Notes for MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

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FUNDAMENTALS FOR FINANCE

C. Numerical Solution of PDEs.

C.05 Convergence of the Numerical Solution.

Consider the following differential equation initial value problem:

$$u'(t) = f(t,u), \quad 0 < t < T, \tag{2.18}$$
$$u(0) = \phi, \tag{2.19}$$

where $f(t,u)$ is nonlinear in u and satisfies the Lipschitz condition given in Section C.04.

Let us use a forward finite difference scheme (it is also called a *forward Euler method,* or *explicit Euler method,* on a uniform mesh over $[0,T]$:

$$u_{i+1} = u_i + hf(t_i, u_i), \quad i = 0, 1, 2, \ldots, N,$$

where

$$h = \frac{T}{N+1} \ll 1.$$

Therefore, we obtain a set of numerical values, or numerical solution

$$\mathcal{A} = \{u_1, u_2, \ldots, u_{N+1}\}.$$

The above values are supposed to be approximations of the true solution values

$$\mathcal{B} = \{u(t_1), u(t_2), \ldots, u(t_{N+1})\},$$

where $t_i = ih,\ i = 1, 2, \ldots, N+1$, and $(N+1)h = T$.

But, how do we know if the above hypothesis is true?

$$\Longleftrightarrow$$

Definition 2.1 (pointwise convergence).
Given any fixed point $\tilde{T} \in (0,T]$. We may construct a uniform mesh with points $t_i,\ i = 0, 1, 2, \ldots, M+1$, where $t_0 = 0,\ t_{M+1} = \tilde{T}$ and $h = \tilde{T}/(M+1)$. We say that a numerical method is convergent if the numerical solution it produces possesses the following feature:

$$\lim_{M\to\infty} \left| u_{M+1} - u(\tilde{T}) \right| = 0.$$

We note that $t_{M+1} = (M+1)h = \tilde{T} \in (0, T]$ in our consideration. Hence, the last limitation can be written equivalently to

$$\lim_{h \to 0^+} \left| u_{M+1} - u(\tilde{T}) \right| = 0.$$

We further note that $\left| u_{M+1} - u(\tilde{T}) \right| = \epsilon_{M+1}$ is actually the absolutely error at the fixed terminal time position $\tilde{T}$. Thus, the above limitations can also be comprised as

$$\lim_{M \to \infty} \epsilon_{M+1} = 0.$$

Theorem 2.1 (convergence for forward Euler).
If the function $f(t, u)$ satisfies the Lipschitz condition with Lipschitz constant $\lambda > 0$, then the forward finite difference scheme is convergent on any uniform mesh.

Proof. This mathematical proof is significantly simple. To this end, recall that

$$u(t_{i+1}) = u(t_i) + hf(t_i, u(t_i)) + \mathcal{O}(h^2), \quad i = 0, 1, 2, \ldots, M,$$

Here we have extended the approximation for $i = 0$. On the other hand, a forward finite difference scheme reads

$$u_{i+1} = u_i + hf(t_i, u_i), \quad i = 0, 1, 2, \ldots, M.$$

A subtraction of the two equations yields

$$u(t_{i+1}) - u_{i+1} = u(t_i) - u_i + h[f(t_i, u(t_i)) - f(t_i, u_i)] + \mathcal{O}(h^2), \quad i = 0, 1, 2, \ldots, M.$$

Without loss of generality, for the estimated quantity $\mathcal{O}(h^2)$, we may assume that there exists a positive constant c_1 such that $|\mathcal{O}(h^2)| \le c_1 h^2$. Now, we have

$$\begin{aligned} |u(t_{i+1}) - u_{i+1}| &\le |u(t_i) - u_i| + h|f(t_i, u(t_i)) - f(t_i, u_i)| + c_1 h^2 \\ &\le |u(t_i) - u_i| + h\lambda |u(t_i) - u_i| + c_1 h^2, \quad i = 0, 1, 2, \ldots, M, \end{aligned}$$

via the Lipschitz condition and triangular inequality. We subsequently find that

$$\begin{aligned} \epsilon_{i+1} &\le \epsilon_i + h\lambda\epsilon_i + c_1 h^2 \\ &= (1 + h\lambda)\epsilon_i + c_1 h^2, \quad i = 0, 1, 2, \ldots, M. \end{aligned}$$

If we claim[1] that there exists a positive constant c_2 such that

$$\epsilon_i \le \frac{c_2}{\lambda} h \left[(1 + h\lambda)^i - 1 \right], \quad i = 0, 1, 2, \ldots, M.$$

Then

$$\epsilon_{i+1} \le (1 + h\lambda) \frac{c_2}{\lambda} h \left[(1 + h\lambda)^i - 1 \right] + c_1 h^2, \quad i = 0, 1, 2, \ldots, M.$$

Set $c = \max\{c_1, c_2\}$. It follows from the above inequality that

$$\epsilon_{i+1} \le \frac{c}{\lambda} h \left[(1 + h\lambda)^{i+1} - 1 \right], \quad i = 0, 1, 2, \ldots, M.$$

May you verify the above?

[1] This can be shown through a mathematical induction. But it is clearly beyond our requirement.

Therefore,

$$\begin{aligned}\epsilon_{M+1} &\le \frac{c}{\lambda} h\left[(1+h\lambda)^{M+1}-1\right] \\ &= \frac{c\tilde{T}}{\lambda(M+1)}\left[\left(1+\frac{\tilde{T}\lambda}{M+1}\right)^{M+1}-1\right].\end{aligned}$$

Recall that

$$\lim_{M\to\infty}\left(1+\frac{\tilde{T}\lambda}{M+1}\right)^{M+1} = e^{\tilde{T}\lambda}.$$

May you verify the above?

We have

$$\begin{aligned}\lim_{M\to\infty}\epsilon_{M+1} &= \lim_{M\to\infty}\frac{c\tilde{T}}{\lambda(M+1)}\lim_{M\to\infty}\left[\left(1+\frac{\lambda\tilde{T}}{M+1}\right)^{M+1}-1\right] \\ &= \lim_{M\to\infty}\frac{c\tilde{T}}{\lambda(M+1)}\left(e^{\tilde{T}\lambda}-1\right) \le \lim_{M\to\infty}\frac{cT}{\lambda(M+1)}e^{\lambda T} = 0.\end{aligned}$$

Since the choose of $\tilde{T}\in(0,T]$ is arbitrary, the forward finite difference scheme is indeed convergent by definition. □

We may also study the convergence of numerical solutions obtained by using other numerical methods, such as the Simpson's algorithm (homework). However, the proofs are in general more complicated and more advanced knowledge is needed. Let's leave the discussions later.

C.06 Systems of Differential Equations.

It is absolutely meaningful to consider multiple options in same time. For instance, if v and w are two market option prices. Their flows may be described by the following initial value problem involving two differential equations:

$$\begin{aligned}v'(t) &= \alpha(t,v,w), \quad 0<t<T, \\ w'(t) &= \beta(t,v,w), \quad 0<t<T, \\ v(0) &= \phi_1, \\ w(0) &= \phi_2,\end{aligned}$$

where $\alpha(t,v,w),\beta(t,v,w)$ are nonlinear in v and w.

The above two equations and two initial conditions can be compressed in the formulation to

$$u'(t) = f(t,u), \quad 0<t<T, \tag{2.20}$$
$$u(0) = \psi, \tag{2.21}$$

where

$$u=\begin{bmatrix} v \\ w\end{bmatrix},\ f(t,u)=\begin{bmatrix}\alpha(t,v,w) \\ \beta(t,v,w)\end{bmatrix}$$

and

$$u'=\begin{bmatrix} v' \\ w'\end{bmatrix}.$$

Such u and f are called two-dimensional (column) vectors. If their elements are real numbers, we say that they belong to the real vector space $\mathbb{R}^2$.

In the general case, we have n-dimensional vectors such as

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad n > 1.$$

Thus, in general, we may assume that (2.20), (2.20) are for n-dimensional vector solutions $u \in \mathbb{R}^n,\ n > 1$. In the sense, (2.20) is considered as a system, rather than an equation.

In circumstances if the function f is linear in u, we may replace (2.20) by the following:

$$u'(t) = Au(t) + g, \tag{2.22}$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \in \mathbb{R}^{n\times n}$$

is an $n \times n$ real matrix and $u, g \in \mathbb{R}^n$. We note that A, g may be functions of t but not u. Further, $Au \in \mathbb{R}^n$ is a vector.

Vectors and matrices are not numbers. They are arrays of numbers. They are rich in many properties for applications. Herewith, let us recall some of the basics.

1. Vector norms: We prefer the so-called ℓ_p norms for a constant $p \geq 1$. To this end, let $u \in \mathbb{R}^n,\ n \geq 1$. We have

$$\|u\|_p = \left(|u_1|^p + |u_2|^p + \cdots + |u_n|^p\right)^{1/p}.$$

Note that absolute values are used. We have following special cases from the above:

$$\begin{aligned} \|u\|_1 &= |u_1| + |u_2| + \cdots + |u_n|, \\ \|u\|_2 &= \left(|u_1|^2 + |u_2|^2 + \cdots + |u_n|^2\right)^{1/2}, \\ \|u\|_\infty &= \max\{|u_i|\}_1^n. \end{aligned}$$

The ℓ_2 vector norm is often referred as the Euclidean norm.

2. Some useful properties of vector norms: For $u, v \in \mathbb{R}^n$, we may prove numerous interesting inequalities such as

$$\begin{aligned} \|u\|_\infty &\leq \|u\|_1 \leq n\|u\|_\infty, \\ \|u\|_\infty &\leq \|u\|_2 \leq \sqrt{n}\|u\|_\infty, \\ \|u\|_2 &\leq \|u\|_1 \leq \sqrt{n}\|u\|_2. \end{aligned}$$

Further,

$$\left|\|u\|_p - \|v\|_p\right| \leq \|u - v\|_p, \quad p \geq 1.$$

3. Matrix norms: Again, we may define ℓ_p norms for $n \times n,\ n \geq 1$, real matrices.
$$\|A\|_p = \sup_{\|u\|_p \neq 0} \frac{\|Au\|_p}{\|u\|_p}, \quad p \geq 1.$$
In particular, we have the following.
$$\begin{aligned} \|A\|_1 &= \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{i,j}|, \\ \|A\|_2 &= \sqrt{\rho(AA^{\intercal})}, \\ \|A\|_\infty &= \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{i,j}|, \end{aligned}$$
where $\rho(AA^{\intercal})$ is the largest eigenvalue of the matrix $AA^{\intercal}$, and $A^{\intercal}$ is the transpose of A.
Since $\rho(M)$ is also called the spectral radius of M, the ℓ_2 matrix norm is often referred to as the spectral norm. In particular, when $A^{\intercal} = A$, we have
$$\rho(A) \; = \; \|A\|_2.$$
We may further show that
$$\begin{aligned} \rho(A) &\leq \|A\|_p, \\ \|AB\|_p &\leq \|A\|_p \|B\|_p \end{aligned}$$
for any $A, B \in \mathbb{R}^{n \times n}$ and $p \geq 1$.

4. Relations between vector and matrix norms:
$$\|Au\|_p \; \leq \; \|A\|_p \|u\|_p, \quad p \geq 1,$$
for all applicable $A \in \mathbb{R}^{n \times n}$ and $u \in \mathbb{R}^n$.

5. Important properties of vectors: inner product (dot product): Let $u, v \in \mathbb{R}^n$. Then
$$< u, v > \; = \; \sum_{i=1}^{n} u_i v_i \; = \; u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \; = \; u^{\intercal} v \; = \; v^{\intercal} u.$$
Apparently, an inner product of two vectors is a number. Let $p > 1$ and $1/p + 1/q = 1$. We have the Hölder inequality:
$$\sum_{i=1}^{n} |u_i v_i| \leq \|u\|_p \times \|v\|_q.$$
Further,
$$| < u, v > | \leq \|u\|_p \|v\|_p, \quad p \geq 1.$$
The above is also called a Hölder inequality. The two inequalities are often referred to as Cauchy-Schwarz inequalities when $p = 2$. A Cauchy-Schwarz inequality when $u = v$ implies that
$$\|u\|_2 \; = \; \sqrt{< u, u >}.$$

$\Longleftrightarrow$

For materials in linear algebra, in particularly vectors and matrices, we would recommend a simple paragraph in Wikipedia:

https://en.wikipedia.org/wiki/Matrix_(mathematics)

For solutions of linear systems such as (2.22), you may

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