Binomial Theorem

Kh notes

Contents

1 Factorial Notation

We define $n!$ as

$$n! = n(n-1)(n-2)...3 \times 2 \times 1$$

And define $0! = 1$

1.1 Ex 1A

2 Binomial Expansions

Pascal's Triangle

1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

In the following Examples we see how the general binomial expansion $(a+b)^n$ may be put to use.

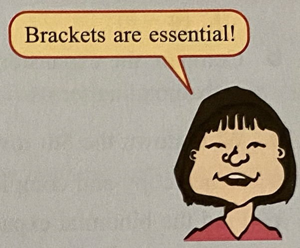
Example 2 **Self Tutor**

Use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ to find the binomial expansion of:

a $(2x+3)^3$ **b** $(x-5)^3$

a In the expansion of $(a+b)^3$ we substitute $a = (2x)$ and $b = (3)$.

$$\begin{aligned}\therefore\ (2x+3)^3 &= (2x)^3 + 3(2x)^2(3) + 3(2x)^1(3)^2 + (3)^3 \\ &= 8x^3 + 36x^2 + 54x + 27\end{aligned}$$



b We substitute $a = (x)$ and $b = (-5)$

$$\begin{aligned}\therefore\ (x-5)^3 &= (x)^3 + 3(x)^2(-5) + 3(x)(-5)^2 + (-5)^3 \\ &= x^3 - 15x^2 + 75x - 125\end{aligned}$$

Example 3 **Self Tutor**

Find the:

a 5th row of Pascal's triangle **b** binomial expansion of $\left(x - \frac{2}{x}\right)^5$.

a

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

the 1st row, for $(a+b)^1$

the 5th row, for $(a+b)^5$

b Using the coefficients obtained in **a**, $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Letting $a = (x)$ and $b = \left(\frac{-2}{x}\right)$,

$$\begin{aligned}&\left(x - \frac{2}{x}\right)^5 \\ &= (x)^5 + 5(x)^4\left(\frac{-2}{x}\right) + 10(x)^3\left(\frac{-2}{x}\right)^2 + 10(x)^2\left(\frac{-2}{x}\right)^3 + 5(x)\left(\frac{-2}{x}\right)^4 + \left(\frac{-2}{x}\right)^5 \\ &= x^5 - 10x^3 + 40x - \frac{80}{x} + \frac{80}{x^3} - \frac{32}{x^5}\end{aligned}$$

2.1 Ex 1B

13 B

1. f) $(2x+5)^3 = (2x)^3 + 3(2x)^2(5) + 3(2x)(5)^2 + (5)^3$

$= 8x^3 + 60x^2 + 150x + 125$

e) $\left(x^2 - \frac{1}{x^2}\right)^3 = (x^2)^3 + 3(x^2)^2\left(-\frac{1}{x^2}\right)^1$

$+ 3(x^2)^1\left(-\frac{1}{x^2}\right)^2 + \left(-\frac{1}{x^2}\right)^3$

$= x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}$

9. b) $(2a+3b)^6$

6	5	4	3
1	6	15	20

$= 20(2a)^3(3b)^{6-3}$

$= 20(2a)^3(3b)^3$

$= 4320a^3b^3$

Ex 1B

1 1 — 1
1 2 1 — 2
1 3 3 1 — 3
1 4 6 4 1 — 4
1 5 10 10 5 1 — 5
1 6 15 20 15 6 1 — 6

6.

a)
$$(1+\sqrt{2})^3 = (1)^3 + 3(1)^2(\sqrt{2}) + 3(1)(\sqrt{2})^2 + (\sqrt{2})^3$$
$$= 1 + 3\sqrt{2} + 6 + 2\sqrt{2}$$
$$= 7 + 5\sqrt{2}$$

c)
$$(2-\sqrt{2})^5 = (2)^5 + 5(2)^4(-\sqrt{2})^1 + 10(2)^3(-\sqrt{2})^2 +$$
$$10(2)^2(-\sqrt{2})^3 + 5(2)^1(-\sqrt{2})^4 + (-\sqrt{2})^5$$
$$= 32 - 80\sqrt{2} + 160 - 80\sqrt{2} + 40 - 4\sqrt{2}$$
$$= 232 - 164\sqrt{2}$$

7.
$$(2+x)^6 = 2^6 + 6(2)^5 x + 15(2)^4 x^2 + 20(2)^3 x^3$$
$$15(2)^2 x^4 + 6(2)^1 x^5 + x^6$$
$$= 64 + 192x + 240x^2 + 160x^3$$
$$+ 60x^4 + 12x^5 + x^6$$

$$(2+0.01)^6 = 64 + 1.92 + 0.0240 + 0.000160$$
$$+ 0.00000060 + 0.000000000012$$
$$+ 0.000000000001$$

64
1.92
0.0240
0.000160
0.00000060
0.0000000012
0.000000000001

65.944160601201

3 Binomial Theorem

Counting:

1. How many ways can the letters AB be organised?:

____

2. How many ways can the letters ABC be organised?:

____

3. How many ways can the letters ABCDE be organised?:

____

1. How many ways can the letters AABBB be organised?:

____

2. How many ways can the letters ABBBB be organised?:

____

If we consider the expansion of $(a+b)^5$
There will be:

- ____________ combinations of $aaaaa$ (a^5)
- ____________ combinations of $aaaab$ (a^4b)
- ____________ combinations of $aaabb$ (a^3b^2)
- ____________ combinations of $aabbb$ (a^2b^3)
- ____________ combinations of $abbbb$ (ab^4)
- ____________ combinations of $bbbbb$ (b^5)

We can see for a string of length **n** using only the letters a and b, the number of possible combinations of a string containing **r** occurrances of a (and consequently $\mathbf{n} - \mathbf{r}$ occurrances of b) will be

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This is a different way of explaining from the book, but the result is the same.
We can now state the binomial theorem that:

$$(a+b)^n = \binom{n}{0}a^n b^0 + \binom{n}{1}a^{n-1}b^1 + ... + \binom{n}{r}a^{n-r}b^r + ... + \binom{n}{n-1}a^1 b^{n-1} + \binom{n}{n}a^0 b^n$$

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + ... + \binom{n}{r}a^{n-r}b^r + ... + \binom{n}{n-1}ab^{n-1} + b^n$$

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r}a^{n-r}b^r \qquad \text{Binomial Theorem}$$

The **general term** or $(r+1)^{th}$ term is:

$$T_{k+1} = \binom{n}{r}a^{n-r}b^r$$

So the first term is: $\quad T_1 = T_{0+1} = \binom{n}{0}a^{n-0}b^0 = a^n$

3.1 Ex 1C

1C

10. $\left(x^2+\frac{1}{ax}\right)^{10} = \sum_{r=0}^{10}\binom{10}{r}(x^2)^{10-r}\cdot\left(\frac{1}{ax}\right)^r$

$= \sum_{r=0}^{10}\binom{10}{r} x^{20-2r}\cdot\frac{1}{a^r x^r}$

$= \sum_{r=0}^{10}\binom{10}{r}\frac{x^{20-3r}}{a^r}$

$x^{11} = x^{20-3r} \Rightarrow 20-3r = 11$

$r = 3$

x^{11} is the fourth term

$T_{3+1} = \binom{10}{3}\frac{x^{11}}{a^3}$

$= \frac{120x^{11}}{a^3}$

$\frac{120}{a^3} = 15$

$\Rightarrow a^3 = \frac{120}{15} = 8$

$a = 2$

11. a) $(x+4)(x-3)^6$

$x.x^3 = x^4$
$4.x^4 = 4x^4$

want x^3 and x^4 terms from $(x-3)^6$

$T_{r+1} = \binom{6}{r}x^{6-r}(-3)^r$

$r=2,\ r=3$

$Ax^4 = xT_{3+1} + 4T_{2+1}$

$= x\binom{6}{3}x^3(-3)^3 + 4\binom{6}{2}x^4(-3)^2$

$= 20\times -27\ x^4 + 4\times15\times9\ x^4$

$= -540x^4 + 540x^4$

$= 0x^4$

12. $(1+kx)^n = 1 - 12x + 60x^2 - \dots, \; n \in \mathbb{Z}^+$

$(1+kx)^n = 1 + \binom{n}{1} kx + \binom{n}{2} k^2x^2 + \dots$

$\binom{n}{1} k = -12$

$\frac{n!}{1!(n-1)!} k = -12$

$nk = -12$

$k = -\frac{12}{n}$

$\binom{n}{2} k^2 = 60$

$\frac{n!}{2!(n-2)!} k^2 = 60$

$\frac{n(n-1)}{2} k^2 = 60$

$n(n-1)k^2 = 120$

$n(n-1)\frac{144}{n^2} = 120$

$k = -\frac{12}{n}$

$144n - 144 = 120n$

$24n = 144$

$n = 6$

and $k = -2$

13.

	Sum
1 1	2
1 2 1	4
1 3 3 1	8
1 4 6 4 1	16
1 5 10 10 5 1	32
1 6 15 20 15 6 1	64

The sum of the numbers in row n of Pascal's Triangle is 2^n

d) $(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} x^r$

$= 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + x^n$

$\Rightarrow (1+1)^n = 1 + \binom{n}{1} + \binom{n}{2} + \dots + 1$

$= 2^n$

$(1+-1)^n = 0^n = 0$

$= 1 - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots (-1)^n$

$= 0$

f) $\sum_{r=0}^{n} \binom{n}{r} 2^r = \sum_{r=0}^{n} 1^{n-r} 2^r = (1+2)^n$

$= 3^n$

14. $$(3+x)^n = 3^n + \binom{n}{1}3^{n-1}x + \binom{n}{2}3^{n-2}x^2 + \binom{n}{3}3^{n-3}x^3 + \dots + \binom{n}{n-1}3^1x^{n-1} + x^n$$

$$(3+1)^n = 3^n + \binom{n}{1}3^{n-1} + \binom{n}{2}3^{n-2} + \binom{n}{3}3^{n-3} + 3n + 1$$

$$= 4^n$$

4 Review Sets

5 Summary formulae

Binomial Theorem	$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$
Combination Definition	$\binom{n}{k} = \frac{n!}{k!(n-k)!}$
First and Last	$\binom{n}{0} = \binom{n}{n} = 1$
A term in the expansion	$T_{k+1} = \binom{n}{k} a^{n-k} b^k$
Pascal's identity	$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
Symmetry	$\binom{n}{k} = \binom{n}{n-k}$

6 Extra questions

1. In the expansion of $(a–3b)^n$, the sum of 9th and 10th term is zero.
   Find the value of $\frac{a}{b}$ in terms of n in terms of n
2. If the coefficient of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in arithmetic sequence, then find the value(s) of n
3. If $x_n - y_n\sqrt{2} = (1-\sqrt{2})^n$, then show that ${x_n}^2 - 2{y_n}^2 = (-1)^n$
4. Find the first three terms in the expansion, in ascending powers of x, of $(1–2x)^5$
5. Given that the coefficient of x^2 in the expansion of $(1+ax+2x^2)(1–2x)^5$ is 12, find the value of the constant a.
6.
   (a) Use the binomial theorem to expand $(a+\sqrt{b})^4$
   (b) Hence, deduce an expression in terms of a and b for $(a+\sqrt{b})^4 + (a-\sqrt{b})^4$
7.
   (a) Write down and simplify the general term in the binomial expansion of $(2x^2 – \frac{d}{x^3})^7$, where d is a constant
   (b) Given that the coefficient of $\frac{1}{x}$ is $-70,000$ find the value of d

1. In the expansion of $(a – 3b)^n$, the sum of 9th and 10th term is zero. Find the value of $\frac{a}{b}$ in terms of n in terms of n

$T_9 = {}^nC_8\, a^{n-8}\, b^8 \quad , \quad T_{10} = {}^nC_9\, a^{n-9}\, b^9$

$T_9 + T_{10} = 0 \quad \Rightarrow \quad T_9 = -T_{10}$

$$\frac{n!}{8!\,(n-8)!}\, a^{n-8}\, b^8 = -\frac{n!}{9!\,(n-9)!}\, a^{n-9}\, b^9$$

$$\Rightarrow \quad \frac{a^{n-8}\, b^8}{a^{n-9}\, b^9} = \frac{-8!\,(n-8)!}{9!\,(n-9)!}$$

${}^nC_n = \frac{n!}{n!\,(n-n)!} = \frac{n!}{n!\,0!} = 1$

${}^nC_0 = \frac{n!}{0!\,(n-0)!} = 1$

$$\Rightarrow \quad \frac{a}{b} = \frac{-8\times7\times\ldots\times1 \times (n-8)(n-9)\ldots\times(n-n)}{9\times8\times\ldots\times1 \times (n-9)\times\ldots\times(n-n)}$$

$$\frac{a}{b} = \frac{-(n-8)}{9}$$

$$= \frac{8-n}{9}$$

2. If the coefficient of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in arithmetic sequence, then find the value(s) of n

$T_6 - T_5 = T_7 - T_6$ Where T_n is the coefficient

$\Rightarrow \quad T_7 - 2T_6 + T_5 = 0$

$T_7 = {}^nC_6 = \frac{n!}{6!\,(n-6)!} \qquad T_6 = {}^nC_5 = \frac{n!}{5!\,(n-5)!} \qquad T_5 = {}^nC_4 = \frac{n!}{4!\,(n-4)!}$

$$\Rightarrow \quad \frac{n!}{6!\,(n-6)!} - \frac{2n!}{5!\,(n-5)!} + \frac{n!}{4!\,(n-4)!} = 0$$

$$\Rightarrow \quad \frac{n!}{6!\,(n-4)!}\left[(n-4)(n-5) - 2.6(n-4) + 6\times5\right] = 0$$

$\Rightarrow \quad (n-4)(n-5) - 12(n-4) + 30 = 0$

$n^2 - 9n + 20 - 12n + 48 + 30 = 0$

$n^2 - 21n + 98 = 0$

$(n-7)(n-14) = 0$

$n = 7 \;,\; n = 14$

3. If $x_n - y_n\sqrt{2} = (1-\sqrt{2})^n$, then show that $x_n{}^2 - 2y_n{}^2 = (-1)^n$

conjugates: $(1-\sqrt{2})(1+\sqrt{2}) = 1-2 = -1$

$\Rightarrow (1-\sqrt{2})^n(1+\sqrt{2})^n = \left[(1-\sqrt{2})(1+\sqrt{2})\right]^n = (-1)^n$

If $x_n - y_n\sqrt{2} = (1-\sqrt{2})^n$

then $(1+\sqrt{2})^n = x_n + y_n\sqrt{2}$

$\Rightarrow (x_n - y_n\sqrt{2})(x_n + y_n\sqrt{2}) = (1-\sqrt{2})^n(1+\sqrt{2})^n$

$\Rightarrow x_n^2 - y_n^2 . 2 = (-1)^n \Rightarrow x_n^2 - 2y_n^2 = (-1)^n$

4. Find the first three terms in the expansion, in ascending powers of x, of $(1–2x)^5$

$$(1-2x)^5 = \sum_{k=0}^{5} {}^5C_k (1)^{5-k}(-2x)^k$$

1 4 6 4 1

1 5 10 10 5 1

$$= \sum_{k=0}^{5} {}^5C_k (-2x)^k$$

$$= 1(-2x)^0 + 5(-2x)^1 + 10(-2x)^2 + 10(-2x)^3 + 5(-2x)^4 + 1(-2x)^5$$

$$= \underline{1 - 10x + 40x^2} - 80x^3 + 80x^4 - 32x^5$$

$$= 1,\ -10x,\ 40x^2$$

5. Given that the coefficient of x^2 in the expansion of $(1 + ax + 2x^2)(1–2x)^5$ is 12, find the value of the constant a.

$(1 + ax + 2x^2)$ $\quad 1\,T_3 + ax\,T_2 + 2x^2\,T_1$ is x^2 term of expansion

0 1 2 3 4

$$T_3 = {}^5C_2 (1)^3(-2x)^2 = 40x^2$$

$$T_2 = {}^5C_1 (1)^4(-2x)^1 = -10x$$

$$T_1 = {}^5C_0 (1)^5(-2x)^0 = 1$$

$$40x^2 + ax(-10x) + 2x^2(1)$$

$$= (42 - 10a)x^2 \qquad 42 - 10a = 12$$

$$a = 3$$

6.

(a) Use the binomial theorem to expand $(a+\sqrt{b})^4$

(b) Hence, deduce an expression in terms of a and b for $(a+\sqrt{b})^4 + (a-\sqrt{b})^4$

$$\overset{\quad 1 \quad\quad 4 \quad\quad 6 \quad\quad 4 \quad\quad 1}{}$$

$$(a+\sqrt{b})^4 = a^4 + 4a^3(\sqrt{b})^1 + 6a^2(\sqrt{b})^2 + 4a(\sqrt{b})^3 + 1(\sqrt{b})^4$$

$$= a^4 + 4a^3\sqrt{b} + 6a^2b + 4ab\sqrt{b} + b^2$$

$$(a-\sqrt{b})^4 = a^4 - 4a^3\sqrt{b} + 6a^2b - 4ab\sqrt{b} + b^2$$

$$(a+\sqrt{b})^4 + (a-\sqrt{b})^4 = 2a^4 + 12a^2b + 2b^2$$

7.

(a) Write down and simplify the general term in the binomial expansion of $(2x^2 - \frac{d}{x^3})^7$, where d is a constant

(b) Given that the coefficient of $\frac{1}{x}$ is $-70,000$ find the value of d

$$\left(2x^2 - \frac{d}{x^3}\right)^7 = \sum_{k=0}^{7} {}^7C_k \, (2x^2)^{7-k} (-dx^{-3})^k$$

$$= \sum_{k=0}^{7} {}^7C_k \, 2^{7-k} x^{14-2k} . (-d)^k x^{-3k}$$

$$= \sum_{k=0}^{7} {}^7C_k \, 2^{7-k} (-d)^k x^{14-5k}$$

$$\frac{1}{x} = x^{-1} \Rightarrow 14 - 5k = -1$$

$$-5k = -15 \Rightarrow k = 3$$

$${}^7C_3 . 2^{7-3} (-d)^3 = -70\,000$$

$$35 . 16 (-d)^3 = -70000$$

$$(-d)^3 = -125$$

$$-d = -5$$

$$d = 5$$