Solution references for homework 5

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Problem 1:

Take any nonzero vector $v = (a, b, c)^\intercal \in \mathbb{R}^3$. We have $a^2 + b^2 + b^2 > 0$. We have

$$\begin{aligned}
v^\intercal A v &= (v^\intercal A)\, v = [(2a-b) \;\; (-a+2b-c) \;\; (-b+2c)] \begin{bmatrix} a \\ b \\ c \end{bmatrix} \\
&= (2a-b)a + (-a+2b-c)b + (-b+2c)c \\
&= 2a^2 - ba - ab + 2b^2 - cb - bc + 2c^2 \\
&= 2a^2 - 2ab + 2b^2 - 2bc + 2c^2 \\
&= a^2 + a^2 - 2ab + b^2 + b^2 - 2bc + c^2 + c^2 \\
&= a^2 + (a-b)^2 + (b-c)^2 + c^2.
\end{aligned}$$

The above result is a sum of squares, and therefore non-negative. Furthermore, it is zero only if $a = b = c = 0$, that is, when v is the zero vector. □

Can you use other definitions to show the same result?

Problem 2:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 \\ 2 & -2 & 1 \\ 1 & 3 & -1 \end{bmatrix}.$$

It follows that

$$B^\intercal = \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 3 \\ 3 & 1 & -1 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
BB^\intercal &= \begin{bmatrix} 10 & 5 & -2 \\ 5 & 9 & -5 \\ -2 & -5 & 11 \end{bmatrix}, \\
B^\intercal B &= \begin{bmatrix} 6 & -1 & 4 \\ -1 & 13 & -5 \\ 4 & -5 & 11 \end{bmatrix}.
\end{aligned}$$

So,

$$BB^\intercal \neq B^\intercal B.$$

Further,

$$\|BB^\intercal\|_\infty = 19 \neq 20 = \|B^\intercal B\|_\infty.$$

If we play further, it can be seen that

$$C = [B^\intercal, B] = B^\intercal B - BB^\intercal = \begin{bmatrix} -4 & -6 & 6 \\ -6 & 4 & 0 \\ 6 & 0 & 0 \end{bmatrix},$$

for which we have eigenvalues:

$$\lambda_1 = -10.111443519657460,\ \lambda_2 = 1.691344780121667,\ \lambda_3 = 8.420098739535792.$$

Therefore

$$\|C\|_2 = \|[B^\intercal, B]\|_2 = \|[B, B^\intercal]\|_2 \approx 10.111443519657460.$$

Do above discussions make a sense?

Now, for $A,\ B$ defined in (1), (2), respectively, we have

$$\begin{aligned} AB &= \begin{bmatrix} 0 & 2 & 5 \\ 2 & -7 & 0 \\ 0 & 8 & -3 \end{bmatrix}, \\ BA &= \begin{bmatrix} 2 & -4 & 6 \\ 6 & -7 & 4 \\ -1 & 6 & -5 \end{bmatrix}. \end{aligned}$$

So,

$$AB \neq BA.$$

That is, $A,\ B$ do not commute.

Further,

$$\|AB\|_\infty = 11 \neq 17 = \|BA\|_\infty.$$

$$D = [A, B] = AB - BA = \begin{bmatrix} -2 & 6 & -1 \\ -4 & 0 & -4 \\ 1 & 2 & 2 \end{bmatrix}.$$

May you verify the above?

<u>Problem 3</u>:

We will discuss this in the next chapter of the class.