Notes for MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

Main text to use: Numerical Partial Differential Equations in Finance Explained by Karel in 't Hout, published by the Springer Nature, 2017

Lecture notes: They will be uploaded frequently to the Moodle system at HKBU

Assessment methods: homework and quizzes (20%); two numerical projects (probably for the weeks of 09/30–10/04 and 11/11–11/15; total 20%); final exam (60%)

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FUNDAMENTALS FOR FINANCE

A. Financial options.

1. a holder;
2. a writer;
3. a contract giving the holder the right but not the obligation;
4. call option (to buy);
5. put option (to sell);
6. European, American and Asian options.

$\Rightarrow$ read carefully Example 1.1.1.

$\Rightarrow$ a fair option value is often difficult to determine...

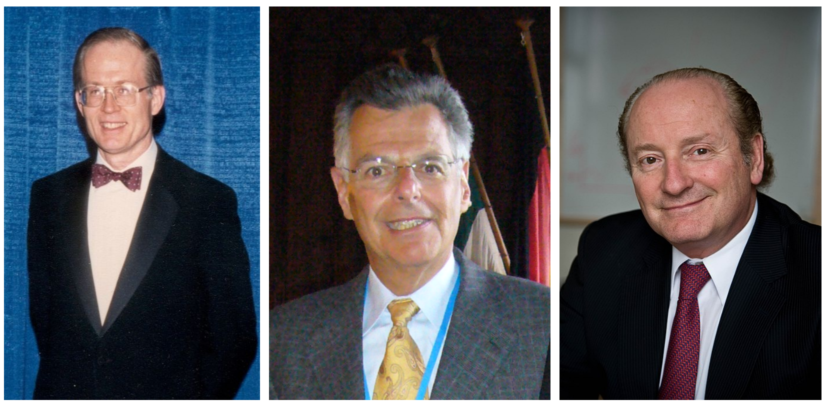
$\Rightarrow$ for the simplicity, we consider European options unless otherwise stated...

B. The Black-Scholes Equation.

1. let S_τ be the price of an asset at time $\tau \in [0, T]$.
2. the price at maturity time T is S_T, which may be greater or less than the strike price K.
3. the fair value of an option can be calculated through $\phi(S_T)$, where ϕ is the payoff function

$$y = \phi(S) = \begin{cases} \max(S-K, 0), & \text{(call)}; \\ \max(K-S, 0), & \text{(put)}, \end{cases} \quad S \geq 0. \tag{1.1}$$

⇒ S_τ is a random variable.



Fischer S. Black, Myron Scholes and Robert C. Merton

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⇒ Black and Scholes' Assumption One (a stochastic process):

$$S_\tau = S_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau\right\}, \quad 0 \leq \tau \leq T, \tag{1.2}$$

where

$\sigma > 0$ is the volatility, μ is the drift rate, and

W_τ is the Brownian motion or Wiener process.

⇒ If $S_0 > 0$, then

$$\ln\frac{S_\tau}{S_0} = \left(\mu - \frac{\sigma^2}{2}\right)\tau + \sigma W_\tau, \quad 0 \leq \tau \leq T.$$

⇒ S_τ has a lognormal distribution and the expected value of S_τ is

$$\mathbb{E}[S_\tau] = S_0 \exp\{\mu\tau\}. \tag{A}$$

⇒ The drift $\mu > 0$ can be viewed as the rate of return on the expected future asset prices.

Black and Scholes' Second Assumption is a consideration of an *risk-free interest rate* r.

Now, let D_0 be our initial investment with a bank and r_y is a risk-free annual interest rate. Recall that we have the following compounded formula for a return at the end of m-th year of our investment:

$$D_m = D_0(1 + r_y)^m, \quad m = 0, 1, 2, \dots$$

If we can compound earnings continuously with a constant risk-free annual interest rate r, then

$$D(\tau) = D_0 \exp\{r\tau\}, \quad \tau \in [0, T],$$

should be our return[1]. Do you think if the above is similar to formula (A)?

Now, here comes Black and Scholes' Third Assumption: our financial option valuation is no-arbitrage. By the way, the principle of no-arbitrage asserts that two securities that provide the same future cash flow and have the same level of risk must sell for the same price. The principle is fundamental in financial engineering.

$\Rightarrow$ It can be proven[2] that there exits a unique deterministic real function

$$u = u(s, t), \quad s, t \in \mathbb{R},$$

such that u is the fair value at time

$$\tau = T - t \tag{B}$$

of a call or put option, if at that time, the asset price equals s, and

$$u_t(s,t) = \frac{\sigma^2 s^2}{2} u_{ss}(s,t) + rsu_s(s,t) - ru(s,t), \quad s > 0,\ 0 < t \le T. \tag{1.3}$$

The above partial differential equation (PDE) is referred as a *Black-Scholes equation.* It was proposed in 1973 and led to a Nobel Prize in Economics in 1997. The equation is, by standard definitions in the PDE theory, a second-order linear parabolic equation. It is also a variable coefficient convection-diffusion-reaction equation from multi-physics point-of-view.

$\Rightarrow$ It is interesting that the drift rate μ in (1.2) does not show in (1.3). This surprising but fundamental fact leads to the so-called risk-neutral option valuation theory in finance.

$\Rightarrow$ Equation (B) is in fact a linear variable change formula, or mapping, which replaces τ by t for the computational convenience and simplicity.

Recall that a fair option value is known at the maturity $\tau = T$, or at $t = 0$ because of (B). We have

$$u(s, 0) = \phi(s), \quad s > 0. \tag{1.4}$$

$\Rightarrow$ The above is called an initial condition for the partial differential equation.

[1] You can prove it utilizing a limit formula in the calculus.

[2] There are different methods that can be used but the mathematical proofs are a bit beyond what we need.

But we still need two more conditions to ensure a unique solution of the Black-Scholes equations. To this end, we have the left-boundary condition

$$u(0,t) = \begin{cases} 0, & \text{(call)}; \\ K\exp\{-rt\}, & \text{(put)}, \end{cases} \quad 0 \le t \le T. \tag{1.5}$$

The right-boundary condition can be more complicated. Herewith we have following typical choices:

$$u(S_{\max},t) = \begin{cases} S_{\max} - K\exp\{-rt\}, & \text{(call)}; \\ 0, & \text{(put)}, \end{cases} \quad 0 \le t \le T, \tag{2.5}$$

or

$$u_s(S_{\max},t) = \begin{cases} 1, & \text{(call)}; \\ 0, & \text{(put)}, \end{cases} \quad 0 \le t \le T. \tag{2.6}$$

⇒ In a standard call or put option, we may have $S_{\max} = \infty$.

⇒ In the theory of partial differential equations, conditions (1.5), (2.5) are called Dirichlet boundary conditions, and (2.6) is called a Neumann boundary condition. Of course, we have other types of boundary conditions.

C. NUMERICAL SOLUTION OF PDES.

First, what is the equation, as well as necessary solution constraints, we must consider?

For $S_{\min} < s < S_{\max},\ 0 < t \le T$ or otherwise specified, we have the following standard **Black-Scholes initial-boundary value problem** (BSIVP):

$$\begin{aligned} u_t(s,t) &= d(s)u_{ss}(s,t) + c(s)u_s(s,t) - r(s)u(s,t), && (2.1)\\ u(s,0) &= \phi(s), && (2.2)\\ u(S_{\min},t) &= \begin{cases} 0, & \text{(call)}; \\ K\exp\{-r(S_{\min})t\}, & \text{(put)}, \end{cases} && (2.3)\\ u(S_{\max},t) &= \begin{cases} S_{\max} - K\exp\{-r(S_{\max})t\}, & \text{(call)}; \\ 0, & \text{(put)}, \end{cases} && (2.4) \end{aligned}$$

where u is the unknown, $S_{\min}, S_{\max}$ are given real values or $\pm\infty$, Functions c, d, r are real-valued with d is always assumed to be nonnegative.

The right-boundary condition (2.4) may sometime be replaced by

$$u_s(S_{\max},t) = \begin{cases} 1, & \text{(call)}; \\ 0, & \text{(put)}. \end{cases} \tag{2.5}$$

Second, how to solve the above BSIVP? Let's fulfill this goal in m-steps with confidence!

References

[1] K. in 't Hout, *Numerical Partial Differential Equations in Finance Explained,* Springer, Antwerp, Belgium, 2017.

[2] R. Pratap, *Getting Started with Matlab: A Quick Introduction for Scientists and Engineers,* 7th Ed., Oxford Univ. Press, 2016.

[3] K. Atkinson and W. Han, *Elementary Numerical Analysis,* 3rd Ed., John Wiley & Sons, Somerset, NJ, 2004.

[4] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge and New York, 2004.

[5] B. Jain and A. Sheng, An exploration of the approximation of derivative functions via finite differences, *Rose-Hulman Undergrd. Math J.,* 8:172-188, 2007.

[6] Khan Academy, *Intro to Matrices,* https://www.khanacademy.org/math/precalculus/x9e81a4f98389efdf:matrices/x9e81a4f98389efdf:mat-intro/a/intro-to-matrices