Notes for MATH 4665/4875/7140/7300, Fall 2019, HKBU

Location for classes: LMC 514, HKBU

Time for classes: Wednesdays 15:30–16:20; Thursdays 13:30–15:20

Instructor: Prof. Tim Sheng

FUNDAMENTALS FOR FINANCE

C. Numerical Solution of PDEs.

C.08 Back to the Partial Differential Equations.

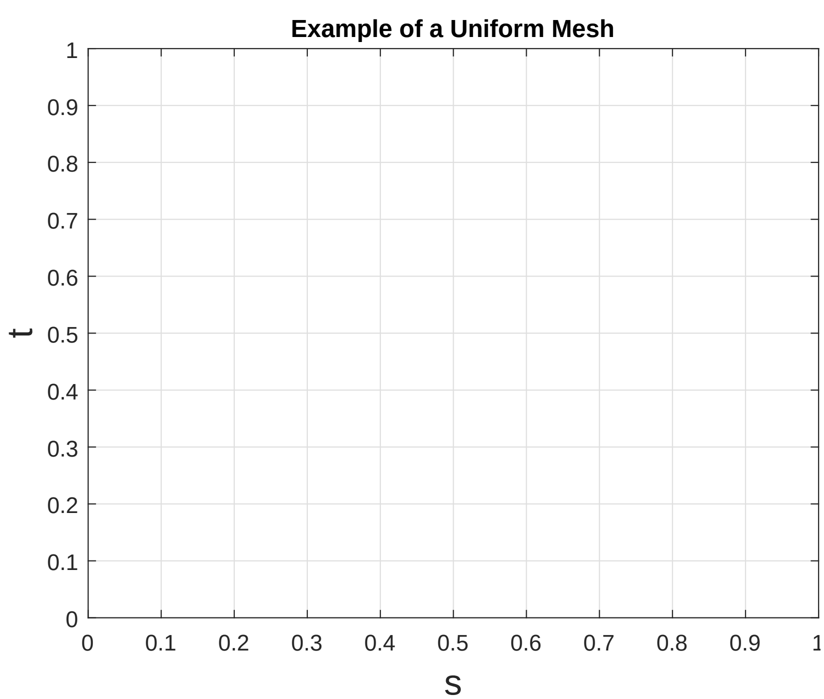
Example.

We consider the following much simplified option price model:

$$u_t(s,t) = u_{ss}(s,t), \quad 0 < s < 1,\ 0 < t < 1, \tag{2.6}$$
$$u(s,0) = \phi(s), \quad 0 \le s \le 1, \tag{2.7}$$
$$u(0,t) = 0, \quad 0 < t < 1, \tag{2.8}$$
$$u(1,t) = 0, \quad 0 < t < 1, \tag{2.9}$$

where u is the option price function of time t and asset price s; ϕ is the given initial price.

Let us adopt a 2-dimensional uniform space-time mesh with $h = \tau = 1/10$.



Now, let us try a semi-discretization procedure via a standard central difference approximation of the second derivative, since it is more accurate:

$$\frac{u(s_{i+1},t) - 2u(s_i,t) + u(s_{i-1},t)}{h^2} = u_{ss}(s_i,t) + \mathcal{O}(h^2). \tag{2.10}$$

To this end, for internal mesh points $s_1, s_2, \ldots, s_9$ in the space, we have

$$\begin{aligned}
u_{ss}(s_1,t) &\approx \frac{u(s_2,t) - 2u(s_1,t) + u(s_0,t)}{h^2},\\
u_{ss}(s_2,t) &\approx \frac{u(s_3,t) - 2u(s_2,t) + u(s_1,t)}{h^2},\\
u_{ss}(s_3,t) &\approx \frac{u(s_4,t) - 2u(s_3,t) + u(s_2,t)}{h^2},\\
\cdots &\quad \cdots\\
\cdots &\quad \cdots\\
u_{ss}(s_8,t) &\approx \frac{u(s_9,t) - 2u(s_8,t) + u(s_7,t)}{h^2},\\
u_{ss}(s_9,t) &\approx \frac{u(s_{10},t) - 2u(s_9,t) + u(s_8,t)}{h^2},
\end{aligned}$$

for which $u(s_0,t) = u(s_{10},t) = 0$.

Apparently, unknown option prices included in approximations are

$$\begin{gathered}
u(s_1,t),\\
u(s_2,t),\\
u(s_3,t),\\
\cdots\\
\cdots\\
u(s_8,t),\\
u(s_9,t).
\end{gathered}$$

We may express them in a vector form

$$u = \begin{bmatrix} u(s_1,t)\\ u(s_2,t)\\ u(s_3,t)\\ \cdots\\ \cdots\\ u(s_8,t)\\ u(s_9,t) \end{bmatrix} = \begin{bmatrix} u_1(t)\\ u_2(t)\\ u_3(t)\\ \cdots\\ \cdots\\ u_8(t)\\ u_9(t) \end{bmatrix} \in \mathbb{R}^9.$$

It follows that

$$u_{ss} = \begin{bmatrix} u_{ss}(s_1,t)\\ u_{ss}(s_2,t)\\ u_{ss}(s_3,t)\\ \cdots\\ \cdots\\ u_{ss}(s_8,t)\\ u_{ss}(s_9,t) \end{bmatrix} = \begin{bmatrix} (u_{ss})_1(t)\\ (u_{ss})_2(t)\\ (u_{ss})_3(t)\\ \cdots\\ \cdots\\ (u_{ss})_8(t)\\ (u_{ss})_9(t) \end{bmatrix} \in \mathbb{R}^9. \tag{2.11}$$

Substitute the central difference approximation into (2.11) we obtain that

$$
\begin{aligned}
u_{ss} &= \begin{bmatrix} \dfrac{u(s_2,t)-2u(s_1,t)}{h^2} \\ \dfrac{u(s_3,t)-2u(s_2,t)+u(s_1,t)}{h^2} \\ \dfrac{u(s_4,t)-2u(s_3,t)+u(s_2,t)}{h^2} \\ \cdots \\ \cdots \\ \dfrac{u(s_9,t)-2u(s_8,t)+u(s_7,t)}{h^2} \\ \dfrac{-2u(s_9,t)+u(s_8,t)}{h^2} \end{bmatrix} \\
&= \frac{1}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \cdots \\ \cdots \\ u_8(t) \\ u_9(t) \end{bmatrix} \\
&= Au, \qquad (2.12)
\end{aligned}
$$

where

$$A = \frac{1}{h^2}\mathrm{tridiag}\{1, \ -2, \ 1\} \in \mathbb{R}^{9\times 9}$$

is a tridiagonal real matrix. A is also a TST matrix (you can verify this)!

Recall the original partial differential equation (2.6), we find that we have the following matrix form system of ordinary differential equations:

$$u'(t) = Au, \qquad (2.13)$$

together with the initial condition in a vector form:

$$u(0) = \phi, \qquad (2.14)$$

where

$$u(0) = \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_3(0) \\ \cdots \\ \cdots \\ u_8(0) \\ u_9(0) \end{bmatrix}, \ \phi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \cdots \\ \cdots \\ \phi_8 \\ \phi_9 \end{bmatrix} \in \mathbb{R}^9.$$

Most importantly, the linear initial-value problem (2.13), (2.14) can now be solved by using our existing numerical methods! The ratio of mesh steps: $\mu = \tau/h^2$, is often referred as the Courant number, or CFL number, after the pioneering work by Richard Courant, Kurt Friedrichs, and Hans Lewy in 1920s.

Of course, recall different approximations of the second derivative, we may derive different systems in addition to (2.13). May you try them?

Frequently used finite difference formulae include the following:

1. Central difference for second order partial derivatives in space.

$$\frac{u(s_{i+1},t) - 2u(s_i,t) + u(s_{i-1},t)}{h^2} = u_{ss}(s_i,t) + \mathcal{O}(h^2). \tag{2.15}$$

2. Forward difference for second order partial derivatives in space.

$$\frac{u(s_{i+2},t) - 2u(s_{i+1},t) + u(s_i,t)}{h^2} = u_{ss}(s_i,t) + \mathcal{O}(h). \tag{2.25}$$

3. Backward difference for second order partial derivatives in space.

$$\frac{u(s_i,t) - 2u(s_{i-1},t) + u(s_{i-2},t)}{h^2} = u_{ss}(s_i,t) + \mathcal{O}(h). \tag{2.26}$$

References

[1] K. in 't Hout, *Numerical Partial Differential Equations in Finance Explained,* Springer, Antwerp, Belgium, 2017 (pages 15-35).

[2] A. Iserles, *A First Course in the Numerical Analysis of Differential Equations,* Cambridge University Press, Cambridge and New York, 2004 (pages 139-146, 251-269, 349-380).