13B

1. d)

Find the equation of the normal to:

$$y = 8\sqrt{x} - \frac{1}{x^2} \quad \text{at } x = 1$$

Point $x = 1$

$$y = 8\sqrt{1} - \frac{1}{1^2} = 7$$

$(1, 7)$

Gradient:

$$y = 8x^{\frac{1}{2}} - x^{-2}$$

$$\frac{dy}{dx} = 8 \cdot \frac{1}{2} x^{-\frac{1}{2}} - -2x^{-3}$$

$$= \frac{4}{x^{1/2}} + \frac{2}{x^3}$$

$$= \frac{4x^{\frac{5}{2}} + 2}{x^3}$$

$$\frac{dy}{dx}\perp = \frac{-x^3}{4x^{5/2}+2}$$

$x = 1$

$$\frac{dy}{dx}\perp = \frac{-1}{4+2} = \frac{-1}{6}$$

$(1, 7)$ and $-\frac{1}{6}$

$$y - 7 = -\frac{1}{6}(x-1)$$

$$y - 7 = -\frac{x}{6} + \frac{1}{6}$$

$$y = -\frac{x}{6} + \frac{43}{6}$$

or $6y + x = 43$

or $x + 6y - 43 = 0$

3. c) Find the equation of the normal to

$$y = e^{2x-1} \quad \text{at } x = 1$$

Point $(1, e^1)$

$$\frac{dy}{dx} = e^{2x-1} \cdot 2$$

$$= 2e^{2x-1}$$

$$\frac{dy}{dx}\perp = \frac{-1}{2e^{2x-1}} = -\frac{e^{1-2x}}{2}$$

$x = 1$

$$\frac{dy}{dx}\perp = \frac{-1}{2e}$$

$$y - e = \frac{-1}{2e}(x-1)$$

$$y = \frac{-x}{2e} + \frac{1}{2e} + e$$

$$y = -\frac{xe^{-1}}{2} + \frac{1+2e^2}{2e}$$

or

$$2ey - 2e^2 = -x + 1$$

$$2ey + x = 1 + 2e^2$$

$$x + 2ey = 1 + 2e^2$$

5. Find the points where the normal to

$$y = x^3 - 2x^2 + 1$$

at $x = 1$ meets the curve again.

Find equation of normal at $x=1$ and then set equal to y, find intersection points.

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{dy}{dx}_{\perp} = \frac{-1}{3x^2 - 4x}$$

$$x = 1 \qquad \frac{dy}{dx}_{\perp} = \frac{-1}{3-4} = 1$$

$$x = 1, \quad y = 1 - 2 + 1 = 0$$

$$y - 0 = 1(x - 1)$$

$$y = x - 1 \quad \text{(normal)}$$

set $\quad x^3 - 2x^2 + 1 = x - 1$

$$\Rightarrow \quad x^3 - 2x^2 - x + 2 = 0$$

We already have one solution $x = 1 \Rightarrow x - 1$ is a factor

$$(x-1)(x^2 - x - 2) = 0$$

← deduce

or

$$\begin{array}{r} x^2 - x - 2 \\ x - 1 \,\overline{)\, x^3 - 2x^2 - x + 2} \\ \underline{x^3 - x^2 } \\ -x^2 - x \\ \underline{-x^2 + x } \\ -2x + 2 \\ \underline{-2x + 2} \\ 0 \end{array}$$

$$(x-1)(x^2 - x - 2)$$

$$\Rightarrow (x-1)(x-2)(x+1) = 0$$

Solution $x = 1, \; x = 2 \quad x = -1$

so new points

$x = 2$

$$y = (2)^3 - 2(2)^2 + 1$$

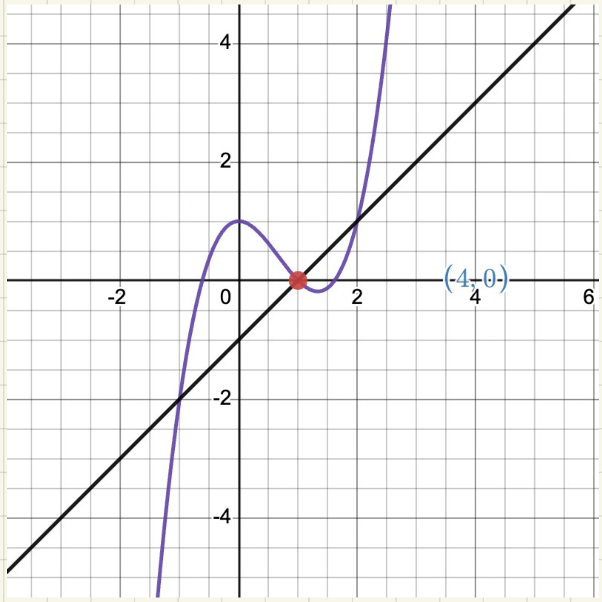
$$= 1 \qquad (2, 1)$$

$x = -1$

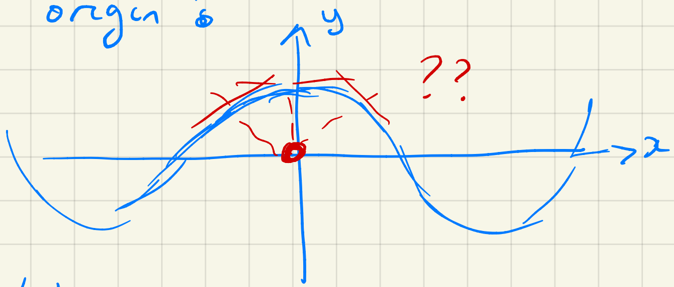
$$y = (-1)^3 - 2(-1)^2 + 1$$

$$= -2 \qquad (-1, -2)$$

points are $(2, 1), \; (-1, -2)$



6. Find the equation of the normal to $f(x) = \cos(x)$ which passes through the origin.



let $x = a$

$\Rightarrow$ $(a, \cos(a))$ is on the curve

$$\frac{dy}{dx} = -\sin(x)$$

$$\frac{dy}{dx}\perp = \frac{1}{\sin(x)}$$

$$\frac{dy}{dx}\perp = \frac{1}{\sin(a)}$$

$$\frac{\cos(a) - 0}{a - 0} = \frac{1}{\sin(a)}$$

$$\sin(a)\cos(a) = a$$
$$2\sin(a)\cos(a) = 2a$$

$$\sin(2a) = 2a$$

Solve $\sin(x) = x$?

We know $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$

$\Rightarrow$ $\sin(x) = x$
when $x = 0$
$a = 0$

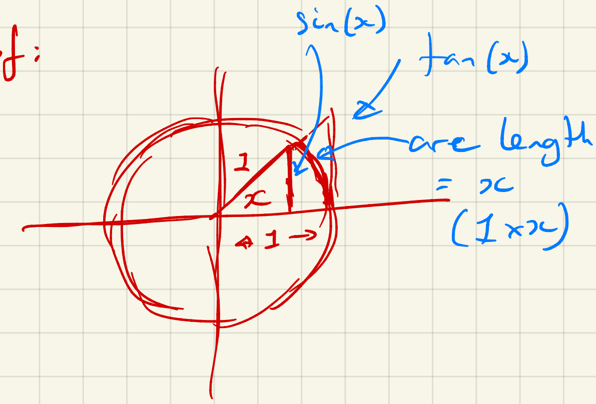
Point $(0, 1)$

Normal line is vertical

$\Rightarrow$ $x = 0$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

Proof:



$$\sin(x) < x < \tan(x)$$

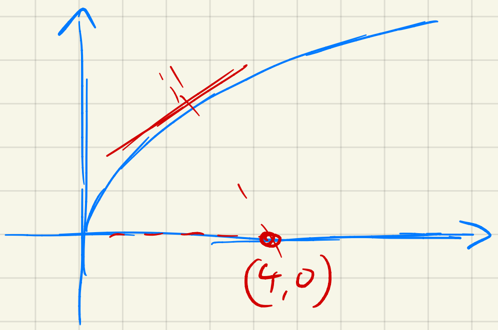
$$\Rightarrow \quad \frac{1}{\sin(x)} > \frac{1}{x} > \frac{\cos(x)}{\sin(x)}$$

$$1 > \frac{\sin(x)}{x} > \cos(x)$$

$$\lim_{x\to 0} \Rightarrow 1 > \frac{\sin(x)}{x} > 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin(x)}{x} = 1$$

7. Find the equation to $y=\sqrt{x}$ from the external point $(4,0)$



$y=\sqrt{x}$, $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$, $\frac{dy}{dx}_{\perp}=-2\sqrt{x}$

let the coordinate on the curve be $(a, \sqrt{a})$

so $\frac{dy}{dx}_{\perp}=-2\sqrt{a}$

and $\frac{\sqrt{a}-0}{a-4}=-2\sqrt{a}$

$\sqrt{a}=-2\sqrt{a}(a-4)$

$\sqrt{a}=-2a\sqrt{a}+8\sqrt{a}$

$2a\sqrt{a}-7\sqrt{a}=0$

$\sqrt{a}(2a-7)=0$

$\sqrt{a}=0$ $\quad 2a-7=0$

$a=\frac{7}{2}$

reject, because derivative is not defined at the end point of a curve

$a=\frac{7}{2}$

Point $\left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$

$\frac{dy}{dx}_{\perp}=-2\sqrt{\frac{7}{2}}$

$y-\sqrt{\frac{7}{2}}=-2\sqrt{\frac{7}{2}}\left(x-\frac{7}{2}\right)$

$y=-2\sqrt{\frac{7}{2}}x+7\sqrt{\frac{7}{2}}+\sqrt{\frac{7}{2}}$

$y=-2\sqrt{\frac{7}{2}}x+8\sqrt{\frac{7}{2}}$

$y=-\sqrt{\frac{4\times7}{2}}x+\sqrt{\frac{7\times64}{2}}$

$=-\sqrt{14}x+\sqrt{7\times32}$

$=-\sqrt{14}x+\sqrt{16\times7\times2}$

$=-\sqrt{14}x+4\sqrt{14}$