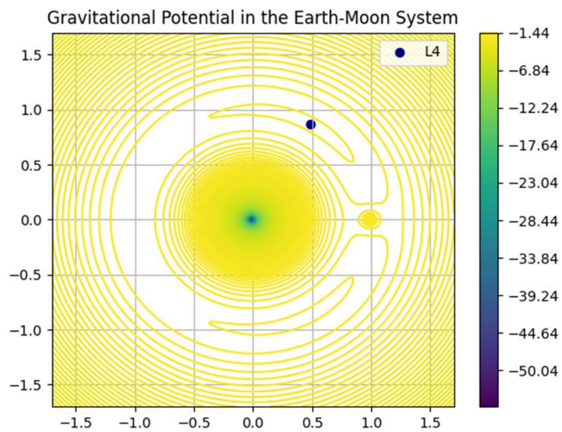

Figure 5. Contour plot showing the gravitational potential of Lagrange Point 4.

Figure 5 was generated with the scaled potential of the Earth-Moon system. The location of L4 is shown and corresponds to the region of low potential that is situation 60° ahead of the Moon on the lunar orbit. The region of white surrounding this dot would allow for some oscillation back and forth around the Lagrange Point, which is shown the solved system of ODEs. However, it can be seen that a lot of energy is necessary to position a satellite or for a naturally occurring mass to situate itself in a location other than the Lagrange points around two significantly more massive bodies.

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