MATH 4875 - Scintillate - Chiu Pit Ho
Final Ass 17212049

1) a) $\Delta(\nabla f(x_i)) = \dfrac{\nabla f(x_{i+1}) - \nabla f(x_i)}{h_{i+1}}$

$= \left[\dfrac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \dfrac{f(x_i) - f(x_{i-1})}{h_i}\right] \Big/ h_{i+1}$

$\rightarrow h_{i+1} \cdot \Delta(\nabla f(x_i)) = \left[\dfrac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \dfrac{f(x_i) - f(x_{i-1})}{h_i}\right]$

$\nabla(\Delta f(x_i)) = \dfrac{\Delta f(x_i) - \Delta f(x_{i-1})}{h_i}$

$= \left[\dfrac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \dfrac{f(x_i) - f(x_{i-1})}{h_i}\right] \Big/ h_i$

$\rightarrow h_i \cdot \nabla(\Delta f(x_i)) = \left[\dfrac{f(x_{i+1}) - f(x_i)}{h_{i+1}} - \dfrac{f(x_i) - f(x_{i-1})}{h_i}\right]$

$||$

$\boxed{h_{i+1} \Delta(\nabla f(x_i))} = \left[\Delta f(x_i) - \nabla f(x_i)\right]$

b) $\left[\Delta f(x_i) - \nabla f(x_i)\right] \approx f'(x_i) + \dfrac{h_{i+1}}{2} f''(x_i) + \dfrac{h_{i+1}^2}{3!} f^3(x_i) + \cdots - \left(f'(x_i) - \dfrac{h_i}{2} f''(x_i) + \dfrac{h_i^2}{3!} f^3(x_i) - \cdots\right)$

$\approx \dfrac{h_{i+1} + h_i}{2!} f''(x_i) + \dfrac{h_{i+1}^2 - h_i^2}{3!} f^3(x_i) + \cdots$

$\rightarrow$ neither $h_{i+1} \Delta(\nabla f(x_i))$ nor $h_i \nabla(\Delta f(x_i))$ approximates $f''(x_i)$

unless $h_{i+1} = h_i = h \rightarrow [\Delta f(x_i) - \nabla f(x_i)] \approx \dfrac{2h}{2!} f''(x_i) + \cdots$

$h \Delta(\nabla f(x_i)) = h \nabla(\Delta f(x_i)) \approx f''(x_i)$

//

2018-2019

2a) $u'(t) = f(u)$ and $u(0) = \phi$

$u'(t) \approx \frac{u(t_{i+1}) - u(t_i)}{h} = f(u_i) + O(h) \rightarrow \boxed{u_{i+1} = u_i + hf(u_i) \text{ and } u_0 = \phi}$

$u_0 = \phi$

$u_1 = \phi + hf(\phi)$

$u_2 = \phi + hf(\phi) + h\{f[\phi + hf(\phi)]\}$

b) In fact, $u(t_{i+1}) = u(t_i) + hf(u(t_i)) + O(h^2)$

$|u(t_{i+1}) - u_{i+1}| \doteq \varepsilon_{i+1}$

$= |u(t_i) - u_i| + h\left|\left[f(u(t_i)) - f(u_i)\right]\right| + O(h^2)$

$\leq \underbrace{|u(t_i) - u_i|}_{\varepsilon_i} + h\lambda \underbrace{|u(t_i) - u_i|}_{\varepsilon_i} + O(h^2)$

$\rightarrow \varepsilon_{i+1} \leq (1 + h\lambda)\varepsilon_i + Ch^2$ for $i = 0, 1, \cdots M+1$

By MI $\rightarrow \varepsilon_{i+1} \leq \frac{C}{\lambda} h\left[(1+h\lambda)^{i+1} - 1\right]$

~~F.E.M is convergent~~ $\cancel{\rightarrow \lim_{M\to\infty} \varepsilon_{M+1} = 0}$

$$\varepsilon_{M+1} \leq \frac{C\tilde{T}}{\lambda(M+1)}\left[\left(1 + \frac{\tilde{T}\lambda}{M+1}\right)^{M+1} - 1\right]$$

$$\rightarrow \lim_{M\to\infty} \varepsilon_{M+1} = \lim_{M\to\infty} \frac{C\tilde{T}}{\lambda(M+1)} \cdot \lim_{M\to\infty}\left[\left(1 + \frac{\lambda\tilde{T}}{M+1}\right)^{M+1} - 1\right]$$

$$= 0 \times (e^{\lambda\tilde{T}} - 1) = 0$$

$\rightarrow$ F.E.M is convergent on any uniform mesh.

3a) ~~Discretization in time~~ : $\cancel{\dfrac{u(s,t_{i+1}) - 2u(s,t) + u(s,t_{i-1})}{h^2}} =$

$$\frac{u(s,t_{i+1}) - u(s,t_{i-1})}{2h} = u_t(s,t_i) + O(h^2)$$

$$\cancel{u = \begin{bmatrix} u(s,t_1) \\ u(s,t_2) \\ \vdots \\ u(s,t_n) \end{bmatrix} = \begin{bmatrix} u_1(s) \\ u_2(s) \\ \vdots \\ u_n(s) \end{bmatrix} \in \mathbb{R}^n}$$

$$\frac{u(s_{i+1},t) - 2u(s_i,t) + u(s_{i-1},t)}{h^2} = u_{ss}(s_i,t) + O(h^2)$$

$$\cancel{u_{ss} = \ldots = \begin{bmatrix} u_{ss1}(s) \\ \vdots \\ u_{ssn}(s) \end{bmatrix}}$$

3)
$$u_{ss} = \begin{pmatrix} \dfrac{u(s_2,t) - 2u(s_1,t) + u(s_0,t)}{h^2} \\ \vdots \\ \dfrac{u(s_n,t) - 2(u_{n-1},t) + u(u_{n-2},t)}{h^2} \end{pmatrix}$$

$$= \frac{1}{h^2} \underbrace{\begin{bmatrix} 1 & -2 & 1 & & & \\ & 1 & -2 & 1 & & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \beta \\ u(t) \\ u_2(t) \\ \vdots \\ u_{n-1}(t) \\ \gamma \end{bmatrix}}_{\vec{u} \in \mathbb{R}^{n+1}}$$

$\vec{u} \in \mathbb{R}^{n+1}$

$A \in \mathbb{R}^{(n+1) \times (n-1)}$

5 $\quad n+1-2 = n-1$

3

$$\therefore u'(t) = \frac{\alpha}{h^2} A \vec{u} \quad \text{with} \quad \vec{u}(0) = \vec{\phi}$$

~~b)~~ Use forward euler's method.

$$u_{i+1} = u_i + \Delta t \cdot \frac{\alpha}{(\Delta x)^2} A u_i \quad \text{with } \vec{u}(0) = \vec{\phi}$$

b) $\dfrac{\alpha \Delta t}{(\Delta x)^2} < \dfrac{1}{2}$

Forward-Time Central-Space method