Solution references for homework 4

MATH 4665/4875/7140/7300, Q. SHENG, HKBU, FALL 2019

Problem 1: Our extended Averaging Formula based on (2.17) should be

$$u(t+h) = u(t) + \frac{h}{2}[f(t, u(t)) + f(t+h, u(t+h))] + \mathcal{O}(h^3), \quad t, t+h \in [0, T].$$

Remove the truncation error $\mathcal{O}(h^3)$. Let $u_i,\ u_{i+1}$ be numerical approximations of $u(t),\ u(t+h)$, we obtain readily that

$$u_{i+1} = u_i + \frac{h}{2}[f(t_i, u_i) + f(t_{i+1}, u_{i+1})), \quad i = 0, 1, \ldots, N.$$

Over a mesh

$$\mathcal{D}_N = \{t_0, t_1, \ldots, t_i, t_{i+1}, \ldots, t_N, t_{N+1}\}.$$

The above is identical to the set of equations given in the problem, where ϕ is the initial value.

Problem 2: the truncation error is given as $\mathcal{O}(h^3)$. To see the actual order of the truncation error, we may rewrite our formula as

$$\frac{u(t+h) - u(t)}{h} = \frac{1}{2}[f(t, u(t)) + f(t+h, u(t+h))] + \mathcal{O}(h^2), \quad t, t+h \in [0, T]. \quad (1)$$

$\Longrightarrow$ But, how can the above be true? Let's go ahead to play with it. To this end, we replace (1) by

$$\frac{u(t+h) - u(t)}{h} = \frac{1}{2}[f(t, u(t)) + f(t+h, u(t+h))] + \mathcal{E}, \quad t, t+h \in [0, T], \quad (2)$$

where $\mathcal{E}$ is yet to be determined.

1. We first observe that the left-hand-side of (1) is a forward finite difference approximation of the derivative $u'(t)$. In other words, we have

   $$\frac{u(t+h) - u(t)}{h} = u'(t) + \mathcal{O}(h), \quad t, t+h \in [0, T].$$

   On the other hand, the averaging value on the right-hand-side of (1) can be replaced by the following via a Taylor expansion:

   $$\begin{aligned} \frac{1}{2}[f(t, u(t)) + f(t+h, u(t+h))] &= \frac{1}{2}[f(t, u(t)) + f(t, u(t)) + \mathcal{O}(h)] \\ &= f(t, u(t)) + \mathcal{O}(h), \quad t, t+h \in [0, T]. \end{aligned}$$

   Combining the above two estimates, unfortunately, we can only say that

   $$\mathcal{E} = \mathcal{O}(h).$$

2. Well, we also see that the left-hand-side of (1) is a backward finite difference approximation of the derivative $u'(t+h)$ in the sense that

   $$\frac{u(t+h) - u(t)}{h} = u'(t+h) + \mathcal{O}(h), \quad t, t+h \in [0, T].$$

Simultaneously,

$$\begin{aligned}\frac{1}{2}[f(t,u(t))+f(t+h,u(t+h))] &= \frac{1}{2}[f(t+h,u(t+h))+f(t+h,u(t+h))+\mathcal{O}(h)]\\ &= f(t+h,u(t+h))+\mathcal{O}(h), \quad t,t+h\in[0,T].\end{aligned}$$

The two estimates suggest that

$$\mathcal{E}=\mathcal{O}(h).$$

3. Finally, let us consider the left-hand-side of (1) to be a central finite difference approximation of the derivative $u'(t+h/2)$. Therefore,

$$\frac{u(t+h)-u(t)}{h}=u'(t+h/2)+\mathcal{O}(h^2), \quad t,t+h\in[0,T].$$

Without loss of generality, we may denote $f(t,u(t))=g(t)$. It is readily to see that

$$\begin{aligned}\frac{1}{2}[g(t)+g(t+h)] &= g(t+h/2)+\mathcal{O}(h^2)\\ &= f(t+h/2,u(t+h/2))+\mathcal{O}(h^2), \quad t,t+h\in[0,T].\end{aligned}$$

Hence,

$$\mathcal{E}=\mathcal{O}(h^2)+\mathcal{O}(h^2)=\mathcal{O}(h^2).$$

In other words, the averaging formula provides a second order approximation to the differential equation

$$u'(t+h/2)=f(t+h/2,u(t+h/2)), \quad t,t+h\in[0,T].$$

Isn't it interesting?

Problem 3: Since $c=\max\{c_1,c_2\}$, we may replace both c_1 and c_2 in the inequalities by c. (*think again, why?*)

Now, substitute the second inequality into the first inequality, we acquire that

$$\begin{aligned}\epsilon_{i+1} &\le (1+h\lambda)\epsilon_i+ch^2\\ &\le (1+h\lambda)\frac{c}{\lambda}h\left[(1+h\lambda)^i-1\right]+ch^2\\ &= \frac{c}{\lambda}h\left[(1+h\lambda)^{i+1}-1-h\lambda\right]+\frac{ch^2\lambda}{\lambda}\\ &= \frac{c}{\lambda}h\left[(1+h\lambda)^{i+1}-1-h\lambda+h\lambda\right]\\ &= \frac{c}{\lambda}h\left[(1+h\lambda)^{i+1}-1\right], \quad i=0,1,2,\dots,M.\end{aligned}$$

Needless to say, the above ensures our result.