

## Properties of curves and applications of differentiation

Kh notes

# Contents

<b>1 Properties of curves</b>	<b>2</b>
1.1 Review Questions . . . . .	2
1.1.1 Some fundamental derivatives: . . . . .	2
1.1.2 Rules of differentiation: . . . . .	2
1.2 Tangents . . . . .	4
1.2.1 Horizontal Tangents . . . . .	5
1.2.2 Ex13A . . . . .	6
1.3 Normals . . . . .	15
1.3.1 Ex 13B . . . . .	15
1.4 Increasing and Decreasing . . . . .	20
1.4.1 Basic sign diagram example . . . . .	20
1.4.2 Sign diagram example with discontinuities . . . . .	22
1.4.3 Ex 13 C . . . . .	22
1.5 Stationary Points . . . . .	27
1.5.1 Turning points (minima, maxima) . . . . .	27
1.5.2 Stationary points of inflection . . . . .	28
1.5.3 Greatest or least values (testing boundaries) . . . . .	29
1.5.4 Ex 13D . . . . .	30
1.6 Shape . . . . .	37
1.6.1 Ex 13E . . . . .	37
1.7 Inflection Points . . . . .	40
1.7.1 Ex 13F . . . . .	40
1.8 Understanding functions and their derivatives . . . . .	45
<b>2 Applications of differentiation</b>	<b>46</b>

# Chapter: 1 Properties of curves

## 1.1 Review Questions

### 1.1.1 Some fundamental derivatives:

Function	Derivative
$f(x) = x^n$	$f'(x) = nx^{n-1} \quad (n \in \mathbb{R})$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \ln x$	$f'(x) = \frac{1}{x}$
$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$
$f(x) = \sin x$	$f'(x) = \cos x$
$f(x) = \cos x$	$f'(x) = -\sin x$
$f(x) = \tan x$	$f'(x) = \sec^2 x$

### 1.1.2 Rules of differentiation:

Chain Rule:

$$y = g(u_{(x)})$$

$$\frac{dy}{dx} = g'(u_{(x)})u'_{(x)}$$

Product Rule:

$$y = u_{(x)}v_{(x)}$$

$$\frac{dy}{dx} = u_{(x)}v'_{(x)} + u'_{(x)}v_{(x)}$$

Quotient Rule:

$$y = \frac{u_{(x)}}{v_{(x)}}$$

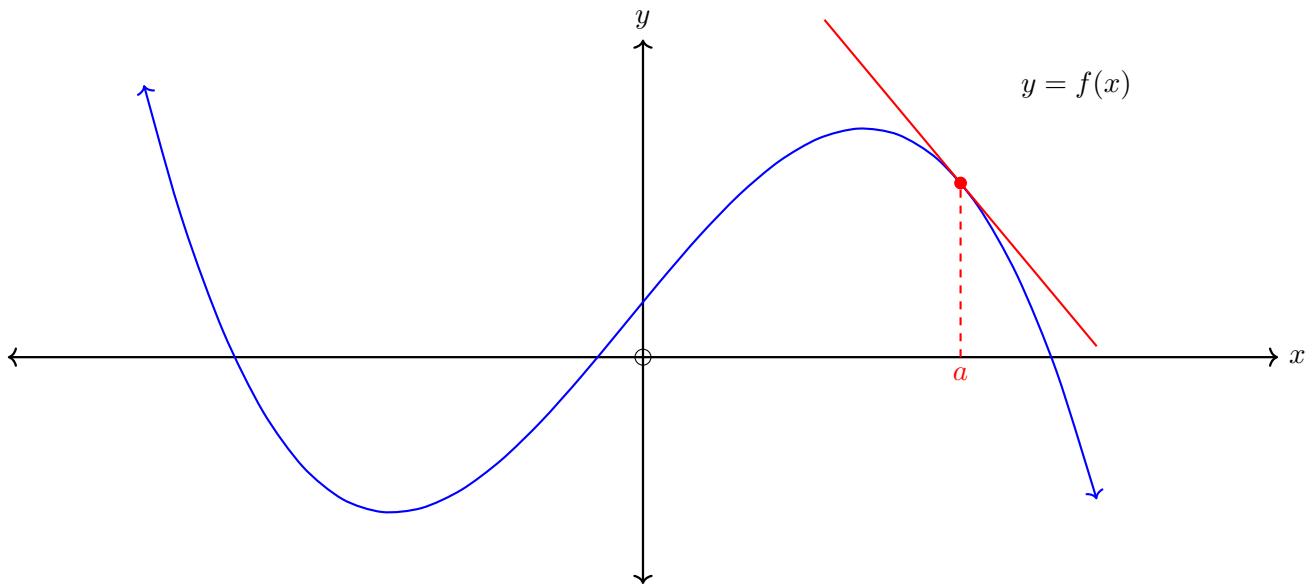
$$\frac{dy}{dx} = \frac{u'_{(x)}v_{(x)} - u_{(x)}v'_{(x)}}{[v_{(x)}]^2}$$

## 1.2 Tangents

- The tangent to a curve at a point A is the best approximating straight line to the curve at point A.
- (Leibniz definition) Tangent to the curve  $y = f(x)$  at the point  $(a, f(a))$  is the line through the infinitely close pair of points either side of  $f(a)$

$$\frac{y - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}$$

- It is a single point of contact with the curve (although it may intersect the curve at some other point)



For the function  $y = f(x)$ , and some  $x = a$

- $(a, f(a))$  is on the curve
- $f'(a)$  is the gradient of the curve at  $x = a$

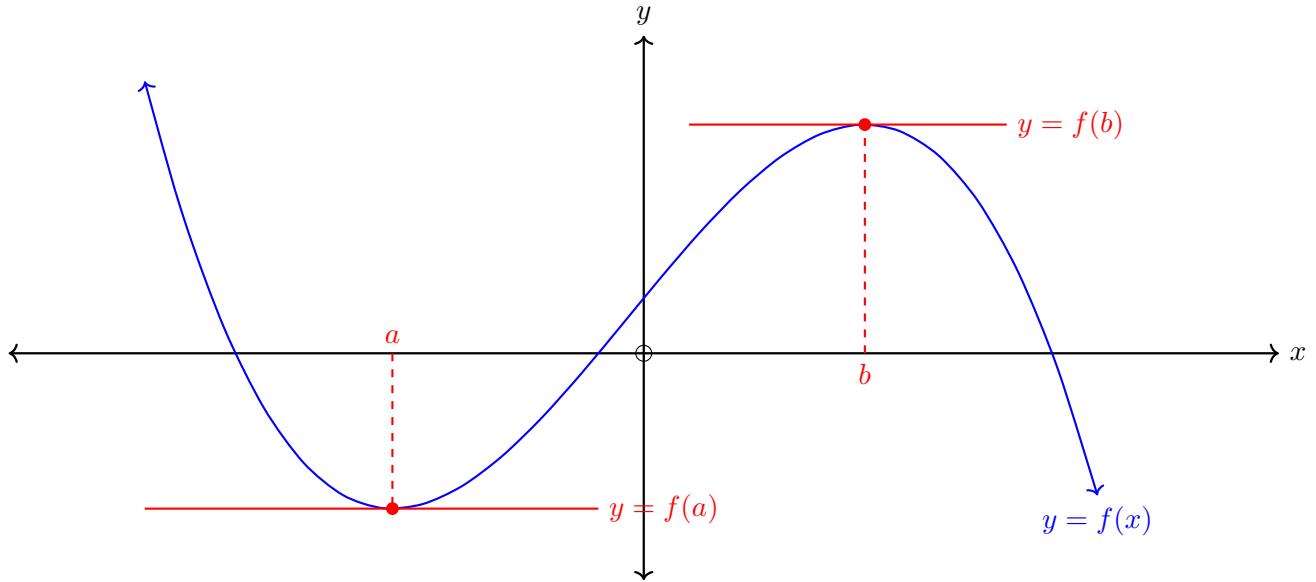
$$\begin{aligned} \frac{y - f(a)}{x - a} &= f'(a) \\ \Rightarrow y &= f'(a)(x - a) + f(a) \end{aligned}$$

Is the equation of the tangent line

### 1.2.1 Horizontal Tangents

Horizontal tangents have a gradient of 0.

These will be very important later, when we investigate stationary points in more detail.



In general, we will need to find where the stationary points are:

---

- Find  $f'(x)$
  - Solve  $f'(x) = 0$
  - Substitute the solution(s) into  $f(x)$  to find the constant terms for the horizontal line equation
-

### 1.2.2 Ex13A

- Tangent equations from root and polynomial functions
- Horizontal tangents
- Natural log and exponent questions
- Concept questions

### 3 worked examples

- Polynomial
- Log or exponential
- Conceptual

Find the equation of the tangent to:

a)  $y = x - 2x^2 + 3$  at  $x=2$

$$\frac{dy}{dx} = 1 - 4x$$

when  $x=2$

$$\frac{dy}{dx} = 1 - 4(2) = -7$$

and  $y = (2) - 2(2)^2 + 3$   
 $= -3$

$$\frac{dy}{dx} = -7 \quad \text{Point } (2, -3)$$

$$y - (-3) = -7(x - 2)$$

$$y + 3 = -7x + 14$$

$$y = -7x + 11$$

d)  $y = \frac{4}{\sqrt{x}}$  at  $(1, 4)$

$$y = 4x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 4 \cdot -\frac{1}{2}x^{-\frac{3}{2}}$$

$$= -2x^{-\frac{3}{2}} = \frac{-2}{x^{\frac{3}{2}}}$$

$x = 1$

$$\frac{dy}{dx} = \frac{-2}{1} = -2$$

Point  $(1, 4)$

$$y - 4 = -2(x - 1)$$

$$y - 4 = -2x + 2$$

$$y = -2x + 6$$

(Horizontal Tangents)

3. c)

Find the equation of horizontal tangents to

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\Rightarrow y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + -\frac{1}{2}x^{-\frac{3}{2}}$$

(factorise using lowest power)

$$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{2}}(x^{\frac{1}{2}} - 1)$$

Set  $\frac{dy}{dx} = 0$

$$\frac{1}{2}x^{-\frac{3}{2}}(x^{\frac{1}{2}} - 1) = 0$$

$$x^{-\frac{3}{2}} \neq 0 \quad x^{\frac{1}{2}} - 1 = 0$$

Find  $y$

$$y = \sqrt{1} + \frac{1}{\sqrt{1}} = 1 + 1$$

$$= 2$$

Horizontal tangent at  $(1, 2)$

Equation:  $y = 2$

$$\frac{dy}{dx} = \frac{1}{2x^{\frac{1}{2}}} - \frac{1}{2x^{\frac{3}{2}}}$$

$$= \frac{x - 1}{2x^{\frac{3}{2}}}$$

$$\frac{x - 1}{2x^{\frac{3}{2}}} = 0 \Rightarrow x - 1 = 0$$

5. Find another tangent to

$$y = 1 - 3x + 12x^2 - 8x^3$$

which is parallel to the tangent at  $(1, 2)$

$$\frac{dy}{dx} = -3 + 24x - 24x^2$$

when  $x=1$

$$\begin{aligned}\frac{dy}{dx} &= -3 + 24(1) - 24(1)^2 \\ &= -3\end{aligned}$$

so gradient is  $-3$

Find all  $x$  values where gradient  $= -3$

$$-3 + 24x - 24x^2 = -3$$

$$24x - 24x^2 = 0$$

$$x - x^2 = 0$$

$$x(1-x) = 0$$

$$x=0 \quad \underbrace{x=1}_{a} \quad ) \text{ already have}$$

Find equation of tangent at  $x=0$

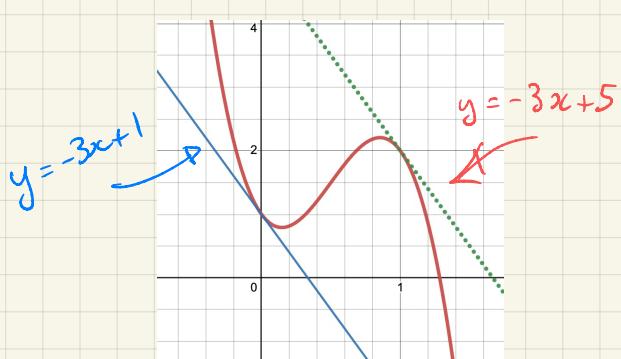
$$y = 1$$

$$y - 1 = -3(x - 0)$$

$$y - 1 = -3x, \quad y = -3x + 1$$

other tangent  $y - 2 = -3(x - 1)$

$$y = -3x + 5$$



6.  $y = x^2 + ax + b$

when  $x=1$  tangent

$$is 2x+y=6$$

$$y = -2x + 6$$

$$\frac{dy}{dx} = -2 \quad y = 4$$

$(1, 4)$  point on curve.

$-2$  gradient at that point.

$$\frac{dy}{dx} = 2x + a$$

$$-2 = 2(1) + a$$

$$a = -4$$

$$y = x^2 - 4x + b$$

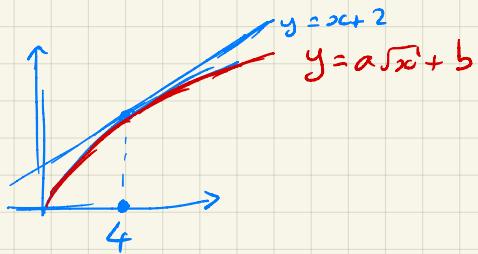
$$4 = (1)^2 - 4(1) + b$$

$$4 = -3 + b$$

$$b = 7.$$

$$\underline{y = x^2 - 4x + 7}$$

7. Find  $a$  and  $b$



What do we know?

- when  $x=4 \frac{dy}{dx} = 1$
- $(4, 4+2) = (4, 6)$  is on  $y = a\sqrt{x} + b$

① Using  $(4, 6)$

$$\begin{aligned} 6 &= a\sqrt{4} + b(4) \\ 6 &= 2a + 4b \\ \Rightarrow 3 &= a + 2b \end{aligned}$$

$$② \frac{dy}{dx} = \frac{a}{2\sqrt{x}} + b$$

$$1 = \frac{a}{2\sqrt{4}} + b$$

$$1 = \frac{a}{4} + b$$

$$4 = a + 4b$$

Equation

$$\begin{aligned} 3 &= a + 2b & ① \\ 4 &= a + 4b & ② \end{aligned}$$

$$② - ① \quad 1 = 2b$$

$$b = \frac{1}{2}$$

$$\begin{aligned} \text{using } ① \quad a &= 3 - 2b \\ &= 3 - 2\left(\frac{1}{2}\right) \\ &= 2 \end{aligned}$$

$$y = 2\sqrt{x} + \frac{x}{2}$$

8. Show equation of

tangent to  $y = 2x^2 - 1$

at  $x=a$  is

$$4ax - y = 2a^2 + 1$$

Find point:

$$x = a$$

$$y = 2a^2 - 1$$

$$(a, 2a^2 - 1)$$

$$\frac{dy}{dx} = 4x$$

$$= 4a$$

$$y - (2a^2 - 1) = 4a(x - a)$$

$$y - 2a^2 + 1 = 4ax - 4a^2$$

$$(-y) \quad -2a^2 + 1 = 4ax - 4a^2 - y$$

$$(+4a^2) \quad 2a^2 + 1 = 4ax - y$$

$$10) \frac{dy}{dx} = \ln \sqrt{x}$$

$$y = \ln(x^{\frac{1}{2}})$$

$$y = \frac{1}{2} \ln(x)$$

$$\frac{dy}{dx} = \frac{1}{2x}$$

$$y = -1$$

$$\frac{1}{2} \ln(x) = -1$$

$$\ln(x) = 2$$

$$x = e^2$$

$$(e^{-2}, -1)$$

$$\frac{dy}{dx} = \frac{1}{2e^{-2}}$$

$$y + 1 = \frac{1}{2e^{-2}}(x - e^{-2})$$

$$y = \frac{x}{2e^{-2}} - \frac{1}{2} - 1$$

$$y = \frac{e^2}{2}x - \frac{3}{2}$$

Using  $\ln x$  and  $e^x$

Find the equation of the tangent to  $y = \ln x$  where  $y = -1$

• Find  $x$

$$\ln x = -1 \quad (\log_e x = 1)$$

$$x = e^{-1}$$

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}$$

$$x = e^{-1}, \quad \frac{dy}{dx} = (e^{-1})^{-1} = e^1 = e$$

Point  $(e^{-1}, -1)$

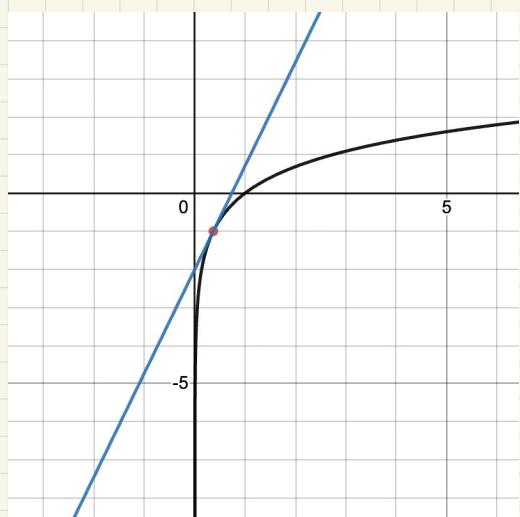
$$\frac{dy}{dx} = e$$

$$y + 1 = e(x - e^{-1})$$

$$y + 1 = ex - e^1 e^{-1}$$

$$y + 1 = ex - 1$$

$$y = ex - 2$$



$$12. f(x) = \ln(x(x-2))$$

Domain of log function  $> 0$

$$\Rightarrow x(x-2) > 0$$

$$\text{solve } x(x-2)=0$$

$$\begin{array}{c} x=0 \\ x=2 \end{array}$$

Domain  $x < 0 \cup x > 2$

$$\{x | x < 0 \cup x > 2\}$$

$$f'(x) = \frac{1}{x(x-2)} \cdot \frac{d}{dx}(x^2 - 2x)$$

$$= \frac{1}{x(x-2)} \cdot 2x - 2$$

$$= \frac{2(x-1)}{x(x-2)}$$

$$= \frac{2x-2}{x^2-2x}$$

$$= \frac{2x-2}{x(x-2)}$$

$$= \frac{2x+x-2}{x(x-2)}$$

$$= \frac{1}{x} + \frac{1}{x-2}$$

$$x=3, f'(x) = \frac{1}{3} + \frac{1}{1} = \frac{4}{3}$$

$$x=3, f(3) = \ln(3(3-2)) = \ln 3$$

$$y - \ln 3 = \frac{4}{3}(x-3)$$

$$y = \frac{4}{3}x - 4 + \ln 3$$

$$13. y = x^2 e^x$$

$$y = uv \quad \frac{dy}{dx} = uv' + u'v$$

(product rule)

$$\begin{aligned} \frac{dy}{dx} &= x^2 e^x + 2x e^x \\ &= x e^x (x+2) \end{aligned}$$

$$x=1$$

$$\frac{dy}{dx} = e(3) = 3e$$

$$x=1, y=e$$

$$\text{Gradient: } 3e$$

$$\text{Point: } (1, e)$$

$$y - e = 3e(x-1)$$

$$y = 3ex - 2e \quad \text{equation of tangent}$$

Find intercepts.

$$x=0 \quad y = -2e$$

$$y=0 \quad x = \frac{2e}{3e}$$

$$x = \frac{2}{3}$$

Intercepts

$$(0, -2e) \quad y\text{-intercept}$$

$$\left(\frac{2}{3}, 0\right) \quad x\text{-intercept}$$

$$14. \quad y = 3xe^{\frac{x}{2}}$$

$$\begin{aligned} \frac{dy}{dx} &= 3x(e^{\frac{x}{2}})' + (3x)' e^{\frac{x}{2}} \\ &= 3x e^{\frac{x}{2}} \cdot \frac{1}{2} + 3e^{\frac{x}{2}} \\ &= 3e^{\frac{x}{2}} \left( \frac{x}{2} + 1 \right) \end{aligned}$$

When  $x = -1$

$$\begin{aligned} \frac{dy}{dx} &= 3e^{-\frac{1}{2}} \left( -\frac{1}{2} + 1 \right) \\ &= \frac{3}{2} e^{-\frac{1}{2}} \end{aligned}$$

$$x = -1, \quad y = -3e^{-\frac{1}{2}}$$

$$y + 3e^{-\frac{1}{2}} = \frac{3e^{-\frac{1}{2}}}{2}(x + 1)$$

$$\begin{aligned} y &= \frac{3e^{-\frac{1}{2}}}{2}x + \frac{3e^{-\frac{1}{2}}}{2} - 3e^{-\frac{1}{2}} \\ &= \frac{3e^{-\frac{1}{2}}}{2}x - \frac{3e^{-\frac{1}{2}}}{2} \end{aligned}$$

$$\text{if } x = 0 \quad y = -\frac{3e^{-\frac{1}{2}}}{2}$$

$$\text{if } y = 0 \quad x = 1$$

$$\frac{3e^{-\frac{1}{2}}}{2} \sqrt{\frac{1}{A}} = \frac{3e^{-\frac{1}{2}}}{4}$$

$$A = \frac{3}{4\sqrt{e}} \text{ units}^2$$

15. Find the equation of the tangent to  $y = \frac{1}{\sin 2x}$  at the point where

$$x = \frac{\pi}{4}$$

$$u = 1, u' = 0 \quad v = \sin 2x, v' = 2\cos(2x) \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\begin{aligned} \frac{dy}{dx} &= -\frac{1 \cdot 2\cos(2x)}{\sin^2(2x)} = -\frac{2\cos(2x)}{\sin^2(2x)} \\ &= -2\cot(2x)\csc(x) \end{aligned}$$

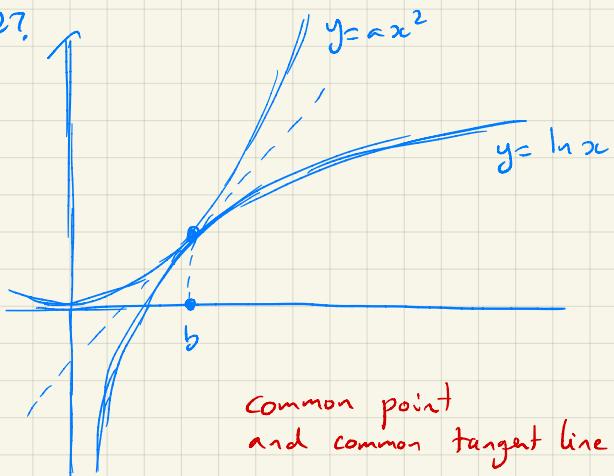
$$\text{when } x = \frac{\pi}{4} \quad \frac{dy}{dx} = -2\cot\left(\frac{\pi}{2}\right) = 0$$

$$\text{when } x = \frac{\pi}{4}, \quad y = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$$

Horizontal tangent

$$y = 1$$

27.



$$\text{Common Points } (b, ab^2)$$

$$\text{a) } (b, \ln b)$$

$$\Rightarrow ab^2 = \ln b$$

Common Tangent

$$\frac{dy}{dx} = 2ax \quad , \quad \frac{dy}{dx} = \frac{1}{x}$$

$$x=b \quad 2ab = \frac{1}{b} \Rightarrow 2ab^2 = 1$$

$$\text{b) } \begin{aligned} ab^2 &= \ln b & \textcircled{1} \\ 2ab^2 &= 1 & \textcircled{2} \end{aligned}$$

$$\begin{aligned} 2ab^2 &= 2\ln b \\ 2ab^2 &= 1 \end{aligned}$$

$$\begin{aligned} 2\ln b &= 1 \\ \ln b &= \frac{1}{2} \end{aligned}$$

$$b = e^{\frac{1}{2}} = \sqrt{e}$$

$$\text{Point } (e^{\frac{1}{2}}, \ln e^{\frac{1}{2}}) = (e^{\frac{1}{2}}, \frac{1}{2})$$

$$\text{c) } a = \frac{1}{2b^2}$$

$$a = \frac{1}{2e}$$

$$\text{d) } a = \frac{1}{2e}, \quad b = \sqrt{e}$$

$$\text{Point } (e^{\frac{1}{2}}, \frac{1}{2})$$

$$\text{Gradient } \frac{dy}{dx} = \frac{1}{b} = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$$

$$y - \frac{1}{2} = e^{-\frac{1}{2}}(x - e^{\frac{1}{2}})$$

$$y - \frac{1}{2} = e^{-\frac{1}{2}}x - 1$$

$$y = 2\sqrt{e}x - \frac{1}{2}$$

$$28. \quad P(x) = ax^2$$

$$P'(x) = 2ax$$

$$x=s \quad (s, as^2) \quad P'(s) = 2as$$

$$y - as^2 = 2as(x - s)$$

$$\begin{aligned} y &= 2asx - 2as^2 + as^2 \\ y &= 2asx - as^2 \end{aligned}$$

$$\begin{aligned} x=t \\ y &= 2atx - at^2 \end{aligned}$$

$$2asx - as^2 = 2atx - at^2$$

$$x(2as - 2at) = as^2 - at^2$$

$$x = \frac{a(s^2 - t^2)}{2a(s-t)}$$

$$= \frac{(s+t)(s-t)}{2(s-t)}$$

$$= \frac{s+t}{2}$$

If perpendicular

$$2as \cdot 2at = -1$$

$$y = 2atx - at^2$$

$$x_c = \frac{s+t}{2}$$

$$y = \frac{2at(s+t)}{2} - at^2$$

$$= ast + at^2 - at^2$$

$$= ast$$

$$t = \frac{-1}{4a^2s}$$

$$y = ast$$

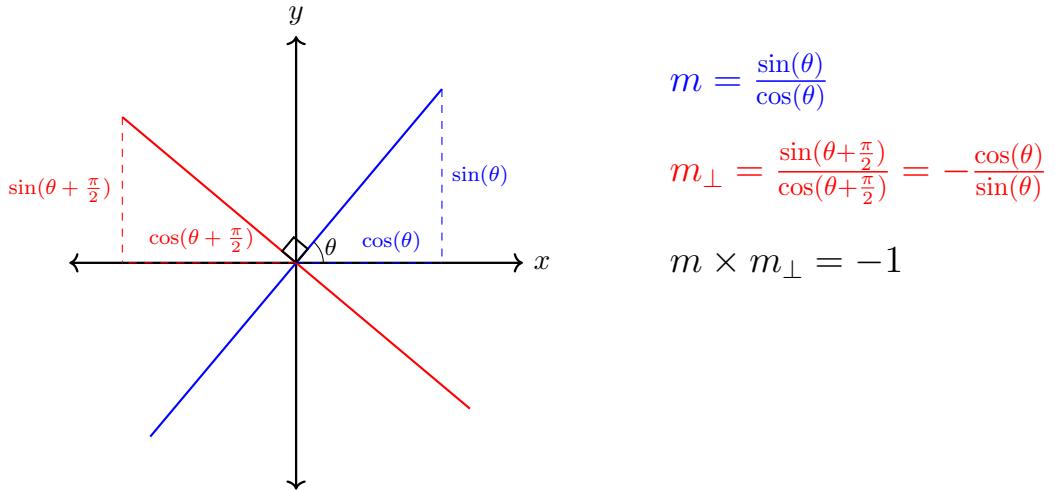
$$= as \cdot \frac{-1}{4a^2s}$$

$$= -\frac{1}{4a}$$

## 1.3 Normals

The product of the gradients of perpendicular lines = -1

There are various proofs, one example below.



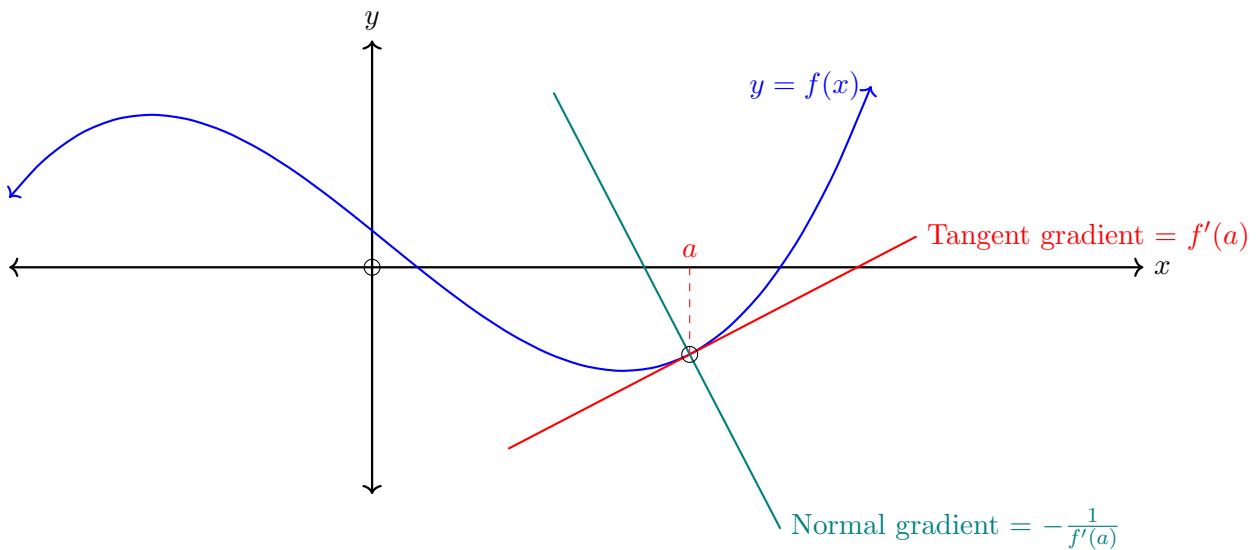
For the function  $y = f(x)$ , and some  $x = a$

- $(a, f(a))$  is on the curve
- $-\frac{1}{f'(a)}$  is the gradient of the **normal to the curve** at  $x = a$

$$\frac{y - f(a)}{x - a} = -\frac{1}{f'(a)}$$

$$\Rightarrow y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Is the equation of the tangent line



### 1.3.1 Ex 13B

13B

1. d)  
Find the equation of the normal to:

$$y = 8\sqrt{x} - \frac{1}{x^2} \text{ at } x=1$$

Point  $x=1$

$$y = 8\sqrt{1} - \frac{1}{1^2} = 7 \quad (1, 7)$$

Gradient:

$$\begin{aligned} y &= 8x^{\frac{1}{2}} - x^{-2} \\ \frac{dy}{dx} &= 8 \cdot \frac{1}{2} x^{-\frac{1}{2}} - -2x^{-3} \\ &= \frac{4}{x^{\frac{1}{2}}} + \frac{2}{x^3} \\ &= \frac{4x^{\frac{5}{2}} + 2}{x^3} \\ \frac{dy}{dx} \perp &= \frac{-x^3}{4x^{\frac{5}{2}} + 2} \end{aligned}$$

$$x=1$$

$$\frac{dy}{dx} \perp = \frac{-1}{4+2} = -\frac{1}{6}$$

$$(1, 7) \text{ and } -\frac{1}{6}$$

$$y-7 = -\frac{1}{6}(x-1)$$

$$y-7 = -\frac{x}{6} + \frac{1}{6}$$

$$y = -\frac{x}{6} + \frac{43}{6}$$

or

$$6y + x = 43$$

$$\text{or } x + 6y - 43 = 0 \quad \circ$$

3. Find the equation of  
c) the normal to  
 $y = e^{2x-1}$  at  $x=1$

Point  $(1, e^1)$

$$\frac{dy}{dx} = e^{2x-1} \cdot 2$$

$$= 2e^{2x-1}$$

$$\frac{dy}{dx} \perp = \frac{-1}{2e^{2x-1}} = -\frac{e^{1-2x}}{2}$$

$$x=1$$

$$\frac{dy}{dx} \perp = -\frac{1}{2e}$$

$$y - e = -\frac{1}{2e}(x-1)$$

$$y = -\frac{x}{2e} + \frac{1}{2e} + e$$

$$y = -\frac{xe^{-1}}{2} + \frac{1+2e^2}{2e}$$

or

$$2ey - 2e^2 = -x + 1$$

$$2ey + x = 1 + 2e^2$$

$$x + 2ey = 1 + 2e^2$$

5. Find the points where the normal to

$$y = x^3 - 2x^2 + 1$$

at  $x=1$  meets the curve again.

Find equation of normal at  $x=1$  and then set equal to  $y$ , find intersection points.

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{dy}{dx} \perp = \frac{-1}{3x^2 - 4x}$$

$$x=1 \quad \frac{dy}{dx} \perp = \frac{-1}{3-4} = 1$$

$$x=1, \quad y = 1 - 2 + 1 = 0$$

$$y - 0 = 1(x-1)$$

$$y = x-1 \quad (\text{normal})$$

$$\text{set } x^3 - 2x^2 + 1 = x-1$$

$$\Rightarrow x^3 - 2x^2 - x + 2 = 0$$

We already have one solution

$$x=1 \Rightarrow x-1 \text{ is a}$$

factor

$$(x-1)(x^2 - x - 2) = 0$$

From deduce

$$\begin{array}{r} x^2 - x - 2 \\ \hline x-1 ) x^3 - 2x^2 - x + 2 \\ x^2 - x^2 \\ \hline -x^2 - x \\ -x^2 + x \\ \hline -2x + 2 \\ -2x + 2 \\ \hline 0 \end{array}$$

$$(x-1)(x^2 - x - 2)^0$$

$$\Rightarrow (x-1)(x-2)(x+1) = 0$$

Solution  $x=1, x=2, x=-1$

so new points

$$x=2 \quad y = (2)^3 - 2(2)^2 + 1$$

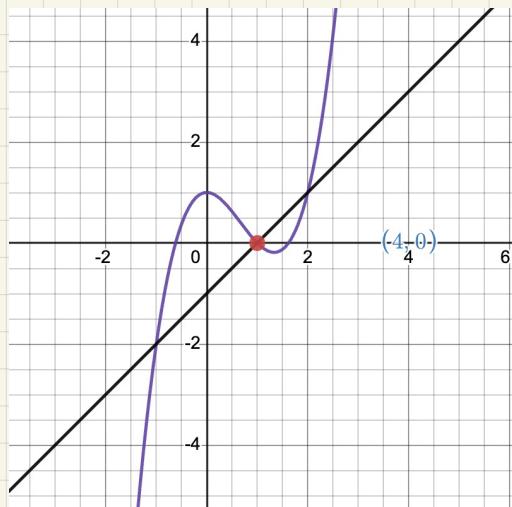
$$= 1 \quad (2, 1)$$

$$x = -1$$

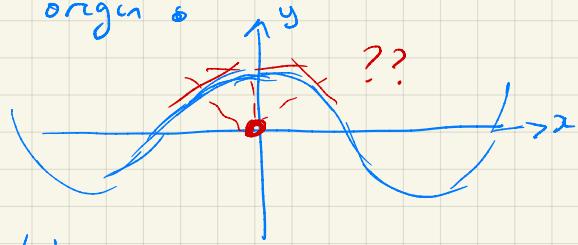
$$y = (-1)^3 - 2(-1)^2 + 1$$

$$= -2 \quad (-1, -2)$$

points are  $(2, 1), (-1, -2)$



6. Find the equation of the normal to  $f(x) = \cos(x)$  which passes through the origin.



let  $x = a$   
 $\Rightarrow (a, \cos(a))$  on the curve

$$\frac{dy}{dx} = -\sin(x)$$

$$\frac{dy}{dx} \perp = \frac{1}{\sin(x)}$$

$$\frac{dy}{dx} \perp = \frac{1}{\sin(a)}$$

$$\frac{\cos(a) - 0}{a - 0} = \frac{1}{\sin(a)}$$

$$\begin{aligned}\sin(a)\cos(a) &= a \\ 2\sin(a)\cos(a) &= 2a\end{aligned}$$

$$\sin(2a) = 2a$$

Solve  $\sin(x) = x$  ?

We know  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$

$$\begin{aligned}\Rightarrow \sin(x) &= x \\ \text{when } x &= 0 \\ a &= 0\end{aligned}$$

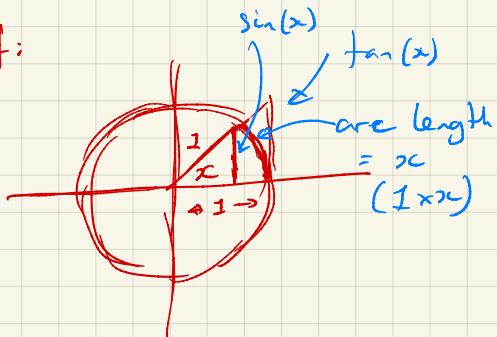
Point  $(0, 1)$

Normal line is vertical

$$\Rightarrow x = 0$$

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

Proof:



$$\sin(x) < x < \tan(x)$$

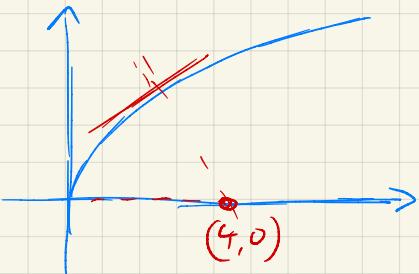
$$\Rightarrow \frac{1}{\sin(x)} > \frac{1}{x} > \frac{\cos(x)}{\sin(x)}$$

$$1 > \frac{\sin(x)}{x} > \cos(x)$$

$$\lim_{x \rightarrow 0} \Rightarrow 1 > \frac{\sin(x)}{x} > 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1$$

7. Find the equation to  $y = \sqrt{x}$   
from the external point  $(4, 0)$



$$y = \sqrt{x}, \frac{dy}{dx} = \frac{1}{2\sqrt{x}}, \frac{dy}{dx} \perp = -2\sqrt{x}$$

let the co ordinate on the  
curve be  $(a, \sqrt{a})$

$$\text{so } \frac{dy}{dx} \perp = -2\sqrt{a}$$

$$\text{and } \frac{\sqrt{a} - 0}{a - 4} = -2\sqrt{a}$$

$$\sqrt{a} = -2\sqrt{a}(a-4)$$

$$\sqrt{a} = -2a\sqrt{a} + 8\sqrt{a}$$

$$2a\sqrt{a} - 7\sqrt{a} = 0$$

$$\sqrt{a}(2a-7) = 0$$

$$\sqrt{a} = 0 \quad 2a-7 = 0$$

$$\xrightarrow{\text{reject}} \quad a = \frac{7}{2}$$

reject, because  
derivative is not  
defined at the  
end point of a  
curve

$$a = \frac{7}{2}$$

$$\text{Point } \left(\frac{7}{2}, \sqrt{\frac{7}{2}}\right)$$

$$\frac{dy}{dx} \perp = -2\sqrt{\frac{7}{2}}$$

$$y - \sqrt{\frac{7}{2}} = -2\sqrt{\frac{7}{2}}(x - \frac{7}{2})$$

$$y = -2\sqrt{\frac{7}{2}}x + 7\sqrt{\frac{7}{2}} + \sqrt{\frac{7}{2}}$$

$$y = -2\sqrt{\frac{7}{2}}x + 8\sqrt{\frac{7}{2}}$$

$$y = -\sqrt{\frac{4 \times 7}{2}}x + \sqrt{\frac{7 \times 64}{2}}$$

$$= -\sqrt{14}x + \sqrt{7 \times 32}$$

$$= -\sqrt{14}x + \sqrt{16 \times 7 \times 2}$$

$$= -\sqrt{14}x + 4\sqrt{14}$$

## 1.4 Increasing and Decreasing

When analysing functions, we are often interested in the intervals across which the function is increasing or decreasing.

- If the gradient is entirely positive on an interval, the function is increasing on that interval.
- If the gradient is entirely negative on an interval, the function is decreasing on that interval.

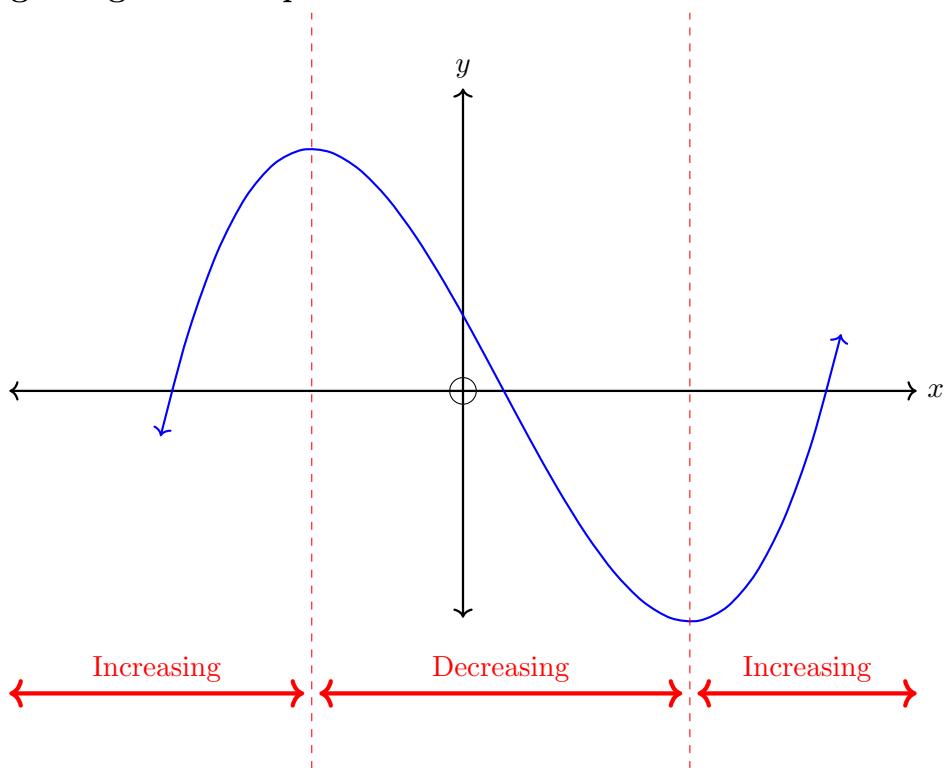
If a function changes from increasing to decreasing, it must pass through a stationary point (or cross a region where the function is not differentiable)

Hence we need to divide up the domain into intervals bounded by stationary points and points where the function is not differentiable.

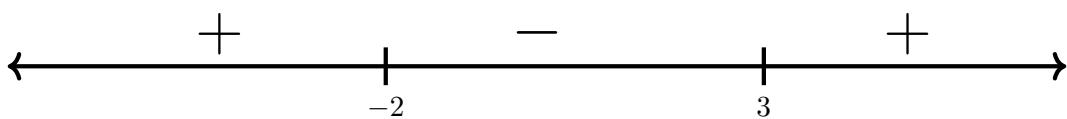
Finding a single gradient on each interval will tell us whether the function is increasing or decreasing.

We can communicate all of this using a **sign diagram**.

### 1.4.1 Basic sign diagram example



Sign diagram:  $f'(x)$  either calculated or observed from a graph



## Worked Example:

Find the intervals where  $f(x)$  is increasing, for the function:  $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$f'(x) = -12x^2 + 30x + 18$$

set  $f'(x) = 0$  (to find stationary points)

$$-12x^2 + 30x + 18 = 0$$

$$-6(2x+1)(x-3) = 0$$

$$x = -\frac{1}{2} \quad x = 3$$

$$\begin{aligned} -6(2x^2 - 5x - 3) &= 0 \\ -6[2x^2 - 6x + x - 3] &= 0 \\ -6[2x(x-3) + (x-3)] &= 0 \end{aligned}$$



Test any  
point in  
each interval  
(choose easy  
points)

Test  $x = -1$

$$\begin{aligned} f'(-1) &= -12(-1)^2 + 30(-1) + 18 \\ &= -24 \end{aligned}$$

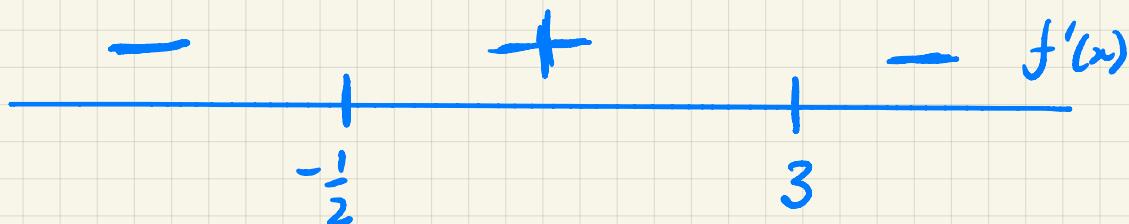
Test  $x = 0$

$$f'(0) = +18$$

Test  $x = 4$

$$\begin{aligned} f'(4) &= -12(4)^2 + 30(4) + 18 \\ &= -54 \end{aligned}$$

Completed Diagram



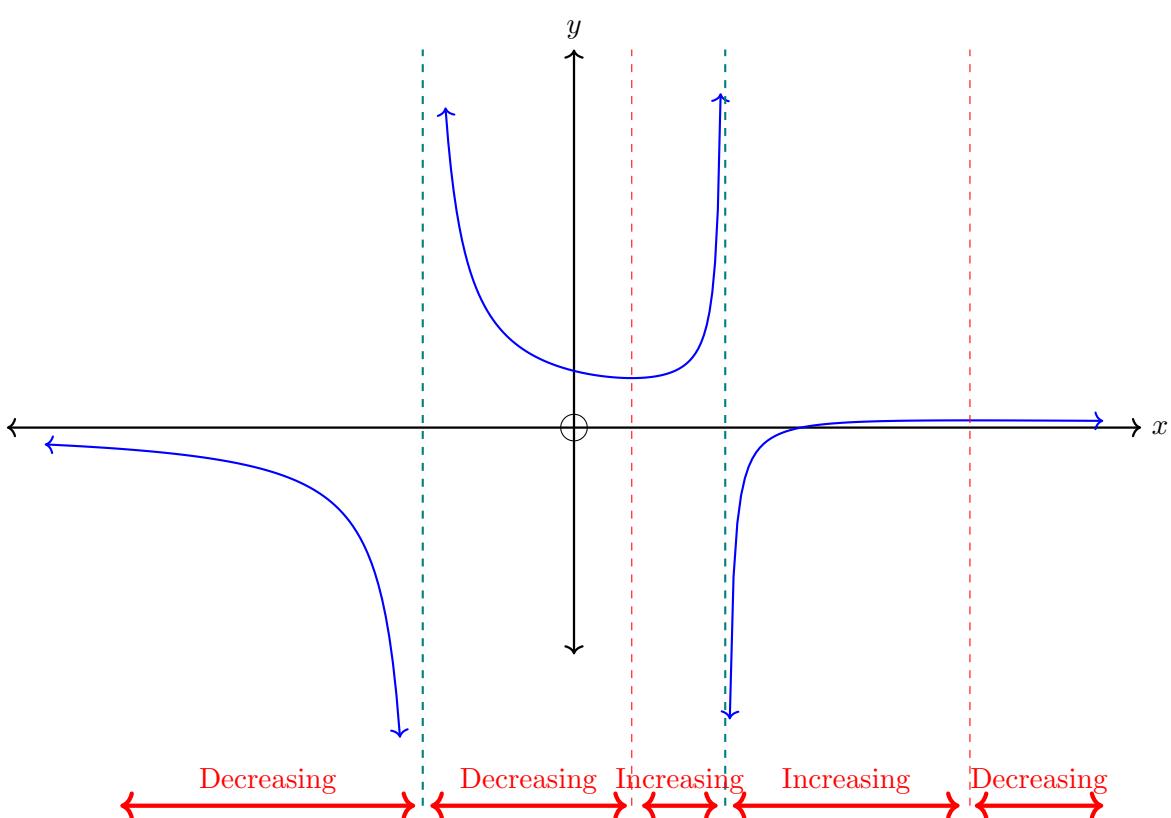
### 1.4.2 Sign diagram example with discontinuities

Below is the graph of

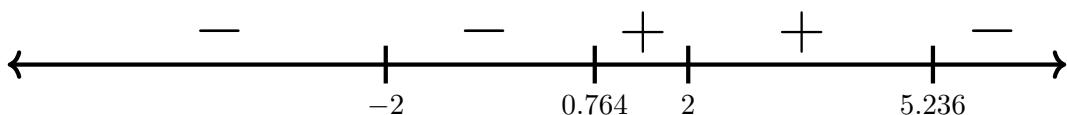
$$y = \frac{x-3}{x^2-4}$$

Because division by 0 is undefined, the graph has vertical asymptotes when  $x^2 - 4 = 0$   
(i.e when  $x = 2$  and  $x = -2$  )

The derivative is  $\frac{dy}{dx} = \frac{-x^2 + 6x - 4}{(x^2 - 4)^2}$  and using a graphics calculator to solve  $-x^2 + 6x - 4 = 0$ , we have stationary points at  $x \approx 0.764$  and  $x \approx 5.236$



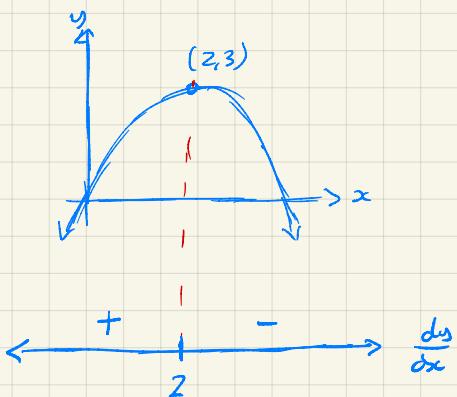
**Sign diagram:**  $f'(x)$  either calculated or observed from a graph



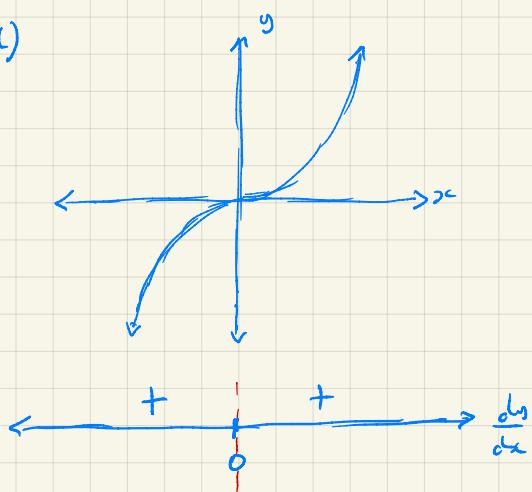
### 1.4.3 Ex 13 C

13c

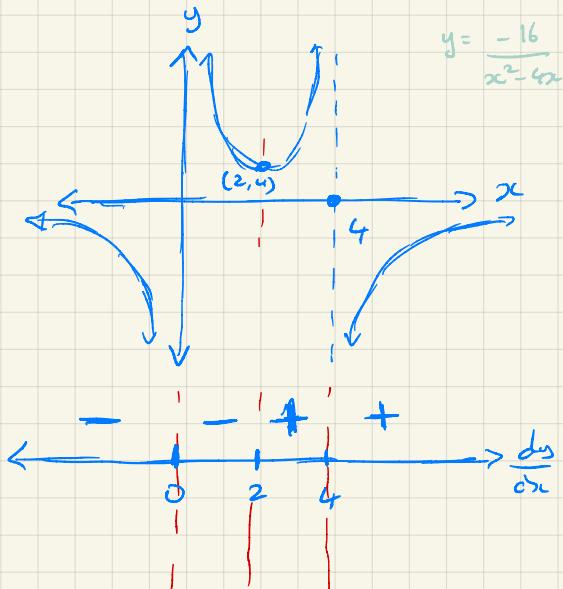
1. c)



d)



f)



$$e. f(x) = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$$

$$f'(x) = 2 - \frac{1}{2}x^{-\frac{3}{2}}$$

$$= -\frac{1}{x^{3/2}} = -\frac{1}{x\sqrt{x}}$$

Domain:  $\{x | x > 0\}$

$\text{if } x < 0 \quad \sqrt{x} \text{ undefined}$   
 $\text{and } \frac{1}{0} \text{ is undefined.}$

$$\text{Test: } f'(1) = -1$$

$$\begin{array}{c} \ominus \\ \oplus \end{array} \longrightarrow f'(x)$$

Function Decreasing on  $(0, \infty)$

$$5. f(x) = x^3 - 3x^2 + 5x + 2$$

$$f'(x) = 3x^2 - 6x + 5$$

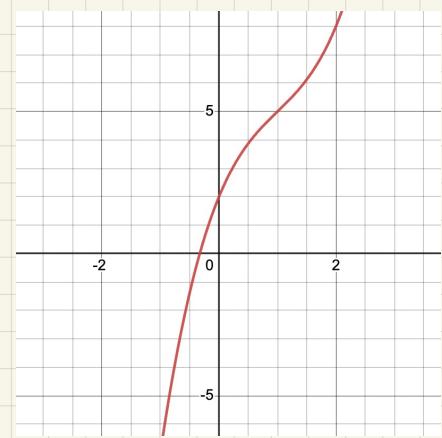
$$f'(x) = 0 \Rightarrow \text{no solution}$$

$$\Delta = (-6)^2 - 4 \times 3 \times 5 \\ = 36 - 60 = -24$$

$f'(x) > 0$  for all  $x$

$$\begin{array}{c} \oplus \\ \oplus \end{array} \longrightarrow f'(x)$$

Function always increasing



$$9. f(x) = \frac{-x^2 + 4x - 7}{x-1}$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$u = -x^2 + 4x - 7, \quad u' = -2x + 4$$

$$v = x-1, \quad v' = 1$$

$$f'(x) = \frac{(-2x+4)(x-1) - (-x^2+4x-7)1}{(x-1)^2}$$

$$= \frac{-2x^2 + 6x - 4 + x^2 - 4x + 7}{(x-1)^2}$$

$$= \frac{-x^2 + 2x + 3}{(x-1)^2} = -\frac{(x^2 - 2x - 3)}{(x-1)^2}$$

$$= -\frac{(x-3)(x+1)}{(x-1)^2}$$

$$f'(x) = 0 \text{ when } x = 3, x = -1$$

also  $f(x)$  undefined when  
 $x-1=0, x=1$

$$\begin{array}{c} - \\ -1 \quad 1 \quad 3 \end{array}$$

$$\text{Test } f'(-2) = -\frac{(-5)(-1)}{(-2-1)^2}$$

$$f'(-2) < 0$$

$$\text{Test } f'(0) = -\frac{(-3)(+1)}{(-1)^2}$$

$$f'(0) > 0$$

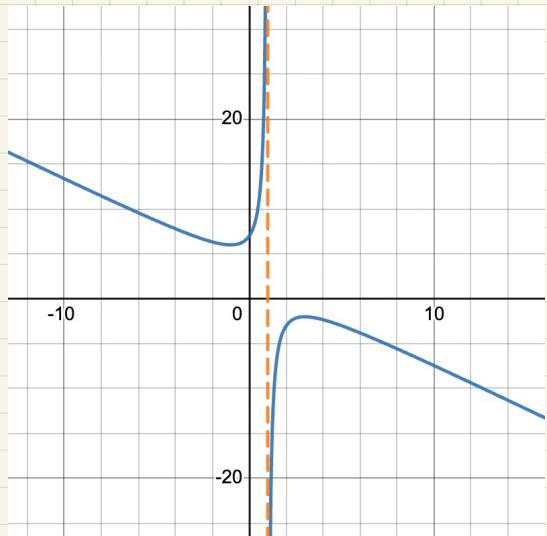
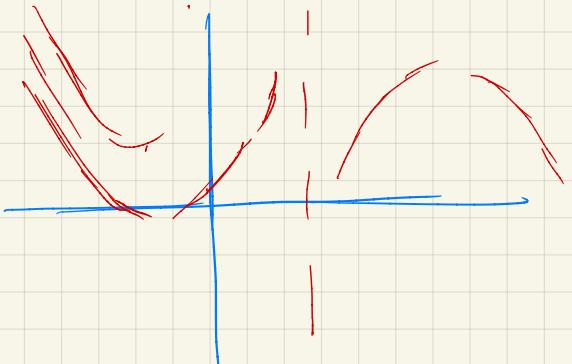
$$\text{Test } f'(2) = -\frac{(-1)(3)}{(2-1)^2}$$

$$f'(2) > 0$$

$$\text{Test } f'(4) = -\frac{(1)(5)}{(4-1)^2}$$

$$f'(4) < 0$$

$$\begin{array}{c} - \quad + \quad + \quad - \\ \leftarrow \quad \quad \quad \quad \rightarrow \end{array} \frac{dy}{dx}$$



II. c)

$$f(x) = 3 + e^{-x}$$

$$f'(x) = -e^{-x}$$

$$f'(x) \neq 0$$

Domain  $\{x \mid x \in \mathbb{R}\}$

$$f'(0) = -e^0 = -1$$

$\xleftarrow{\hspace{1cm}}$   $\xrightarrow{\hspace{1cm}}$   
Decreasing everywhere

d)  $f(x) = xe^x$

$$(uv)' = uv' + u'v$$

$$f'(x) = xe^x + 1e^x$$

$$= e^x(x+1)$$

$$e^x(x+1) = 0$$

$$e^x \neq 0 \quad x+1=0$$

$$x = -1$$

$$\text{Test } f'(0) = e^0(0+1) = 1$$

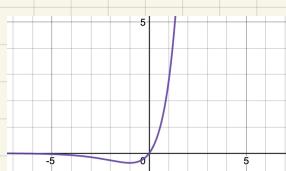
$$f'(0) > 0$$

$$\text{Test } f'(-2)$$

$$f'(-2) = e^{-2}(-2+1)$$

$$f'(-2) < 0$$

$\begin{array}{c} - \\ \hline -1 & + \end{array}$



f)  $f(x) = x^3 \ln x$

Domain  $\{x \mid x > 0\}$

$$(uv)' = uv' + u'v$$

$$f'(x) = x^3 \cdot \frac{1}{x} + 3x^2 \ln x$$

$$= x^2 + 3x^2 \ln x$$

$$f'(x) = 0$$

$$\Rightarrow x^2 = 0 \quad , \quad x = 0$$

(outside domain)

$$1 + 3 \ln x = 0$$

$$3 \ln x = -1$$

$$\ln x = -\frac{1}{3}$$

$$x = e^{-\frac{1}{3}} \quad x \approx 0.717$$



Test  $x = 0.5$

$$f'(0.5) = (0.5)^2 (1 + 3 \ln 0.5)$$

$$= (0.5)^2 (-1.079)$$

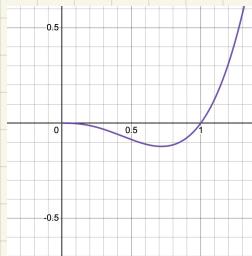
$$f'(0.5) < 0$$

Test  $x = 1$

$$f'(1) = 1^2 (1 + 3 \ln 1)$$

$$= 1^2 (1)$$

$$f'(1) > 0$$



$$k) f(x) = \ln(x^2 + 4)$$
$$x^2 + 4 > 0 \quad \forall x \in \mathbb{R}$$

$\Rightarrow$  Domain  $\{x \mid x \in \mathbb{R}\}$

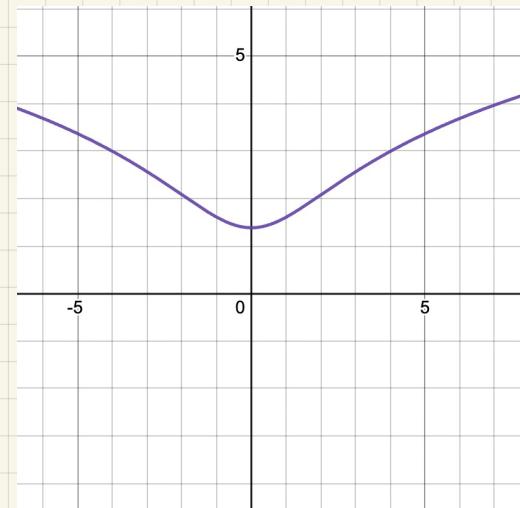
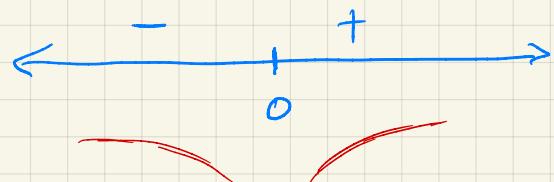
$$(g(u))' = g'(u) \cdot u'$$

$$\begin{aligned}f'(x) &= \frac{1}{x^2 + 4} \cdot 2x \\&= \frac{2x}{x^2 + 4}\end{aligned}$$

$$f'(x) = 0 \Rightarrow x = 0$$

$$\text{Test } x = -1 \quad f'(-1) = \frac{-2}{5} < 0$$

$$\text{Test } x = 1 \quad f'(1) = \frac{2}{5} > 0$$

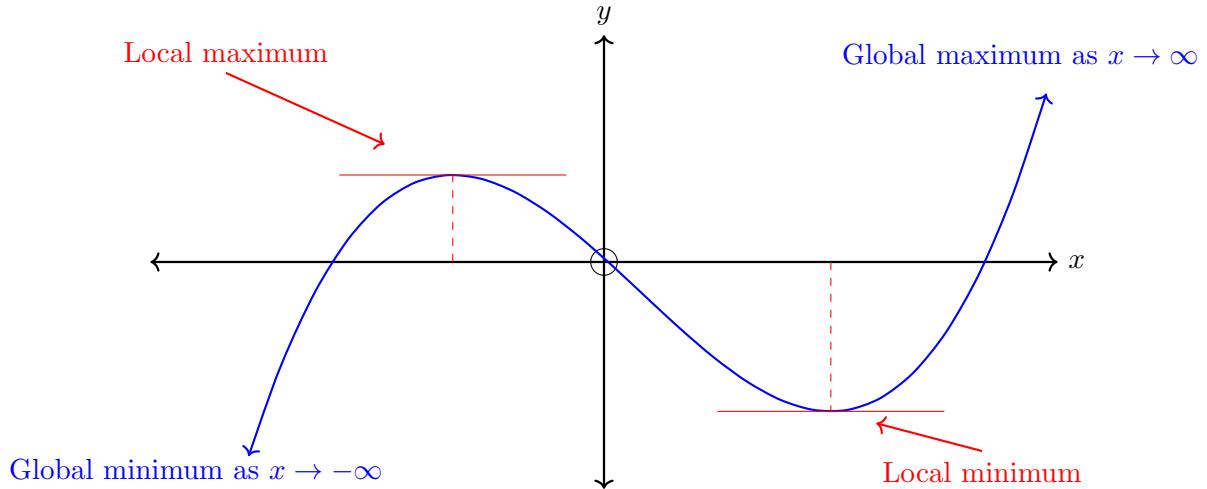


## 1.5 Stationary Points

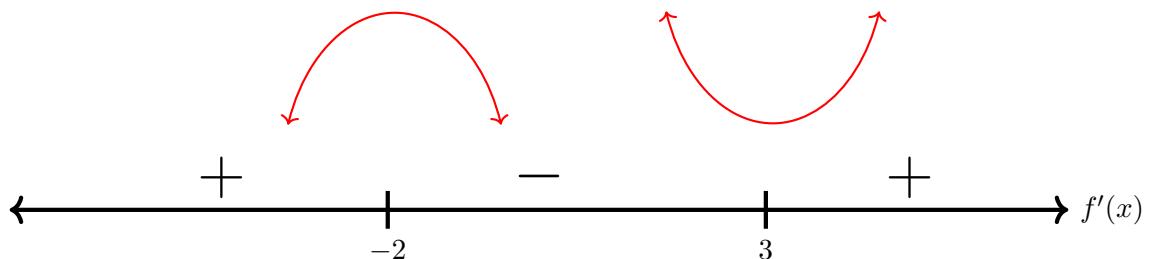
### 1.5.1 Turning points (minima, maxima)

From our 'standard' graph we can identify a couple of key features.

$$f(x) = x^3 - 1.5x^2 - 18x + 1$$



If we do not have the graph we can classify the stationary points using a sign diagram.

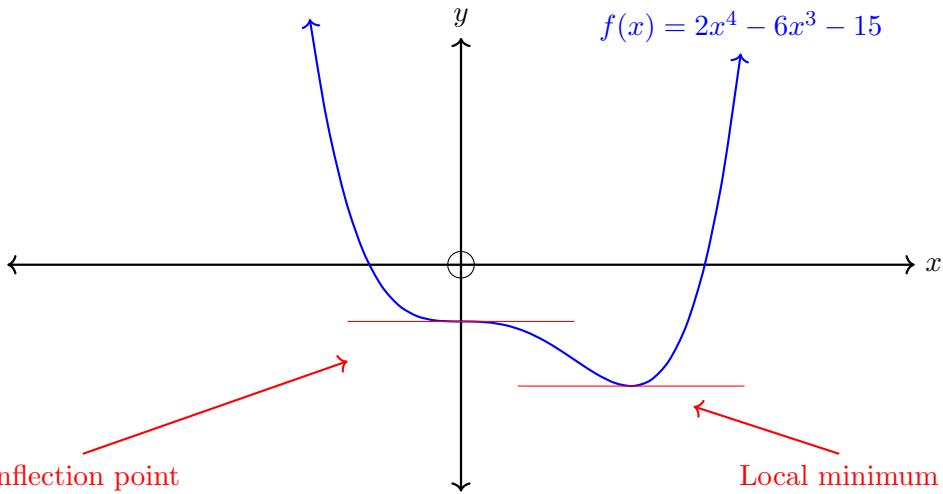


### 1.5.2 Stationary points of inflection

Stationary points are not necessarily local maxima or minima.

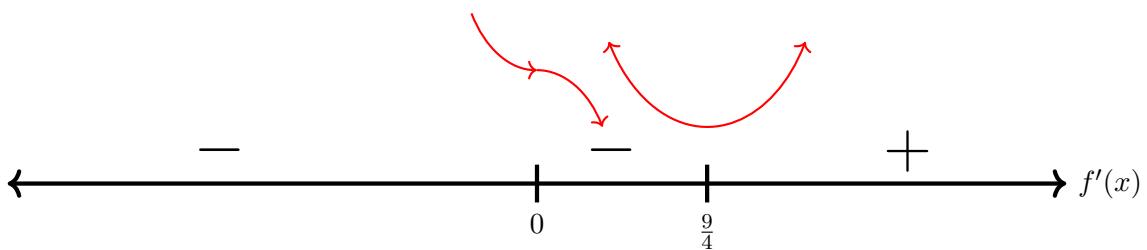
They may be **inflection** points.

This is where the shape of the curve is changing, but the gradient is not changing sign.



**Constructing a sign diagram:**

$$\begin{aligned}
 f(x) &= 2x^4 - 6x^3 - 15 \\
 f'(x) &= 8x^3 - 18x^2 \\
 &= 2x^2(4x - 9) \\
 \text{Set } f'(x) &= 0 \\
 \Rightarrow 2x^2(4x - 9) &= 0 \\
 x = 0 \text{ and } x &= \frac{9}{4}
 \end{aligned}$$



If the sign of the gradient repeats across a stationary point, we will have an inflection point.

Note: Inflection points can be at stationary points, or not. Consequently we classify the one above as a **stationary inflection point**

Please note: Summary chart in the textbook.

Stationary point where $f'(a) = 0$	Sign diagram of $f'(x)$ near $x = a$	Shape of curve near $x = a$
local maximum	$\leftarrow + \frac{1}{a} - \frac{f'(x)}{x}$	
local minimum	$\leftarrow - \frac{1}{a} + \frac{f'(x)}{x}$	
stationary inflection	$\leftarrow + \frac{1}{a} + \frac{f'(x)}{x}$ $\leftarrow - \frac{1}{a} - \frac{f'(x)}{x}$	

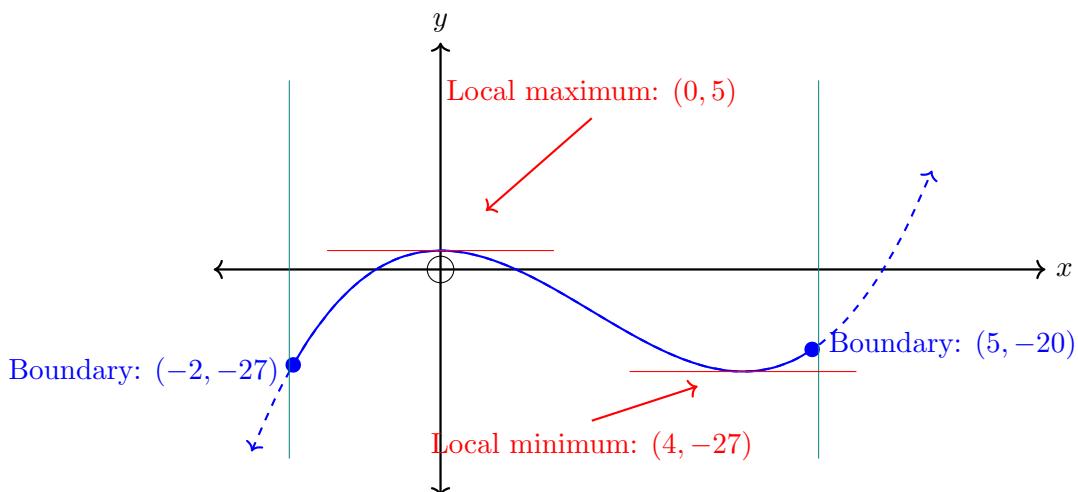
### 1.5.3 Greatest or least values (testing boundaries)

If we are given a function which is bounded on an interval, and are finding the greatest/least values on that interval: we need to

- find the values of the stationary points
- find the values at the end points of the function (as these could be greater/less than the stationary point values).

Following the example from the book using

$$y = x^3 - 6x^2 + 5 \text{ On the interval } -2 \leq x \leq 5$$



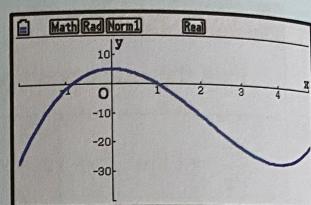
#### Example 13

Find the greatest and least value of  $y = x^3 - 6x^2 + 5$  on the interval  $-2 \leq x \leq 5$ .

$$\begin{aligned} \text{Now } \frac{dy}{dx} &= 3x^2 - 12x \\ &= 3x(x - 4) \end{aligned}$$

$$\therefore \frac{dy}{dx} = 0 \text{ when } x = 0 \text{ or } 4.$$

The sign diagram of  $\frac{dy}{dx}$  is:



$\therefore$  there is a local maximum at  $x = 0$ , and a local minimum at  $x = 4$ .

Critical value ( $x$ )	$y$
-2 (end point)	-27
0 (local maximum)	5
4 (local minimum)	-27
5 (end point)	-20

If an interval is given, we must also check the value of the function at the end points.



The greatest of these values is 5 when  $x = 0$ .

The least of these values is -27 when  $x = -2$  and when  $x = 4$ .

14 Find the greatest and least value of

#### 1.5.4 Ex 13D

13 D

S.

Find and classify stationary points

g)  $f(x) = x - \sqrt{x}$

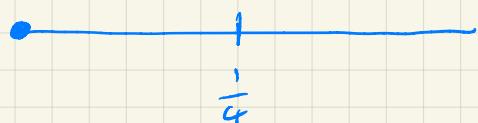
Domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$

$$f'(x) = 1 - \frac{1}{2\sqrt{x}}$$

$$1 - \frac{1}{2\sqrt{x}} = 0 \Rightarrow 2\sqrt{x} = 1$$

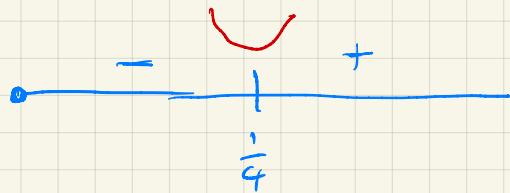
$$\sqrt{x} = \frac{1}{2}$$

$$x = \frac{1}{4}$$



$$\text{Test } f'(1) = 1 - \frac{1}{2} = \frac{1}{2} \quad f'(1) > 0$$

$$\text{Test } f'\left(\frac{1}{16}\right) = 1 - \frac{1}{\frac{2}{4}} = 1 - 2 = -1 \quad f'\left(\frac{1}{16}\right) < 0$$



Local minimum at  $x = \frac{1}{4}$

$$f\left(\frac{1}{4}\right) = \frac{1}{4} - \sqrt{\frac{1}{4}} = -\frac{1}{4}$$
$$\left(\frac{1}{4}, -\frac{1}{4}\right)$$

Also note:  $f(x) = 0$

when  $x - \sqrt{x} = 0$

$$\sqrt{x}( \sqrt{x} - 1) = 0$$

$$x = 0 \quad x = 1$$

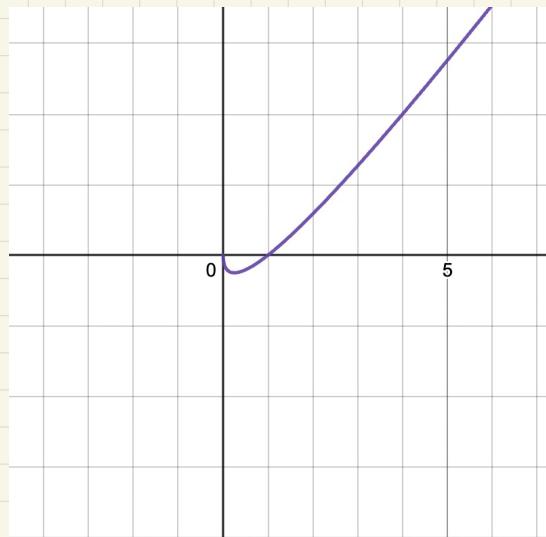
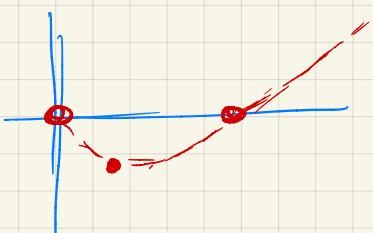
$x$ -intercepts  $(0, 0) \quad (1, 0)$

for  $\forall x, x > 1$

$$x > \sqrt{x}$$

$$\Rightarrow \lim_{x \rightarrow \infty} x - \sqrt{x} = \infty$$

but increasing quite slowly



7. d)

$$y = e^{-x}(x+2)$$

$$\frac{dy}{dx} = e^{-x} \cdot 1 + -e^{-x}(x+2)$$

$$= e^{-x} - e^{-x}(x+2)$$

$$= e^{-x}(1 - (x+2))$$

$$= e^{-x}(-x-1)$$

$$= -e^{-x}(x+1)$$

$$\frac{dy}{dx} = 0 \quad x = -1$$

$$x = -2 \quad \frac{dy}{dx} = -e^2(-2+1) \\ > 0$$

$$x = 0 \quad \frac{dy}{dx}(0) = -e^0(0+1) \\ < 0$$

$$+ \quad | \quad - \\ -1$$

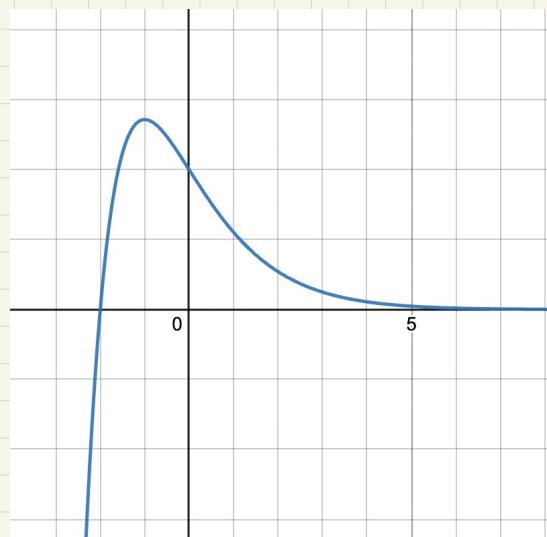
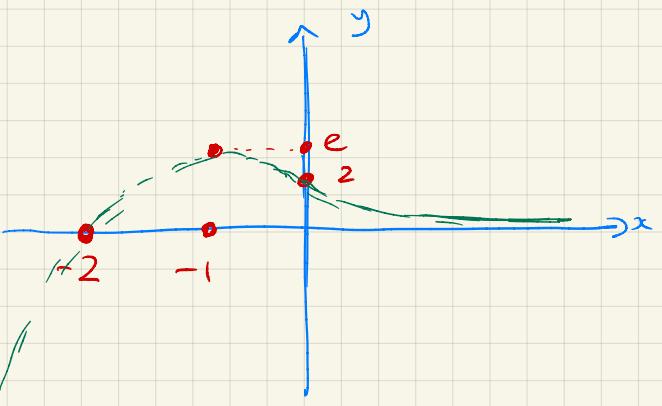
$$y(-1) = e^1(1) = e$$

$$\text{set } y = 0 \Rightarrow x = -2$$

$$x = 0 \quad y = 2$$

$$\lim_{x \rightarrow -\infty} e^{-x}(x+2) = -\infty$$

$$\lim_{x \rightarrow \infty} e^{-x}(x+2) = 0$$



$$10. \quad y = \frac{e^{ax}}{bx}$$

has stationary point at  
 $(\frac{1}{3}, \frac{e}{2})$

Find  $a, b$   
 Classify the stationary point.

$$\frac{e}{2} = \frac{e^{\frac{a}{3}}}{b}$$

$$\frac{3}{3} = \frac{a}{3}$$

$$\frac{be}{b} = e^{\frac{a}{3}}$$

$$b = 6e^{\frac{a}{3}-1}$$

$$u = e^{ax} \quad u' = ae^{ax}$$

$$v = bx \quad v' = b$$

$$\frac{dy}{dx} = \frac{ae^{ax}(bx) - be^{ax}}{(bx)^2}$$

$$= \frac{be^{ax}(ax-1)}{(bx)^2}$$

$$= \frac{e^{ax}(ax-1)}{b^2x^2}$$

$$\frac{dy}{dx}(\frac{1}{3}) = 0$$

$$\Rightarrow \left( \frac{a}{3} - 1 \right) = 0$$

$$a = 3$$

$$b = 6e^{\frac{3}{3}-1}$$

$$= 6$$

$$y = \frac{e^{3x}}{6x}$$

$$\frac{dy}{dx} = \frac{e^{3x}(3x-1)}{6x^2}$$

Function is undefined at  $x=0$



Test:  $x = -1$

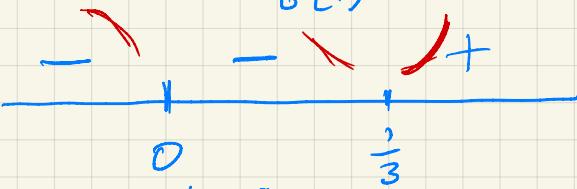
$$\frac{dy}{dx}(-1) = \frac{e^{-3}(-4)}{6(-1)^2} < 0$$

Test:  $x = \frac{1}{4}$

$$\frac{dy}{dx}(\frac{1}{4}) = \frac{e^{3/4}(\frac{3}{4}-1)}{6(\frac{1}{4})^2} < 0$$

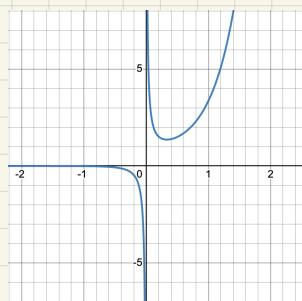
Test:  $x = 1$

$$\frac{dy}{dx}(1) = \frac{e^3(3-1)}{6(1)^2} > 0$$



$$\lim_{x \rightarrow -\infty} \frac{e^x}{x} = -\infty \quad \lim_{x \rightarrow 0^+} \frac{e^x}{x} = +\infty$$

$$\lim_{x \rightarrow 0^-} \frac{e^x}{x} = -\infty \quad \lim_{x \rightarrow +\infty} \frac{e^x}{x} = +\infty$$



12. f)

$$f(x) = \sin(2x) + 2\cos(x)$$

$$f'(x) = 2\cos(2x) - 2\sin(x)$$

$$f'(x) = 0$$

$$\Rightarrow 2\cos(2x) - 2\sin(x) = 0$$

Trig identity:

$$\cos(2x) = 1 - 2\sin^2(x)$$

(chapter 9: page 238)

$$2(1 - 2\sin^2(x)) - 2\sin(x) = 0$$

$$-4\sin^2(x) - 2\sin(x) + 2 = 0$$

$$-2(2\sin^2(x) + \sin(x) - 1) = 0$$

$$2y^2 + y - 1 \quad \begin{matrix} -2 \\ +2 \end{matrix} \quad \begin{matrix} \times \\ -1 \end{matrix}$$

$$2y^2 + 2y - y - 1 \quad \begin{matrix} +1 \\ \end{matrix} \quad \begin{matrix} \oplus \\ \end{matrix}$$

$$2y(y+1) - 1(y+1) = (2y-1)(y+1)$$

$$-2(2\sin(x)-1)(\sin(x)-1) = 0$$

$$\sin(x) = \frac{1}{2} \Rightarrow \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

Domain  $0 \leq x \leq 2\pi$

$$\frac{\pi}{6} + 2k\pi \Rightarrow \frac{\pi}{6} \text{ in } D$$

$$\pi - \frac{\pi}{6} + 2k\pi \Rightarrow \frac{5\pi}{6} \text{ in } D$$

$$\sin(x) = -1 \Rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$$

$$-\frac{\pi}{2} + 2k\pi \Rightarrow \frac{3\pi}{2} \text{ in } D$$

Solutions:  $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

$$f'(x) = 2\cos(2x) - 2\sin(x)$$



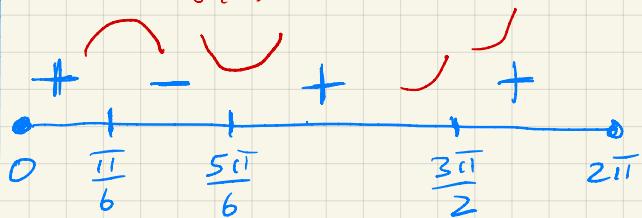
Test  $x=0$ , Test  $x=\frac{\pi}{2}$   
 $f'(0)=2 > 0$        $f'\left(\frac{\pi}{2}\right)=-4 < 0$

Test  $x=\pi$

$$f'(\pi) = 2 > 0$$

Test  $x=2\pi$

$$f'(2\pi) = 2 > 0$$



Local maximum at

$$x = \frac{\pi}{6}, f\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) + 2\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \approx 2.598$$

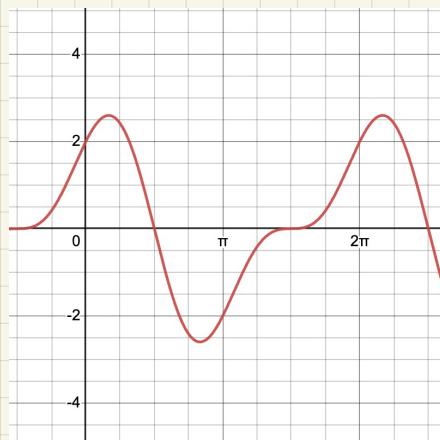
Local minimum at  $x = \frac{5\pi}{6}$

$$f\left(\frac{5\pi}{6}\right) = \sin\left(\frac{5\pi}{3}\right) - 2\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$$

Stationary Inflection point at  $x = \frac{3\pi}{2}$

$$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) - 2\cos\left(\frac{3\pi}{2}\right) = 0$$

$$\left(\frac{3\pi}{2}, 0\right)$$



16.

$$\begin{aligned}
 f(x) &= \sin(x) \cos(2x) \\
 f'(x) &= \sin(x)(-2\sin(2x)) \\
 &\quad + \cos(x)\cos(2x) \\
 &= -2\sin(x)\sin(2x) + \cos(x)\cos(2x) \\
 &= -2\sin(x)(2\sin(2x)\cos(x)) \\
 &\quad + \cos(x)(2\cos^2(x)-1) \\
 &= -4\sin^2(x)\cos(x) \\
 &\quad + 2\cos^3(x) - \cos(x) \\
 &= -4(1-\cos^2(x))\cos(x) \\
 &\quad + 2\cos^3(x) - \cos(x) \\
 &= -4\cos(x) + 4\cos^3(x) \\
 &\quad + 2\cos^3(x) - \cos(x) \\
 &= 6\cos^3(x) - 5\cos(x)
 \end{aligned}$$

$$f'(x) = 0$$

$$\Rightarrow 6\cos^3(x) - 5\cos(x) = 0$$

$$\cos(x)(6\cos^2(x) - 5) = 0$$

$$\cos(x) = 0$$

$$\cos(x) = \pm \sqrt{\frac{5}{6}}$$

$$\text{Domain } \{x \mid 0 \leq x \leq \pi\}$$

which is Domain of  $\cos^{-1}(y)$

$$\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} \approx 1.571$$

$$\cos(x) = \pm \sqrt{\frac{5}{6}}$$

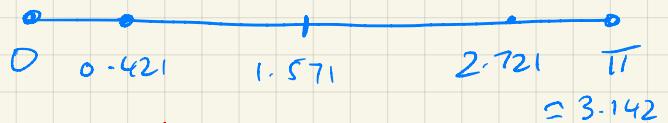
$$x = \cos^{-1}\left(\pm \sqrt{\frac{5}{6}}\right) \approx 0.421$$

$$\cos(x) = -\sqrt{\frac{5}{6}}$$

$$x = \cos^{-1}\left(-\sqrt{\frac{5}{6}}\right)$$

$$\approx 2.721$$

$$( \text{or } \pi - 0.421 = 2.721 )$$

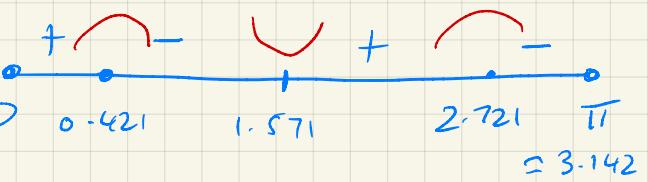


$$\text{Test } f'(0) = 1 > 0$$

$$\text{Test } f'(1) = -1.755 < 0$$

$$\text{Test } f'(2) = 1.648 > 0$$

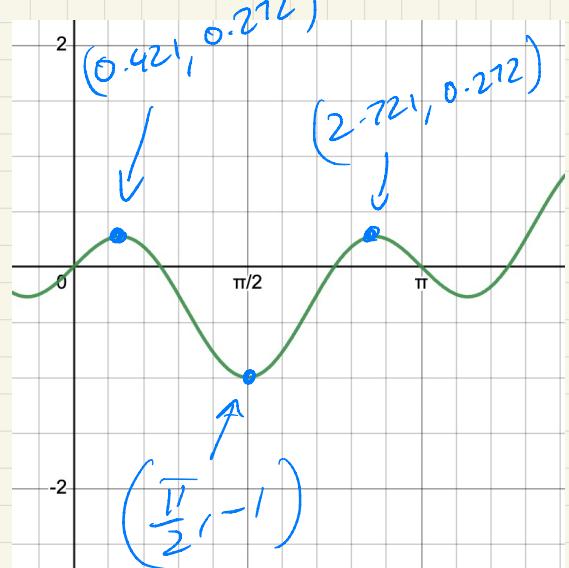
$$\text{Test } f'(\pi) = -1 < 0$$



$$f(0.421) = 0.272$$

$$f\left(\frac{\pi}{2}\right) = -1$$

$$f(2.721) = 0.272$$



18. Prove  $\frac{\ln(x)}{x} \leq \frac{1}{e}$

for all  $x > 0$

Create a sign diagram and hopefully identify a maximum value for  $\frac{\ln x}{x}$

let  $f(x) = \frac{\ln(x)}{x}$      $u = \ln x$      $u' = \frac{1}{x}$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \ln x = 0$$

$$\ln x = 1$$

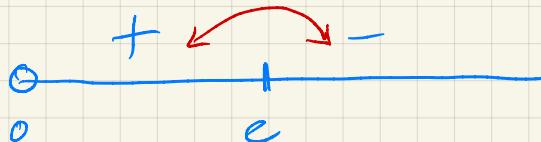
$$x = e$$

Test  $x = 1$

$$f'(1) = \frac{1-0}{1^2} > 0$$

Test  $x = e^2$

$$f'(e^2) = \frac{1 - 2\ln e}{e^4} = \frac{-1}{e^4} < 0$$



Global maximum at  $x = e$

$$f(e) = \frac{\ln e}{e} = \frac{1}{e}$$

$$(e, \frac{1}{e})$$

So  $\frac{1}{e}$  is largest possible value of  $\frac{\ln x}{x}$

$$\Rightarrow \frac{\ln x}{x} \leq \frac{1}{e}$$

19.  $f(x) = x - \ln(x)$   $D \{ x > 0 \}$

a) show only local minimum.

b) prove  $\ln(x) < x - 1 \quad \forall x > 0$

$$f'(x) = 1 - \frac{1}{x}$$

$$f'(x) = 0$$

$$1 - \frac{1}{x} = 0 \quad x = 1$$



$$f'\left(\frac{1}{2}\right) = 1 - 2 = -1 < 0$$

$$f'(2) = 1 - \frac{1}{2} = \frac{1}{2} > 0$$

local minimum at  $(1, 1)$

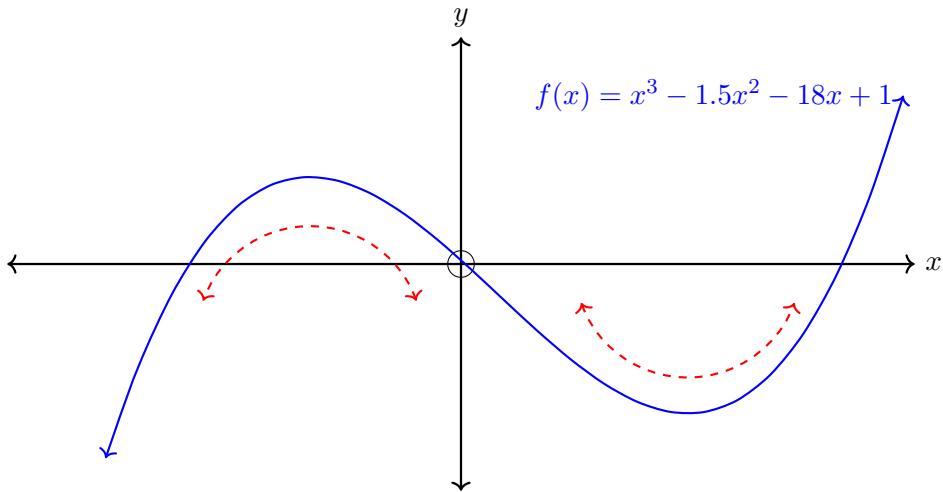
$$\Rightarrow x - \ln(x) \geq 1 \quad \forall x > 0$$

$$x - 1 \geq \ln(x)$$

$$\ln(x) \leq x - 1$$

## 1.6 Shape

Looking at the shape of the graph below, we can see that over a certain interval, the curve is turning downwards **it is concave down**, and on then it changes to be turning upwards **concave up**.



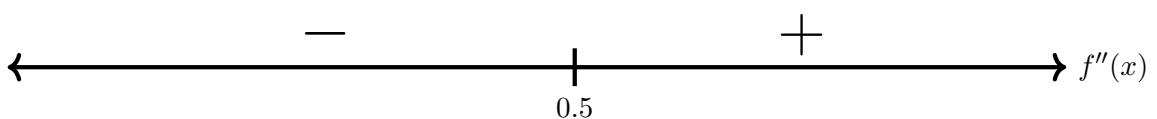
If the curve is concave down the **gradient** is decreasing, so we consider the second derivative

Leibnitz	or	Newton
$\frac{d}{dx} \left( \frac{dy}{dx} \right) < 0$		$(f'(x))' < 0$
$\Rightarrow \frac{d^2y}{dx^2} < 0$		$\Rightarrow f''(x) < 0$

Similarly, if the curve is concave up the **gradient** is increasing, so

$$\frac{d^2y}{dx^2} > 0 \quad \text{or} \quad f''(x) > 0$$

We can construct a sign diagram for (in this case)  $f''(x) = 6x - 3$ , and  $f''(x) = 0$  when  $x = \frac{1}{2}$



We can also observe that there is an **non-stationary inflection point** at  $x = \frac{1}{2}$ ,  $f\left(\frac{1}{2}\right) = -\frac{37}{4}$

### 1.6.1 Ex 13E

13 E

3 b

Determine the shape of

$$y = -2(x-3)(x+1)$$

$$\begin{aligned} \frac{dy}{dx} &= -2[(x-3)1 + 1(x+1)] \\ &= -2(2x-2) \\ &= -4x+4 \end{aligned}$$

$$\frac{d^2y}{dx^2} = -4 \quad \frac{d^2y}{dx^2} < 0 \text{ always}$$

$$\xleftarrow{\text{---}} \xrightarrow{\frac{d^2y}{dx^2}}$$

function is concave down always.

4 f

$$f(x) = x^4 - 12x^3$$

$$f'(x) = 4x^3 - 36x^2$$

$$4x^3 - 36x^2 = 0$$

$$4x^2(x-9) = 0$$

$$x=0 \quad x=9$$

$$\begin{array}{c} - \\ \hline - & - & + \\ \hline 0 & & 9 \end{array}$$

$$f''(x) = 12x^2 - 72x$$

$$f''(x) = 0$$

$$12x^2 - 72x = 0$$

$$12x(x-6) = 0$$

$$x=0 \quad x=6$$

$$\begin{array}{c} + & - & + \\ \hline 0 & & 6 \end{array}$$

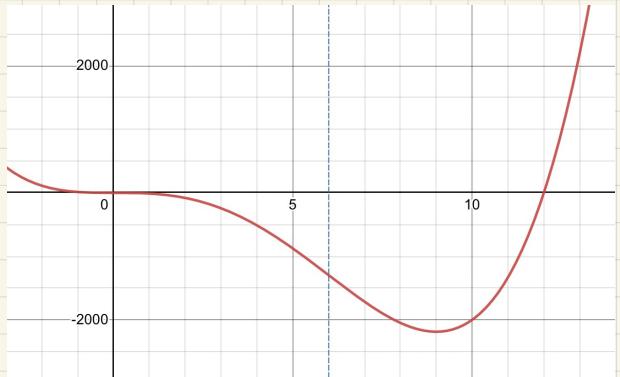
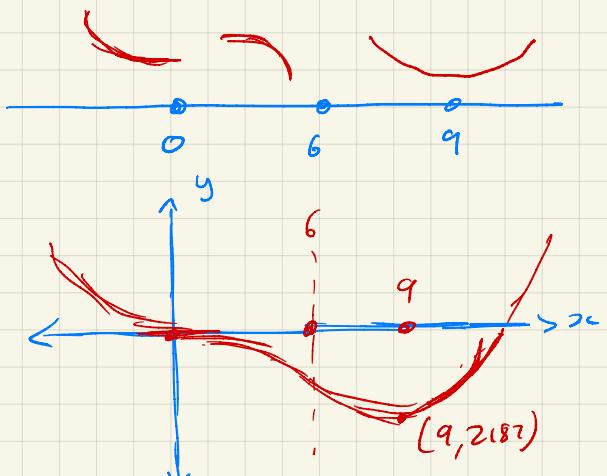
$f(x)$  is decreasing for  $x < 9$

$f(x)$  is increasing for  $x > 9$

$f(x)$  is concave up  $x < 0$

concave down  $0 < x < 6$

concave up  $x > 6$



$$7. f(x) = \frac{e^x}{x}$$

a) When  $x=0$ ,  $f(x)$  is undefined  
 $\Rightarrow$   $y$ -intercept does not exist.

for  $x$ -intercepts  $\frac{e^x}{x} = 0$

but  $e^x \neq 0$ , so no  $x$ -intercepts

b) as  $x \rightarrow \infty$

$e^x$  gets very large

$\frac{1}{x}$  gets very small

but  $e^x$  is an exponential

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{e^x}{x} = +\infty$$

and  $\lim_{x \rightarrow -\infty} \frac{e^x}{x} = 0$

c)  $f(x) = \frac{e^x}{x}$

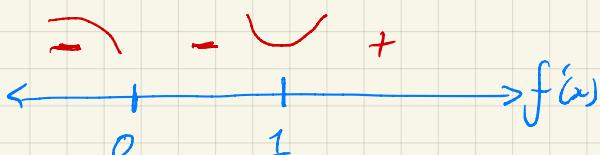
$$f'(x) = \frac{xe^x - e^x \cdot 1}{x^2}$$

$$= \frac{e^x(x-1)}{x^2}$$

$$f'(x) = 0$$

$$\Rightarrow e^x(x-1) = 0$$

$x=1$  is a stationary pt  $(1, e)$



$(1, e)$  is a stationary point and is a local minimum.

$$f'(x) = \frac{xe^x - e^x}{x^2}$$

$$u = xe^x - e^x$$

$$u' = xe^x + e^x - e^x$$

$$= xe^x$$

$$v = x^2$$

$$v' = 2x$$

$$f''(x) = \frac{xe^x(x^2) - (xe^x - e^x)2x}{x^4}$$

$$= \frac{x^3e^x - 2x^2e^x + 2xe^x}{x^4}$$

$$= \frac{xe^x(x^2 - 2x + 2)}{x^4}$$

$$xe^x(x^2 - 2x + 2) = 0$$

$$x=0 \quad \text{no sol.}$$

so no inflection points.

but need to look at either side of undefined

$$\begin{array}{c|cc} - & + & \\ \hline & & f''(x) \end{array}$$

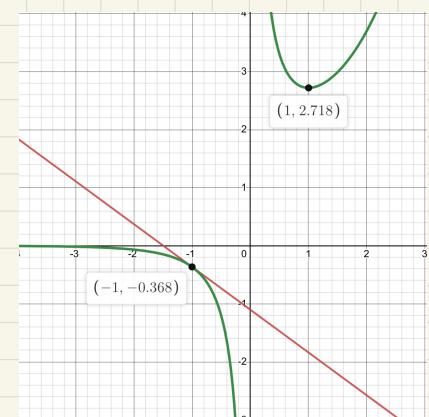
$$\begin{array}{ll} \text{Concave down} & 0 \\ x < 0 & \end{array} \quad \begin{array}{ll} \text{Concave up} & 0 \\ x > 0 & \end{array}$$

tangent at  $x=-1$

$$f'(-1) = -\frac{2}{e} \quad f(-1) = -\frac{1}{e}$$

$$y + \frac{1}{e} = -\frac{2}{e}(x+1)$$

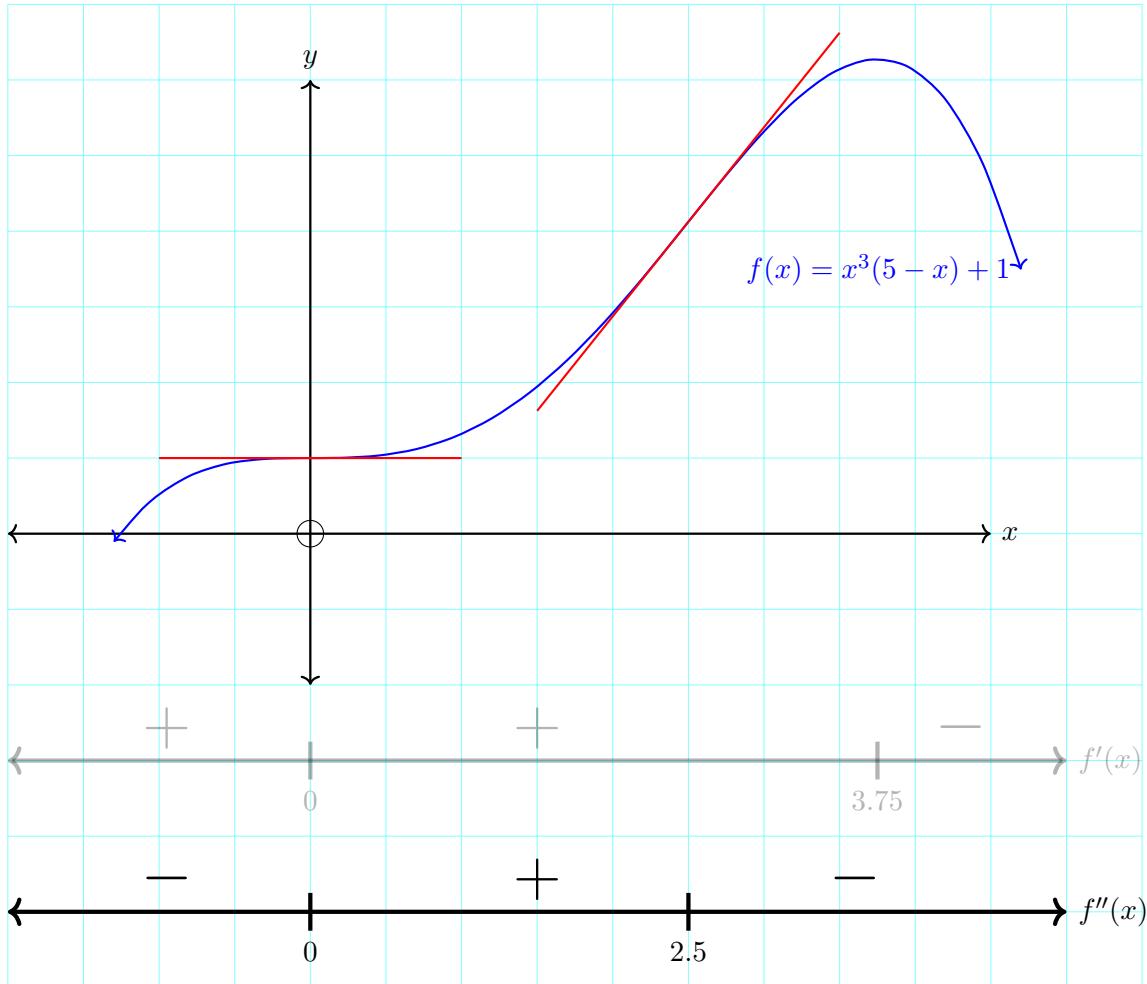
$$y = -\frac{2x}{e} - \frac{3}{e} \quad \text{or} \quad 2x + ey = -3$$



## 1.7 Inflection Points

At an inflection point:

- the tangent line crosses the curve
- the shape of the curve is changing from concave up to concave down (or vice versa).



We can identify inflection points by

- Solving  $f''(x) = 0$
- Then using a sign diagram to verify that the points we have found are actually inflection points.  
The sign diagram must **change signs** across the 0 for the points to be inflection points.

### 1.7.1 Ex 13F

Find stationary and inflection points for  
 $y = x^3(5-x)$

$$y = 5x^3 - x^4$$

$$\frac{dy}{dx} = 15x^2 - 4x^3$$

$$15x^2 - 4x^3 = 0$$

$$x^2(15 - 4x) = 0$$

$$x=0 \quad x = \frac{15}{4} = 3.75$$

Stationary points at  $(0, 0)$  and  $(3.75, 65.918)$

$$\frac{d^2y}{dx^2} = 30x - 12x^2$$

$$= 6x(5 - 2x)$$

If  $x=0$

$$\frac{d^2y}{dx^2} = 0$$

If  $x = 3.75$

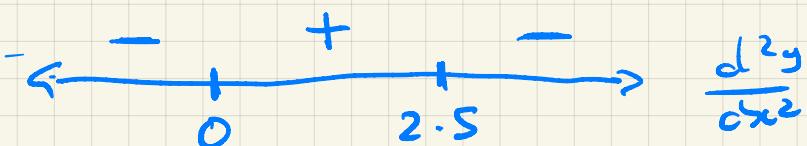
$$\frac{d^2y}{dx^2} = -56.25 \quad \curvearrowleft$$

unknown at this stage.

$(3.75, 65.918)$  is a local max.

Sign diagram for  $\frac{d^2y}{dx^2}$        $6x(5-2x)=0$

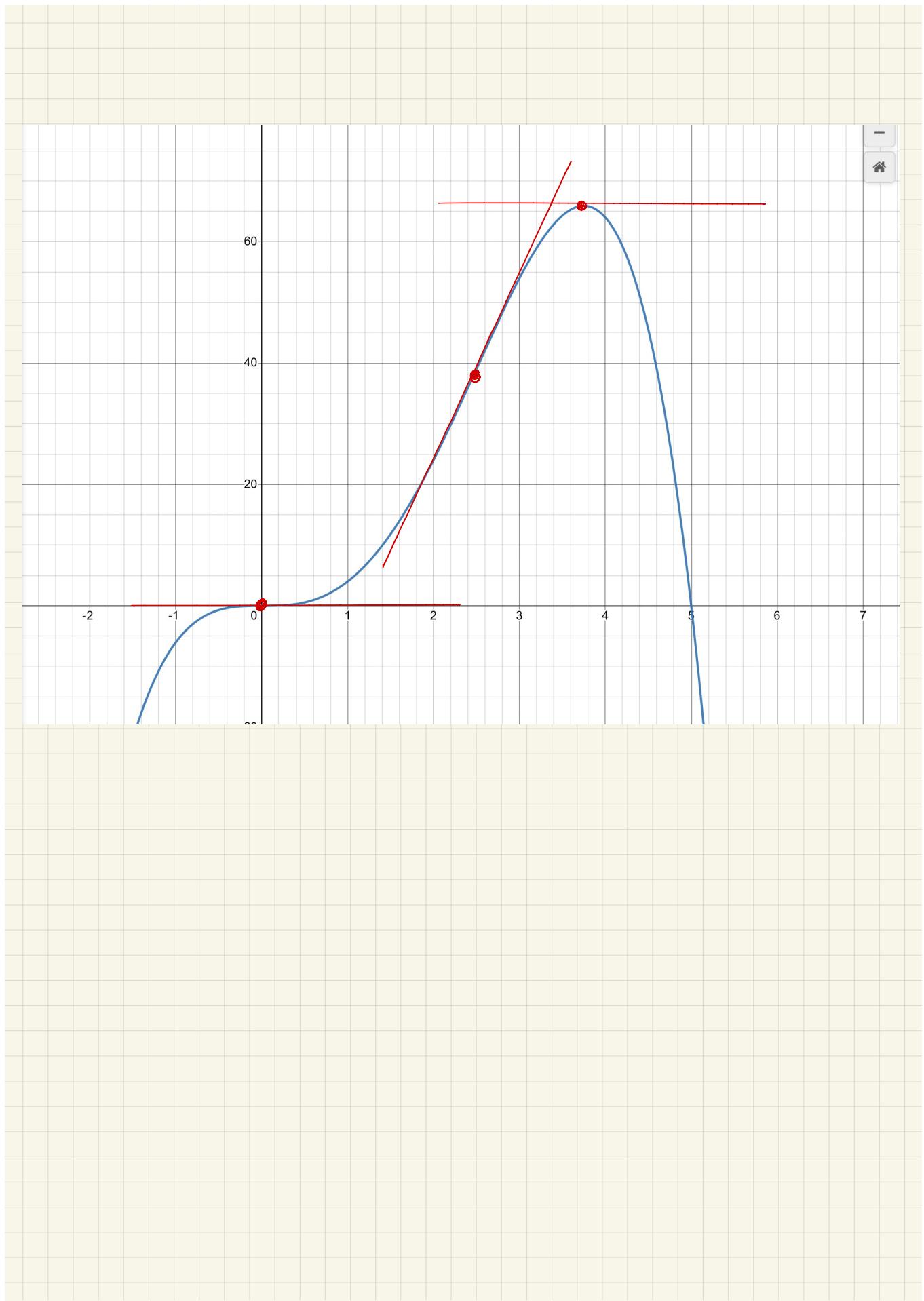
$$x=0 \quad x=2.5$$



Change of sign at  
 $x=0$

Change of sign at  
 $2.5$

So  $(0,0)$  is a stationary inflection point      so  $(2.5, 39.0625)$  is a non-stationary inflection point



Find stationary and inflection points for:

$$f(x) = \frac{x^6}{30} + 6.4x$$

$$f'(x) = \frac{6x^5}{30} + 6.4 = \frac{x^5}{5} + 6.4$$

$$\frac{x^5}{5} + 6.4 = 0$$

$$x^5 = -32$$

$$x = \sqrt[5]{-32} = -2$$

Stationary point at  $(-2, -10\frac{2}{3})$

$$f''(x) = \frac{5x^4}{5} = x^4$$

Sign diagram  
not required.

$$f''(-2) = 16 > 0$$

Stationary point is a local minimum.

Set  $f''(x) = 0$

$$\Rightarrow x = 0$$

could be an inflection point

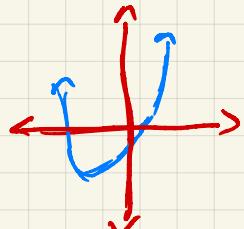
but  $\begin{array}{c} + \\ \hline + & + \\ \hline 0 \end{array} f''(x)$

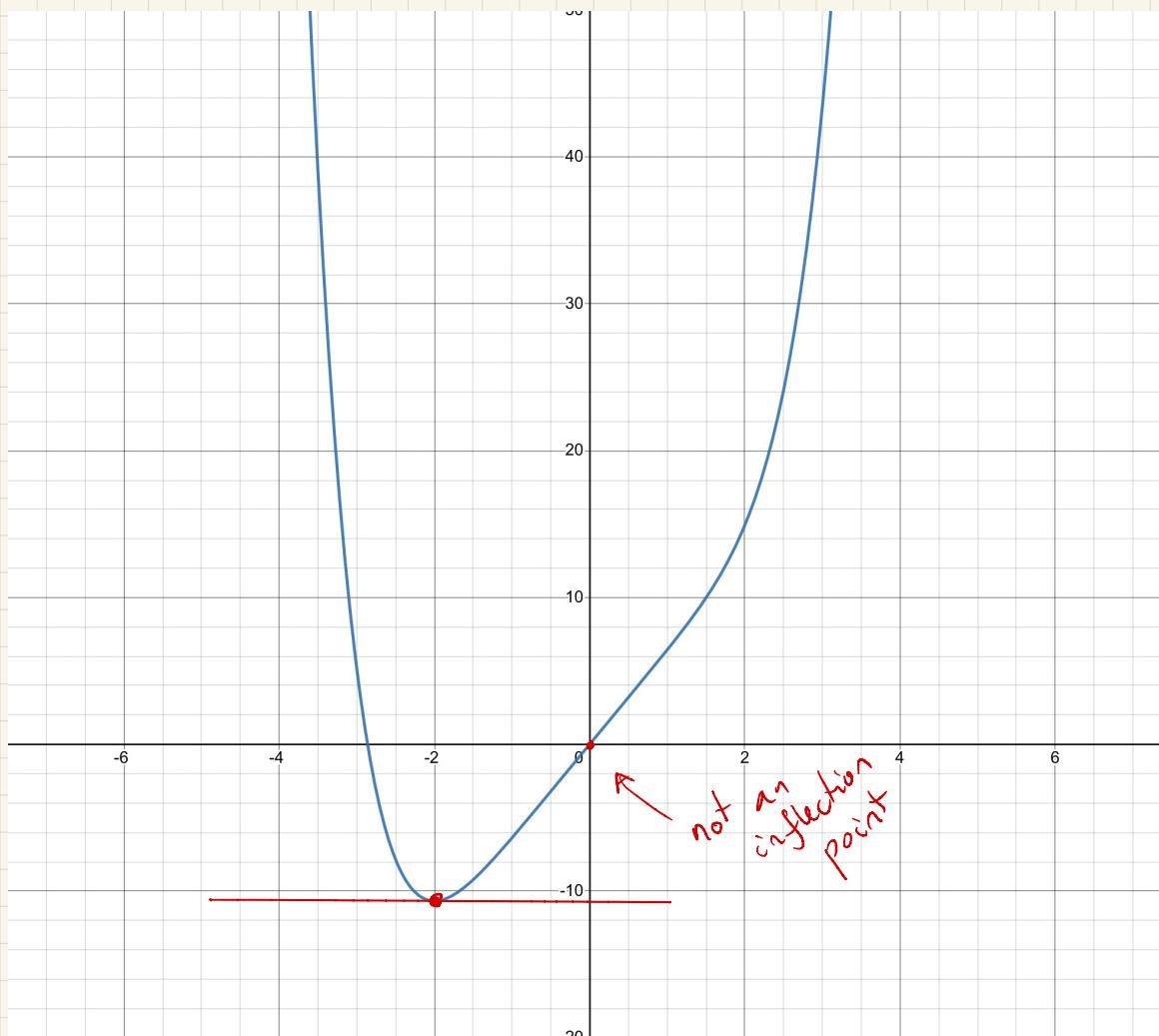
Does not change sign

so  $(0, 0)$  is not an inflection point.

A local minimum at  $(-2, -10\frac{2}{3})$

No inflection points.





## 1.8 Understanding functions and their derivatives

## Chapter: 2 Applications of differentiation

$$\text{The equation is: } 9a - 4 = 14 + 3a$$

$$\text{Subtract } 3a: \quad 6a - 4 = 14$$

$$\text{Subtract } 4: \quad 6a = 18$$

$$\text{Divide by } 6: \quad a = 3$$

$$A\widehat{B}C$$

$$A\widehat{B}CC$$

$$A\dot{B}C$$

$$N\tilde{a}$$

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

