Ch-6 Spatial diffusion of ⓝ

Spatial distribution of ⓝ becomes important as reactors considered become more realistic.

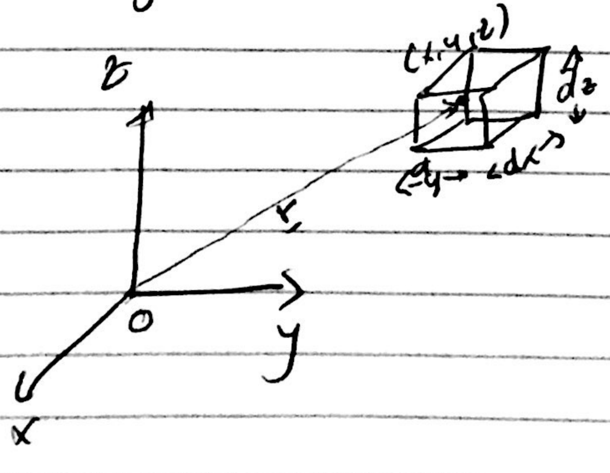
Consider - one group / monoenergetic / integrated in energy ⓝ distribution model
ⓝ flux & XS are already averaged over energy.

We derive and solve steady state diffusion equation numerically and analytically. We look at critical & subcritical assemblies.

ⓝ Diffusion equation.

- Consider ⓝ balance condition for incremental volume. Then use Fick's law to get diffusion eqn.

Consider an infinitesimal incremental volume. $dV = dx\,dy\,dz$ centered at point $\vec{r} = (x, y, z)$



Under steady state, conservation requires that

loss rate = production rate

⇒ leakage rate + absorption rate = source emission rate + fission ⓝ gen. rate.

Recall:

ϕ = scalar flux → $v\,n(t)$ [@ / cm^2 / s]
→ total distance travelled during 1 s by all @ located inside a unit volume.

Σ_x: macroscopic xs → [/cm]
→ probability law of flight of a @ undergoing a reaction of type x.

$\Sigma_x \phi$ = probable # of collisions of type x / s / cm^3 of all @

$\Sigma_x \phi\, dx\,dy\,dz$ = probable # of collisions of type x/s in volume $dV = dx\,dy\,dz$.

Production rate:

source @ emission rate = $S'''(x,y,z)\,dx\,dy\,dz$ ← @ src emission rate density

fission @ generation rate = $\underbrace{\nu \underbrace{\Sigma_f(x,y,z)\,\phi(x,y,z)}_{\text{total \# of probable fissions per s per cc}}}_{\text{total \# of @ generated from "prompt" fission per s}} \underbrace{dx\,dy\,dz}_{\text{total volume}}$

↑ # @ per fission

Loss rate:

@ absorption rate = $\underbrace{\underbrace{\Sigma_a(x,y,z)\,\phi(x,y,z)}_{\text{total \# of probable absorption reactions per s per cc}} \underbrace{dx\,dy\,dz}_{\text{total volume of incremental vol}}}_{\text{total \# of @ absorbed per s.}}$

Leakage rate:

In order to evaluate leakage, we look at what comes in and what goes out thro the surfaces of the incremental volume.

Recall current:

$J_x(x,y,z)$ = net # of ⊖/cm²/s passing thro y-z plane in the x direction at (x,y,z).

$J_y(x,y,z)$ = net # of ⊖/cm²/s passing thro x-z plane in the y dirⁿ at (x,y,z)

$J_z(x,y,z)$ = net # of ⊖/cm²/s passing thro x-y plane in the z dirⁿ at (x,y,z).

For the cubic volume dV, net # of ⊖ passing thro dV we define

$J_x(x+\frac{1}{2}dx, y, z)\,dy\,dz$ ← ⊖ passing thro front face
$J_y(x, y+\frac{1}{2}dy, z)\,dx\,dz$ ← ⊖ passing thro right face
$J_z(x, y, z+\frac{1}{2}dz)\,dx\,dy$ ← ⊖ passing thro top face

$-J_x(x-\frac{1}{2}dx, y, z)\,dy\,dz$ ← ⊖ passing thro back face
$-J_y(x, y-\frac{1}{2}dy, z)\,dx\,dz$ ← ⊖ passing thro left face
$-J_z(x, y, z+\frac{1}{2}dz)\,dx\,dy$ ← ⊖ passing thro bottom face

net leakage out of system is then

$$\left(J_x(x+\tfrac{1}{2}dx, y, z) - J_x(x-\tfrac{1}{2}dx, y, z)\right) dy\, dz$$
$$\left(J_y(x, y+\tfrac{1}{2}dy, z) - J_y(x, y-\tfrac{1}{2}dy, z)\right) dx\, dz$$
$$\left(J_z(x, y, z+\tfrac{1}{2}dz) - J_z(x, y, z-\tfrac{1}{2}dz)\right) dx\, dy$$

$$= \frac{J_x(x+\tfrac{1}{2}dx, y, z) - J_x(x-\tfrac{1}{2}dx, y, z)}{dx} dx\, dy\, dz$$

$$+ \frac{J_y(x, y+\tfrac{1}{2}dy, z) - J_y(x, y-\tfrac{1}{2}dy, z)}{dy} dx\, dy\, dz$$

$$+ \frac{J_z(x, y, z+\tfrac{1}{2}dz) - J_z(x, y, z-\tfrac{1}{2}dz)}{dz} dx\, dy\, dz$$

Now, let $\left.\begin{matrix} dx \to 0 \\ dy \to 0 \\ dz \to 0 \end{matrix}\right\} \Rightarrow$ above differences become partial derivatives.

$\Rightarrow$ leakage term becomes:

$$\left(\frac{\partial}{\partial x} J_x(x,y,z) + \frac{\partial}{\partial y} J_y(x,y,z) + \frac{\partial}{\partial z} J(x,y,z)\right) dx\, dy\, dz$$

Now, put all terms together in the balance conservation equation & cancel out $dx\, dy\, dz$ on both sides

$$\boxed{\frac{\partial}{\partial x} J_x(\underline{r}) + \frac{\partial}{\partial y} J_y(\underline{r}) + \frac{\partial}{\partial z} J_z(\underline{r}) + \Sigma_a(\underline{r})\phi(\underline{r}) = S'''(\underline{r}) + \nu \Sigma_f(\underline{r})\phi(\underline{r})}$$

Note that

$$\underline{J}(\underline{r}) = J_x(\underline{r})\,\hat{i} + J_y(\underline{r})\,\hat{j} + J_z(\underline{r})\,\hat{k}$$

& $$\underline{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

$\therefore$ leakage term becomes $\underline{\nabla}\cdot\underline{J}(\underline{r})$ & balance equation becomes,

$$\boxed{\nabla\cdot\underline{J}(\underline{r}) + \Sigma_a(\underline{r})\phi(\underline{r}) = S'''(\underline{r}) + \nu\Sigma_f(\underline{r})\phi(\underline{r})}$$

In order to get diffusion equation, we use ~~arrive~~ at the ~~following~~ Fick's law:

$$\underline{J}(\underline{r}) = -D(\underline{r})\,\underline{\nabla}\phi(\underline{r}).$$

$\Rightarrow$ We get diffusion equ.

$$\boxed{\nabla\cdot D(\underline{r})\nabla\phi(\underline{r}) + \Sigma_a(\underline{r})\phi(\underline{r}) = S'''(\underline{r}) + \nu\Sigma_f(\underline{r})\phi(\underline{r})}$$

All that is fine but what is ~~~~ ~~~~ $D(\underline{r})$?

To find $D(\underline{r})$ we go back to transport equation for 1D (like we saw before for HW3)

We have, 1D transport equation:

$$\mu \frac{\partial}{\partial x} \psi(x,\mu) + \Sigma_t(x)\psi(x,\mu) = \frac{1}{2}\Sigma_s(x)\phi(x) + \frac{1}{2}S(x).$$

$$\phi(x) = \int_{-1}^{1} d\mu \, \psi(x,\mu)$$

$$J(x) = \int_{-1}^{1} d\mu \, \mu \, \psi(x,\mu).$$

Recall: we take 0th & 1st moment of the 1D eqn w/ isotropic scattering ~~transport~~ to get diffusion equation where

$$-\frac{\partial}{\partial x}\frac{1}{3\sigma_t}\frac{\partial}{\partial x}\phi + \Sigma_a\phi = \nu\Sigma_f\phi + S$$

$$\Rightarrow \boxed{D = \frac{1}{3\sigma_t}}$$

Note: We are implicitly saying that

$$\boxed{\psi = \frac{1}{2}\phi(x) + \frac{3}{2}\mu J(x)}$$

when we make diffusion approximation.

→ Angular flux is linear in angle.

In general, angular flux $\psi(x,\mu)$ can be expanded in Legendre polynomials. ~~$\psi(x,\mu) = \sum_{l=0} P_l(\mu)\phi_l(x)$~~

$$\boxed{\psi(x,\mu) = \sum_{l=0}^{L} \frac{2l+1}{2} P_l(\mu)\phi_l(x)}$$

– for diffusion we chop series at $L = 1$.

1D diffusion equation:

$$\boxed{-\frac{\partial}{\partial x} D \frac{\partial}{\partial x} \phi(x) + \Sigma_a(x)\phi(x) = S(x) + \nu\Sigma_f(x)\phi(x)}$$

~~Validity of diffusion equation.~~

Boundary conditions.

① Reflecting condition (Symmetry condition)

Net current at the boundary = 0.

⇒ net current crossing boundary = 0.

Suppose we consider 1D system,

$$J_x(\text{boundary}) = -D\frac{d}{dx}\phi(x)\Big|_{x=x_L/x_R} = 0$$

$$\Rightarrow \frac{d}{dx}\phi(x)\Big|_{x=x_{L/R}} = 0$$

Similarly, for other boundaries, $\frac{d}{dy}\phi(y)\Big|_{y=y_{\text{boundary}}} = 0$

& $\frac{d}{dz}\phi(z)\Big|_{z=z_{\text{boundary}}} = 0$.

~~at corners/faces more than one~~

For 2D, 3D, at faces & corners, more than one of the above conditions may apply simultaneously (we'll talk more later)

Vacuum boundaries

This boundary essentially means nothing is coming in thru the boundaries.

⇒ $\longrightarrow |$ $\qquad\qquad | \longleftarrow$

~~@~~

"Coming in" thru that boundary surface = 0.

In order to mathematically represent this condition, we introduce partial currents.

$$J(x) = \int_{-1}^{1} d\mu\, \mu\, \psi(x,\mu)$$

$$= \int_{-1}^{0} d\mu\, \mu\, \psi(x,\mu) + \int_{0}^{1} d\mu\, \mu\, \psi(x,\mu)$$

$$= \int_{0}^{1} d\mu\, \mu\, \psi(x,\mu) - \int_{-1}^{0} d\mu\, |\mu|\, \psi(x,\mu)$$

partial currents.

$$= J_x^{+}(x) - J_x^{-}(x)$$

$J^{+} \longrightarrow |$ $\qquad | \longleftarrow J^{-}$

Moreover, in order to evaluate partial currents

$$J^{+}(x) = \int_0^1 \mu\, \psi(x,\mu)\, d\mu$$

$$= \int_0^1 \mu \left[\frac{1}{2}\phi(x) + \frac{3}{2}\mu J(x) \right] d\mu$$

$$= \frac{\phi(x)}{2} \underbrace{\int_0^1 \mu\, d\mu}_{\left[\frac{\mu^2}{2}\right]_0^1 = \frac{1}{2}} + \frac{3}{2} J(x) \underbrace{\int_0^1 \mu^2 d\mu}_{\left[\frac{\mu^3}{3}\right]_0^1 = \frac{1}{3}}$$

$$\boxed{J^{+}(x) = \frac{\phi(x)}{4} + \frac{1}{2} J(x)}$$

Similarly, $\boxed{J^{-}(x) = \frac{\phi(x)}{4} - \frac{1}{2} J(x)}$

For vacuum boundaries, then,

$$J^{+}(x_L) = 0 = \frac{\phi(x)}{4} + \frac{1}{2} J(x)$$

$$0 = \frac{\phi(x)}{4} - \frac{1}{2} D \frac{d\phi}{dx}$$

$$\Rightarrow \boxed{D \frac{d\phi}{dx}\bigg|_{x_L} = \frac{\phi(x)}{2}}$$

Similarly at right boundary,

$$J^{-}(x_r) = 0 = \frac{\phi(x_r)}{4} - \frac{1}{2} D \frac{d}{dx} \phi_x \bigg|_{x_R}$$