

Ch-5 Lec 2

①

Multiplying Systems Behavior.

Definition of criticality comes by setting $S(t) \rightarrow 0$.

$$\frac{dn(t)}{dt} = \left(\frac{k_1}{e}\right)n(t).$$

$$\Rightarrow n(t) = n_0 e^{(k_1)t/e}$$

For a system to be critical, $k=1$

Subcritical if $k < 1$

Supercritical if $k > 1$.

In order to incorporate effects of source, let $S(t) = S_0$ & $n_0 > 0$.
then we get the following IVP.

$$\begin{aligned} \frac{dn(t)}{dt} &= S_0 + \left(\frac{k_1}{e}\right)n(t) \\ n_0 &> 0 \end{aligned} \quad \left. \begin{array}{l} \text{HW - 4.3} \\ \text{Solve the IVP \&} \\ \text{show soln to be} \end{array} \right\}$$

$$n(t) = \frac{e S_0}{k_1} \left(\exp \left(\frac{k_1}{e} t \right) - 1 \right)$$

Supercritical ($k > 1$)

- (i) population rises at an increasing rate.
- exponential at long times

Subcritical ($k < 1$)

$$n(t) = \frac{e S_0}{k_1} \left(\exp \left(\frac{k_1}{e} t \right) - 1 \right)$$

begin by switching sign:

$$n(t) = \frac{150}{1-k} \left(1 - \exp\left(-\left(\frac{(k-1)k}{2}\right)t\right) \right)$$

then for long times, the exponential term decays away

$$\boxed{n_{\infty} = \frac{150}{1-k}}$$

Critical: $k=1$

$$n(t) = \frac{150}{k-1} \left(\exp\left(\frac{k-1}{2}\right) t^2 - 1 \right)$$

Hmm $k=1$ so what do we do?

We expand the exponential in Taylor series.

$$\exp(x) = 1 + x + \frac{1}{2}x^2 + \dots$$

$$\Rightarrow \exp\left(\frac{k-1}{2}t\right) = 1 + \frac{k-1}{2}t + \frac{1}{2}\left(\frac{k-1}{2}\right)^2 t^2 + \dots$$

then

$$n(t) = \frac{150}{k-1} \left[1 + \frac{k-1}{2}t + \left(\frac{k-1}{2}\right)^2 t^2 + \dots - 1 \right]$$

$$= \frac{150}{k-1} \left[\frac{t}{2} + \frac{(k-1)^2}{2} t^2 + \dots \right]$$

Now, we don't have $k_{\text{I}} \rightarrow k_{\text{II}} \rightarrow \dots$ in denominator anymore.

$$\frac{d}{dt} n(t) = S_0 t$$

then $\textcircled{1}$ population increases linearly in presence of a ~~time-dependent~~ constant-in-time source.

At small $\textcircled{2}$ lifetimes controlling reactions is very difficult. Fortunately there are delayed $\textcircled{2}$ - which have ~~been~~ been neglected till now. We consider delayed $\textcircled{2}$ next.

Delayed $\textcircled{2}$ kinetics

- More than 99% of fission $\textcircled{2}$ are produced instantaneously at the time of fission.
- Remaining fraction β , comes from decay of fission products. We call these $\textcircled{3}$ precursors.
- We incorporate this into account.

Going back to the balance equation:

$$\frac{dn(t)}{dt} = \begin{aligned} & \# \text{Source } \textcircled{2} \text{ produced } I_s \\ & + \# \text{fission } \textcircled{2} \text{ produced } I_s \text{ (prompt)} \\ & + \# \text{delayed } \textcircled{2} \text{ produced } I_s \\ & - \# \text{absorbed } I_s \\ & - \# \text{leaking from system } I_s \end{aligned} \quad \left. \begin{aligned} & \} \text{production rate} \\ & \} \text{loss rate} \end{aligned} \right.$$

Balance Equation:

$$\frac{dn}{dt} = \text{production rate} - \text{loss rate}$$

= # @ from sources

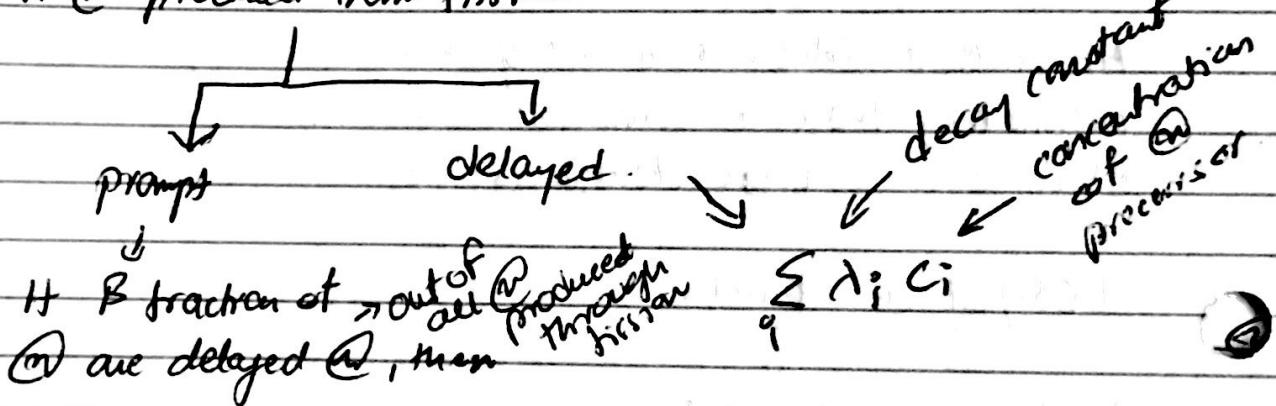
- Σ (6)

+ # @ from fission (delayed + prompt) - τ_{BD}

- # @ lost from absorptions - $\int \Sigma \kappa n(t) dt$

- # @ lost to leakage is. - $\int \Sigma \kappa_n(t) dt$

@ produced from fission.



(1- β) fraction of @ are prompt!

Then rate of @ prompt @ production

is

$$(1-\beta) \nu \sum \lambda_i n(t).$$

There can literally be lots & lots of @ precursors and each can be treated separately. However, in practice, these @ precursors are lumped into groups, where,

$$\beta = \sum_{i=1}^6 \beta_i$$

If the half-life of i^{th} group is $t_{1/2,i}$, then average half-life of delayed @ is

$$t_{1/2} = \frac{1}{\beta} \sum_{i=1}^6 \beta_i t_{1/2,i}$$

< weighted sum.

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McGraw-Hill

$$t_{1/2} = \frac{0.693}{\lambda_i}$$

Then the average decay constant is denoted by

$$\frac{1}{\lambda} = \frac{1}{\beta} \sum_{i=1}^6 \beta_i \frac{1}{\lambda_i}$$

~~Processor average delayed @ lifetime is average lifetime + average time before either a delayed @~~

Recall $\ell \rightarrow$ prompt @ lifetime \rightarrow average lifetime of ℓ produced instantaneously at the time of fission.

Define $\ell_d \rightarrow$ average delayed @ lifetime \rightarrow average time interval b/w fission and time at which delayed ℓ are absorbed or leaked out from system, then

$$\ell_d = \frac{\text{avg.}}{\text{time after fission b/w fission \& delayed @ birth}} + \text{avg. @ lifetime for prompt } \ell$$

(assuming delayed ℓ live ~~the~~ similar life)

$$= \cancel{\ell} + \cancel{\ell_{\text{prompt}}} + \ell$$

$$\boxed{\ell_d = \frac{1}{\lambda} + \ell}$$

Now, incorporating both delayed @ & prompt @, average @ lifetime

$$\begin{aligned}\bar{\lambda} &= (1-\beta)\lambda + \beta\lambda_d = \lambda - \lambda\beta + \beta\left(\lambda + \frac{1}{\tau}\right) \\ &= \lambda - \lambda\beta + \lambda\beta + \frac{\beta}{\tau} \\ \boxed{\bar{\lambda}} &= \lambda + \frac{\beta}{\tau}\end{aligned}$$

of total @

Although delayed @ are a small fraction, they dominate average @ lifetime as $\beta/\tau \gg 1$.

Typically $\beta \approx 0.65$: (for U-235)

HW 4.4 Calculate λ for U-235. Use data from table 5-1 in textbook.

Going back to the balance equation, we have

$$\frac{d}{dt} n(t) = S(t) + (1-\beta)\nu \sum E_j \bar{\lambda}_j n(t) + \sum \lambda_i C_i$$

$$- \sum E_i \bar{\lambda}_i n(t) - \Gamma \sum E_i n(t)$$

↑
lacks file
bug in textbook

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Now, in order to get $C_i(t)$, we have additional equations.

Precursor concentration basically comes from fission product production & is lost to decay of these fission prod.

then,

$$\frac{d}{dt} C_i(t) = \begin{matrix} \text{of } i\text{-type} \\ \# \text{ precursors produced/s} \end{matrix} - \begin{matrix} \text{of } i\text{-type} \\ \# \text{ precursors lost/s} \end{matrix}$$

← \leftarrow fission

← \leftarrow decay.

$$\text{precursors of type } i, \text{ produced per s} = \beta_i \underbrace{\sum_j \varepsilon_j \sqrt{n(t)}}_P$$

β_i fraction of precursor produced per prompt γ produced/
 ε_j rate of prompt γ produced/
 $n(t)$ produced

$$\boxed{\frac{d}{dt} C_i(t) = \beta_i \sum_j \varepsilon_j \sqrt{n(t)} - \lambda_i C_i(t)}$$

Together then, the reactor kinetics ~~two~~ equations are

$$\frac{d}{dt} n(t) = S(t) + (1 - \rho) \sum_j \varepsilon_j \sqrt{n(t)} + \sum_i \lambda_i C_i(t) - \sum_a \varepsilon_a \sqrt{n(t)}$$

$\leftarrow \sum_a \varepsilon_a \sqrt{n(t)}$

$$\frac{d}{dt} C_i(t) = \beta_i \sum_j \varepsilon_j \sqrt{n(t)} - \lambda_i C_i(t) \quad i = 1 \dots 6$$

Now, if we look at the I^+ kinetics equation,

$$\frac{dn(t)}{dt} = S(t) + (1-\beta)\nu \sum_i \bar{v}_i n(t) + \sum_i \lambda_i c_i(t) - \epsilon a \bar{v} n(t) - \Gamma \epsilon a \bar{v} n(t)$$

$$\hookrightarrow \frac{dn(t)}{dt} = S(t) + \sum_i \lambda_i c_i(t) + (1-\beta)\nu \sum_i \bar{v}_i n(t) - \epsilon a \bar{v} n(t) - \Gamma \epsilon a \bar{v} n(t).$$

$$= \cancel{S(t)} + \cancel{\sum_i \lambda_i c_i(t)} + \cancel{(1-\beta)\nu \sum_i \bar{v}_i n(t)} - \cancel{\epsilon a \bar{v} n(t)}$$

$$= S(t) + \sum_i \lambda_i c_i(t) + (1-\beta)\nu \sum_i \bar{v}_i n(t) - \epsilon a \bar{v} n(t) - \left(1 - \frac{P_{NL}}{P_{NL}}\right) \epsilon a \bar{v} n(t).$$

$$\hookrightarrow P_{NL} \frac{dn(t)}{dt} = P_{NL} S(t) + P_{NL} \sum_i \lambda_i c_i(t) + P_{NL} \cancel{(1-\beta)\nu \sum_i \bar{v}_i n(t)} - \epsilon a \bar{v} n(t)$$

$$= P_{NL} S(t) + P_{NL} \sum_i \lambda_i c_i(t) + \left((1-F) P_{NL} \frac{\nu \epsilon}{\epsilon a} - 1 \right) \bar{v} n(t)$$

$$= P_{NL} S(t) + P_{NL} \sum_i \lambda_i c_i(t) + \frac{((1-F) P_{NL} K_{\infty} - 1) n(t)}{K_{\infty}}$$

$$\Rightarrow \boxed{\frac{dn(t)}{dt} = S(t) + \sum_i \lambda_i c_i(t) + \frac{((1-F) K_{\infty} - 1) n(t)}{K_{\infty}}} \quad)$$

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Moreover,

$$\begin{aligned}\frac{d}{dt} c_i(t) &= \beta_i \sum_{j \neq i} \frac{\lambda_j}{\lambda_i} n(t) - \lambda_i c_i(t) \\ &= \beta_i \frac{\sum_{j \neq i} \lambda_j n(t)}{\sum_{j \neq i} \lambda_j} - \lambda_i c_i(t) \\ &= \beta_i \frac{k_{so} p_{n,i} n(t)}{k_{so} p_{n,i}} - \lambda_i c_i(t)\end{aligned}$$

~~$\frac{dc_i(t)}{dt} = \beta_i \frac{k}{e} n(t) - \lambda_i c_i(t)$~~

Then we have kinetics equation:

$$\frac{dn(t)}{dt} = s(t) + \frac{(1-\beta) k}{e} n(t) + \sum_i \lambda_i c_i(t)$$

$$\frac{dc_i(t)}{dt} = \beta_i \frac{k}{e} n(t) - \lambda_i c_i(t)$$

In order to obtain steady state solution, we begin by setting $s(t) \rightarrow s_0$. Then set derivatives to 0 to get $\vec{n}(t) \rightarrow \vec{n}_c$

$$0 = s_0 + \frac{(1-\beta) k}{e} n_c + \sum_i \lambda_i c_i$$

$$\& 0 = \beta_i \frac{k}{e} n_c - \lambda_i c_i \quad i = 1 - 6.$$

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Now,

$$\lambda_i c_i = \beta_i \frac{k_n}{\ell}$$

$$\Rightarrow c_i = \frac{\beta_i}{\lambda_i} \frac{k_n}{\ell}$$

Then Substituting in

$$0 = S_0 + \frac{(1-\beta)k-1}{\ell} n + \sum_i \lambda_i c_i$$

$$\Rightarrow S_0 + \frac{(1-\beta)k-1}{\ell} n + \sum_i \lambda_i c_i$$

$$= S_0 + \frac{(1-\beta)k-1}{\ell} n + \sum_i \frac{\lambda_i}{\beta_i} \frac{\beta_i k_n}{\ell}$$

$$= S_0 + \frac{(1-\beta)k-1}{\ell} n + \frac{k_n}{\ell} \underbrace{\sum_i \beta_i}_{\beta}$$

$$= S_0 + \frac{(1-\beta)k-1}{\ell} n + k_n \frac{k_n}{\ell} \beta$$

$$D = S_0 + \left(\frac{k-1}{\ell} \right) n$$

$$\Leftrightarrow n = \frac{S_0}{1-k}$$

Then, for situations where source is present,
 $n = \frac{1 + \omega}{1 - \kappa}$ returns the ans. Only for $\kappa < 1$.

i.e. system is sub critical.

If $\kappa = 1$, any value of n satisfies the equation
~~then~~

Moreover, this is same solⁿ as before - just for prompt ω .
Therefore, presence of delayed ω has no effect on the requirement for steady state ω distribution.

Reactivity formulation for kinetics equation.

Time dependent reactor behavior is sensitive to deviations of κ about 1. So we introduce reactivity: ρ .

$$\rho = \frac{\kappa - 1}{\kappa}.$$

When $\rho > 0 \rightarrow$ super critical

$\rho = 0 \rightarrow$ critical

$\rho < 0 \rightarrow$ sub critical.

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let $\lambda = \ell/k \rightarrow$ prompt generation time

using n & p , we rewrite kinetics equation:

$$\text{II} \quad \frac{d}{dt} n(t) = s(t) + \frac{(1-p)k-1}{\ell} n(t) + \sum_i \lambda_i c_i$$

~~$$\frac{d}{dt} n(t) = s(t) + \frac{k-1-k^{\beta}}{\ell} n(t) + \sum_i \lambda_i c_i$$~~

$$= s(t) + \frac{k-1-k^{\beta}}{\ell k} n(t) + \sum_i \lambda_i c_i$$

$$= s(t) + \frac{k-1}{k} - \frac{k^{\beta}}{k} \frac{n(t)}{n(t)} + \sum_i \lambda_i c_i(t)$$

$$\xrightarrow{\rho} \frac{k-1}{k} - \frac{k^{\beta}}{k} \xrightarrow{\lambda \rightarrow \ell/k}$$

$$\boxed{\frac{d}{dt} n(t) = s(t) + \frac{\rho - \beta}{\lambda} n(t) + \sum_i \lambda_i c_i(t)}$$

Similarly,

$$\frac{d}{dt} c_i(t) = \beta_i \frac{k}{\ell} n(t) - \lambda_i c_i(t) \quad i=1 \dots 6$$

$$\xrightarrow{\lambda \rightarrow \ell/k}$$

becomes

$$\boxed{\frac{d}{dt} n(t) = \frac{\beta_i}{\lambda} n(t) - \lambda_i c_i(t)}$$

for many reactor cases.

Moreover, we can rewrite steady state soln of kinetics equation:

$$0 = S_0 + \frac{k_1 n}{\lambda} = S_0 + \frac{k_1 n}{k \frac{\lambda}{\kappa}} = S_0 + \frac{P n}{\kappa}$$

or

$$n = \frac{S_0 \kappa}{P}$$

$$C_i = \left(\frac{\beta_i}{\lambda_i \kappa} \right) n$$

$\frac{n}{P}$ much
less

In many cases, $\frac{\lambda_i \kappa}{\beta_i} \ll C_i \Rightarrow$ under steady state condition, $C_i \gg n$.

Therefore, # of fission products in reactor is much larger than # of n .

Let's use one delayed group approximation:

To obtain this, we lump all 6 precursor groups into one

$$C(T) = \sum_{i=1}^6 C_i(T)$$

We also replace λ_i by average of λ_i , where

$$\bar{\lambda} = \frac{1}{P} \sum_{i=1}^6 \beta_i \frac{1}{\lambda_i}$$

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We can reduce the kinetics equation system to

$$\frac{d}{dt} n(t) = S(t) + \frac{(\rho - \rho)}{\lambda} n(t) + \lambda C(t)$$

$$\frac{d}{dt} C(t) = \frac{\beta}{\lambda} n(t) - \lambda C(t).$$

~~HWS Q's~~

Say $\rho = 0.1$ ~~5.1.2 from Textbook~~.

~~(a) Find β for $t=1000$~~

~~(b) find~~

Optional problem
for extra credit.

Say $\beta = 0.007$

$$\lambda = 0.5 \times 10^{-5} \text{ s}$$

$$\lambda = 0.08 \text{ s}^{-1}$$

$$S(t) = S_0 = 8000 @ 1s$$

$$C_0 = 0$$

$$n_0 = 0$$

PLOT $n(t)$ & $C(t)$ for $t = 0$ to ~~3600~~ 3600 s.

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Reactivity formulation.

$$\lambda = \frac{Q}{V}$$

$$\frac{dn(t)}{dt} = S(t) + \frac{(P-\beta)}{\lambda} n(t) + \sum \lambda_i c_i(t)$$

$$\frac{dc_i(t)}{dt} = \frac{\beta_i}{\lambda} n(t) - \lambda_i c_i(t).$$

Solution A Reactor Period.

In order to determine reactor period by setting source in kinetics eqn. to 0.

Then we look for a solution of seven equation set in the form

$$n(t) = A \exp(\omega t)$$

$$c_i(t) = B_i \exp(\omega t) \quad i=1 \dots 6.$$

A, B_i & ω are constants.

Inserting these expressions in kinetics equations ~~rearrange~~ by setting $S(t) \rightarrow 0$ returns:

$$\left\{ \begin{array}{l} \frac{d}{dt} A \exp(\omega t) = \frac{(P-\beta)}{\lambda} A \exp(\omega t) + \sum \lambda_i B_i \exp(\omega t) \\ \frac{d}{dt} B_i \exp(\omega t) = \frac{\beta_i}{\lambda} A \exp(\omega t) - \lambda_i B_i \exp(\omega t). \end{array} \right.$$

$$\left\{ \begin{array}{l} A \omega \exp(\omega t) = A \frac{P-\beta}{\lambda} \exp(\omega t) + \sum \exp(\omega t) \sum \lambda_i B_i \\ B_i \omega \exp(\omega t) = \frac{\beta_i}{\lambda} A \exp(\omega t) - \lambda_i B_i \exp(\omega t). \end{array} \right.$$

cancel

$$\left\{ \begin{array}{l} A \omega = \frac{P-\beta}{\lambda} A + \sum \lambda_i B_i \\ \omega B_i = \frac{\beta_i}{\lambda} A - \lambda_i B_i \end{array} \right.$$

Use 2nd eq. & write B_i in terms of A to get

$$\omega B_i + \lambda_i B_i = \frac{\rho_i}{\lambda} A$$

$$\therefore B_i (\omega + \lambda_i) = \frac{\rho_i}{\lambda} A$$

$$\therefore B_i = \frac{\rho_i A}{\lambda (\omega + \lambda_i)}$$

Now, substitute that in the 1st equation:

$$\omega A = \cancel{\rho - P} \frac{P - F}{\lambda} A + \sum_i \frac{\lambda_i B_i A}{\lambda (\omega + \lambda_i)}$$

$$\omega A = \cancel{\frac{P - F}{\lambda}} A + \cancel{\frac{A}{\lambda}} \sum_i \frac{\lambda_i B_i}{\omega + \lambda_i}$$

$$\therefore \omega = \cancel{\frac{P - F}{\lambda}} + \cancel{\frac{1}{\lambda}} \sum_i \frac{\lambda_i B_i}{\omega + \lambda_i}$$

Solve for P now,

$$P = \cancel{\frac{F}{\lambda}} + \cancel{\frac{1}{\lambda}} \sum_i \frac{\lambda_i B_i}{\omega + \lambda_i}$$

$$\therefore P = \cancel{\frac{F}{\lambda}} - \sum_i \frac{\lambda_i B_i}{\omega + \lambda_i} + \cancel{\frac{1}{\lambda}} \sum_i \frac{\lambda_i B_i}{\omega + \lambda_i}$$

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$$= \frac{\rho}{\lambda} + \frac{1}{\lambda} \sum_i \left(\frac{\lambda_i \beta_i}{\omega + \lambda_i} - \beta_i \right)$$

$$= \frac{\rho}{\lambda} + \frac{1}{\lambda} \sum_i \frac{\alpha_i \beta_i - \beta_i (\omega + \lambda_i)}{\omega + \lambda_i}$$

$$= \frac{\rho}{\lambda} + \frac{1}{\lambda} \sum_i \frac{-\beta_i \cancel{\omega}}{\omega + \lambda_i}$$

$$\Rightarrow \omega = \frac{\rho}{\lambda} - \sum_i \frac{\beta_i \cancel{\omega}}{\omega + \lambda_i}$$

$$\Rightarrow \omega \lambda = \rho - \sum_i \frac{\beta_i \omega}{\omega + \lambda_i}$$

$$\Rightarrow \boxed{\omega \left(\lambda + \sum_i \frac{\beta_i}{\omega + \lambda_i} \right) = \rho}$$

This is called m hoen equation. Since units of ω are commonly taken in terms of inverse hours.

In order to solve the above m hoen equation, we use graphical method.

Plot $f(\omega) = \omega \left(\lambda + \sum_i \frac{\beta_i}{\omega + \lambda_i} \right)$ on the vertical axis

ω on the horizontal axis.

then choose reactivity insertion ρ as I draw a horizontal line. The roots of m hoen equation are where that the

horizontal line for P crosses $f(\omega)$.

The solution looks like Fig 5.3 in textbook.

We observe that for any specific value of P , there are 7 solutions (because 7 equations are represented in the matrix equation),

The 7 solutions, $\omega_1 > \omega_2 > \dots > \omega_7$ regardless of whether the reactivity is positive or negative.

Accordingly the population takes the following form.

$$n(t) = \sum_{i=1}^7 A_i \exp(\omega_i t)$$

Figure 5.3 indicates that ω_1

Now, since ω_1 is the largest, other terms in the exponential sum decay away more rapidly yet yielding an asymptotic solution in the form

$$\boxed{n(t) \approx A_1 \exp(\omega_1 t)}$$

Now comparing to expected exponential sum $A_1 \exp(\omega_1 t)$ we see that

$$\text{reactor period } T = \frac{1}{\omega_1}$$

The condition $P = \beta$ defines prompt critical, where chain reaction is sustainable with delayed τ .

Reactivity can be measured in dollars where $\$ = \beta/\rho$.

For small reactivity, ~~$\omega_i \ll \lambda_i$~~ $\omega_i \ll \lambda_i$ for all i .

Further we see that when we plot ω vs. $f(\omega)$, we then eliminate ω from the denominator of the ~~first~~ whom equation & get

$$P = \left(\lambda + \sum_i \frac{\beta_i}{\lambda_i} \right) \omega_1$$

Then $\frac{P}{\lambda} = \frac{1}{\beta} \sum_i \frac{\beta_i}{\lambda_i}$ from previous definition of

$$\frac{1}{\lambda} = \frac{1}{\beta} \sum_i \frac{\beta_i}{\lambda_i}$$

we have $P = \left(\lambda + \frac{P}{\lambda} \right) \omega_1$

$$\Rightarrow \omega_1 = \frac{P}{\lambda + \frac{P}{\lambda}} \Rightarrow T = \frac{\lambda + \frac{P}{\lambda}}{P}$$

Generally $\lambda = \frac{\ell}{k} \ll \beta_i / \lambda$

Then $T \approx \beta / P$

for small reactivities, reactor period \rightarrow almost completely governed by delayed γ properties $\beta \gg \lambda$.

Above prompt critical, reactor period is very small.. Therefore ω_1 is very large.

2) $\omega_1 \gg d$; ~~(need to plot)~~ (Pys7.)

then inhom equation reduces to

$$\rho = \Lambda \omega_1 + \beta \quad \Rightarrow \quad \omega_1 = \frac{\rho - \beta}{\Lambda}$$

then

$$\boxed{T = \frac{\Lambda}{\beta - \beta}}$$

- independent of delayed γ half-lives.