1 General Arithmetic

Without a calculator, work out the following

1. i) $79 + 44 =$ 143 iii) $125 + 189 =$ 314 v) $48 - 100 =$ -52

   ii) $98 + 63 =$ 161 iv) $98 - 57 =$ 41 vi) $61 - 88 =$ -27

2. i) $15 \times 9 =$ 135 iii) $-21 \times 11 =$ -221 v) $38 \times 24 =$ 912

   ii) $19 \times -7 =$ -133 iv) $-35 \times 14 =$ -490 vi) $45 \times 16 =$ 720

3. i) $1.5 \times 2.5 =$ 3·75 iii) $-2.1 \times 1.1 =$ -2·21 v) $6 \times -2.45 =$ -14·70

   ii) $1.8 \times -1.7 =$ -3·06 iv) $-3.5 \times -1.2 =$ 4·2 vi) $8 \times 9.9 =$ 7·92

4. i) $456 \div 4 =$ 114 iii) $\frac{90}{5} =$ 18 v) $14\overline{)4914}$ 351

   ii) $7\overline{)161}$ = 23 iv) $312 \div 8 =$ 39 vi) $\frac{112}{7} =$ 16

5. For these give the answer with a remainder.
   For example $28 \div 3 = 9 \;\; R\ 1$ (9 remainder 1)

   i) $100 \div 8 =$ 12 R 4 iii) $97 \div 14 =$ 6 R 13 v) $125 \div 6 =$ 20 R 5

   ii) $115 \div 3 =$ 38 R 1 iv) $78 \div 9 =$ 8 R 6 vi) $77 \div 16 =$ 4 R 13

2 Modular Arithmetic

2.1 Definition

When an integer A is divided by a positive integer N, it has a remainder R from the division. The value of R can be 0 and $0 \leq R < N$ (R will be greater than or equal to 0 and less than N). We can define an operation:

$$A \bmod N = R$$

Where R is the remainder when A is divided by N.

Let's look at some examples:

$$5 \bmod 2 = 1$$

because $5 \div 2$ has a remainder of 1

$$8 \bmod 3 = 2$$

because $8 \div 3$ has a remainder of 2

$$39 \bmod 7 = 4$$

because $39 \div 7$ has a remainder of 4

$$99 \bmod 13 = 8$$

because $99 \div 13$ has a remainder of 8

$$26 \bmod 13 = 0$$

because $26 \div 13$ has a remainder of 0

$$7 \bmod 8 = 7$$

because $7 \div 8$ has a remainder of 7

2.1.1 Without a calculator work out the following:

i) $16 \bmod 5 =$ 1

ii) $19 \bmod 2 =$ 1

iii) $11 \bmod 11 =$ 0

iv) $38 \bmod 4 =$ 2

v) $79 \bmod 6 =$ 1

vi) $90 \bmod 8 =$ 2

vii) $49 \bmod 7 =$ 0

viii) $69 \bmod 6 =$ 3

ix) $88 \bmod 7 =$ 4

x) $76 \bmod 3 =$ 1

xi) $90 \bmod 10 =$ 0

xii) $101 \bmod 3 =$ 2

2.2 Modulus Relationships

There are some interesting modulus relationships:

$$A \bmod N + B \bmod N = (A + B) \bmod N$$

$$A \bmod N \times B \bmod N = (A \times B) \bmod N$$

2.2.1 Exercise:

1. Make up a number of test cases and demonstrate to yourself that the above relationships are true.

2. Using what you have learned above, work out:

   i) $(16 \times 90) \bmod 5 =$ 16 Mod 5 × 90 Mod 5 = 1 × 0 = 0

   ii) $(19 \times 89) \bmod 2 =$ 19 Mod 2 × 89 Mod 2 = 1 × 1 = 1

   iii) $(11 \times 87) \bmod 11 =$ 0

   iv) $(38 \times 61) \bmod 4 =$ 2 × 1 = 2

   v) $(79 \times 5) \bmod 6 =$ 1

   vi) $(90 \times 8) \bmod 8 =$ 0

   vii) $(49 \times 712872818) \bmod 7 =$ 0

   viii) $(69 \times 7) \bmod 6 =$ 3

   ix) $(88 \times 91) \bmod 7 =$ 0

   x) $(76 \times 2) \bmod 3 =$ 2

   xi) $(90 \times 80) \bmod 10 =$ 0

   xii) $(101 \times 100) \bmod 3 =$ 2

3. Using what you have learned above answer the following questions.

   i) What is the remainder when 48×40 is divided by 13?

   ii) What is the remainder when $37 \times 41 \times 90$ is divided by 8?

   iii) A number has prime factors $2, 2, 7, 7, 11, 13$ what is the remainder when the number is divided by 3

   iv) For the same number above, what is the remainder when the number is divided by 5 ?

3. i) (48 × 40) Mod 13 = 48 Mod 13 × 40 Mod 13
= 9 × 1
= 9

ii) (37 × 41 × 90) Mod 8 = 37 Mod 8 × 41 Mod 8 × 90 Mod 8
= (5 × 1 × 2) Mod 8
= (10) Mod 8
= 2

iii) (2 × 2 × 1 × 1 × 2 × 1) Mod 3 = 8 Mod 3
= 2

iv) (2 × 2 × 2 × 2 × 1 × 3) Mod 5 = 48 Mod 5
= 3

2.3 Divisibility Test for 3

You probably know that if we want to know if a number is divisible by 3, we can add up the digits and if this sum is divisible by 3, then the number is divisible by 3.

But **why** is this true? This section is designed to give you an idea about how we can prove that this is the case.

Complete the following:

1. 1 mod 3 = 1
2. 10 mod 3 = 1
3. 100 mod 3 = 1
4. 1000 mod 3 = 1

1. The number $492 = 4 \times 100 + 9 \times 10 + 2$
   Show that $(4 \times 100 + 9 \times 10 + 2) \bmod 3 = (4 + 9 + 2) \bmod 3$
2. How can you tell that this number is divisible by 3??
3. What is the reminder when 83674 is divided by 3 ?
4. Using a 4 digit number $PQRS$ can you **prove** that if the number is divisible by 3, then the sum of its digits is also divisible by 3?

2.4 General Questions

1. What is the unit digit for the number 123^{456}?
2. Prove that if $a^2 + b^2$ is a multiple of 3 then a and b are multiples of 3.

$\Rightarrow (4 \times 100 + 9 \times 10 + 2) \text{ Mod } 3$

$= (4 \times 100) \text{ Mod } 3 + (9 \times 10) \text{ Mod } 3 + 2 \text{ Mod } 3$

$= 4 \text{ Mod } 3 \times 1 + 9 \text{ Mod } 3 \times 1 + 2 \text{ Mod } 3$

$= (4 + 9 + 2) \text{ Mod } 3$

2. $(4+9+2) \text{ Mod } 3 = 0$

3. $(8+3+6+7+4) \text{ Mod } 3 = 28 \text{ Mod } 3$
$= 1$

4. $PQRS = P \times 1000 + Q \times 100 + R \times 10 + S$
$(PQRS) \text{ Mod } 3 = 0$
$\Rightarrow (P+Q+R+S) \text{ Mod } 3 = 0$

5. $123^{456} \text{ Mod } \cancel{3}0$
$= 123 \text{ Mod } \cancel{3}0 \times \ldots \times 123 \text{ Mod } \cancel{3}0$
$= (123 \text{ Mod } 10)^{456}$
$= (3 \text{ Mod } 10)^{456} = 3^{456} \text{ Mod } 10$
$\cancel{(3 \text{ Mod } 10)^1 = 3}$ $\quad 3 \text{ Mod } 10 = 3$
$\cancel{3 \text{ Mod } 10}$ $\quad 3^2 \text{ Mod } 10 = 9$
$3^3 \text{ Mod } 10 = 7$
$\rightarrow 3^4 \text{ Mod } 10 = 1$
$3^5 \text{ Mod } 10 = 3$
$\vdots$
4 goes into 456
So remainder is 1.

2. $(a \times a + b \times b) \text{ Mod } 3 = 0$
$\Rightarrow (a \times a) \text{ Mod } 3 + (b \times b) \text{ Mod } 3 = 0$
$\Rightarrow a \text{ Mod } 3 = 0$ and $b \text{ Mod } 3 = 0$

3 Exponents

1. In the questions below find the value for the given letters (x or y or both)

   i) $2^5 \times (1+2) = 2^x + 2^y$

   ii) $2^3 \times 2^4 = 2^x$

   iii) $(4 \times 3^4)^2 = y \times 3^x$

   iv) $32 \times 2^6 = 2^x$

   v) $(3^2 + 3^2 + 3^2 + 3^2)^2 = x \times 3^y$

   vi) $12^3 = 2^y \times 3^x$

2. Find the prime factors of $2^{302} + 2^{303}$

3. Find the prime factors of $2^{21} + 2^{17}$

4. Which pocket money system would you like (over a month only !!)?
   Explain why.

   (a) \$10 each day.

   (b) \$3 on the first day, \$3.50 on the second and increasing by 50 cents each day.

   (c) 1 cent on the first day, 2 cents on the second day, 4 cents on the third, doubling each day.

5. Which is larger $2^7 + 2^6$ or 2^8 ?

1i) $x = 5$
$y = 6$

ii) $x = 7$

iii) $y = 16$
$x = 12$

iv) $32 \times 2^6 = 2^5 \times 2^6$
$= 2^{11}$
$x = 11$

v) $(4 \times 3^2)^2 = x \times 3^y$
$= 16 \times 3^4$
$x = 16$
$y = 4$

2. $2^{302}(1+2^1)$
$= 2^{302} \times 3$
factors 2 and 3

3. $2^{17}(2^4+1)$
$2^{17} \times 17$ factors
2 and 17

4. c) ✓ last day get 2^{29} cents

5. 2^8

3.1 Working in base 2

Please note that $2^1 = 2$ and $2^0 = 1$ (the reason for the second of these can be explained in class).

Any number can be written as sums of powers of 2.

For example:

$$7 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$$

$$12 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$$

$$18 = 1 \times 2^4 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$$

3.1.1 For the following, fill in the blanks and remember that they can only be 0 or 1.

i) $14 = \underline{0} \times 2^4 + \underline{1} \times 2^3 + \underline{1} \times 2^2 + \underline{0} \times 2^1 + \underline{0} \times 2^0$

ii) $19 = \underline{1} \times 2^4 + \underline{0} \times 2^3 + \underline{0} \times 2^2 + \underline{1} \times 2^1 + \underline{1} \times 2^0$

iii) $9 = \underline{0} \times 2^4 + \underline{1} \times 2^3 + \underline{0} \times 2^2 + \underline{0} \times 2^1 + \underline{1} \times 2^0$

iv) $25 = \underline{1} \times 2^4 + \underline{1} \times 2^3 + \underline{0} \times 2^2 + \underline{0} \times 2^1 + \underline{1} \times 2^0$

v) $31 = \underline{1} \times 2^4 + \underline{1} \times 2^3 + \underline{1} \times 2^2 + \underline{1} \times 2^1 + \underline{1} \times 2^0$

vi) $20 = \underline{1} \times 2^4 + \underline{0} \times 2^3 + \underline{1} \times 2^2 + \underline{0} \times 2^1 + \underline{0} \times 2^0$

If we just take the multiplying numbers (0,1) in order and drop any leading zeros, we can represent a number in **base 2**.
Base 2 is one of the most simple ways of representing numbers as it only uses two symbols 0 and 1. In order they represent place values of powers of 2 (instead of powers of 10 in our decimal system).

$$7 \equiv 111 \quad (base\ 2)$$

because $7 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0$

$$12 \equiv 1100 \quad (base\ 2)$$

because $12 = 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0$

$$18 \equiv 1010 \quad (base\ 2)$$

because $18 = 1 \times 2^4 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

The little sign $\equiv$ means "equivalent to". So the first statement above says 7 (in base 10) is equivalent to 111 (in base 2).

Counting from 1 to 8:

$1 \equiv 1 \quad (b\ 2)$
$2 \equiv 10 \quad (b\ 2)$
$3 \equiv 11 \quad (b\ 2)$
$4 \equiv 100 \quad (b\ 2)$
$5 \equiv 101 \quad (b\ 2)$
$6 \equiv 110 \quad (b\ 2)$
$7 \equiv 111 \quad (b\ 2)$
$8 \equiv 1000 \quad (b\ 2)$

3.1.2 Complete the following

1.

i) $9 \equiv \underline{1001} \quad (b\ 2)$

ii) $10 \equiv \underline{1010} \quad (b\ 2)$

iii) $15 \equiv \underline{1111} \quad (b\ 2)$

iv) $24 \equiv \underline{11000} \quad (b\ 2)$

v) $35 \equiv \underline{100011} \quad (b\ 2)$

vi) $69 \equiv \underline{\quad} \quad (b\ 2)$
1000101

2.

i) 9 $\equiv 1001 \quad (b\ 2)$

ii) 13 $\equiv 1101 \quad (b\ 2)$

iii) 10 $\equiv 1010 \quad (b\ 2)$

iv) 32 $\equiv 10000 \quad (b\ 2)$

v) 21 $\equiv 10101 \quad (b\ 2)$

vi) 15 $\equiv 11111 \quad (b\ 2)$

Let's consider introducing a decimal (or more correctly) a binary point.

What do you think 1.1 (base 2) means ? What about 1.01 (base 2)?

3.1.3 Complete the following

1.

i) 1.5 $\equiv 1.1 \quad (b\ 2)$

ii) 1.25 $\equiv 1.01 \quad (b\ 2)$

iii) 2.5 $\equiv 10.1 \quad (b\ 2)$

iv) 4.75 $\equiv 100.11 \quad (b\ 2)$

v) $3\frac{5}{16}$ $\equiv 11.0101 \quad (b\ 2)$

vi) $1\frac{15}{16}$ $\equiv 1.1111 \quad (b\ 2)$

2.

i) $1.5 \equiv$ 1.1 $(b\ 2)$

ii) $1.25 \equiv$ 1.01 $(b\ 2)$

iii) $1.75 \equiv$ 1.11 $(b\ 2)$

iv) $1.125 \equiv$ 1.001 $(b\ 2)$

v) $1.375 \equiv$ 1.011 $(b\ 2)$

vi) $2.625 \equiv$ 1.101 $(b\ 2)$

4 Rounding Numbers

1. i) Round 38 to the nearest multiple of 10 40
   ii) Round 47 to the nearest multiple of 6 48
   iii) Round 345 to the nearest multiple of 101 303
   iv) Round $\frac{3}{4}$ to the nearest multiple of $\frac{1}{3}$ $\frac{8}{12} = \frac{2}{3}$
   v) Round $\frac{3}{4}$ to the nearest multiple of $\frac{1}{9}$ $\frac{28}{36} = \frac{7}{9}$
   vi) Round $\frac{31}{15}$ to the nearest multiple of $\frac{1}{7}$ $\frac{210}{105} = 2$

2. A particular computer language can add, subtract, multiply and divide numbers (+, -,*,/). It can also round any decimal or fraction to the nearest integer.
   For example *round*(4.3)=4 or *round*($\frac{3}{4}$)=1

   Using the information above:

   i) Write a step-by-step process for the computer to round a number N, to the nearest 10. Create a set of sensible tests to show that your process works.

   ii) Write a step by step process to round a number N to the nearest multiple of M. Create a set of sensible tests to show that your process works.

i) $10 \times \text{Round}\left(\frac{N}{10}\right)$

ii) $M \times \text{Round}\left(\frac{M}{10}\right)$