EED 4106 Antenna and Propagation Laboratory

Preliminary Report - Lab 3

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Before deriving the equation of loop antennas, first consider the Bessel function. Bessel's equation often appears if a problem shows cylindrical symmetry (2). The Bessel equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2) y = 0 \qquad (1)$$

Considering the Frobenius Theory, its solution is in the form:

$$y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \qquad (2)$$

By substituting the Eq(2) into Eq(1) and some mathematical manipulations particular solution of Bessel Function, or the Bessel function of the first kind of order n $J_n(x)$, can be found as

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n+m)!} \qquad (3)$$

The general solution of the Bessel function, or the Bessel function of second kind of order

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \qquad (4)$$

There are many useful Bessel function integrals, the one will be used in this context is

$$J_n(x) = \frac{j^{-n}}{\pi} \int_0^{\pi} \cos(n\phi)\, e^{jx\cos\phi}\, d\phi \qquad (5)$$

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Radiated Fields

In order to calculate radiated field, first potential function must be calculated by using current spatial distribution $I_e(x',y',z')$ which is

$$I_e(x',y',z') = \hat{a}_x I_x(x',y',z') + \hat{a}_y I_y(x',y',z') + \hat{a}_z I_z(x',y',z') \quad (6)$$

By using cartesian to spherical coordinate transformation matrix,

$$I_e = \hat{a}_r I_\phi \sin\theta \sin(\phi-\phi') + \hat{a}_\theta I_\phi \cos\theta \sin(\phi-\phi') + \hat{a}_\phi I_\phi \cos(\phi-\phi') \quad (7)$$

where (r',θ',ϕ') is source point, (r,θ,ϕ) is observation point.

The distance R, from the source to the observation point

$$R = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \quad (8)$$

where (x',y',z') is source point in cartesian coordinates and (x,y,z) is the observation point in cartesian coordinates.

$$x = r.\sin\theta\cos\phi \quad (9a)$$

$$y = r\sin\theta.\sin\phi \quad (9.b)$$

$$z = r\cos\theta \quad (9.c)$$

$$x^2+y^2+z^2 = r \quad (9.d)$$

$$x' = a\cos\phi' \quad (9.e)$$

$$y' = a\sin\phi' \quad (9.f)$$

$$z' = 0 \quad (9.g)$$

$$x'^2+y'^2+z'^2 = a^2 \quad (9.h)$$

It is possible to redefine R vector in spherical coordinates such that

$$R = \sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')} \quad (10)$$

and differential length can also be redefined as

$$dl' = a\,d\phi' \quad (11)$$

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Potential function in cartesian form

$$A(x,y,z) = \frac{\mu}{4\pi} \int_C I_e(x',y',z') \frac{e^{-jkR}}{R} \, dl' \qquad (12)$$

In order to transform the potential function from cartesian to spherical coordinates, use Equation (9), Equation (10) and Equation (11).

$$A_\phi = \frac{a\mu}{4\pi} \int_0^{2\pi} I_\phi \cos(\phi-\phi') \frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos(\phi-\phi')}} \, d\phi' \qquad (13)$$

Current distrubution I_ϕ is constant, radiated field does not depend on observation angle ϕ. For simplicity ϕ is chosen as "0",

$$A_\phi = \frac{a\mu I_0}{4\pi} \int_0^{2\pi} \cos\phi' \frac{e^{-jk\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}}}{\sqrt{r^2+a^2-2ar\sin\theta\cos\phi'}} \, d\phi' \qquad (14)$$

The integral in Equation (14) is quite complex. If the far field characteristic of antenna is under investigation, then small radius approximation is not needed.

For the far field characteristics Eq. (10) can be approximated such that

$$R = \sqrt{r^2+a^2-2ar\sin\theta.\cos\phi'} \simeq \sqrt{r^2-2ar\sin\theta\cos\phi'} \quad \text{for } r \gg a \qquad (15)$$

By using binomial expansion, reduction can be made, so

$$R \simeq r\sqrt{1-\frac{2a}{r}\sin\theta.\cos\phi'} = r - a\sin\theta\cos\phi' = r - a\cos\psi_0$$

$$R \simeq r \qquad \text{for phase \& amplitude terms}$$

since

$$\cos\psi_0 = \hat{a}_\rho' \cdot \hat{a}_r \Big|_{\phi=0} = (\hat{a}_x\cos\phi' + \hat{a}_y\sin\phi') \cdot (\hat{a}_x\sin\theta\underbrace{\cos\phi}_{1} + \hat{a}_y\sin\theta\underbrace{\sin\phi}_{0} + \hat{a}_z\cos(\theta))$$

$$= \sin\theta\cos\phi'$$

As a result, the integral given in Equation (14) can be simplified to,

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$$A_\phi = \frac{a\mu I_0}{4\pi r}\int_0^{2\pi} \cos\phi\, e^{-jk(r - a\sin\theta\cos\phi')}\, d\phi'$$

$$A_\phi = \frac{a\mu I_0 e^{-jkr}}{4\pi r}\int_0^{2\pi} \cos\phi\, e^{+jka\sin\theta\cos\phi'}\, d\phi' \qquad (16)$$

Separate the integral into two parts,

$$A_\phi \simeq \frac{a\mu I_0 e^{-jkr}}{4\pi r}\left[\int_0^{\pi} \cos\phi' e^{+jka\sin\theta\cos\phi'}\, d\phi' + \int_{\pi}^{2\pi} \cos\phi' e^{+jka\sin\theta\cos\phi'}\, d\phi'\right]$$

In the second integral change the variable as

$$\phi' = \phi'' + \pi$$

then,

$$A_\phi \simeq \frac{a\mu I_0 e^{-jkr}}{4\pi r}\left[\int_0^{\pi} \cos\phi' e^{+jka\sin\theta\cos\phi'}\, d\phi' - \int_0^{\pi} \cos\phi'' e^{-jka\sin\theta\cos\phi''}\, d\phi''\right] \qquad (17)$$

Now the integrals are similar to one, obtained in Equation (5),

$$A_\phi \simeq \frac{a\mu I_0 e^{-jkr}}{4\pi r}\left(\pi j J_1(ka\sin\theta) - \pi j J_1(-ka\sin\theta)\right) \qquad (18)$$

The useful property of first kind of Bessel function is

$$J_n(-z) = (-1)^n (J_n(z))$$

$$A_\phi \simeq \frac{a\mu I_0 e^{-jkr}}{4\pi r}\, \pi j \left(J_1(ka\sin\theta) + J_1(ka\sin\theta)\right)$$

$$\simeq j\,\frac{a\mu I_0 e^{-jkr}}{2r}\, J_1(ka\sin\theta) \qquad (19)$$

By using the potential function, in Equation (19), $\vec{E}$ and $\vec{H}$ fields can be found for the far field case such that,

$$\begin{aligned} E_r &\simeq 0 \\ E_\theta &\simeq -jwA_\theta \quad \Rightarrow \vec{E}_A = -jw\vec{A} \\ E_\phi &\simeq -jwA_\phi \end{aligned} \qquad (20)$$

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$$\left.\begin{aligned} H_r &\simeq 0 \\ H_\theta &\simeq +j\frac{\omega}{\eta}A_\phi = -\frac{E_\phi}{\eta} \\ H_\phi &\simeq -j\frac{\omega}{\eta}A_\theta = +\frac{E_\theta}{\eta} \end{aligned}\right\} H_A \simeq \frac{\hat{a}_r}{\eta}\times\vec{E}_A = -j\frac{\omega}{\eta}\hat{a}_r\times\vec{A} \qquad (21)$$

By considering Equation (19)(20)(21),

$$E_r \simeq E_\theta = 0$$

$$E_\phi \simeq \frac{ak\eta I_0 e^{-jkr}}{2r} J_1(ka\sin\theta) \qquad (22)$$

$$H_r \simeq H_\phi = 0$$

$$H_\theta \simeq -\frac{E_\phi}{\eta} = -\frac{akI_0 e^{-jkr}}{2r} J_1(ka\sin\theta) \qquad (23)$$

Small loop approximation can be made for the case $\left(a < \frac{\lambda}{6\pi}\right)$. In order to achieve this, Bessel function of the first kind of order n should be expanded by using Equation (3)

$$J_1(ka\sin\theta) = \frac{1}{2}(ka\sin\theta) - \frac{1}{16}(ka\sin\theta)^3 + \ldots$$

For small values of the ka $\left(ka < \frac{1}{3}\right)$

$$J_1(ka\sin\theta) \simeq \frac{ka\sin\theta}{2}$$

Then

$$E_r \simeq E_\theta = 0$$

$$E_\phi \simeq \frac{a^2\omega\mu k I_0 e^{-jkr}}{4r}\sin\theta = \eta\frac{a^2k^2I_0e^{-jkr}}{4r}\sin\theta \qquad (24)$$

$$H_r \simeq H_\phi = 0$$

$$H_\theta \simeq -\frac{a^2\omega\mu k I_0 e^{-jkr}}{4\eta r}\sin\theta = -\frac{a^2k^2I_0e^{-jkr}}{4r}\sin\theta \qquad (25)$$

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Time-Average Power Density

Average power density can be calculated as

$$W_{av} = \frac{1}{2} Re\left[\vec{E} \times \vec{H}^*\right] \qquad (26)$$

By considering Equation (22), (23),

$$W_{av} = \frac{1}{2} Re\left[\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*\right] = \hat{a}_r \frac{1}{2\eta} |E_\phi|^2$$

$$= \hat{a}_r \frac{1}{2\eta} \frac{a^2 k^2 \eta^2 I_0^2}{4r^2} J_1^2(ka\sin\theta)$$

$$= \hat{a}_r \frac{a^2 k^2 \eta I_0^2}{8r^2} J_1^2(ka\sin\theta) \qquad (27)$$

Consider the Equation (26) and Equation (24)(25), to calculate the time average power density for very small loop

$$W_{av} = \frac{1}{2} Re\left[\hat{a}_\phi E_\phi \times \hat{a}_\theta H_\theta^*\right] = \frac{1}{2} \hat{a}_r \left(\eta \frac{a^2 k^2 I_0 e^{-jkr}}{4r}\right)\left(\frac{a^2 k^2 I_0 e^{jkr}}{4r}\right) \sin^2\theta$$

$$= \frac{1}{2} \hat{a}_r \, \eta \frac{(ak)^4 |I_0|^2}{16r^2}$$

$$W_r = \frac{\eta (ak)^4 |I_0|^2}{32 r^2} \sin^2\theta \qquad (28)$$

Total Radiated Power

$$P_{rad} = \iint_S \vec{W}_{av} \cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi} \frac{a^2 k^2 \eta I_0^2}{8r^2} J_1^2(ka\sin\theta)\, r d\theta\, r\sin\theta d\phi$$

$$= \eta \frac{a^2 k^2 I_0^2 \, 2\pi}{8} \int_0^{\pi} J_1^2(ka\sin\theta) \cdot \sin\theta d\theta \qquad (29)$$

Radiation Resistance

The relation between radiated power and radiation resistance can be given as

$$P_{rad} = \frac{1}{2} R_{rad} |I_0|^2$$

$$\Rightarrow R_{rad} = \frac{2P_{rad}}{|I_0|^2} \qquad (30)$$

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By using the Equation (29) and Equation (30)

$$R_{rad} = \frac{2}{I_o^2} \eta \frac{a^2 k^2 I_o^2 . 2\pi}{8} \int_0^{\pi} J_1^2(ka \sin\theta) \sin\theta \, d\theta$$

$$= \eta \frac{(a.k)^2 \pi}{2} \int_0^{\pi} J_1^2(ka \sin\theta) \sin\theta \, d\theta \qquad (31)$$

Consider the Equation (28), first calculate the radiated power

$$P_{rad} = \iint_S \vec{W}_{av} \cdot d\vec{s} = \int_0^{2\pi}\int_0^{\pi} \frac{\eta (ak)^4 |I_o|^2}{32 r^2} \sin^2\theta \, r^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi \frac{\eta (ak)^4 |I_o|^2}{32} \int_0^{\pi} \sin^3\theta \, d\theta$$

$$P_{rad} = \eta \left(\frac{\pi}{12}\right) (ka)^4 |I_o|^2 \qquad (32)$$

Then using the Equation (30),

$$R_{rad} = \frac{2 P_{rad}}{|I_o|^2} = \frac{2}{|I_o|^2} \eta \left(\frac{\pi}{12}\right) (ka)^4 |I_o|^2 = \frac{(ka)^4 \pi \eta}{6} = 20\pi^2 (ka)^4$$

(33)

Directivity

Consider the Equation (29). The integral can be calculated as,

$$\int_0^{\pi} J_1^2(ka \sin\theta) \sin\theta \, d\theta = \frac{1}{ka} \int_0^{2ka} J_2(x) dx \simeq \frac{1}{ka}$$

Then the Equation (29) becomes

$$P_{rad} = \eta \frac{a^2 k^2 I_o^2 2\pi}{8\, ka}$$

The maximum radiation intensity occurs when $ka \sin\theta = 1.84$. Then $J_1(1.84) = 0.582$

$$U|_{max} = \frac{a^2 k^2 \eta I_o^2}{8} . (0.582)^2 \qquad \because \text{ Equation (27)}$$

$$D_o = 4\pi \frac{a^2 k^2 \eta I_o^2}{8} . (0.582)^2 \frac{1}{\eta \frac{a^2 k^2 I_o^2 2\pi}{8\, ka}} = 2ka\,(0.582)^2$$

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Directivity of small loop, consider Equation (28),

$$U_{max}\Big|_{\theta=\frac{\pi}{2}} = \frac{\eta (ak)^4 |I_0|^2}{32}$$

Radiated power is in Equation (32),

$$D_0 = 4\pi \, \frac{\eta (ak)^4 |I_0|^2}{32} \cdot \frac{1}{\eta \frac{\pi}{12} (ka)^4 |I_0|^2} = \frac{48}{32} = \frac{3}{2} \, //$$

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