13D

5.

Find and classify stationary points

g) $f(x) = x - \sqrt{x}$

Domain $\{x \mid x \geq 0,\ x \in \mathbb{R}\}$

$$f'(x) = 1 - \frac{1}{2\sqrt{x}}$$

$$1 - \frac{1}{2\sqrt{x}} = 0 \implies 2\sqrt{x} = 1$$

$$\sqrt{x} = \frac{1}{2}$$

$$x = \frac{1}{4}$$

$\frac{1}{4}$

Test $f'(1) = 1 - \frac{1}{2} = \frac{1}{2}$ $\quad f'(1) > 0$

Test $f'\left(\frac{1}{16}\right) = 1 - \frac{1}{\frac{2}{4}} = 1 - 2 = -1$

$f'\left(\frac{1}{16}\right) < 0$

$-$ $\quad$ $+$

$\frac{1}{4}$

Local minimum at $x = \frac{1}{4}$

$$f\left(\frac{1}{4}\right) = \frac{1}{4} - \sqrt{\frac{1}{4}} = -\frac{1}{4}$$

$$\left(\frac{1}{4}, -\frac{1}{4}\right)$$

Also note: $f(x) = 0$

when $x - \sqrt{x} = 0$

$$\sqrt{x}\left(\sqrt{x} - 1\right) = 0$$

$$x = 0 \qquad x = 1$$

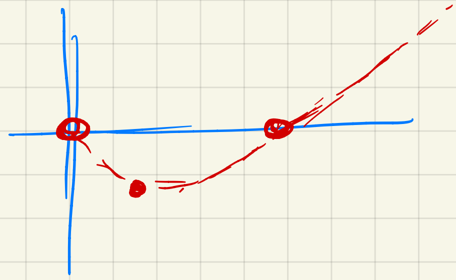
x-intercepts $(0,0)$ $\ (1,0)$

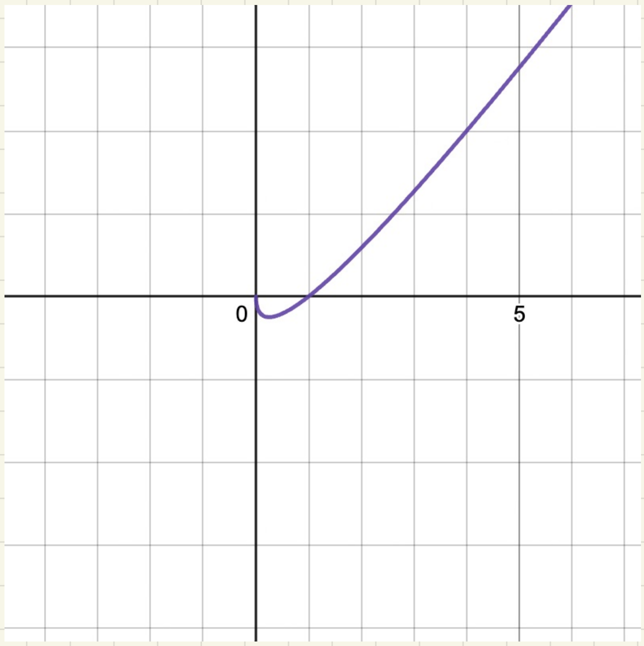
for $\forall x,\ x > 1$

$$x > \sqrt{x}$$

$$\implies \lim_{x \to \infty} x - \sqrt{x} = \infty$$

but increasing quite slowly





7. d)

$y = e^{-x}(x+2)$

$\frac{dy}{dx} = e^{-x} \cdot 1 + -e^{-x}(x+2)$

$= e^{-x} - e^{-x}(x+2)$

$= e^{-x}(1-(x+2))$

$= e^{-x}(-x-1)$

$= -e^{-x}(x+1)$

$\frac{dy}{dx} = 0 \qquad x = -1$

-1

$x = -2 \qquad \frac{dy}{dx}(-2) = -e^{2}(-2+1) > 0$

$x = 0 \qquad \frac{dy}{dx}(0) = -e^{0}(0+1) < 0$

$+ \qquad -$

-1

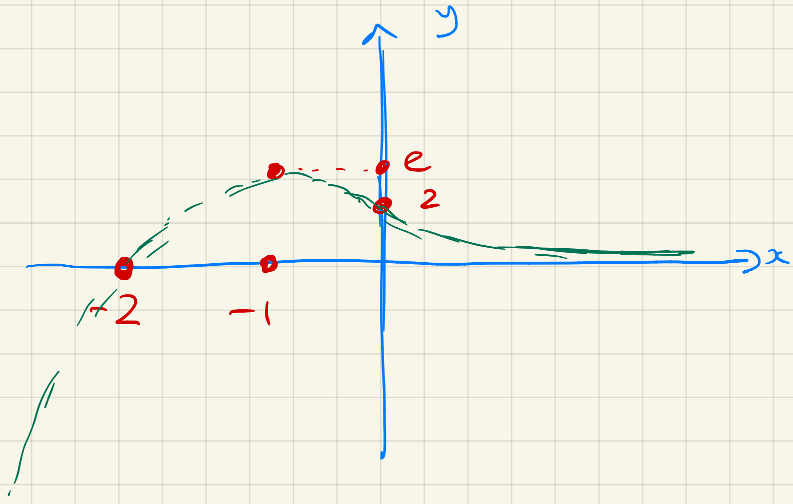
$y(-1) = e^{1}(1) = e$

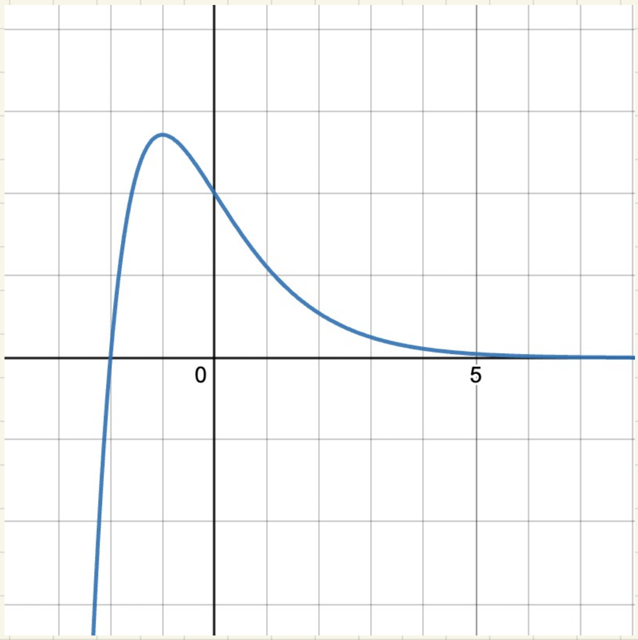
set $y = 0 \Rightarrow x = -2$

$x = 0 \qquad y = 2$

$\lim_{x \to -\infty} e^{-x}(x+2) = -\infty$

$\lim_{x \to \infty} e^{-x}(x+2) = 0$





10. $y = \dfrac{e^{ax}}{bx}$

has stationary point at $\left(\frac{1}{3}, \frac{e}{2}\right)$

Find a, b

Classify the stationary point.

$$\frac{e}{2} = \frac{e^{\frac{a}{3}}}{\frac{b}{3}}$$

$$\frac{be}{6} = e^{\frac{a}{3}}$$

$$b = 6e^{\frac{a}{3}-1}$$

$u = e^{ax}$ $\quad u' = ae^{ax}$

$v = bx$ $\quad v' = b$

$$\frac{dy}{dx} = \frac{ae^{ax}(bx) - be^{ax}}{(bx)^2}$$

$$= \frac{be^{ax}(ax-1)}{(bx)^2}$$

$$= \frac{e^{ax}(ax-1)}{bx^2}$$

$$\frac{dy}{dx}\left(\frac{1}{3}\right) = 0$$

$$\Rightarrow \left(\frac{a}{3} - 1\right) = 0$$

$$a = 3$$

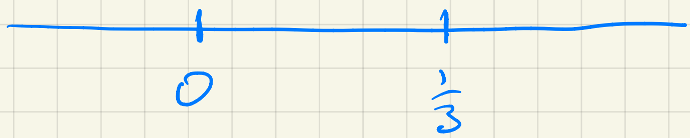
$$b = 6e^{\frac{3}{3}-1}$$

$$= 6$$

$$y = \frac{e^{3x}}{6x}$$

$$\frac{dy}{dx} = \frac{e^{3x}(3x-1)}{6x^2}$$

Function is undefined at $x=0$



Test : $x = -1$

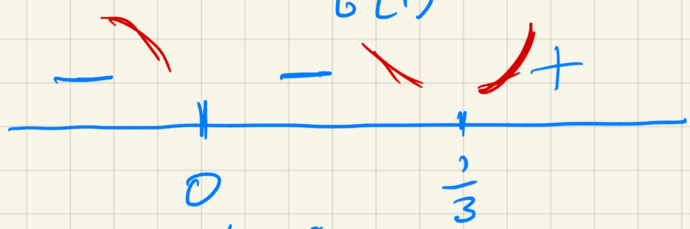
$$\frac{dy}{dx}(-1) = \frac{e^{-3}(-4)}{6(-1)^2} < 0$$

Test : $x = \frac{1}{4}$

$$\frac{dy}{dx}\left(\frac{1}{4}\right) = \frac{e^{3/4}\left(\frac{3}{4}-1\right)}{6\left(\frac{1}{4}\right)^2} < 0$$

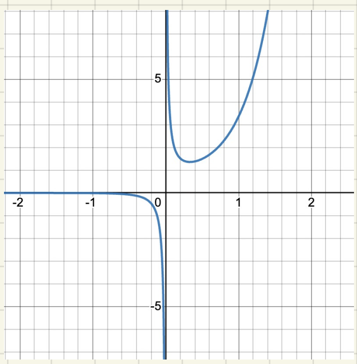
Test: $x = 1$

$$\frac{dy}{dx}(1) = \frac{e^{3}(3-1)}{6(1)^2} > 0$$



$\lim\limits_{x \to -\infty} \frac{e^x}{x} = {}^-0$ $\quad \lim\limits_{x \to 0^+} \frac{e^x}{x} = +\infty$

$\lim\limits_{x \to {}^-0} \frac{e^x}{x} = -\infty$ $\quad \lim\limits_{x \to +\infty} \frac{e^x}{x} = +\infty$



12. f)

$f(x) = \sin(2x) + 2\cos(x)$

$f'(x) = 2\cos(2x) - 2\sin(x)$

$f'(x) = 0$

$\Rightarrow 2\cos(2x) - 2\sin(x) = 0$

Trig identity:
$\cos(2\theta) = 1 - 2\sin^2(\theta)$
(chapter 9: page 238)

$2(1 - 2\sin^2(x)) - 2\sin(x) = 0$

$-4\sin^2(x) - 2\sin(x) + 2 = 0$

$-2(2\sin^2(x) + \sin(x) - 1) = 0$

$2y^2 + y - 1$ $\quad -2 \quad (\times)$

$2y^2 + 2y - y - 1$ $\quad +2 \; -1 \quad +1 \quad (+)$

$2y(y+1) - 1(y+1) = (2y-1)(y+1)$

$-2(2\sin(x) - 1)(\sin(x) - 1) = 0$

$\sin(x) = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Domain $0 \le x \le 2\pi$

$\frac{\pi}{6} + 2k\pi \Rightarrow \frac{\pi}{6}$ in D

$\pi - \frac{\pi}{6} + 2k\pi \Rightarrow \frac{5\pi}{6}$ in D

$\sin(x) = -1 \Rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$

$-\frac{\pi}{2} + 2k\pi \Rightarrow \frac{3\pi}{2}$ in D

Solutions: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

$f'(x) = 2\cos(2x) - 2\sin(x)$

$0 \quad \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{3\pi}{2} \quad 2\pi$

Test $x = 0$ $\quad f'(0) = 2 > 0$, Test $x = \frac{\pi}{2}$ $\quad f'\left(\frac{\pi}{2}\right) = -4 < 0$

Test $x = \pi$ $\quad f'(\pi) = 2 \quad > 0$

Test $x = 2\pi$ $\quad f'(2\pi) = 2 \quad > 0$

$+ \quad - \quad + \quad +$

$0 \quad \frac{\pi}{6} \quad \frac{5\pi}{6} \quad \frac{3\pi}{2} \quad 2\pi$

Local maximum at $x = \frac{\pi}{6}$ $\quad f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \simeq 2.598$

$\left(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}\right)$

Local minimum at $x = \frac{5\pi}{6}$

$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{5\pi}{3}\right) - 2\cos\left(\frac{5\pi}{6}\right)$

$= -\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

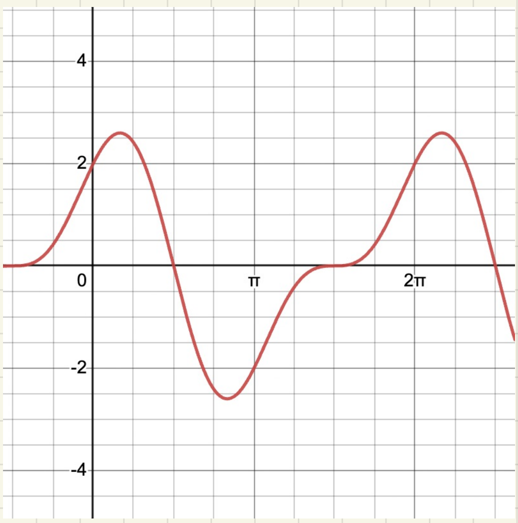
$\left(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2}\right)$

Stationary Inflection point at $x = \frac{3\pi}{2}$

$f\left(\frac{3\pi}{2}\right) = \sin(3\pi) - 2\cos\left(\frac{3\pi}{2}\right)$

$= 0$

$\left(\frac{3\pi}{2}, 0\right)$



16.

$$f(x) = \sin(x)\cos(2x)$$

$$f'(x) = \sin(x)\left(-2\sin(2x)\right) + \cos(x)\cos(2x)$$

$$= -2\sin(x)\sin(2x) + \cos(x)\cos(2x)$$

$$= -2\sin(x)\left(2\sin(x)\cos(x)\right) + \cos(x)\left(2\cos^2(x)-1\right)$$

$$= -4\sin^2(x)\cos(x) + 2\cos^3(x) - \cos(x)$$

$$= -4\left(1-\cos^2(x)\right)\cos(x) + 2\cos^3(x) - \cos(x)$$

$$= -4\cos(x) + 4\cos^3(x) + 2\cos^3(x) - \cos(x)$$

$$= 6\cos^3(x) - 5\cos(x)$$

$$f'(x) = 0$$

$$\Rightarrow 6\cos^3(x) - 5\cos(x) = 0$$

$$\cos(x)\left(6\cos^2(x) - 5\right) = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\cos(x) = \pm\sqrt{\frac{5}{6}}$$

Domain $\{x \mid 0 \le x \le \pi\}$

which is Domain of $\cos^{-1}(y)$

$$\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} \simeq 1.571$$

$$\cos(x) = +\sqrt{\frac{5}{6}}$$

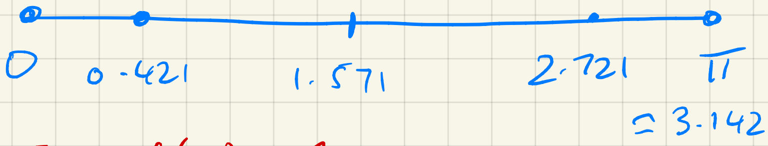
$$x = \cos^{-1}\left(+\sqrt{\frac{5}{6}}\right) \simeq 0.421$$

$$\cos(x) = -\sqrt{\frac{5}{6}}$$

$$x = \cos^{-1}\left(-\sqrt{\frac{5}{6}}\right)$$

$$\simeq 2.721$$

$(\text{or } \pi - 0.421 = 2.721)$

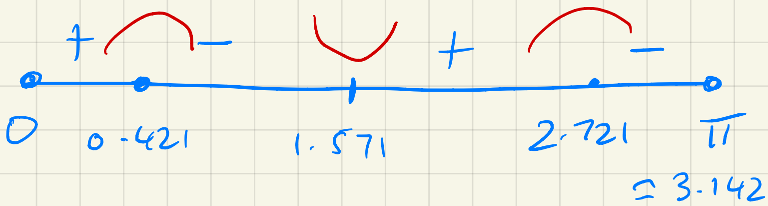


Test $f'(0) = 1 > 0$

Test $f'(1) = -1.755 < 0$

Test $f'(2) = 1.648 > 0$

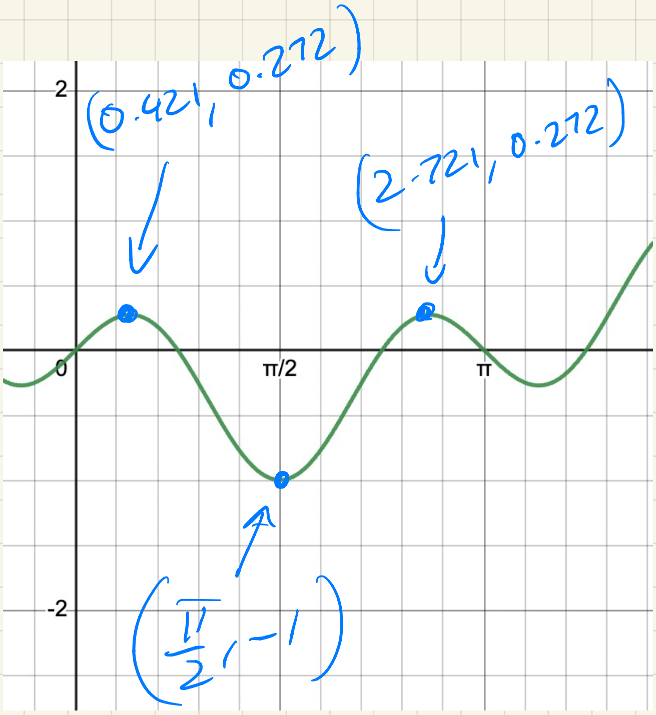
Test $f'(\pi) = -1 < 0$



$$f(0.421) = 0.272$$

$$f\left(\frac{\pi}{2}\right) = -1$$

$$f(2.721) = 0.272$$



18. Prove $\frac{\ln(x)}{x} \leq \frac{1}{e}$

for all $x > 0$

Create a sign diagram and hopefully identify a maximum value for $\frac{\ln x}{x}$

let $f(x) = \frac{\ln(x)}{x}$ $\qquad u = \ln x \quad v = x$

$\qquad u' = \frac{1}{x} \quad v' = 1$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2}$$

$f'(x) = 0 \Rightarrow 1 - \ln x = 0$

$\ln x = 1$

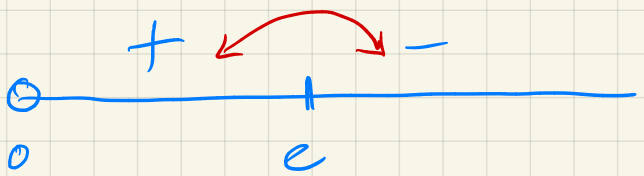
$x = e$

Test $x = 1$

$f'(1) = \frac{1 - 0}{1^2} > 0$

Test $x = e^2$

$$f'(e^2) = \frac{1 - 2\ln e}{e^4} = \frac{-1}{e^4} < 0$$



Global maximum at $x = e$

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

$$\left(e, \frac{1}{e}\right)$$

So $\frac{1}{e}$ is largest possible value of $\frac{\ln x}{x}$

$$\Rightarrow \frac{\ln x}{x} \leq \frac{1}{e}$$

19. $f(x) = x - \ln(x)$ $\quad D\{x > 0\}$

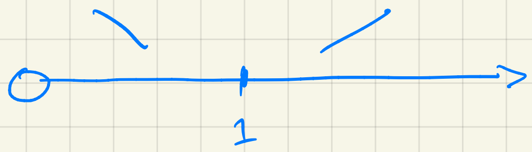
a) show only local minimum.

b) Prove $\ln(x) < x - 1 \quad \forall\, x > 0$

$f'(x) = 1 - \frac{1}{x}$

$f'(x) = 0$

$1 - \frac{1}{x} = 0 \qquad x = 1$



$f'(\frac{1}{2}) = 1 - 2 = -1 \quad < 0$

$f'(2) = 1 - \frac{1}{2} = \frac{1}{2} \quad > 0$

local minimum at $(1, 1)$

$\Rightarrow x - \ln(x) \geq 1 \quad \forall\, x < 0$

$x - 1 \geq \ln(x)$

$\ln(x) \leq x - 1$