Sequences and Series

Kh notes

Contents

Introduction Problem

OPENING PROBLEM **THE LEGEND OF SISSA IBN DAHIR**

Around 1260 AD, the Kurdish historian Ibn Khallikān recorded the following story about Sissa ibn Dahir and a chess game against the Indian King Shihram.

King Shihram was a tyrant king, and his subject Sissa ibn Dahir wanted to teach him how important all of his people were. He invented the game of chess for the king, and the king was greatly impressed. He insisted on Sissa ibn Dahir naming his reward, and the wise man asked for one grain of wheat for the first square, two grains of wheat for the second square, four grains of wheat for the third square, and so on, doubling the wheat on each successive square on the board.

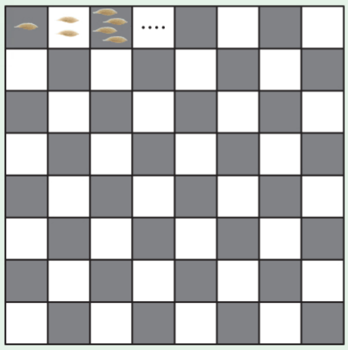
The king laughed at first and agreed, for there was so little grain on the first few squares. By halfway he was surprised at the amount of grain being paid, and soon he realised his great error: that he owed more grain than there was in the world.

Things to think about:

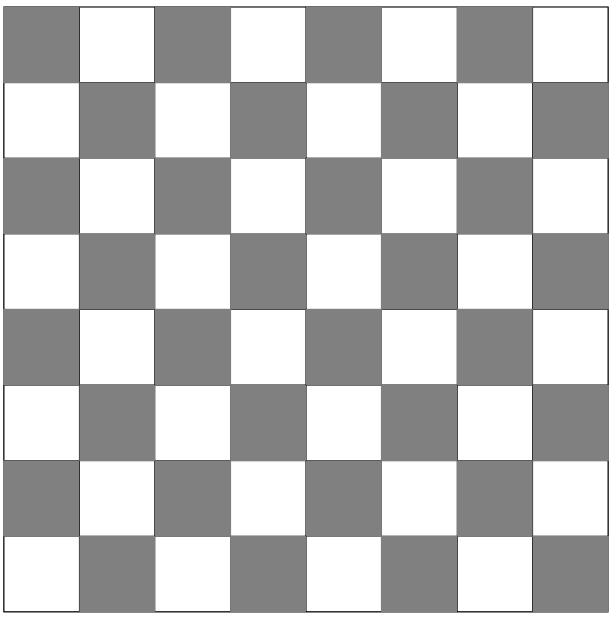
a How can we describe the number of grains of wheat for each square?

b What expression gives the number of grains of wheat for the nth square?

c Find the total number of grains of wheat that the king owed.



Question: How do we go about solving this problem mathematically?



Grains of rice problem.

How do we define this problem mathematically?

- Create a number sequence for each square: $1, 2, 4, 8, 16, ..., 2^{62}, 2^{63}$
- Consider relations between the number in each square: $s_2 = 2s_1$, or $s_9 = 2^8$
- Create a formula for each square: $s_n = 2^{n-1}$
- Communicate the idea of summing : $1 + 2 + 4 + 8 + 16, +...+, 2^{62} + 2^{63} = \sum_{i=1}^{64} 2^{i-1}$
- Actually solve the problem $1 + 2 + 4 + 8 + 16, +...+, 2^{62} + 2^{63} =?$

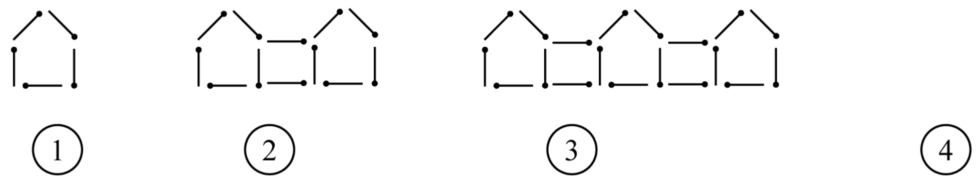
Consider:

$$\begin{aligned}
1 + 1 + 2 + 4 + 8 &= 2 + 2 + 4 + 8 \\
&= 2(1 + 1 + 2 + 4) \\
&= 2(2 + 2 + 4) \\
&= 2(2(1 + 1 + 2)) \\
&= 2(2(2 + 2)) \\
&= 2(2(2(1 + 1))) \\
&= 2(2(2(2))) = 2^4
\end{aligned}$$

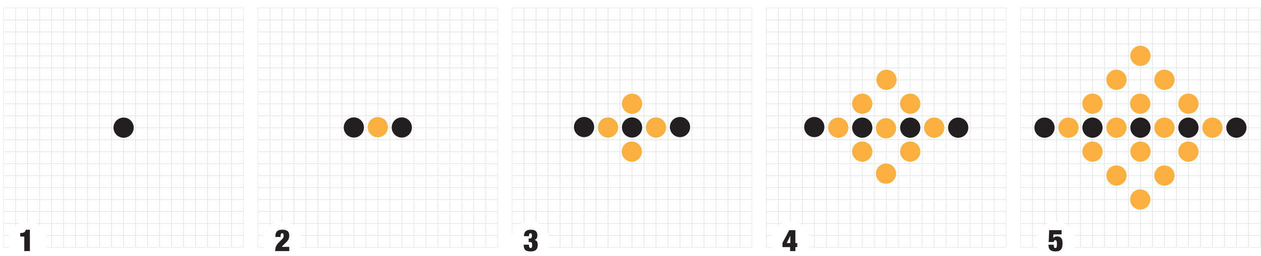
A number sequence is an ordered list of numbers defined by a rule:

A couple of examples from your past

A linear sequence



A quadratic sequence



- We need to be able to descibe the sequence:
  What is the first term? How do the terms change?
- We define u_n to be the n^{th} term of the sequence
- Unless otherwise stated u_1 is the first term.
- $\{u_n\}$ represents the whole sequence.
- In general we use an **explicit formula** to define the sequence for example: $\{2n-3\}$

Ex 5A

- 1: b, d
- 2: c
- 6
- 7: d,e,h
- 9: a,e,f
- 10: g,h,i

1 Arithmetic Sequences

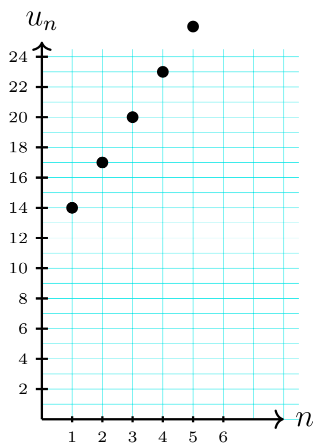
An example of an arithmetic sequence is:

$$14, 17, 20, 23, ...$$

An arithmetic sequence has a common difference between the terms (in this case 3)

- We need to be able to descibe the sequence:
  The sequence has a first term of 14 and increases by 3 each time
- We define d as the common difference ($d = u_{n+1} - u_n$).

Note: the sequence is linear (so the difference is the gradient).



$u_4 = 23$
$u_3 = 20$
$u_2 = 17$
$u_1 = 14$

The middle of any three consectutive terms is the arithmetic mean of the other two terms.

General term formula:

$$u_n = u_1 + (n - 1)d$$

(this can then be simplified to a '$y = mx + c$' form)

1.1 Ex 5B.1

- Finding terms
  - ⋆ 3: a, e, 4 , 6, 7
- Working with k
  - ⋆ 12: a, d, g
- Find the general term
  - ⋆ 13
- Inserting terms between
  - ⋆ 14, 15

Example Questions: 5B.1

7. A sequence is defined by
$$u_n = \frac{71-7n}{2}$$

a) prove the sequence is arithmetic

Show $u_{n+1} - u_n = d$

$$u_{n+1} - u_n = \frac{71-7(n+1)}{2} - \frac{71-7n}{2}$$
$$= \frac{71-7n-7-71+7n}{2}$$
$$= \frac{-7}{2} = -3.5$$

So $u_{n+1} - u_n = -3.5$ for all $n \in \mathbb{Z}^+$

b) $u_n = \frac{71}{2} - \frac{7n}{2}$

$u_1 = \frac{71}{2} - \frac{7}{2} = \frac{64}{2} = 32$

$d = -\frac{7}{2}$

c) $u_{75} = \frac{71-7(75)}{2} = -227$

d) $u_n < -200$

$$\frac{71-7n}{2} < -200$$
$$71-7n < -400$$
$$7n > 71+400$$
$$7n > 471$$
$$n > 67\tfrac{2}{7}$$

for $n \geq 68$ $u_n < -200$

check $u_{68} = \frac{71-7(68)}{2}$
$= -202.5$

12. f) Find k given the consecutive arithmetic terms:
$$\underset{u_1}{2k+18},\ \underset{u_2}{-2-k},\ \underset{u_3}{2k+2}$$

$$u_2 - u_1 = u_3 - u_2$$

$$-2-k-(2k+18) = 2k+2-(-2-k)$$
$$-2-k-2k-18 = 2k+2+2+k$$
$$-20-3k = 3k+4$$
$$6k = -24$$
$$k = -4$$

check:
$10,\ 2,\ -6$
$d = -8$

Other notes:
Could have:
$$\frac{2k+2+2k+18}{2} = -2-k$$
$$4k+20 = -4-2k$$
$$6k = -24$$

or

$$u_2 - u_1 = u_3 - u_2$$

$$2u_2 - u_1 = u_3$$
$$2(-2-k) - (2k+18) = 2k+2$$
$$-4-2k-2k-18 = 2k+2$$
$$-22-4k = 2k+2$$
$$6k = -24$$

13. Find the general term u_n for the arithmetic sequence $u_5 = -2$ and $u_{12} = -12\frac{1}{2}$

$u_5 = a + 4d$
$u_{12} = a + 11d$

$a + 4d = -2$ ①
$a + 11d = -\frac{25}{2}$ ②

$-7d = +\frac{21}{2}$ ① − ②

$d = +\frac{21}{2} \times -\frac{1}{7} = -\frac{3}{2}$

from ① $a = -2 - 4d$
$= -2 - 4\left(-\frac{3}{2}\right)$
$= -2 + 6 = 4$

$u_n = 4 - (n-1)\frac{3}{2}$

Test: $u_5 = 4 - (5-1)\frac{3}{2}$
$= 4 - 6 = -2$
$u_{12} = 4 - (12-1)\frac{3}{2}$
$= 4 - \frac{33}{2}$
$= \frac{8-33}{2}$
$= -\frac{25}{2} = -12\frac{1}{2}$

15. Insert six numbers between -1 and 32 so all 8 are in an arithmetic sequence

$a = -1$

$u_8 = -1 + 7d$

$-1 + 7d = 32$

$7d = 33$

$d = \frac{33}{7}$

$u_n = -1 + (n-1)\frac{33}{7}$

or $d = \frac{u_8 - u_1}{7} = \frac{33}{7}$

1.2 Ex 5B.2

- 2,3,5

2 Geometric Sequences

An example of a geometric sequence is:

$$3, 6, 12, 24, 48, \ldots$$

In this case the difference between each term is not constant, but we can observe that each term is $2\times$ the previous term. So we have

$$\begin{aligned} u_{n+1} &= 2u_n \\ \Rightarrow \quad \frac{u_{n+1}}{u_n} &= 2 \end{aligned}$$

and

$$\begin{aligned} u_1 &= 3 \\ u_2 &= 2 \times 3 \\ u_3 &= 2 \times 2 \times 3 \\ u_4 &= 2 \times 2 \times 2 \times 3 \\ \Rightarrow \quad u_n &= 3 \times (2)^{n-1} \end{aligned}$$

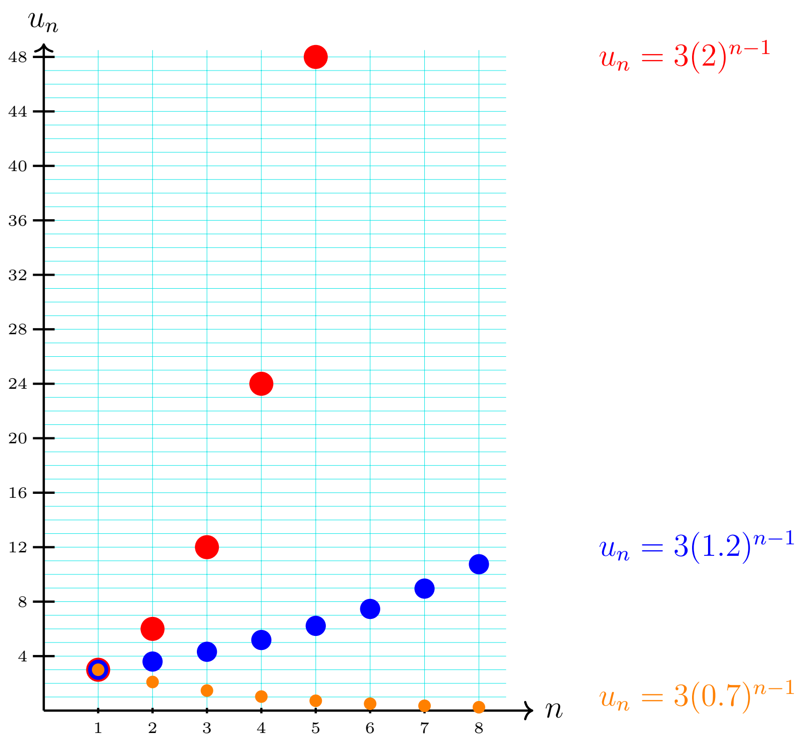
A geometric sequence has a **common ratio** between the terms. In general:

$$\frac{u_{n+1}}{u_n} = r$$

The first term is (generally) u_1

$$u_n = u_1 r^{n-1}$$

Behaviour of some sequences with different values of r



2.1 Ex 5C

- Finding common ratio
- Showing a sequence is geometric
- Finding the general formula and terms.
- k problems
- Finding the general formula, given two terms

Example Questions.

Ex 5C

5.

a) Show that the sequence

$$12, -6, 3, -\frac{3}{2}, \ldots$$

is geometric

b) Find u_n and u_{13}

a) $\frac{-6}{12} = -\frac{1}{2}$

and each term is

$-\frac{1}{2} \times$ the previous term.

b) $u_n = 12\left(-\frac{1}{2}\right)^{n-1}$

$$u_{13} = 12\left(-\frac{1}{2}\right)^{12}$$

$$= \frac{12}{2^{12}}$$

$$= \frac{2^2 \times 3}{2^{12}}$$

$$= \frac{3}{2^{10}} = \frac{3}{1024}$$

10. The first 3 terms of a geometric sequence are

$$k-1, 6, 3k$$

a) Find possible terms for k

b) For each k find the next term in the sequence.

a) $\frac{6}{k-1} = \frac{3k}{6}$

$$36 = 3k(k-1)$$
$$36 = 3k^2 - 3k$$
$$3k^2 - 3k - 36 = 0$$
$$k^2 - k - 12 = 0$$
$$(k-4)(k+3) = 0$$
$k = 4$ or $k = -3$

b) sequences

$k = 4$

$r = \frac{6}{3} = 2$, $u_1 = k - 1 = 3$

$$u_n = 3(2)^{n-1}$$

$$u_4 = 3(2)^3 = 24$$

$k = -3$

$r = \frac{6}{-4} = -\frac{3}{2}$, $u_1 = k - 1 = -4$

$$u_n = -4\left(-\frac{3}{2}\right)^{n-1}$$

$$u_4 = -4\left(-\frac{3}{2}\right)^3$$

$$= -4 \times \frac{-27}{8}$$

$$= -\frac{27}{2} = -13\frac{1}{2}$$

11. Find the general term, u_n of the geometric sequence which has

$u_3 = 8$, $u_6 = -1$

$u_n = u_1 r^{n-1}$

$8 = u_1 r^2$ ①

$-1 = u_1 r^5$ ②

$\frac{②}{①}$ $\frac{u_1 r^5}{u_1 r^2} = \frac{-1}{8}$

$r^3 = -\frac{1}{8}$

$r = \sqrt[3]{-\frac{1}{8}} = -\frac{1}{2}$

$u_1 = \frac{8}{r^2} = \frac{8}{\left(-\frac{1}{2}\right)^2}$

$= 8 \times \frac{4}{1} = 32$

$u_n = 32\left(-\frac{1}{2}\right)^{n-1}$

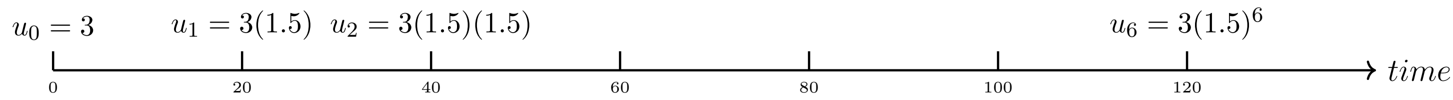
3 Growth and Decay

Starter Questions:

1. A school had 1200 students and a year later this has increased by 8% . How many students are now in the school?
2. Mary buys a car for \$40,000 and in one year its price has decreased by 12%. What is the value of it now?
3. The population of Sydney is currently 5.2 million. If it increases at a rate of 1.25% annually, what will the population be after 3 years?

Problem:

3 fruit flies arrive at fruit bowl that has been left out on the kitchen bench. The number of fruit flies increases by 50% every 20 minutes.
How many fruit flies will there be after 2 hours?



Because we want to have a starting value at 'time 0' we start our sequence at u_0 and have the general term:

$$u_n = 3(1.5)^n$$

And more generally, in **growth and decay** problems we use

$$u_n = u_0 r^n$$

3.1 Ex 5D

4 Financial Mathematics

4.1 Compound Interest

$$u_n = u_0(1+i)^n$$

u_0 Initial Investment (Principal)

i Interest rate per compounding period

n Number of periods

u_n The final value of the investment

4.1.1 Ex 5E.1

4.2 Inflation

4.2.1 Ex 5E.2

4.3 Real Value of an Investment

4.3.1 Ex 5E.3

4.4 Depreciation

4.4.1 Ex 5E.4

4.5 Using Financial Models

4.5.1 Ex 5E.5

5 Series

5.1 Sigma Notation

5.1.1 Ex 5F

5.2 Arithmetic Series

5.2.1 Ex 5G

5.3 Finite Geometric Series

5.3.1 Ex 5H

5.4 Infinite Geometric Series

5.4.1 Ex 5I