Rearranging quadratic equations

Quadratic equations can only be solved by factorising if one side is 0. To achieve this the usual rules of clearing fractions, and "change sides, change operations" apply.

Example Solve the equation $x^2 = 5 - 4x$

Answer Ensure all terms end up on the left of the = sign:

$$\begin{aligned} x^2 &= 5 - 4x \\ x^2 + 4x - 5 &= 0 \quad \text{(rearranging)} \\ (x-1)(x+5) &= 0 \quad \text{(factorising)} \\ x &= 1 \text{ or } {}^{-}5 \end{aligned}$$

Example Solve $(x-2)(x+1) = 10$

Answer Expand the brackets first:

$$\begin{aligned} (x-2)(x+1) &= 10 \\ x^2 - 2x + x - 2 &= 10 \\ x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \\ x &= 4 \text{ or } {}^{-}3 \end{aligned}$$

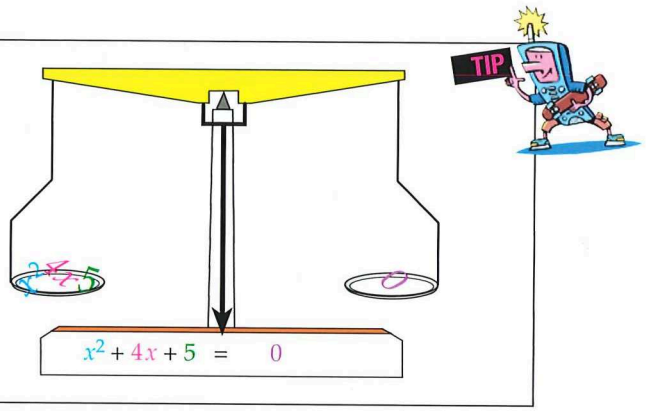
Example Solve the equation $x - 1 = \frac{6}{x}$

Answer It is not immediately obvious that this is a quadratic equation, but the x^2 term typical of a quadratic appears when the fraction is cleared by multiplying through by x:

$$\begin{aligned} x - 1 &= \frac{6}{x} \\ x(x-1) &= 6 \\ x^2 - x &= 6 \\ x^2 - x - 6 &= 0 \\ (x-3)(x+2) &= 0 \\ x &= 3 \text{ or } {}^{-}2 \end{aligned}$$

5

Solve quadratic equations by placing all terms on the left, and 0 on the right. Then factorise, and solve each factor separately.



exercise 5.03

Solve these quadratic equations.

1 $x^2 + 4x = 12$
2 $x^2 = x + 2$
3 $x^2 + 6 = 7x$
4 $x^2 + 2x = 3$
5 $x^2 - 12 = x$
6 $x^2 = 3x$
7 $x^2 = 5x + 6$
8 $x^2 = 36$
9 $x^2 - x = 20$
10 $x^2 - 21 = 4x$
11 $x^2 = x + 30$
12 $x^2 + 20 = 9x$
13 $x^2 + 8x - 5 = 4$
14 $x^2 + 1 = 50$
15 $x(x - 2) = 15$
16 $(x - 3)^2 = 25$
17 $5x = 14 - x^2$
18 $160 - 5x^2 = 140$
19 $(x - 4)(x - 6) = 8$
20 $(x + 4)^2 = 9$
21 $x(x - 3) = 5x$
22 $2x(4 - x) = 6$
23 $2x(x - 1) = x^2 + 8$
24 $x + 3 = \frac{12}{x + 4}$
25 $\frac{5x}{x - 1} = x + 8$
26 $(x - 3)(x + 2) = 4x$
27 $(x + 2)(x + 7) = (x + 5)(x + 3)$
28 $3x^2 = 13x + 10$
29 $2x + 1 = \frac{27}{x - 1}$
30 $2x = \frac{3 - 5x}{x}$
31 $(2x - 3)^2 = 16$

Forming quadratic equations

Example A negative number has 2 added to it, and is then squared, giving a result of 9.

What is the number?

What positive number has the same property?

Answer Suppose the number is x. Then adding 2 and squaring is $(x + 2)^2$.

Solve the equation $(x + 2)^2 = 9$

$$(x + 2)^2 = 9$$
$$(x + 2)(x + 2) = 9$$
$$x^2 + 4x + 4 = 9$$
$$x^2 + 4x - 5 = 0$$
$$(x + 5)(x - 1) = 0$$
$$x = -5 \text{ or } 1$$

The negative number is -5. The positive number with the same property is 1.

exercise 5.04

1 When a number is squared, and then 1 is subtracted, the result is 8.
   a Write this information down as a quadratic equation.
   b Solve the equation to work out the two possible numbers.

2 A number is squared, and then added to the original number. The result is 20.
   a Write this information down as a quadratic equation.
   b Solve the equation to work out the two possible numbers.

3 When a number, x, is multiplied by a number 4 less than x, the result is 12. What two numbers have this property?

4 Squaring a number gives the same result as multiplying the number by 8, and then subtracting 12.
   a Write this information down as a quadratic equation.
   b Solve the equation to work out the two possible numbers.

5 Squaring a number and then adding on 5 times the original number gives a result of 6.
   a Write this information down as a quadratic equation.
   b Solve the equation to work out the positive number that has this property.

6 When the result of adding 2 to a number is multiplied by the result of subtracting 2 from the same number the answer is 21. What two numbers have this property?

7 The number 3 has the property that when 4 is added to it, and the result is squared, the answer is 49. There is also a *negative* number that has this property. Form a quadratic equation and solve it to find the negative number.

8 Two numbers have a difference of 3. One is x, the other is $x + 3$. Form a quadratic equation and solve it to find the two numbers if the sum of their squares is 29.

Applications

Example John is 6 years old and Amy is 2 years old. In how many years will the product of their ages be 96?

Answer Let x = the required number of years.

At that time John's age will be $6 + x$, and Amy's age will be $2 + x$.

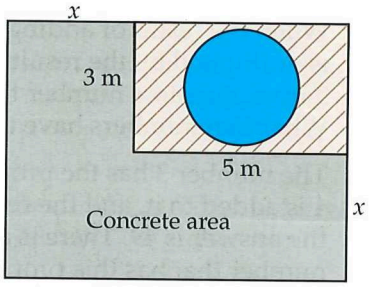
The product means the ages are *multiplied*.

$$(6 + x)(2 + x) = 96 \text{ or } (x + 6)(x + 2) = 96$$
$$x^2 + 8x + 12 = 96$$
$$x^2 + 8x - 84 = 0$$
$$(x + 14)(x - 6) = 0$$
$$x = -14 \text{ or } 6$$

Because the question is about ages we use the positive answer only – that is, in 6 years' time.

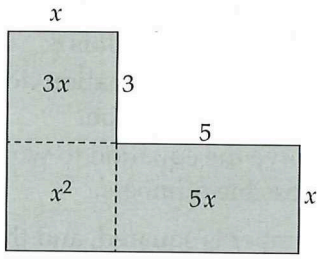
5

Example A landscape designer is planning to lay concrete on two sides of a spa-pool deck. The deck measures 3 m by 5 m, and the concrete will extend the same distance, x, out from the sides of the deck as shown. The designer will have enough material to make a concrete shape with an area of 48 m^2.



Form a quadratic equation and solve it to work out the distance, x, that the concrete extends from the sides of the deck.

Answer Split the concrete area into three sections:



The total area is
$x^2 + 3x + 5x = x^2 + 8x$.

This area must equal 48 m^2.

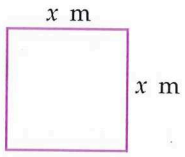
The equation is:

$$x^2 + 8x = 48$$
$$x^2 + 8x - 48 = 0$$
$$(x + 12)(x - 4) = 0$$
$$x = -12 \text{ or } 4$$

The concrete extends 4 m from the sides of the deck.

exercise 5.05

1 A square measures x metres by x metres. The number of metres for its perimeter is 12 less than the number of m^2 in its area.



The problem is to work out the size of the square. x can be calculated by solving the equation $x^2 - 12 = 4x$.

- **a** What term in the equation represents the perimeter?
- **b** What term in the equation represents the area?
- **c** Solve the equation, giving both of the two solutions.
- **d** How long is the side of the square?

2 The sum of the first n odd counting numbers is n^2.

- **a** Write down the first seven odd counting numbers.
- **b** Add together the numbers in part **a**.
- **c** Write down a quadratic equation that shows how many odd numbers are needed to add to 100.

3 The motion of a snowball rolling down a hill can be modelled by the equation $d = t^2 + 6t$ where d is the distance travelled in metres and t is the time in seconds from when the snowball is pushed. Solve the equation $t^2 + 6t = 72$ to calculate how long the snowball will take to roll 72 m down the hill.

4 a Write down a pair of consecutive numbers.

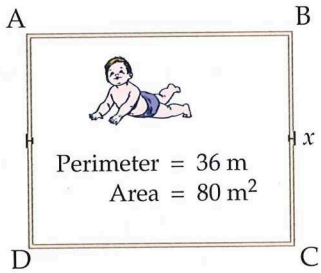
b 73 is one of a pair of consecutive numbers. What are the two possible values for the other number in the pair?

c x is the smaller one from a pair of consecutive numbers. Express the other number in terms of x.

d Two consecutive numbers multiply to give 42. Write down a quadratic equation for this information.

e Solve the equation from part **d** to find the pair of consecutive numbers that multiply to 42. Note that there are two pairs: one both positive, one both negative.

5 The diagram shows a rectangular play-pen labelled ABCD. The perimeter is 36 m, and the area is 80 m^2.



a What do the lengths AB and BC add up to? (Hint: the perimeter is 36.)

b Write AB in terms of x and the answer to part **a**.

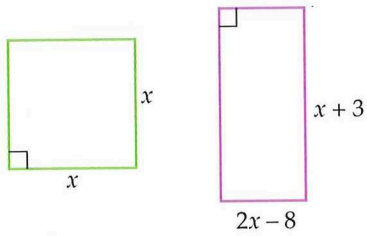
c Explain why the area of the play-pen is given by AB × BC.

d Express the area in terms of x.

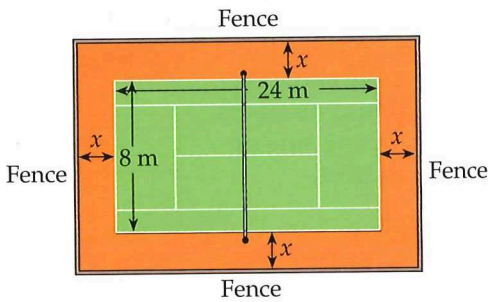
e Form a quadratic equation and solve it to calculate x.

f What are the measurements of the play-pen?

6 The area of this square is the same as the area of the rectangle. Solve a quadratic equation to calculate x, and hence give the area.



7 The rules of tennis say that a singles court should measure 8 m by 24 m. A tennis club decide to construct a fence to make a rectangular enclosure around a singles court. The fence is the same distance everywhere, x, from the outside lines of the court. Tournament play requires that there be an unobstructed area outside the court but inside the enclosure that is 2.75 times the area of the court.



a Explain why the total enclosed area must be 720 m^2.

b Complete this equation for the area of the enclosure:

$(2x + 8)$ ( ) = 720

c When the equation in part **b** is expanded the result is:

$4x^2 + 64x + 192 = 720$

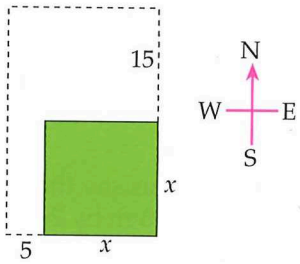
Simplify this equation by rearranging and then dividing through by 4 so that it is written in the form:

x^2 _____ = 0

d Solve the equation and hence write down the distance of the fence from the outside lines of the court.

5

8 The local authority has decided to extend a children's playground, which at present is square. The new playground will be rectangular, extended 5 m to the west and 15 m to the north. The area of the extended playground will be 875 m^2.



a This information can be used to write a quadratic equation:
$(x + 5)(x + 15) = 875$
Explain what x represents in this situation.

b Write down some working to show that the equation in part **a** can be simplified to give:
$x^2 + 20x - 800 = 0$

c Solve the equation and hence write down the present area of the playground.

INVESTIGATION

Street numbering

The houses in most residential streets are numbered according to a strict pattern. On one side of the street the houses have odd numbers, and on the other side the houses have even numbers.

1 Complete this table showing house numbers on one side of a street.

Position of house	Number on letter box
1st	2
2nd	4
3rd	
4th	
…	…
18th	
	76
nth	

2 Write down a formula, in terms of n, that will give the number of the house immediately after the nth house on this side of the street.

3 The numbers of two consecutive houses on this side of the street multiply to 224. Show that the numbers of these two houses could be found by solving the equation: $4n^2 + 4n = 224$. Show your working.

4 Solve this quadratic equation to find the two house numbers.

9 One side of this postage stamp is 1 cm longer than the other. The area of the stamp is 12 cm^2.

- **a** If the shorter side is x, write down an expression for the longer side.
- **b** Write down, as simply as possible, an expression for the area in terms of x.
- **c** Form a quadratic equation from the given information.
- **d** Solve the equation to work out the lengths of the sides of the stamp.

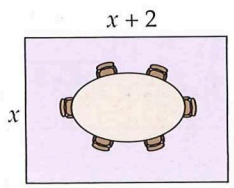
10 Jonah uses a tennis racket to hit a stone straight up in the air. The height, h, in metres, of the stone after t seconds is given by the formula $h = 35t - 5t^2$.

- **a** How high above the ground is the stone after 1 second?
- **b** Solve a quadratic equation to work out the two times when the stone is 60 m above the ground.
- **c** Explain why the solution to the equation $35t - 5t^2 = 0$ tells us when the stone returns to the ground.
- **d** Solve the equation $35t - 5t^2 = 0$. There are two solutions: explain the meaning of each.

11 A group of n friends decide to send each other Christmas cards at the end of the year. They save money by all buying their cards from UNICEF. Renee is going to organise the purchase.

- **a** Explain why two friends would need a total of two cards.
- **b** Three friends would need a total of six cards. Call the three friends A, B, and C and explain why they would need six cards by writing down who sends who a card.
- **c** How many cards would a group of four friends need to buy?
- **d** To work out the total number of cards eight friends would need to buy, Renee works out the calculation 8×7. Complete this sentence which explains her reasoning:
  "There are _____ of us altogether, and each of us will need to send _____ cards, so we will need a total of _____ cards".
- **e** Write down a formula for the total number of cards n friends will need.
- **f** Use your answer in part **e** to work out the total number of cards five people would need to buy.
- **g** Renee eventually purchases 110 cards for the group of n people. Form a quadratic equation that uses this information.
- **h** Solve the equation to work out n, the number of people in the group.

12 This is a plan of a dining room where the length is 2 m longer than the width. The area of the room is 35 m^2. Form a quadratic equation and solve it to work out the width of the room.



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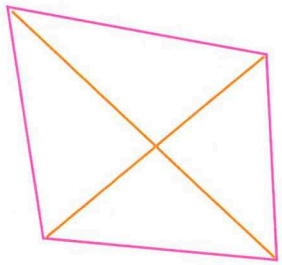
13 The diagram shows a quadrilateral with two *diagonals*.



a Draw a pentagon, with all of its diagonals.

b Draw a six-sided polygon, with all of its diagonals.

c Fill in the blank spaces in this table.

Polygon	Number of sides, n	Number of diagonals, d
Quadrilateral		2
Pentagon		
	6	

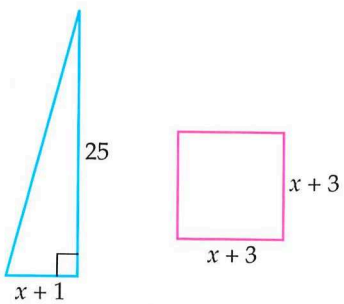
d The formula for the number of diagonals in a polygon with n sides is

$$d = \frac{1}{2}n(n-3).$$

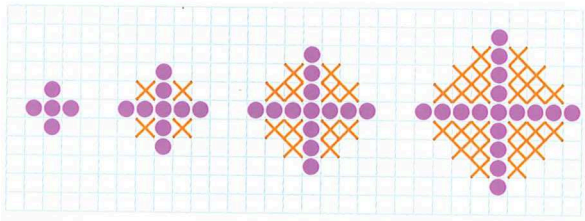
How many diagonals can be drawn in an octagon (eight sides)?

e Form a quadratic equation and solve it to work out the number of sides that a polygon with 35 diagonals has.

14 The triangle and the square have the same area. Calculate the value of x.



15 Tukutuku patterns are used to decorate panels in a whanau on a marae. This tukutuku pattern has some squares covered with round paua shells arranged in rows, and other squares that are stitched in the shape of a cross.



a Copy and complete the six gaps in this table that shows the number of round shells and crosses in each pattern.

Shape number (n)	1	2	3	4	5
Number of round shells (r)	5	9			
Number of crosses (c)		4	12		

b The formula for the number of round shells can be written as $r = ? \times n + ?$. Write down this formula, replacing the question marks with numbers.

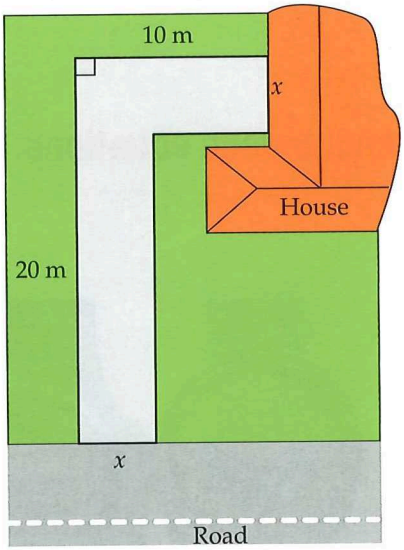
c The formula for the number of crosses for shape number n is $2n(n-1)$. Use this formula to calculate the number of crosses in shape number 8.

d Write down working to show that the formula for the total number of squares covered (that is, by both round shells and crosses) is $2n^2 + 2n + 1$.

e One of these shapes has 181 squares covered. This information can be used to write down the equation $2n^2 + 2n + 1 = 181$. Copy and continue with this working to solve the equation and explain what the solution represents.

$$2n^2 + 2n + 1 = 181$$
$$2n^2 + 2n = 180$$
$$n^2 + n = 90$$

16 The access to George's house is along a right of way with a sharp right-angled corner as shown on the plan.



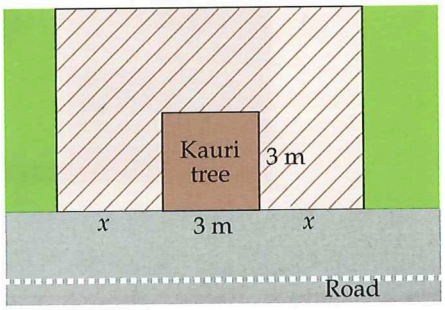
George has been told that he will have to pay for 104 m^2 of asphalt to surface the right of way. He wants to check this estimate by calculating the width, x, of the right of way. George does this by splitting the right of way into two rectangles, and writing down their areas. One of the areas is $x \times 20$. The other is $(10 - x) \times x$.

a Make a rough copy of the plan and on it draw a line that shows George's two rectangles.

b Use the two area expressions to write down a single, simplified expression for the area of the right of way.

c Let the area expression equal 104. Solve this equation to work out the width of the right of way.

17 A wooden viewing platform is to be built on three sides of a kauri tree as shown in the diagram to protect its roots.



The outside rails of the platform will be the same distance, x, from the inside rails all the way around. Calculations show that the area of the platform will be 68 m^2. The problem is to work out the distance, x, between the inside rails and outside rails.

a What is the area of the part between the platform and the road enclosing the kauri tree?

b Explain why the area of the platform can be written as:
Area $= (2x + 3)(x + 3) - 9$.

c Simplify the expression for the area in part **b**.

d The required distance, x, is one of the solutions to the equation $2x^2 + 9x - 68 = 0$. One of the factors of this quadratic is $(2x + 17)$. What is the other factor?

e Hence give the required distance.

exercise 5.03 (page 68)

1 $x = -6$ or $x = 2$
2 $x = -1$ or $x = 2$
3 $x = 1$ or $x = 6$
4 $x = -3$ or $x = 1$
5 $x = -3$ or $x = 4$
6 $x = 0$ or $x = 3$
7 $x = -1$ or $x = 6$
8 $x = -6$ or $x = 6$
9 $x = -4$ or $x = 5$
10 $x = -3$ or $x = 7$
11 $x = -5$ or $x = 6$
12 $x = 4$ or $x = 5$
13 $x = -9$ or $x = 1$
14 $x = -7$ or $x = 7$
15 $x = -3$ or $x = 5$
16 $x = -2$ or $x = 8$
17 $x = -7$ or $x = 2$
18 $x = 2$ or $x = -2$
19 $x = 2$ or $x = 8$
20 $x = -7$ or $x = -1$
21 $x = 0$ or $x = 8$
22 $x = 1$ or $x = 3$
23 $x = -2$ or $x = 4$
24 $x = -7$ or $x = 0$
25 $x = -4$ or $x = 2$
26 $x = -1$ or $x = 6$
27 $x = 1$
28 $x = \frac{-2}{3}$ or $x = 5$
29 $x = \frac{-7}{2}$ or $x = 4$
30 $x = -3$ or $x = \frac{1}{2}$
31 $x = \frac{-1}{2}$ or $x = \frac{7}{2}$

exercise 5.04 (page 69)

1 a $x^2 - 1 = 8$
b $x = 3$ or $x = -3$

2 a $x^2 + x = 20$
b $x = -5$ or $x = 4$

3 -2 and 6

4 a $x^2 = 8x - 12$
b $x = 2$ or $x = 6$

5 a $x^2 + 5x = 6$
b $x = 1$

6 -5 and 5
7 $(x + 4)^2 = 49$; -11
8 There are two answers: 2 and 5; and -2 and -5

5

exercise 5.05 (page 70)

1 a $4x$
b x^2
c $x = -2$ or $x = 6$
d 6 m

2 a 1, 3, 5, 7, 9, 11, 13
b 49
c $n^2 = 100$

3 6 seconds

4 a 4, 5 or 37, 38, for example
b 72 or 74
c $x + 1$
d $x(x + 1) = 42$
e 6, 7 or -7, -6

5 a 18
b $18 - x$
c Area of rectangle is base (AB) × height (BC)
d Area = $x(18 - x)$
e $x(18 - x) = 80$; $x = 8$ or 10
f 10 m by 8 m

6 $x^2 = (2x - 8)(x + 3)$; $x = 6$; area = 36

7 a Area of court = $8 \times 24 = 192$ m^2
Area outside court and inside enclosure = $2.75 \times 192 = 528$ m^2
Total area = $192 + 528 = 720$ m^2
b $(2x + 8)(2x + 24) = 720$
c $x^2 + 16x - 132 = 0$
d $x = -22$ or 6; that is, $x = 6$ m (distance from outside lines to fence)

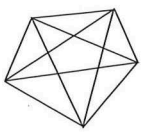
8 a x is the length of one side of the playground before it is extended.
b $(x+5)(x+15) = 875$
$x^2 + 5x + 15x + 75 = 875$
$x^2 + 20x + 75 = 875$
$x^2 + 20x - 800 = 0$
c $x = 20$ or -40; that is, $x = 20$ m and the present area is $20^2 = 400$ m^2

9 a $x + 1$
b $x(x + 1) = x^2 + x$
c $x^2 + x = 12$
d 3 cm, 4 cm

10 a 30 m
b $35t - 5t^2 = 60$
$5t^2 - 35t + 60 = 0$
$t^2 - 7t + 12 = 0$
$(t - 3)(t - 4) = 0$
$t = 3$ seconds and 4 seconds
c Because when the height is 0 the stone must be on the ground.
d $35t - 5t^2 = 0$
$5t(7 - t) = 0$
$t = 0$ or 7
$t = 0$ gives the time when the stone is first hit, $t = 7$ seconds gives the time when the stone *returns* to the ground.

11 a Each friend sends a card to the other.
b A to B; A to C; B to A; B to C; C to A; C to B
c 12
d 8, 7, 56
e $n(n - 1)$
f 20
g $n(n - 1) = 110$
h $n^2 - n - 110 = 0$
$(n - 11)(n + 10) = 0$
$n = 11, -10$
i.e. 11 people in the group

12 $x(x + 2) = 35$; $x = 5$ m

13 a 
b 

c

Polygon	Number of sides, n	Number of diagonals
Quadrilateral	4	2
Pentagon	5	5
Hexagon	6	9

d 20

e $\frac{1}{2}n(n-3) = 35$

$n^2 - 3n - 70 = 0$

$(n-10)(n+7) = 0$

$n = 10, -7$

$n = 10$ sides

14 $x = 7$

15 a

Shape number (n)	1	2	3	4	5
Number of round shells (r)	5	9	13	17	21
Number of crosses (c)	0	4	12	24	40

b $r = 4n + 1$

c 112

d $4n + 1 + 2n(n-1) = 2n^2 - 2n + 4n + 1$

$= 2n^2 + 2n + 1$

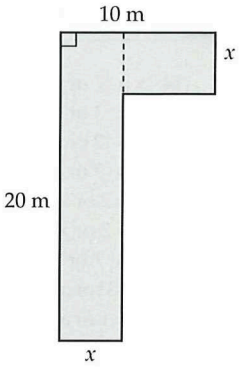
e $n^2 + n - 90 = 0$

$(n-9)(n+10) = 0$

$n = 9, -10$

That is, shape number 9 has 181 squares covered altogether.

16 a



b $x \times 20 + (10 - x) \times x$

$= 20x + 10x - x^2$

$= 30x - x^2$

c $30x - x^2 = 104$

$x^2 - 30x + 104 = 0$

$(x-4)(x-26) = 0$

Width of right of way = 4 m

17 a $9\ m^2$

b Area of platform = area of rectangle − area of square

$= (2x + 3)(x + 3) - 3 \times 3$

$= (2x + 3(x + 3) - 9$

c $2x^2 + 9x + 9 - 9 = 2x^2 + 9x$

d $(x - 4)$

e 4 m

INVESTIGATION

Street numbering

(page 72)

1

Position of house	Number on letter box
1st	2
2nd	4
3rd	6
4th	8
...	...
18th	36
38th	76
nth	$2n$

2 $2n + 2$

3 $2n(2n + 2) = 224$

$4n^2 + 4n = 224$

4 Divide through by 4:

$n^2 + n = 56$

$n^2 + n - 56 = 0$

$(n + 8)(n - 7) = 0$

$n = -8$ or 7

The two house numbers are 14 and 16.