Properties of curves and applications of differentiation

Kh notes

Contents

Chapter: 1 Properties of curves

1.1 Review Questions

1.1.1 Some fundamental derivatives:

Function	Derivative
$f(x) = x^n$	$f'(x) = nx^{n-1} \quad (n \in \mathbb{R})$
$f(x) = e^x$	$f'(x) = e^x$
$f(x) = \ln x$	$f'(x) = \frac{1}{x}$
$f(x) = \sqrt{x}$	$f'(x) = \frac{1}{2\sqrt{x}}$
$f(x) = \sin x$	$f'(x) = \cos x$
$f(x) = \cos x$	$f'(x) = -\sin x$
$f(x) = \tan x$	$f'(x) = \sec^2 x$

1.1.2 Rules of differentiation:

Chain Rule:

$$y = g(u_{(x)})$$
$$\frac{dy}{dx} = g'(u_{(x)})u'_{(x)}$$

Product Rule:

$$y = u_{(x)}v_{(x)}$$
$$\frac{dy}{dx} = u_{(x)}v'_{(x)} + u'_{(x)}v_{(x)}$$

Quotient Rule:

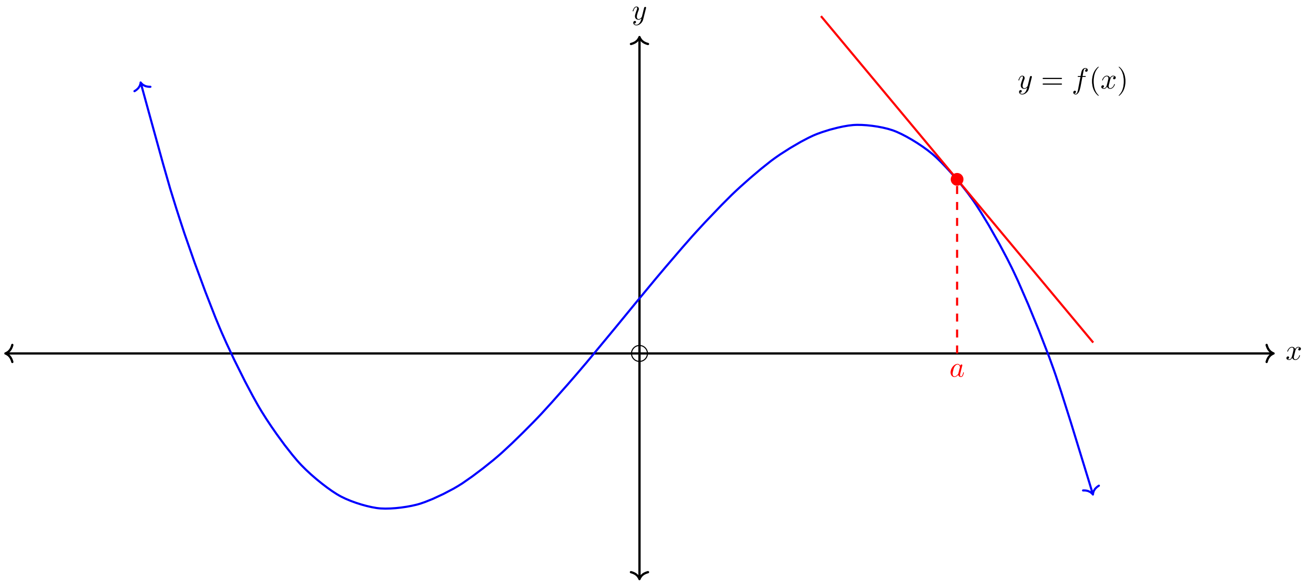
$$y = \frac{u_{(x)}}{v_{(x)}}$$
$$\frac{dy}{dx} = \frac{u'_{(x)}v_{(x)} - u_{(x)}v'_{(x)}}{[v_{(x)}]^2}$$

1.2 Tangents

- The tangent to a curve at a point A is the best approximating straight line to the curve at point A.
- (Leibniz definition) Tangent to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through the infinitely close pair of points either side of $f(a)$

$$\frac{y - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- It is a single point of contact with the curve (although it may intersect the curve at some other point)



For the function $y = f(x)$, and some $x = a$

- $(a, f(a))$ is on the curve
- $f'(a)$ is the gradient of the curve at $x = a$

$$\frac{y - f(a)}{x - a} = f'(a)$$

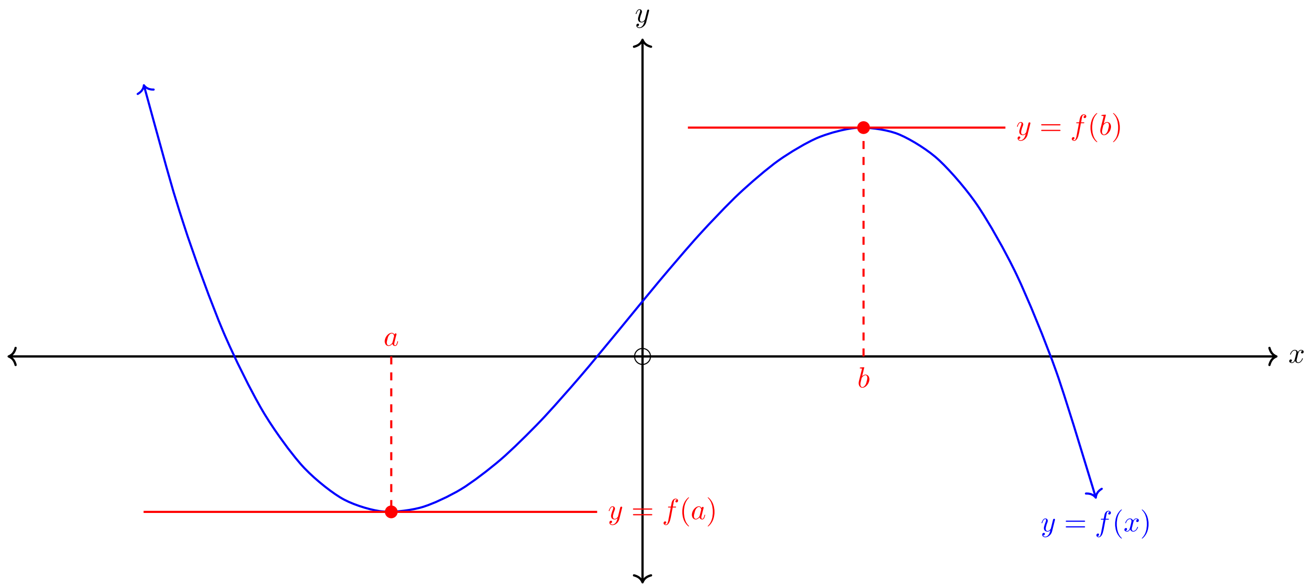
$$\Rightarrow \qquad y = f'(a)(x - a) + f(a)$$

Is the equation of the tangent line

1.2.1 Horizontal Tangents

Horizontal tangents have a gradient of 0.

These will be very important later, when we investigate stationary points in more detail.



In general, we will need to find where the stationary points are:

- Find $f'(x)$
- Solve $f'(x) = 0$
- Substitue the solution(s) into $f(x)$ to find the constant terms for the horizontal line equation

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1.2.2 Ex13A

- Tangent equations from root and polynomial functions
- Horzontal tangents
- Natural log and exponent questions
- Concept questions

3 worked examples

- Polynomial
- Log or exponential
- Conceptual

2. Find the equation of the tangent to:

a) $y = x - 2x^2 + 3$ at $x = 2$

$\frac{dy}{dx} = 1 - 4x$

when $x = 2$

$\frac{dy}{dx} = 1 - 4(2) = -7$

and $y = (2) - 2(2)^2 + 3$
$= -3$

$\frac{dy}{dx} = -7$ Point $(2, -3)$

$y - -3 = -7(x - 2)$

$y + 3 = -7x + 14$

$y = -7x + 11$

d) $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$

$y = 4x^{-\frac{1}{2}}$

$\frac{dy}{dx} = 4 \cdot -\frac{1}{2}x^{-\frac{3}{2}}$

$= -2x^{-\frac{3}{2}} = \frac{-2}{x^{3/2}}$

$x = 1$

$\frac{dy}{dx} = \frac{-2}{1} = -2$

Point $(1, 4)$

$y - 4 = -2(x - 1)$

$y - 4 = -2x + 2$

$y = -2x + 6$

(Horizontal Tangents)

3. c) Find the equation of horizontal tangents to

$y = \sqrt{x} + \frac{1}{\sqrt{x}}$

$\Rightarrow y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} + -\frac{1}{2}x^{-\frac{3}{2}}$

(factorise using lowest power)

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{3}{2}}(x^1 - 1)$

Set $\frac{dy}{dx} = 0$

$\frac{1}{2}x^{-\frac{3}{2}}(x - 1) = 0$

$x^{-\frac{3}{2}} \neq 0$ $x - 1 = 0$
$x = 1$

Find y

$y = \sqrt{1} + \frac{1}{\sqrt{1}} = 1 + \frac{1}{1}$

$= 2$

Horizontal tangent at $(1, 2)$

Equation: $y = 2$

$\frac{dy}{dx} = \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$

$= \frac{x - 1}{2x^{3/2}}$

$\frac{x-1}{2x^{3/2}} = 0 \Rightarrow x - 1 = 0$
$x = 1$

5. Find another tangent to

$y = 1 - 3x + 12x^2 - 8x^3$

which is parallel to the tangent at $(1, 2)$

$\frac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$

$\frac{dy}{dx} = -3 + 24(1) - 24(1)^2$

$= -3$

so gradient is -3

Find all x values where gradient $= -3$

$-3 + 24x - 24x^2 = -3$

$24x - 24x^2 = 0$

$x - x^2 = 0$

$x(1 - x) = 0$

$x = 0$ $\quad$ $x = 1$ already have

Find equation of tangent at $x = 0$

$y = 1$

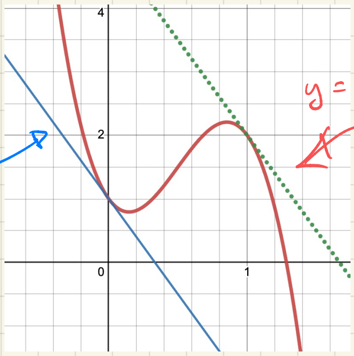
$y - 1 = -3(x - 0)$

$y - 1 = -3x$, $y = -3x + 1$

other tangent $y - 2 = -3(x - 1)$

$y = -3x + 5$

$y = -3x+1$



$y = -3x+5$

6. $y = x^2 + ax + b$

when $x = 1$ tangent is $2x + y = 6$

$y = -2x + 6$

$\frac{dy}{dx} = -2$ $\qquad$ $y = 4$

$(1, 4)$ point on curve.

-2 gradient at that point.

$\frac{dy}{dx} = 2x + a$

$-2 = 2(1) + a$

$a = -4$

$y = x^2 - 4x + b$

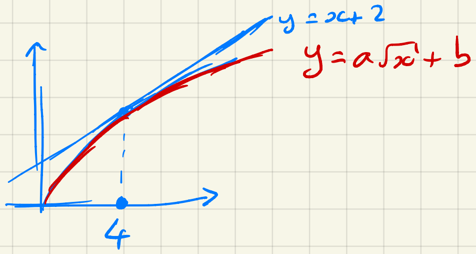
$4 = (1)^2 - 4(1) + b$

$4 = -3 + b$

$b = 7$

$y = x^2 - 4x + 7$

7. Find a and b



What do we know?

- when $x = 4$ $\frac{dy}{dx} = 1$
- $(4, 4+2) = (4, 6)$ is on $y = a\sqrt{x} + bx$

① using $(4, 6)$

$$6 = a\sqrt{4} + b(4)$$
$$6 = 2a + 4b$$
$$\Rightarrow 3 = a + 2b$$

② $\frac{dy}{dx} = \frac{a}{2\sqrt{x}} + b$

$$1 = \frac{a}{2\sqrt{4}} + b$$

$$1 = \frac{a}{4} + b$$

$$4 = a + 4b$$

Equations

$$3 = a + 2b \quad ①$$
$$4 = a + 4b \quad ②$$

② - ① $1 = 2b$

$b = \frac{1}{2}$

using ① $a = 3 - 2b$

$$= 3 - 2\left(\tfrac{1}{2}\right)$$

$$= 2$$

$$y = 2\sqrt{x} + \frac{x}{2}$$

8. Show equation of tangent to $y = 2x^2 - 1$ at $x = a$ is

$$4ax - y = 2a^2 + 1$$

Find point:

$x = a$

$y = 2a^2 - 1$

$(a, 2a^2 - 1)$

$$\frac{dy}{dx} = 4x$$

$$= 4a$$

$$y - (2a^2 - 1) = 4a(x - a)$$

$$y - 2a^2 + 1 = 4ax - 4a^2$$

$(-y)$ $-2a^2 + 1 = 4ax - 4a^2 - y$

$(+4a^2)$ $2a^2 + 1 = 4ax - y$

10d

$y = \ln\sqrt{x}$

$y = \ln(x^{\frac{1}{2}})$

$y = \frac{1}{2}\ln(x)$

$\frac{dy}{dx} = \frac{1}{2x}$

$y = -1$

$\frac{1}{2}\ln(x) = -1$

$\ln(x) = -2$

$x = e^{-2}$

$(e^{-2}, -1)$

$\frac{dy}{dx} = \frac{1}{2e^{-2}}$

$y+1 = \frac{1}{2e^{-2}}(x - e^{-2})$

$y = \frac{x}{2e^{-2}} - \frac{1}{2} - 1$

$y = \frac{e^{2}}{2}x - \frac{3}{2}$

Using $\ln x$ and e^{x}

Find the equation of the tangent to $y = \ln x$ where $y = -1$

- Find x

$\ln x = -1 \quad (\log_e x = 1)$

$x = e^{-1}$

- $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$

$x = e^{-1}, \quad \frac{dy}{dx} = (e^{-1})^{-1} = e^{1}$

$= e$

Point $(e^{-1}, -1)$

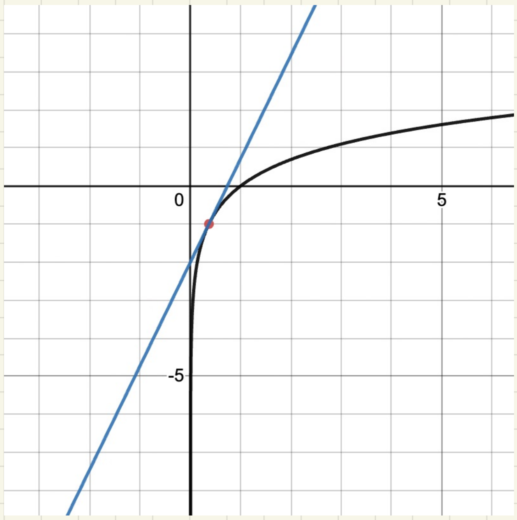
$\frac{dy}{dx} = e$

$y - -1 = e(x - e^{-1})$

$y + 1 = ex - e^{1}e^{-1}$

$y + 1 = ex - 1$

$y = ex - 2$



12. $f(x) = \ln\left(x(x-2)\right)$

Domain of log function > 0

$\Rightarrow \quad x(x-2) > 0$

solve $\quad x(x-2) = 0$

$x = 0 \quad x = 2$

$+ \qquad - \qquad +$

Domain $\quad x < 0 \;\cup\; x > 2$

$\{x \mid x < 0 \;\cup\; x > 2\}$

$$f'(x) = \frac{1}{x(x-2)} \cdot \frac{d}{dx}\left(x^2 - 2x\right)$$

$$= \frac{1}{x(x-2)} \cdot 2x - 2$$

$$= \frac{2(x-1)}{x(x-2)}$$

$$= \frac{2x-2}{x^2-2x}$$

$$= \frac{2x-2}{x(x-2)}$$

$$= \frac{x + x - 2}{x(x-2)}$$

$$= \frac{1}{x} + \frac{1}{x-2}$$

$x = 3, \quad f'(x) = \frac{1}{3} + \frac{1}{1} = \frac{4}{3}$

$x = 3, \quad f(3) = \ln\left(3(3-2)\right) = \ln 3$

$(3, \ln 3)$

$$y - \ln 3 = \frac{4}{3}(x-3)$$

$$y = \frac{4}{3}x - 4 + \ln 3$$

13.

$$y = x^2 e^x$$

$y = uv \qquad \frac{dy}{dx} = uv' + u'v$ (product rule)

$$\frac{dy}{dx} = x^2 e^x + 2x e^x$$

$$= x e^x (x+2)$$

$x = 1$

$$\frac{dy}{dx} = e(3) = 3e$$

$x = 1, \quad y = e$

Gradient: $3e$

Point: $(1, e)$

$$y - e = 3e(x-1)$$

$$y = 3ex - 2e$$ equation of tangent

Find intercepts.

$x = 0 \qquad y = -2e$

$y = 0 \qquad x = \frac{2e}{3e}$

$$x = \frac{2}{3}$$

Intercepts

$(0, -2e)$ y-intercept

$\left(\frac{2}{3}, 0\right)$ x-intercept.

14. $y = 3xe^{\frac{x}{2}}$

$$\frac{dy}{dx} = 3x\left(e^{\frac{x}{2}}\right)' + (3x)'e^{\frac{x}{2}}$$

$$= 3x\,e^{\frac{x}{2}} \cdot \frac{1}{2} + 3e^{\frac{x}{2}}$$

$$= 3e^{\frac{x}{2}}\left(\frac{x}{2} + 1\right)$$

When $x = -1$

$$\frac{dy}{dx} = 3e^{-\frac{1}{2}}\left(\frac{-1}{2} + 1\right)$$

$$= \frac{3}{2}e^{-\frac{1}{2}}$$

$x = -1$, $y = -3e^{-\frac{1}{2}}$

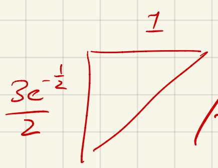
$$y + 3e^{-\frac{1}{2}} = \frac{3e^{-\frac{1}{2}}}{2}(x+1)$$

$$y = \frac{3e^{-\frac{1}{2}}}{2}x + \frac{3e^{-\frac{1}{2}}}{2} - 3e^{-\frac{1}{2}}$$

$$= \frac{3e^{-\frac{1}{2}}}{2}x - \frac{3e^{-\frac{1}{2}}}{2}$$

if $x = 0$ $y = -\frac{3e^{-\frac{1}{2}}}{2}$

if $y = 0$ $x = 1$



$$A = \frac{3e^{-\frac{1}{2}}}{4}$$

$$A = \frac{3}{4\sqrt{e}} \text{ units}^2$$

15. Find the equation of the tangent to $y = \frac{1}{\sin 2x}$ at the point where $x = \frac{\pi}{4}$

$u = 1$, $u' = 0$
$v = \sin 2x$ $v' = 2\cos(2x)$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

$$\frac{dy}{dx} = -\frac{1 \cdot 2\cos(2x)}{\sin^2(2x)} = \frac{-2\cos(2x)}{\sin^2(2x)}$$

$$= -2\cot(x)\operatorname{cosec}(x)$$

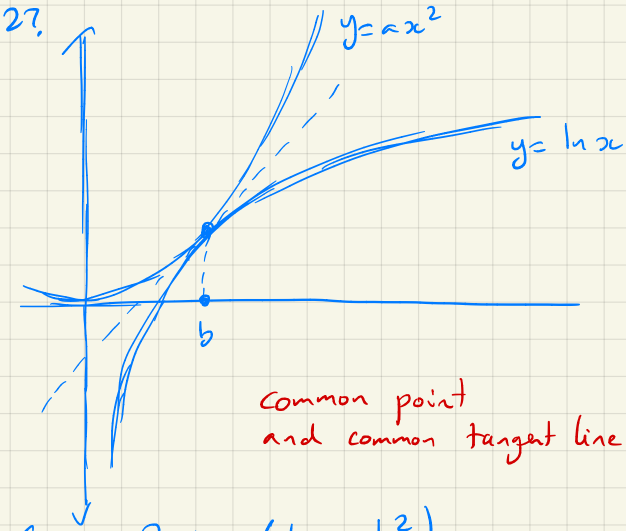
when $x = \frac{\pi}{4}$ $\frac{dy}{dx} = \frac{-2\cos\left(\frac{\pi}{2}\right)}{\sin^2\left(\frac{\pi}{2}\right)} = 0$

when $x = \frac{\pi}{4}$, $y = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

Horizontal tangent

$y = 1$

27.



Common Point (b, ab^2)

a) $(b, \ln b)$

$\Rightarrow \quad ab^2 = \ln b$

Common Tangent

$\frac{dy}{dx} = 2ax \quad , \quad \frac{dy}{dx} = \frac{1}{x}$

$x = b \qquad 2ab = \frac{1}{b} \quad \Rightarrow \quad 2ab^2 = 1$

b) $ab^2 = \ln b$ ①

$2ab^2 = 1$ ②

$2ab^2 = 2\ln b$

$2ab^2 = 1$

$2\ln b = 1$

$\ln b = \frac{1}{2}$

$b = e^{\frac{1}{2}} = \sqrt{e}$

Point $(e^{\frac{1}{2}}, \ln e^{\frac{1}{2}}) = (e^{\frac{1}{2}}, \frac{1}{2})$

c) $a = \frac{1}{2b^2}$

$a = \frac{1}{2e}$

d) $a = \frac{1}{2e} \quad , \quad b = \sqrt{e}$

Point $(e^{\frac{1}{2}}, \frac{1}{2})$

Gradient $\frac{dy}{dx} = \frac{1}{b} = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$

$y - \frac{1}{2} = e^{-\frac{1}{2}}(x - e^{\frac{1}{2}})$

$y - \frac{1}{2} = e^{-\frac{1}{2}}x - 1$

$y = x\sqrt{e} - \frac{1}{2}$

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28. $P(x) = ax^2$

$p'(x) = 2ax$

$x = s \qquad (s, as^2) \quad p'(s) = 2as$

$y - as^2 = 2as(x - s)$

$y = 2asx - 2as^2 + as^2$

$y = 2asx - as^2$

$x = t$

$y = 2atx - at^2$

$2asx - as^2 = 2atx - at^2$

$x(2as - 2at) = as^2 - at^2$

$x = \frac{a(s^2 - t^2)}{2a(s-t)}$

$= \frac{(s+t)(s-t)}{2(s-t)}$

$= \frac{s+t}{2}$

If perpendicular

$$2as \cdot 2at = -1$$

$$y = 2atx - at^2$$

$$x = \frac{s+t}{2}$$

$$y = \frac{2at(s+t)}{2} - at^2$$

$$= ast + at^2 - at^2$$

$$= ast$$

$$t = \frac{-1}{4a^2s}$$

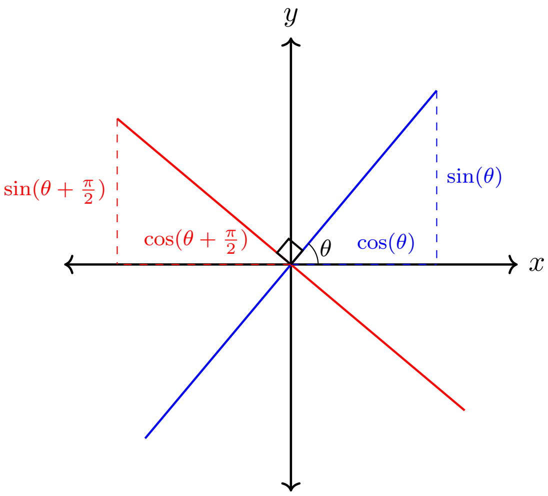
$$y = ast$$

$$= as \cdot \frac{-1}{4a^2s}$$

$$= \frac{-1}{4a}$$

1.3 Normals

The product of the gradients of perpedicular lines = -1
There are various proofs, one example below.



$$m = \frac{\sin(\theta)}{\cos(\theta)}$$

$$m_\perp = \frac{\sin(\theta+\frac{\pi}{2})}{\cos(\theta+\frac{\pi}{2})} = -\frac{\cos(\theta)}{\sin(\theta)}$$

$$m \times m_\perp = -1$$

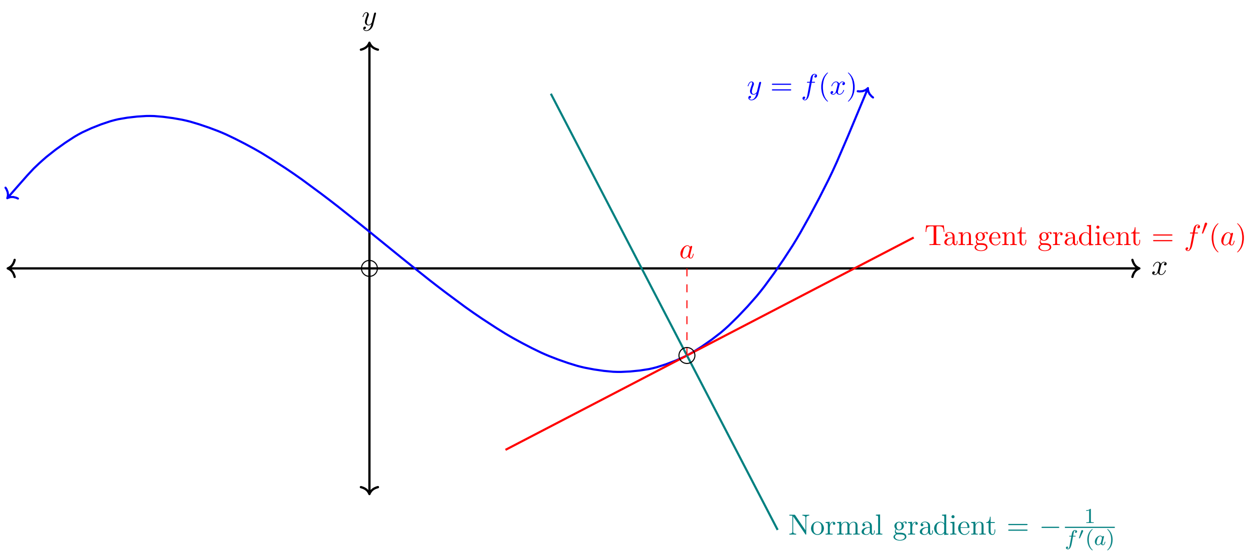
For the function $y = f(x)$, and some $x = a$

- $(a, f(a))$ is on the curve
- $-\dfrac{1}{f'(a)}$ is the gradient of the **normal to the curve** at $x = a$

$$\frac{y - f(a)}{x - a} = -\frac{1}{f'(a)}$$

$$\Rightarrow \qquad y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Is the equation of the tangent line



1.3.1 Ex 13B

13B

1. d) Find the equation of the normal to:

$$y = 8\sqrt{x} - \frac{1}{x^2} \quad \text{at } x = 1$$

Point $x = 1$

$$y = 8\sqrt{1} - \frac{1}{1^2} = 7$$

$(1,7)$

Gradient:

$$y = 8x^{\frac{1}{2}} - x^{-2}$$

$$\frac{dy}{dx} = 8 \cdot \frac{1}{2} x^{-\frac{1}{2}} - -2x^{-3}$$

$$= \frac{4}{x^{1/2}} + \frac{2}{x^3}$$

$$= \frac{4x^{\frac{5}{2}} + 2}{x^3}$$

$$\frac{dy}{dx}_\perp = \frac{-x^3}{4x^{5/2}+2}$$

$x = 1$

$$\frac{dy}{dx}_\perp = \frac{-1}{4+2} = \frac{-1}{6}$$

$(1,7)$ and $-\frac{1}{6}$

$$y - 7 = -\frac{1}{6}(x-1)$$

$$y - 7 = -\frac{x}{6} + \frac{1}{6}$$

$$y = -\frac{x}{6} + \frac{43}{6}$$

or

$$6y + x = 43$$

or $x + 6y - 43 = 0$:)

3. c) Find the equation of the normal to

$$y = e^{2x-1} \quad \text{at } x = 1$$

Point $(1, e^1)$

$$\frac{dy}{dx} = e^{2x-1} \cdot 2$$

$$= 2e^{2x-1}$$

$$\frac{dy}{dx}_\perp = \frac{-1}{2e^{2x-1}} = -\frac{e^{1-2x}}{2}$$

$x = 1$

$$\frac{dy}{dx}_\perp = \frac{-1}{2e}$$

$$y - e = \frac{-1}{2e}(x-1)$$

$$y = \frac{-x}{2e} + \frac{1}{2e} + e$$

$$y = \frac{-xe^{-1}}{2} + \frac{1+2e^2}{2e}$$

or

$$2ey - 2e^2 = -x + 1$$

$$2ey + x = 1 + 2e^2$$

$$x + 2ey = 1 + 2e^2$$

5. Find the points where the normal to

$$y = x^3 - 2x^2 + 1$$

at $x=1$ meets the curve again.

Find equation of normal at $x=1$ and then set equal to y, find intersection points.

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{dy}{dx}_{\perp} = \frac{-1}{3x^2-4x}$$

$x=1$ $\quad \frac{dy}{dx}_{\perp} = \frac{-1}{3-4} = 1$

$x=1$, $y = 1-2+1 = 0$

$$y - 0 = 1(x-1)$$

$$y = x - 1 \quad \text{(normal)}$$

set $\quad x^3 - 2x^2 + 1 = x - 1$

$$\Rightarrow \quad x^3 - 2x^2 - x + 2 = 0$$

We already have one solution $x=1 \Rightarrow x-1$ is a factor

$$(x-1)(x^2 - x - 2) = 0$$

← deduce

or

$$
\begin{array}{r}
x^2 - x - 2 \\
x-1 \overline{) x^3 - 2x^2 - x + 2} \\
\underline{x^3 - x^2} \quad\quad\quad \\
-x^2 - x \quad\; \\
\underline{-x^2 + x} \quad\; \\
-2x + 2 \\
\underline{-2x + 2} \\
0
\end{array}
$$

$$(x-1)(x^2 - x - 2)$$

$$\Rightarrow (x-1)(x-2)(x+1) = 0$$

Solution $x=1$, $x=2$ $\quad x=-1$

so new points

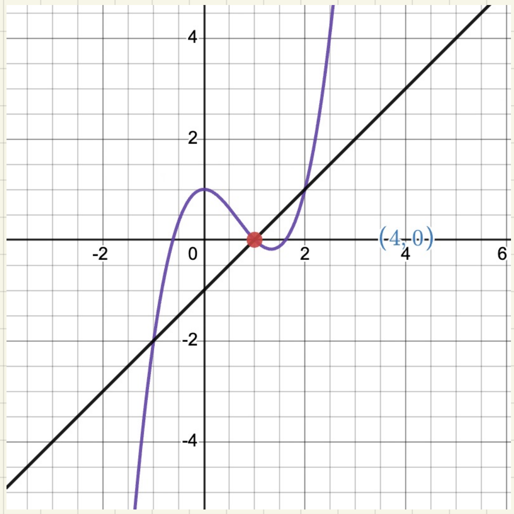
$x=2$

$$y = (2)^3 - 2(2)^2 + 1 = 1 \qquad (2, 1)$$

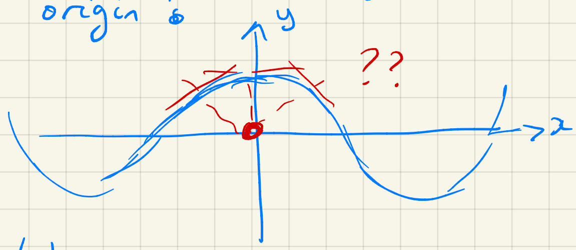
$x = -1$

$$y = (-1)^3 - 2(-1)^2 + 1 = -2 \qquad (-1, -2)$$

points are $(2, 1)$, $(-1, -2)$



6. Find the equation of the normal to $f(x) = \cos(x)$ which passes through the origin.



let $x = a$

$\Rightarrow (a, \cos(a))$ is on the curve

$$\frac{dy}{dx} = -\sin(x)$$

$$\frac{dy}{dx}\perp = \frac{1}{\sin(x)}$$

$$\frac{dy}{dx}\perp = \frac{1}{\sin(a)}$$

$$\frac{\cos(a) - 0}{a - 0} = \frac{1}{\sin(a)}$$

$$\sin(a)\cos(a) = a$$
$$2\sin(a)\cos(a) = 2a$$

$$\sin(2a) = 2a$$

Solve $\sin(x) = x$?

We know $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$

$\Rightarrow \sin(x) = x$
when $x = 0$
$a = 0$

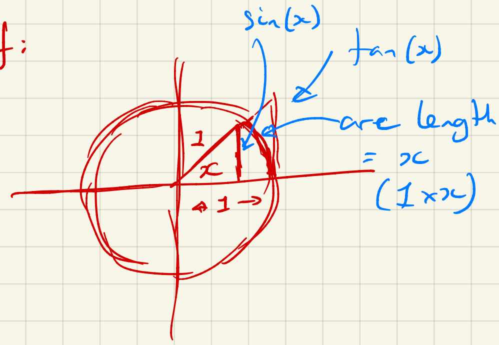
Point $(0, 1)$

Normal line is vertical

$\Rightarrow x = 0$

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

Proof:



$$\sin(x) < x < \tan(x)$$

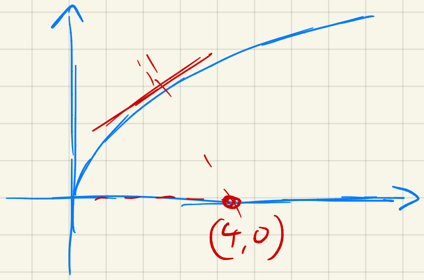
$$\Rightarrow \frac{1}{\sin(x)} > \frac{1}{x} > \frac{\cos(x)}{\sin(x)}$$

$$1 > \frac{\sin(x)}{x} > \cos(x)$$

$$\lim_{x \to 0} \Rightarrow 1 > \frac{\sin(x)}{x} > 1$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin(x)}{x} = 1$$

7. Find the equation to $y=\sqrt{x}$ from the external point $(4,0)$



$y=\sqrt{x}$, $\frac{dy}{dx}=\frac{1}{2\sqrt{x}}$, $\frac{dy}{dx}_{\perp}=-2\sqrt{x}$

let the coordinate on the curve be $(a,\sqrt{a})$

so $\frac{dy}{dx}_{\perp}=-2\sqrt{a}$

and $\frac{\sqrt{a}-0}{a-4}=-2\sqrt{a}$

$\sqrt{a}=-2\sqrt{a}(a-4)$

$\sqrt{a}=-2a\sqrt{a}+8\sqrt{a}$

$2a\sqrt{a}-7\sqrt{a}=0$

$\sqrt{a}(2a-7)=0$

$\sqrt{a}=0$ $\quad 2a-7=0$

$a=\frac{7}{2}$

reject, because derivative is not defined at the end point of a curve

$a=\frac{7}{2}$

Point $\left(\frac{7}{2},\sqrt{\frac{7}{2}}\right)$

$\frac{dy}{dx}_{\perp}=-2\sqrt{\frac{7}{2}}$

$y-\sqrt{\frac{7}{2}}=-2\sqrt{\frac{7}{2}}\left(x-\frac{7}{2}\right)$

$y=-2\sqrt{\frac{7}{2}}x+7\sqrt{\frac{7}{2}}+\sqrt{\frac{7}{2}}$

$y=-2\sqrt{\frac{7}{2}}x+8\sqrt{\frac{7}{2}}$

$y=-\sqrt{\frac{4\times7}{2}}x+\sqrt{\frac{7\times64}{2}}$

$=-\sqrt{14}x+\sqrt{7\times32}$

$=-\sqrt{14}x+\sqrt{16\times7\times2}$

$=-\sqrt{14}x+4\sqrt{14}$

1.4 Increasing and Decreasing

When analysing functions, we are often interested in the intervals across which the function is increasing or decreasing.

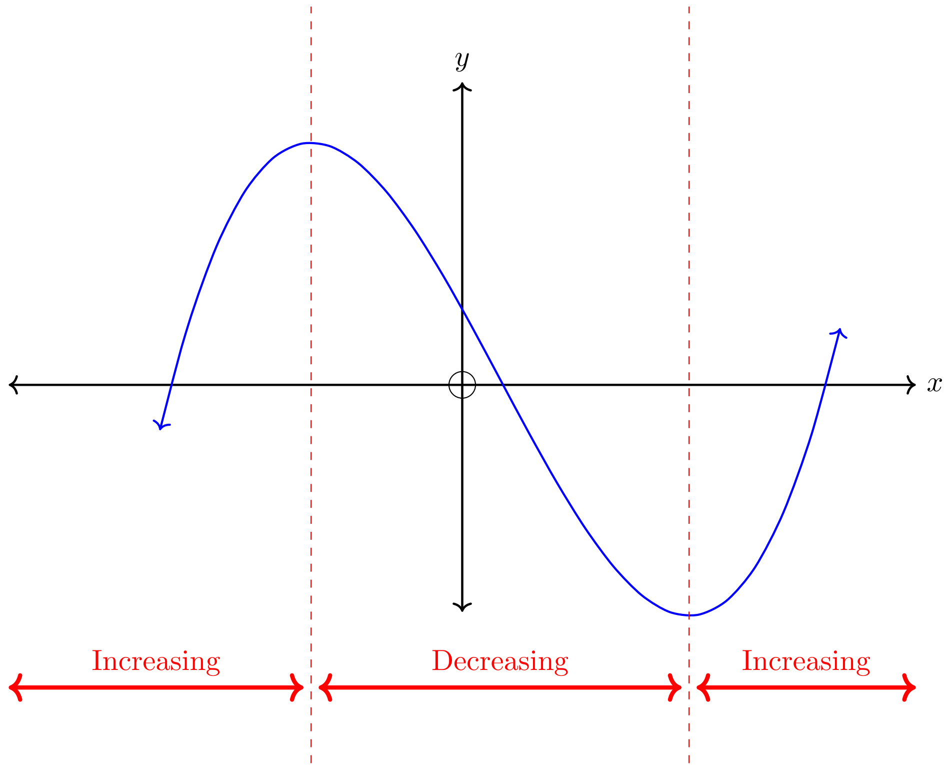
- If the gradient is entirely positive on an interval, the function is increasing on that interval.
- If the gradient is entirely negative on an interval, the function is decreasing on that interval.

If a function changes from increasing to decreasing, it must pass through a stationary point (or cross a region where the function is not differentiable)

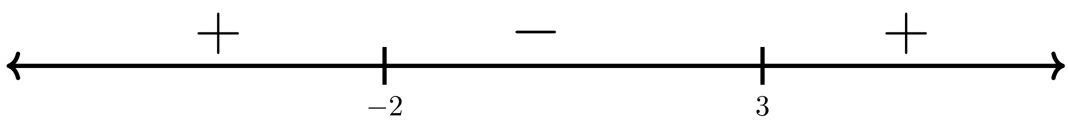
Hnece we need to divide up the domain into intervals bounded by stationary points and points where the function is not differentiable.
Finding a single gradient on each interval will tell us whether the function is increasing or decreasing.

We can communicate all of this using a **sign diagram**.

1.4.1 Basic sign diagram example



Sign diagram: $f'(x)$ either calculated or observed from a graph



Worked Example:

Find the intervals where $f(x)$ is increasing, for the function: $f(x) = -4x^3 + 15x^2 + 18x + 3$

$$f'(x) = -12x^2 + 30x + 18$$

set $f'(x) = 0$ (to find stationary points)

$$-12x^2 + 30x + 18 = 0$$

$$-6(2x+1)(x-3) = 0$$

$$x = -\frac{1}{2} \qquad x = 3$$

$$-6(2x^2 - 5x - 3) = 0$$

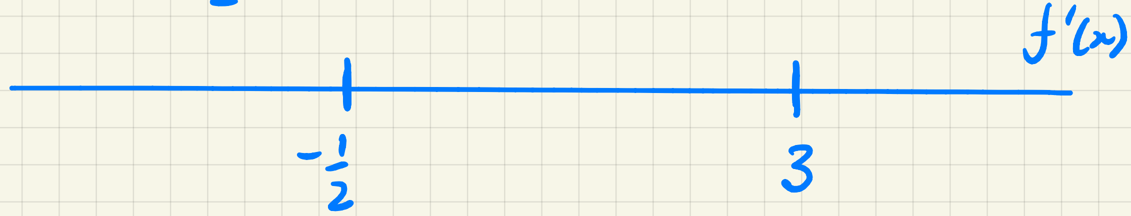
$$-6[2x^2 - 6x + x - 3] = 0$$

$$-6[2x(x-3) + (x-3)] = 0$$

$-6 \quad \times$

$-6 \quad +1$

-5



Test any point in each interval (choose easy points)

Test $x = -1$

$$f'(-1) = -12(-1)^2 + 30(-1) + 18 = -24$$

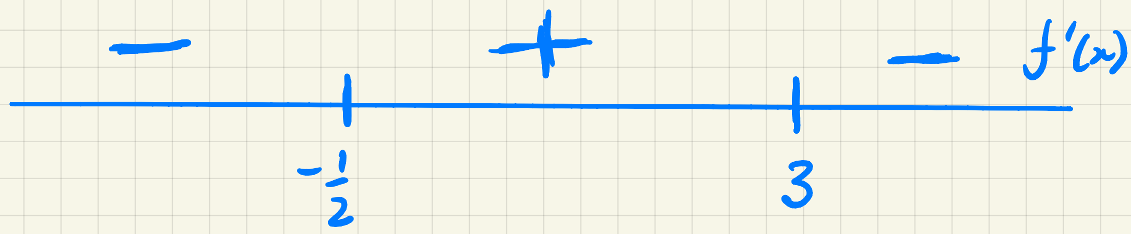
Test $x = 0$

$$f'(0) = +18$$

Test $x = 4$

$$f'(4) = -12(4)^2 + 30(4) + 18 = -54$$

Completed Diagram



1.4.2 Sign diagram example with discontinuites

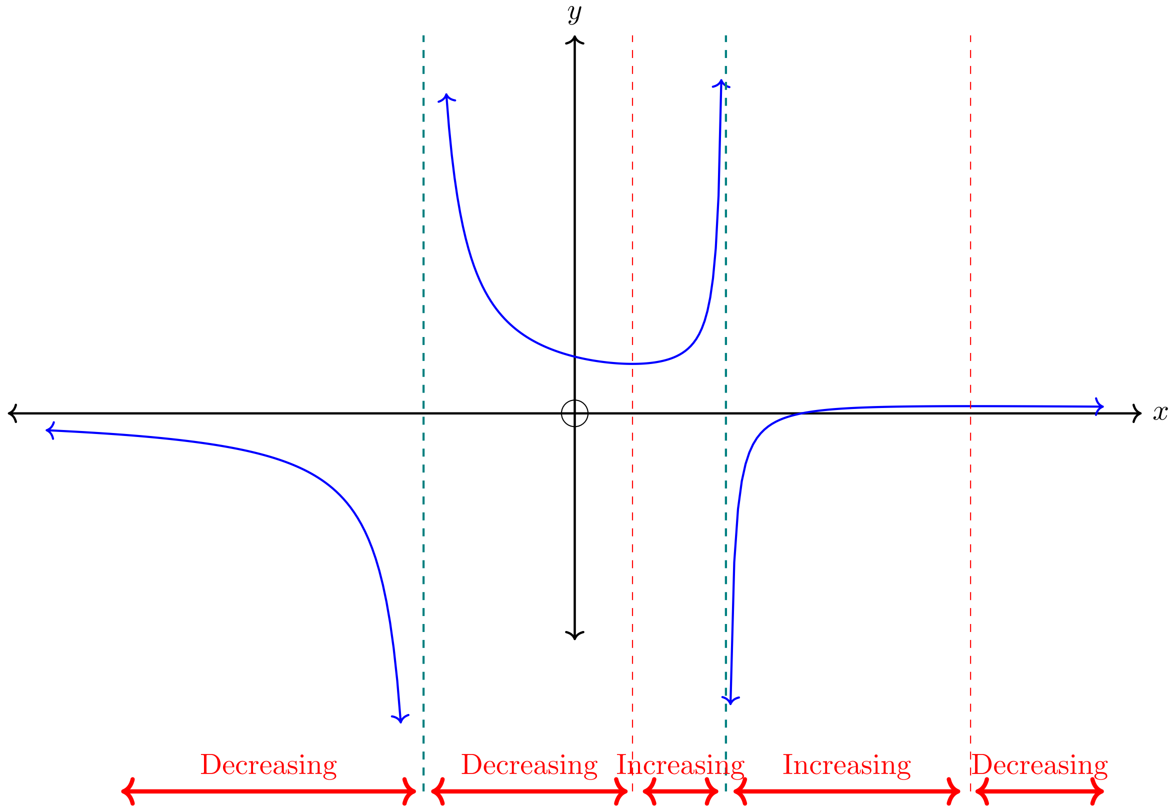
Below is the graph of

$$y = \frac{x-3}{x^2-4}$$

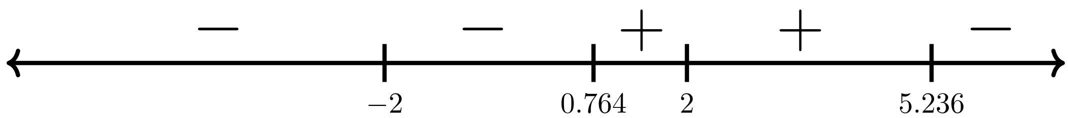
Because division by 0 is undefined, the graph has vertical asymptotes when $x^2 - 4 = 0$
(i.e when $x = 2$ and $x = -2$)

The derivative is $\dfrac{dy}{dx} = \dfrac{-x^2+6x-4}{(x^2-4)^2}$ and using a graphics calulator to solve $-x^2+6x-4=0$,

we have stationary points at $x \approx 0.764$ and $x \approx 5.236$



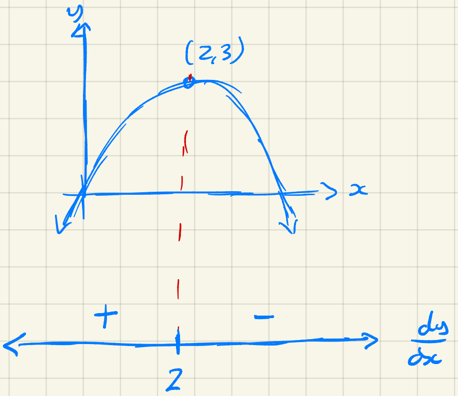
Sign diagram: $f'(x)$ either calculated or observed from a graph



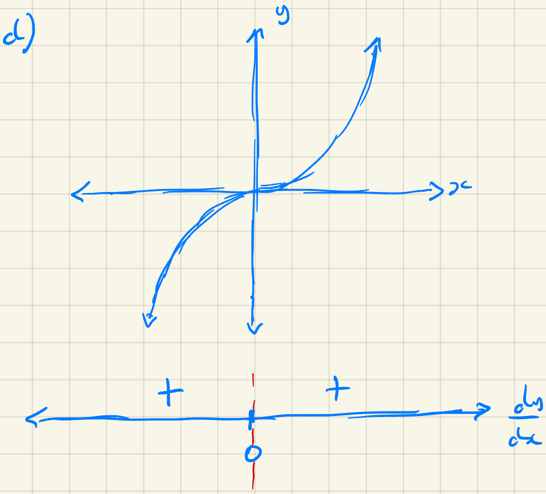
1.4.3 Ex 13 C

13c

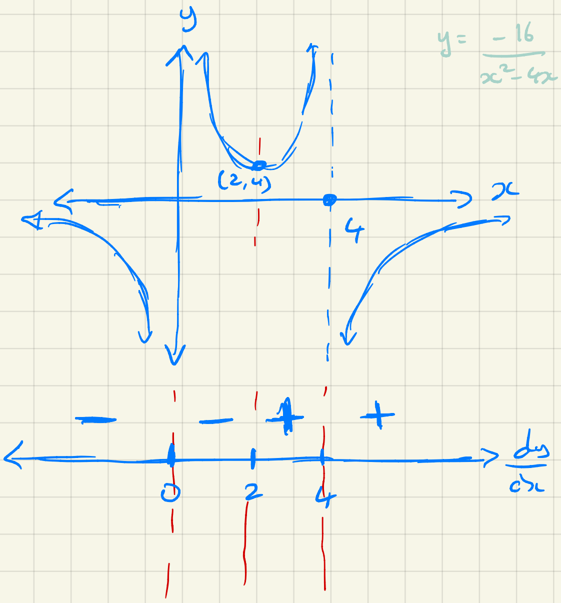
1. c)



d)



f)



4.
e. $f(x) = \frac{2}{\sqrt{x}} = 2x^{-\frac{1}{2}}$

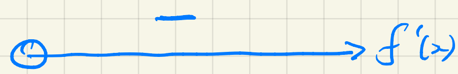
$f'(x) = 2 \cdot -\frac{1}{2} x^{-\frac{3}{2}}$

$= -\frac{1}{x^{3/2}} = -\frac{1}{x\sqrt{x}}$

Domain: $\{x \mid x > 0\}$

if $x < 0$ $\sqrt{x}$ undefined
and $\frac{1}{0}$ is undefined.

Test: $f'(1) = -1$



Function Decreasing on $(0, \infty)$

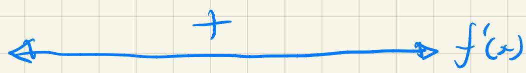
5. $f(x) = x^3 - 3x^2 + 5x + 2$

$f'(x) = 3x^2 - 6x + 5$

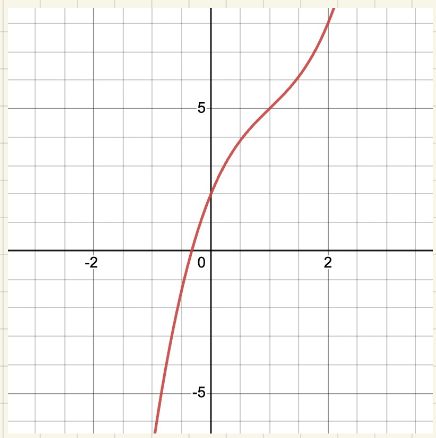
$f'(x) = 0 \implies$ no solution

$\Delta = (-6)^2 - 4 \times 3 \times 5$
$= 36 - 60 = -24$

$f'(x) > 0$ for all x



Function always increasing



9. $f(x) = \dfrac{-x^2+4x-7}{x-1}$

$\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

$u = -x^2+4x-7$, $u' = -2x+4$

$v = x-1$, $v' = 1$

$f'(x) = \dfrac{(-2x+4)(x-1) - (-x^2+4x-7)1}{(x-1)^2}$

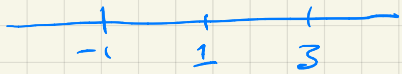
$= \dfrac{-2x^2+6x-4+x^2-4x+7}{(x-1)^2}$

$= \dfrac{-x^2+2x+3}{(x-1)^2} = -\dfrac{(x^2-2x-3)}{(x-1)^2}$

$= -\dfrac{(x-3)(x+1)}{(x-1)^2}$

$f'(x) = 0$ when $x=3$, $x=-1$

also $f(x)$ undefined when $x-1=0$, $x=1$



Test $f'(-2) = -\dfrac{(-5)(-1)}{(-2-1)^2}$

$f'(-2) < 0$

Test $f'(0) = -\dfrac{(-3)(+1)}{(-1)^2}$

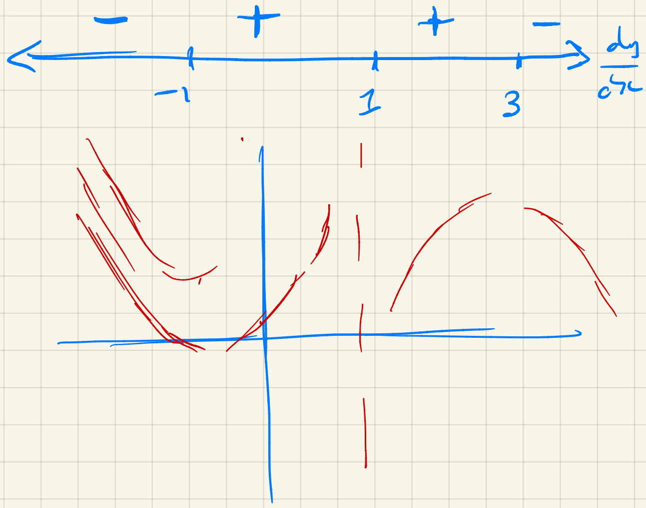
$f'(0) > 0$

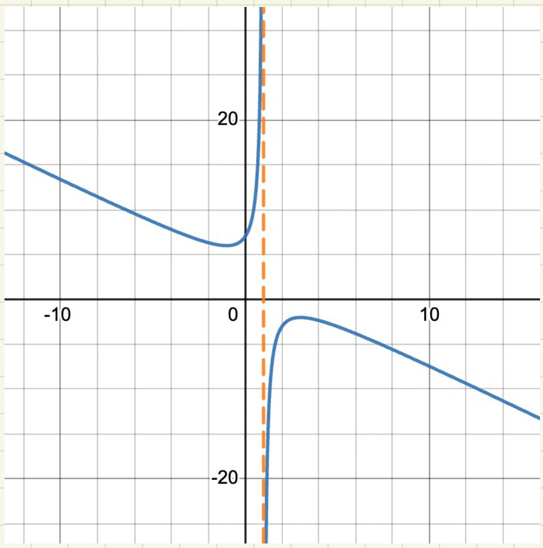
Test $f'(2) = -\dfrac{(-1)(3)}{(2-1)^2}$

$f'(2) > 0$

Test $f'(4) = -\dfrac{(1)(5)}{(4-1)^2}$

$f'(4) < 0$





11. c)

$f(x) = 3 + e^{-x}$

$f'(x) = -e^{-x}$

$f'(x) \neq 0$

Doman $\{x \mid x \in \mathbb{R}\}$

$f'(0) = -e^{0} = -1$

$-$

Decreasing everywhere

d) $f(x) = xe^{x}$

$(uv)' = uv' + u'v$

$f'(x) = xe^{x} + 1e^{x}$

$= e^{x}(x+1)$

$e^{x}(x+1) = 0$

$e^{x} \neq 0 \quad x+1=0$

$x = -1$

Test $f'(0) = e^{0}(0+1) = 1$

$f'(0) > 0$

Test $f'(-2)$

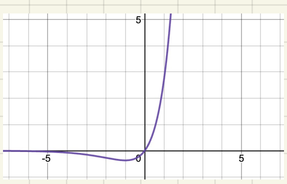
$f'(-2) = e^{-2}(-2+1)$

$= -e^{-2}$

$f'(-2) < 0$

$- \qquad +$

-1



f) $f(x) = x^3 \ln x$

Domain $\{x \mid x > 0\}$

$(uv)' = uv' + u'v$

$f'(x) = x^3 \cdot \frac{1}{x} + 3x^2 \ln(x)$

$= x^2 + 3x^2 \ln x$

$= x^2(1 + 3\ln x)$

$f'(x) = 0$

$\Rightarrow \quad x^2 = 0 \quad , \; x = 0$ (outside domain)

$1 + 3\ln x = 0$

$3\ln x = -1$

$\ln x = -\frac{1}{3}$

$x = e^{-\frac{1}{3}} \qquad x \simeq 0.717$

$0 \qquad 0.717$

Test $x = 0.5$

$f'(0.5) = (0.5)^2(1 + 3\ln(0.5))$

$= (0.5)^2(-1.079)$

$f'(0.5) < 0$

Test $x = 1$

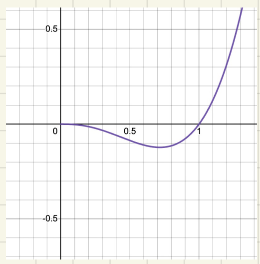
$f'(1) = 1^2(1 + 3\ln 1)$

$= 1^2(1)$

$f'(1) > 0$

$- \qquad +$

$0 \qquad 0.717$



k) $f(x) = \ln(x^2+4)$

$x^2+4 > 0 \quad \forall x \in \mathbb{R}$

$\Rightarrow$ Domain $\{x \mid x \in \mathbb{R}\}$

$(g(u))' = g'(u) \cdot u'$

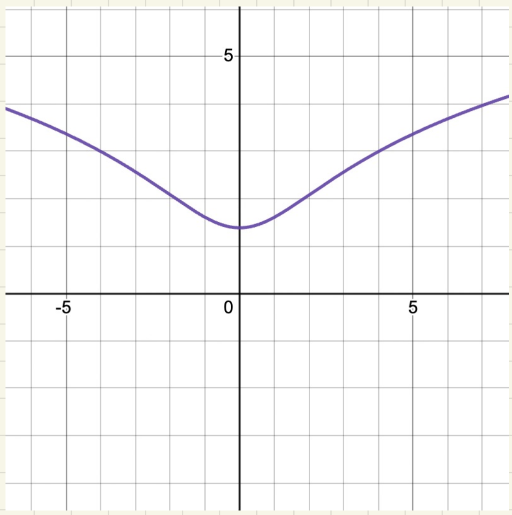
$$f'(x) = \frac{1}{x^2+4} \cdot 2x$$

$$= \frac{2x}{x^2+4}$$

$f'(x) = 0 \Rightarrow x = 0$

Test $x=-1$ $f'(-1) = \frac{-2}{5} < 0$

Test $x=1$ $f'(1) = \frac{2}{5} > 0$

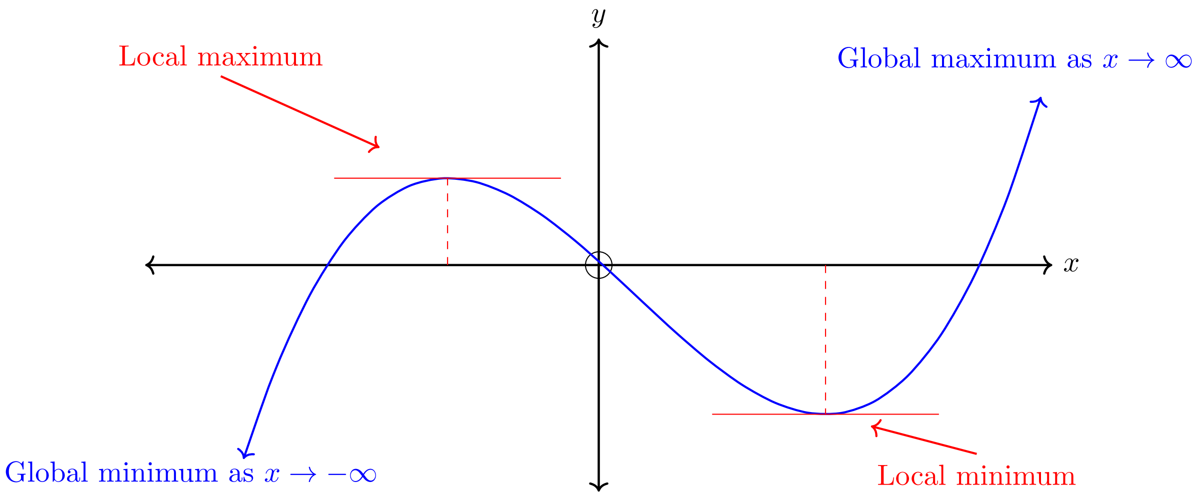


1.5 Stationary Points

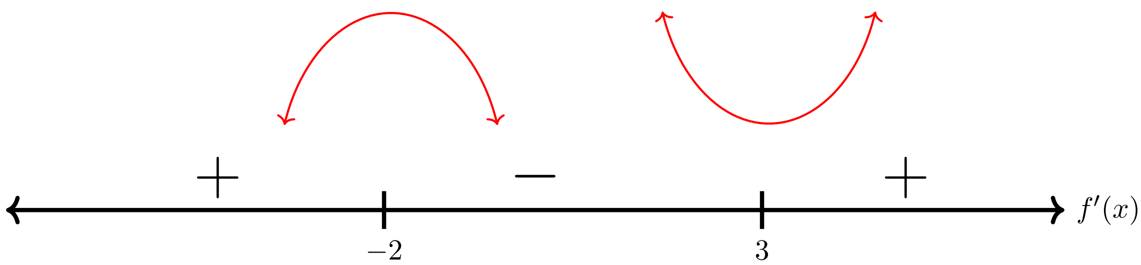
1.5.1 Turning points (minima, maxima)

Frrom our 'standard' graph we can identify a couple of key features.

$$f(x) = x^3 - 1.5x^2 - 18x + 1$$



If we do not have the graph we can classify the stationary points using a sign diagram.

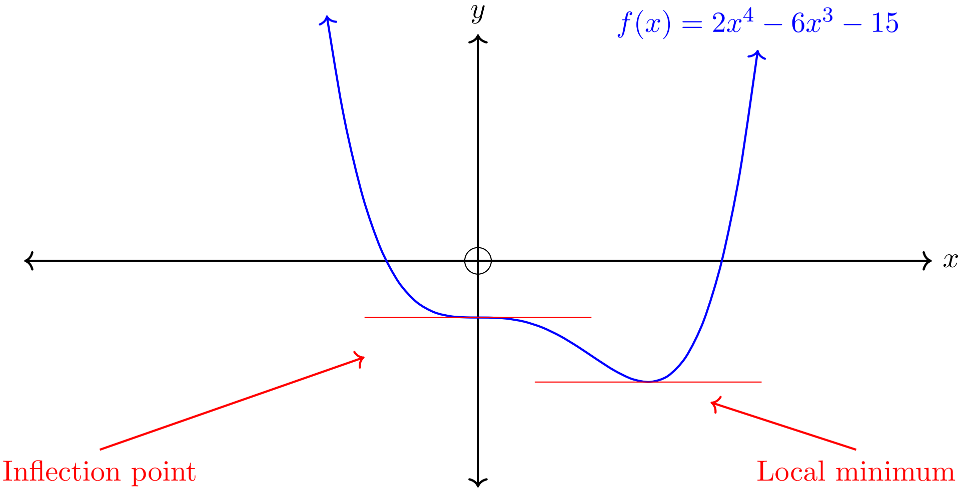


1.5.2 Stationary points of inflection

Stationary points are not necessarily local maxima or minima.

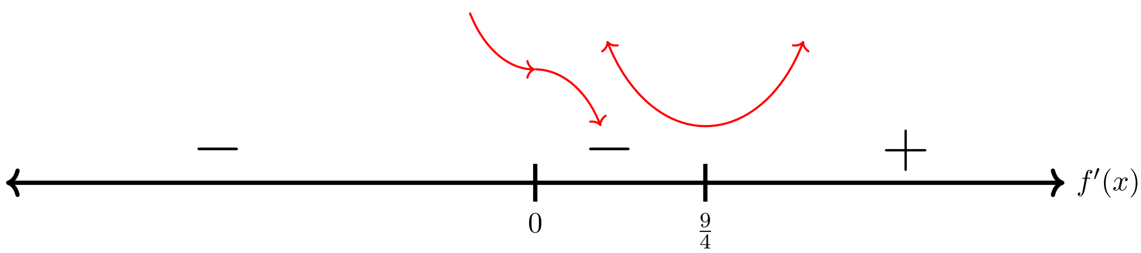
They may be **inflection** points.

This is where the shape of the curve is changing, but the gradient is not changing sign.



Constructing a sign diagram:

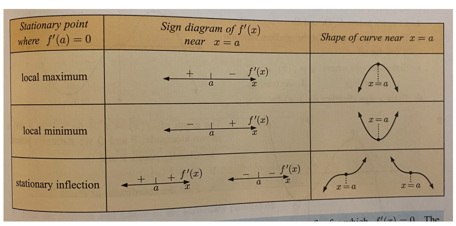
$$
\begin{aligned}
f(x) &= 2x^4 - 6x^3 - 15 \\
f'(x) &= 8x^3 - 18x^2 \\
&= 2x^2(4x-9) \\
\text{Set} \qquad f'(x) &= 0 \\
\Rightarrow \qquad 2x^2(4x-9) &= 0 \\
x = 0 \text{ and } x &= \frac{9}{4}
\end{aligned}
$$



If the sign of the gradient repeats across a staionary point, we will have an inflection point.

Note: Inflection points can be at stationary points, or not. Consequently we classify the one above as a **stationary inflection point**

Please note: Summary chart in the textbook.



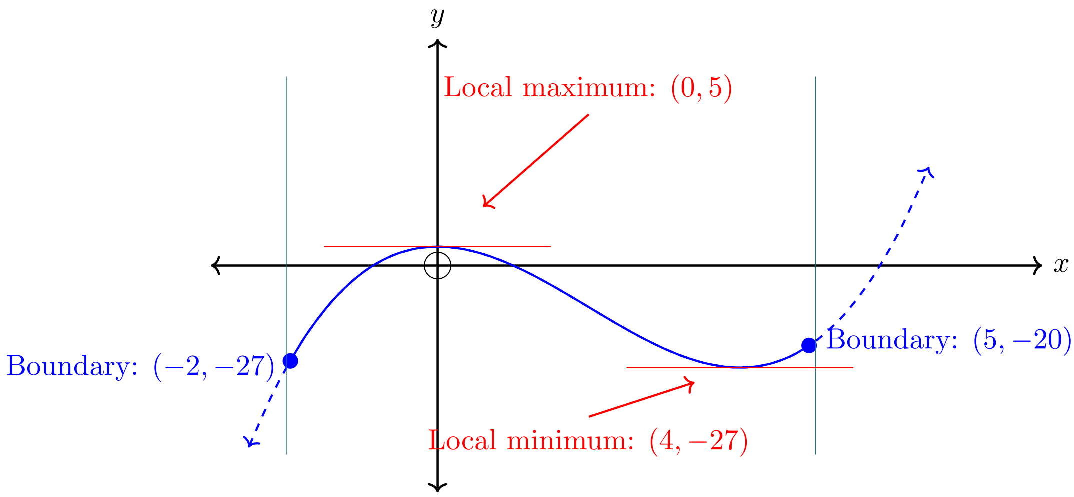
1.5.3 Greatest or least values (testing boundaries)

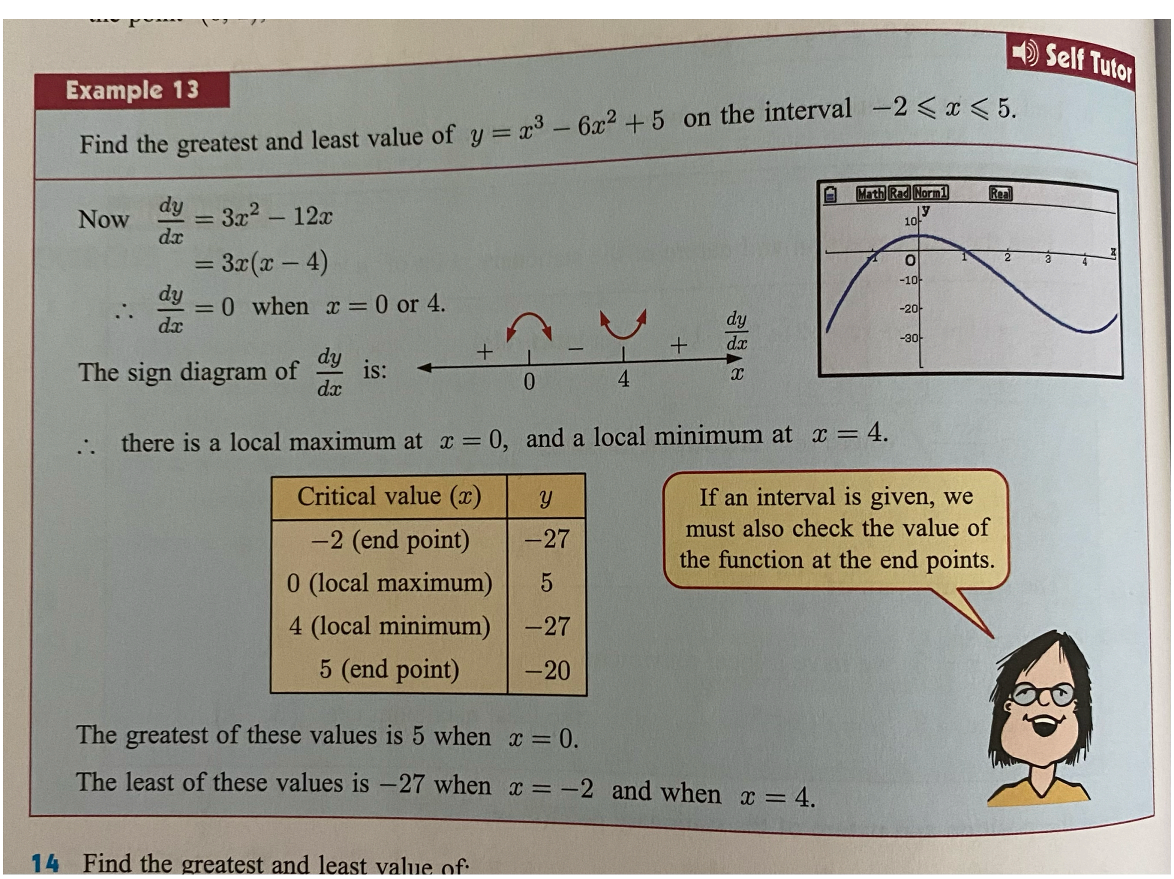
If we are given a function which is bounded on an interval, and are finding the greatest/least values on that interval: we need to

- find the values of the stationary points
- find the values at the end points of the function (as these could be geater/less than the staionary point values).

Following the example from the book using

$$y = x^3 - 6x^2 + 5 \quad \text{On the interval} \quad -2 \leqslant x \leqslant 5$$





Example 13

Self Tutor

Find the greatest and least value of $y = x^3 - 6x^2 + 5$ on the interval $-2 \leqslant x \leqslant 5$.

Now $\dfrac{dy}{dx} = 3x^2 - 12x$

$= 3x(x-4)$

$\therefore \dfrac{dy}{dx} = 0$ when $x = 0$ or 4.

The sign diagram of $\dfrac{dy}{dx}$ is: $+$ | 0 | $-$ | 4 | $+$ $\quad \dfrac{dy}{dx}$, x

$\therefore$ there is a local maximum at $x = 0$, and a local minimum at $x = 4$.

Critical value (x)	y
-2 (end point)	-27
0 (local maximum)	5
4 (local minimum)	-27
5 (end point)	-20

If an interval is given, we must also check the value of the function at the end points.

The greatest of these values is 5 when $x = 0$.

The least of these values is -27 when $x = -2$ and when $x = 4$.

14 Find the greatest and least value of:

1.5.4 Ex 13D

13 D

5.

Find and classify stationary points

g) $f(x) = x - \sqrt{x}$

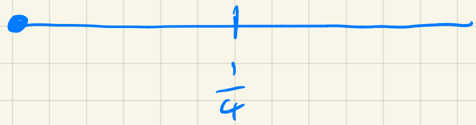
Domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$

$$f'(x) = 1 - \frac{1}{2\sqrt{x}}$$

$$1 - \frac{1}{2\sqrt{x}} = 0 \quad \Rightarrow 2\sqrt{x} = 1$$

$$\sqrt{x} = \frac{1}{2}$$

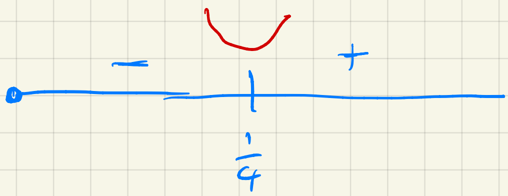
$$x = \frac{1}{4}$$



Test $f'(1) = 1 - \frac{1}{2} = \frac{1}{2}$ $\quad f'(1) > 0$

Test $f'\left(\frac{1}{16}\right) = 1 - \frac{1}{\frac{2}{4}} = 1 - 2 = -1$

$f'\left(\frac{1}{16}\right) < 0$



Local minimum at $x = \frac{1}{4}$

$$f\left(\frac{1}{4}\right) = \frac{1}{4} - \sqrt{\frac{1}{4}} = -\frac{1}{4}$$

$$\left(\frac{1}{4}, -\frac{1}{4}\right)$$

Also note: $f(x) = 0$

when $x - \sqrt{x} = 0$

$$\sqrt{x}\left(\sqrt{x} - 1\right) = 0$$

$$x = 0 \quad x = 1$$

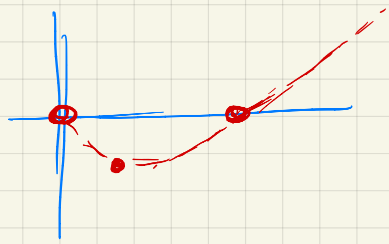
x-intercepts $(0,0)$ $(1,0)$

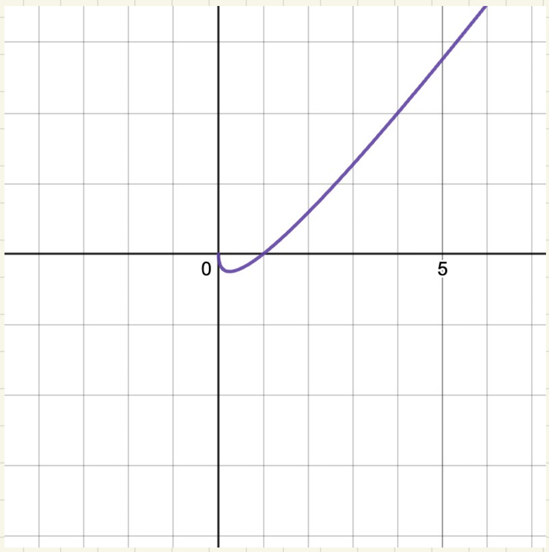
for $\forall x$, $x > 1$

$$x > \sqrt{x}$$

$$\Rightarrow \lim_{x \to \infty} x - \sqrt{x} = \infty$$

but increasing quite slowly





7. d)

$y = e^{-x}(x+2)$

$\frac{dy}{dx} = e^{-x}\cdot 1 + -e^{-x}(x+2)$

$= e^{-x} - e^{-x}(x+2)$

$= e^{-x}(1-(x+2))$

$= e^{-x}(-x-1)$

$= -e^{-x}(x+1)$

$\frac{dy}{dx} = 0 \qquad x = -1$

-1

$x = -2 \qquad \frac{dy}{dx}_{(-2)} = -e^{2}(-2+1)$

> 0

$x = 0 \qquad \frac{dy}{dx}_{(0)} = -e^{0}(0+1)$

< 0

$+ \qquad -$

-1

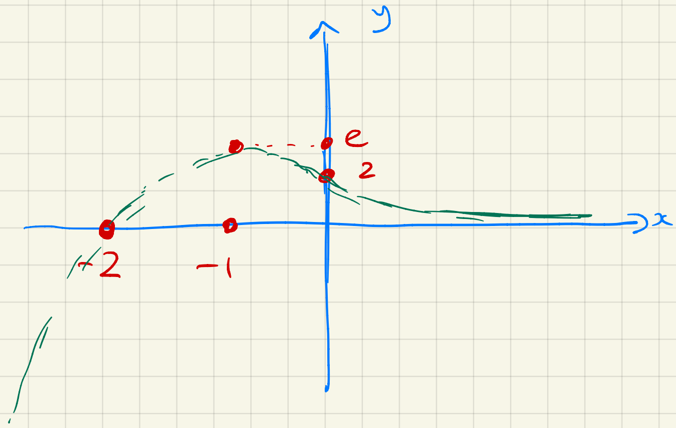
$y(-1) = e^{1}(1) = e$

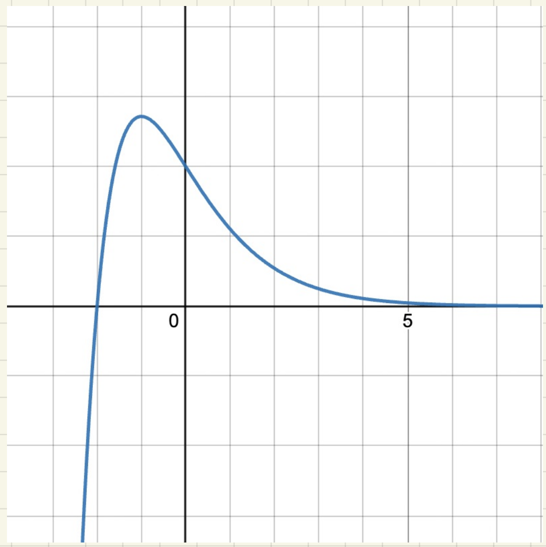
set $y = 0 \Rightarrow x = -2$

$x = 0 \qquad y = 2$

$\lim_{x \to -\infty} e^{-x}(x+2) = -\infty$

$\lim_{x \to \infty} e^{-x}(x+2) = 0$





10. $y = \dfrac{e^{ax}}{bx}$

has stationary point at $\left(\frac{1}{3}, \frac{e}{2}\right)$

Find a, b

Classify the stationary point.

$$\frac{e}{2} = \frac{e^{\frac{a}{3}}}{\frac{b}{3}}$$

$$\frac{be}{6} = e^{\frac{a}{3}}$$

$$b = 6e^{\frac{a}{3}-1}$$

$u = e^{ax}$ $\quad u' = ae^{ax}$

$v = bx$ $\quad v' = b$

$$\frac{dy}{dx} = \frac{ae^{ax}(bx) - be^{ax}}{(bx)^2}$$

$$= \frac{be^{ax}(ax-1)}{(bx)^2}$$

$$= \frac{e^{ax}(ax-1)}{bx^2}$$

$$\frac{dy}{dx}\left(\frac{1}{3}\right) = 0$$

$$\Rightarrow \left(\frac{a}{3} - 1\right) = 0$$

$$a = 3$$

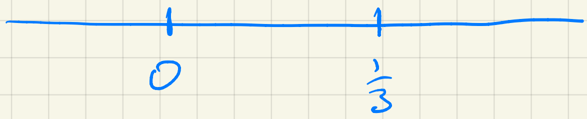
$$b = 6e^{\frac{3}{3}-1}$$

$$= 6$$

$$y = \frac{e^{3x}}{6x}$$

$$\frac{dy}{dx} = \frac{e^{3x}(3x-1)}{6x^2}$$

Function is undefined at $x=0$



Test: $x = -1$

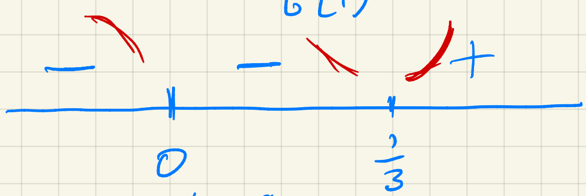
$$\frac{dy}{dx}(-1) = \frac{e^{-3}(-4)}{6(-1)^2} < 0$$

Test: $x = \frac{1}{4}$

$$\frac{dy}{dx}\left(\frac{1}{4}\right) = \frac{e^{3/4}\left(\frac{3}{4}-1\right)}{6\left(\frac{1}{4}\right)^2} < 0$$

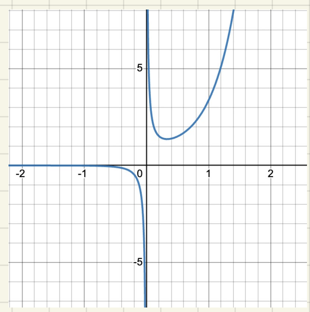
Test: $x = 1$

$$\frac{dy}{dx}(1) = \frac{e^{3}(3-1)}{6(1)^2} > 0$$



$\lim\limits_{x \to -\infty} \frac{e^x}{x} = {}^-0$ $\quad \lim\limits_{x \to 0^+} \frac{e^x}{x} = +\infty$

$\lim\limits_{x \to {}^-0} \frac{e^x}{x} = -\infty$ $\quad \lim\limits_{x \to +\infty} \frac{e^x}{x} +\infty$



12. f)

$f(x) = \sin(2x) + 2\cos(x)$

$f'(x) = 2\cos(2x) - 2\sin(x)$

$f'(x) = 0$

$\Rightarrow 2\cos(2x) - 2\sin(x) = 0$

Trig identity:
$\cos(2\theta) = 1 - 2\sin^2(\theta)$
(chapter 9: page 238)

$2(1 - 2\sin^2(x)) - 2\sin(x) = 0$

$-4\sin^2(x) - 2\sin(x) + 2 = 0$

$-2(2\sin^2(x) + \sin(x) - 1) = 0$

$2y^2 + y - 1$ $\quad -2 \quad (\times)$

$\quad +2 \quad -1$

$2y^2 + 2y - y - 1$ $\quad +1 \quad (+)$

$2y(y+1) - 1(y+1) = (2y-1)(y+1)$

$-2(2\sin(x) - 1)(\sin(x) - 1) = 0$

$\sin(x) = \frac{1}{2} \Rightarrow \sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$

Domain $0 \le x \le 2\pi$

$\frac{\pi}{6} + 2k\pi \Rightarrow \frac{\pi}{6}$ in D

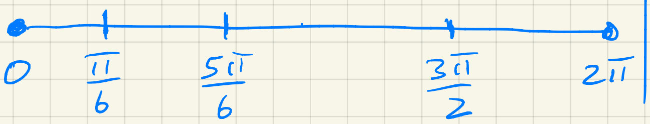
$\pi - \frac{\pi}{6} + 2k\pi \Rightarrow \frac{5\pi}{6}$ in D

$\sin(x) = -1 \Rightarrow \sin^{-1}(-1) = -\frac{\pi}{2}$

$-\frac{\pi}{2} + 2k\pi \Rightarrow \frac{3\pi}{2}$ in D

Solutions: $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

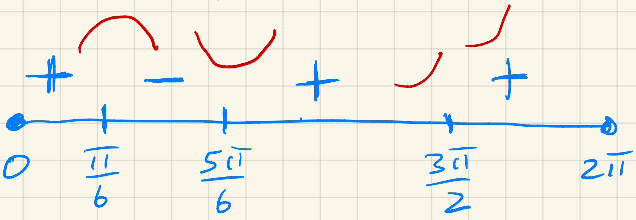
$f'(x) = 2\cos(2x) - 2\sin(x)$



Test $x=0$, $f'(0) = 2 > 0$, Test $x = \frac{\pi}{2}$, $f'(\frac{\pi}{2}) = -4 < 0$

Test $x = \pi$
$f'(\pi) = 2 > 0$

Test $x = 2\pi$
$f'(2\pi) = 2 > 0$



Local maximum at
$x = \frac{\pi}{6}$ $f(\frac{\pi}{6}) = \sin(\frac{\pi}{3}) + 2\cos(\frac{\pi}{6})$

$= \frac{\sqrt{3}}{2} + \frac{2\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$

$(\frac{\pi}{6}, \frac{3\sqrt{3}}{2}) \simeq 2.598$

Local minimum at $x = \frac{5\pi}{6}$

$f(\frac{5\pi}{6}) = \sin(\frac{5\pi}{3}) - 2\cos(\frac{5\pi}{6})$

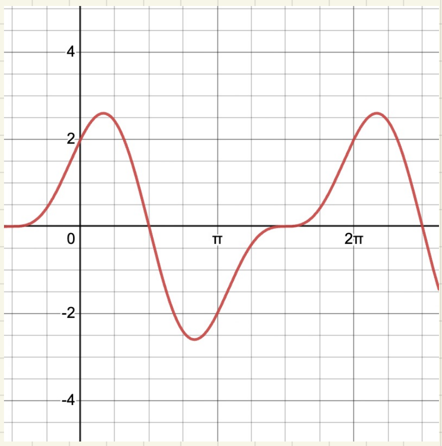
$= -\frac{\sqrt{3}}{2} - \frac{2\sqrt{3}}{2} = -\frac{3\sqrt{3}}{2}$

$(\frac{5\pi}{6}, -\frac{3\sqrt{3}}{2})$

Stationary Inflection point at $x = \frac{3\pi}{2}$

$f(\frac{3\pi}{2}) = \sin(3\pi) - 2\cos(\frac{3\pi}{2})$
$= 0$

$(\frac{3\pi}{2}, 0)$



16.

$$f(x) = \sin(x)\cos(2x)$$

$$f'(x) = \sin(x)\left(-2\sin(2x)\right) + \cos(x)\cos(2x)$$

$$= -2\sin(x)\sin(2x) + \cos(x)\cos(2x)$$

$$= -2\sin(x)\left(2\sin(x)\cos(x)\right) + \cos(x)\left(2\cos^2(x) - 1\right)$$

$$= -4\sin^2(x)\cos(x) + 2\cos^3(x) - \cos(x)$$

$$= -4\left(1-\cos^2(x)\right)\cos(x) + 2\cos^3(x) - \cos(x)$$

$$= -4\cos(x) + 4\cos^3(x) + 2\cos^3(x) - \cos(x)$$

$$= 6\cos^3(x) - 5\cos(x)$$

$$f'(x) = 0$$

$$\Rightarrow 6\cos^3(x) - 5\cos(x) = 0$$

$$\cos(x)\left(6\cos^2(x) - 5\right) = 0$$

$$\Rightarrow \cos(x) = 0$$

$$\cos(x) = \pm\sqrt{\frac{5}{6}}$$

Domain $\{x \mid 0 \le x \le \pi\}$

which is Domain of $\cos^{-1}(y)$

$$\cos(x) = 0 \Rightarrow x = \frac{\pi}{2} \simeq 1.571$$

$$\cos(x) = +\sqrt{\frac{5}{6}}$$

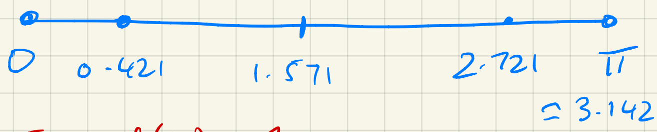
$$x = \cos^{-1}\left(+\sqrt{\frac{5}{6}}\right) \approx 0.421$$

$$\cos(x) = -\sqrt{\frac{5}{6}}$$

$$x = \cos^{-1}\left(-\sqrt{\frac{5}{6}}\right)$$

$$\simeq 2.721$$

(or $\pi - 0.421 = 2.721$)

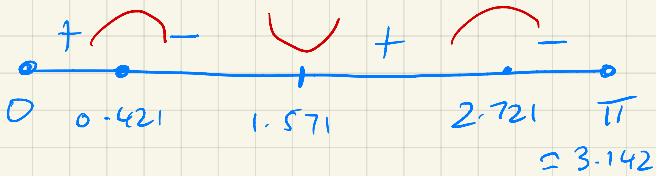


Test $f'(0) = 1 > 0$

Test $f'(1) = -1.755 < 0$

Test $f'(2) = 1.648 > 0$

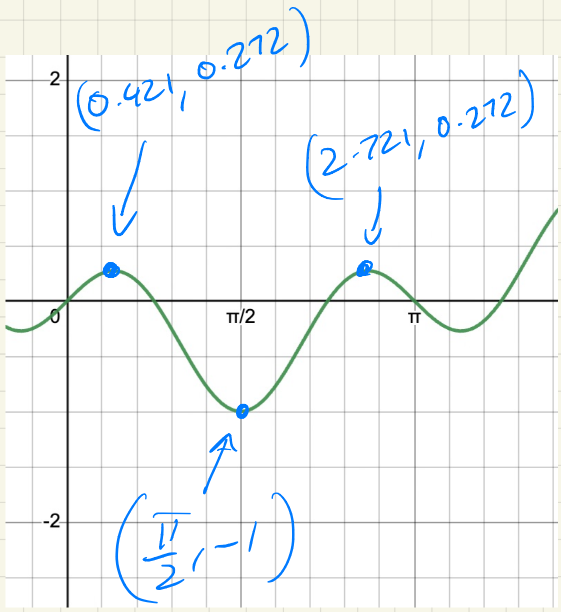
Test $f'(\pi) = -1 < 0$



$$f(0.421) = 0.272$$

$$f\left(\frac{\pi}{2}\right) = -1$$

$$f(2.721) = 0.272$$



18. Prove $\frac{\ln(x)}{x} \leq \frac{1}{e}$

for all $x > 0$

Create a sign diagram and hopefully identify a maximum value for $\frac{\ln x}{x}$

let $f(x) = \frac{\ln(x)}{x}$ $\quad u = \ln x \quad v = x$

$\quad u' = \frac{1}{x} \quad v' = 1$

$$f'(x) = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2}$$

$f'(x) = 0 \Rightarrow 1 - \ln x = 0$

$\ln x = 1$

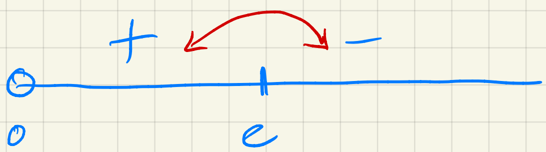
$x = e$

Test $x = 1$

$f'(1) = \frac{1-0}{1^2} > 0$

Test $x = e^2$

$f'(e^2) = \frac{1 - 2\ln e}{e^4} = \frac{-1}{e^4} < 0$



Global maximum at $x = e$

$f(e) = \frac{\ln e}{e} = \frac{1}{e}$

$(e, \frac{1}{e})$

So $\frac{1}{e}$ is largest possible value of $\frac{\ln x}{x}$

$\Rightarrow \frac{\ln x}{x} \leq \frac{1}{e}$

19. $f(x) = x - \ln(x)$ $\quad D\{x > 0\}$

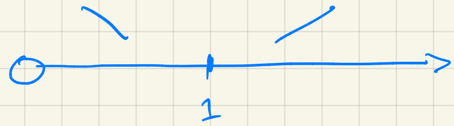
a) show only local minimum.

b) prove $\ln(x) < x - 1 \quad \forall x > 0$

$f'(x) = 1 - \frac{1}{x}$

$f'(x) = 0$

$1 - \frac{1}{x} = 0 \quad x = 1$



$f'(\frac{1}{2}) = 1 - 2 = -1 \quad < 0$

$f'(2) = 1 - \frac{1}{2} = \frac{1}{2} \quad > 0$

local minimum at $(1, 1)$

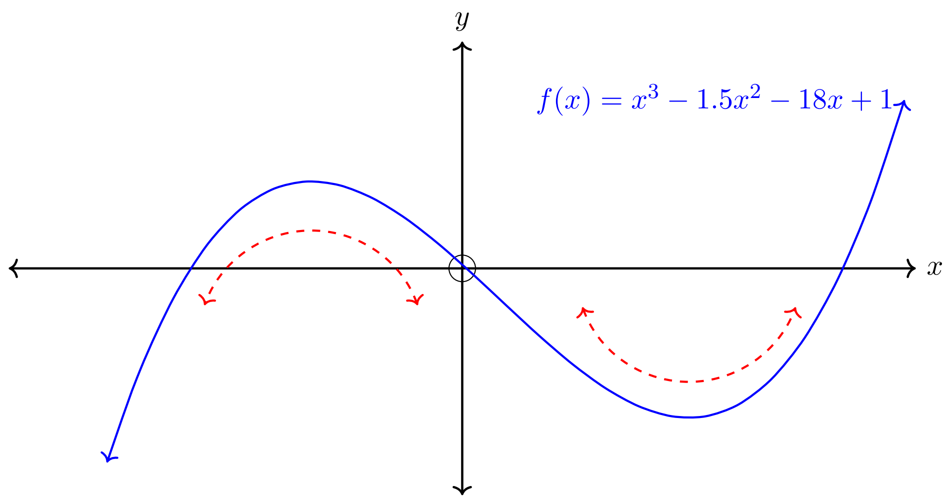
$\Rightarrow x - \ln(x) \geq 1 \quad \forall x < 0$

$x - 1 \geq \ln(x)$

$\ln(x) \leq x - 1$

1.6 Shape

Looking at the shape of of the graph below, we can see that over a certain interval, the curve is turning downwards **it is concave down**, and on then it changes to be turning upwards **concave up**.



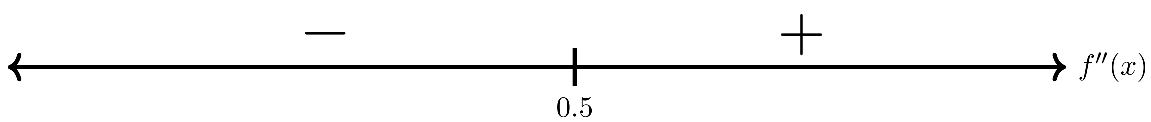
If the curve is <u>concave down</u> the **gradient** is <u>decreasing</u>, so we consider the <u>second derivative</u>

Leibniz	or	Newton
$\frac{d}{dx}(\frac{dy}{dx}) < 0$		$(f'(x))' < 0$
$\Rightarrow \quad \frac{d^2y}{dx^2} < 0$		$\Rightarrow \quad f''(x) < 0$

Similarly, if the curve is <u>concave up</u> the **gradient** is <u>increasing</u>, so

$$\frac{d^2y}{dx^2} > 0 \qquad \text{or} \qquad f''(x) > 0$$

We can contruct a sign diagram for (in this case) $f''(x) = 6x - 3$, and $f''(x) = 0$ when $x = \frac{1}{2}$



We can also observe that there is an **non-stationary inflection point** at $x = \frac{1}{2}$, $f(\frac{1}{2}) = -\frac{37}{4}$

1.6.1 Ex 13E

13 E

3 b

Determine the shape of

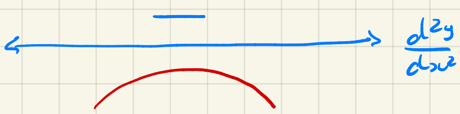
$y = -2(x-3)(x+1)$

$\frac{dy}{dx} = -2\left[(x-3)1 + 1(x+1)\right]$

$= -2(2x-2)$

$= -4x+4$

$\frac{d^2y}{dx^2} = -4 \qquad \frac{d^2y}{dx^2} < 0$ always



function is concave down always.

4 f

$f(x) = x^4 - 12x^3$

$f'(x) = 4x^3 - 36x^2$

$4x^3 - 36x^2 = 0$

$4x^2(x-9) = 0$

$x=0 \qquad x=9$

$- \qquad - \qquad +$

$0 \qquad 9$

$f''(x) = 12x^2 - 72x$

$f''(x) = 0$

$12x^2 - 72x = 0$

$12x(x-6) = 0$

$x=0 \qquad x=6$

$f''(-1) = 84 > 0$

$f''(3) = -108 < 0$

$f''(9) = 324 > 0$

$+ \qquad - \qquad +$

$0 \qquad 6$

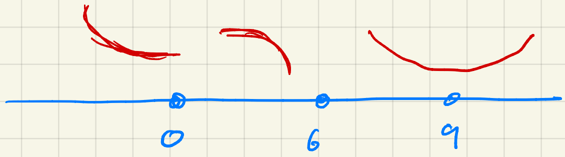
$f(x)$ is decreasing for $x < 9$

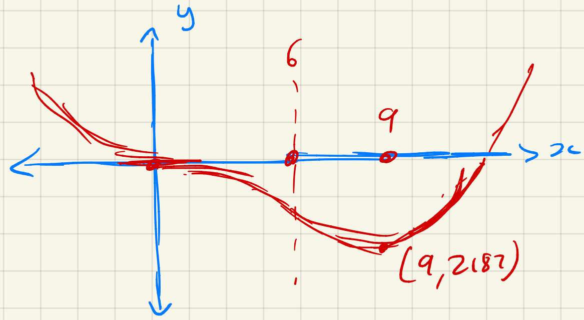
$f(x)$ is increasing for $x > 9$

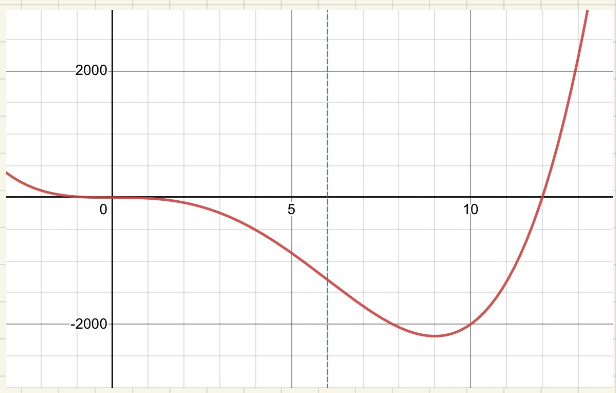
$f(x)$ is concave up $x < 0$

concave down $0 < x < 6$

concave up $x > 6$







7. $f(x) = \dfrac{e^x}{x}$

a) When $x=0$, $f(x)$ is undefined
$\Rightarrow$ y-intercept does not exist.

for x-intercepts $\dfrac{e^x}{x} = 0$

but $e^x \neq 0$, so no x-intercepts

b) as $x \to \infty$

e^x gets very large

$\frac{1}{x}$ gets very small

but e^x is an exponential

$\Rightarrow \lim\limits_{x \to \infty} \dfrac{e^x}{x} = +\infty$

and $\lim\limits_{x \to -\infty} \dfrac{e^x}{x} = 0$

c) $f(x) = \dfrac{e^x}{x}$

$f'(x) = \dfrac{xe^x - e^x \cdot 1}{x^2}$

$= \dfrac{e^x(x-1)}{x^2}$

$f'(x) = 0$

$\Rightarrow e^x(x-1) = 0$

$x=1$ is a stationary pt $(1, e)$

$-$ $-$ $+$

0 1 $f'(x)$

$(1, e)$ is a stationary point and is a local minimum.

$f'(x) = \dfrac{xe^x - e^x}{x^2}$

$u = xe^x - e^x$

$u' = xe^x + e^x - e^x$

$= xe^x$

$v = x^2$

$v' = 2x$

$f''(x) = \dfrac{xe^x(x^2) - (xe^x - e^x)2x}{x^4}$

$= \dfrac{x^3e^x - 2x^2e^x + 2xe^x}{x^4}$

$= \dfrac{xe^x(x^2 - 2x + 2)}{x^4}$

$xe^x(x^2 - 2x + 2) = 0$

$x = 0$ $\quad$ no sol.

so no inflection points.

but need to look at either side of undefined

$-$ $+$

0 $f''(x)$

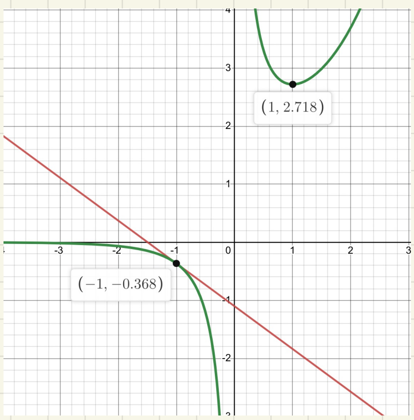
Concave down $x < 0$ $\quad$ Concave up $x > 0$

tangent at $x = -1$

$f'(-1) = -\dfrac{2}{e}$ $\quad$ $f(-1) = -\dfrac{1}{e}$

$y + \frac{1}{e} = -\frac{2}{e}(x+1)$

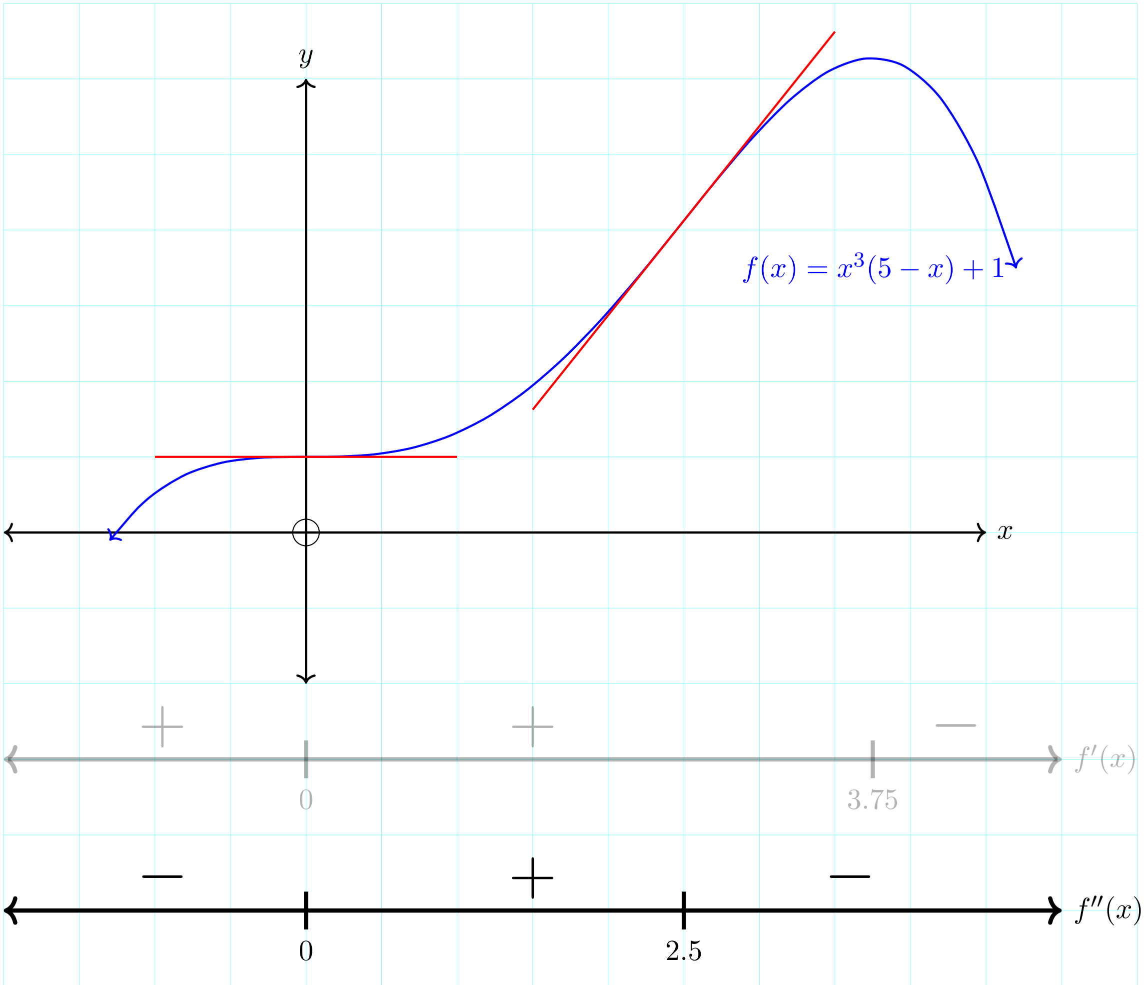
$y = -\dfrac{2x}{e} - \dfrac{3}{e}$ or $2x + ey = -3$



1.7 Inflection Points

At an inflection point:

- the tangent line crosses the curve
- the shape of the curve is changing from concave up to concave down (or vice versa).



We can identify inflection points by

- Solving $f''(x) = 0$
- Then using a sign diagram to verify that the points we have found are actually inflection points. The sign diagram must **change signs** across the 0 for the points to be inflection points.

1.7.1 Ex 13F

Find stationary and inflection points for

$$y = x^3(5-x)$$

$$y = 5x^3 - x^4$$

$$\frac{dy}{dx} = 15x^2 - 4x^3$$

$$15x^2 - 4x^3 = 0$$

$$x^2(15-4x) = 0$$

$$x = 0 \qquad x = \frac{15}{4} = 3.75$$

Stationary points at $(0,0)$ and $(3.75, 65.918)$

$$\frac{d^2y}{dx^2} = 30x - 12x^2$$

$$= 6x(5-2x)$$

If $x = 0$

$$\frac{d^2y}{dx^2} = 0$$

unknown at this stage.

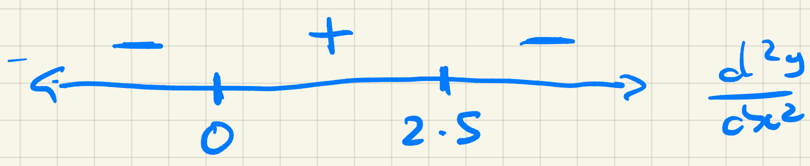
If $x = 3.75$

$$\frac{d^2y}{dx^2} = -56.25$$

$(3.75, 65.918)$ is a local max.

Sign diagram for $\frac{d^2y}{dx^2}$ $\qquad 6x(5-2x) = 0$

$x = 0 \qquad x = 2.5$

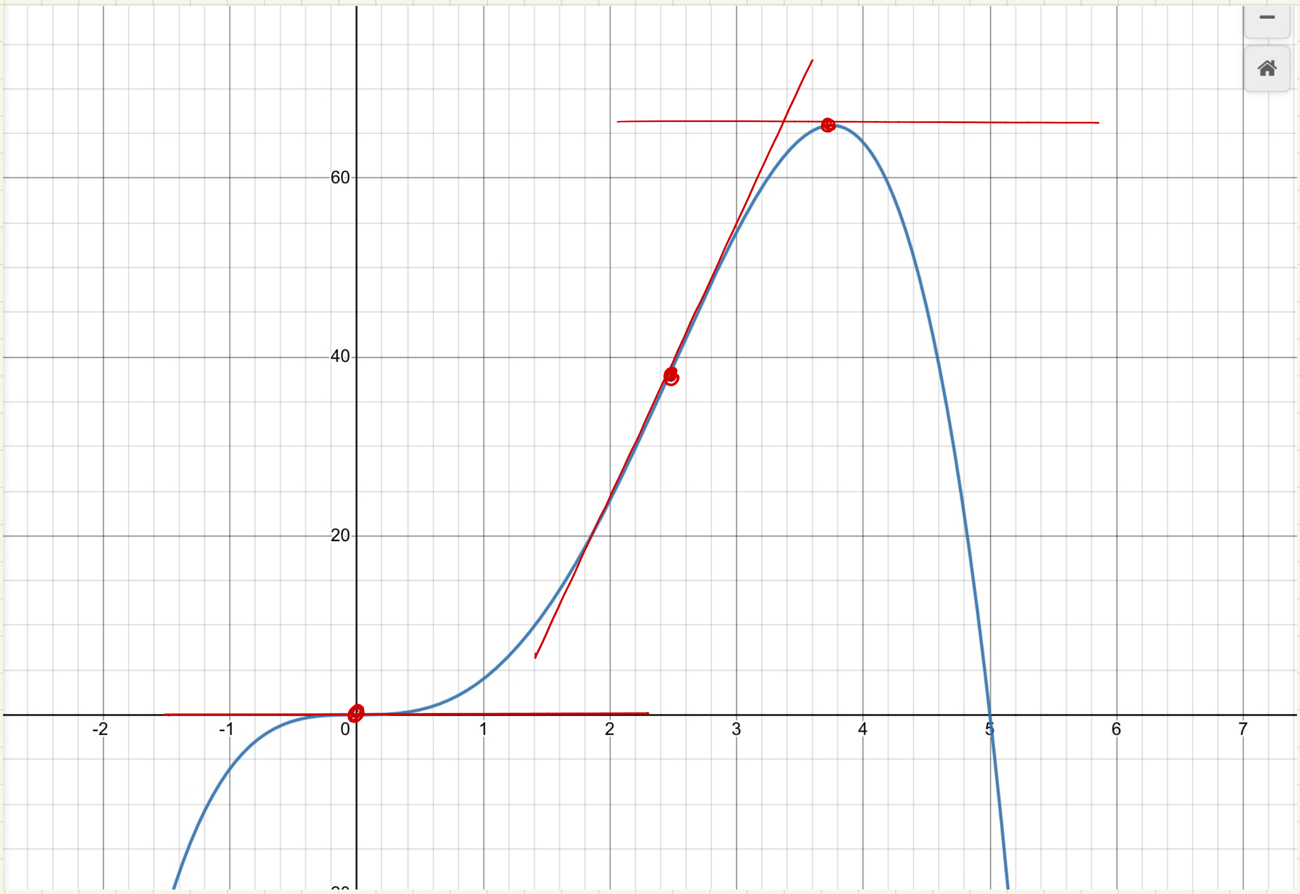


Change of sign at $x = 0$

So $(0,0)$ is a stationary inflection point

Change of sign at 2.5

so $(2.5, 39.0625)$ is a non-stationary inflection point



60
40
20
-2
-1
0
1
2
3
4
5
6
7

Find stationary and inflection points for:

$$f(x) = \frac{x^6}{30} + 6.4x$$

$$f'(x) = \frac{6x^5}{30} + 6.4 = \frac{x^5}{5} + 6.4$$

$$\frac{x^5}{5} + 6.4 = 0$$

$$x^5 = -32$$

$$x = \sqrt[5]{-32} = -2$$

Stationary point at $\left(-2, -10\tfrac{2}{3}\right)$

$$f''(x) = \frac{5x^4}{5} = x^4$$

Sign diagram not required.

$$f''(-2) = 16 \quad > 0$$

Stationary point is a local minimum.

Set $f''(x) = 0$

$\Rightarrow \quad x = 0$

could be an inflection point

but

$+ \quad | \quad +$ $\quad f''(x)$

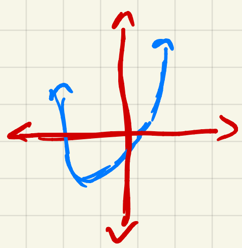
$\quad\quad 0$

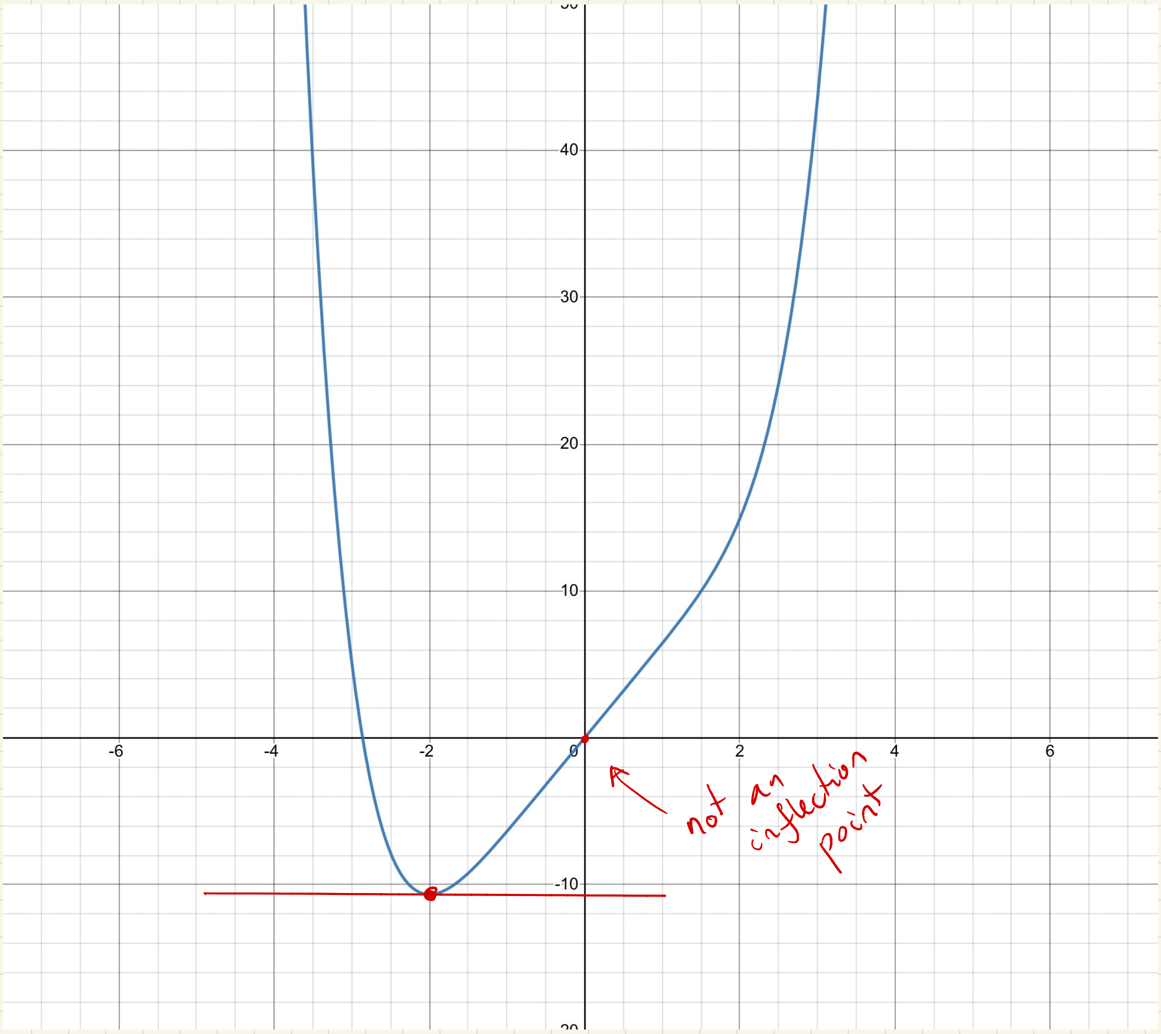
Does not change sign

So $(0,0)$ is <u>not</u> an inflection point.

A local minimum at $\left(-2, -10\tfrac{2}{3}\right)$

No inflection points.





-6
-4
-2
0
2
4
6
40
30
20
10
-10
not an inflection point

1.8 Understanding functions and their derivatives

Chapter: 2 Applications of differentiation

$$\begin{aligned}
\text{The equation is:} \qquad 9a - 4 &= 14 + 3a \\
\text{Subtract } 3a\text{:} \qquad 6a - 4 &= 14 \\
\text{Subtract 4:} \qquad 6a &= 18 \\
\text{Divide by 6:} \qquad a &= 3
\end{aligned}$$

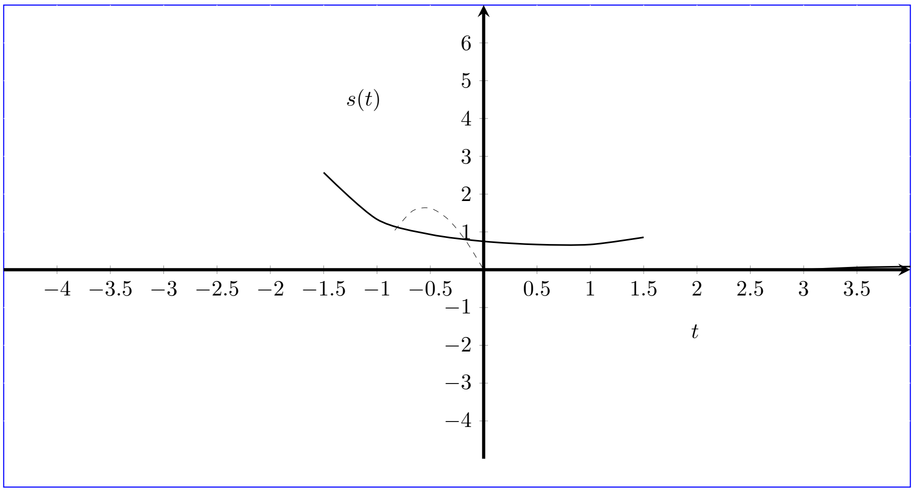
$$A\widehat{B}C$$

$$A\widehat{BC}C$$

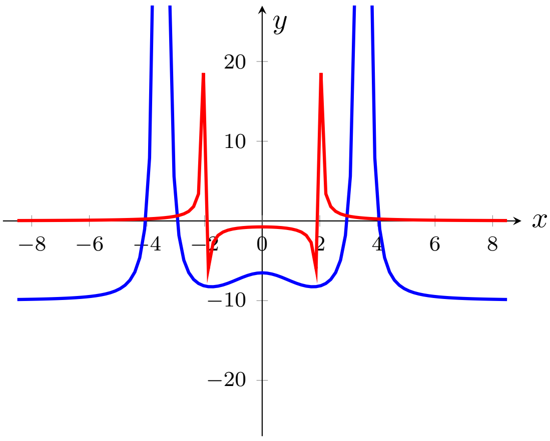
$$A\hat{B}C$$

$$N\tilde{a}$$

$$X \sim \mathcal{N}(\mu,\, \sigma^2)$$



$s(t)$
t



y
x