Notes on September 10, 2019

MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

Main text to use: Numerical Partial Differential Equations in Finance Explained by Karel in 't Hout, published by the Springer Nature, 2017

Lecture notes: They will be uploaded frequently to the Moodle system at HKBU

Assessment methods: homework and quizzes (20%); two numerical projects (probably for the weeks of 09/30–10/04 and 11/11–11/15; total 20%); final exam (60%)

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Ideas of a discretization process: from intervals to meshes

A. One-dimensional intervals.

Given an interval $I = [a, b]$, where $b > a$. We have following steps:

1. Choose a sufficiently large integer $N \gg 0$.
2. Select a set of N values $\mathcal{D} = \{x_1, x_2, \ldots, x_N\}$ for which

   $$a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N < x_{N+1} = b.$$

   We require that

   $$0 < h_i = x_i - x_{i-1} \ll 1, \; i = 1, 2, \ldots, N+1.$$

3. Denote that

   $$\bar{\mathcal{D}} = \{x_0, x_1, x_2, \ldots, x_N, x_{N+1}\}$$

   as the closure of $\mathcal{D}$.

Apparently,

$$\mathcal{D} \subset \bar{\mathcal{D}} \subset I.$$

Both $\bar{\mathcal{D}}$, $\mathcal{D}$ are meshes, or grids. We call $\mathcal{D}$ as the set of internal mesh points, and $\bar{\mathcal{D}} \setminus \mathcal{D} = \{x_0, x_{N+1}\}$ as the set of boundary mesh points. Positive values h_i, $i = 1, 2, \ldots, N+1$, are mesh steps.

Definition 1. Meshes $\bar{\mathcal{D}}$, $\mathcal{D}$ are uniform if

$$h_i = h = (b-a)/(N+1), \quad i = 1, 2, \ldots, N+1.$$

Otherwise the meshes are nonuniform.

Definition 2. For any $x_i \in \mathcal{D}$, we may define following forward, backward and central differences of function $y = f(x)$, $x \in I$:

$$\begin{aligned} \Delta f(x_i) &= \frac{f(x_{i+1}) - f(x_i)}{h_{i+1}}, \\ \nabla f(x_i) &= \frac{f(x_i) - f(x_{i-1})}{h_i}, \\ \delta f(x_i) &= \frac{f(x_{i+1}) - f(x_{i-1})}{h_{i+1} + h_i}. \end{aligned}$$

Theorem 1. *Assume $y = f''(x)$ be continuous on (a, b). Then*

$$\begin{aligned} \Delta f(x_i) &= f'(x_i) + \mathcal{O}(h_{i+1}), \\ \nabla f(x_i) &= f'(x_i) + \mathcal{O}(h_i), \\ \delta f(x_i) &= f'(x_i) + \begin{cases} \mathcal{O}(\tilde{h}_i) & \text{if } h_{i+1} \neq h_i \\ \mathcal{O}(\tilde{h}_i^2) & \text{if } h_{i+1} = h_i, \end{cases} \end{aligned}$$

where $\tilde{h}_i = \max\{h_{i+1}, h_i\}$ and $x_i \in \mathcal{D}$.

B. <u>Two-dimensional domains.</u>

Given a rectangular domain $D = [a, b] \times [c, d]$, where $b > a$, $d > c$. We consider the following:

1. Choose sufficiently large integers N, $M \gg 0$.
2. Select a set of NM points $\mathcal{D}_2 = \{(x_i, y_j)\}_{1 \leq i \leq N;\, 1 \leq j \leq M}$ for which

$$\begin{aligned} a = x_0 < x_1 < x_2 < \ldots < x_{N-1} < x_N < x_{N+1} = b; \\ c = y_0 < y_1 < y_2 < \ldots < y_{M-1} < y_M < y_{M+1} = d. \end{aligned}$$

   We require that

$$\begin{aligned} 0 < h_i = x_i - x_{i-1} \ll 1,\ i = 1, 2, \ldots, N+1; \\ 0 < k_j = y_j - y_{j-1} \ll 1,\ j = 1, 2, \ldots, M+1. \end{aligned}$$

3. Denote that

$$\bar{\mathcal{D}}_2 = \{(x_i, y_j)\}_{0 \leq i \leq N+1;\, 0 \leq j \leq M+1}$$

   as the closure of $\mathcal{D}_2$.

Apparently,

$$\mathcal{D}_2 \subset \bar{\mathcal{D}}_2 \subset D.$$

Both $\bar{\mathcal{D}}_2,\ \mathcal{D}_2$ are two-dimensional meshes, or grids. We call $\mathcal{D}_2$ as the set of internal mesh points, and $\bar{\mathcal{D}}_2 \setminus \mathcal{D}_2 = \{(x_i, y_j)\}_{i=0,N+1;\, j=0,M+1}$ as the set of boundary mesh points. Positive values $h_i,\ i = 1, 2, \ldots, N+1;\ k_j,\ j = 1, 2, \ldots, M+1$, are mesh steps.

Definition 3. Meshes $\bar{\mathcal{D}}_2,\ \mathcal{D}_2$ are uniform if

$$\begin{aligned} h_i = h = (b-a)/(N+1), &\quad i = 1, 2, \ldots, N+1; \\ k_j = k = (d-c)/(M+1), &\quad j = 1, 2, \ldots, M+1. \end{aligned}$$

Otherwise the meshes are nonuniform.

Definition 4. For any $(x_i, y_j) \in \mathcal{D}_2$, we may define following forward, backward and central differences of function $z = f(x,y),\ (x,y) \in D$:

$$\begin{aligned}
\Delta_x f(x_i, y_j) &= \frac{f(x_{i+1}, y_j) - f(x_i, y_j)}{h_{i+1}}, \\
\nabla_x f(x_i, y_j) &= \frac{f(x_i, y_j) - f(x_{i-1}, y_j)}{h_i}, \\
\delta_x f(x_i, y_j) &= \frac{f(x_{i+1}, y_j) - f(x_{i-1}, y_j)}{h_{i+1} + h_i}; \\
\Delta_y f(x_i, y_j) &= \frac{f(x_i, y_{j+1}) - f(x_i, y_j)}{k_{j+1}}, \\
\nabla_y f(x_i, y_j) &= \frac{f(x_i, y_j) - f(x_i, y_{j-1})}{k_j}, \\
\delta_y f(x_i, y_j) &= \frac{f(x_i, y_{j+1}) - f(x_i, y_{j-1})}{k_{j+1} + k_j}.
\end{aligned}$$

Theorem 2. *Assume $z = f(x,y)$ be second order continuously differentiable on $(a,b) \times (c,d)$. Then*

$$\begin{aligned}
\Delta_x f(x_i, y_j) &= f_x(x_i, y_j) + \mathcal{O}(h_{i+1}), \\
\nabla_x f(x_i, y_j) &= f_x(x_i, y_j) + \mathcal{O}(h_i), \\
\delta_x f(x_i, y_j) &= f_x(x_i, y_j) + \left\{ \begin{array}{ll} \mathcal{O}(\tilde{h}_i) & \text{if } h_{i+1} \neq h_i \\ \mathcal{O}(\tilde{h}_i^2) & \text{if } h_{i+1} = h_i, \end{array} \right. ,
\end{aligned}$$

where $\tilde{h}_i = \max\{h_{i+1}, h_i\}$ and $(x_i, y_j) \in \mathcal{D}_2$. Further,

$$\begin{aligned}
\Delta_y f(x_i, y_j) &= f_y(x_i, y_j) + \mathcal{O}(k_{j+1}), \\
\nabla_y f(x_i, y_j) &= f_y(x_i, y_j) + \mathcal{O}(k_j), \\
\delta_y f(x_i, y_j) &= f_y(x_i, y_j) + \left\{ \begin{array}{ll} \mathcal{O}(\tilde{k}_j) & \text{if } k_{j+1} \neq k_j \\ \mathcal{O}(\tilde{k}_j^2) & \text{if } k_{j+1} = k_j, \end{array} \right. ,
\end{aligned}$$

where $\tilde{k}_j = \max\{k_{j+1}, k_j\}$ and $(x_i, y_j) \in \mathcal{D}_2$.

C. Simulation experiments.

$\hookrightarrow$ Given an interval $I = [0, 1]$.

$\hookrightarrow$ Given a function $f(x) = \sin(\pi x),\ x \in I$.

$\hookrightarrow$ We have a sample MATLAB program **matlab002.m** which

1. calculates a set of N randomly distributed internal mesh points

$$\mathcal{D} = \{x_1, x_2, \ldots, x_N\} \subset I.$$

2. calculates corresponding $h_1, h_2, \ldots, h_{N+1}$.

3. calculates forward finite difference values

$$\Delta f(x_1), \Delta f(x_2), \ldots, \Delta f(x_N).$$

4. plots the approximations $(x_i, \Delta f(x_i)),\ i = 1, 2, \ldots, N$, together with the true values $(x_i, f'(x_i)),\ i = 1, 2, \ldots, N$, in the same frame.

5. calculates and simulates the absolute error

$$\text{error}_a(x_i) = |\Delta f(x_i) - f'(x_i)|,\ i = 1, 2, \ldots, N.$$

6. calculates and simulates the relative error

$$\text{error}_r(x_i) = \frac{|\Delta f(x_i) - f'(x_i)|}{|f'(x_i)|},\ i = 1, 2, \ldots, N,$$

as far as the denominators are nontrivial. In most cases, relative errors provide the information of the numerical error in percentile.

QUESTION: Can you repeat the above experiments for the backward difference formula?

References

[1] K. in 't Hout, *Numerical Partial Differential Equations in Finance Explained,* Springer, Antwerp, Belgium, 2017.

[2] K. Atkinson and W. Han, *Elementary Numerical Analysis,* 3rd Ed., John Wiley & Sons, Somerset, NJ, 2004.

[3] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge and New York, 2004.

[4] B. Jain and A. Sheng, An exploration of the approximation of derivative functions via finite differences, *Rose-Hulman Undergrd. Math J.,* 8:172-188, 2007.