Notes for MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

Main text to use: Numerical Partial Differential Equations in Finance Explained by Karel in 't Hout, published by the Springer Nature, 2017

Lecture notes: They will be uploaded frequently to the Moodle system at HKBU

Assessment methods: homework and quizzes (20%); two numerical projects (probably for the weeks of 09/30–10/04 and 11/11–11/15; total 20%); final exam (60%)

Instructor: Prof. Tim Sheng

Tim's office: FSC 1108, HKBU

Office hours: Wednesdays 09:00–11:00 & 13:00–14:00; Thursdays 09:00–12:00 (or by appointment)

Email address: QSheng100@hkbu.edu.hk

URL: http://www.math.hkbu.edu.hk/people/faculty/

Teaching Assistant: Mr. Xuelei Lin

Xuelei's office: FSC 1101, HKBU

FUNDAMENTALS FOR FINANCE

C. Numerical Solution of PDEs.

Recall that, for $S_{\min} < s < S_{\max},\ 0 < t \le T$ or otherwise specified, we have the following standard Black-Scholes initial-boundary value problem (BSIVP):

$$
\begin{aligned}
u_t(s,t) &= d(s)u_{ss}(s,t) + c(s)u_s(s,t) - r(s)u(s,t), &(2.1)\\
u(s,0) &= \phi(s), &(2.2)\\
u(S_{\min},t) &= \begin{cases} 0, & \text{(call)};\\ K\exp\{-r(S_{\min})t\}, & \text{(put)}, \end{cases} &(2.3)\\
u(S_{\max},t) &= \begin{cases} S_{\max} - K\exp\{-r(S_{\max})t\}, & \text{(call)};\\ 0, & \text{(put)}, \end{cases} &(2.4)
\end{aligned}
$$

where u is the unknown, $S_{\min}, S_{\max}$ are given real values or $\pm\infty$, Functions c, d, r are real-valued with d is always assumed to be nonnegative.

The right-boundary condition (2.4) may sometime be replaced by

$$u_s(S_{\max}, t) = \begin{cases} 1, & \text{(call)}; \\ 0, & \text{(put)}. \end{cases} \tag{2.5}$$

C.01 Simplified Model Equation One.

Set $d(s) \equiv c(s) \equiv r(s) \equiv 0$ in (2.1). We have

$$u_t(s,t) = 0.$$

Since s is not effective in the above equation, we may remove it and take the initial condition (2.2), that is,

$$u'(t) = 0, \quad 0 < t < T, \tag{2.6}$$

$$u(0) = \phi, \tag{2.7}$$

where ϕ is a constant. Equations (2.6), (2.7) form a typical linear homogeneous ordinary differential equation initial value problem.

Integrate (2.6) we acquire that

$$\int_0^t u'(\tau)d\tau = \int_0^t 0 d\tau, \quad 0 < t < T.$$

According to the fundamental theorem of calculus, we arrive at

$$u(t) - u(0) = 0, \quad 0 < t < T.$$

Therefore

$$u(t) \equiv \phi, \quad 0 < t < T,$$

in the consideration of the initial condition (2.7).

C.02 Simplified Model Equation Two.

Consider a linear nonhomogeneous ordinary differential equation initial value problem:

$$u'(t) = c, \quad 0 < t < T, \tag{2.8}$$

$$u(0) = \phi, \tag{2.9}$$

where c is a nontrivial constant.

An integration of (2.6)

$$\int_0^t u'(\tau)d\tau = \int_0^t c d\tau, \quad 0 < t < T,$$

yields that

$$u(t) = ct + \phi, \quad 0 < t < T.$$

This is, needless to say, the unique true solution of the initial value problem.

C.03 SIMPLIFIED MODEL EQUATION THREE.

Consider a linear ordinary differential equation initial value problem with a variable forcing term:

$$
\begin{aligned}
u'(t) &= f(t), \quad 0 < t < T, &(2.10)\\
u(0) &= \phi, &(2.11)
\end{aligned}
$$

where $f(t)$ is an integrable function.

An integration of (2.10),

$$\int_0^t u'(\tau)d\tau = \int_0^t f(\tau)d\tau = \psi(t), \quad 0 < t < T,$$

yields that

$$u(t) = \psi(t) + \phi, \quad 0 < t < T,$$

with $\psi(0) = 0$.

Apparently, the function $\psi(t)$ may not exist in finite form, especially in situations when strong stochasticities are involved.

Can we solve (2.10), (2.11) *by using the finite difference approximations we studied in previous weeks?*

The answer is, of course, affirmative.

To this end, for example, we may introduce an uniform mesh $\mathcal{D}$ over the temporal interval $[0, T]$, and adopt a forward difference formula for approximating the derivative. These yield the following from (2.10), (2.11):

$$
\begin{aligned}
\Delta u(t_i) &= \frac{u(t_{i+1}) - u(t_i)}{h} = f(t_i) + \mathcal{O}(h), \quad t_i \in \mathcal{D},\\
u(t_0) &= \phi,
\end{aligned}
$$

where $0 < h \ll 1$ and $(N+1)h = T$.

Rewrite the first equation above to the following:

$$u(t_{i+1}) - u(t_i) = hf(t_i) + \mathcal{O}(h^2), \quad i = 1, 2, 3, \ldots, N.$$

That is,

$$u(t_{i+1}) = u(t_i) + hf(t_i) + \mathcal{O}(h^2), \quad i = 1, 2, 3, \ldots, N. \tag{2.12}$$

Let $i = 1$ in (2.12) and drop the truncation error term $\mathcal{O}(h^2)$. We have

$$u(t_{1+1}) = u(t_2) = u(t_1) + hf(t_1).$$

But we know neither $u(t_2)$ nor $u(t_1)$. The above equation is unsolvable! What can we do now?

Of course we are intelligent!

Let $i = 0$ in (2.12) and drop the truncation error term $\mathcal{O}(h^2)$. Now we observe that

$$u(t_{0+1}) = u(t_1) = u(t_0) + hf(t_0).$$

This equation is solvable. We have a deal!

Denote $u_{i+1} \approx u(t_{i+1}),\ u_i \approx u(t_i)$. We obtain from the above discussion our first finite difference scheme:

$$u_{i+1} = u_i + hf(t_i), \quad i = 0, 1, 2, \ldots, N;\ u_0 = \phi. \tag{2.13}$$

Therefore we have solved (2.10)**,** (2.11) **without using an integral!** More importantly, we may let a computer complete all calculations incurred without touching a piece of paper.

The forward finite difference scheme (2.13) is named after the Swiss mathematician Leonhard Euler (1707-1783) due to his brilliant research work related.

$\Longleftrightarrow$

If we use the backward finite difference formula instead, we may obtain from (2.10), (2.11) the following:

$$\begin{aligned}\nabla u(t_i) &= \frac{u(t_i)-u(t_{i-1})}{h} = f(t_i)+\mathcal{O}(h), \quad t_i\in\mathcal{D},\\ u(t_0) &= \phi,\end{aligned}$$

where again $0<h\ll 1$ and $(N+1)h=T$.

The above gives us a backward finite difference scheme

$$u_{i+1}=u_i+hf(t_{i+1}), \quad i=0,1,2,\ldots,N;\ u_0=\phi. \tag{2.14}$$

$\Rightarrow$ What is the difference between (2.13) and (2.14)? Can you find it out?

$\Longleftrightarrow$

Here we have following fundamental concerns.

1. Can we use the central difference formula for solving (2.10), (2.11) in a similar way? Why or why not?
2. Can we solve (2.10), (2.11) on a mesh $\mathcal{D}$ without using any of the finite difference formulae introduced?

To answer the first question, we may have

$$\begin{aligned}\delta u(t_i) &= \frac{u(t_{i+1})-u(t_{i-1})}{2h} = f(t_i)+\mathcal{O}(h^2), \quad t_i\in\mathcal{D}, \qquad (2.15)\\ u(t_0) &= \phi,\end{aligned}$$

which gives

$$u_{i+1}=u_{i-1}+hf(t_i), \quad i=1,2,3,\ldots,N;\ u_0=\phi.$$

The above means that

$$\begin{aligned}u_0 &= \phi,\\ u_1 &= ?\\ u_2 &= u_0+2hf(t_1),\\ u_3 &= u_1+2hf(t_2) \;=\; ?\\ u_4 &= u_2+2hf(t_3),\\ u_5 &= u_3+2hf(t_5) \;=\; ?\\ u_6 &= u_4+2hf(t_6),\\ &\cdots\cdots\end{aligned}$$

The second, fourth, and sixth equations above are apparently unsolvable. In other words, we can only compute numerical solution values

$$u_0,u_2,u_6,\ldots,u_{2i},\ldots$$

by using the central finite difference approximation of u'. The numerical solution is expected to be better since the order of truncation error shown in (2.15) is actually of second.

Do you think the crisis can be resolved readily by using a finer mesh for which $\tilde{h} = h/2$ is utilized? Why or why not?

To answer the second question, we may consider the integral formula for solving (2.10), (2.11):

$$\int_0^t u'(\tau)d\tau = \int_0^t f(\tau)d\tau, \quad 0 < t < T.$$

Apparently, a numerical quadrature may be employed. To avoid the use of composite formulations, we may rewrite the above to a compact form, that is,

$$\int_t^{t+h} u'(\tau)d\tau = \int_t^{t+h} f(\tau)d\tau, \quad t, t+h \in [0,T].$$

The above is equivalent to

$$u(t+h) = u(t) + \int_t^{t+h} f(\tau)d\tau, \qquad t, t+h \in [0,T]. \tag{2.16}$$

We obtain immediately the following according to single variable calculus.

i) Left-end formula

$$u(t+h) = u(t) + hf(t) + \mathcal{O}(h^2), \qquad t, t+h \in [0,T];$$

ii) Right-end formula

$$u(t+h) = u(t) + hf(t+h) + \mathcal{O}(h^2), \qquad t, t+h \in [0,T];$$

iii) Averaging formula

$$u(t+h) = u(t) + \frac{h}{2}[f(t) + f(t+h)] + \mathcal{O}(h^3), \qquad t, t+h \in [0,T]. \tag{2.17}$$

Let $t = t_i, t+h = t_{i+1},\ i = 0,1,2,\ldots,N$. Drop all truncation errors. Do you think if the first two formulae above recover the forward and backward finite difference formulae we have had, while the averaging formula appears to be new?

Can you derive more computational algorithms based on (2.16)*?*

C.04 Simplified Model Equation Four.

Now, we consider a different ordinary differential equation initial value problem:

$$u'(t) = f(t,u), \quad 0 < t < T, \tag{2.18}$$
$$u(0) = \phi, \tag{2.19}$$

where $f(t,u)$ is a nice integrable function with two variables (one of them is the unknown).

But, what do we mean a “nice” function? Herewith, we do mean that the following Lipschitz condition holds:

$$|f(t,u) - f(t,v)| \le \lambda |u - v|,$$

for all possible real-valued functions u and v. Here $\lambda > 0$ is a constant independent of choices of u and v. λ is frequently called a Lipschitz constant.

The Lipschitz condition is crucial for a guarantee that ensures a unique solution of (2.18), (2.19) in the theory of differential equations.

The function $f(t, u)$ can also be extremely "rich" in the sense that

1. The initial value problem (2.18), (2.19) is linear if f is linear in u, that is,

$$f(t, u) = \alpha(t)u + \beta(t), \qquad 0 < t < T,$$

   where α, β are integrable functions. A linear f satisfies the Lipschitz condition.

2. The initial value problem (2.18), (2.19) is nonlinear if f is nonlinear in u, that is, the above linear structure does not exist.

Let $\mathcal{D}$ be a mesh over $(0, T)$ with h as the uniform step. Replace the derivative in (2.18), (2.19) by forward, backward and central differences, respectively. We acquire the following results.

i) Forward approximation

$$u(t_{i+1}) = u(t_i) + hf(t_i, u(t_i)) + \mathcal{O}(h^2), \qquad t_i, t_{i+1} \in \bar{\mathcal{D}};$$

ii) Backward approximation

$$u(t_{i+1}) = u(t_i) + hf(t_{i+1}, u(t_{i+1})) + \mathcal{O}(h^2), \qquad t_i, t_{i+1} \in \bar{\mathcal{D}};$$

iii) Central approximation

$$u(t_{i+1}) = u(t_{i-1}) + 2hf(t_i, u(t_i)) + \mathcal{O}(h^3), \qquad t_{i-1}, t_{i+1} \in \bar{\mathcal{D}}.$$

Drop the error terms, we observe that

i) Forward finite difference scheme

$$u_{i+1} = u_i + hf(t_i, u_i), \qquad t_i, t_{i+1} \in \bar{\mathcal{D}};$$

ii) Backward finite difference scheme

$$u_{i+1} = u_i + hf(t_{i+1}, u_{i+1}), \qquad t_i, t_{i+1} \in \bar{\mathcal{D}};$$

iii) Central finite difference scheme

$$u_{i+1} = u_{i-1} + 2hf(t_i, u_i), \qquad t_{i-1}, t_{i+1} \in \bar{\mathcal{D}}.$$

Can we effectively solve the above equations (by using a computer)?

$$\Longleftrightarrow$$

Case I. When f is linear in u. In the situation we have $f(t, u) = \alpha(t)u(t) + \beta(t)$. For the forward scheme, it follows immediately that

$$\begin{aligned}
u_0 &= \phi, \\
u_1 &= u(t_0) + h(\alpha(t_0)u(t_0) + \beta(t_0)) &= u_0 + h(\alpha_0 u_0 + \beta_0), \\
u_2 &= u(t_1) + h(\alpha(t_1)u(t_1) + \beta(t_1)) &= u_1 + h(\alpha_1 u_1 + \beta_1), \\
u_3 &= u(t_2) + h(\alpha(t_2)u(t_2) + \beta(t_2)) &= u_2 + h(\alpha_2 u_2 + \beta_2), \\
u_4 &= u(t_3) + h(\alpha(t_3)u(t_3) + \beta(t_3)) &= u_3 + h(\alpha_3 u_3 + \beta_3), \\
u_5 &= u(t_4) + h(\alpha(t_4)u(t_4) + \beta(t_4)) &= u_4 + h(\alpha_4 u_4 + \beta_4), \\
&\ldots\ldots
\end{aligned}$$

where $\alpha_i = \alpha(t_i),\ \beta_i = \beta(t_i)$ for all appropriate indexes i. The recursive formulae offer explicitly numerical solution values $u_0, u_1, u_2, \ldots, u_{N+1}$ very effectively!

By the same token, for the backward scheme, we may obtain the following relations which are very interesting:

$$\begin{aligned}
u_0 &= \phi,\\
u_1 &= u_0 + h(\alpha_1 u_1 + \beta_1),\\
u_2 &= u_1 + h(\alpha_2 u_2 + \beta_2),\\
u_3 &= u_2 + h(\alpha_3 u_3 + \beta_3),\\
u_4 &= u_3 + h(\alpha_4 u_4 + \beta_4),\\
u_5 &= u_4 + h(\alpha_5 u_5 + \beta_5),\\
&\ldots\ldots
\end{aligned}$$

Let us solve the second equation in the list first. To this end, we have

$$u_1 = u_0 + h\alpha_1 u_1 + h\beta_1.$$

Thus,

$$u_1 - h\alpha_1 u_1 = u_0 + h\beta_1.$$

Then,

$$(1 - h\alpha_1)u_1 = u_0 + h\beta_1.$$

So,

$$u_1 = (u_0 + h\beta_1)/(1 - h\alpha_1)$$

given that $1 - h\alpha_1 \neq 0$. This seems to be easy to accommodate, since we can always select, say, a sufficiently small step $h > 0$.

It is not difficult (why?) to notice that we have a general formula for the solutions of the list of backward scheme equations:

$$u_{i+1} = (u_i + h\beta_{i+1})/(1 - h\alpha_{i+1}), \quad i = 0, 1, 2, \ldots, N, \tag{2.20}$$

with $u_0 = \phi$, and assuming that $1 - h\alpha_{i+1} \neq 0,\ i = 0, 1, 2, \ldots, N$.

As for the central finite difference scheme, we may have

$$\begin{aligned}
u_0 &= \phi,\\
u_1 &= \ ?\\
u_2 &= u_0 + 2h(\alpha_1 u_1 + \beta_1) \ = \ ?\\
u_3 &= u_1 + 2h(\alpha_2 u_2 + \beta_2) \ = \ ?\\
u_4 &= u_2 + h(\alpha_3 u_3 + \beta_3) \ = \ ?\\
u_5 &= u_3 + h(\alpha_4 u_4 + \beta_4) \ = \ ?\\
&\ldots\ldots
\end{aligned}$$

Would you have any suggestions for solving the above?

$\Longleftrightarrow$

Case II. When f is nonlinear in u. In the situation we follow the definition of a forward scheme to find that

$$\begin{aligned}
u_0 &= \phi, \\
u_1 &= u(t_0) + hf(t_0, u(t_0)) = u_0 + hf(t_0, u_0), \\
u_2 &= u(t_1) + hf(t_1, u(t_1)) = u_1 + hf(t_1, u_1), \\
u_3 &= u(t_2) + hf(t_2, u(t_2)) = u_2 + hf(t_2, u_2), \\
u_4 &= u(t_3) + hf(t_3, u(t_3)) = u_3 + hf(t_3, u_3), \\
u_5 &= u(t_4) + hf(t_4, u(t_4)) = u_4 + hf(t_4, u_4), \\
& \quad ......
\end{aligned}$$

Apparently, the above recursive formulae are explicitly and solvable! Therefore numerical solution values $u_0, u_1, u_2, \ldots, u_{N+1}$ can be computed in order and effectively!

Unfortunately, the real trouble starts when a backward formulation is utilized. In the circumstance, we arrive at the following implicit recursive algorithms:

$$\begin{aligned}
u_0 &= \phi, \\
u_1 &= u(t_0) + hf(t_1, u(t_1)) = u_0 + hf(t_1, u_1), \\
u_2 &= u(t_1) + hf(t_2, u(t_2)) = u_1 + hf(t_2, u_2), \\
u_3 &= u(t_2) + hf(t_3, u(t_3)) = u_2 + hf(t_3, u_3), \\
u_4 &= u(t_3) + hf(t_4, u(t_4)) = u_3 + hf(t_4, u_4), \\
u_5 &= u(t_4) + hf(t_5, u(t_5)) = u_4 + hf(t_5, u_5), \\
& \quad ......
\end{aligned}$$

But what is the problem here? To see this, let us take the second equation from the above list:

$$u_1 = u_0 + hf(t_1, u_1), \tag{2.21}$$

where $u_0 = \phi$ is the given initial value and u_1 is the unknown. But $f(t_1, u_1)$ is nonlinear with respect to its second variable, that is, u_1, on the right-hand-side of the equation.

A particular example is

$$f(t, u) = \exp(t^2u^2)\sin(3\pi u).$$

In the case (2.21) becomes

$$u_1 = u_0 + \exp(t_1^2 u_1^2)\sin(3\pi u_1).$$

Equivalently,

$$\phi + e^{h^2u_1^2}\sin(3\pi u_1) - u_1 = 0. \tag{$*$}$$

Can we solve the above nonlinear algebraic equation for u_1?

On the other hand, utilizing the central difference formula, we may derive another set of implicit iterative equations as follows.

$$\begin{aligned}
u_0 &= \phi, \\
u_1 &= ? \\
u_2 &= u_0 + 2hf(t_1, u_1), \\
u_3 &= u_1 + 2hf(t_3, u_3), \\
u_4 &= u_2 + 2hf(t_4, u_4), \\
u_5 &= u_3 + 2hf(t_5, u_5), \\
& \quad ......
\end{aligned}$$

Again, the numerical solution value u_1 is not defined. But we notice that if it can be evaluated in certain way, then the rest of recursive procedure becomes executable! Can you now suggest a reasonable resolution for the entire solution process?

$\Longleftrightarrow$

Now, recall our averaging formula (2.17). A very careful calculation (homework) may show that

$$\begin{aligned}
u_0 &= \phi, \\
u_1 &= u_0 + \frac{h}{2}[f(t_0, u_0) + f(t_1, u_1)], \\
u_2 &= u_1 + \frac{h}{2}[f(t_1, u_1) + f(t_2, u_2)], \\
u_3 &= u_2 + \frac{h}{2}[f(t_2, u_2) + f(t_3, u_3)], \\
u_4 &= u_3 + \frac{h}{2}[f(t_3, u_3) + f(t_4, u_4)], \\
u_5 &= u_4 + \frac{h}{2}[f(t_4, u_4) + f(t_5, u_5)], \\
& \quad \ldots\ldots
\end{aligned}$$

Again, they are hard to solve if not impossible to solve! To see this, for example, we again pick up the second equation from above and rewrite it to

$$2u_1 - hf(t_1, u_1) = 2u_0 + hf(t_0, u_0).$$

Now, take a test function $f(t, u) = \exp(t^2 u^2)\sin(3\pi u)$. It follows that

$$2u_1 - he^{h^2u_1^2}\sin(3\pi u_1) = 2\phi + he^{0^2\phi^2}\sin(3\pi\phi) = 2\phi + h\sin(3\pi\phi).$$

The above can be written into a "standard" nonlinear algebraic equation

$$2u_1 - he^{h^2u_1^2}\sin(3\pi u_1) - 2\phi - h\sin(3\pi\phi) = 0 \qquad (**)$$

for possible roots.

$\Longleftrightarrow$

But, nevertheless, we can see from equations $(*)$ and $(**)$ that both backward and Simpson's methods share a key feature. That is, the numerical solution of an equation in the form of

$$f(x) = 0, \qquad (2.22)$$

where $f = f(x)$ is a given nonlinear algebraic function.

$\Longleftrightarrow$

For solutions of nonlinear algebraic equations, there has been a tremendous amount of publications. Herewith we would recommend discussions provided by

https://www.cs.usask.ca/˜spiteri/M211/notes/chapter4.pdf

with supporting documents in

https://www.cs.usask.ca/˜spiteri/M211/notes/

References

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