COMP 3490 Assignment 2
Fall 2024 Written Solutions

Here are the intersection and normal vector formulas for some common shapes.

Plane

A plane can be described by the equation

$$Ax + By + Cz + D = 0.$$

A plane can also be described by specifying a normal vector $\vec{N} = (A, B, C)$ and a point $\vec{P} = (p_x, p_y, p_z)$. The point $\vec{X} = (x, y, z)$ is on the plane if and only if

$$\vec{N} \cdot (\vec{X} - \vec{P}) = 0.$$

Given a ray with parametric equation

$$\vec{R}(t) = \vec{E} + t\vec{d},$$

this ray intersects the plane if and only if

$$\vec{N} \cdot \vec{d} \neq 0.$$

Assuming this condition holds, the intersection point occurs at

$$t = -\frac{\vec{N} \cdot (\vec{E} - \vec{P})}{\vec{N} \cdot \vec{d}}.$$

The normal vector at every point on the plane is simply $\vec{N}$.

Cylinder

The equation of an infinite cylinder that is parallel to the y-axis, centered on the point $(C_x, 0, C_z)$, with radius r, is

$$(x - C_x)^2 + (z - C_z)^2 = r^2.$$

Consider a ray with parametric equation

$$\vec{R}(t) = \vec{E} + t\vec{d}.$$

This ray intersects the cylinder if and only if the quadratic equation

$$At^2 + Bt + C = 0$$

has real solutions, where:

$$\begin{aligned} A &= d_x^2 + d_z^2 \\ B &= 2d_x(E_x - C_x) + 2d_z(E_z - C_z) \\ C &= (E_x - C_x)^2 + (E_z - C_z)^2 - r^2 \end{aligned}$$

Given a point $P = (x, y, z)$ on the cylinder, a normal vector to the cylinder at P is

$$\vec{N} = (x - C_x, 0, z - C_z).$$

Cone

The equation of an infinite cone that is parallel to the y-axis and has vertex (V_x, V_y, V_z) takes the general form

$$(x - V_x)^2 + (z - V_z)^2 = c^2(y - V_y)^2.$$

The parameter c describes the slope of the cone. The circle that lies at height h above the vertex has radius ch.

Consider a ray with parametric equation

$$\vec{R}(t) = \vec{E} + t\vec{d}.$$

This ray intersects the cone if and only if the quadratic equation

$$At^2 + Bt + C = 0$$

has real solutions, where:

$$\begin{aligned} A &= d_x^2 + d_z^2 - c^2 d_y^2 \\ B &= 2d_x(E_x - V_x) + 2d_z(E_z - V_z) - 2c^2 d_y(E_y - V_y) \\ C &= (E_x - V_x)^2 + (E_z - V_z)^2 - c^2(E_y - V_y)^2 \end{aligned}$$

Given a point $P = (x, y, z)$ on the cone, a normal vector to the cone at P is

$$\vec{N} = (x - V_x, -c^2(y - V_y), z - V_z).$$

Hyperboloid

The equation of an infinite hyperboloid that is parallel to the y-axis and centered on the point (C_x, C_y, C_z) takes the general form

$$(x - C_x)^2 + (z - C_z)^2 = a(y - C_y)^2 + r^2.$$

The parameter a describes how quickly the hyperboloid widens as it climbs. The circle that lies at height h above the center point has radius $\sqrt{ah^2 + r^2}$.

Consider a ray with parametric equation

$$\vec{R}(t) = \vec{E} + t\vec{d}.$$

This ray intersects the cone if and only if the quadratic equation

$$At^2 + Bt + C = 0$$

has real solutions, where:

$$\begin{aligned} A &= d_x^2 + d_z^2 - a d_y^2 \\ B &= 2d_x(E_x - C_x) + 2d_z(E_z - C_z) - 2a d_y(E_y - C_y) \\ C &= (E_x - C_x)^2 + (E_z - C_z)^2 - a(E_y - C_y)^2 - r^2 \end{aligned}$$

Given a point $P = (x, y, z)$ on the hyperboloid, a normal vector to the hyperboloid at P is

$$\vec{N} = (x - C_x, -a(y - C_y), z - C_z).$$

Circle

The simplest way to implement a planar circle is to define the plane that contains the circle, along with a center point $\vec{C}$ and a radius r. To determine whether a ray intersects the circle, first determine whether the ray intersects the plane. If so, the parameter corresponding to the intersection point is

$$t_I = -\frac{\vec{N} \cdot (\vec{E} - \vec{P})}{\vec{N} \cdot \vec{d}},$$

as discussed for planes above, and the point of intersection itself is located at

$$\vec{R}(t_I) = \vec{E} + t_I \vec{d}.$$

This point of intersection lies within the circle if

$$\left| \vec{E} + t_I \vec{d} - \vec{C} \right| \leq r.$$

A normal vector to any point on the circle is simply $\vec{N}$, the normal vector to the plane.

Square

The simplest type of square to implement lies in a plane parallel to one of the coordinate planes, and has edges that are axis-aligned. Such a square can be defined by specifying a plane, the coordinates of one of the vertices, and a side length. As an example, consider a square that lies in the plane $z = c$. Let $\vec{P} = (P_x, P_y, c)$ be the lower left vertex of the square, and let L be its side length.

To determine whether a ray intersects the square, first determine whether it intersects the plane $z = c$. Assuming it does, and assuming t_I is the parameter corresponding to the point of intersection, then the point of intersection itself has the form

$$\vec{R}(t_I) = \vec{E} + t_I \vec{d} = (R_x, R_y, c).$$

This point of intersection lies within the square if

$$P_x \leq R_x \leq P_x + L \qquad \text{and} \qquad P_y \leq R_y \leq R_y + L.$$

A normal vector to any point on the square is simply the normal vector to the plane.