SHIVAM PATEL DS203 Assignment 1 200070077

Ans1.(i) By total probability law, we have

$$P(E) = \sum_{i=1}^{n} P(E|F_i)\, P(F_i)$$

E → event that sample is defective

$F_i$ → event that sample is from $i^{th}$ factory.

$$\therefore P(E) = \frac{80}{100} \times \frac{30}{100} + \frac{20}{100} \times \frac{10}{100} = 0.26.$$

ii) probability that defective sample is from company A →

we apply Bayes rule

$$P(F_i|E) = \frac{P(E|F_i)\, P(F_i)}{\sum_{j=1} P(E|F_j)\, P(F_j)}$$

$$= \frac{\frac{80}{100} \times \frac{30}{100}}{\frac{80}{100} \times \frac{30}{100} + \frac{20}{100} \times \frac{10}{100}} = 0.923$$

Ans 2. (i) P( first access attempt fails) → P(X)

By law of total probability, we have that

P(X) = Prob. that webserver is not working
+ Prob that server is working but rejected

$$= 0.2 + 0.8 \times (1-0.9) = 0.28.$$

ii) P (server is working | first access attempt fail) → P(X)

By Bayes' Theorem, we have

$$P(X) = \frac{P(\text{server is working} \cap \text{failed attempt})}{P(\text{failed attempt})}$$

$$= \frac{0.8 \times 0.01}{0.2 + 0.8 \times 0.01} = 0.285.$$

iii) P ( second access attempt fails | first attempt fails) → P(X)

these are not independent events as if webserver is not working, then second attempt will always fail.

$$P_X \to \frac{P(\text{second attempt fails} \cap \text{first attempt fails})}{P(\text{first attempt fails})}$$

= ~~(P(E₁ = server not working) P₁(E₁)P₂ E₁) + (P(E₂ = server working)~~

$$= \frac{P(E_1 = \text{server working})\, P_1(E_1)\, P_2(E_2) + P(E_2 = \text{server not working}) \times P_1(E_2)\, P_2(E_2)}{P(\text{first attempt failed})}$$

$$= \frac{0.8 \times (0.1) \times (0.1) + 0.2 \times 1 \times 1}{0.28\cancel{0}}$$

$= \cancel{0.7298}$. $= 0.742$

(iv) $P(\text{server is working} \mid 1^{st}$ & $2^{nd}$ attempts fail$) = \dfrac{0.8 \times 0.1 \times 0.1}{0.8 \times 0.1 \times 0.1 + 0.2 \times 1 \times 1}$

$= \underline{\underline{0.03846}}$

Ans 3. Probability that at least one is a six

$$P(E) = 1 - P(E^c)$$

Prob that no faces have six $= P(E^c) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$

$\therefore P(E) = \dfrac{11}{36}$

Sample space $= \{1,2,3,4,5,6\}^2 = \Omega$

$|\Omega| = 36$

No of events with different faces $= |\Omega| - |\{(1,1),(2,2),(3,3)$
$(4,4),(5,5)\ (6,6)\}|$

$= 30$.

Prob. that atleast one is a six when two faces are different $= P(F \mid G)$

$F \to$ event of six occuring

$G \to$ unique faces.

$$P(F|G) = \frac{P(F\cap G)}{P(G)} = \frac{\left|\begin{Bmatrix}(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5) \\ (1,6)\ (2,6)\ (3,6)\ (4,6)\ (5,6)\end{Bmatrix}\right|}{30}$$

$$= \frac{10}{30} = \underline{0.33}.$$

$\therefore\ P(F|G) = 0.33.$

Ans 4. By Bayes theorem, we have that

$$P(F_j|E) = \frac{P(E\cap F_j)}{P(E)} = \frac{P(E|F_j)\ P(F_j)}{\sum_{i=1}^{n} P(E|F_i)\ P(F_i)}$$

Prob of choosing color blind

$$= P(\text{male})\ P(\text{male}|\text{color blind}) + P(\text{female})\ P(\text{female}|\text{color blind})$$

$$= \frac{1}{2}\times\frac{5}{100} + \frac{1}{2}\times\frac{1}{100} = 0.03$$

Prob of a chosen color blind person being male

$$= \frac{P(\text{male})\ P(\text{male}|\text{color blind})}{P(\text{color blind})} = \frac{\frac{1}{2}\times\frac{5}{100}}{0.03}$$

$$= 0.833.$$

Ans5. (a) condition of independence,

$P(A)$ and $P(B)$ are independent if and only if

~~P(A)~~ $P(A \cap B) = P(A) \times P(B)$.

$(P(A), P(B) > 0)$

For an event to be independent of itself.

$P(E \cap E) = P(E) \times P(E)$

$\therefore P(E) = (P(E))^2$

$\therefore \boxed{P(E) = 1}$.

But when $\boxed{P(E) = 0}$ we can say the event E does not occur at all, hence it is vacuously independent.

(b) $P(A) = 0.3$ $\quad P(B) = 0.4$ $\quad$ ~~P(A)~~ A & B are independent.

$\therefore P(A \cap B) = P(A)\,P(B) = 0.12$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 - 0.12$$
$$= 0.58.$$

($\because$ A & B are independent)

When A & B are mutually exclusive, the $A \cap B = \emptyset$

$\therefore P(A \cap B) = 0$

$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.3 + 0.4 = 0.7$.

(c) $P(A) = 0.6$ $P(B) = 0.8$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

If A & B are independent, then $P(A \cup B)$

$= 0.6 + 0.8 - 0.6 \times 0.8 = 0.92 \leq 1$

Hence, A & B can be independent.

But if A & B are mutually exclusive, then $P(A \cap B) = 0$ and,

$P(A \cup B) = P(A) + P(B) - 0 = 0.6 + 0.8 = 1.4 \not\leq 1$

Hence they cannot be mutually exclusive.

Ans6. (1.) $F(x) = \begin{cases} e^{-x^2/4} & x < 0 \\ 1 - e^{-x^2/4} & x \geq 0 \end{cases}$

This is not a valid CDF, as CDF is always non decreasing. But

$F(x) = 1 - e^{-x^2/4}$ for $x \geq 0$ is decreasing in nature.

(2) $F(x) = \begin{cases} 0 & x < 0 \\ 0.5 + e^{-x} & 0 \leq x < 3 \\ 1 & x \geq 3 \end{cases}$

This is not a valid CDF, as a valid CDF function assumes values only $\leq 1$.

Ans 6) Part 2 contd.

But for $F(x) = 0.5 + e^{-x}$ at $x=0$,

we see that CDF = 1.5.

Hence this is invalid CDF.

3) $F(x) = \begin{cases} 0 & x < 0 \\ 0.5 + x/20 & 0 \leq x \leq 10 \\ 1 & x \geq 10 \end{cases}$

This is a valid CDF function.

$$P(x^2 > 5) = P(x < -\sqrt{5}) + P(x > \sqrt{5})$$

$$= 0 + \left( (P(x < \infty)) - (P(x \leq \sqrt{5})) \right)$$

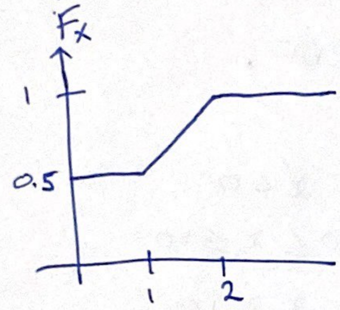
$$= 1 - (0.5 + \sqrt{5}/20)$$

$$= 0.388.$$

Ans 7. (1) $P(X \leq 0.8)$

$P(X \leq 0.8) = F_X(0.8)$

where $F_X$ is the given CDF. –




$$F_X = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leq x \leq 1 \\ x/2 & 1 < x \leq 2 \\ 1 & x > 2 \end{cases}$$

$F_X(0.8) = 0.5$

2) $E(X) = \int x f_X(x)\, dx$

But here, we have a step up in the CDF. So, that would be considered along with discrete calculations.

$$f(x) = \frac{d(F(x))}{dx} = \begin{cases} \text{PMF} \rightarrow 0.5 & x = 0 \\ \text{PDF} \rightarrow 0.5 & 1 \leq x \leq 2 \end{cases}$$

$\therefore E(X) = 0 \times 0.5 + \int_1^2 0.5x\, dx$

$= 0 + 0.75 = 0.75.$

$\therefore E(X) = 0.75$

Ans 7 contd.

3) $Var(X) = E(x^2) - (E(x))^2$

$E(x) = 0.75$ $\qquad (E(x))^2 = 0.5625$

$E(x^2) = \int x^2 f(x)\,dx$

$= (0^2) \times (0.5) + \int_1^2 \frac{x^2}{2} dx$

$= 0 + \frac{7}{6} = 1.166$

$Var\ x = 1.166 - \cancel{0.75}\ 0.5625 = \cancel{0.416}\ 0.6041$

$\boxed{\therefore\ Var(x) = 0.6041}$

Ans 8. PDF $f(x) = \begin{cases} ce^{-2x} & \text{if } 0 \leq x < \infty \\ 0 & \text{if } x < 0 \end{cases}$

As the $\int_{-\infty}^{\infty} f(x)\,dx = 1$ for any PDF,

~~So~~ $\cancel{\int_{0}^{\infty} ce^{-2x}\,dx = ce^{-2x}}$

So $\int_{-\infty}^{0} 0\,dx + \int_0^{\infty} ce^{-2x}\,dx = 1$

$\therefore \frac{c}{2} = 1 \qquad \therefore \underline{c = 2}$

$\therefore \; F(x) = \begin{cases} 2e^{-2x} & 0 \le x < \infty \\ 0 & \text{if } x < 0 \end{cases}$

$$P\{x > 2\} = \int_2^\infty 2e^{-2x}\,dx = e^{-4} = \frac{1}{e^4}.$$

$\therefore \; P\{x>2\} = \frac{1}{e^4}$

Ans 9. Random variable X denotes the No. of heads appearing in three tosses. X can take values $\{0,1,2,3\}$

$P(X=0) = {}^3C_0 \times 0.3 \times 0.3 \times 0.3 = 0.027 \times 1 = 0.027$

$P(X=1) = {}^3C_1 \times 0.7 \times 0.3 \times 0.3 = 0.063 \times 3 = 0.189$

$P(X=2) = {}^3C_2 \times 0.7 \times 0.7 \times 0.3 = 0.147 \times 3 = 0.441$

$P(X=3) = {}^3C_3 \times 0.7 \times 0.7 \times 0.7 = 0.34 \times 1 = 0.34.$

$\therefore \; PMF = \begin{cases} 0.027 & x=0 \\ 0.189 & x=1 \\ 0.441 & x=2 \\ 0.34 & x=3 \end{cases}$

Ans 10.

$$f_X(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

~~Find~~ $P(X \geq 0.4 \mid X \leq 0.8)$

$$= \frac{P(0.4 \le X \le 0.8)}{P(X \le 0.8)}$$

But we know that $P(X \in A) = \int_{x \in A} f_X(x)\,dx$

$$\therefore \quad P(0.4 \le X \le 0.8) = \int_{0.4}^{0.8} 2x\,dx$$

$$= 0.48$$

$$P(X \le 0.8) = \int_{0}^{0.8} 2x\,dx = 0.64$$

$$\therefore \quad P(x \geq 0.4 \mid x \le 0.8) = \frac{P(0.4 \le X \le 0.8)}{P(X \le 0.8)}$$

$$= \frac{0.48}{0.64} = \frac{3}{4}.$$

Ans11. ~~Let~~ $X$ is exponentially distributed random variable with parameter $\lambda$.

$$\therefore \quad f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$P(X > a+b \mid X > a) = \frac{P(X > a+b)}{P(X > a)}$$

as $(X > a+b) \cap (X > a) = (X > a+b)$

$$\therefore \quad P(X > a+b) = \int_{a+b}^{\infty} \lambda e^{-\lambda x} \, dx$$

$$= e^{-\lambda(a+b)}$$

similarly $P(X > a) = e^{-\lambda a}$

$$\therefore \quad P(X > a+b \mid X > a) = \frac{e^{-\lambda(a+b)}}{e^{-\lambda a}} = e^{-\lambda b}$$

Ans 12 $E \to$ all 5 tossed coins land heads

$$I_E = \begin{cases} 1 & \text{if } E \text{ occurs} \\ 0 & \text{if } E^c \text{ occurs.} \end{cases}$$

$I_E$ equals 1 when event E occurs, i.e, for all 5 coins coming heads {H, H, H, H, H} is the outcome when $I_E = 1$.

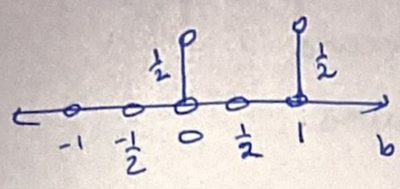
$$P(I_E = 1) = {}^5C_5 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{32}.$$

Ans 13. PDF $F(b) = \begin{cases} 0 & b < 0 \\ \frac{1}{2} & 0 \le b < 1 \\ 1 & 1 \le b < \infty \end{cases}$

~~that~~ PMF of X can be calculated by the difference of $(F(x^+) - F(x^-))$ where $x$ represents a step in the PDF.

In this case, step-ups in PDF occur at $b = 0$ and $b = 1$.

$$\therefore \quad \text{PMF} = P(x) = \begin{cases} F(0^+) - F(0^-) = \frac{1}{2} & X = 0 \\ F(1^+) - F(1^-) = \frac{1}{2} & X = 1 \end{cases}$$




PMF of F(b)

Ans 14. As selection is being done with replacement, hence probability of selecting each coloured ball remains same throughout the experiment.

3 white balls, 3 black balls

$P(\text{white}) = \frac{3}{3+3} = \frac{1}{2}$ $\quad$ $P(\text{black}) = \frac{3}{3+3} = \frac{1}{2}$

Let X ~~denote~~ be a RV denoting No of white balls in the drawing/selection of first four balls.

PMF of X is given by →

$X=0 \quad {}^4C_0 \times \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{16}$

$X=1 \quad {}^4C_1 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$X=2 \quad {}^4C_2 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{3}{8}$

$X=3 \quad {}^4C_3 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$X=4 \quad {}^4C_4 \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$

Prob that exactly two balls are drawn

$= P(X=2) = \frac{3}{8}$

Ans 15. Let us assume an arbitrary series of coin flips, with $r^{th}$ head ending the series.

$$\underbrace{HT ---- TT ---- TH --- HT}_{n-1 \text{ tosses.}} \; \underbrace{H}_{r^{th} \text{ head}}$$

$P(X=n)$ – Here, we need to find the probability of getting the $(r-1)$ heads and $((n-1)-(r-1))$ tails in the initial $(n-1)$ tosses, and multiply it with prob of getting heads in $n^{th}$ toss.

Here, Probability distribution of such a binomial system is given as →

$$\cancel{{}^{n-1}C_{r-1} (p)^{(r-1)} \; (1-p)}$$

$$\left( {}^{n-1}C_{r-1} (p)^{r-1} (1-p)^{[(n-1)-(r-1)]} \right) \times p$$

$$= {}^{n-1}C_{r-1} \; p^{r} (1-p)^{n-r}. \qquad n \geq r.$$

Hence the required probability is →

$$P(X=n) = {}^{n-1}C_{r-1} \; p^{r} (1-p)^{n-r}$$

Ans16. A Poisson distribution is given as →

~~PMF (λ) = e⁻~~

$$PMF(X=i) = \frac{e^{-\lambda}\lambda^i}{i!}$$ for poisson parameter $\lambda$.

In the given case, $\lambda = 1$.

Probability that there is at least one error on the Page $= P(E) = 1 - P(E^c)$

$P(E^c)$ = no error on the page

$$= PMF(X=0) = \frac{e^{-1}(1)^0}{0!} = \frac{1}{e}$$

$$\therefore \quad P(E) = 1 - \frac{1}{e} = \frac{e-1}{e} = 0.632$$

Hence, probability of having at least one error on the page is $\frac{e-1}{e} = 0.632$