Notes for MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

Instructor: Prof. Tim Sheng

FUNDAMENTALS FOR FINANCE

C. NUMERICAL SOLUTION OF PDES.

C.07 STABILITY OF THE NUMERICAL SOLUTION.

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Example. Consider

$$\begin{aligned} y' &= \Lambda y, \quad t > 0, && (2.27)\\ y(0) &= \phi, && (2.28) \end{aligned}$$

where

$$\Lambda = \begin{bmatrix} -100 & 1 \\ 0 & -1/10 \end{bmatrix}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}.$$

It can be shown in linear algebra that

$$\begin{aligned} \Lambda &= \begin{bmatrix} -100 & 1 \\ 0 & -1/10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 999/10 \end{bmatrix} \begin{bmatrix} -100 & 0 \\ 0 & -1/10 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 999/10 \end{bmatrix}^{-1} \\ &= VDV^{-1}. \end{aligned}$$

In fact, the symbolic solution of (2.27), (2.28) is actually

$$y = \exp\{t\Lambda\}\phi, \quad t \geq 0,$$

which looks very neat!

But what is $\exp\{t\Lambda\} = e^{t\Lambda}$? Recall what we have discussed about "matrix functions", the matrix exponential function can be defined as

$$\begin{aligned} e^{t\Lambda} &= I + t\Lambda + \frac{1}{2!}(t\Lambda)^2 + \frac{1}{3!}(t\Lambda)^3 + \cdots + \frac{1}{k!}(t\Lambda)^k + \cdots \\ &= I + t\Lambda + \frac{t^2}{2!}\Lambda^2 + \frac{t^3}{3!}\Lambda^3 + \cdots + \frac{t^k}{k!}\Lambda^k + \cdots \\ &= I + tVDV^{-1} + \frac{t^2}{2!}(VDV^{-1})^2 + \frac{t^3}{3!}(VDV^{-1})^3 + \cdots + \frac{t^k}{k!}(VDV^{-1})^k + \cdots \\ &= I + tVDV^{-1} + \frac{t^2}{2!}VDV^{-1}VDV^{-1} + \frac{t^3}{3!}VDV^{-1}VDV^{-1}VDV^{-1} + \cdots \\ &\quad + \frac{t^k}{k!}VDV^{-1}VDV^{-1}\cdots VDV^{-1} + \cdots \\ &= I + tVDV^{-1} + \frac{t^2}{2!}VD^2V^{-1} + \frac{t^3}{3!}VD^3V^{-1} + \cdots + \frac{t^k}{k!}VD^kV^{-1} + \cdots \\ &= V\left(I + tD + \frac{t^2}{2!}D^2 + \frac{t^3}{3!}D^3 + \cdots + \frac{t^k}{k!}D^k + \cdots\right)V^{-1} \\ &= Ve^{tD}V^{-1} \in \mathbb{R}^{2\times 2}. \end{aligned}$$

In fact, we have already learned from previous investigations that there exist two-dimensional vectors $x_1,\ x_2$ depending only on ϕ_1, ϕ_2 such that the exact solution of (2.27), (2.28) can be written as

$$y(t) = e^{-100t}x_1 + e^{-t/10}x_2, \quad t \geq 0. \tag{2.29}$$

We may notice that:

1. both terms in (2.29) decay as $t \to +\infty$;
2. the decay of the first term in (2.29) is much faster than the second term as $t \to +\infty$.

But, on the other hand, we wish to check out if a numerical method can be employed for solving (2.27), (2.28) successfully. Of course, the numerical method to use must be proven to be convergent. It is definitely desirable that numerical methods can be used for solving more complicated option trading problems later on.

To this end, let us consider a forward Euler method for its simplicity and guaranteed convergence in computations. A straightforward uniform mesh with mesh step size $h,\ 0 < h \ll 1$, is used.

We thus obtain from (2.27), (2.28) that

$$y_1 = y_0 + h\Lambda y_0 = (I + h\Lambda)y_0.$$

We note here that $y_0, y_1 \in \mathbb{R}^2$ are vectors. Further,

$$y_2 = y_1 + h\Lambda y_1 = (I + h\Lambda)y_1 = (I + h\Lambda)^2 y_0.$$

and then,

$$y_3 = y_2 + h\Lambda y_2 = (I + h\Lambda)y_2 = (I + h\Lambda)^3 y_0.$$

In general we may expect that a numerical solution of (2.27), (2.28) generated by a forward Euler scheme looks like:

$$\begin{aligned} y_n &= y_{n-1} + h\Lambda y_{n-1} \ = \ (I + h\Lambda)y_{n-1} \ = \ (I + h\Lambda)^n y_0 \\ &= (I + hVDV^{-1})^n y_0 \ = \ V(I + hD)^n V^{-1} y_0 \\ &= V \begin{bmatrix} 1 - 100h & 0 \\ 0 & 1 - h/10 \end{bmatrix}^n V^{-1} y_0 \\ &= V \begin{bmatrix} (1 - 100h)^n & 0 \\ 0 & (1 - h/10)^n \end{bmatrix} V^{-1} y_0, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

because D is a diagonal matrix.

We may rearrange the above equation as

$$V^{-1} y_n = \begin{bmatrix} (1 - 100h)^n & 0 \\ 0 & (1 - h/10)^n \end{bmatrix} V^{-1} y_0, \quad n = 1, 2, 3, \ldots$$

Then we denote

$$w_k = V^{-1} y_k, \quad k = 0, 1, 2, \ldots$$

hence the above recursive relations can be reformulated to

$$w_n = \begin{bmatrix} (1 - 100h)^n & 0 \\ 0 & (1 - h/10)^n \end{bmatrix} w_0, \quad n = 1, 2, 3, \ldots \tag{2.30}$$

♡♡♡ Now, what can we say?

1. Let $h = 1/10 \ll 1$. From (2.30) we have

$$w_n = \begin{bmatrix} (1-10)^n & 0 \\ 0 & (1-1/100)^n \end{bmatrix} w_0, \quad n = 1, 2, 3, \ldots$$

We have

$$\begin{aligned} \|D^n\|_1 &= 9^n \to \infty, \\ \|D^n\|_2 &= 9^n \to \infty, \\ \|D^n\|_\infty &= 9^n \to \infty \end{aligned}$$

as $n \to \infty$. The above indicate that

$$\|w_n\|_p \to \infty, \quad p = 1, 2, \infty,$$

as $n \to \infty$. Equivalently,

$$\|y_n\|_p \to \infty, \quad p = 1, 2, \infty,$$

as $n \to \infty$, since V^{-1} is a constant matrix. This result clearly violates the basic property of the true solution that it converges to a zero vector as $t \to \infty$.

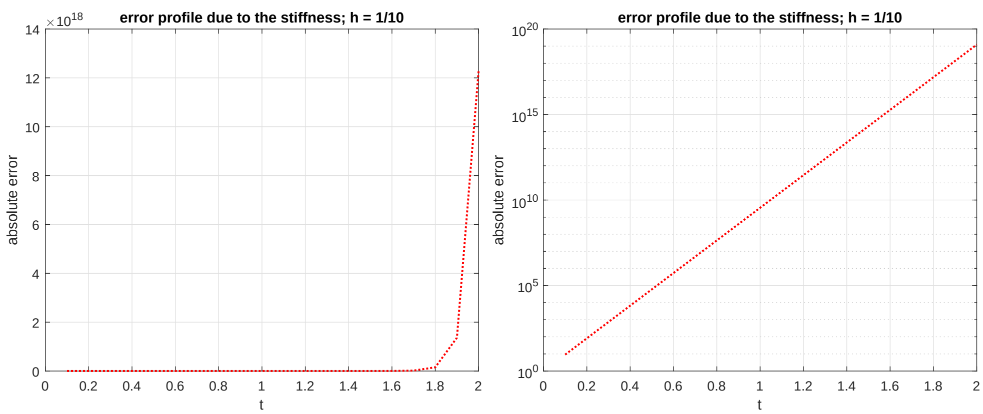

Fig 2.1: absolute error between the exact and numerical solutions (forward Euler method and Euclidean norm are used)

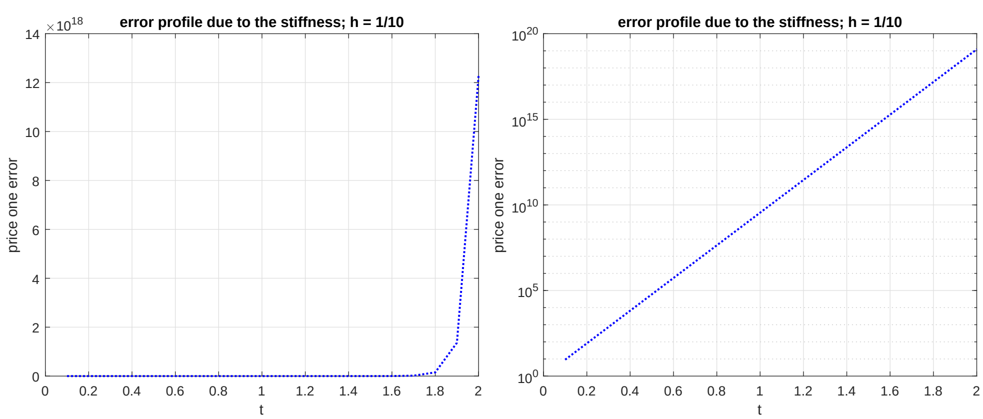

Fig 2.2: absolute error between the exact and numerical solutions (first price values; forward Euler method is used)

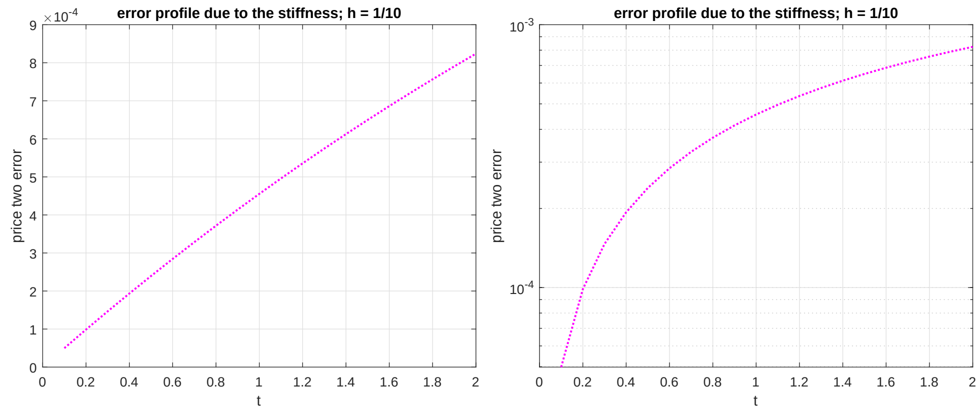

Fig 2.3: absolute error between the exact and numerical solutions (second price values; forward Euler method is used)

2. Let $h = 1/100 \ll 1$. From (2.30) we have

$$w_n = \begin{bmatrix} 0 & 0 \\ 0 & (1 - 1/1000)^n \end{bmatrix} w_0, \quad n = 1, 2, 3, \ldots$$

We have

$$\begin{aligned} \|D^n\|_1 &= \left(\frac{999}{1000}\right)^n \to 0, \\ \|D^n\|_2 &= \left(\frac{999}{1000}\right)^n \to 0, \\ \|D^n\|_\infty &= \left(\frac{999}{1000}\right)^n \to 0 \end{aligned}$$

as $n \to \infty$.

The above analysis indicates that the numerical solution preserves the basic feature of the true solution, that is, the solution should decay to a zero vector as $n \to 0$. The the following absolute error plots, we are again convinced that the numerical solution is reliable.

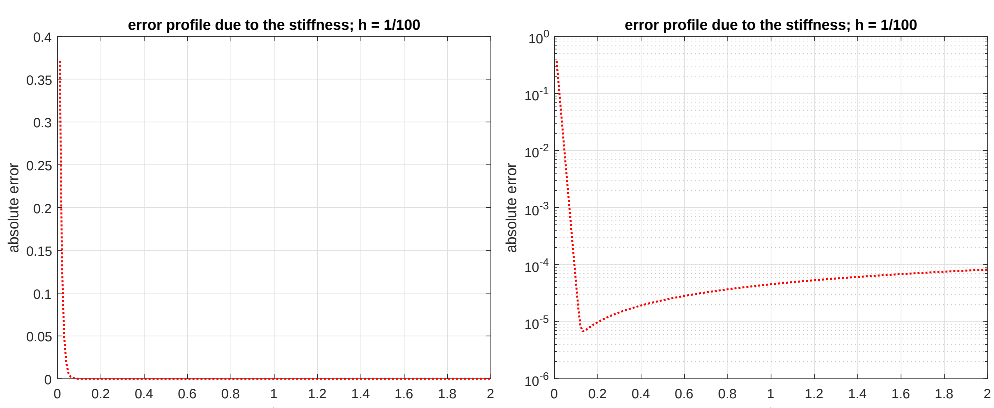

Fig 2.4: absolute error between the exact and numerical solutions (forward Euler method and Euclidean norm are used)

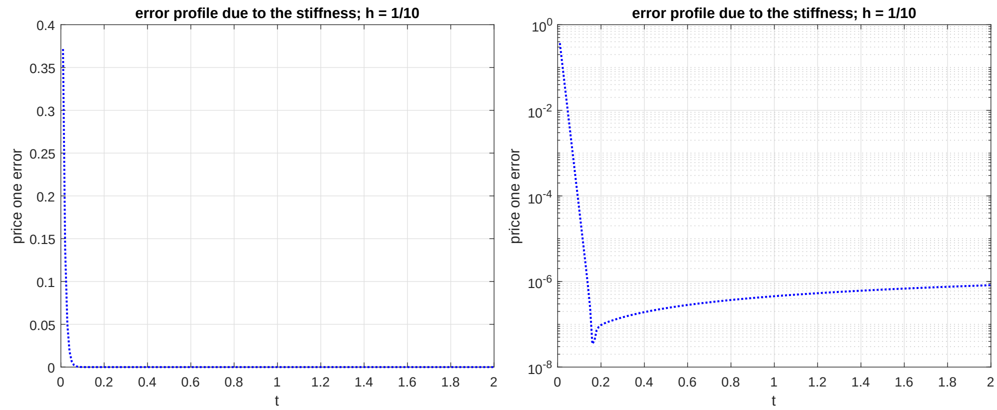

Fig 2.5: absolute error between the exact and numerical solutions (first price values; forward Euler method is used)

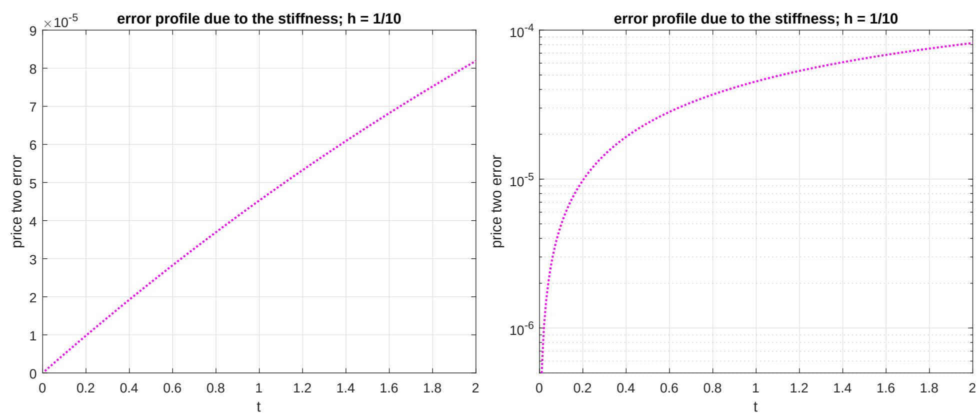

Fig 2.6: absolute error between the exact and numerical solutions (second price values; forward Euler method is used)

3. Can you choose different h values for more hands-on numerical experiments?

♡♡♡ Needless to say that our mathematical analysis and computational experiments reveal that h must be chosen to be sufficiently small in the computation when the differential equation given is stiff, that is, the coefficient matrix is ill-conditioned. Otherwise the numerical solution can be spurious or wrong.

♡♡♡ It is found that there will be no such issues if a backward Euler method is used for solving (2.27), (2.28):

$$(I - h\Lambda)y_n = y_{n-1}, \quad n = 1, 2, \ldots$$

♡♡♡ They indicate that some regulations must be introduced for numerical methods.

For the purpose, let us consider the following general linear financial system

$$u'(t) = Au(t) + g, \quad t > 0, \tag{2.22}$$

together with an initial condition

$$u(0) = u_0,$$

where

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{bmatrix} \in \mathbb{R}^{n\times n}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n-1} \\ u_n \end{bmatrix} \in \mathbb{R}^n, \quad n \gg 1.$$

Let us assume that a particular numerical method is applied for solving the above problem.

But, the question may be, can we just use any numerical method available, such as the forward Euler method?

A quick answer may be NO. In fact, we may have seen from the last problem in Homework V. We have seen that the numerical solution blows up in finite steps when a forward Euler scheme is utilized.

Such a solution blowing up, or instability, is apparently related to the stiffness of the system (2.22).

We use the following to describe the stability capability of a numerical method.

First, we define an extremely simple scalar linear "testing problem" equipped with a complex number $\lambda \in \mathbb{C}$, *that is,*

$$u'(t) = \lambda u(t), \quad t > 0, \tag{2.31}$$

$$u(0) = 1. \tag{2.32}$$

Second, we introduce the following two definitions.

$$\Longleftrightarrow$$

Definition 2.1 (Stability domain).
Suppose that a particular numerical method is used for solving the linear initial value problem (2.31), (2.32). We may obtain that

$$u_n = K(z)u_{n-1}, \quad n = 1, 2, \ldots,$$

where $K(z)$ is in general a nonlinear function of $z = h\lambda$ in which $0 < h \ll 1,\ \lambda \in \mathbb{C}$. We call the set

$$\mathcal{D} = \{z \in \mathbb{C} : \ |K(z)| < 1\} \subseteq \mathbb{C}$$

as the stability domain of the numerical method.

For example, let us consider the forward Euler method. It follows immediately that

$$\begin{aligned} u_n &= u_{n-1} + h\lambda u_{n-1} = (1 + h\lambda)u_{n-1} = (1 + z)u_{n-1}, \quad n = 1, 2, \ldots, \\ u_0 &= 1. \end{aligned}$$

Therefore,

$$\mathcal{D}_{\mathsf{F.Euler}} = \{z \in \mathbb{C} : \ |1 + z| < 1\} \subseteq \mathbb{C}.$$

Another example can be the backward Euler method. For which we have

$$\begin{aligned} u_n &= u_{n-1} + h\lambda u_n, \quad n = 1, 2, \ldots, \\ u_0 &= 1. \end{aligned}$$

Equivalently,

$$\begin{aligned} u_n - h\lambda u_n &= (1-h\lambda)u_n = (1-z)u_n = u_{n-1}, \quad n = 1,2,\ldots, \\ u_0 &= 1. \end{aligned}$$

Subsequently,

$$\begin{aligned} u_n &= \frac{1}{1-z}u_{n-1}, \quad n = 1,2,\ldots, \\ u_0 &= 1. \end{aligned}$$

So the stability domain of backward Euler method is

$$\mathcal{D}_{\mathsf{B.Euler}} = \{z \in \mathbb{C} : \ |1/(1-z)| < 1\} \subseteq \mathbb{C}.$$

We may further show that for a trapezoidal finite difference method, which is often called a Crank–Nicolson method, has the following stability domain:

$$\mathcal{D}_{\mathsf{C-N}} = \left\{ z \in \mathbb{C} : \ \left| \frac{1+z/2}{1-z/2} \right| < 1 \right\} \subseteq \mathbb{C}.$$

Can we graph the aforementioned three stability domains on a complex plane?

Definition 2.2 (A-Stability).
A numerical method is called A-stable if

$$\mathbb{C}^- \subseteq \mathcal{D},$$

where $\mathbb{C}^- = \{z \in \mathbb{C} : \ \mathrm{Re}(z) < 0\}$ is the strictly left half of the complex plane and $\mathcal{D}$ is the stability domain of the numerical method considered.

It can be shown that both backward Euler and Crank–Nicolson methods are A-stable, while the forward Euler method is not.

Theorem 2.2 (linear stability).
If a linear system is stiff, then an A-stable numerical method should be used to avoid nonphysical blow-ups or instabilities due to stochastic disturbances in computations.

C.08 Back to the Partial Differential Equations.

Recall our standard Black-Scholes initial-boundary value problem (BSIVP):

$$u_t(s,t) = d(s)u_{ss}(s,t) + c(s)u_s(s,t) - r(s)u(s,t) + \psi(\varrho), \tag{2.1}$$

$$u(s,0) = \phi(s), \tag{2.2}$$

$$u(S_{\min}, t) = \begin{cases} 0, & \text{(call);} \\ K\exp\{-r(S_{\min})t\}, & \text{(put),} \end{cases} \tag{2.3}$$

$$u(S_{\max}, t) = \begin{cases} S_{\max} - K\exp\{-r(S_{\max})t\}, & \text{(call);} \\ 0, & \text{(put),} \end{cases} \tag{2.4}$$

where u is the unknown, $S_{\min}, S_{\max}$ are given real values or $\pm\infty$, Functions c, d, r are real-valued with d is always assumed to be nonnegative. The source function $\psi(\varrho)$ is stochastic.

Further, the initial data $\phi(s)$ from markets are often *nonsmooth,* that is, there are undesirable jumps and/or discontinuities within domains of definitions.

The right-boundary condition (2.4) may sometime be replaced by

$$u_s(S_{\max}, t) = \begin{cases} 1, & \text{(call)}; \\ 0, & \text{(put)}. \end{cases} \tag{2.5}$$

Without loss of generality, we may consider a much simplified model problem for option prices:

$$\begin{aligned} u_t(s,t) &= d\, u_{ss}(s,t) + c\, u_s(s,t), \quad S_{\min} < s < S_{\max},\ t > 0, && (2.6)\\ u(s,0) &= \phi(s), \quad S_{\min} < s < S_{\max}, && (2.7)\\ u(S_{\min}, t) &= a, \quad t > 0, && (2.8)\\ u(S_{\max}, t) &= b, \quad t > 0, && (2.9) \end{aligned}$$

where u is the option price, ϕ is the initial price function given, $S_{\min}, S_{\max}$ are real values. Further, a, b, c, d are real constants and d is always assumed to be positive.

Further, the boundary condition (2.9) may be replaced by

$$u_s(S_{\max}, t) = c, \quad t > 0, \tag{2.10}$$

in some cases.

The option model (2.6)-(2.10) is, of course, a typical linear initial-boundary value problem from the mathematical sense. The derivative boundary condition (2.5) or (2.10) is often called a *Neumann boundary condition,* while boundary conditions without derivatives, such as (2.3), (2.4), (2.8) and (2.9), are called *Dirichlet boundary conditions.*

References

[1] K. in 't Hout, *Numerical Partial Differential Equations in Finance Explained,* Springer, Antwerp, Belgium, 2017 (pages 15-35).

[2] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge and New York, 2004 (pages 139-146, 251-269, 349-380).