Diffusion Approximation

$$\underbrace{\text{leakage} + \text{absorption}}_{\text{loss}} = \underbrace{\text{source} + \text{fission}}_{\text{gain}}$$

$\Rightarrow \quad div(\vec{J}(\vec{r})) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$

Now, by fick's law (3D), we have $\vec{J}(\vec{r}) = -D(\vec{r})\vec{\nabla}\phi(\vec{r})$

$\Rightarrow$ the equation now becomes,

$$div(-D(\vec{r})\vec{\nabla}\phi(\vec{r})) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

$div(\vec{x}) = \vec{\nabla}\cdot\vec{x}$

above eqn is written as (as in laws)

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

Now as derived previously, $D = \dfrac{1}{3\Sigma_{tr}}$; $\Sigma_{tr} = \Sigma_t - \underbrace{\bar{\mu}}_{\text{avg scattering angle}}\Sigma_s$

Averaging diffusion coefficient can be accomplished by first using techniques to average Σ_{tr} & then $\bar{D} = \dfrac{1}{3\bar{\Sigma}_{tr}}$

SOLUTION OF DIFFUSSION EQUATION

Diffussion equation.

$$-\vec{\nabla}\cdot D(\vec{r})\vec{\nabla}\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

i.e.

$$-D(\vec{r})\vec{\nabla}\cdot\vec{\nabla}\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

OR

$$\boxed{-D(\vec{r})\nabla^2\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})}$$

$$\nabla^2 = \vec{\nabla}\cdot\vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

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DIFFUSSION EQUATION FOR NON MULTIPLYING SYSTEMS - "PLANE GEOMETRY"

First consider case of a uniform medium with no fissionable material. ie a non multiplying system. $\Sigma_f = 0$ & D & Σ_a are constants.

Hence the diffusion equation,

$$-D(\vec{r})\nabla^2\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

becomes

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a\phi(\vec{r}) = S'''(\vec{r}) \qquad (\because \Sigma_a,\ D = \text{constant}\ \Sigma_f = 0)$$

Dividing the equation by D,

$$-\nabla^2\phi(\vec{r}) + \frac{\Sigma_a}{D}\phi(\vec{r}) = \frac{S'''(\vec{r})}{D}$$

Let $\sqrt{\dfrac{D}{\Sigma_a}} = L$, L = diffusion length (magnitude of displacement of neutron before absorption from birth)

$$\Rightarrow \boxed{-\nabla^2\phi(\vec{r}) + \frac{1}{L^2}\phi(\vec{r}) = \frac{S'''(\vec{r})}{D}}$$

Source free example

Consider a simple problem in plane geometry where flux varies so slowly in y & z direction that it can be ignored, allowing us to eliminate y & z derivatives from ∇^2. We may also set source $= 0$.

Hence now, the diffusion eqn (above) becomes

$$-\frac{d^2}{dx^2}\phi(x) + \frac{1}{L^2}\phi(x) = 0$$

or

$$\underbrace{\frac{d^2}{dx^2}\phi(x) - \frac{1}{L^2}\phi(x) = 0.}_{\text{Second order differential eqn}}$$

Homogeneous

MATH SUPPORT

Linear second order ODE w/ constant coefficients (solving)

$$y'' + Ay' + By = 0$$ ← homogeneous

Assume general solⁿ $y = C_1 y_1 + C_2 y_2$
where y_1 & y_2 are solutions
Initial condition are satisfied by choosing C_1 & C_2.

— To solve the ODE, find 2 solutions (independent)

Basic method

Try $y = e^{rt}$ (t = independent variable, y - dependent)
plug in to above ODE,

$$r^2 e^{rt} + Ar e^{rt} + B e^{rt} = 0.$$

$$\boxed{r^2 + Ar + B = 0.}$$

Characteristic equation of system

Case 1: Roots $r_1 \neq r_2$ (real)
General solution $\boxed{y = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$

Ex

$$y'' + 4y' + 3y = 0.$$

$r^2 + 4r + 3 = 0$ ← characteristic eqn

$\Rightarrow (r+3)(r+1) = 0$

Solⁿ. $y = c_1 e^{-3t} + c_2 e^{-t}$

Initial conditions: $y(0) = 1$; $y'(0) = 0$

$y' = -3c_1 e^{-3t} - c_2 e^{-t}$ ← $t = 0$ by $y = c_1 e^0 + c_2 e^0$ at $t=0$

$\Rightarrow 0 = -3c_1 - c_2$; $1 = c_1 + c_2$ $\Rightarrow$ $c_1 = -1/2$; $c_2 = 3/2$

$\Rightarrow$ Solⁿ is $\underline{y = \frac{-1}{2} e^{-3t} + \frac{3}{2} e^{-t}}$

CASE 2 Complex roots

$r = a \pm bi$

We get a complex solution

$y = e^{(a+bi)t}$

theorem. If $u+iv$ is a complex solution to a real differential equation to $y'' + Ay' + By = 0$, then u & v are real solutions

Proof:

$(u+iv)'' + A(u+iv)' + B(u+iv) = 0$

$\Rightarrow \underbrace{u'' + Au' + Bu}_{\text{real part}} + i\underbrace{(v'' + Av' + Bv)}_{\text{imaginary part}} = 0$

$\quad\quad = 0 \quad\quad\quad\quad = 0$

only that would make it no 0

that makes u & v a sol^{n}s.

case 2 sol^n

$y = e^{at+ibt} = e^{at}(\cos bt)$ real part

$e^{at}(\sin bt)$ imaginary part

sol^n. $y = e^{at}(C_1 \cos bt + C_2 \sin bt)$

example

$y'' + 4y' + 5y = 0$

char eqn

$r^2 + 4r + 5 = 0$

$\Rightarrow r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{4 \pm \sqrt{-4}}{2} = -2 \pm i$

$\Rightarrow e^{(-2+i)t}$

$e^{-2t}\cos t, \quad e^{-2t}\sin t$

$\Rightarrow y = e^{-2t}(C_1 \cos t + C_2 \sin t)$

Case 3 Critically damped

$s^2 + As + B = 0$ has 2 equal roots

$(s+a)^2 = 0$

$\Rightarrow s^2 + 2As + A^2 = 0$

$\Rightarrow$ ODE looks like $y'' + 2Ay' + A^2 y = 0$

Sol^n

$y = e^{-at}$

Know 1 sol^n to $y'' + py' + qy = 0$, then there is another $y = y_1 u$

unknown sol^n:

$a^2 \times]$ $\quad y = e^{-at} u$

$2a \times]$ $\quad y' = -aue^{-at} + e^{-at}u'$

$1 \times]$ $\quad y'' = a^2 e^{-at} u - 2a e^{-at} u' + e^{-at} u''$

$0 = 0 + 0 + e^{-at} u''$

$\Rightarrow e^{-at} u'' = 0$

$\Rightarrow u = C_1 t + C_2$ (just t would be enough)

$y_2 = e^{-at} t$

$y_1 = e^{-at}$

GOING BACK TO OUR EXAMPLE.

Recall

equation was

$$\frac{d^2}{dx^2}\phi(x) - \frac{1}{L^2}\phi(x) = 0. \qquad (\text{sol}^n \text{ of form } \phi(x) = e^{rx})$$

The above is a linear ODE of 2^{nd} order

Characteristic equation is $r^2 - \frac{1}{L^2} = 0$

$$\Rightarrow \left(r - \frac{1}{L}\right)\left(r + \frac{1}{L}\right) = 0$$

$$\Rightarrow r = \pm\frac{1}{L} \rightarrow r_1, r_2$$

Equation has distinct & real roots, hence the solution looks like

$$\phi(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

$$\boxed{\phi(x) = C_1 e^{x/L} + C_2 e^{-x/L}}$$

Now to find C_1 & C_2, we need boundary conditions & initial conditions $\boxed{\downarrow \text{Initial \& Boundary conditions} \downarrow}$

Here, suppose the domain is semiinfinite ie $0 \leq x \leq \infty$

$\phi(0) = \phi_0$ (from left)

No neutrons enter from right as medium is semiinfinite hence each neutron that entered can safely be assumed to be absorbed at the right end ie at ∞. $\Rightarrow \phi(\infty) = 0$

Now applying boundary conditions. $\phi(0) = \phi_0$; $\phi(\infty) = 0$ to $\phi(x)$.

$$\Rightarrow \phi(0) = C_1 e^{0/L} + C_2 e^{-0/L}$$

$$\Rightarrow \phi_0 = C_1 + C_2$$

$$\phi(\infty) = \underbrace{C_1 e^{\infty/L}}_{\to\infty} + \underbrace{C_2 e^{-\infty/L}}_{\to 0} = 0$$

because $e^{\infty} = \infty$, $e^{-\infty} = 0$

$\rightarrow$ this eqn satisfied only if $\boxed{C_1 = 0}$

Now from $C_1 = 0$

$$\Rightarrow \boxed{C_2 = \phi_0}$$

Now substituting C_1 & C_2 in $\phi(x)$ eqn.

$$\phi(x) = C_1 e^{+x/L} + C_2 e^{-x/L}$$
$$= 0 + C_2 e^{-x/L}$$
$$\boxed{\phi(x) = \phi_0 e^{-x/L}}$$

Uniform source example:

We next examine the case with uniform source $S'''(\vec{r}) \to S_0''$. In plane geometry reduces from

$$-\nabla^2 \phi(\vec{r}) + \frac{1}{L^2}\phi(\vec{r}) = \frac{1}{D} S'''(\vec{r})$$

to

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{L^2}\phi(x) = \frac{1}{D} S_0''$$

(non homogeneous 2nd order differential eqn

MATH SUPPORT

2nd order linear ODE linear in y, y', y''

Sol^n method:

Find 2 independent sol^n - y_1 & y_2.

Independent means $y_2 \neq c y_1$; $y_1 \neq c' y_2$

Then all sol^n are linear combination with constant coefficients

$$y = C_1 y_1 + C_2 y_2 \quad - \text{WHY?}$$

Q2 why are these all the soln s?

Q1 Why are $C_1 y_1 + C_2 y_2$ sol^{n}s ?

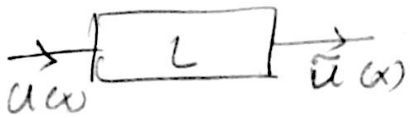
Q-1 Superposition principle. y_1 & y_2 are sol^n of
homogeneous ODE

$\Rightarrow$ $c_1 y_1 + c_2 y_2$ is a sol^n (Linear combination of y_1 & y_2)

Proof: $y'' + py' + qy = 0$

$D^2 y + p D y + q y = 0 \rightarrow \underbrace{(D^2 + pD + q)}_{\text{Linear operator } L} y = 0 \Rightarrow Ly = 0.$

differentiated operator



Linear operator - $L(u_1(x) + u_2(x)) = L(u_1(x)) + L(u_2(x))$

$L(c\,u(x)) = c\,L(u(x))$

Ex. D is linear because $D(u_1 + u_2) = Du_1 + Du_2$

Proof of superposition.

ODE : $Ly = 0$

If L is linear,

$L(c_1 y_1 + c_2 y_2) = L(c_1 y_1) + L(c_2 y_2)$

$= c_1 L(y_1) + c_2 L(y_2)$

$L(y_1) = 0$, $L(y_2) = 0$ as y_1 & y_2 are solutions

$\Rightarrow L(y_1), L(y_2) = 0$

$= 0 + 0$

$= 0.$

Hence $c_1 y_1 + c_2 y_2$ is also a solution.

Q-2 Solving the initial value problem (find sol^n with given initial values at a pt.)

Thm. $\{c_1 y_1 + c_2 y_2\}$ is enough to satisfy any initial values

Proof.

$$y(x_0) = a \\ y'(x_0) = b$$ } initial values

simultaneous linear eqns

$y = c_1 y_1 + c_2 y_2$
$y' = c_1 y_1' + c_2 y_2'$
} plug $x = x_0 \Rightarrow$
$c_1 y_1(x_0) + c_2 y_2(x_0) = a$
$c_1 y_1'(x_0) + c_2 y_2'(x_0) = b$

c_1 & c_2 are unknown variables → need to find

The system of linear eqns is solvable for c_1 & c_2 if

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}_{x=x_0} \neq 0$$

← called Wronskian $= W(y_1, y_2)$ · fn of x

Thm: If y_1, y_2 are solutions to ODE (linear, homogeneous, 2nd order), either $W(y_1, y_2) \equiv 0$ (for all values of x) or $W(y_1, y_2)$ is never 0. (for all x)

$\{c_1 y_1 + c_2 y_2\} = \{c_1' u_1 + c_2' u_2\}$ – u_1, u_2 – any other pair of independent solutions

$u_1 = \bar{c}_1 y_1 + \bar{c}_2 y_2$
$u_2 = \bar{\bar{c}}_1 y_1 + \bar{\bar{c}}_2 y_2$

Finding normalized solutions (at 0; could be x_0)

Y_1, Y_2

Y_1: $Y_1(0) = 1$, $Y_1'(0) = 0$

Y_2: $Y_2(0) = 0$, $Y_2'(0) = 1$

$y'' + y = 0$

$y_1 = \cos x = Y_1$
$y_2 = \sin x = Y_2$
} because they fit

$y'' - y = 0$

$y_1 = e^x$
$y_2 = e^{-x}$
} General soln. $c_1 e^x + c_2 e^{-x} = y$, $y' = c_1 e^x - c_2 e^{-x}$

Y_1: $c_1 + c_2 = 1$
$c_1 - c_2 = 0$
$\Rightarrow c_1 = c_2 = \frac{1}{2}$

$Y_1 = \frac{e^x + e^{-x}}{2} = \cosh x$

Y_2 -- put in initial conditions

$$Y_2 = \frac{e^x - e^{-x}}{2} = \sinh x$$

Y_1, Y_2 normalized at 0

soln to IVP ODE + $y(0) = a$
$y'(0) = b$

is

$$\boxed{y_0 Y_1 + y_0' Y_2}$$

Existence & Uniqueness theorem

$y'' + py' + qy = 0$ p, q are continuous for all x

There is one & only one soln satisfying given initial values
such that $y(0) = A$, $y'(0) = B$

Want all solutions to ODE

Claim $\{c_1 Y_1 + c_2 Y_2\}$ are **all** the solutions

Proof

Given soln $u(x)$, $u(0) = u_0$, $u'(0) = u_0'$ then
$u_0 Y_1 + u_0' Y_2$ satisfies these initial values

IN HOMOGENEOUS ODE 2nd order

$$\boxed{y'' + p(x)y' + q(x)y = f(x)}$$

$f(x)$: input signal / forcing term / driving term /

Soln $y(x)$ called response / output etc

$y'' + p(x)y' + q(x)y = 0$ associated homogeneous eqn aka reduced equation

sol^n $y_c = C_1 y_1 + C_2 y_2$

= solution to associated homogeneous equation aka complementary solution

Example:

$mx'' + bx' + kx = f(t)$

$mx'' = -kx - bx' + f(t)$

Thm:

$Ly = f(x)$ $y'' + p(x)y' + q(x)y = f(x)$

L is a linear operator

Solution:

$y_p + y_c$

i.e.

$y = y_p + y_c = \underbrace{y_p}_{\text{particular eqn to } Ly = f(x)} + \underbrace{C_1 y_1 + C_2 y_2}_{\text{solutions of associated homogeneous equation}}$

any one sol^n

$y'' + Ay' + By = f(t)$

$y = y_p + \underbrace{C_1 y_1 + C_2 y_2}_{\text{use initial conditions}}$

Q when does $C_1 y_1 + C_2 y_2 \to 0$ as $t \to \infty$ for all C_1, C_2

If this is so, ODE is called stable

$y = \underbrace{y_p}_{\text{steady state sol}^n} + \underbrace{C_1 y_1 + C_2 y_2}_{\text{transient}}$

roots of characteristic eqn	Soln	stability condition
real & distinct	$c_1 e^{r_1 t} + c_2 e^{r_2 t}$	$r_1 < 0, r_2 < 0$
$r_1 = r_2$	$(c_1 + c_2 t) e^{rt}$	$r_1 < 0$
$r = a \pm bi$	$e^{at}(c_1 \cos bt + c_2 \sin bt)$	$a < 0$

ODE is stable if all characteristic roots have -ve real part

Finding particular solution

$$y'' + Ay' + By = f(x)$$

Find a particular soln y_p — complementary sol

General soln $y = y_p + y_c = y_p + c_1 y_1 + c_2 y_2$

particular sol - anything that satisfies given equation

(Back to diffusion theory)

Uniform source example in a non multiplying system

$S'''(\vec{x}) \rightarrow S_0'''$

In plane geometry, equation reduces to (as shown previously)

$$-\frac{d^2\phi(x)}{dx^2} + \frac{1}{L^2}\phi(x) = \frac{1}{D} S_0'''$$

The above equation is 2nd order linear inhomogeneous ODE

Its general solution is of the form $y = y_p + y_c = \underbrace{y_p}_{\text{particular solution.}} + \underbrace{C_1 y_1 + C_2 y_2}_{\text{solution to related homogeneous sol}^n}$

First finding y_p, the particular solution,

Now since source is uniform, the particular solution is a constant, Hence its derivative is a constant.

let ϕ_p be that constant (particular sol^n)

$$\Rightarrow -\frac{d^2\phi_p}{dx^2} + \frac{1}{L^2}\phi_p = \frac{1}{D} S_0'''$$

$$\Rightarrow 0 + \phi_p = \frac{L^2}{D} S_0''' = \frac{D/\Sigma_a}{D} S_0''' = \frac{1}{\Sigma_a} S_0'''$$

$$\Rightarrow y_p = \phi_p = \frac{1}{\Sigma_a} S_0'''$$

For y_c,

the corresponding homogeneous equation is $-\frac{d^2\phi(x)}{dx^2} + \frac{1}{L^2}\phi(x) = 0$

or $\frac{d^2\phi(x)}{dx^2} - \frac{1}{L^2}\phi(x) = 0$

We know its solution to be $C_1 e^{x/L} + C_2 e^{-x/L}$

Now to find the constants C_1 & C_2, we apply boundary conditions to the general equation sol^n

$$y = \phi(x) = y_p + y_c = \frac{S_0'''}{\Sigma_a} + C_1 e^{x/L} + C_2 e^{-x/L}$$

Suppose uniform source is distributed throughout a slab extending betn $-a \leq x \leq a$, and that we specify 2 boundary conditions as the flux vanishing at the slab ie $\phi(\pm a) = 0$ Hence now applying that boundary condition

$$\phi(-a) = 0 = C_1 e^{-a/L} + C_2 e^{a/L} + \frac{S_0'''}{\Sigma_a} \quad ①$$

$$\phi(+a) = 0 = C_1 e^{a/L} + C_2 e^{-a/L} + \frac{S_0'''}{\Sigma_a} \quad ②$$

Now, the 2 equations are identical & if they are both going to be equal to same number, the equations are equal too & that gives $C_1 = C_2$ (which is the only condition under which both eqns could be 0)

Hence now using that to solve for C_1 using 2nd eqn,

$$C_1 e^{a/L} + C_1 e^{-a/L} + \frac{S_0}{\Sigma_a} = 0$$

$$C_1 (e^{a/L} + e^{-a/L}) = -\frac{S_0''}{\Sigma_a}$$

$$\Rightarrow C_1 = C_2 = -(e^{a/L} + e^{-a/L})^{-1} \frac{S_0''}{\Sigma_a}$$

$$= -\frac{1}{2} (\cosh a/L)^{-1} \frac{S_0'''}{\Sigma_a}.$$

Now substituting that in our general sol^n, we have

$$\phi(x) = \frac{S_0'''}{\Sigma_a} + \frac{-1}{\frac{1}{2}\cosh(a/L)} \frac{S_0''}{\Sigma_a} e^{x/L} - \frac{1}{\frac{1}{2}\cosh(a/L)} \frac{S_0'''}{\Sigma_a} e^{-x/L}$$

$$= \frac{S_0'''}{\Sigma_a} \left(1 - \frac{\frac{1}{2}\cosh(x/L)}{\frac{1}{2}\cosh(a/L)} \right)$$

$$\boxed{\phi(x) = \frac{S_0'''}{\Sigma_a} \left(1 - \frac{\cosh(x/L)}{\cosh(a/L)} \right)}$$

BOUNDARY CONDITIONS

While solving the diffusion equation, which is 2nd order differential eqn, the soln contains 2 arbitrary constants (for 1-D). 2 boundary conditions are required to determine these coefficients. Often these boundary conditions accurately represent the physical situation at hand.

A particularly useful concept is that of <u>partial currents</u>.

Recall

$J(x)$ is net number of neutrons per cm² per s crossing a plane perpendicular to x axis in +ve x dirn. This may be divided into partial currents, $J_x^+(x)$ & $J_x^-(x)$ of neutrons travelling in +ve x & -ve x direction.

$$J_x(x) = J_x^+(x) - J_x^-(x)$$

Now as shown before,

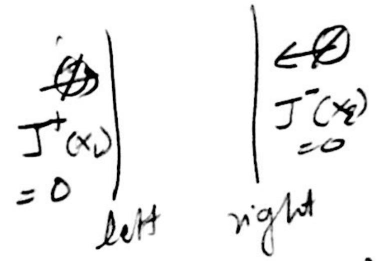
$$\boxed{J_x^{\pm}(x) = \frac{1}{4}\phi(x) \mp \frac{1}{2}D\frac{d}{dx}\phi(x)}$$

VACUUM BOUNDARIES

– Consider a surface across which no neutrons enter. This would be the case if a vacuum containing no neutron sources extended to infinity. Hence we refer to such surfaces at boundaries as vacuum boundaries.

If the boundary is on left, $J_x^+(x_l) = 0$

If the boundary is on right, $J^-(x_R) = 0$



The condition on right, using above eqn may be written as

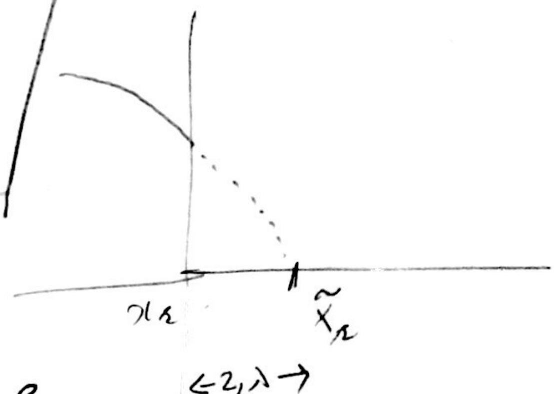
$$J_x^-(x_R) = 0 = \frac{1}{4}\phi(x_R) + \frac{1}{2}D\left(\frac{d}{dx}\phi(x)\right)_{x_R} = \frac{1}{4}\phi(x_R) - \frac{1}{2}D\left|\frac{d}{dx}\phi(x)\right|_{x_R}$$

Now, it must be noted that derivative is -ve since flux reduces along the distance as no flux enters

Now, for an isotropic scatter, $D = \frac{1}{3\Sigma_T}$ & $\lambda = mfp = \frac{1}{\Sigma_T}$

Hence $$0 = \frac{1}{4}\phi(x_R) - \frac{1}{2}\frac{1}{3\Sigma_T}\left|\frac{d}{dx}\phi(x)\right|_{x_R}$$

$$\Rightarrow \boxed{\frac{\phi(x_r)}{\left|\frac{d}{dx}\phi(x)\right|_{x_r}} = \frac{2}{3}\lambda = \frac{2}{3\Sigma_t}}$$ ①



Annual interpretation of the situation is shown in adjoining figure.

Now, extrapolating flux linearly, flux goes to 0 when it is $\frac{2}{3}\lambda$ outside boundary

Thus $\frac{2}{3}\lambda$ is referred to as extrapolation distance

$$\frac{\phi(x)-\phi(0)}{\underbrace{x_r - x_{\phi_0}}_{\text{extrapolation distance}}} = \frac{d}{dx}\phi(x) = \frac{3}{2\lambda}\phi(x_r) \quad (\text{by ①})$$

$$\rightarrow \frac{\phi(x_r)}{x_r - x_0} = \frac{3}{2\lambda}\phi(x_r) \quad \Rightarrow \quad \frac{2\lambda}{3} = x_r - x_0.$$

Hence x_0 is $\frac{2\lambda}{3}$ units to the right of x_r as shown in figure

When a vacuum boundary is encountered, simply adjusting the dimensions & using zero flux boundary condition is more straight forward. Thus for problem with left & right vacuum boundaries x_ℓ & x_r.

$$\boxed{\begin{aligned} x_\ell &: \phi\left(x_\ell - \tfrac{2}{3}\lambda\right) = 0 = \phi(\tilde{x}_\ell) \\ x_r &: \phi\left(x_r + \tfrac{2}{3}\lambda\right) = 0 = \phi(\tilde{x}_r) \end{aligned}} \quad \begin{aligned} &(x_\ell - 2\lambda/3 = \tilde{x}_\ell) \\ &(x_r + \tfrac{2}{3}\lambda = \tilde{x}_r) \end{aligned}$$

(FOR VACUUM BOUNDARIES

if λ is too small, $\boxed{\phi(x_r) \approx 0\ ;\ \phi(x_\ell) \approx 0}$

REFLECTED BOUNDARIES

If the net current J_x is known at a boundary. It may be used as a condition. Current conditions appear most frequently as a result of problem symmetry.

Here the net number of neutrons crossing boundary vanishes

$$\boxed{J_x(0) = -D\frac{d}{dx}(\phi(x))\Big|_{x=0} = 0.}$$

$$\boxed{\frac{d}{dx}\phi(x)\Big|_{x=0} = 0}$$

SURFACE SOURCES AND ALBEDOS

Partial currents are particularly useful in specifying boundary conditions in situations where surfaces are bombarded by neutrons. To illustrate this consider a source free sol^n. For this sol^n to hold, neutrons must be being supplied from left. We can relate the boundary flux ϕ_0 to a surface source of s'' neutrons/cm²/s noting that no. of neutrons entering from left is

$$J_x^+(0) = s''$$

Now, we know that $J_x^+(x) = \frac{1}{4}\phi(x) - \frac{1}{2}D\frac{d}{dx}\phi(x)$

& $\phi(x) = \phi_0 e^{-x/L}$

substitute

$\Rightarrow \frac{d}{dx}\phi(x) = -\frac{1}{L}\phi_0 e^{-x/L}$

$$\Rightarrow J_x^+(x) = \frac{1}{4}\phi_0 e^{-x/L} - \frac{1}{2}\frac{D}{(-L)}\phi_0 e^{-x/L} = \phi_0 e^{-x/L}\left(\frac{1}{4} + \frac{D}{2L}\right)$$

Now we know that $J_x^+(0) = s''$

substitute

$$\Rightarrow J_x^+(0) = s'' = \left(\frac{1}{4} + \frac{D}{2L}\right)\phi_0 e^{0/L}$$

$$\rightarrow \frac{s''}{\frac{1}{4} + \frac{D}{2L}} = \phi_0$$

$$\Rightarrow \boxed{\phi(x) = \frac{s''}{\left(\frac{1}{4} + \frac{D}{2L}\right)} e^{-x/L}}$$

ALBEDO

The ratio of exiting to entering neutrons is termed albedo. It may be expressed in terms of partial currents as

$$\alpha = \frac{J_x^-(0)}{J_x^+(0)}$$

Here.
We know that $J_x^+(0) = \left(\frac{1}{4} + \frac{D}{2L}\right) \phi_0 e^{-x/L}$

Similarly $J_x^-(0) = \left(\frac{1}{4} - \frac{D}{2L}\right) \phi_0 e^{-x/L}$

$$\therefore \alpha = \frac{J_x^-(0)}{J_x^+(0)} = \frac{\text{outgoing } J}{\text{incoming } J} = \frac{\frac{1}{4} - \frac{D}{2L}}{\frac{1}{4} + \frac{D}{2L}} = \boxed{\frac{1 - \frac{2D}{L}}{1 + \frac{2D}{L}}}$$

Thus in a semiinfinite medium $\alpha S''$ neutrons/s cm^2 will be reemitted from the surface, while remaining $(1-\alpha)S''$ neutrons will be absorbed within the medium.

INTERFACE CONDITIONS

If more than 1 region is present, with different CCs, the diffusion equation solution will have 2 arbitrary constants for each region. Thus 2 conditions are required for each region. Thus 2 conditions are required at each interface.

The conditions are - both current and flux are continuous.
Hence for an interface at x_0.

$$\boxed{\phi(x_{0-}) = \phi(x_{0+})} \quad \& \quad \boxed{D(x_{0-}) \frac{d}{dx} \phi(x)\Big|_{x_{0-}} = D(x_{0+}) \frac{d}{dx} \phi(x)\Big|_{x_{0+}}}$$

where x_{0-} & x_{0+} indicate evaluation immediately to left & right of x_0.

An exception is if there is a localized source present. If an infinitely thin source emits S_{pl}'' neutrons/cm^2/s along the interface, neutron balance at interface is

$$J(x_{0-}) + S_{pl}'' = J(x_{0+}) \quad \text{or}$$

$$\boxed{-D(x_{0-}) \frac{d}{dx} \phi(x)\Big|_{x_{0-}} + S_{pl}'' = -D(x_{0+}) \frac{d}{dx} \phi(x)\Big|_{x_{0+}}}$$

BOUNDARY CONDITIONS IN OTHER GEOMETRIES

The boundary conditions are applicable in cylindrical and spherical as well as cartesian geometries: x is simply replaced by the direction normal to surface.

In spherical or cylindrical geometries, boundary conditions at the origin or center line must be treated differently

For a point source emitting S_p neutrons/s at center of a sphere, S_p

$$S_p = \lim_{r \to 0} \left(4\pi r^2 J_r(r)\right)$$ for a line source emitting S_ℓ neutrons/cm/s along centerline of a cylinder, $S_\ell = \lim_{r \to 0} [2\pi r J_r(r)]$

If there are no sources present, it is required that flux be finite at $r=0$

NON MULTIPLYING SYSTEMS - SPHERICAL GEOMETRY

① Consider a spherical geometry problem with a source at center of sphere in infinite medium

Recall the diffusion eqn

$$\nabla^2 \phi(\vec{r}) - \frac{1}{L^2} \phi(\vec{r}) = -\frac{1}{D} S'''(\vec{r})$$

Now, replace ∇^2 by its 1D spherical form,

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \phi(r) - \frac{1}{L^2} \phi(r) = -\frac{1}{D} S'''(r)$$

POINT SOURCE EXAMPLE

Consider a point source of strength S_p concentrated at the origin of an infinite medium extending from $r=0$ to $r=\infty$

For $r>0$, $S'''(r)=0$.

Hence,

$$\frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \phi(r) - \frac{1}{L^2} \phi(r) = 0$$

Now, let $\phi(r) = \frac{1}{r} \varphi(r)$ ~~[illegible]~~ flux

$\therefore \quad \frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\phi(r) - \frac{1}{L^2}\phi(r) = 0$ becomes

$\frac{1}{r^2}\frac{d}{dr} r^2 \frac{d}{dr}\frac{1}{r}\psi(r) - \frac{1}{L^2}\frac{\psi(r)}{r} = 0$ which simplifies to

$\frac{d^2}{dr^2}\psi(r) - \frac{1}{L^2}\psi(r) = 0.$

Above is a 2^{nd} order linear differential equation with a general soln that looks like $y = C_1 y_1 + C_2 y_2 \quad y_1 = e^{r_1 r} \quad y_2 = e^{r_2 r}$

The characteristic eqn is $r^2 - \frac{1}{L^2} = 0 \Rightarrow \left(r + \frac{1}{L}\right)\left(r - \frac{1}{L}\right) = 0.$

$\Rightarrow \quad r_{1,2} = \pm\frac{1}{L}.$

$\Rightarrow \quad y = C_1 e^{r/L} + C_2 e^{-r/L} = \psi(r)$

Now, $\psi(r) = \phi(r)\, r$

$\Rightarrow \quad r\,\phi(r) = C_1 e^{r/L} + C_2 e^{-r/L}$

$\Rightarrow \quad \boxed{\phi(r) = \frac{C_1}{r} e^{r/L} + \frac{C_2}{r} e^{-r/L}}$

Now, we shall employ boundary conditions to determine C_1 & C_2
At ∞, $\phi(r) = \phi(\infty) = 0$. The second boundary condition is ~~flux~~ at the origin. In the limit as $r \to 0$, the current

$J_r(r) = -D\frac{d}{dr}\phi(r)$ emerging from a small sphere with surface area $= 4\pi r^2$, must be equal to source strength

$\underbrace{S_p}_{\text{total neutrons etc}} = \lim_{r \to 0} \underbrace{4\pi r^2}_{\text{total area}} \underbrace{J_r(r)}_{\text{neutrons from unit area per unit time.}}$

Now from the 1^{st} boundary condition,

$\phi(\infty) = 0$, we find that

$\phi(\infty) = \frac{C_1}{r} e^{\infty/L} + C_2 e^{-\infty/L} = 0 \Rightarrow C_1 = 0$ as $\phi(\infty) \to \infty$ otherwise

$$\therefore \phi(r) = \frac{C_2}{r} e^{-r/L}$$

Now to find C_2, apply 2nd boundary condition, at origin

$$S_p = \lim_{r\to 0} 4\pi r^2 J_r(r) = \lim_{r\to 0} 4\pi r^2 \left(-D \frac{d}{dr}\phi(r)\right)$$

$$= -D \lim_{r\to 0} 4\pi r^2 \frac{d}{dr}\phi(r)$$

substitute

Now $\phi(r) = \frac{C_2}{r} e^{-r/L}$

$$\Rightarrow S_p = -D \lim_{r\to 0} 4\pi r^2 \frac{d}{dr}\left(\frac{C_2}{r} e^{-r/L}\right)$$

Substitute

$$\frac{d}{dr}\frac{C_2}{r} e^{-r/L} = C_2\left[\frac{-1}{Lr}e^{-r/L} - \frac{1}{r^2}e^{-r/L}\right]$$

Substitute

$$\Rightarrow S_p = -D(4\pi) \lim_{r\to 0} r^2\left[-C_2\left(\frac{1}{Lr}e^{-r/L} - \frac{1}{r^2}e^{-r/L}\right)\right]$$

$$\Rightarrow S_p = 4\pi C_2 D \lim_{r\to 0}\left(\frac{r e^{-r/L}}{L} + e^{-r/L}\right)$$

$$\Rightarrow S_p = 4\pi C_2 D\,(0+1)$$

$$\Rightarrow S_p = 4\pi C_2 D$$

$$\Rightarrow \boxed{C_2 = \frac{S_p}{4\pi D}}$$

$$\boxed{\phi(r) = \frac{S_p}{4\pi D r} e^{-r/L}}$$

All neutrons produced by point source must be absorbed in infinite medium. Taking an incremental volume as $dV = 4\pi r^2 dr$, we show that

$$\int_{\text{all space}} \Sigma_a \phi(r)\, dV = S_p.$$

TWO REGION EXAMPLE

In this problem, we shall consider 2 regions and a distributed source. It illustrates the treatment of boundary condition at origin as well as interface condition.

① Consider a sphere of radius R with material properties D & Σ_a contains a uniform source S_0'''.

② The sphere is surrounded by a second source free medium with properties $\hat{D}$ & $\hat{\Sigma}_a$, it extends to $r=\infty$. Our objective is to determine neutron flux.

For ① region, the diffusion eqn takes the form

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{L^2}\phi(r) = \frac{1}{D}S_0''' \qquad 0 \le r \le R$$

For ② region, diffusion eqn is of form

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{\hat{L}^2}\phi(r) = 0 \qquad R \le r \le \infty$$

For region ①,

$$\phi(r) = \phi_p(r) + \phi_c(r)$$

For a uniform source $\frac{d}{dr}\phi(r) = 0$

$$\Rightarrow \frac{1}{L^2}\phi_p(r) = \frac{1}{D}S_0''' \Rightarrow \phi_p(r) = \frac{L^2}{D}S_0''' = \frac{S_0'''}{\Sigma_a}$$

$\phi_c(r)$ = sol^n to corresponding homogeneous sol^n which was previously seen to be $\phi_c(r) = C_1 e^{r/L} + C_2 e^{-r/L}$

$\Rightarrow$ General sol^n $\phi(r) = \phi_p(r) + \phi_c(r)$ $\qquad (0 \le r \le R)$

$$\boxed{\phi(r) = \frac{S_0''}{\Sigma_a} + C_1 e^{r/L} + C_2 e^{-r/L}} \qquad ①$$

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For the 2nd region,
as solved previously, solution is of the form,

$$\boxed{\phi(r) = C_1' e^{r/L} + C_2' e^{-r/L}} \quad ② \quad (R < r \leq \infty)$$

Now, to solve for the arbitrary constants, we apply boundary conditions

1) $0 \leq \phi(0) < \infty$ - which means we have a finite flux at origin
(since no point source at origin)

This condition is applicable to eqn ①
Applying it to

$$\phi(r) = \frac{S_0'''}{\Sigma_a} + \frac{C_1}{r} e^{+r/L} + \frac{C_2}{r} e^{-r/L}$$

Now apply boundary condition

$$\phi(0) = \lim_{r \to 0} \left(\frac{S_0'''}{\Sigma_a} + \underbrace{\left(\frac{C_1}{r}\right) e^{r/L} + \left(\frac{C_2}{r}\right) e^{-r/L}}\right)$$

$e^{0/L} = 1$, $e^{-0/L} = 1$

Now we see the 2 terms have r in denominator that would make sol^n ∞ as $r \to 0$. So we need them to cancel each other out (ie $\frac{C_1}{r}$ should cancel $\frac{C_2}{r}$) For that $C_1 = -C_2$ or $C_2 = -C_1$.

$$\boxed{C_2 = -C_1} \Rightarrow \boxed{\phi(r) = \frac{2C_1}{r} \sinh(r/L) + \frac{S_0'''}{\Sigma_a} \quad (0 \leq r \leq R)} \; -③$$

2) $\phi(\infty) = 0$. - which means, flux eventually goes to 0 at ∞.
This boundary condition must be applied to the ② eqn

$$\phi(\infty) = C_1' e^{r/L} + C_2' e^{-r/L} = 0$$

$$\Rightarrow C_1' e^{\infty/L} + C_2' e^{-\infty/L} = 0$$

$$C_1' e^{\infty/L} + 0 = 0$$

$\Rightarrow$ C_1' must be 0 to balance $e^{\infty/L}$ term

$$\Rightarrow \boxed{C_1' = 0} \quad \text{II}$$

$$\Rightarrow \boxed{\phi(r) = C_2' e^{-r/L} \quad (R < r \leq \infty)} \; -④$$

3) $\Phi(R_-) = \Phi(R_+)$ This is also called the interface condition which means flux to left at R = flux to right at R

$\Rightarrow$ I = II at R

$$\Rightarrow \frac{2C_1}{R}\sinh(R/L) + \frac{S_0'''}{\Sigma_a} = \frac{C_2'}{R} e^{(-R/L)} \text{ at } R = R$$

$$\Rightarrow \boxed{\frac{2C_1}{R}\sinh(R/L) + \frac{S_0'''}{\Sigma_a} = \frac{C_2'}{R} e^{-R/L}} \quad \text{III}$$

4) $D\frac{d}{dR}\Phi(R)\Big|_{R_-} = \hat{D}\frac{d}{dR}\Phi(R)\Big|_{R_+}$ Another interface condition implying equal partial currents at R.

0 as constant

$$\Rightarrow D\frac{d}{dR}\left(\frac{2C_1}{R}\sinh(R/L) + \frac{S_0'''}{\Sigma_a}\right)$$

$$= \hat{D}\frac{d}{dR}\left(\frac{C_2'}{R}e^{-R/L}\right)$$

$$\Rightarrow 2DC_1\left[\frac{d}{dR}\frac{1}{R}\sinh(R/L)\right] = DC_2'\left[\frac{d}{dR}\frac{1}{R}e^{-R/L}\right]$$

$$\Rightarrow \boxed{2DC_1\left[\frac{1}{RL}\cosh(R/L) - \frac{1}{R}\sin(R/L)\right] = -\hat{D}C_2'\left(\frac{1}{R^?} + \frac{1}{R^2}\right)e^{-R/L}} \quad \text{IV}$$

Solving III & IV for C_1 & C_2' and inserting these results in 3 & 4 results in sol^n

$$\phi(R) = \left[1 - C\frac{R\sinh(R/L)}{R\sinh(R/L)}\right]\frac{S_0'''}{\Sigma_a} \qquad 0 \le R \le R$$

$$\phi(R) = (1-C)\frac{S_0'''}{\Sigma_a}\frac{R}{R}e^{-(R-R)/L} \qquad R < R \le \infty$$

$$C = \left[1 + \frac{D}{\hat{D}}\frac{(R/L)\coth(R/L) - 1}{R/L + 1}\right]^{-1}$$

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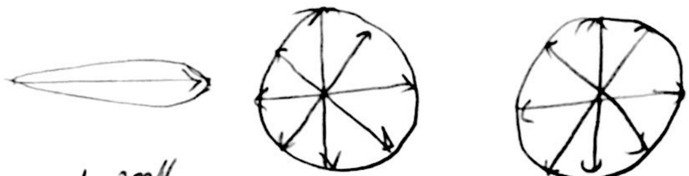
THE VALIDITY OF DIFFUSSION APPROXIMATION



does not work works works (leakage → 0 though)

DIFFUSSION LENGTH

The physical interpretation of diffusion can be gained by examining mean squared distance that neutron diffuses betn birth and death to do this.

assume neutron was born at $r=0$ & calculate the mean squared distance weighed by absorption rate.

$$\overline{r^2} = \frac{\int r^2 \Sigma_a \phi(r)\, dV}{\int \Sigma_a \phi(r)\, dV}$$

Next, recall flux distribution of a point source is of the form $\phi(r) = \frac{S_p}{4\pi D r} e^{-r/L}$ & $dV = 4\pi r^2 dr$.

$$\overline{r^2} = \frac{\int_0^\infty r^2\, \Sigma_a \frac{S_p}{4\pi r D} e^{-r/L}\, 4\pi r^2\, dr}{\int_0^\infty \Sigma_a \frac{S_p}{4\pi r D}\, 4\pi r^2 e^{-r/L}\, dr}$$

constant, take outside integral & cancel

$$\Rightarrow \overline{r^2} = \frac{\int_0^\infty r^3 e^{-r/L}\, dr}{\int_0^\infty r e^{-r/L}\, dr} = 6L^2$$

$$\Rightarrow \boxed{L = \frac{1}{\sqrt{6}} \sqrt{\overline{r^2}} = 0.408 \sqrt{\overline{r^2}}}$$

Thus diffusion length is proportional to rms distance diffused by neutron betn birth & absorption

Now recall that $\lambda = \frac{1}{\Sigma}$ is the mean free path and average distance betn scatter collisions.

For isotropic scatter, $D = \frac{1}{3\Sigma_{tr}} = \frac{1}{3\Sigma_t} = \frac{\lambda}{3}$

$$c = \frac{\Sigma_s}{\Sigma_t} = \frac{\Sigma_s}{\Sigma_s + \Sigma_a}$$

$\Rightarrow \quad c(\Sigma_s + \Sigma_a) = \Sigma_s$

$\Rightarrow \quad \Sigma_a = \frac{(1-c)}{c}\Sigma_s = \frac{(1-c)\Sigma_s}{\frac{\Sigma_s}{\Sigma_t}} = (1-c)\Sigma_t$

$\Rightarrow \quad \boxed{\Sigma_a = (1-c)\Sigma_t = \frac{(1-c)}{\lambda}}$

We know that $L = \sqrt{\frac{D}{\Sigma_a}}$

$$= \left[\frac{D}{\frac{(1-c)}{\lambda}}\right]^{1/2} = \left[\frac{\lambda/3 \, \lambda}{(1-c)}\right]^{1/2} = \left[\frac{\lambda^2}{3(1-c)}\right]^{1/2}$$

$$\boxed{L = \frac{\lambda}{\sqrt{3(1-c)}}}$$ (valid if $c > 0.7$)

UNCOLLIDED FLUX REVISITED

Further insights on validity of diffusion equation may be gained by comparing uncollided flux from point source to total (i.e. collided + uncollided) flux attributable to the source.

Recall from Ch. 2 that uncollided flux of a pt source is

$$\phi_u(r) = \frac{S_p}{4\pi r^2} e^{-\Sigma r} = \frac{S_p}{4\pi r^2} e^{-r/\lambda}$$

, Now comparing it with diffusion eqn for pt source

$$\phi(r) = \frac{S_p}{4\pi D r} e^{-r/L}$$

λ = m.f.p.

Regardless of crossection, the uncollided flux drops off as $\frac{1}{r^2}$, while total flux drops only at $\frac{1}{r}$

while mfp appear in exponential of the uncollided flux, while diffusion length - which is larger - appear in diffusion length of total flux. Thus total flux decays more slowly with distance.

2nd point of comparison betn uncollided and diffusing neutrons ?? rms distance to first collision & absorption, respectively. To obtain rms distance travelled by uncollided neutrons, we again consider pt. source at origin but employ first collision rate as weight instead of absorption rate.

sub. $\phi_u = \frac{S_p}{4\pi r^2} e^{-r/\lambda}$

$$\overline{r_u^2} = \frac{\int r^2 \Sigma_t \phi_u(r) dV}{\int \Sigma_t \phi_u(r) dV}$$

$$\overline{r_u^2} = \frac{\int_0^\infty r^2 \Sigma_t \frac{S_p}{4\pi r^2} e^{-r/\lambda} 4\pi r^2 dr}{\int_0^\infty \Sigma_t \frac{S_p}{4\pi r^2} e^{-r/\lambda} 4\pi r^2 dr}$$

$$= \frac{\int_0^\infty r^2 e^{-r/\lambda} dr}{\int_0^\infty e^{-r/\lambda} dr}$$

$$\Rightarrow \overline{r_u^2} = 2\lambda^2 \Rightarrow \lambda = 0.707\sqrt{\overline{r_u^2}}$$

meaning that mfp is proportional to rms before making its first collision. The ratio of 2 rms distances.

$$\frac{\overline{r^2}}{\overline{r_u^2}} = \frac{6L^2}{2\lambda^2} = \frac{3L^2}{\lambda^2} = \frac{3\lambda^2}{3(1-c)\lambda^2} = \frac{1}{1-c}$$

$$\Rightarrow \boxed{\frac{\overline{r^2}}{\overline{r_u^2}} = \frac{1}{1-c}}$$

As the distance from source increases, importance of uncollided source rapidly diminishes provided that c, the ratio of scattering to total cross section, is sufficiently close to 1

If not, then on average the neutron will not make enough scatters before absorption for the diffusion eqn to be valid

Examining angular distributions of neutrons further illuminates the distinction betn uncollided & diffusing neutrons. The uncollided neutrons at a pt. same distance r from point source can travel in single dirn, radially outward, while diffusing neutrons will be dispersed at an angle.

Hence for diffusion to give a reasonable accuracy, only a small fraction of neutron population at a point can remain uncollided. This condition typically exists only in situations where $c = \Sigma_s/\Sigma_t$ is greater than about 0.7.

$\boxed{\text{MULTIPLYING SYSTEMS}}$ – fission term present

SUBCRITICAL ASSEMBLIES

spherical geometry

In a multiplying systems, $\nu\Sigma_f > 0$ because fissionable material is present.

For a uniform system with a uniform source S_0''', case is space independent

the main diffusion equation

$$-D\nabla^2\phi(\vec{r}) + \Sigma_a(\vec{r})\phi(\vec{r}) = S'''(\vec{r}) + \nu\Sigma_f(\vec{r})\phi(\vec{r})$$

$$\downarrow$$

$$-\nabla^2\phi(\vec{r}) + \frac{1}{D}\Sigma_a(\vec{r})\phi(\vec{r}) = \frac{S_0'''(\vec{r})}{D} + \frac{\nu}{D}\Sigma_f(\vec{r})\phi(\vec{r})$$

$\downarrow$ 1D spherical geometry becomes

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{D}\Sigma_a\phi(r) = \frac{S_0'''}{D} + \frac{\nu}{D}\Sigma_f\phi(r)$$

$$\frac{\nu\Sigma_f}{\Sigma_a} = k_\infty \quad \text{as} \quad L^2 = D/\Sigma_a \quad D = L^2\Sigma_a \Rightarrow \frac{\nu\Sigma_f}{D} = \frac{\nu\Sigma_f}{L^2\Sigma_a} = \frac{k_\infty}{L^2}$$

$$\downarrow$$

$$\boxed{-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{L^2}\phi(r) = \frac{S_0'''}{D} + \frac{1}{L^2}k_\infty\phi(r)}$$

$\Rightarrow$

$$\boxed{-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{L^2}(1-k_\infty)\phi(r) = \frac{S_0''}{D}}$$ (2)

The above is a 2^{nd} order inhomogeneous linear differential ODE of degree 2.

Its sol^n is of the ~~form~~ form

$y = y_p + y_c$ Here $\phi(r) = \phi_p(r) + \phi_c(r)$

y_c = sol^n to corresponding homogeneous eqn i.e

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) + \frac{1}{L^2}(1-k_\infty)\phi(r) = 0$$ (1)

As before,

let $\phi_c(r) = \frac{1}{r}\psi(r)$

$\Rightarrow$ eqn (1) becomes

$$\frac{d^2}{dr^2}\psi(r) - \frac{1}{L^2}(1-k_\infty)\psi(r) = 0.$$

the above eqn has a characteristic eqn,

$$r^2 - \frac{1}{L^2}(1-k_\infty) = 0$$

$$\Rightarrow \left(r + \frac{1}{L}\sqrt{(1-k_\infty)}\right)\left(r - \frac{1}{L}\sqrt{1-k_\infty}\right) = 0$$

$$\Rightarrow r_{1,2} = \pm\frac{1}{L}\sqrt{1-k_\infty}$$

$\Rightarrow$ the sol^n of homogeneous eqn looks like

$$\psi_c(r) = C_1 e^{\frac{r}{L}\sqrt{1-k_\infty}} + C_2 e^{-r/L\sqrt{1-k_\infty}} \quad \& \quad \phi_c(r) = \frac{\psi(r)}{r}$$

$$\Rightarrow \boxed{\phi_c(r) = \frac{C_1}{r}e^{r/L\sqrt{1-k_\infty}} + \frac{C_2}{r}e^{-r/L\sqrt{1-k_\infty}}}$$ (3)

solving for particular sol^n.

For a uniform source, particular sol^n is a constant. Hence derivative term in eqn ① vanishes

& so $\frac{1}{L^2}(1-k_\infty)\,\phi_p(r) = \frac{S_0'''}{D}$

$$\Rightarrow \boxed{\phi_p(r) = \frac{S_0'''}{D}\frac{L^2}{(1-k_\infty)} = \frac{S_0'''\,D/\Sigma_a}{D\,(1-k_\infty)} = \frac{S_0'''}{\Sigma_a(1-k_\infty)}}$$

Now

$$\phi(r) = \phi_c(r) + \phi_p(r)$$

$$\Rightarrow \boxed{\phi(r) = \frac{C_1}{r}e^{+r/L(\sqrt{1-k_\infty})} + \frac{C_2}{r}e^{-r/L(\sqrt{1-k_\infty})} + \frac{S_0'''}{\Sigma_a(1-k_\infty)}}$$

We must apply boundary conditions to solve for C_1 & C_2

Now since we have a uniform source & no point source, we have a finite flux at the origin.

ie $\phi(0) \Rightarrow$ finite

this is possible only if $C_2 = -C_1$ (exponential terms go to 1 & for ϕ to be finite $1/r$ terms must cancel out)

$$\Rightarrow \phi(r) = \frac{C_1}{r}e^{r/L\sqrt{1-k_\infty}} - \frac{C_1}{r}e^{-r/L(1-k_\infty)} + \frac{S_0'''}{\Sigma_a(1-k_\infty)}$$

$$= \frac{C_1}{r}\left[\left(e^{r/L\sqrt{1-k_\infty})} - e^{-r/L\sqrt{1-k_\infty}}\right)\right] + \frac{S_0'''}{\Sigma_a(1-k_\infty)}$$

$$= \frac{2C_1}{r}\sinh\left(\frac{r\sqrt{1-k_\infty}}{L}\right) + \frac{S_0'''}{\Sigma_a(1-k_\infty)} \quad ⑦$$

Take $\tilde{R}$ as extrapolated radius of sphere when $\phi \to 0$

$\Rightarrow \phi(\tilde{R}) = 0$

Hence from eqn (1)

$$\phi(\tilde{R}) = 0 = \frac{2C_1}{\tilde{R}} \sinh\left(\frac{\tilde{R}\sqrt{1-k_\infty}}{L}\right) + \frac{S_0'''}{\Sigma_a(1-k_\infty)}$$

Now solving for C_1,

$$C_1 = \frac{-\tilde{R}\, S_0'''}{2\,\Sigma_a(1-k_\infty)\sinh\left(\frac{\tilde{R}\sqrt{1-k_\infty}}{L}\right)}$$

↓ substitute in eqn of $\phi(r)$ general soln

$$\boxed{\phi(r) = \frac{S_0'''}{\Sigma_a(1-k_\infty)}\left[1 - \frac{\tilde{R}\sinh\left(r\sqrt{1-k_\infty}/L\right)}{r\,\sinh\left(\tilde{R}\sqrt{1-k_\infty}/L\right)}\right]}$$

For $k_\infty < 1$ system must be subcritical even if there were no neutron leakage out of sphere (particular & general soln eqns remain the same)

Now for $\boxed{k_\infty > 1}$ corresponding homogeneous eqn ②

$$\frac{d^2}{dr^2}\phi(r) - \frac{1}{L^2}(1-k_\infty)\phi(r) = 0$$

Now $k_\infty > 1$

$$\Rightarrow \frac{d^2}{dr}\phi(r) + \frac{1}{L^2}(k_\infty - 1)\phi(r) = 0$$

Here characteristic eqn takes form

$$r^2 + \frac{k_\infty - 1}{L^2} = 0$$

$$\Rightarrow r_{1,2} = \pm\frac{\sqrt{k_\infty - 1}}{L}\, i$$

Let $\frac{k_\infty - 1}{L} = B$

$$\Rightarrow r_{12} = B\,i$$

We know that for complex roots of characteristic eqn $a \pm bi$, soln is of the form $e^{at}(C_1 \sin bt + C_2 \cos bt)$

Here $r_{1,2} = 0 \pm Bi$

$\Rightarrow$ Soln is of the form $e^{0t}(C_1 \sin Br + C_2 \cos Br)$

$$\Rightarrow \boxed{\varphi_c(r) = C_1 \sin Br + C_2 \cos Br} \qquad \left(B = \frac{\sqrt{K_\infty - 1}}{L}\right)$$

Now, $\frac{\varphi(r)}{r} = \phi(r)$

$$\Rightarrow \phi_c(r) = \frac{C_1}{r} \sin Br + \frac{C_2}{r} \cos Br.$$

$$\boxed{= \frac{C_1}{r} \sin\left(\frac{\sqrt{K_\infty - 1}}{L} r\right) + \frac{C_2}{r} \cos\left(\frac{\sqrt{K_\infty - 1}}{L} r\right)}$$

Solving for particular soln,

For uniform source particular soln is a constant

$\Rightarrow$ derivative term goes to 0.

$$\Rightarrow 0 + \frac{1}{L^2}(K_\infty - 1)\,\phi_p(r) = -\frac{S_0'''}{D}$$

$$\Rightarrow \boxed{\phi_p(r) = -\frac{S_0'''}{K_\infty - 1} L^2/D = -\frac{S_0'''}{\Sigma_a(K_\infty - 1)}}$$

Hence $\phi(r)$'s general soln is

$$\phi(r) = \phi_p(r) + \phi_c(r)$$

$$\boxed{\phi(r) = -\frac{S_0'''}{\Sigma_a(K_\infty - 1)} + \frac{C_1}{r} \sin\left(\frac{r\sqrt{K_\infty - 1}}{L}\right) + \frac{C_2}{r} \cos\left(\frac{r}{L}\sqrt{K_\infty - 1}\right)}$$

To solve for C_1 & C_2, apply boundary conditions.

① $\phi(0)$ must be finite

$$\Rightarrow \lim_{r\to 0} \phi(r) = \lim_{r\to 0}\left[\frac{-S_0'''}{\Sigma_a(K_\infty-1)} + \frac{C_1}{r}\sin(Br) + \frac{C_2}{r}\cos(Br)\right] \text{ is finite}$$

$$= \frac{-S_0'''}{\Sigma_a(K_\infty-1)} + \lim_{r\to 0}\frac{C_1}{r}\sin(Br) + \lim_{r\to 0}\frac{C_2}{r}\cos(Br)$$

$$\lim_{r\to 0}\frac{\sin Br}{r} = B \quad , \lim_{r\to 0}\frac{C_2}{r}\cos Br = \infty \quad \text{as } \cos Br \to 1 \text{ at } r\to 0$$

$\Rightarrow$ for sol^n to be finite, $C_2 = 0$.

$$\Rightarrow \boxed{C_2 = 0}$$

$$\boxed{\phi(r) = \frac{-S_0'''}{(K_\infty-1)\Sigma_a} + \frac{C_1}{r}\sin(Br)}$$

We determine C_1 by requiring the above eq^n to meet $\phi(\tilde{R}) = 0$

$$\therefore \phi(\tilde{R}) = \frac{-S_0'''}{\Sigma_a(K_\infty-1)} + \frac{C_1}{\tilde{R}}\sin(B\tilde{R}) = 0$$

substitute.

$$\Rightarrow \frac{C_1}{\tilde{R}}\sin(B\tilde{R}) = \frac{S_0'''}{\Sigma_a(K_\infty-1)}$$

$$\Rightarrow C_1 = \frac{S_0'''\tilde{R}}{\sin(B\tilde{R})\,\Sigma_a(K_\infty-1)}$$

$$\phi(r) = \frac{S_0'''\tilde{R}\,\sin B(r)}{\sin(B\tilde{R})\Sigma_a(K_\infty-1)\,r} - \frac{S_0'''}{\Sigma_a(K_\infty-1)}$$

$$\boxed{\phi(r) = \frac{S_0'''}{\Sigma_a(K_\infty-1)}\left[\frac{\tilde{R}\sin(Br)}{r\sin(B\tilde{R})} - 1\right]}$$

$$\left(B = \sqrt{\frac{K_\infty-1}{L}} \cdot r\right)$$

Scanned with CamScanner

Similar analysis can be done for different geometries where.

Geometry	Dim.	B_g^2	~~$\phi(x)$~~
Infinite slab	thickness ~~x~~ a	$\left(\frac{\pi}{a}\right)^2$	$\cancel{A\cos}\left(\frac{\pi x}{a}\right)$
3D Rectangle	$a \times b \times c$	$\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2+\left(\frac{\pi}{c}\right)^2$	
∞ Cylinder	radius R	$\left(\frac{2.405}{R}\right)^2$	
Finite Cylinder	radius R height H	$\left(\frac{2.405}{R}\right)^2+\left(\frac{\pi}{H}\right)^2$	
Sphere	radius R	$\left(\frac{\pi}{R}\right)^2$.	

Reactor is critical when ~~TH~~ $B_m = B_g$..

If reactor is smaller than critical size for a given material

$B_g > B_m$. → reactor is subcritical.

If $B_g < B_m$ → reactor is critical.

Numerical criticality search.

$$-\frac{\partial}{\partial x} D \frac{\partial}{\partial x}\phi + \Sigma_a \phi = \frac{\nu \Sigma_f}{k}\phi .$$

(i) $$M\phi = \frac{1}{k} F\phi$$

If ϕ is assumed to be known, & if we let $S = F\phi$,

inverse power iteration

$$M\phi = \frac{S}{k} \Rightarrow \boxed{M\phi^{n+1} = \frac{S^n}{k^n}}$$

looks just like our steady state ~~src~~ eqn which we can solve ~~by discreti~~ numerically - FD / ~~F~~EM etc.

Soln

1. Discretise domain
2. find matrix M, F & vector ϕ.
3. Start iteration. Assume $\phi^{(0)}$, $k^{(0)} \rightarrow$ get $S^{(0)}$ (generalize to $\phi^{(n)}$, $k^{(n)}$, $S^{(n)}$)

$$\cancel{M\phi^{(1)} = S^{(0)} k^{(0)} \Rightarrow M\phi^{(n+1)} =}$$

4. Solve for $\phi^{(1)}$: $\phi^{(1)} = M^{-1} \frac{S^{(0)}}{k^{(0)}}$

5. $\Rightarrow$ generalize to $\boxed{\phi^{(n+1)} = M^{-1} \frac{S^{(n)}}{k^{(n)}}}$

5. Calculate $k^{(1)} \rightarrow$ generalize to k^n.

eqn: $M\phi = \frac{1}{k} S$.

M is a matrix, ϕ is a vector, S is a vector.
S is a scalar.

To find k, we integrate eqn over space

$$\int dx \, M\phi = \frac{1}{k} \int dx \, S$$

$$\therefore \quad k \cong \int dx \, S \Big/ \int dx \, M\phi \quad \Rightarrow \quad \underline{\underline{k^{n+1} = \frac{\int dx \, S^{n+1}}{\int dx \, M\phi^{n+1}}}}$$

Moreover, by iteration eqn.

$$M\phi^{n+1} = \frac{1}{k^{(n)}} S^{(n)}$$

$$\therefore \quad \boxed{k^{(n+1)} = \int dx \, S^{(n+1)} \Big/ \frac{1}{k^{(n)}} \int dx \, S^{(n)}}$$

Now that we know k^{n+1},

(5) Update $\cancel{\frac{s^{n+1}}{k^{n+1}}}$ ~~& find~~ $\cancel{\phi^{n+1}}$ ... go back ~~to (4)~~.

check for convergence in k & ϕ.

If not converged, go back to (4).

Show Ryan's code.