EED4106 Antennas and Propagation Experiment 4

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1- Radiation Field:

According to the Figure 1, given in Experiment 4 laboratory paper, there is an image and a source antenna due to the infinite perfect electric conductor.

Source antena for field characteristic is called as direct component. It can be seen in below,

$$E_\psi^d = j\eta \frac{kI_o l e^{-jkr_1}}{4\pi r_1} \sin\psi_1 \qquad (1)$$

Image antena for field characteristic is called as reflected component. It can also be seen in below,

$$E_\psi^r = jR_v \eta \frac{kI_o l e^{-jkr_2}}{4\pi r_2} \sin\psi_2$$

Since electric dipole is placed horizontally,

$$E_\psi^r = -j\eta \frac{kI_o l e^{-jkr_2}}{4\pi r_2} \sin\psi \qquad (2)$$

ψ angle is measured from the y-axis to the observation point. Consider this

$$\cos\psi = \hat{a}_y \cdot \hat{a}_r = \hat{a}_y(\hat{a}_x \sin\theta\cos\phi + \hat{a}_y \sin\theta\sin\phi + \hat{a}_z \cos\theta) = \sin\theta\sin\phi$$

Using trigonometric identity,

$$\sin\psi = \sqrt{1-\cos^2\psi} = \sqrt{1-\sin^2\theta\sin^2\phi} \qquad (3)$$

The approximation for the far-field observations are the ones for phase term

$$r_1 \simeq r - h\cos\theta$$
$$r_2 \simeq r + h\cos\theta \qquad (4a)$$

and the ones for amplitude term

$$r \simeq r_1 \simeq r_2 \qquad (4b)$$

Add the equation (1) and (2) up, (considering equation (3a) and (3b))

$$E_\psi = j\eta \frac{kI_0 L}{4\pi r} \sin\psi \left(e^{-jkr_1} - e^{-jkr_2} \right)$$

$$= j\eta \frac{kI_0 L}{4\pi r} \sqrt{1-\sin^2\theta \sin^2\phi} \left(e^{-jk(r-h\cos\theta)} + e^{-jk(r+h\cos\theta)} \right)$$

$$= j\eta \frac{kI_0 L}{4\pi r} e^{-jkr} \sqrt{1-\sin^2\theta \sin^2\phi} \left(e^{+jkh\cos\theta} - e^{-jkh\cos\theta} \right)$$

$$E_\psi = j\eta \frac{kI_0 L}{4\pi r} e^{-jkr} \underbrace{\underbrace{\sqrt{1-\sin^2\theta\sin^2\phi}}_{\sin\psi}}_{\text{element factor}} \underbrace{2j\sin(kh\cos\theta)}_{\text{array factor}} \qquad \because \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

2- Radiated Power

In order to obtain radiated power, time average power density need to be find.

$$W_{av} = \frac{1}{2\eta} |E_\psi|^2 = \frac{(k_0 L)^2 I_0^2}{32\pi^2 r^2} \eta \left(1-\sin^2\theta\sin^2\phi\right) . 4\sin^2(kh\cos\theta)$$

$$P_{rad} = \int_0^{2\pi}\int_0^{\pi/2} W_{av}\, r^2 \sin^2\theta \, d\theta \, d\phi$$

$$= \frac{(k_0 L)^2 I_0^2}{32\pi^2} \eta \int_0^{2\pi}\int_0^{\pi/2} \left(1-\sin^2\theta\sin^2\phi\right) 4\sin^2(kh\cos\theta) = \frac{(k_0 L)^2 I_0^2}{8\pi^2} \eta . M$$

In order to evaluate the integral above, apply change of variable method,

$$\cos\theta = u \Rightarrow -\sin\theta d\theta = du \quad \text{and} \quad \sin^2\theta \triangleq 1-u^2$$

The boundaries of "θ" integral must change due to change of variable.

lower boundary $\theta = 0^\circ \Rightarrow u = 1$

upper boundary $\theta = 90^\circ \Rightarrow u = 0$

New form of integral,

$$M = \int_0^{2\pi}\int_0^{1} \left(1-(1-u^2)\sin^2\phi\right)\sin^2(kh\cos\theta)\,du\,d\phi$$

$$= \underbrace{\int_0^{2\pi}\int_0^{1}\left[\sin^2(khu)\,du\,d\phi\right]}_{A_1} - \underbrace{\int_0^{2\pi}\int_0^{1}(1-u^2)\sin^2\phi\,\sin^2(kh\cos\theta)\,du\,d\phi}_{-A_2} = A_1 + (-A_2)$$

$$A_1 = \int_0^{2\pi} d\phi \int_0^{1} \sin^2(khu)\,du = 2\pi \int_0^{1} \frac{1-\cos(2khu)}{2}\,du \qquad \because \sin^2 x \triangleq \frac{1-\cos 2x}{2}$$

$$= \pi\left[\left[u\right]_0^1 - \left[\frac{\sin(2khu)}{2kh}\right]_0^1\right] = \pi.\left(1-\frac{\sin 2kh}{2kh}\right)$$

$$-A_2 = -\int_0^{2\pi}\int_0^{1}(1-u^2)\sin^2(khu)\sin^2\phi\,du\,d\phi = -\int_0^{2\pi}\sin^2\phi\,d\phi\int_0^{1}(1-u^2)\sin^2(khu)\,du$$

$$= -\pi.\left[\int_0^1 \sin^2(khu)\,du - \int_0^1 u^2\sin^2(khu)\,du\right]$$

$$= -\pi\left[\frac{1}{2} - \frac{\sin(2kh)}{4kh} - \underbrace{\int_0^1 u^2\sin^2(khu)\,du}_{A_{21}}\right]$$

In order to find A_2, integration by parts method will be used for A_{21}

$$\int t\,dv = tv - \int v\,dt$$

$$A_{21} = \int_0^1 \underbrace{u^2}_{t}\,\underbrace{\sin^2(khu)\,du}_{dv}$$

$$\Rightarrow \quad t = u^2 \qquad v = \frac{u}{2} - \frac{\sin(2khu)}{4kh}$$

$$dt = 2u\,du \qquad dv = \sin^2(khu)\,du$$

Applying integration by parts for the expression, A_{21},

$$A_{21} = \int_0^1 \underbrace{u^2}_{t} \underbrace{\sin^2(khu)du}_{dv} = t^2\left(\frac{1}{2} - \frac{\sin(2kht)}{4kh}\right)\Big|_0^1 - \int_0^1 \left[\frac{t}{2} - \frac{\sin(2kht)}{4kh}\right] 2t\,dt$$

$$= \frac{1}{2} - \frac{\sin(2kh)}{4kh} - \int_0^1 \left[t^2 - 2t\,\frac{\sin(2kht)}{4kh}\right] dt$$

$$= \frac{1}{2} - \frac{\sin(2kh)}{4kh} - \frac{1}{3} + \underbrace{\int_0^1 2t\,\frac{\sin(2kht)}{4kh}\,dt}_{A_{211}}$$

Apply integration by parts for the expression A_{211}

$$A_{211} = \int_0^1 \underbrace{2t}_{\alpha}\, \underbrace{\frac{\sin(2kht)}{4kh}}_{V} \Rightarrow \quad \alpha = 2t \quad V = \frac{-\cos(2kht)}{8(kh)^2}$$

$$d\alpha = 2dt \qquad dv = \frac{\sin(2kht)}{4kh}dt$$

$$= \frac{-2t\cos(2kht)}{8(kh)^2}\Big|_0^1 - \int_0^1 \frac{-\cos(2kht)}{8(kh)^2}\,2dt$$

$$= -\frac{\cos(2kh)}{(2kh)^2} + \left[\frac{\sin(2kht)}{(2kh)^3}\right]_0^1 = -\frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}$$

$$A_{21} = \frac{1}{2} - \frac{\sin(2kh)}{4kh} - \frac{1}{3} + \left(-\frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right)$$

$$-A_2 = -\pi\left[\cancel{\frac{1}{2}} - \cancel{\frac{\sin(2kh)}{4kh}} - \left(\cancel{\frac{1}{2}} - \cancel{\frac{\sin(2kh)}{4kh}} - \frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right)\right]$$

$$-A_2 = \pi\left(-\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right)$$

$$M = A_1 + (-A_2) = \pi\left(1 - \frac{\sin(2kh)}{2kh}\right) + \pi\left(-\frac{1}{3} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3}\right)$$

$$M = \pi \left(\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right)$$

Finally radiated power can be calculated such that,

$$P_{rad} = \frac{k_0^2 L^2 I_0^2}{8\pi^2} \eta . \pi \left(\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right)$$

$$= \frac{(2\pi)^2}{\lambda^2} \frac{L^2 I_0^2}{8\pi^2} \eta \pi (\ldots)$$

$$= \eta \frac{\pi}{2} \left(\frac{L I_0}{\lambda} \right)^2 \left(\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right)$$

3_ Radiation Resistance

Radiation resistance is derived from radiated power so

$$R_{rad} = \frac{2 P_{rad}}{|I_0|^2}$$

$$R_{rad} = \eta \pi \left(\frac{L}{\lambda} \right)^2 \left(\frac{2}{3} - \frac{\sin(2kh)}{2kh} - \frac{\cos(2kh)}{(2kh)^2} + \frac{\sin(2kh)}{(2kh)^3} \right)$$

Using the expansions of the sine & cosine functions, when kh approaches to zero, radiation resistance becomes

$$R_{rad} = \eta \pi \left(\frac{L}{\lambda} \right)^2 \left[\frac{2}{3} - \frac{2}{3} + \frac{8}{15} \left(\frac{2kh}{\lambda} \right)^2 \right] = \eta \frac{32\pi^3}{15} \left(\frac{L}{\lambda} \right)^2 \left(\frac{h}{\lambda} \right)^2$$

4_ Radiation Intensity & Directivity

The radiation intensity is given by

$$U \simeq \frac{r^2}{2\eta} |E_\varphi|^2 = \frac{\eta}{2} \left| \frac{I_0 L}{\lambda} \right|^2 (1 - \sin^2\theta \sin^2\phi) \sin^2(kh\cos\theta)$$

The maximum value of radiation intensity depends on the value of kh (whether $kh \leqslant \frac{\pi}{2}$, $h \leqslant \frac{\lambda}{4}$; $kh > \frac{\pi}{2}$, $h > \frac{\lambda}{4}$). It can be shown that the maximum of radiation intensity, considering the conditions mentioned above,

$$U_{max} = \begin{cases} \frac{\eta}{2}\left|\frac{I_0 L}{\lambda}\right|^2 \sin^2(kh) & kh \leq \frac{\pi}{2} \quad (h \leq \lambda/4) \\ & (\theta = 0^\circ) \\ \frac{\eta}{2}\left|\frac{I_0 L}{\lambda}\right|^2 & kh > \frac{\pi}{2} \quad (h > \lambda/4) \\ & [\phi = 0^\circ \text{ and } \sin(kh\cos\theta_{max}) = 1 \\ & \text{or } \theta_{max} = \cos^{-1}\left(\frac{\pi}{2kh}\right) \end{cases}$$

Directivity formula can be found from the equation,

$$D_0 = \frac{4\pi U_{max}}{P_{rad}}$$

where

$$P_{rad} = \frac{1}{2}|I_0|^2 R_{rad} \rightarrow$$

The conditions which are used in the calculation of U_{max} are still valid for directivity.

Consider the case where $kh \leq \frac{\pi}{2}$ and $h \leq \lambda/4$

$$D_{0_1} = 4\pi \frac{\cancel{\eta/2}\frac{\cancel{I_0^2} L^2}{\lambda^2}\sin^2 kh}{\cancel{1/2}\cancel{|I_0|^2} R_{rad}} = \frac{4\cancel{\pi}\cancel{\eta}\sin^2(kh)\frac{\cancel{L^2}}{\cancel{\lambda^2}}}{\cancel{\eta}\cancel{\pi}\cancel{\left(\frac{L}{\lambda}\right)^2} R(kh)} = \frac{4\sin^2(kh)}{R(kh)}$$

and $kh > \frac{\pi}{2}$ $(h > \lambda/4)$ case

$$D_{0_2} = \frac{4\pi\ \cancel{\eta/2}\frac{\cancel{I_0^2} L^2}{\lambda^2}}{\cancel{1/2}\cancel{|I_0|^2} R_{rad}} = \frac{4\pi\eta\frac{L^2}{\lambda^2}}{\eta\pi\left(\frac{L}{\lambda}\right)^2 R(kh)} = \frac{4}{R(kh)}$$

As a result

$$D_0 = \begin{cases} \frac{4\sin^2(kh)}{R(kh)} & kh \leq \frac{\pi}{2} \quad (h \leq \lambda/4) \\ \frac{4}{R(kh)} & kh > \frac{\pi}{2} \quad (h > \lambda/4) \end{cases}$$