Solution references for week 3

MATH 4665/4875/7140/7300, Fall 2019, HKBU

Problem 1: Recall the standard Taylor expansion

$$\begin{aligned} f(x\pm h) &= \frac{(\pm h)^0}{0!}f^{(0)}(x)+\frac{(\pm h)^1}{1!}f^{(1)}(x)+\frac{(\pm h)^2}{2!}f^{(2)}(x)+\frac{(\pm h)^3}{3!}f^{(3)}(x) \\ &\quad +\frac{(\pm h)^4}{4!}f^{(4)}(x)+\frac{(\pm h)^5}{5!}f^{(5)}(x)+\cdots \\ &= f(x)\pm hf'(x)+\frac{h^2}{2!}f''(x)\pm\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(4)}(x)\pm\frac{h^5}{5!}f^{(5)}(x)+\cdots \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta f(x_i) &= \frac{f(x_{i+1})-f(x_i)}{h_{i+1}} = \frac{f(x_i+h_{i+1})-f(x_i)}{h_{i+1}} \\ &= \Big[f(x_i)+h_{i+1}f'(x_i)+\frac{h_{i+1}^2}{2!}f''(x_i)+\frac{h_{i+1}^3}{3!}f'''(x_i) \\ &\quad +\frac{h_{i+1}^4}{4!}f^{(4)}(x_i)+\frac{h_{i+1}^5}{5!}f^{(5)}(x_i)+\cdots-f(x_i)\Big]/h_{i+1} \\ &= f'(x_i)+\frac{h_{i+1}}{2!}f''(x_i)+\frac{h_{i+1}^2}{3!}f'''(x_i)+\frac{h_{i+1}^3}{4!}f^{(4)}(x_i)+\frac{h_{i+1}^4}{5!}f^{(5)}(x_i)+\cdots \end{aligned}$$

By the same token,

$$\begin{aligned} \nabla f(x_i) &= \frac{f(x_i)-f(x_{i-1})}{h_i} = \frac{f(x_i)-f(x_i-h_i)}{h_i} \\ &= \Big[f(x_i)-f(x_i)+h_if'(x_i)-\frac{h_i^2}{2!}f''(x_i)+\frac{h_i^3}{3!}f'''(x_i) \\ &\quad -\frac{h_i^4}{4!}f^{(4)}(x_i)+\frac{h_i^5}{5!}f^{(5)}(x_i)-\cdots\Big]/h_i \\ &= f'(x_i)-\frac{h_i}{2!}f''(x_i)+\frac{h_i^2}{3!}f'''(x_i)-\frac{h_i^3}{4!}f^{(4)}(x_i)+\frac{h_i^4}{5!}f^{(5)}(x_i)-\cdots \end{aligned}$$

It follows immediately that

$$\begin{aligned} \Delta f(x_i)-\nabla f(x_i) &= f'(x_i)+\frac{h_{i+1}}{2!}f''(x_i)+\frac{h_{i+1}^2}{3!}f'''(x_i)+\frac{h_{i+1}^3}{4!}f^{(4)}(x_i) \\ &\quad +\frac{h_{i+1}^4}{5!}f^{(5)}(x_i)+\cdots \\ &\quad -f'(x_i)+\frac{h_i}{2!}f''(x_i)-\frac{h_i^2}{3!}f'''(x_i)+\frac{h_i^3}{4!}f^{(4)}(x_i)-\frac{h_i^4}{5!}f^{(5)}(x_i)+\cdots \\ &= \frac{h_{i+1}+h_i}{2!}f''(x_i)+\frac{h_{i+1}^2-h_i^2}{3!}f'''(x_i)+\frac{h_{i+1}^3+h_i^3}{4!}f^{(4)}(x_i) \\ &\quad +\frac{h_{i+1}^4-h_i^4}{5!}f^{(5)}(x_i)+\cdots \end{aligned}$$

Now,

$$
\begin{aligned}
\frac{\Delta f(x_i) - \nabla f(x_i)}{(h_{i+1}+h_i)/2} &= f''(x_i) + \frac{h_{i+1}^2 - h_i^2}{3(h_{i+1}+h_i)} f'''(x_i) + \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)} f^{(4)}(x_i) \\
&\quad + \frac{h_{i+1}^4 - h_i^4}{60(h_{i+1}+h_i)} f^{(5)}(x_i) + \cdots \\
&= f''(x_i) + \frac{h_{i+1} - h_i}{3} f'''(x_i) + \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)} f^{(4)}(x_i) \\
&\quad + \frac{h_{i+1}^4 - h_i^4}{60(h_{i+1}+h_i)} f^{(5)}(x_i) + \cdots
\end{aligned}
$$

Based on the above, we may state the following

(a) The formula is an approximation of the second derivative at x_i since the absolute error

$$
\begin{aligned}
\left| \frac{\Delta f(x_i) - \nabla f(x_i)}{(h_{i+1}+h_i)/2} - f''(x_i) \right| &\le \left| \frac{h_{i+1} - h_i}{3} f'''(x_i) \right. \\
&\quad \left. + \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)} f^{(4)}(x_i) + \frac{h_{i+1}^4 - h_i^4}{60(h_{i+1}+h_i)} f^{(5)}(x_i) + \cdots \right| \\
&\le \left| \frac{h_{i+1} - h_i}{3} f'''(x_i) \right| + \left| \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)} f^{(4)}(\xi) \right| \\
&\le \frac{|h_{i+1} - h_i|}{3} \left| f'''(x_i) \right| + \frac{h_{i+1}^3 + h_i^3}{12(h_{i+1}+h_i)} \left| f^{(4)}(\xi) \right|, \quad x_i < \xi < x_{i+1},
\end{aligned}
$$

due to the remainder theorem.

(b) The approximation is only first order, that is,

$$
\left| \frac{\Delta f(x_i) - \nabla f(x_i)}{(h_{i+1}+h_i)/2} - f''(x_i) \right| = \mathcal{O}(h),
$$

where $h = \max\{h_{i+1}, h_i\}$ if $h_{i+1} \neq h_i$.

(c) The approximation is second order, that is,

$$
\left| \frac{\Delta f(x_i) - \nabla f(x_i)}{(h_{i+1}+h_i)/2} - f''(x_i) \right| = \frac{h^3 + h^3}{12(h+h)} \left| f^{(4)}(\xi) \right| = \frac{h^2}{12} \left| f^{(4)}(\xi) \right| = \mathcal{O}(h^2)
$$

if $h_{i+1} = h_i = h$.

<u>Problem 2</u>: Standard calculus problems.

<u>Problem 3:</u>

(a) We have

$$
g'(x_n) = 2\sin(x_n)\cos(x_n) = \sin(2x_n) = \sin\left(\frac{10\pi}{8}\right).
$$

(b) Use a point-slope form of linear equation:

$$
\frac{y - g(x_n)}{x - x_n} = g'(x_n).
$$

(c) Let $y = 0$ in the above equation and solve it for the x value.