Find the equation of the tangent to:

2.

a) $y = x - 2x^2 + 3$ at $x = 2$

$$\frac{dy}{dx} = 1 - 4x$$

when $x = 2$

$$\frac{dy}{dx} = 1 - 4(2) = -7$$

and $y = (2) - 2(2)^2 + 3$
$= -3$

$\frac{dy}{dx} = -7$ Point $(2, -3)$

$$y - -3 = -7(x - 2)$$

$$y + 3 = -7x + 14$$

$$y = -7x + 11$$

d) $y = \frac{4}{\sqrt{x}}$ at $(1, 4)$

$$y = 4x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = 4 \cdot -\frac{1}{2} x^{-\frac{3}{2}}$$

$$= -2x^{-\frac{3}{2}} = \frac{-2}{x^{3/2}}$$

$x = 1$

$$\frac{dy}{dx} = \frac{-2}{1} = -2$$

Point $(1, 4)$

$$y - 4 = -2(x - 1)$$

$$y - 4 = -2x + 2$$

$$y = -2x + 6$$

(Horizontal Tangents)

3. c)

Find the equation of horizontal tangents to

$$y = \sqrt{x} + \frac{1}{\sqrt{x}}$$

$$\Rightarrow \quad y = x^{\frac{1}{2}} + x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{1}{2}} + -\frac{1}{2} x^{-\frac{3}{2}}$$

(factorise using lowest power)

$$\frac{dy}{dx} = \frac{1}{2} x^{-\frac{3}{2}} (x^1 - 1)$$

Set $\frac{dy}{dx} = 0$

$$\frac{1}{2} x^{-\frac{3}{2}} (x - 1) = 0$$

$x^{-\frac{3}{2}} \neq 0$ $\quad x - 1 = 0$
$x = 1$

Find y

$$y = \sqrt{1} + \frac{1}{\sqrt{1}} = 1 + \frac{1}{1}$$

$$= 2$$

Horizontal tangent at $(1, 2)$

Equation: $y = 2$

$$\frac{dy}{dx} = \frac{1}{2x^{1/2}} - \frac{1}{2x^{3/2}}$$

$$= \frac{x - 1}{2x^{3/2}}$$

$\frac{x - 1}{2x^{3/2}} = 0 \Rightarrow x - 1 = 0$
$x = 1$

5. Find another tangent to

$y = 1 - 3x + 12x^2 - 8x^3$

which is parallel to the tangent at $(1, 2)$

$\frac{dy}{dx} = -3 + 24x - 24x^2$

when $x = 1$

$\frac{dy}{dx} = -3 + 24(1) - 24(1)^2$

$= -3$

so gradient is -3

Find all x values where gradient $= -3$

$-3 + 24x - 24x^2 = -3$

$24x - 24x^2 = 0$

$x - x^2 = 0$

$x(1-x) = 0$

$x = 0$ $\quad x = 1$ already have

Find equation of tangent at

$x = 0$

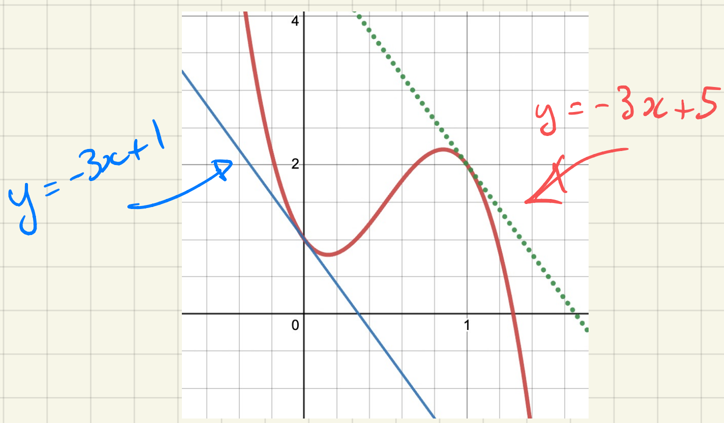
$y = 1$

$y - 1 = -3(x - 0)$

$y - 1 = -3x$, $y = -3x + 1$

other tangent $y - 2 = -3(x - 1)$

$y = -3x + 5$



6. $y = x^2 + ax + b$

when $x = 1$ tangent is $2x + y = 6$

$y = -2x + 6$

$\frac{dy}{dx} = -2$ $\qquad y = 4$

$(1, 4)$ point on curve.

-2 gradient at that point.

$\frac{dy}{dx} = 2x + a$

$-2 = 2(1) + a$

$a = -4$

$y = x^2 - 4x + b$

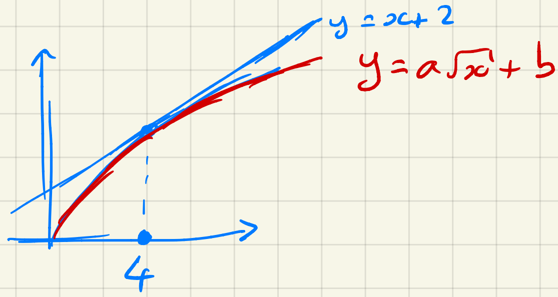
$4 = (1)^2 - 4(1) + b$

$4 = -3 + b$

$b = 7$.

$y = x^2 - 4x + 7$

7. Find a and b



What do we know?

- when $x = 4$ $\frac{dy}{dx} = 1$
- $(4, 4+2) = (4, 6)$ is on $y = a\sqrt{x} + bx$

① using $(4, 6)$

$$6 = a\sqrt{4} + b(4)$$
$$6 = 2a + 4b$$
$$\Rightarrow 3 = a + 2b$$

② $\frac{dy}{dx} = \frac{a}{2\sqrt{x}} + b$

$$1 = \frac{a}{2\sqrt{4}} + b$$
$$1 = \frac{a}{4} + b$$
$$4 = a + 4b$$

Equations

$$3 = a + 2b \quad ①$$
$$4 = a + 4b \quad ②$$

② - ① $1 = 2b$

$$b = \frac{1}{2}$$

using ① $a = 3 - 2b$

$$= 3 - 2\left(\frac{1}{2}\right)$$
$$= 2$$

$$y = 2\sqrt{x} + \frac{x}{2}$$

8. Show equation of tangent to $y = 2x^2 - 1$ at $x = a$ is

$$4ax - y = 2a^2 + 1$$

Find point:

$$x = a$$
$$y = 2a^2 - 1$$
$$(a, 2a^2 - 1)$$

$$\frac{dy}{dx} = 4x$$
$$= 4a$$

$$y - (2a^2 - 1) = 4a(x - a)$$
$$y - 2a^2 + 1 = 4ax - 4a^2$$
$$(-y) \quad -2a^2 + 1 = 4ax - 4a^2 - y$$
$$(+4a^2) \quad 2a^2 + 1 = 4ax - y$$

10 d

$y = \ln\sqrt{x}$

$y = \ln\left(x^{\frac{1}{2}}\right)$

$y = \frac{1}{2}\ln(x)$

$\frac{dy}{dx} = \frac{1}{2x}$

$y = -1$

$\frac{1}{2}\ln(x) = -1$

$\ln(x) = 2$

$x = e^{2}$

$(e^{-2}, -1)$

$\frac{dy}{dx} = \frac{1}{2e^{-2}}$

$y + 1 = \frac{1}{2e^{-2}}\left(x - e^{-2}\right)$

$y = \frac{x}{2e^{-2}} - \frac{1}{2} - 1$

$y = \frac{e^{2}}{2}x - \frac{3}{2}$

Using $\ln x$ and e^{x}

Find the equation of the tangent to $y = \ln x$ where $y = -1$

- Find x

$\ln x = -1 \quad (\log_e x = 1)$

$x = e^{-1}$

- $\frac{dy}{dx} = \frac{1}{x} = x^{-1}$

$x = e^{-1}, \quad \frac{dy}{dx} = (e^{-1})^{-1} = e^{1}$

$= e$

Point $(e^{-1}, -1)$

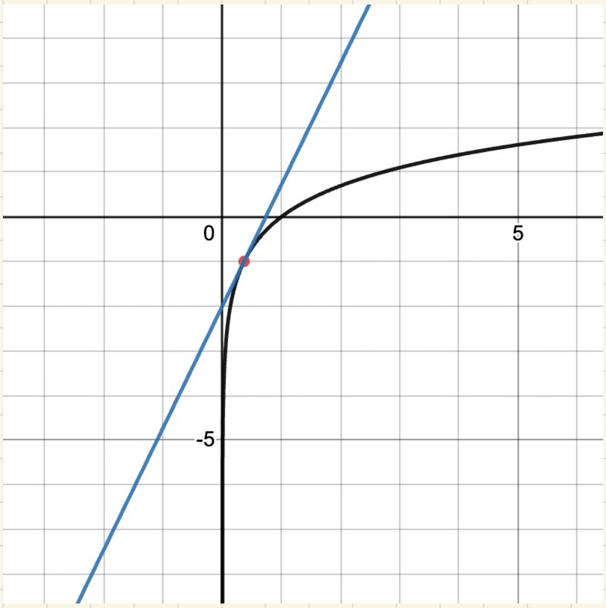
$\frac{dy}{dx} = e$

$y - -1 = e(x - e^{-1})$

$y + 1 = ex - e^{1}e^{-1}$

$y + 1 = ex - 1$

$y = ex - 2$

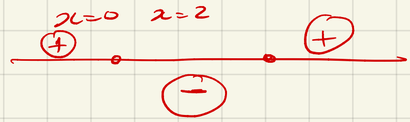


12. $f(x) = \ln(x(x-2))$

Domain of log function > 0

$\Rightarrow \quad x(x-2) > 0$

solve $\quad x(x-2) = 0$



Domain $\quad x < 0 \ \cup \ x > 2$

$\{x \mid x < 0 \ \cup \ x > 2\}$

$$f'(x) = \frac{1}{x(x-2)} \cdot \frac{d}{dx}(x^2 - 2x)$$
$$= \frac{1}{x(x-2)} \cdot 2x - 2$$
$$= \frac{2(x-1)}{x(x-2)}$$
$$= \frac{2x-2}{x^2-2x}$$
$$= \frac{2x-2}{x(x-2)}$$
$$= \frac{x + x - 2}{x(x-2)}$$
$$= \frac{1}{x} + \frac{1}{x-2}$$

$x = 3, \quad f'(x) = \frac{1}{3} + \frac{1}{1} = \frac{4}{3}$

$x = 3, \quad f(3) = \ln(3(3-2)) = \ln 3$

$(3, \ln 3)$

$$y - \ln 3 = \frac{4}{3}(x - 3)$$
$$y = \frac{4}{3}x - 4 + \ln 3$$

13. $$y = x^2 e^x$$

$y = uv \qquad \frac{dy}{dx} = uv' + u'v$ (product rule)

$$\frac{dy}{dx} = x^2 e^x + 2xe^x$$
$$= xe^x(x+2)$$

$x = 1$

$$\frac{dy}{dx} = e(3) = 3e$$

$x = 1, \quad y = e$

Gradient: $3e$

Point: $(1, e)$

$$y - e = 3e(x-1)$$
$$y = 3ex - 2e$$ equation of tangent

Find intercepts.

$x = 0 \qquad y = -2e$

$y = 0 \qquad x = \frac{2e}{3e}$

$x = \frac{2}{3}$

Intercepts

$(0, -2e)$ y-intercept

$(\frac{2}{3}, 0)$ x-intercept.

14. $y = 3xe^{\frac{x}{2}}$

$\frac{dy}{dx} = 3x\left(e^{\frac{x}{2}}\right)' + (3x)' e^{\frac{x}{2}}$

$= 3x e^{\frac{x}{2}} \cdot \frac{1}{2} + 3e^{\frac{x}{2}}$

$= 3e^{\frac{x}{2}}\left(\frac{x}{2} + 1\right)$

When $x = -1$

$\frac{dy}{dx} = 3e^{-\frac{1}{2}}\left(\frac{-1}{2} + 1\right)$

$= \frac{3}{2} e^{-\frac{1}{2}}$

$x = -1, \quad y = -3e^{-\frac{1}{2}}$

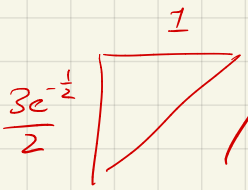
$y + 3e^{-\frac{1}{2}} = \frac{3e^{-\frac{1}{2}}}{2}(x+1)$

$y = \frac{3e^{-\frac{1}{2}}}{2}x + \frac{3e^{-\frac{1}{2}}}{2} - 3e^{-\frac{1}{2}}$

$= \frac{3e^{-\frac{1}{2}}}{2}x - \frac{3e^{-\frac{1}{2}}}{2}$

if $x = 0$ $\quad y = -\frac{3e^{-\frac{1}{2}}}{2}$

if $y = 0$ $\quad x = 1$



$A = \frac{3e^{-\frac{1}{2}}}{4}$

$A = \frac{3}{4\sqrt{e}}$ units^2

15. Find the equation of the tangent to $y = \frac{1}{\sin 2x}$ at the point where $x = \frac{\pi}{4}$

$u = 1, u' = 0$

$v = \sin 2x \quad v' = 2\cos(2x)$

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

$\frac{dy}{dx} = -\frac{1 \cdot 2\cos(2x)}{\sin^2(2x)} = \frac{-2\cos(2x)}{\sin^2(2x)}$

$= -2\cot(x)\operatorname{cosec}(x)$

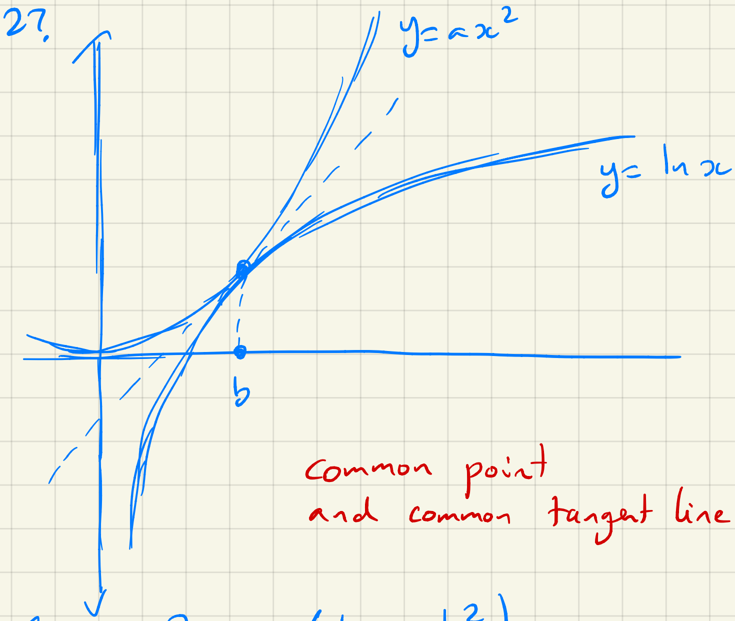
when $x = \frac{\pi}{4}$ $\quad \frac{dy}{dx} = \frac{-2\cos\left(\frac{\pi}{2}\right)}{\sin^2\left(\frac{\pi}{2}\right)} = 0$

when $x = \frac{\pi}{4}$, $y = \frac{1}{\sin\left(\frac{\pi}{2}\right)} = 1$

Horizontal tangent

$y = 1$

27.



common point
and common tangent line

Common Point (b, ab^2)

a) $(b, \ln b)$

$\Rightarrow \quad ab^2 = \ln b$

Common Tangent

$\frac{dy}{dx} = 2ax \quad , \quad \frac{dy}{dx} = \frac{1}{x}$

$x = b \qquad 2ab = \frac{1}{b} \quad \Rightarrow 2ab^2 = 1$

b) $ab^2 = \ln b \quad ①$
$2ab^2 = 1 \quad ②$

$2ab^2 = 2\ln b$
$2ab^2 = 1$

$2\ln b = 1$
$\ln b = \frac{1}{2}$
$b = e^{\frac{1}{2}} = \sqrt{e}$

Point $(e^{\frac{1}{2}}, \ln e^{\frac{1}{2}}) = (e^{\frac{1}{2}}, \frac{1}{2})$

c) $a = \frac{1}{2b^2}$
$a = \frac{1}{2e}$

d) $a = \frac{1}{2e} \quad , \quad b = \sqrt{e}$

Point $(e^{\frac{1}{2}}, \frac{1}{2})$

Gradient $\frac{dy}{dx} = \frac{1}{b} = \frac{1}{\sqrt{e}} = e^{-\frac{1}{2}}$

$y - \frac{1}{2} = e^{-\frac{1}{2}}(x - e^{\frac{1}{2}})$
$y - \frac{1}{2} = e^{-\frac{1}{2}}x - 1$

$y = x\sqrt{e} - \frac{1}{2}$

---

28. $P(x) = ax^2$

$p'(x) = 2ax$

$x = s \quad (s, as^2) \quad p'(s) = 2as$

$y - as^2 = 2as(x - s)$

$y = 2asx - 2as^2 + as^2$
$y = 2asx - as^2$

$x = t$
$y = 2atx - at^2$

$2asx - as^2 = 2atx - at^2$

$x(2as - 2at) = as^2 - at^2$

$x = \frac{a(s^2 - t^2)}{2a(s - t)}$

$= \frac{(s+t)(s-t)}{2(s-t)}$

$= \frac{s+t}{2}$

If perpendicular

$2as \cdot 2at = -1$

$y = 2atx - at^2$

$x = \frac{s+t}{2}$

$y = \frac{2at(s+t)}{2} - at^2$

$= ast + at^2 - at^2$

$= ast$

$t = \frac{-1}{4a^2 s}$

$y = ast$

$= as \cdot \frac{-1}{4a^2 s}$

$= \frac{-1}{4a}$