Solution references for weeks 1 and 2

MATH 4665/4875/7140/7300, Fall 2019, HKBU

Week 1

Problems 1–3: Standard calculus problems.

Problem 1: The MVT is used to prove statements about a function on an interval starting from local hypotheses about derivatives at points of the interval.

More precisely, if f is a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Problem 4: use Taylor series expansions to deliver the order of the error. In fact, similar to Theorem 3.1 in our Notes on 09/05, for a nonuniform mesh $\mathcal{D}$ we know that the forward, backward, and central differences are first order approximations of the derivative function f' on $\mathcal{D}$. That is,

$$D_+ f(t_k) - f'(t_k) = \frac{h_k}{2} f''(\xi_1), \tag{3.1}$$

$$D_- f(t_k) - f'(t_k) = -\frac{h_{k-1}}{2} f''(\zeta_1), \tag{3.2}$$

$$\delta f(t_k) - f'(t_k) = \frac{1}{2(h_k + h_{k-1})} \left[h_k^2 f''(\xi_2) - h_{k-1}^2 f''(\zeta_2) \right], \tag{3.3}$$

where $t_{k+1} > \xi_\ell > t_k,\ t_k > \zeta_\ell > t_{k-1},\ \ell = 1, 2.$

For example, the last equation can be extended to

$$\delta f(t_k) - f'(t_k) = \frac{1}{2}\left(h_k - h_{k-1}\right) f''(t_k) + \frac{1}{6(h_k + h_{k-1})}\left[h_k^3 f'''(\xi_3) - h_{k-1}^3 f''(\zeta_3) \right]$$

which offers our 4(b) result.

Week 2

Problem 1: consider the solution to the last problem.

Problem 2: you only to understand and modify the second sample program and make sure it works.

Problem 3: Use Taylor expansions again repeatedly. You may need to replace notations in (4.1), (4.2) by our new notations in notes on 9/5:

$$\begin{aligned} D_+(D_- f(t_k)) &= \frac{D_- f(t_{k+1}) - D_- f(t_k)}{h_k} = \left[\frac{f(t_{k+1}) - f(t_k)}{h_k} - \frac{f(t_k) - f(t_{k-1})}{h_{k-1}} \right] / h_k \\ &= \frac{h_{k-1} f(t_{k+1}) - (h_k + h_{k-1}) f(t_k) + h_k f(t_{k-1})}{h_k^2 h_{k-1}}. \end{aligned} \tag{4.1}$$

By the same token,

$$\begin{aligned} D_-(D_+ f(t_k)) &= \frac{D_+ f(t_k) - D_+ f(t_{k-1})}{h_{k-1}} = \left[\frac{f(t_{k+1}) - f(t_k)}{h_k} - \frac{f(t_k) - f(t_{k-1})}{h_{k-1}} \right] / h_{k-1} \\ &= \frac{h_{k-1} f(t_{k+1}) - (h_k + h_{k-1}) f(t_k) + h_k f(t_{k-1})}{h_k h_{k-1}^2}. \end{aligned} \tag{4.2}$$