Chapter 1 Problems:

[1.1] The following isotopes frequently appear in reactor cores. What are their chemical symbols and names
a. $^{90}_{38}??$, b. $^{91}_{40}??$ c. $^{137}_{55}??$, d. $^{157}_{64}??$, e. $^{178}_{72}??$, f. $^{137}_{93}??$, g. $^{241}_{95}??$

a. $^{90}_{38}??$, Sr, strontium
b. $^{91}_{40}??$, Zr, zirconium
c. $^{137}_{55}??$, Cs, cesium
d. $^{157}_{64}??$, Gd, gadolinium
e. $^{178}_{72}??$, Hf, hafnium
f. $^{137}_{93}??$, Np, neptunium
g. $^{241}_{95}??$ Am, americium

[1.2] There are several possible modes of disintegration for the unstable nucleus ${}^{27}_{13}Al$. Complete the following reactions: ${}^{27}_{13}Al \rightarrow ? + {}^{1}_{0}n$, ${}^{27}_{13}Al \rightarrow ? + {}^{1}_{1}p$, ${}^{27}_{13}Al \rightarrow ? + {}^{2}_{1}H$, ${}^{27}_{13}Al \rightarrow ? + {}^{4}_{2}He$

$$
\begin{aligned}
&{}^{27}_{13}Al \rightarrow {}^{26}_{13}Al + {}^{1}_{0}n\,, \\
&{}^{27}_{13}Al \rightarrow {}^{26}_{12}Mg + {}^{1}_{1}p\,, \\
&{}^{27}_{13}Al \rightarrow {}^{25}_{12}Mg + {}^{2}_{1}H\,, \\
&{}^{27}_{13}Al \rightarrow {}^{23}_{11}Na + {}^{4}_{2}He
\end{aligned}
$$

[1.3] Complete the following reactions ${}^{9}_{?}Be + {}^{4}_{2}He \rightarrow ? + {}^{1}_{1}H$, ${}^{60}_{?}Co \rightarrow ? + {}^{0}_{-1}e$, ${}^{7}_{3}Li + {}^{1}_{1}H \rightarrow ? + {}^{4}_{2}He$, ${}^{10}_{5}B + {}^{4}_{2}He \rightarrow ? + {}^{1}_{1}H$

${}^{9}_{4}Be + {}^{4}_{2}He \rightarrow {}^{12}_{5}B + {}^{1}_{1}H$, ${}^{60}_{27}Co \rightarrow {}^{60}_{28}Ni + {}^{0}_{-1}e$,

${}^{7}_{3}Li + {}^{1}_{1}H \rightarrow {}^{4}_{2}He + {}^{4}_{2}He$, ${}^{10}_{5}B + {}^{4}_{2}He \rightarrow {}^{12}_{6}C + {}^{1}_{1}H$

[1.4] What target isotope must be used for forming the compound nucleus $^{60}_{28}Ni$ if the incident projective is (a) an alpha particle, (b) a proton, or (c) a neutron.

(a) $^{56}_{26}Fe$ (b) $^{59}_{27}Co$ (c) $^{59}_{28}Ni$

[1.5] The average kinetic energy of a fission neutron is 2.0 MeV. Defining the kinetic energy as $E_{total} - m_0 c^2$, what is the percent error introduced into the kinetic energy from using Eq. (1.12) instead of Eq. (1.9)

Using Eq. (1.10), we obtain from Eq. (1.9):

$$K.E. = E_{total} - m_0 c^2 = \left[\frac{1}{\sqrt{1-(v/c)^2}} - 1 \right] m_0 c^2$$

From Eq. (1.12): $K.E.' = \frac{1}{2} m_0 v^2$

Thus the ratio of the two is $K.E.'/K.E. = \frac{1}{2} \left[\frac{1}{\sqrt{1-(v/c)^2}} - 1 \right]^{-1} (v/c)^2$

We must next determine the neutron velocity. Assuming that it is much less than the speed of light we may take $v = \sqrt{K.E.'\,2/m_0}$

Since 1 eV = 1.602x10^{-19} J. , and from Appendix E the rest mass of a neutron is m_0 =1.675x10^{-27} Kg , we have

$$v = \sqrt{2 \cdot K.E.'/m_0} = \sqrt{2 \cdot 2 \cdot 10^6 \cdot 1.602 \cdot 10^{-19} / 1.675 \cdot 10^{-27}} = \sqrt{3.826 \cdot 10^{14}} = 1.956 \cdot 10^7 \, m/s$$

$v/c = 1.956 \cdot 10^7 / 3.0 \cdot 10^8 = 0.062$ and therefore $(v/c)^2 = 0.00425$

Thus $\frac{1}{\sqrt{1-(v/c)^2}} - 1 = 0.002132$ and

$$K.E.'/K.E. = \frac{1}{2} [0.002132]^{-1} \, 0.00425 = 0.9967$$

or only about 0.3 % difference.

[1.6] Consider the following nuclear and chemical reactions:

a. A uranium-235 nucleus fissions as a result of being bombarded by a slow neutron. If the energy of fission is 200 MeV, approximately what fraction of the reactants mass is converted to energy?
b. A carbon-12 atom undergoes combustion following collision with an oxygen-16 molecule, forming carbon dioxide. If four eV or energy are released, approximately what fraction of the reactants mass is converted to energy?

a. Since one atomic mass unit is equivalent to 931.5 MeV. The energy equivalent of the reactants' is $236x931.5 = 219.8\ x10^3$ MeV. The fraction of energy converted is thus $200/219.8\ x10^3 = 0.91\ x10^{-3}$ or about 0.1%.

b. The energy equivalent of the reactants is $(12+2x16)x931.5 = 40.99x10^3$ MeV or $40.99x10^9$ eV Thus the fraction of energy converted is $4/40.99x10^9 = 0.097x10^{-9}$ or about 10^{-8} %.

[1.7] a. If plutonium-239 captures two neutrons followed by a beta decay, what isotope is produced. b. If plutonium-239 captures three neutrons, followed by two beta decays, what isotope is produced?

a. $^{239}_{94}Pu + ^{1}_{0}n \rightarrow ^{240}_{94}Pu + ^{1}_{0}n \rightarrow ^{241}_{94}Pu \xrightarrow{\beta} ^{241}_{95}Am$ Thus americium-240

b. $^{239}_{94}Pu + ^{1}_{0}n \rightarrow ^{240}_{94}Pu$ $^{240}_{94}Pu + ^{1}_{0}n \rightarrow ^{241}_{94}Pu$,

$$^{241}_{94}Pu + ^{1}_{0}n \rightarrow ^{242}_{94}Pu \xrightarrow{\beta} ^{242}_{95}Am \xrightarrow{\beta} ^{242}_{96}Cm$$

Thus curium-242

[1.8] To first approximation a nucleus may be considered to be a sphere with the radius in cm given by $R = 1.25\cdot 10^{-13} A^{1/3}$ cm, where A is the atomic mass number. What are the radii of a. hydrogen, b. carbon-12, c. xenon-140, d. uranium-238?

a. hydrogen, $R = 1.25\cdot 10^{-13} 1^{1/3} = 1.25\cdot 10^{-13}$ cm
b. carbon-12, $R = 1.25\cdot 10^{-13} 12^{1/3} = 2.86\cdot 10^{-13}$ cm
c. xenon-140, $R = 1.25\cdot 10^{-13} 140^{1/3} = 6.49\cdot 10^{-13}$ cm
d. uranium-238 $R = 1.25\cdot 10^{-13} 238^{1/3} = 7.75\cdot 10^{-13}$ cm

[1.9] A reactor operates at a power of 10^3 MW(t) for one year. Calculate the power from decay heat

a. one day following shutdown, 24x60x60 = 86,400 s
b. one month following shutdown, 30.5x = 2.64×10^6 s
c. one year following shutdown 365.25x = 31.6×10^6 s

Equation (1.27??) $P_d(t) = 0.0622 P_o \left[t^{-0.2} - (t_o + t)^{-0.2} \right]$ t in sec.

a. $P_d(1day) = 0.0622 \cdot 10^3 \left[86400^{-0.2} - \left(31.6 \cdot 10^6 + 86400\right)^{-0.2} \right] = 4.44\text{MW}$

b. $P_d(1month) = 0.0622 \cdot 10^3 \left[\left(2.64 \cdot 10^6\right)^{-0.2} - \left(31.6 \cdot 10^6 + 2.64 \cdot 10^6\right)^{-0.2} \right] = 1.30\text{MW}$

c. $P_d(1year) = 0.0622 \cdot 10^3 \left[\left(31.6 \cdot 10^6\right)^{-0.2} - \left(31.6 \cdot 10^6 + 31.6 \cdot 10^6\right)^{-0.2} \right] = 0.255\text{MW}$

[1.10] In Eq. (1.28) the uranium-239 and neptunium-239 both undergo beta decay with half-lives of 23.4 m and 2.36 d respectively. If neutron bombardment in a reactor causes uranium-239 to be produced at a constant rate, how long will it take plutonium-239 to reach a. ½ of its saturation activity, b. 90% of its saturation activity, c. 99% of its saturation activity? (assume that plutonium-239 undergoes no further reactions.)

Since the half life of uranium-239 is very small compared to that of neptunium-239, to first approximation we can assume that the uranium decay is instantaneous. The decay rate of neptunium is $\lambda = 0.693/t_{1/2} = 0.693/2.36d = 0.294d^{-1}$ which we may use directly in Eq. (1.42)

a. $[1-\exp(-\lambda t] = 0.5$ or $t = -(1/\lambda)\ln(0.5) = -(1/0.294)\ln(0.5) = 2.36d$ (obviously)

b. $[1-\exp(-\lambda t] = 0.90$ or $t = -(1/\lambda)\ln(0.10) = -(1/0.294)\ln(0.10) = 7.83d$

c. $[1-\exp(-\lambda t] = 0.99$ or $t = -(1/\lambda)\ln(0.01) = -(1/0.294)\ln(0.01) = 15.7d$

[1.11] Uranium-238 has a half-life of 4.51x10^9 yr, while the half-life of uranium-235 is only 7.13x10^8 yr. Thus since the earth was formed 4.5 billion years ago, this isotopic abundance of uranium-235 has been steadily decreasing.

a. What was the enrichment of uranium when the earth was formed?
b. How long ago was the enrichment 4%?

At any time we may define atom enrichment as $e(t) = N^{25}(t)/\left[N^{25}(t) + N^{28}(t)\right]$ or

$$e(t) = 1/\left[1 + N^{28}(t)/N^{25}(t)\right]$$

Conversely the ratio of isotopes is

$$N^{28}(t)/N^{25}(t) = \left[\frac{1}{e(t)} - 1\right]. \qquad *$$

We may also use the decay properties to write this equation as

$$N^{28}(0)\exp(-\lambda^{28}t)/N^{25}(0)\exp(-\lambda^{25}t) = \left[\frac{1}{e(t)} - 1\right] \text{ or}$$

$$N^{28}(0)/N^{25}(0) = \left[\frac{1}{e(t)} - 1\right]\exp(-(\lambda^{25} - \lambda^{28})t)$$

Taking time in billions of years, we have

$\lambda^{28} = 0.693/t_{1/2} = 0.693/4.51 = 0.153$ and $\lambda^{25} = 0.693/t_{1/2} = 0.693/0.713 = 0.972$

Hence

$$N^{28}(0)/N^{25}(0) = \left[\frac{1}{e(t)} - 1\right]\exp(-0.819t) \qquad **$$

a. To find the isotope ratio at the time the earth was formed, we just take t = 4.5: and use present day enrichment as 0.007 (0.7%):

$$N^{28}(0)/N^{25}(0) = \left[\frac{1}{0.007} - 1\right]\exp(-0.819 \cdot 4.5) = 3.56$$

Thus the enrichment at the time the earth was formed was

$e(0) = 1/\left[1 + N^{28}(0)/N^{25}(0)\right] = 1/(1+3.56) = 0.219$ or about 22%!

b. Suppose we now let t =0 correspond not to the earths formation but to the time when the enrichment was 4%. Combining Eqs. * and ** we have:

$\left[\frac{1}{e(0)} - 1\right] = \left[\frac{1}{e(t)} - 1\right]\exp(-0.819t)$ or substituting for the enrichments then & now:

$\left[\frac{1}{0.04} - 1\right] = \left[\frac{1}{0.007} - 1\right]\exp(-0.819t)$ we may solve to t to obtain

t = 2.17 billion years ago

[1.12] How many Ci of radium-226 are needed in the reaction given in Eqs. (1.31) and (1.32) to produce 10^6 neutrons/s?

For each alpha decay we may assume that one neutron is produced by the second reaction thus we need 10^6 decays/s or (10^6 decays/s)/ ($3.7x10^{10}$decays/s/Ci) = $27x10^{-6}$ Ci

[1.13] Suppose that a specimen is placed in a reactor, and neutron bombardment causes a radioisotope to be produced at a rate of 2 x 10^{12} nuclei/s. The radioisotope has a half-life of two weeks. How long should the specimen be irradiated to produce 25 Ci of the radioisotope?

To produce 35 Ci , we must have $\lambda N(t) = 25 \cdot 3.7 \cdot 10^{10} = 92.5 \cdot 10^{10}$ decays/s

Thus from Eq. (1.42) $92.5 \cdot 10^{10} = 2 \cdot 10^{12}[1 - \exp(-\frac{0.693}{2.0}t)]$ Solving for time (in weeks) we have $0.4625 = [1 - \exp(-0.3465t)]$: $t = -\ln(0.5375)/0.3465 = 1.79$ weeks or 12.54 days.

[1.14] The decay constant for the radioactive antimony isotope $^{124}_{51}Sb$ is $1.33x10^{-7}$ s^{-1}.

a. What is its half-life in years?
b. How many years would it take for it to decay to 0.01% of its initial value?
c. If it were produced at a constant rate, how many years would it take to reach 95% of its saturation value?

a. $t_{1/2} = 0.693/\lambda = 0.693/1.33\cdot10^{-7}s^{-1} = 5.2\cdot10^{6}s$
60x60x24x365.25 s = $31.56x10^{6}$s= 1 year Thus
$t_{1/2} = 5.2\cdot10^{6}s/(31.6\cdot10^{6}s/yr) = 0.164yrs$

b First find the decay constant in yr^{-1}: $\lambda = 1.33\cdot10^{-7}s^{-1}\cdot31.6\cdot10^{6}s/yr = 4.20/yr$. Now with time in years: $0.01\cdot10^{-2} = \exp(-4.2t)$ or $t = -\ln(10^{-4})/4.2 = 2.19yrs$

c. From Eq. (1.42): $0.95 = 1 - \exp(-4.2t)$ Thus
$t = -\ln(0.05)/4.2 = 0.713yrs = 8.56months$

[1.15] Approximately what mass of cobalt-60, which has a half-life of 5.26 yr, will have the same number of curies as 10 gm of stronsium-90, which has a half-life of 28.8 yr?

Let c=>cobalt & s=>strontium, for this problem. Then $\lambda_c N_c = \lambda_s N_s \propto$ # of Ci
$N = V \cdot N'''$, where V is volume and N''' is atom density

$N''' = \rho N_o / A$ Thus $\lambda_c (V_c \rho_c) N_o / A_c = \lambda_s (V_s \rho_s) N_o / A_s$,

But mass is $m = \rho V$, hence $\lambda_c m_c N_o / A_c = \lambda_s m_s N_o / A_s$ or

$$m_c = \frac{\lambda_s A_c}{\lambda_c A_s} m_s = \frac{t_{1/2,c} A_c}{t_{1/2,s} A_s} m_s = \frac{5.26 \cdot 60}{28.8 \cdot 90} \cdot 10 = 1.22 \text{ g}$$

[1.16] Ninety percent of an isotope decays in three hours

a. What fraction decays in six hours?
b. What is the half-life?
c. If the isotope is produced in a reactor at the rate of 10^9 nuclei per hour, after a long time how many nuclei will be present in the reactor?

First find the decay constant: $0.10 = \exp(-\lambda \cdot 3)$ Therefore $\lambda = -\ln(0.1)/3 = 0.768\,hr^{-1}$

a. The fraction left is $\exp(-0768 \cdot 6) = 0.00997$ Thus the fraction that has decayed is 1.0-0.00997=0.99

b. $t_{1/2} = 0.693/\lambda = 0.693/(0.768\,hr^{-1}) = 0.903hr$

c. Taking $t = \infty$ in Eq. (1.42) we have

$N(\infty) = A_o / \lambda = 10^9 hr^{-1} /(0.903hr^{-1}) = 1.107 \cdot 10^9 nuclei$

[1.17] A fission product "A" with a half-life of two weeks is produced at the rate of $5.0\text{x}10^8$ nuclei/sec in a reactor.

a. What is the saturation activity in disintegrations /sec.?
b. What is the saturation activity in Curies?
c. How long after the startup of the reactor will 90 percent of the saturation activity be reached?
d. If the fission product undergoes decay A -> B -> C, where B also has a 2 week half-life, what will be the activity of B after two weeks.

First find the decay rate $\lambda = 0.693 / t_{1/2} = 0.693/(2.0wks) = 0.3465wk^{-1} =$

a. At saturation (see Eq. (1.42) production = disintegration. Thus $5.0\text{x}10^8$ nuclei/sec

b. $5.0\text{x}10^8/3.7\text{x}10^{10} = 1.35\text{x}10^{-2}$ Ci

c. From Eq. (1.42) : $0.90 = 1 - \exp(-\lambda t)$. Or
$t = -\ln(0.10)/\lambda = 2.302/0.3465 = 6.64wks$

d. The differential equation governing A is Eq. (1.42) The differential equation governing B is

$$\frac{d}{dt}N_B(t) = \lambda N_A(t) - \lambda N_B(t)$$

Combining this with Eq. (1.42) for $N_A(t)$ yields

$$\frac{d}{dt}N_B(t) = A_0[1-\exp(-\lambda t)] - \lambda N_B(t)$$

We may solve this equation using the same integrating factor technique applied in Eqs. (1.39) to (1.41). Multiplying by $\exp(\lambda t)$ the preceding equation becomes

$$\frac{d}{dt}[N_B(t)\exp(\lambda t)] = A_0[\exp(\lambda t) - 1]$$

Integrating between 0 and t, we have

$$N_B(t)\exp(\lambda t) - N_B(0)\exp(\lambda 0) = A_0[\frac{1}{\lambda}[\exp(\lambda t) - \exp(\lambda 0)] - t]$$

Since $N_B(0) = 0$ and $\exp(\lambda 0) = 1$, we obtain

$$\lambda N_B(t) = A_0[1-(1+\lambda t)\exp(-\lambda t)]$$

After 2 wks $\lambda t = \frac{0.693}{2}\cdot 2 = 0.693$ Thus

$$\lambda N_B(2) = 5.0\cdot 10^8\ [1-(1+.693)\exp(-.693)] = 0.7675\cdot 10^8 \text{ disintigration/s}$$

or $0.7675\text{x}10^8/3.7\text{x}10^{10} = 2.074\ \text{x}10^{-3}$ Ci

[1.18] Suppose the radioactive cobalt and strontium sources in problem [15] are allowed to decay for 10 years. It is found that after 10 years 1.0 Ci of cobalt-60 remains. How many Ci of stronsium-90 will remain?

Let c=>cobalt & s=>strontium, for this problem. Then initially at t = 0
$\lambda_c N_c(0) = \lambda_s N_s(0) \propto$ # of Ci With t = 10 yr: $1.0\,Ci = \lambda_c N_c(0)\exp(-\lambda_c t)$ and therefore $\lambda_c N_c(0) = \exp(\lambda_c t)\cdot 1.0\,Ci$

The number of Ci of stonsium-90 after 10 years will be $\lambda_s N_s(0)\exp(-\lambda_s t)$. But since $\lambda_c N_c(0) = \lambda_s N_s(0)$, the number of Ci will be

$$\exp[(\lambda_c - \lambda_s)t]\cdot 1.0\,Ci = \exp\left[0.693\left(1/t_{1/2c} - 1/t_{1/2s}\right)t\right]\cdot 1.0\,Ci$$

$$= \exp\left[0.693\left(1/5.26 - 1/28.8\right)10\right]\cdot 1.0\,Ci = 4.73Ci$$

[1.19] Polonium-210 decays to lead-206 by emitting an alpha particle with a half-life of 138 days, and an energy of 5.305 MeV

a. How many Curies are there in one gm of pure polonium?
b. How many watts of heat are produced by one gm of polonium?

a. The number of atoms in one gram of an isotope is N_0 / A and therefore the number of Ci in one gram is $(\lambda N_0 / A)/(3.7 \cdot 10^{10}) = (0.693 N_0 / t_{1/2} A)/(3.7 \cdot 10^{10})$ where the units of λ must be s^{-1}. With $t_{1/2} = 138 d \cdot 86400 s / d = 11.9 \cdot 10^{6} s$ Thus $(0.693 \cdot 6.023 \cdot 10^{23} / 11.9 \cdot 10^{6} \cdot 210)/(3.7 \cdot 10^{10}) = 4.51 \cdot 10^{3} Ci$
b. Since there are $(4.51 \cdot 10^{3} Ci) \cdot (3.7 \cdot 10^{10} \text{disintigrations} / s / Ci) = 16.7 \cdot 10^{13} \text{disintigrations} / s$ each Each alpha dissipates 5.305 MeV, and since 1.0 Mev = 1.6x10^{-13} J, we have 16.7x10^{13}x5.305x1.6x10^{-13} = 142 J/s = 142 W. (It would melt!)

[1.20] Consider the fission product chain $A \xrightarrow{\beta} B \xrightarrow{\beta} C$ with decay constants λ_A and λ_B A reactor is started up at $t = 0$ and produces fission product A at a rate of A_o thereafter: Assuming that B and C are not produced directly from fission:

a. Find $N_A(t)$ and $N_B(t)$
b. What are $N_A(\infty)$ and $N_B(\infty)$?

a. $N_A(t)$ is determined from Eq. (1.42):

$$N_A(t) = \frac{A_0}{\lambda_A}[1 - \exp(-\lambda_A t)]$$

Since each decay of isotope A creates and isotope B, the concentration of isotope B is governed by Eq. (1.45):

$$\frac{d}{dt}N_B(t) = \lambda_A N_A(t) - \lambda_B N_B(t)$$

Combining this with Eq. (1.42) for $N_A(t)$ yields

$$\frac{d}{dt}N_B(t) = A_0[1 - \exp(-\lambda_A t)] - \lambda_B N_B(t)$$

We may solve this equation using the same integrating factor technique applied in Eqs. (1.39) to (1.41). Multiplying by $\exp(\lambda_B t)$ the preceding equation becomes

$$\frac{d}{dt}[N_B(t)\exp(\lambda_B t)] = A_0[1 - \exp(-\lambda_A t)]\exp(\lambda_B t)$$

Integrating between 0 and t, we have

$$N_B(t)\exp(\lambda_B t) - N_B(0)\exp(\lambda_B 0) = A_0\left\{\frac{1}{\lambda_B}[\exp(\lambda_B t) - 1] - \frac{1}{\lambda_B - \lambda_A}[\exp((\lambda_B - \lambda_A)t) - 1]\right\}$$

Since $N_B(0) = 0$ and $\exp(\lambda_B 0) = 1$, we obtain

$$N_B(t) = A_0\left\{\frac{1}{\lambda_B}[1 - \exp(-\lambda_B t)] - \frac{1}{\lambda_B - \lambda_A}[\exp(-\lambda_A t) - \exp(-\lambda_B t)]\right\}$$

b. Setting $t = \infty$ In these equations yields

$$N_A(\infty) = \frac{A_0}{\lambda_A} \qquad N_B(\infty) = \frac{A_0}{\lambda_B}$$

or

$$\lambda_A N_A(\infty) = \lambda_B N_B(\infty) = A_0$$

Chapter 2 Problems:

[2.1] Neutrons impinge on a material with a cross section of $\Sigma = 0.8\ cm^{-1}$. How thick must the material be if no more than 5.0% of the neutrons are to penetrate the material without making a collision? What fraction of the neutrons make their first collision within the first 2.0 cm. of the material?

From Eq. (2.4) $0.05 = 1.0\exp(-0.8x)$ or $x = -\ln(0.05)/0.8 = 3.74cm$
The fraction colliding is just 1- fraction penetrating without making a collision. Thus
$1 - \exp(-0.8 \cdot 2.0) = 0.798$

[2.2] The uncollided flux at a distance *r* from a point source emitting is given by Eq. (2.9):

a. If you are 1 m away from a very small 1 Curie source of neutrons. What is the flux of neutrons in n/cm^2/s , neglecting scattering and absorption in air.
b. If a shield is placed between you and the source, what absorption cross section would be required to reduce the flux by a factor of 10?
c. Suppose the shield made of the material specified in part b. is only 0.5 m thick. How far must you be from the source, for the flux to be reduced by the same amount as in part b.

a. 1 Ci = 3.7 x 10^{10} n/s; 1m= 100 cm

$$I(100) = \frac{\exp(-0)}{4\pi 100^2} 3.7\cdot 10^{10} = 294\cdot 10^3 \, n/cm^2 s$$

b. $\exp(-\Sigma\cdot 100) = 1/10$ Therefore $\Sigma = \frac{1}{100}\ln(10) = 0.023\,cm^{-1}$

c. $\frac{\exp(-\Sigma\cdot 50)}{4\pi R^2} S = \frac{\exp(-\Sigma\cdot 100)}{4\pi 100^2} S$ Hence: $R^2 = 10^4 \exp(\Sigma\cdot 50)$

$R = 179\,cm = 1.79\,m$

[2.3] A material has a neutron cross section of 3.50×10^{-24} cm^2/nuclei, and contains 4.20×10^{23} nuclei/cm^3:

a. What is the macroscopic cross section?
b. What is the mean free path?
c. If neutrons impinge perpendicularly on a slab of the material, which is 3.0 cm thick, what fraction of them will penetrate the slab without making a collision?
d. What fraction of the neutrons in part c will collide in the slab before penetrating a distance of 1.5 cm?

$$\sigma = 3.5 \cdot 10^{-24}\ cm^2 \qquad N''' = 4.2 \cdot 10^{23} / cm^3$$

a. $\sum = \sigma N''' = 3.5 \cdot 10^{-24} \cdot 4.2 \cdot 10^{23} = 1.47 cm^{-1}$

b. $\lambda = 1/\Sigma = 0.68\ cm$

c. $\exp(-\Sigma \cdot 3) = \exp(-1.47 \cdot 3) = 0.0122$

d. $1 - \exp(-\Sigma \cdot 1.5) = 1 - \exp(-1.47 \cdot 1.5) = 0.890$

[2.4] A boiling water reactor operates at 1000 psi. At that pressure the density of water and of steam are respectively 0.74 gm/cm^3 and 0.036 gm/cm^3. The microscopic cross section of *H* and *O* 2200 cm/s are 38 b and 4.2 *x* 10^{-5} b ,

a. What is the macroscopic total cross section of the water?
b. What is the macroscopic total cross section of the steam?
c. If, on average, 40% of the volume is occupied by steam then what is the macroscopic total cross section of the steam-water mixture?
d. What is the macroscopic total cross section of water under atmospheric conditions at room temperature?

. $\sigma_{H_2O} = 2\sigma_H + \sigma_O = 2\cdot 38\cdot 10^{-24} + 4.2\cdot 10^{-29} = 76\cdot 10^{-24} cm^2$

$$\Sigma \ = N'''\sigma_{H_2O} = \frac{\rho N_o}{A}\sigma_{H_2O} = \rho\left(\frac{N_o}{A}\sigma_{H_2O}\right)$$

$$\left(\frac{N_o}{A}\sigma_{H_2O}\right) = \frac{0.6023\cdot 10^{24}}{18}\cdot 76\cdot 10^{-24} = 2.54 cm^2 / g$$

a. Water $\rho_w = 0.74 g / cm^3$ $\qquad \Sigma_w = \rho_w\left(\frac{N_o}{A}\sigma_{H_2O}\right) = 0.74\cdot 2.54 = 1.88 cm^{-1}$

b. Steam $\rho_w = 0.036 g / cm^3$ $\qquad \Sigma_s = \rho_s\left(\frac{N_o}{A}\sigma_{H_2O}\right) = 0.036\cdot 2.54 = 0.0914 cm^{-1}$

[2.5] Determine:

a. The fraction of fission neutrons born with energy of less than 0.1 MeV.
b. The fraction of fission neutrons born with energy greater than 10 MeV.

Although is possible to work this problem analytically, it is simpler just to perform the integrations numerically. Thus

a. $$\int_0^{0.1} \chi(E)dE = \int_0^{0.1} 0.453\exp(-1.036E)\sinh(\sqrt{2.29E})dE = 0.014$$

and

b. $$\int_{10}^{\infty} \chi(E)dE == 1 - \int_0^{10} \chi(E)dE$$
$$= 1 - \int_0^{10} 0.453\exp(-1.036E)\sinh(\sqrt{2.29E})dE = 1 - .9999 \approx 0$$

[2.6] Neutrons are distributed in the Maxwell-Boltzmann distribution given by Eq. (2.34):

a. Verify Eq. (2.35),
b. b. Verify Eq. (2.33),
c. c. Determine the most probably neutron energy.

a. $\int_0^\infty M(E)dE = \frac{2\pi}{(\pi kT)^{3/2}} \int_0^\infty E^{1/2} \exp(-E/kT)dE$

let $x^2 = E/kT$ and hence $E = kTx^2$, $dE = 2kTxdx$ and $\sqrt{E} = \sqrt{kT}x$. The integral then becomes $\int_0^\infty M(E)dE = \frac{2\pi}{(\pi kT)^{3/2}} \cdot 2(kT)^{3/2} \int_0^\infty x^2 \exp(-x^2)dx$. However from Appendix E we have $\int_0^\infty x^2 \exp(-x^2)dx = \sqrt{\pi}/4$ Thus $\int_0^\infty M(E)dE = 1$

b. $\int_0^\infty EM(E)dE = \frac{2\pi}{(\pi kT)^{3/2}} \int_0^\infty E^{3/2} \exp(-E/kT)dE$ Again take therefore $x^2 = E/kT$.

We have $\int_0^\infty EM(E)dE = \frac{2\pi}{(\pi kT)^{3/2}} \cdot 2(kT)^{5/2} \int_0^\infty x^4 \exp(-x^2)dx$

Appendix E we have $\int_0^\infty x^4 \exp(-x^2)dx = 3\sqrt{\pi}/8$ Thus $\int_0^\infty EM(E)dE = \frac{3}{2} kT$

c. The most probable value occurs when $M(E)$ is a maximum. Thus take

$$\frac{d}{dE} M(E) = \frac{2\pi}{(\pi kT)^{3/2}} \frac{d}{dE} E^{1/2} \exp(-E/kT) = 0$$

$$\frac{d}{dE} E^{1/2} \exp(-E/kT) = \frac{1}{2} E^{-1/2} \exp(-E/kT) - \frac{E^{1/2}}{kT} \exp(-E/kT) = 0$$

Canceling the exponentials, we obtain $E = \frac{1}{2} kT$.

[2.7] How may parts per million of boron must be dissolved in water at room temperature to double its absorption cross section for thermal neutrons?

We have $N^B\sigma_a^B + N^{H_2O}\sigma_a^{H_2O} = 2\cdot N^{H_2O}\sigma_a^{H_2O}$ or $N^B / N^{H_2O} = \sigma_a^{H_2O} / \sigma_a^B$ Using the absoption cross section data from Appendix E, we have $N^B / N^{H_2O} = 0.2948/767 = 384\cdot 10^{-6}$ This would be 384 ppma, or parts per million by atom. However, ppm is normally measured as a mass ratio. Thus since boron has only 10.85 atomic mass units while water has 18. the ratio of masses will be $\frac{10.82}{18}384\cdot 10^{-6} = 231\cdot 10^{-6}$ or 231 ppm

[2.8] What is the total macroscopic thermal cross section of uranium dioxide (UO_2) that has been enriched to 4%? Assume $\sigma^{25} = 607.5$ b , $\sigma^{28} = 11.8$ b , $\sigma^{O} = 3.8$ b , and that UO_2 has a density of 10.5gm/cm^3.

Use Eq. (2.26) to calculate the composite microscopic cross section of the uranium:

$$\sigma^U = \tilde{e}\sigma^{25} + (1-\tilde{e})\sigma^{28} = 0.04 \cdot 607.5 + 0.96 \cdot 11.8 = 17.7b$$

The microscopic cross section of UO2 is thus $\sigma^{UO_2} = \sigma^U + 2\sigma^O = 17.7 + 2 \cdot 3.8 = 25.3b$

The macroscopic cross section is

$\Sigma^{UO_2} = \dfrac{\rho N_o}{A}\sigma^{UO_2} = \dfrac{10.5 \cdot 0.6023 \cdot 10^{24}}{238 + 2 \cdot 16} 25.3 \cdot 10^{-24} = 0.593\, cm^{-1}$ where we have rounded the molecular weight of the enriched uranium to 238.

[2.9] In the Breit-Wigner formula for the capture cross section show that Γ is equal to the width of the resonance at half height. What, if any, assumptions must you make to obtain this result?

The formula is given by Eq. (2.41). We assume that the $(E_r / E)^{1/2}$ changes little over the width of the resonance, and therefore we set it equal to one. The formula then becomes

$$\sigma_\gamma(E) \approx \sigma_o \frac{\Gamma_\gamma}{\Gamma} \frac{1}{1 + 4(E - E_r)^2 / \Gamma^2}$$

The peak of the resonance occurs at $E = E_r$ and therefore $\sigma_\gamma(E_r) = \sigma_o \dfrac{\Gamma_\gamma}{\Gamma}$. The half height of the resonance thus occurs at the energies for which $\dfrac{1}{2}\sigma_o \dfrac{\Gamma_\gamma}{\Gamma} = \sigma_o \dfrac{\Gamma_\gamma}{\Gamma} \dfrac{1}{1 + 4(E - E_r)^2 / \Gamma^2}$ and consequently $(E - E_r)^2 = \Gamma^2 / 4$. Thus the Half heights occur at $E = E_r \pm \Gamma / 2$, giving a width at half height of Γ.

[2.10] Verify Eqs. (2.46) and (2.56)

First to obtain Eq. (2.46) we solve Eq. (2.44) for V: $V = (v - v')/A$ and plug into Eq. (2.45): $v^2 - v'^2 = \frac{1}{A}(v - v')^2$. Let $x = v'/v$. We may then write $1 - x^2 = \frac{1}{A}(1-x)^2$ or $(A+1)x^2 - 2x - (A-1) = 0$. Solving the quadratic equation:
$x = \frac{2 \pm \sqrt{4 + 4(A+1)(A-1)}}{2(A+1)} = \frac{1 \pm A}{A+1}$. We must take the minus sign. Otherwise V=0.

Thus $E'/E = (v'/v)^2 = x^2 = \frac{(1-A)^2}{(A+1)^2} = \frac{(A-1)^2}{(A+1)^2}$

For Eq. (2.56) Let $x = E'/E$ And therefore $dx = dE'/E$ The integral becomes
$\xi = \frac{1}{1-\alpha}\int_\alpha^1 \ln(1/x)dx = -\frac{1}{1-\alpha}\int_\alpha^1 \ln(x)dx$ From The integral table in appendix E we have
$\int_\alpha^1 \ln(x)dx = \left[x\ln x - x\right]_\alpha^1 = -1 - \alpha\ln\alpha + \alpha$ Thus $\xi = 1 + \frac{1}{1-\alpha}\alpha\ln\alpha$

[2.11] Boron is frequently used as a material to shield against thermal neutrons. Using the data in Appendix E Estimate the thickness of boron required to reduce the intensity of a neutron beam by a factors 100, 1,000, 10,000 and 100,000.

Since the scattering cross section of boron is only 4.27 b while the absorption cross section is 767 barns. We will assume that it is a pure absorber. Its density is 2.45 gm/cm^3. Thus $\Sigma = \frac{\rho N_o}{A}\sigma = \frac{2.45 \cdot 0.6023 \cdot 10^{24}}{10.82} \cdot 767 \cdot 10^{-24} = 105\ cm^{-1}$. Let *a* be the attenuation factor. Thus $1/a = \exp(-\Sigma x)$ or $x = -\ln(1/a)/\Sigma = \ln(a)/\Sigma$. Thus

$$x == \ln(a)/\Sigma = \ln(100)/105 = 0.044cm$$
$$x == \ln(a)/\Sigma = \ln(1000)/105 = 0.065cm$$
$$x == \ln(a)/\Sigma = \ln(10000)/105 = 0.088cm$$
$$x == \ln(a)/\Sigma = \ln(100000)/105 = 0.110cm$$

[2.12] A 5.0 cm thick layer of purely absorbing material is found to absorb 99.90 percent of a neutron beam. The material is known to have a density of $4.0x10^{22}$ nuclei/cm^3. Determine:

a. the macroscopic cross section,
b. the mean free path,
c. the microscopic cross section.
d. Is the cross section as big as a barn?

a. From Eqs. (2.4) & (2.5): $0.001 = \exp(-\Sigma \cdot 5.0)$. Thus the macroscopic cross section is $\Sigma = \ln(0.001)/5.0 = 1.38 cm^{-1}$.

b From Eq. (2.7) the mean free path is $\lambda = 1/\Sigma = 1/1.38$ = 0.724 cm

c $\sigma = \Sigma / N = 1.38/(4.0 \cdot 10^{22}) = 0.345 \cdot 10^{-22} cm^2$

d Since one barn = 10^{-24} cm^2 $\sigma = 0.345 \cdot 10^{-22} cm^2 \cdot 10^{24} b/cm^2 = 34.5b$ Definitely bigger than a barn.

[2.13] Equal volumes of graphite and iron are mixed together. Fifteen percent of the volume of the resulting mixture is occupied by air pockets. Find the total macroscopic cross section given the following data σ_C = 4.75 b , σ_{Fe} = 10.9 b, ρ_c = 1.6 gm/cm^3, ρ_{Fe} = 7.7 gm/cm^3. Is it reasonable to neglect the cross section of air? Why?

The density of air is too small to contribute significantly to the cross section. Therefore

$$\frac{V_{Fe}+V_c}{V}=0.85 \quad \text{but } V_{Fe}=V_c \quad \text{Thus } \frac{V_{Fe}}{V}=\frac{V_c}{V}=0.425$$

$$\Sigma \ =\frac{V_{Fe}}{V}\frac{\rho_{Fe}N_o}{A_{Fe}}\sigma_{Fe}+\frac{V_C}{V}\frac{\rho_C N_o}{A_C}\sigma_C$$

$$\Sigma=.425\frac{7.87*0.6023\cdot 10^{24}}{55.85}\cdot 10.9\cdot 10^{-24}+.425\frac{1.6*0.6023\cdot 10^{24}}{12.01}\cdot 4.75\cdot 10^{-24}=.555cm^{-1}$$

[2.14] Neutrons scatter elastically at 1.0 MeV. After one scattering collision, determine fraction of the neutrons will have energy of less than 0.5 MeV if they scatters from

a. hydrogen,
b. deuterium,
c. carbon-12,
d. uranium-238.

Let $E = 1.0 MeV \qquad E_1 = 0.9 MeV \quad \frac{E_1}{E} = 0.9$

$$fr = \begin{cases} \int_{\alpha E}^{E_1} \frac{1}{(1-\alpha)E} dE' = \frac{E_1 - \alpha E}{1-\alpha} = \frac{1}{1-\alpha}\left(\frac{E_1}{E} - \alpha\right) & if \quad \alpha < \frac{E_1}{E} \\ 0 \quad if \quad \alpha \geq \frac{E_1}{E} \end{cases}$$

$$\alpha = \left(\frac{A-1}{A+1}\right)^2$$

a. hydrogen $\quad \alpha = 0 \qquad fr = \frac{1}{1}(0.9 - 0.0) = 0.9$

b. deuterium $\quad \alpha = 0.111 \qquad fr = \frac{1}{0.889}(0.9 - 0.111) = 0.886$

c. carbon-12 $\quad \alpha = 0.716 \qquad fr = \frac{1}{0.284}(0.9 - .716) = 0.648$

d. uraniium-238 $\quad \alpha = 0.983 \qquad fr = 0 \qquad$ since $\alpha > E_1 / E$

[2.15] What is the *minimum* number of elastic scattering collisions required to slow a neutron down from 1.0 MeV to 1.0 eV in

a. deuterium,
b. carbon-12,
c. iron-56, and
d. uranium-238?

The minimum number of collision will result if the neutron looses the maximum amount of energy with each collision; That is From Eq. (2.46) $E = \alpha E'$. Thus we have $1.0 eV = \alpha^N 10^6 eV$ where N is the minimum number of collisions. Solving we have $N = -\ln(10^6)/\ln(\alpha) = -13.8/\ln(\alpha)$. Thus with $\alpha = (A-1)^2/(A+1)^2$

a.	deuterium,	$\alpha =$	0.111	$\ln(\alpha) =$ -2.20	N = 6.28 $\rightarrow$	7
b.	carbon-12,	$\alpha =$	0.716	$\ln(\alpha) =$ -0.334	N = 41.3 $\rightarrow$	42
c.	iron-56, and	$\alpha =$	0.931	$\ln(\alpha) =$ -0.0714	N = 193.2 $\rightarrow$	194
d.	uranium-238?	$\alpha =$	0.983	$\ln(\alpha) =$ -0.0168	N = 821.1 $\rightarrow$	822

[2.16] Using the macroscopic scattering cross sections in Appendix Table II-3, calculate the slowing down decrement for UO_2, where U is natural uranium. Does the presence of oxygen have a significant effect on the slowing down decrement?

First note that $\alpha^{25} = (235-1)^2/(235+1)^2 = 0.9831$

$\alpha^{28} = (238-1)^2/(238+1)^2 = 0.9833$

Assume alpha for the uranium isotopes is that of uranium-238

$$\xi_U = 1 + \frac{0.9833}{1-.9833}\ln(0.9833) = 0.00840$$

$\alpha^{o} = (16-1)^2/(16+1)^2 = 0.0.8823$

$$\xi_O = 1 + \frac{0.8823}{1-.8823}\ln(0.8823) = 0.0613$$

Note these numbers can be obtained more quickly using the approximation of Eq. (2.57)

$$\xi_{UO_2} = \frac{\xi_U N_U \sigma_s^U + \xi_O N_O \sigma_s^O}{N_U \sigma_s^U + N_O \sigma_s^O} \quad \text{Since } N_O = 2N_U:$$

$$\xi_{UO_2} = \frac{\xi_U \sigma_s^U + 2\xi_O \sigma_s^O}{\sigma_s^U + 2\sigma_s^O} = \frac{0.0084\cdot 9.146 + 2\cdot 0.0613\cdot 3.761}{9.146 + 2\cdot 3.761} = 0.0322$$

Thus ξ_{UO_2} is significantly larger than ξ_U as a result of the oxygen.

[2.17] Prove that $\xi = 1$ for hydrogen.

Let $x = 1/\alpha$ Then from Eq. (2.56) $\xi = 1 + \frac{1}{1-x}\ln(x)$ Since for hydrogen $\alpha \to 0$, $x \to \infty$

Thus $\xi = 1 + \lim_{x\to\infty} \frac{1}{1-x}\ln(x)$ Use L'Hospital's rule

$$\lim_{x\to\infty} \frac{1}{1-x}\ln(x) = \lim_{x\to\infty} \frac{\frac{d}{dx}\ln(x)}{\frac{d}{dx}(1-x)} = \lim_{x\to\infty} \frac{1/x}{-1} = 0 \quad \text{Thus} \quad \xi = 1$$

[2.18] Show that for elastic scattering

a. $\overline{E-E'} \equiv \int (E-E')p(E\rightarrow E')dE'$. is equal to $\frac{1-\alpha}{2}E$

b. Evaluate $\overline{E-E'}$ for ordinary water.

a. Using Eq. (2.47) we have

$$\overline{E-E'} \equiv \int_{\alpha E}^{E} (E-E')\frac{1}{(1-\alpha)E}dE' = \frac{1}{(1-\alpha)}\int_{\alpha E}^{E} dE' - \frac{1}{(1-\alpha)E}\int_{\alpha E}^{E} E'dE'$$

Performing the integrals:

$$\overline{E-E'} = \frac{1}{(1-\alpha)}(E-\alpha E) - \frac{1}{(1-\alpha)E}\frac{1}{2}(E^2-\alpha^2E^2) = \left(1-\frac{1}{2}\frac{1-\alpha^2}{1-\alpha}\right)E$$

But since $1-\alpha^2 = (1+\alpha)(1-\alpha)$ We have $\overline{E-E'} = \frac{1-\alpha}{2}E$

b For water, we must combine Eqs. (2.47), (2.53) and Assuming the cross sections are energy independent we have

$$\overline{E-E'} = \frac{1}{\Sigma_s}\sum_i \Sigma_{si}\int (E-E')p_i(E\rightarrow E')dE'.$$

$$\overline{E-E'} = \frac{1}{\Sigma_s}\sum_i \Sigma_{si}\int_{\alpha_i E}^{E} (E-E')\frac{1}{(1-\alpha_i)E}dE'.$$

But from part a. we know that the integrals are just equal to $\frac{1-\alpha_i}{2}E$. Hence

$$\overline{E-E'} = \frac{1}{\Sigma_s}\sum_i \Sigma_{si}\frac{1}{2}(1-\alpha_i)E$$

Since we may write the cross section for water as $\Sigma_s = N(2\sigma_s^H + \sigma_s^O)$, $\Sigma_s^H = N2\sigma_s^H$ and $\Sigma_s^O = N\sigma_s^O$ we may reduce this expression to

$$\overline{E-E'} = \frac{1}{2\sigma_s^H + \sigma_s^O}\left[2\sigma_s^H(1-\alpha_H) + \sigma_s^O(1-\alpha_O)\right]\frac{1}{2}E$$

We know that $\alpha_H = 0$ and $\alpha_0 = (15/17)^2 = 0.779$ Thus

$$\overline{E-E'} = \frac{2\sigma_s^H + 0.221\sigma_s^O}{2\sigma_s^H + \sigma_s^O}\frac{1}{2}E$$ Taking $\sigma_s^H = 20b$ and $\sigma_s^O = 3.8b$,

$$\overline{E-E'} = \frac{2\cdot 20 + 0.221\cdot 3.8}{2\cdot 20 + 3.8}\frac{1}{2}E = 0.466E$$ Which is fairly close to the value of $0.5E$ for pure hydrogen.

Chapter 3 Problems;

[3.1] Verify Eqs. (3.23) and (3.25).

Substitute Eq. (3.23) into (3.22):

$$C/E = \int_{E}^{E/\alpha} \frac{1}{(1-\alpha)E'}\frac{C}{E'}dE' = \frac{C}{(1-\alpha)}\int_{E}^{E/\alpha} \frac{1}{E'^2}dE' = \frac{C}{(1-\alpha)}\left[-\frac{1}{E'}\right]_{E}^{E/\alpha} = \frac{C}{E}$$

Thus Eq. (3.22) is satisfied.

Now plug Eq. (3.23) into Eq. (3.24):

$$q = \int_{E}^{E/\alpha}\left[\int_{\alpha E'}^{E} \frac{1}{(1-\alpha)E'}\frac{C}{E'}dE''\right]dE' = \frac{C}{(1-\alpha)}\int_{E}^{E/\alpha}\frac{1}{E'^2}\left[\int_{\alpha E'}^{E} dE''\right]dE'$$

perform the E'' integral:

$$q == \frac{C}{(1-\alpha)}\int_{E}^{E/\alpha}\frac{1}{E'^2}(E-\alpha E')dE' = \frac{C}{(1-\alpha)}\left\{E\left[-\frac{1}{E'}\right]_{E}^{E/\alpha} - \alpha\left[\ln(E')\right]_{E}^{E/\alpha}\right\}$$

$$q = \frac{C}{(1-\alpha)}\left\{-\alpha + 1 - \alpha[\ln(E) - \ln(\alpha) - \ln(E)]\right\} = \left[1 + \frac{\alpha\ln(\alpha)}{(1-\alpha)}\right]C$$

[3.2] Show that in Eq. (3.31) the normalization condition $\int_0^{E_o} s(E)dE = 1$ must be obeyed. Hint: Note that $\int_0^{E_o} p(E' \to E)dE = 1$ for $E' \leq E_o$.

Write $\Sigma_s(E' \to E) = \Sigma_s(E')p(E' \to E)$

$$\Sigma_t(E)\varphi(E) = \int_0^{E_o} \Sigma_s(E')p(E' \to E)\varphi(E')dE' + s(E)q_o$$

Next integrate E between 0 and E_o:

$$\int_0^{E_o} \Sigma_t(E)\varphi(E)dE = \int_0^{E_o} \Sigma_s(E')\left[\int_0^{E_o} p(E' \to E)dE\right]\varphi(E')dE' + \int_0^{E_o} s(E)dEq_o$$

Since the bracketed term is equal to one, (because there is no up-scatter past E_o)and $\Sigma_a(E) = \Sigma_t(E) - \Sigma_s(E)$, we have

$\int_0^{E_o} \Sigma_a(E)\varphi(E)dE = \int_0^{E_o} s(E)dEq_o$ But all of the neutrons slowing down past E_o must be absorbed at lower energies. Thus $\int_0^{E_o} \Sigma_a(E)\varphi(E)dE = q_o$ and hence we must have $\int_0^{E_o} s(E)dE = 1$.

[3.3] In Eq.(3.31) suppose that the neutron slowing down past E_o is due entirely from elastic scattering from a single nuclide with *A*>1, and with no absorption for $E > E_o$. Show that $s(E)$ then takes the form

$$s(E) = \begin{cases} \dfrac{1}{(1-\alpha)\xi}\left(\dfrac{1}{E_o} - \dfrac{\alpha}{E}\right), & \alpha E_o < E < E_o \\ 0, & E < \alpha E_o \end{cases}$$

First write Eq. (3.32) as $s(E)q_o = \int_{E_o}^{\infty} \Sigma_s(E')p(E' \to E)\varphi(E')dE'$. For a pure scatter we my simplify Eq. (3.29) to $\varphi(E) = \dfrac{q_o}{\xi\Sigma_s(E)E}$. Thus $s(E) = \int_{E_o}^{\infty} p(E' \to E)\dfrac{1}{\xi E'}dE'$ We may rewrite the scattering probability of Eq. (2.47) as $p(E' \to E) = \dfrac{1}{1-\alpha}\cdot\dfrac{1}{E'}$ for $E < E' < E/\alpha$ Thus

$$s(E) = \int_{E_o}^{E/\alpha} \frac{1}{1-\alpha}\frac{1}{\xi E'^2}dE' = \frac{1}{1-\alpha}\frac{1}{\xi}\left[-\frac{1}{E'}\right]_{E_o}^{E/\alpha} = \frac{1}{1-\alpha}\frac{1}{\xi}\left[\frac{1}{E_o} - \frac{\alpha}{E}\right].$$

Below αE_o the source must vanish, since a neutron cannot scatter elastically form $E > E_o$ to $E < \alpha E_o$. Note that you can show that this expression for *S*(*E*) satisfies the normalization condition of problem [3.2].

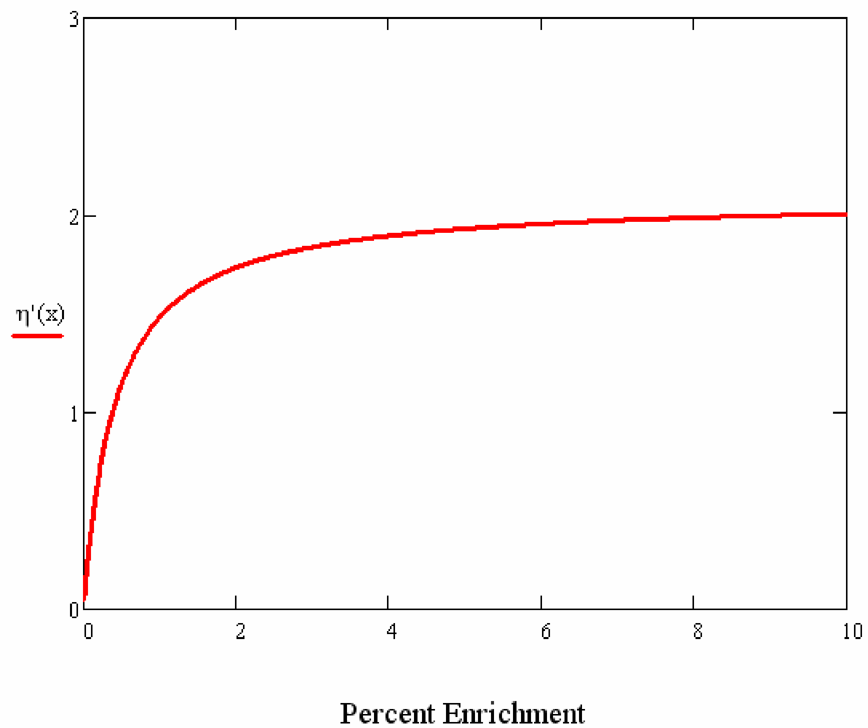
[3.4] For thermal neutrons calculate $\bar{\eta}$ as a function of uranium enrichment and plot your results. Use the uranium data from the following table:

	ν	σ_f (barns)	σ_a (barns)
Uranium-235	2.43	505	591
Plutonium-239	2.90	698	973
Uranium-238	---	0	2.42

We have $\bar{\eta} = \dfrac{(\nu N \sigma_f)^{25}}{(N\sigma_a)^{25} + (N\sigma_a)^{28}} = \dfrac{\nu^{25}\sigma_f^{\ 25}}{\sigma_a^{\ 25}} \dfrac{1}{1 + (N^{28}/N^{25})\sigma_a^{\ 28}/\sigma_a^{\ 25}}$

Since the enrichment is $\tilde{e} = N^{25}/(N^{25} + N^{28})$ we have

$$\bar{\eta} = \frac{\nu^{25}\sigma_f^{\ 25}}{\sigma_a^{\ 25}} \frac{1}{1 + (\tilde{e}^{-1} - 1)\sigma_a^{\ 28}/\sigma_a^{\ 25}} = 2.08 \frac{1}{1 + 0.00409 \cdot (\tilde{e}^{-1} - 1)}$$



[3.5] Suppose a new isotope is discovered with a '1/*E*' absorption cross section given by $\Sigma_a(E) = (E_o / E)\Sigma_a(E_0)$. Determine the energy-averaged cross section if the isotope is placed the thermal flux distribution given by Eq. (3.35??).

$$\bar{\Sigma}_a = \frac{\int_0^\infty \Sigma_a(E\ )\varphi_M(E)dE}{\int_0^\infty \varphi_M(E)dE}$$

The Maxwell Boltzmann distribution may be written, from Eq. (3.35??) to be

$$\varphi_M(E) = AE\exp(-E/kT) \text{ where } A = \frac{2\pi n'''}{(\pi kT)^{3/2}}\left(\frac{2}{m}\right)^{1/2}$$

$$\text{Thus } \bar{\Sigma}_a = \frac{\int_0^\infty (E_o / E)\Sigma_a(E_o)AE\exp(-E/kT)dE}{\int_0^\infty AE\exp(-E/kT)dE} = E_o\Sigma_a(E_o)\frac{\int_0^\infty \exp(-E/kT)dE}{\int_0^\infty E\exp(-E/kT)dE}$$

Change variables: $x = E/kT$:

$$\int_0^\infty E\exp(-E/kT)dE = (kT)^2\int_0^\infty x\exp(-x)dx = (kT)^2$$

$$\int_0^\infty \exp(-E/kT)dE = kT\int_0^\infty \exp(-x)dx = kT$$

Hence: $\bar{\Sigma}_a = \left(\frac{E_o}{kT}\right)\Sigma_a(E_o)$

[3.6] In the wide resonance approximation (Also called narrow resonance infinite mass approximation because the fuel is assumed to have an infinite mass) $A^f \to \infty$ and thus $\alpha^f \to 1$ in the first integral on the left of Eq. (3.28) , while the remaining approximations are the same as in narrow resonance approximation. Determine $\varphi(E)$ through the resonance. How does it differ from Eq. (3.29)? In which case is there more energy self shielding?

Since the domain of integration between E and E/α^f in the first integral of Eq. (3.28) becomes infinitely narrow, we may approximate $\Sigma_s^f(E')\varphi(E')/E' \to \Sigma_s^f(E)\varphi(E)/E$ within this interval. We thus have

$$\lim_{\alpha^f \to 1} \int_E^{E/\alpha^f} \frac{1}{(1-\alpha^f)E'} \Sigma_s^f(E')\varphi(E')dE' = \lim_{\alpha^f \to 1} \frac{\Sigma_s^f(E)\varphi(E)}{E} \int_E^{E/\alpha^f} \frac{1}{(1-\alpha^f)} dE'$$

$$= \frac{\Sigma_s^f(E)\varphi(E)}{E} \lim_{\alpha^f \to 1} \frac{1}{(1-\alpha^f)} \left(E/\alpha^f - E \right)$$

$$= \Sigma_s^f(E)\varphi(E) \lim_{\alpha^f \to 1} \frac{1}{(1-\alpha^f)} \left(1/\alpha^f - 1 \right) = \Sigma_s^f(E)\varphi(E) \lim_{\alpha^f \to 1} \frac{1}{\alpha^f} = \Sigma_s^f(E)\varphi(E)$$

Substituting this limiting case for the first integral and Eq. (3.26) for the second integral reduces Eq. (3.28) to

$$\Sigma_t(E)\varphi(E) = \Sigma_s^f(E)\varphi(E) + \frac{1}{\xi E} q \quad \text{or}$$

$$\varphi(E) = \frac{1}{\xi \left[\Sigma_t(E) - \Sigma_s^f(E) \right] E} q$$

Since $\Sigma_t(E) - \Sigma_s^f(E)$ is smaller that $\Sigma_t(E)$ appearing in the denominator of Eq. (3.29), the flux is larger, and therefore there is less self-shielding in this approximation.

[3.7] Lethargy defined as $u = \ln(E_o / E)$ is often used in neutron slowing down problems; lethargy increases as energy decreases. Noting the following transformations: $\varphi(E)dE = -\varphi(u)du \quad p(E \to E')dE' = -p(u \to u')du'$ and $\Sigma_x(E) = \Sigma_x(u)$,

a. Show that $p(E \to E')$ given by Eq. (2.47) becomes

$$p(u \to u') = \begin{cases} \dfrac{1}{1-\alpha}\exp(u - u'), & u \le u' \le u + \ln(1/\alpha) \\ 0, & \text{otherwise} \end{cases},$$

b. Express Eq. (3.22) in terms of *u*.

First note that $E = E_o \exp(-u)$ and thus $dE / du = -E_o \exp(-u)$

a. $p(u \to u') = -p(E \to E')dE' / du' = p(E \to E')E_o \exp(-u')$

Substituting from Eq. (2.47):

$$p(u \to u') = \frac{1}{1-\alpha} \cdot \frac{1}{E} E_o \exp(-u') = \frac{1}{1-\alpha}\exp(u - u')$$

Finally when $E' = E$, $u' = u$ and when $E' = \alpha E$,

$u' = \ln(E_o / \alpha E) = \ln(E_o / E) - \ln(\alpha) = u + \ln(1/\alpha)$

b. First note that $\varphi(E) = -\varphi(u)du / dE = \varphi(u)\dfrac{1}{E_o}\exp(u)$. Thus Eq. (3.22) may be written as

$$\Sigma_s(u)\varphi(u)\frac{1}{E_o}\exp(u) = \frac{1}{1-\alpha}\int_E^{E/\alpha} \frac{1}{E'}\Sigma_s(E')\varphi(E')dE' = -\frac{1}{1-\alpha}\int_u^{u-\ln(1/\alpha)} \frac{\exp(u')}{E_o}\Sigma_s(u')\varphi(u')du'$$

Thus $\Sigma_s(u)\varphi(u) = \dfrac{1}{1-\alpha}\displaystyle\int_{u-\ln(1/\alpha)}^{u} \exp(u' - u)\Sigma_s(u')\varphi(u')du'$

[3.8] Making a change of variables from energy to speed, show that Eq. (2.40) becomes

$$p(\mathrm{v} \to \mathrm{v}') = \begin{cases} \dfrac{2\mathrm{v}'}{(1-\alpha)\mathrm{v}^2}, & \mathrm{v}\sqrt{\alpha} \le \mathrm{v}' \le \mathrm{v} \\ 0, & \text{otherwise} \end{cases},$$

Since $E \equiv \tfrac{1}{2} m\mathrm{v}^2$ and $\mathrm{v} = \sqrt{2E/m}$, we have $dE/d\mathrm{v}=m\mathrm{v}$

$$p(\mathrm{v} \to \mathrm{v}') = p(E \to E')dE'/d\mathrm{v}' = \frac{1}{(1-\alpha)E} dE'/d\mathrm{v}' = \frac{1}{(1-\alpha)\tfrac{1}{2} m\mathrm{v}^2} m\mathrm{v}'$$

or $$p(\mathrm{v} \to \mathrm{v}') = \frac{2\mathrm{v}'}{(1-\alpha)\mathrm{v}^2}$$

For the limits, when $E' = E$, $\mathrm{v}'=\mathrm{v}$ and when $E' = \alpha E$, $\mathrm{v}'=\sqrt{\alpha}\mathrm{v}$

[3.9] Suppose that the Maxwell-Boltzmann distribution, Eq. (2.34), represents the neutron density in Eqs. (3.43) and (3.44):

a. Find the value of $\bar{\mathrm{v}}$.
b. If we define $\bar{E} \equiv \frac{1}{2} m \bar{\mathrm{v}}^2$, show that $\bar{E} = 1.273 kT$.
c. Why is your result different from the average energy of $\frac{3}{2} kT$ given by Eq. (2.33)?

For part a, we substitute Eq. (2,34) into Eq. (3.44), along with $\mathrm{v} = \sqrt{2E/m}$:

$$\bar{\mathrm{v}} = \int_0^\infty \sqrt{2E/m} M(E) dE \Big/ \int_0^\infty M(E) dE$$ and Eq. (2.36) sets the denominator equal to one. Thus $\bar{\mathrm{v}} = \int_0^\infty \sqrt{2E/m} M(E) dE = \sqrt{2/m} \dfrac{2\pi}{(\pi kT)^{3/2}} \int_0^\infty E \exp(-E/kT) dE$

Let $x = E/kT$ Then

$$\bar{\mathrm{v}} == \sqrt{2/m} \frac{2(kT)^{1/2}}{\pi^{1/2}} \int_0^\infty x \exp(-x) dx == 2\sqrt{2kT/\pi m}$$

For part b, we have $\bar{E} \equiv \frac{1}{2} m \left(2\sqrt{2kT/\pi m} \right)^2 = \dfrac{4}{\pi} kT = 1.273 kT$

For c, we note that $\bar{E} \equiv \frac{1}{2} m \bar{\mathrm{v}}^2$ holds only if all of the neutrons have the same speed and energy.

[3.10] A power reactor is cooled by heavy water, (D_2O) but a leak causes a 1.0 atom % contamination of the coolant with light water(H_2O). Determine the resulting percentage increase or decrease in the coolant's

a. slowing down decrement
b. slowing down power
c. slowing down ratio.

All the needed data is in Table 3.1, provided we take $\Sigma_{si} = (\xi_i \Sigma_{si}) / \xi_i$ yielding

$$\Sigma_s^{D_2O} = 0.353, \qquad \Sigma_s^{H_2O} = 1.38$$

and $\Sigma_{ai} = \xi_i \Sigma_{si} / (\xi_i \Sigma_{si} / \Sigma_{ai})$

$$\Sigma_a^{D_2O} = 8.57 \cdot 10^{-6} \qquad \Sigma_a^{H_2O} = 0.022$$

For a.: From Eq. (2.61) the averaged slowing down decrement is

$$\bar{\xi} = \frac{1}{\Sigma_s} \sum_i \xi_i \Sigma_{si}$$

For 1% contamination, the number densities and thus the macroscopic cross sections of heavy water and water are replaced by 0.99 and 0.01 of their nominal values. Thus

$$\bar{\xi} = \frac{\xi_{D_2O} 0.99 \Sigma_s^{D_2O} + \xi_{H_2O} 0.01 \Sigma_s^{H_2O}}{0.99 \Sigma_s^{D_2O} + 0.01 \Sigma_s^{H_2O}}$$

$$\bar{\xi} = \frac{0.51 \cdot 0.99 \cdot 0.353 + 0.93 \cdot 0.01 \cdot 1.38}{0.99 \cdot 0.353 + 0.01 \cdot 1.38} = \frac{0.191}{0.363} = 0.53$$

For b. We again use Eq. (2.61);

$$\bar{\sigma}_{aT}(T) = \left(T_o / T\right)^{1/2} \bar{\sigma}_{aT}(T_o)$$

For c.

$$\bar{\xi} \Sigma_s / \Sigma_a = \sum_i \xi_i \Sigma_{si} / (0.99 \Sigma_a^{D_2O} + 0.01 \Sigma_a^{H_2O}) = \frac{0.19}{0.99 \cdot 8.57 \cdot 10^{-6} + 0.01 \cdot 0.022} = 833$$

Thus while the contamination has only small effects on the slowing down decrement and power, it decreases the slowing down ratio substantially as a result of the much larger absorption cross section of water.

[3.11] Using the data in Appendix E calculate the microscopic absorption cross section of water, averaged over a thermal neutron spectrum:

a. at room temperature.
b. at 300 ^{o}C, which is a typical operating temperature for a water-cooled reactor.

The microscopic cross section may be considered 1/v.

a. The value in the appendix is tat room temperature of $T_o = 293.61\,^oK$ .The value is 0.5896 b

b. Temperature corrections for 1/v cross sections are given by the expression following Eq. (3.59):

$$\bar{\sigma}_{aT}(T) = \left(T_o/T\right)^{1/2}\bar{\sigma}_{aT}(T_o)$$

Since 300 ^{0}C = 300 + 273 ^{0}K = 573 ^{0}K

$$\bar{\sigma}_{aT}(573) = \left(293.6/573\right)^{1/2} 0.5896 = 0.422\,b$$

[3.12] Repeat problem [3.11] for heavy water

The microscopic cross section may be considered 1/v.

a. The value in the appendix is tat room temperature of $T_o = 293.61\ ^oK$ .The value is 0.0010 b

b. Temperature corrections for 1/v cross sections are given by the expression following Eq. (3.59):

$$\bar{\sigma}_{aT}(T) = \left(T_o/T\right)^{1/2} \bar{\sigma}_{aT}(T_o)$$

Since 300 ^{0}C = 300 + 273 ^{0}K = 573 ^{0}K

$$\bar{\sigma}_{aT}(573) = \left(293.6/573\right)^{1/2} 0.0010 = 7.2 \cdot 10^{-4}\ b$$

Chapter 4 Problems:

[4.1] A reactor is to be built with fuel rods of 1.2 cm in diameter, and a liquid moderator with a 2:1 volume ratio of moderator to fuel. What will the distance between nearest fuel centerlines be

a. for a square lattice?
b. for a hexagonal lattice?

Note: solution is presently for
1:1 volume ratio

Let r be the radius of the fuel rod or r = 0.6 cm and $V_f / V_m = 1.0$ The fuel to moderator volume ratio.

a. Let l be the lattice pitch, which is also the distance between nearest centerline:

$$V_f / V_m = \frac{\pi r^2}{l^2 - \pi r^2} \text{ or } l = \sqrt{\pi (V_m / V_f + 1)}\, r = \sqrt{3.141(1+1)}\, 0.6 = 1.504 cm$$

b. Let l be the base of an equilateral triangle. Then the height will be $h = \frac{\sqrt{3}}{2} l$. The area of the hexagon will be that of six equilateral triangles or

$$A_{hex} = 6 \cdot \frac{1}{2} l \cdot \frac{\sqrt{3}}{2} l = \frac{3\sqrt{3}}{2} l^2 \text{ while } V_f / V_m = \frac{\pi r^2}{A_{hex} - \pi r^2} = \frac{\pi r^2}{\frac{3\sqrt{3}}{2} l^2 - \pi r^2}$$

Hence $l = 3^{-3/4} \sqrt{2\pi (V_m / V_f + 1)}\, r$ However the distance between centerlines is

$$\sqrt{3} l = 3^{-1/4} \sqrt{2\pi (V_m / V_f + 1)}\, r = 3^{-1/4} \sqrt{2 \cdot 3.141(1+1)}\, 0.6 = 1.616$$

[4.2] In a fast reactor designers often want to minimize the coolant to fuel volume ratio to minimize the amount of neutron slowing down. From a geometric point of view what is the theoretical limit on the smallest ratio of coolant to fuel volume that can be obtained

a. with a square lattice,
b. with a hexagonal lattice?

For part a: The fuel diameter *d* can be no larger than the lattice pitch *l*. Thus

$$V_c / V_f \geq \frac{l^2 - \pi d^2 / 4}{\pi d^2 / 4} = \frac{4}{\pi} - 1 = 0.273$$

For part b. the diameter can be no greater than the face to face distance of the hexagon. If *s* is the side length of the hexagon, its area is six equilateral triangles, each with an area of $(\sqrt{3}/4)s^2$ or $(3\sqrt{3}/2)s^2$ the side to side distance is twice the height of one of these triangles or $\sqrt{3}s$ hence $d = \sqrt{3}s$. We have

$$V_c / V_f \geq \frac{3\sqrt{3}s^2 / 2 - \pi d^2 / 4}{\pi d^2 / 4} = \frac{3\sqrt{3}s^2 / 2}{\pi 3 s^2 / 4} - 1 = \frac{2\sqrt{3}}{\pi} - 1 = 0.1027$$

[4.3] A sodium-cooled fast reactor is fueled with PuO_2, mixed with depleted UO_2. the structural material is iron. Averaged over the spectrum of fast neutrons, the microscopic cross sections and densities are the following:

	σ_f b	σ_a b	σ_t b	ρ g/cm^3
PuO_2	1.95	2.40	8.6	11.0
UO_2	0.05	0.404	8.2	11.0
Na	-	0.0018	3.7	0.97
Fe	-	0.0087	3.6	7.87

The fuel is 15% PuO_2 and 85% UO_2 by volume. The volumetric composition of the core is 30% fuel, 50% coolant and 20% structural material. Calculate k_∞ assuming that the values of ν for plutonium and uranium in the fast spectrum are 2.98 and 2.47, respectively, and that the cross sections of oxygen can be neglected. What fraction of the mass of the core does the fuel account for?

In this core we know that $V = V_f + V_{Na} + V_{Fe}$ and are given: $V_f / V = 0.30$, $V_{Na} / V = 0.50$, $V_{Fe} / V = 0.20$.

We also know that $V_f = V_{PuO_2} + V_{UO_2}$ and are given $V_{PuO_2} / V_f = 0.15$ and $V_{UO_2} / V_f = 0.85$

Using volume weighting we may write

$$k_\infty = \frac{\underline{\nu}\Sigma_f}{\Sigma_a} = \frac{(V_f / V)\underline{\nu}\Sigma_f^f}{(V_f / V)\Sigma_a^f + (V_{Na} / V)\Sigma_a^{Na} + (V_{Fe} / V)\Sigma_a^{Fe}} = \frac{0.30\underline{\nu}\Sigma_f^f}{0.30\Sigma_a^f + 0.50\Sigma_a^{Na} + 0.20\Sigma_a^{Fe}}$$

Using volume weighting for the fissile and fertile material in the fuel:

$$\nu\Sigma_f^f = (V_{PuO_2} / V_f)\nu^{49}\Sigma_f^{PuO_2} + (V_{UO_2} / V_f)\nu^{28}\Sigma_f^{UO_2} = 0.15\nu^{49}\Sigma_f^{PuO_2} + 0.85\nu^{28}\Sigma_f^{UO_2}$$

and

$$\Sigma_a^f = (V_{PuO_2} / V_f)\Sigma_a^{PuO_2} + (V_{UO_2} / V_f)\Sigma_a^{UO_2} = 0.15\Sigma_a^{PuO_2} + 0.85\Sigma_a^{UO_2}$$

Combining equations we then have for the multiplication

$$k_\infty = \frac{0.045\nu^{49}\Sigma_f^{PuO_2} + 0.255\nu^{28}\Sigma_f^{UO_2}}{0.045\Sigma_a^{PuO_2} + 0.255\Sigma_a^{UO_2} + 0.50\Sigma_a^{Na} + 0.20\Sigma_a^{Fe}}$$

Using $\Sigma = N\sigma = \dfrac{\rho N_o}{A}\sigma$ and ignoring the cross sections of oxygen, we have

$$k_\infty = \frac{0.045\dfrac{\rho_{PuO_2}}{A_{PuO_2}}N_o\nu^{49}\sigma_f^{49} + 0.255\dfrac{\rho_{UO_2}}{A_{UO_2}}N_o\nu^{28}\sigma_f^{28}}{0.045\dfrac{\rho_{PuO_2}}{A_{PuO_2}}N_o\sigma_a^{49} + 0.255\dfrac{\rho_{UO_2}}{A_{UO_2}}N_o\sigma_a^{28} + 0.50\dfrac{\rho_{Na}}{A_{Na}}N_o\sigma_a^{Na} + 0.20\dfrac{\rho_{Fe}}{A_{Fe}}N_o\sigma_a^{Fe}}$$

Note that $A_{PuO_2} = 239 + 2*16 = 271$, $A_{UO_2} = 238 + 2*16 = 270$, $A_{Na} = 23$, $A_{Fe} = 55.85$.

Thus canceling N_o from numerator and denominator, and using the densities and cross sections given we have

$$k_\infty = \frac{0.045\frac{11}{271}2.98\cdot 1.95 + 0.255\frac{11}{270}2.47\cdot 0.05}{0.045\frac{11}{271}2.40 + 0.255\frac{11}{270}0.404 + 0.50\frac{0.97}{23}0.0018 + 0.20\frac{7.87}{55.85}0.0087}$$

$$k_\infty = \frac{0.011 + 0.001283}{0.004384 + 0.004197 + 0.003796\cdot 10^{-2} + 0.002452\cdot 10^{-1}}$$

$$k_\infty = 1.342$$

[4.4] Suppose the nonleakage probability for a sodium cooled fast reactor specified in problem [4.3] is 0.90. Using the data from problem [4.3], adjust the volume fractions of PuO_2 and UO_2 in the fuel so that k = 1.0. What is the % PuO_2 in the fuel by volume?

From Eq. (3.2) $k_\infty = k / P_{NL} = 1.0/0.9 = 1.111$

In this core we know that $V = V_f + V_{Na} + V_{Fe}$ and are given: $V_f / V = 0.30$, $V_{Na}/V = 0.50$, $V_{Fe}/V = 0.20$.

We also know that $V_f = V_{PuO_2} + V_{UO_2}$

Let $x = V_{PuO_2}/V_f$ and $1 - x = V_{UO_2}/V_f$

Using volume weighting we may write

$$k_\infty = \frac{\underline{\nu\Sigma}_f}{\Sigma_a} = \frac{(V_f/V)\underline{\nu\Sigma}_f^f}{(V_f/V)\Sigma_a^f + (V_{Na}/V)\Sigma_a^{Na} + (V_{Fe}/V)\Sigma_a^{Fe}} = \frac{0.30\underline{\nu\Sigma}_f^f}{0.30\Sigma_a^f + 0.50\Sigma_a^{Na} + 0.20\Sigma_a^{Fe}}$$

Using volume weighting for the fissile and fertile material in the fuel:

$$\nu\Sigma_f^f = x\nu^{49}\Sigma_f^{PuO_2} + (1-x)\nu^{28}\Sigma_f^{UO_2}$$

and

$$\Sigma_a^f = x\Sigma_a^{PuO_2} + (1-x)\Sigma_a^{UO_2}$$

Combining equations we then have for the multiplication

$$k_\infty = \frac{0.30[x \cdot \nu^{49}\Sigma_f^{PuO_2} + (1-x)\nu^{28}\Sigma_f^{UO_2}]}{0.30[x \cdot \Sigma_a^{PuO_2} + (1-x)\Sigma_a^{UO_2}] + 0.50\Sigma_a^{Na} + 0.20\Sigma_a^{Fe}}$$

Using $\Sigma = N\sigma = \dfrac{\rho N_o}{A}\sigma$ and ignoring the cross sections of oxygen, we have

$$k_\infty = \frac{0.3[x\dfrac{\rho_{PuO_2}}{A_{PuO_2}}N_o\nu^{49}\sigma_f^{49} + (1-x)\dfrac{\rho_{UO_2}}{A_{UO_2}}N_o\nu^{28}\sigma_f^{28}]}{0.3[x\dfrac{\rho_{PuO_2}}{A_{PuO_2}}N_o\sigma_a^{49} + (1-x)\dfrac{\rho_{UO_2}}{A_{UO_2}}N_o\sigma_a^{28}] + 0.50\dfrac{\rho_{Na}}{A_{Na}}N_o\sigma_a^{Na} + 0.20\dfrac{\rho_{Fe}}{A_{Fe}}N_o\sigma_a^{Fe}}$$

Note that $A_{PuO_2} = 239 + 2*16 = 271$, $A_{UO_2} = 238 + 2*16 = 270$, $A_{Na} = 23$, $A_{Fe} = 55.85$.

Thus canceling N_o from numerator and denominator, and using the densities and cross sections given we have

$$k_\infty = \frac{0.3[x\dfrac{11}{271}2.98 \cdot 1.95 + (1-x)\dfrac{11}{270}2.47 \cdot 0.05]}{0.3[x\dfrac{11}{271}2.40 + (1-x)\dfrac{11}{270}0.404] + 0.50\dfrac{0.97}{23}0.0018 + 0.20\dfrac{7.87}{55.85}0.0087} = 1.111$$

$$\frac{0.07076x + 0.001509(1-x)}{0.02922x + 0.004938(1-x) + 0.003796 \cdot 10^{-2} + 0.002452 \cdot 10^{-1}} = 1.111$$

Simplify:

$$\frac{0.06925x + 0.001509}{0.02428x + 0.005221} = 1.111$$

Solve for x:

$$x = 0.102$$

Thus the fuel should be 10.2% PuO_2 and 89.8% UO_2 by volume.

[4.5] Verify Eq. (4.18).

Substitute Eq. (4.2) into Eq. (4.1):

$$k_\infty = \frac{\int_0^\infty [(V_f/V)\nu\Sigma_f^f(E) + (V_c/V)\nu\Sigma_f^c(E) + (V_{st}/V)\nu\Sigma_f^{st}(E)]\varphi(E)dE}{\int_0^\infty [(V_f/V)\Sigma_a^f(E) + (V_c/V)\Sigma_a^c(E) + (V_{st}/V)\Sigma_a^{st}(E)]\varphi(E)dE},$$

Multiply numerator and denominator by V and note that $\nu\Sigma_f^c(E) = \nu\Sigma_f^{st}(E) = 0$:

$$k_\infty = \frac{V_f \int_0^\infty \nu\Sigma_f^f(E)\varphi(E)dE}{V_f \int_0^\infty \Sigma_a^f(E)\varphi(E)dE + V_c \int_0^\infty \Sigma_a^c(E)\varphi(E)dE + V_{st} \int_0^\infty \Sigma_a^{st}(E)\varphi(E)dE}$$

[4.6] A pressurized water reactor has 3% enriched UO_2 fuel pins that are 1.0 cm in diameter and have a density of 11.0 gm/cm^3. The moderator to fuel volume ratio is 2:1. Calculate η_T, *p*, *f*, and k_∞ at room temperature under the assumptions that $\varepsilon = 1.24$, the thermal disadvantage factor $\varsigma = 1.16$, and the Dancoff correction increases the fuel diameter for the resonance integral calculation by 10%.

Let *N* be the number density of UO2. From Eq. (4.49), and the enrichment is $\tilde{e} = 0.03$

$$\eta_T = \frac{\nu\Sigma_f^f}{\Sigma_a^f} = \frac{\tilde{e}N\nu^{25}\sigma_f^{25}}{\tilde{e}N\sigma_a^{25} + (1-\tilde{e})N\sigma_a^{28} + 2N\sigma_a^O} = \frac{\tilde{e}\nu^{25}\sigma_f^{25}}{\tilde{e}\sigma_a^{25} + (1-\tilde{e})\sigma_a^{28} + 2\sigma_a^O}$$

Using data from Table 3.2 and oxygen data from Table E-3

$$\eta_T = \frac{0.03\cdot 2.43\cdot 505}{0.03\cdot 591 + (1-0.03)2.42 + 2\cdot 0.0002} = 1.82$$

From Eq. (4.40)

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s}I\right]$$

For the fuel $A = 238 + 2\cdot 16 = 270$,

$$N_f = \rho N_o / A = 11.0\cdot 0.6023\cdot 10^{24}/270 = 0.0245\cdot 10^{24} \text{ gm/cm}^3$$

From Table 4.3 for UO2

$$I = 4.45 + 26.6\sqrt{4/\rho D} = 4.45 + 26.6\sqrt{4/(11\cdot 1.1)} = 19.74\, b$$

where we have increased the diameter by 10%: Taking $\xi\Sigma_s = 1.28$ for water from Table 3.1, we obtain

$$p = \exp\left[-\left(\frac{1}{2}\right)\frac{0.0245\cdot 10^{24}}{1.28}19.74\cdot 10^{-24}\right] = 0.828$$

From Eq. (4.55)

$$f = \frac{1}{1 + \varsigma(V_m/V_f)(N_m/N_f)(\bar{\sigma}_{aT}^m/\bar{\sigma}_{aT}^f)}$$

$$N_m/N_f = \frac{\rho_m N_o/A_m}{\rho_f N_o/A_f} = \frac{\rho_m A_f}{\rho_f A_m} = \frac{1.0\cdot 270}{11\cdot 18} = 1.364$$

Taking the absorption cross sections from Table E-3:

$$f = \frac{1}{1 + 1.16\cdot 2\cdot 1.364(0.5896/6.540)} = 0.778$$

Finally,

$$k_\infty = \varepsilon p f \eta_T = 1.24\cdot 0.828\cdot 0.778\cdot 1.82 = 1.45$$

[4.7] Suppose the fuel rods from problem [4.6] are to be used in a D_2O moderated reactor.

a. What volume ratio of moderator to fuel is required to give the same value of *p* as for the H_2O lattice in problem [4.6]? (assume no Dancoff correction)
b. What volume ratio of moderator to fuel is required to give the same value of *f* as for the H_2O lattice in problem [4.6]? (assume ς is unchanged)

For part a: From Eq. (4.40) the resonance escape probability is

$$p = \exp[-(V_f / V_m)\frac{N_f}{\xi\Sigma_s} I] \quad \text{Thus}$$

$$(V_f / V_m)\frac{\cancel{N_f}}{\xi\Sigma_s}\cancel{I}\bigg|_{D_2O} = (V_f / V_m)\frac{\cancel{N_f}}{\xi\Sigma_s}\cancel{I}\bigg|_{H_2O} \quad \text{or using the data from Table 3.1}$$

$$(V_f / V_m)\frac{1}{0.18}\bigg|_{D_2O} = (1/2)\frac{1}{1.28} \qquad \text{Thus } \frac{V_m}{V_f}\bigg|_{D_2O} = \frac{2\cdot 1.28}{0.18} = 14.2$$

For part b: From Eq. (4.48) the thermal utilization is

$$f = \frac{1}{1+\varsigma\frac{V_m N_m}{V_f N_f}\frac{\sigma_a^m}{\sigma_a^f}} \quad \text{With the thermal disadvantage factor held constant:}$$

$$\frac{V_m N_m}{V_f \cancel{N_f}}\frac{\sigma_a^m}{\cancel{\sigma_a^f}}\bigg|_{D_2O} = \frac{V_m N_m}{V_f \cancel{N_f}}\frac{\sigma_a^m}{\cancel{\sigma_a^f}}\bigg|_{H_2O} \text{ or } \frac{V_m}{V_f}\bigg|_{D_2O} = 2\frac{N_{H_2O}\sigma_a^{H_2O}}{N_{D_2O}\sigma_a^{D_2O}} = 2\frac{1.0\cdot 0.5896}{1.1\cdot 0.0010} = 1072$$

where the number density and cross section data is from Table E-3.
These results explain, in part, why heavy water reactors have much larger moderator to fuel volume ratios than light water reactors

[4.8] A reactor lattice consists of uranium rods in a heavy water moderator. The heavy water is replaced by light water

- a Would the resonance escape probability increase or decrease? Why?
- b Would the thermal utilization increase or decrease? Why?
- c. What would you expect the net effect on k_∞ to be? Why?

Part a: From Eq. (4.40): $p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s} I\right]$

The only quantity changing is the slowing down power, $\xi\Sigma_s$. From Table 3.1 we see that its value is larger for water than for heavy water. Thus the resonance escape probability will increase.

Part b. From Eq. (4.55): $f = \dfrac{1}{1+\varsigma(V_m/V_f)(N_m/N_f)(\bar{\sigma}_{aT}^m/\bar{\sigma}_{aT}^f)}$

The major change will be in the moderator thermal absorption cross section $\bar{\sigma}_{aT}^m$. The thermal disadvantage factor will change less. From Table E-3, we seen that $\bar{\sigma}_{aT}^m$ is much smaller for heavy water than for water. Thus the thermal utilization will decrease.

Part c. Because of its very small thermal abortion cross section, heavy water is considered to be the best moderator; reactors can be built using natural uranium if heavy water is the moderator, but not with ordinary water. Thus the net effect of replacing heavy water with water would be to decrease the value of k_∞.

[4.9] Suppose the volume ratio of coolant to fuel is increased in a pressurized water reactor:

a. Will the fast fission factor increase, decrease or remain unchanged? Why?
b. Will the resonance escape probability increase, decrease or remain unchanged? Why?
c. Will the thermal utilization increase, decrease or remain unchanged? Why?
d. Will the value of η_T increase, decrease or remain unchanged? Why?

Part a: The fast fission factor should show a slite decrease, since once neutrons escape the fuel rod in which they were born, the increase in moderator will be more effective in slowing the neutrons down below the energy threshold were fast fission can take place in uranium-238 before they enter another fuel rod. This effect is not included in the simple approximation given by Eq. (4.56)

Part b: The resonance escape probability should increase, since the additional moderator will more effectively slow down neutrons, causing decreased capture in the uranium-238 resonances. This can be seen in Eq. (4.40), since the volume ratio V_f / V_m will decrease, causing p to increase.

Part c: The thermal utilization will decrease, since the added moderator will cause added thermal neutron absorption in the moderator. This is indicated in Eq. (4.55) by the increase in V_m / V_f, which causes f to decrease.

Part d: As indicated by Eq. (4.49), η_T depends only on fuel cross sections. Thus changes in moderator ratios will have only secondary effects, cause by shifts in the thermal neutron spectrum $, \varphi_f(E)$

[4.10]* Using the data from problem [4.6], vary the coolant/fuel volume ratio between 0.5 and 2.5 and plot vs. V_m / V_f :

a. the resonance escape probability
b. the thermal utilization
c. k_∞
d. Determine the moderator/fuel volume ratio that yields the largest k_∞ .
e. What is the largest value of k_∞ ?

You may assume that changes in the fast fission factor and the thermal disadvantage factor are negligible.

From Eq. (4.49), and the enrichment is $\tilde{e} = 0.03$

$$\eta_T = \frac{\nu\Sigma_f^f}{\Sigma_a^f} = \frac{\tilde{e}N\nu^{25}\sigma_f^{25}}{\tilde{e}N\sigma_a^{25} + (1-\tilde{e})N\sigma_a^{28} + 2N\sigma_a^O} = \frac{\tilde{e}\nu^{25}\sigma_f^{25}}{\tilde{e}\sigma_a^{25} + (1-\tilde{e})\sigma_a^{28} + 2\sigma_a^O}$$

Using data from Table 3.2 and oxygen data from Table E-3

$$\eta_T = \frac{0.03 \cdot 2.43 \cdot 505}{0.03 \cdot 591 + (1-0.03)2.42 + 2 \cdot 0.0002} = 1.82$$

From Eq. (4.40)

$$p = \exp\left[-\left(\frac{V_f}{V_m}\right)\frac{N_f}{\xi\Sigma_s} I\right]$$

For the fuel $A = 238 + 2 \cdot 16 = 270$,

$$N_f = \rho N_o / A = 11.0 \cdot 0.6023 \cdot 10^{24} / 270 = 0.0245 \cdot 10^{24} \text{ gm/cm}^3$$

From Table 4.3 for UO2

$$I = 4.45 + 26.6\sqrt{4/\rho D} = 4.45 + 26.6\sqrt{4/(11 \cdot 1.1)} = 19.74\, b$$

where we have increased the diameter by 10%: Taking $\xi\Sigma_s = 1.28$ for water from Table 3.1, we obtain

$$p = \exp\left[-\left(\frac{V_m}{V_f}\right)^{-1} \frac{0.0245 \cdot 10^{24}}{1.28} 19.74 \cdot 10^{-24}\right]$$

or

$$p = \exp\left[-0.378(V_m / V_f)^{-1}\right]$$

From Eq. (4.55)

$$f = \frac{1}{1 + \varsigma(V_m / V_f)(N_m / N_f)(\bar{\sigma}_{aT}^m / \bar{\sigma}_{aT}^f)}$$

$$N_m / N_f = \frac{\rho_m N_o / A_m}{\rho_f N_o / A_f} = \frac{\rho_m A_f}{\rho_f A_m} = \frac{1.0 \cdot 270}{11 \cdot 18} = 1.364$$

With the thermal disadvantage factor $\varsigma = 1.16$ (from problem [4.6]) and taking the absorption cross sections from Table E-3:

$$f = \frac{1}{1 + 1.16 \cdot (V_m / V_f) \cdot 1.364(0.5896 / 6.540)}$$

or

$$f = \frac{1}{1 + 0.1426(V_m / V_f)}$$

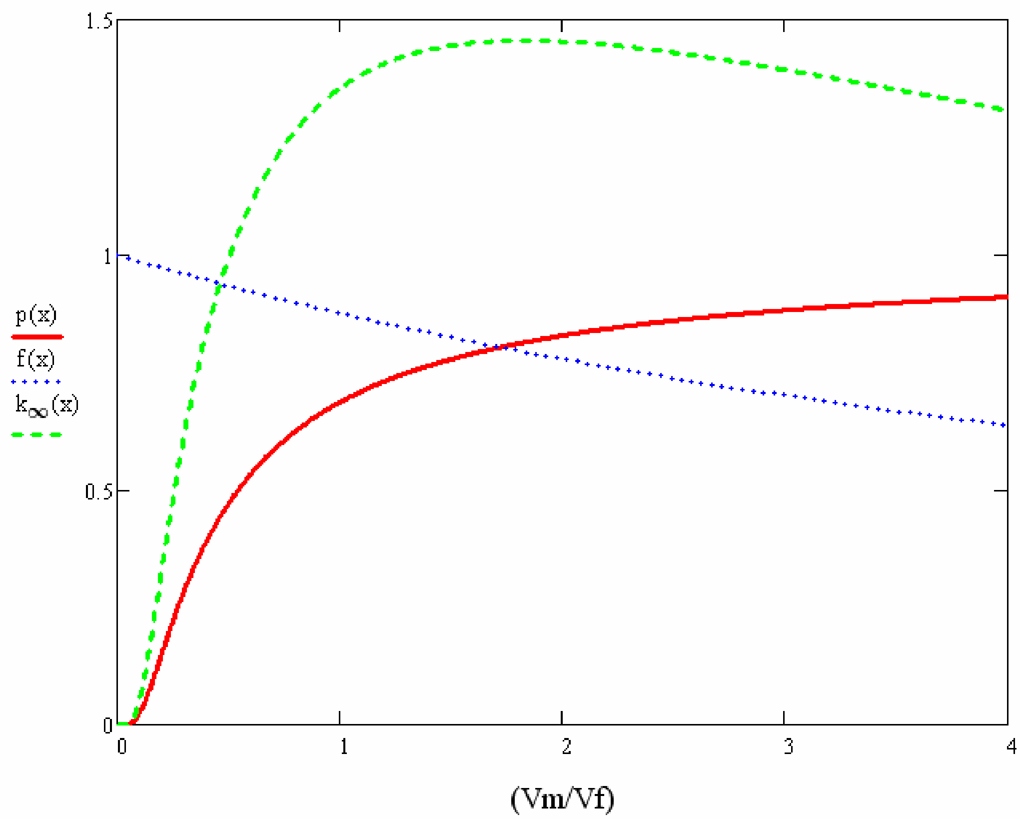
Finally, with $\varepsilon = 1.24$ (form problem [4.6], we combine the above equations to obtain:

$$k_\infty = \varepsilon p f \eta_T$$
$$= 1.24 \cdot \exp\left[-0.378(V_m / V_f)^{-1}\right] \cdot \frac{1}{1 + 0.1426(V_m / V_f)} \cdot 1.82$$

or

$$k_\infty = 2.257 \frac{\exp\left[-0.378(V_m / V_f)^{-1}\right]}{1 + 0.1426(V_m / V_f)}$$

The plots of *p*, *f* and k_∞ are as follows:



We obtain a maximum value of k_∞ of $k_\infty\big|_{Max}$ = 1.46

At a volume ratio of V_m / V_f = 1.82

[4.11] A reactor designer decides to replace uranium with UO_2 fuel in a water cooled reactor, keeping the enrichment, fuel diameter and water to fuel volume ratios the same.

a. Will *p* increase, decrease or remain unchanged? Why?
b Will *f* increase, decrease or remain unchanged? Why?
c. Will η_T increase, decrease or remain unchanged? Why?

Part a; As indicated by Table 4.3 the resonance integral increases when replacing U with UO_2. This effect is magnified by the lower density of UO_2.

Part b. The thermal utilization may decrease slightly as a result of the lower density of UO_2. The thermal absorption cross section of oxygen is so small that it has little effect.

Part c. Since the thermal absorption cross section of oxygen is so small it causes only a very slight decrease in η_T. The decrease in density cancels out of the definition of η_T.

[4.12] The fuel for a thermal reactor has the following composition by atom ratio: 2% uranium-235, 1 % plutonium-239 and 97% uranium-238. Calculate the value of η_T to be used for this fuel in the four factor formula. (Use the data given for problem [3.4].)

$$\eta_T = \frac{\nu\Sigma_f^f}{\Sigma_a^f} = \frac{0.02N\nu^{25}\sigma_f^{25} + 0.01N\nu^{49}\sigma_f^{49}}{0.02N\sigma_a^{25} + 0.01N\sigma_a^{49} + 0.97N\sigma_a^{28}}$$

$$\eta_T = \frac{0.02 \cdot 2.43 \cdot 505 + 0.01 \cdot 2.90 \cdot 698}{0.02 \cdot 591 + 0.01 \cdot 973 + 0.97 \cdot 2.42}$$

$$\eta_T = \frac{24.54 + 20.24}{11.82 + 9.73 + 2.35}$$

$$\eta_T = 1.87$$

Chapter 5 Problems:

[5.1] What is l_∞ of 0.0253 eV thermal neutron in graphite ($\Sigma_a = 0.273 \cdot 10^{-3} cm^{-1}$)?
What is l_∞ in of 1 MeV fast neutron in iron ($\Sigma_a = 0.738 \cdot 10^{-3} cm^{-1}$)?
(Note that a 0.0253 eV neutron has a speed of 2200m/s)

Since $l_\infty = 1 / v\Sigma_a$ For the thermal neutron

$$l_\infty = 1 / 2.2 \cdot 10^5 cm\ s^{-1} \cdot 0.273 \cdot 10^{-3} cm^{-1} = 0.0167 s$$

For the fast neutron, we must first calculate the speed. Since kinetic energy is proportional to speed squared $E_2/E_1 = v_2^2/v_1^2$ or $v_2 = v_1\sqrt{E_2/E_1}$. Thus

$$v_2 = 2.2 \cdot 10^5 cm\ s^{-1}\sqrt{10^6 eV/0.0253 eV} = 13.8 \cdot 10^8 cm/s$$ Thus

$$l_\infty = 1 / 13.8 \cdot 10^8 cm\ s^{-1} \cdot 0.738 \cdot 10^{-3} cm^{-1} = 0.98 \cdot 10^{-6} s$$

[5.2] A power reactor is fueled with slightly enriched uranium. At the end of core life 30% of the power comes from the fissioning of the built up plutonium-239. Calculate the effective value of β . at the beginning and at the end of core life; determine the percent increase or decrease.

The power produced is proportional to the macroscopic fission cross section. Thus $\Sigma_f = \Sigma_f^U + \Sigma_f^{Pu}$. Thus $\Sigma_f^{Pu} / \Sigma_f = 0.30$ and $\Sigma_f^{Pu} / \Sigma_f^U = 0.30/0.70 = 0.429$. In Eq. (5.37) β appears in the term $\beta\nu\Sigma_f$. Dividing the number of delayed neutrons into uranium and plutonium contributions:

$$\beta\nu\Sigma_f = \beta_U \nu_U \Sigma_f^U + \beta_{Pu}\nu_{Pu}\Sigma_f^{Pu} \text{ and also } \nu\Sigma_f = \nu_U\Sigma_f^U + \nu_{Pu}\Sigma_f^{Pu} \text{ Thus}$$

$$\beta = \frac{\beta_U \nu_U \Sigma_f^U + \beta_{Pu}\nu_{Pu}\Sigma_f^{Pu}}{\nu_U\Sigma_f^U + \nu_{Pu}\Sigma_f^{Pu}} = \frac{\beta_U + \beta_{Pu}(\nu_{Pu}/\nu_U)(\Sigma_f^{Pu}/\Sigma_f^U)}{1 + (\nu_{Pu}/\nu_U)(\Sigma_f^{Pu}/\Sigma_f^U)}$$

From Table 5.1 we take $\nu_U = 2.43$, $\nu_{Pu} = 2.90$, $\beta_U = 0.00650$ and $\beta_{Pu} = 0.00210$. Thus

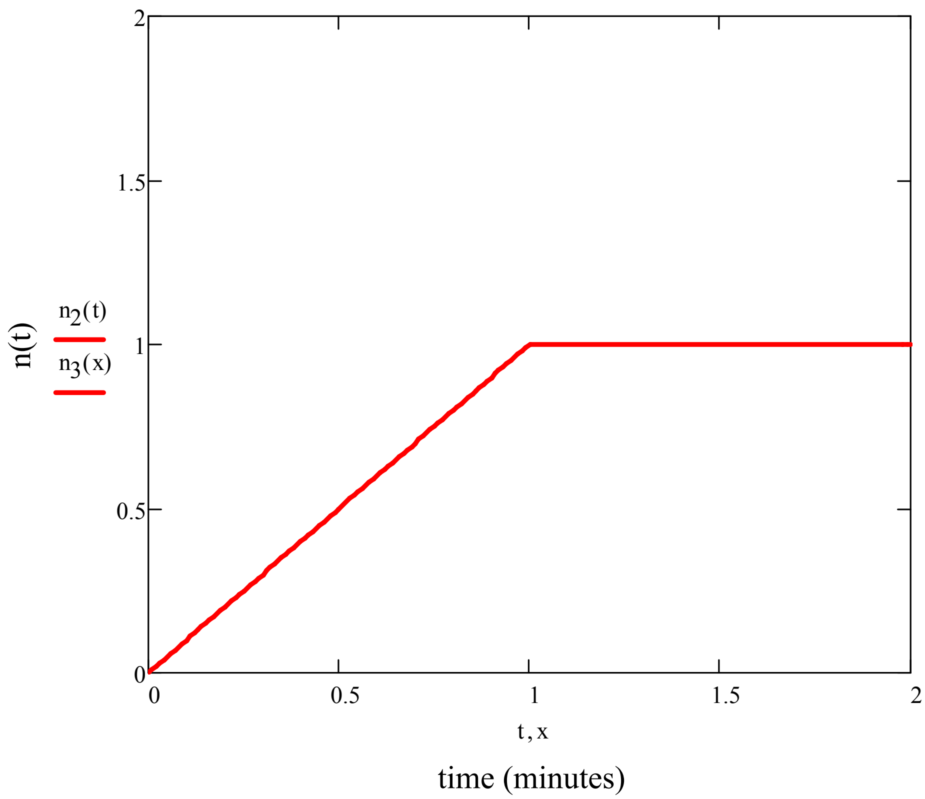
$$\beta = \frac{0.00650 + 0.00210(2.90/2.43)(0.429)}{1 + (2.90/2.43)(0.429)} = 0.00501$$

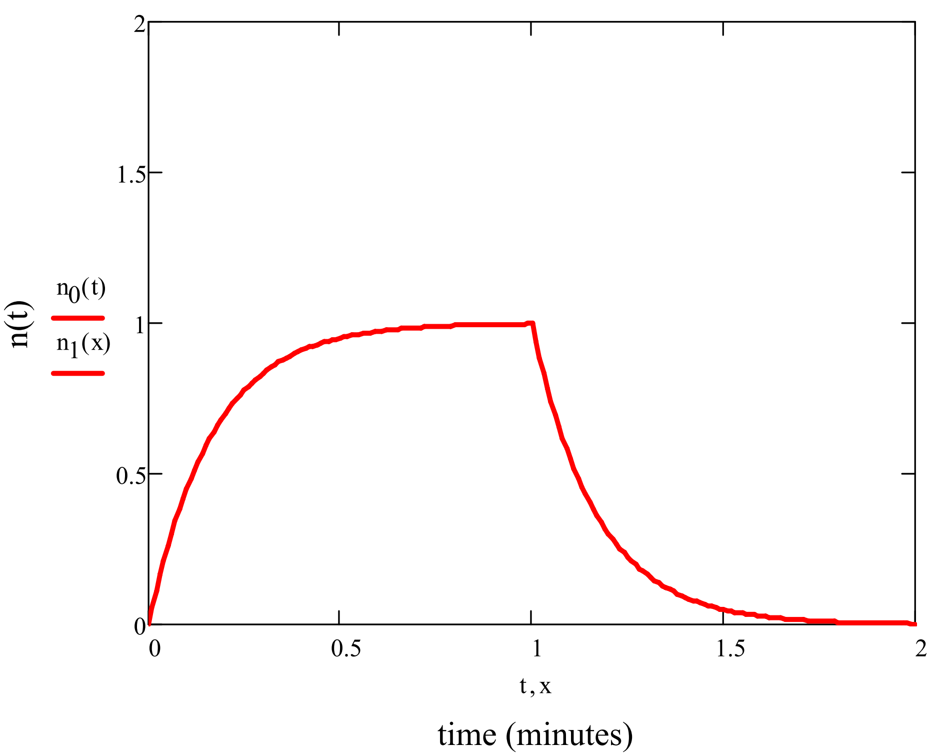
Thus $\beta/\beta_U = 0.00501/0.00650 = 0.77$ or a 23% decrease.

Simply taking $\beta = 0.7\beta_U + 0.3\beta_{Pu}$ yields β=0.515, which is quite close.
Note that we have used the same value of ν_U for the small amount of fast fission in uranium-238 and for the uranium-235 fission. Distinguishing between the two would have made an even smaller difference in the result.

[5.3] At $t = 0$ there are no neutrons in a reactor. A neutron source is inserted into the reactor at $t = 0$, but then withdrawn at $t = 1$ min. Sketch the neutron population for $0 \le t \le 2$ min:

a. for a subcritical reactor,
b. for a critical reactor.





[5.4] Suppose that a fissile material is discovered for which all of the neutrons are prompt. The neutron population is then governed by Eq. (5.22). Furthermore suppose that a reactor fueled with this material has a prompt neutron lifetime of 0.002 s.

a. If the reactor is initially critical, and there is no source present, what period should the reactor be put on if it is to triple its power in ten seconds?
b. What is the reactivity ρ needed in part a?

Solution:

a. Since the reactor period is determined from $n(t) = n(0)\exp(t/T)$ or
   $T = t/(\ln n(t)/n(0))$ $\quad T = 10/(\ln 3) = 9.1$ sec.
b. Since $n(t) = n(0)\exp(\rho/\Lambda)$, $k-1 = (\ln n(t)/n(0))l/t = \ln(3)\cdot 0.002/10 = 0.00022$
   Thus k=1.00022 and $\rho = 0.00022$

[5.5] Show that Eq. (5.47) and (5.48) result from inserting the definitions of ρ and Λ into Eqs. (5.40) and (5.41)

Substituting $l = \Lambda k$ Into Eqs. (5.40)

$$\frac{d}{dt}n(t) = S(t) + \frac{1}{\Lambda}[1-\beta-\frac{1}{k}]n(t) + \sum_i \lambda_i C_i(t)$$

Since $\rho = \frac{k-1}{k} = 1 - \frac{1}{k}$, we have $[1-\beta-\frac{1}{k}] = [1-\frac{1}{k}-\beta] = \rho - \beta$

and Eq. (5.40) becomes

$$\frac{d}{dt}n(t) = S(t) + \frac{1}{\Lambda}(\rho-\beta)n(t) + \sum_i \lambda_i C_i(t)$$

Likewise, substituting $l = \Lambda k$ into Eq. (5.41) yields

$$\frac{d}{dt}C_i(t) = \beta_i \frac{1}{\Lambda}n(t) - \lambda_i C_i(t) .$$

[5.6] A thermal reactor fueled with uranium operates at 1.0 W. The operator is to increase the power to 1.0 kW over a two-hour span of time.

a. What reactor period should she put the reactor on?
b. How many cents of reactivity must be present to achieve the period in part a.?

a. $\exp(12/T) = 10^3$. Thus $T = 2/\ln(10^3) = 0.290 hr = 17.4 \min$

c. Since the period is very long the positive reactivity must be quite small relative to $\beta =$, we may use the approximation from p 82: $T \approx \beta/(\rho\lambda)$ or $\rho \approx \beta/(\lambda T)$ Using Eq. (5.34) and Table 5.1 ,

	I	lamda	beta	beta/lamda
	1	0.0124	0.000215	0.01733871
	2	0.0305	0.001424	0.046688525
	3	0.111	0.001274	0.011477477
	4	0.301	0.002568	0.008531561
	5	1.14	0.000748	0.00065614
	6	3.01	0.000273	9.06977E-05
total			0.006502	0.084783111
	1/lamda=sum/beta			13.03954341
	lamda =			0.076689802

$\lambda \approx 0.077\ s^{-1}$ Hence

$\rho/\beta \approx 1/(0.077\ s^{-1} \cdot 17.4 \min \cdot 60 s/\min) = 0.0124\ dollars = 1.24 cents$

[5.7] A thermal reactor fueled with uranium operates at 1.0 W. The operators put in on a 15 min period. How long will the reactor take to reach a power of 1.0 MW?
We have simply, $n(t) = n(0)\exp(t/T)$ or

$$t = T(\ln n(t)/n(0)) = 15 \cdot \ln(10^6) = 207\text{min.} = 3.45 \text{ hrs.}$$

[5.8] Show that Eq. (5.53) follows from Eq. (5.51)

First regroup terms in Eq. (5.51):

$$\omega = \frac{\rho}{\Lambda} + \frac{1}{\Lambda}\left(\sum_i \frac{\beta_i \lambda_i}{\omega + \lambda_i} - \beta\right) = \frac{\rho}{\Lambda} + \frac{1}{\Lambda}\left(\sum_i \frac{\beta_i \lambda_i}{\omega + \lambda_i} - \sum_i \beta_i\right)$$

$$= \frac{\rho}{\Lambda} + \frac{1}{\Lambda}\sum_i \left(\frac{\beta_i \lambda_i}{\omega + \lambda_i} - \beta_i\right) = \frac{\rho}{\Lambda} + \frac{1}{\Lambda}\sum_i \frac{1}{\omega + \lambda_i}\left(\beta_i \lambda_i - \beta_i(\omega + \lambda_i)\right)$$

$$\omega = \frac{\rho}{\Lambda} + \frac{1}{\Lambda}\sum_i \frac{\beta_i}{\omega + \lambda_i}\omega$$

Solving for ρ:

$$\rho = \left(\Lambda + \sum_i \frac{\beta_i}{\omega + \lambda_i}\right)\omega$$

[5.9] Find the periods for reactors fuel by uranium-235, plutonium-239, and uranium-233 if

a. one cent of reactivity as added to the critical systems
b. one cent of reactivity is withdrawn for the critical systems.

Since only one cent of reactivity is involved Eq. (5.57) is a reasonable approximation for both positive and negative periods of parts a & b respectively. Table 5.1 gives the delayed neutron fractions. However, we must use Eq. (5.32) or (5.34) to calculate λ.

	beta(I)			beta(I)/lamda(I)		
half-life (s)	U-233	U-235	Pu-239	U-233	U-235	Pu-239
56	0.00023	0.00021	0.00007	0.018586	0.016970	0.005657
23	0.00078	0.00142	0.00063	0.025887	0.047128	0.020909
6.2	0.00064	0.00128	0.00044	0.005726	0.011452	0.003937
2.3	0.00074	0.00257	0.00069	0.002456	0.008530	0.002290
0.61	0.00014	0.00075	0.00018	0.000123	0.000660	0.000158
0.23	0.00008	0.00027	0.00009	0.000027	0.000090	0.000030
sum:	0.00261	0.0065	0.0021	0.052805	0.084829	0.032981
1/lamda:				20.231765	13.050638	15.705009
lamda				0.049427	0.076625	0.063674

Part a: $T = \beta/(\rho\lambda) = \beta/(0.01\beta\lambda) = 100/\lambda$

U-233 $T = 100/0.0494 = 2023\ s = 33.7\,\text{min}.$

U-235 $T = 100/0.0766 = 1305\ s = 21.8\,\text{min}.$

Pu-239 $T = 100/0.0637 = 1570\ s = 26.2\,\text{min}.$

Part b $T = \beta/(\rho\lambda) = -\beta/(0.01\beta\lambda) = -100/\lambda$

U-233 $T = -100/0.0494 = -2023\ s = -33.7\,\text{min}.$

U-235 $T = -100/0.0766 = -1305\ s = -21.8\,\text{min}.$

Pu-239 $T = -100/0.0637 = -1570\ s = -26.2\,\text{min}.$

[5.10] The one delayed group approximation results from lumping all six precursors into one, $C(t)=\sum_{i=1}^{6}C_i(t)$ and replacing the λ_i by the average value defined by Eq. (5.34). Equations (5.47) and (5.48) then reduce to the one delayed group equations:

$$\frac{d}{dt}n(t)=S(t)+\frac{(\rho-\beta)}{\Lambda}n(t)+\lambda C(t)$$

and

$$\frac{d}{dt}C(t)=\frac{\beta}{\Lambda}n(t)-\lambda C(t).$$

With these equations, consider a critical reactor that is initially operating with a neutron population of $n(0)$ and for which $S(t)=0$. At $t=0$ a step reactivity change ρ is made. Using the assumptions that

$$\frac{1}{\lambda\Lambda}|\beta-\rho| \gg 1 \text{ and } \quad \frac{\beta}{\lambda\Lambda} \gg 1$$

a. show that

$$n(t)=n(0)\left\{\frac{\rho}{\rho-\beta}\exp\left[\frac{\rho-\beta}{\Lambda}t\right]+\frac{\beta}{\beta-\rho}\exp\left[\frac{\lambda\rho}{\beta-\rho}t\right]\right\}$$

b. Show that for long times the solution is independent of λ when $\rho>\beta$ and independent of Λ when $0<\rho<\beta$.

c. Taking $\beta=0.007$, $\Lambda=5\cdot 10^{-5}$ s. and $\lambda=0.08$ s.$^{-1}$, make a graph of the reactivity in dollars vs. the reactor period for reactivities between –2 dollars and +2 dollars. Indicate the region or regions on the graph where you expect the results from part a. to be poor.

Part a: With $S(t)=0$, substitute $n(t)=n\exp(\omega t)$ and $C(t)=C\exp(\omega t)$ into the two differential equations. We obtain

$$\omega n=\frac{(\rho-\beta)}{\Lambda}n+\lambda C$$

and

$$\omega C=\frac{\beta}{\Lambda}n-\lambda C.$$

Eliminating C between the equations, we obtain

$$\omega n=\frac{(\rho-\beta)}{\Lambda}n+\frac{\lambda}{\omega}\left(\frac{\rho}{\Lambda}-\omega\right)n$$

from which we obtain a quadratic equation for ω:

$$\Lambda\omega^2+(\Lambda\lambda+\beta-\rho)\omega-\lambda\rho=0$$

However, since $\frac{1}{\lambda\Lambda}|\beta-\rho| \gg 1$ we may simplify the quadratic to

$$\Lambda\omega^2+(\beta-\rho)\omega-\lambda\rho=0$$

Which has solutions of

$$\omega = \frac{1}{2\Lambda}\left[-(\beta-\rho) \pm \sqrt{(\beta-\rho)^2 + 4\lambda\Lambda\rho}\,\right]$$
$$= \frac{(\beta-\rho)}{2\Lambda}\left[-1 \pm \sqrt{1+4\lambda\Lambda\rho(\beta-\rho)^{-2}}\,\right]$$

but since $\lambda\Lambda(\beta-\rho)^{-1} \ll 1$, $\sqrt{1+4\lambda\Lambda\rho(\beta-\rho)^{-2}} \approx 1 + 2\lambda\Lambda\rho(\beta-\rho)^{-2}$ and hence the two solutions are

$$\omega_+ = \lambda\rho/(\beta-\rho) \text{ and } \omega_- = (\rho-\beta)/\Lambda \text{ and thus}$$
$$n(t) = n_+\exp(\omega_+ t) + n_-\exp(\omega_- t)$$

and

$$C(t) = C_+\exp(\omega_+ t) + C_-\exp(\omega_- t)$$

To determine $n_\pm$ we note that for a critical initial condition: $C(0) = \dfrac{\beta}{\lambda\Lambda}n(0)$ and thus

$$n(0) = n_+ + n_-$$

and recalling that $C = \dfrac{1}{\omega}\left(\dfrac{\rho}{\Lambda} - \omega\right)n = \left(\dfrac{\rho}{\Lambda\omega} - 1\right)n$

$$\frac{\beta}{\lambda\Lambda}n(0) = \left(\frac{\rho}{\Lambda\omega_+} - 1\right)n_+ + \left(\frac{\rho}{\Lambda\omega_-} - 1\right)n_-$$

Adding equations, and inserting the values of $\omega_\pm$:

$$\left(\frac{\beta}{\lambda\Lambda} + 1\right)n(0) = \frac{(\beta-\rho)}{\lambda\Lambda}n_+ + \frac{\beta}{\rho-\beta}n_-$$

Now since $\dfrac{\beta}{\lambda\Lambda} \gg 1$, and $\dfrac{|(\beta-\rho)|}{\lambda\Lambda} \gg 1$, while at the same time $\dfrac{\beta}{\rho-\beta}$ cannot be large without violating the constraint that $\rho \not\approx \beta$, we have $\dfrac{\beta}{\lambda\Lambda}n(0) = \dfrac{(\beta-\rho)}{\lambda\Lambda}n_+$ or

$$n_+ = \frac{\beta}{\beta-\rho}n(0) \text{ and hence } n_- = n(0) - n_+ = \frac{\rho}{\rho-\beta}n(0)$$

Combining equations for $n_\pm$ and $\omega_\pm$ we obtain the equation specified in part a.

For part b. note that when $\rho > \beta$ the argument of the first exponential is positive, while that of the second exponential is negative. Thus the second term dies away, leaving

$$n(t) \simeq n(0)\frac{\rho}{\rho-\beta}\exp\left[\frac{\rho-\beta}{\Lambda}t\right], \qquad \rho > \beta \quad \text{which is independent of } \lambda.$$

Conversely when $\rho < \beta$ the argument of the second exponential is positive, while that of the second is negative, causing the first to die away, and

$$n(t) \simeq n(0)\frac{\beta}{\beta-\rho}\exp\left[\frac{\lambda\rho}{\beta-\rho}t\right], \qquad \rho < \beta \quad \text{which is independent of } \Lambda.$$

For part c: Approximation is poor when ρ is in the vicinity of β.

[5.11] Using the kinetics equations with one-delayed group from problem [10]:

a. find the reactor period when $\rho = \beta$, simplifying your answer by assuming $\lambda\Lambda / \beta \ll 1$.

b. Calculate the reactor period for $\beta = 0.007$, $\Lambda = 5 \cdot 10^{-5}$ s, and $\lambda = 0.08$ s^{-1}.

For $\rho = \beta$, and S=0the first of the pair of equations reduces to

$\frac{d}{dt} n(t) = \lambda C(t)$ Let $n(t) = A\exp(\omega t)$ and $C(t) = B\exp(\omega t)$ The two equations then reduce to $\omega A = \lambda B$ and $\omega B = \frac{\beta}{\Lambda} A - \lambda CB$ Eliminating A and B between then gives the quadratic equation $\omega^2 + \lambda\omega - \lambda\beta / \Lambda = 0$ which has the solutions:

$\omega = \frac{-\lambda \pm \sqrt{\lambda^2 + 4\lambda\beta/\Lambda}}{2} = \frac{\lambda}{2}\left(-1 \pm \sqrt{1 + 4\beta/\lambda\Lambda}\right)$ The period corresponds to the inverse of the largest ω. Thus

$$T = \frac{2}{\lambda}\left(-1 + \sqrt{1 + 4\beta/\lambda\Lambda}\right)^{-1}$$

Since $\lambda\Lambda/\beta \ll 1$, we have $\beta/\lambda\Lambda \gg 1$ and

$-1 + \sqrt{1 + 4\beta/\lambda\Lambda} \approx -1 + 2\sqrt{\beta/\lambda\Lambda} \approx 2\sqrt{\beta/\lambda\Lambda}$ Hence

$$T \approx \sqrt{\Lambda/\lambda\beta}$$

For the data given, $\lambda\Lambda/\beta = 0.08 \cdot 5 \cdot 10^{-5} / 0.007 = 0.000571 \ll 1$ Thus

$$T \approx \sqrt{5 \cdot 10^{-5} / 0.08 \cdot 0.007} = 0.299\ s$$

[5.12] By differentiating the kinetics equations with one group of delayed neutrons (given in problem [10] and then letting $\Lambda \to 0$:

a. Show that $C(t)$ can be eliminated to obtain

$$\frac{d}{dt}n(t) = \frac{\lambda}{\beta-\rho}\left(\rho + \frac{1}{\lambda}\frac{d\rho}{dt}\right)n(t)$$

which is referred to as the zero lifetime or prompt jump approximation.

b. For a step change in reactivity of $|\rho| \ll \beta$, find the zero lifetime approximation to the reactor period.

For part a, assume there is no source, and differentiate the first equation from problem [10]:

$$\frac{d^2}{dt^2}n = \frac{n}{\Lambda}\frac{d}{dt}\rho + \frac{(\rho-\beta)}{\Lambda}\frac{d}{dt}n + \lambda\frac{d}{dt}C$$

Now substitute the second equation from problem [10] to eliminate the derivative of *C*:

$$\frac{d^2}{dt^2}n = \frac{n}{\Lambda}\frac{d}{dt}\rho + \frac{(\rho-\beta)}{\Lambda}\frac{d}{dt}n + \frac{\lambda\beta}{\Lambda}n - \lambda^2 C$$

Then solve the first equation from problem [10] for C in order to eliminate C from this equation

$$\frac{d}{dt}n(t) - \frac{(\rho-\beta)}{\Lambda}n(t) = \lambda C(t), \text{ and hence}$$

$$\frac{d^2}{dt^2}n = \frac{n}{\Lambda}\frac{d}{dt}\rho + \frac{(\rho-\beta)}{\Lambda}\frac{d}{dt}n + \cancel{\frac{\lambda\beta}{\Lambda}}n - \lambda\frac{d}{dt}n + \lambda\frac{(\rho-\cancel{\beta})}{\Lambda}n$$

Hence

$$\Lambda\frac{d^2}{dt^2}n = \left[(\rho-\beta) - \Lambda\lambda\right]\frac{d}{dt}n + \left(\lambda\rho + \frac{d}{dt}\rho\right)n$$

Setting Λ equal to zero we have:

$$\frac{d}{dt}n = \frac{1}{(\beta-\rho)}\left(\lambda\rho + \frac{d}{dt}\rho\right)n$$

For part b, $|\rho| \ll \beta$ and $\frac{d}{dt}\rho=0$ Thus.

$$\frac{d}{dt}n = \frac{\lambda\rho}{\beta}n \quad \text{or} \quad T = \frac{\beta}{\lambda\rho}$$

[5.13] A critical reactor operates at a power level of 80 W. Dropping a control rod into the core causes the flux to undergo a sudden decrease to 60 W. How many dollars is the control rod worth? .

From Eq. (3.63) $|\rho| / \beta = \frac{n_0}{n_1} - 1 = \frac{80}{60} - 1 = 0.333\$$

[5.14] Estimate the period of the reactor from the curve without feedback in Fig. 5.6. Suppose you wanted to put the reactor on a period of one minute. What reactivity would you insert?

From the logarithmic plot without feedback in Fig. 5.14, the neutron population increases from approximately 1.1 to 1,000 in about 625 s. Thus

$$10^3/1.1 = \exp(625/T) \text{ or } T = 625/\ln(10^3/1.1) = 91\,\text{s}$$

Since this is for a reactivity of 0.1\$, Eq. (5.57) holds, and we can write $\dfrac{\rho}{\beta} \approx \dfrac{1}{T\lambda}$. Thus

$$\frac{\rho_2}{\beta} \Big/ \frac{\rho_1}{\beta} \approx \frac{T_2}{T_1}\,, \quad \frac{\rho_2}{\beta} \approx \frac{T_1}{T_2}\frac{\rho_1}{\beta} = \frac{91}{60}0.1\$ = 0.152\$$$

[5.15]* Solve the kinetics equations numerically with one group of delayed neutrons, given in problem [5.10], using the data $\beta = 0.007$, $\Lambda = 5 \cdot 10^{-5}$ s. and $\lambda = 0.08$ s.$^{-1}$

a. For a step insertion of $+0.25$ dollars between 0 and 5 s.
b. For a step insertion of -0.25 dollars between 0 and 5 s.

Assume that the reactor is initially in the critical state, and plot $n(t)/n(0)$ for each case.

The kinetics equations are set equal to zero to find the initial conditions.

$$c := \frac{\beta \cdot P_0}{\lambda \cdot \Lambda} = 1.75 \times 10^{3}$$

Use a program such as Mathematica, Matlab, or in this case Mathcad 14 to solve the pair of differential equations.

It is important to note that due to the effects of prompt neutrons, it will be best to solve the differential equations using a stiff method, in this case, the Radau function in Mathcad.

Dividing the results by the initial condition standardizes the results to obtain the graph below

a)

$$\rho_1 := 0.25 \cdot \beta = 1.75 \times 10^{-3}$$

$$D_1(t,y) := \begin{bmatrix} \dfrac{(\rho_1 - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\dfrac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

$$Y := \mathrm{Radau}\left(\frac{y}{P_0}, 0, 50, 10000, D_1\right)$$

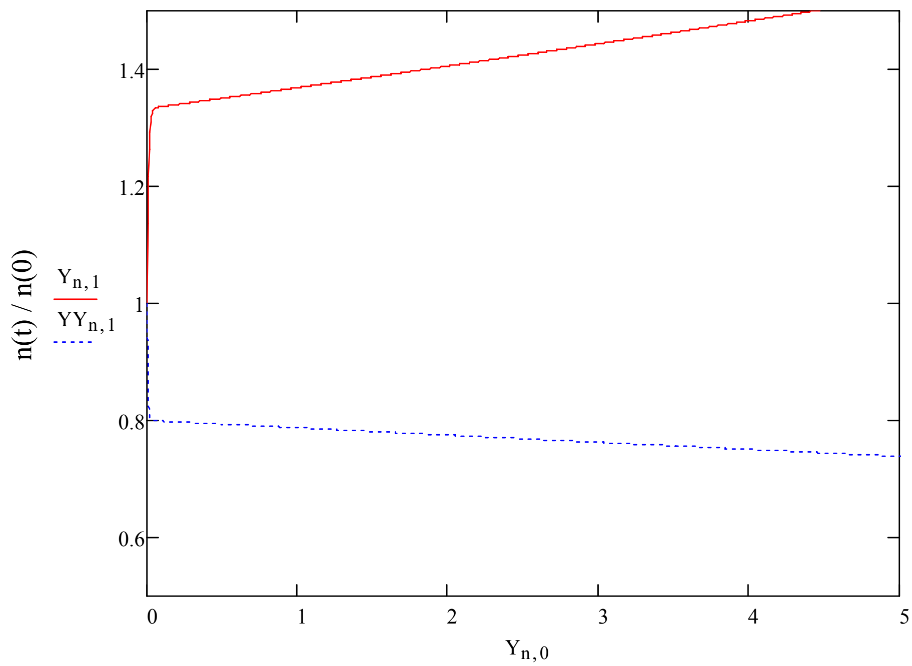
b)

$$\rho_2 := -0.25 \cdot \beta = -1.75 \times 10^{-3}$$

$$D_2(t,y) := \begin{bmatrix} \dfrac{(\rho_2 - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\dfrac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

$$YY := \mathrm{Radau}\left(\frac{y}{P_0}, 0, 50, 10000, D_2\right)$$

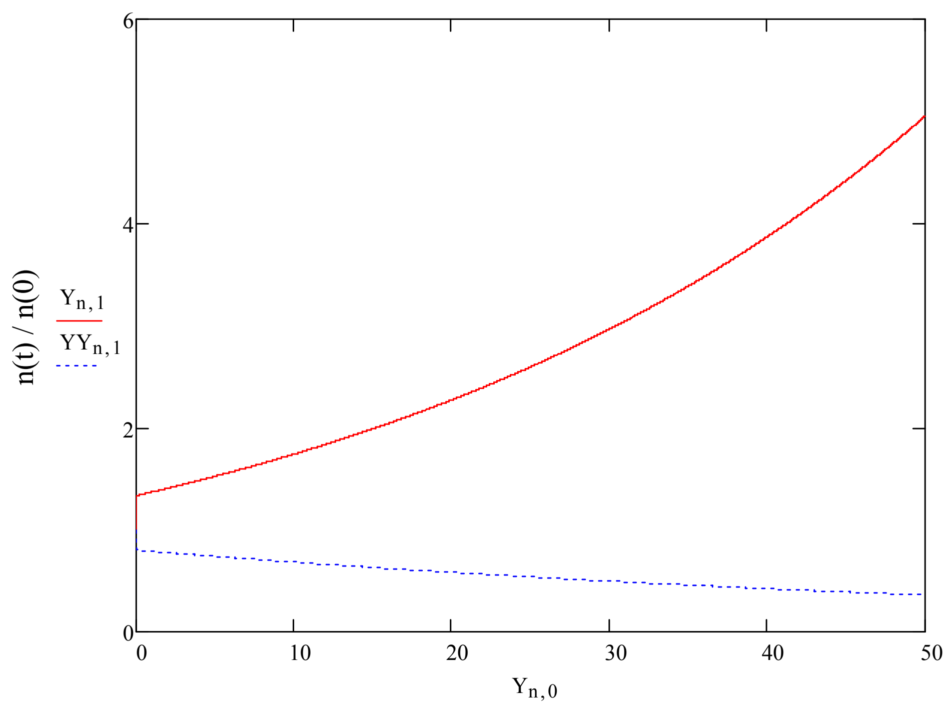
$$n := 0, 1 .. 1000C$$



time, seconds

+0.25 \$
-0.25 \$

However, taking into account the reactor's period is 50 seconds, the plot will look like the one shown below.



time, seconds

+0.25 \$
-0.25 \$

[5.16]* Repeat problem [5.15] with six groups of delayed neutrons, employing the data for uranium-235 given in Table 5.1 , and a prompt generation time of $50x10^{-6}$ s.

Using equation (5.33) and the Approximate Half-Life in seconds from Table 5.1, we calculate

$$\lambda_i := \frac{0.693}{t_{0.5i}}$$

Approximate Half-Life (sec)
56
23
6.2
2.3
0.61
0.23

$$\lambda_1 := \frac{.693}{56} = 0.01237 \qquad \beta_1 := 0.00021$$

$$\lambda_2 := \frac{.693}{23} = 0.03013 \qquad \beta_2 := 0.00142$$

$$\lambda_3 := \frac{.693}{6.2} = 0.11177 \qquad \beta_3 := 0.00128$$

$$\lambda_4 := \frac{.693}{2.3} = 0.3013 \qquad \beta_4 := 0.00257$$

$$\lambda_5 := \frac{.693}{0.61} = 1.13607 \qquad \beta_5 := 0.00075$$

$$\lambda_6 := \frac{.693}{0.23} = 3.01304 \qquad \beta_6 := 0.00027$$

$$\beta := \beta_1 + \beta_2 + \beta_3 + \beta_4 + \beta_5 + \beta_6 \qquad \Lambda := 50 \cdot 10^{-6}$$

$$\beta = 6.5 \times 10^{-3} \qquad P_0 := 1.0$$

$$c_1 := \beta_1 \cdot \frac{P_0}{(\lambda_1 \cdot \Lambda)} = 339.394 \qquad c_2 := \beta_2 \cdot \frac{P_0}{(\lambda_2 \cdot \Lambda)} = 942.569$$

$$c_3 := \beta_3 \cdot \frac{P_0}{(\lambda_3 \cdot \Lambda)} = 229.033 \qquad c_4 := \beta_4 \cdot \frac{P_0}{(\lambda_4 \cdot \Lambda)} = 170.592$$

$$c_5 := \beta_5 \cdot \frac{P_0}{(\lambda_5 \cdot \Lambda)} = 13.203 \qquad c_6 := \beta_6 \cdot \frac{P_0}{(\lambda_6 \cdot \Lambda)} = 1.792$$

$$y := \begin{pmatrix} P_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{pmatrix}$$

$\rho_1 := 0.25 \cdot \beta = 1.625 \times 10^{-3}$

$$D_1(t,y) := \begin{bmatrix} \dfrac{(\rho_1 - \beta)}{\Lambda} y0 + \lambda_1 \cdot y1 + \lambda_2 \cdot y2 + \lambda_3 \cdot y3 + \lambda_4 \cdot y4 + \lambda_5 \cdot y5 + \lambda_6 \cdot y6 \\ \left(\dfrac{\beta_1}{\Lambda}\right) \cdot y0 - \lambda_1 \cdot y1 \\ \left(\dfrac{\beta_2}{\Lambda}\right) \cdot y0 - \lambda_2 \cdot y2 \\ \left(\dfrac{\beta_3}{\Lambda}\right) \cdot y0 - \lambda_3 \cdot y3 \\ \left(\dfrac{\beta_4}{\Lambda}\right) \cdot y0 - \lambda_4 \cdot y4 \\ \left(\dfrac{\beta_5}{\Lambda}\right) \cdot y0 - \lambda_5 \cdot y5 \\ \left(\dfrac{\beta_6}{\Lambda}\right) \cdot y0 - \lambda_6 \cdot y6 \end{bmatrix}$$

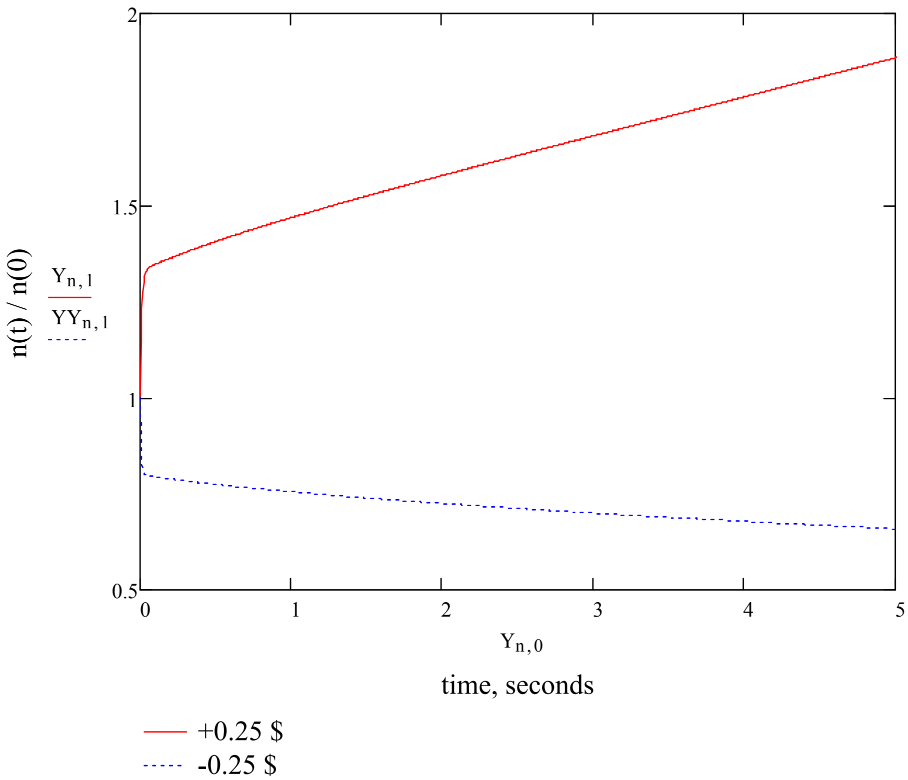
$Y := \text{Radau}(y, 0, 60, 10000, D_1)$

$\rho_2 := -0.25 \cdot \beta = -1.625 \times 10^{-3}$

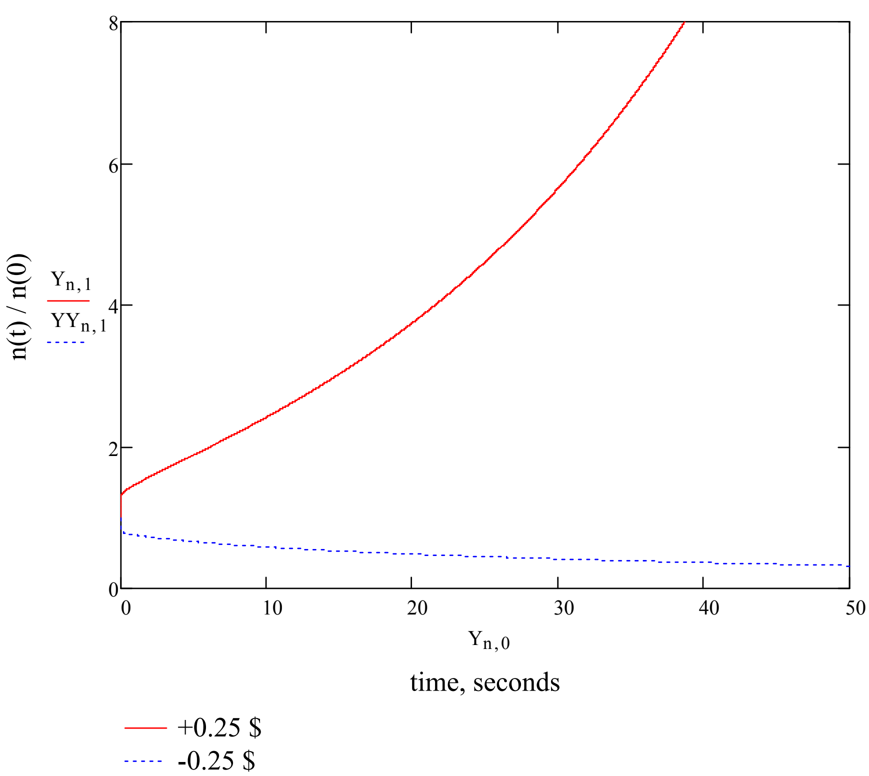
$$D_2(t,y) := \begin{bmatrix} \dfrac{(\rho_2 - \beta)}{\Lambda} y0 + \lambda_1 \cdot y1 + \lambda_2 \cdot y2 + \lambda_3 \cdot y3 + \lambda_4 \cdot y4 + \lambda_5 \cdot y5 + \lambda_6 \cdot y6 \\ \left(\dfrac{\beta_1}{\Lambda}\right) \cdot y0 - \lambda_1 \cdot y1 \\ \left(\dfrac{\beta_2}{\Lambda}\right) \cdot y0 - \lambda_2 \cdot y2 \\ \left(\dfrac{\beta_3}{\Lambda}\right) \cdot y0 - \lambda_3 \cdot y3 \\ \left(\dfrac{\beta_4}{\Lambda}\right) \cdot y0 - \lambda_4 \cdot y4 \\ \left(\dfrac{\beta_5}{\Lambda}\right) \cdot y0 - \lambda_5 \cdot y5 \\ \left(\dfrac{\beta_6}{\Lambda}\right) \cdot y0 - \lambda_6 \cdot y6 \end{bmatrix}$$

$YY := \text{Radau}(y, 0, 60, 10000, D_2)$

$n := 0 .. 10000$



However, taking into account the reactor's period, the plot will look like:



[5.17]* Consider a subcritical reactor described by the pair of kinetics equations in problem [5.10], and with $\beta = 0.007$, $\Lambda = 5 \cdot 10^{-5}$ s. and $\lambda = 0.08$ s.$^{-1}$. The subcritical system is in steady state equilibrium with reactivity $\rho = -10$ cents and a time-independent source S_o. Then, at $t = 0$ the source is removed. Determine $n(t)$ for $t \geq 0$ and plot your result, normalized to S_o.

$$\rho_1 := -.1 \cdot \beta = -7 \times 10^{-4}$$

$$P_0 := 1.0$$

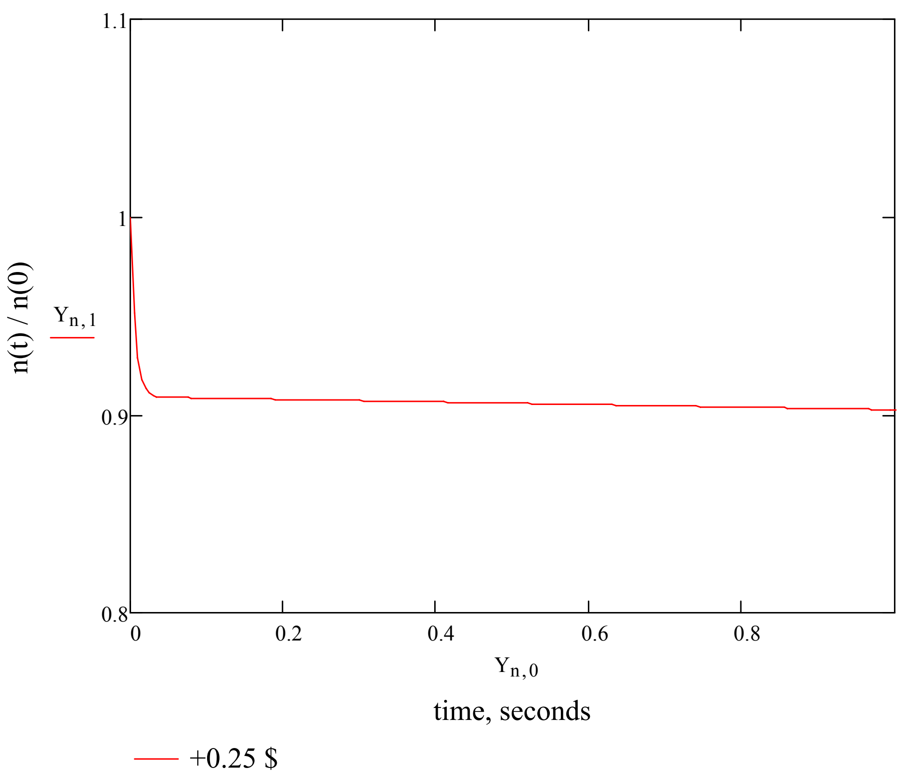
$$c := \frac{\beta \cdot P_0}{\lambda \cdot \Lambda} = 1.75 \times 10^3$$

$$y := \begin{pmatrix} P_0 \\ c \end{pmatrix}$$

$$D_1(t, y) := \begin{bmatrix} \frac{(\rho_1 - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\frac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

$$Y := \mathrm{Radau}\left(\frac{y}{P_0}, 0, 50, 10000, D_1\right)$$

$$n := 0, 1 .. 1000($$



[5.18]* Solve Eqs. (5.47) and (5.48) using the uranium-235 data from table 5.1 and a prompt generation time of $50\text{x}10^{-6}$ s. for the following three ramp insertions.

a. $\rho(t) = 0.25\beta t$
b. $\rho(t) = 0.5\beta t$
c. $\rho(t) = 1.0\beta t$

Normalizing your results to $n(0) = 1$, make linear plots and for each case determine: (1) at what time does the neutron population reach $n(t)/n(0) = 1{,}000$ and (2) What is the value of $n(t)/n(0)$ at the point in time when the system reaches prompt critical?

1)

a)

$$\rho_1(t) := 0.25 \cdot \beta \cdot t$$

$$D_1(t,y) := \begin{bmatrix} \frac{(\rho_1(t) - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\frac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

$$Y := \text{Radau}\left(\frac{y}{P_0}, 0, 7, 10000, D_1\right)$$

b)

$$\rho_2(t) := 0.5 \cdot \beta \cdot t$$

$$D_2(t,y) := \begin{bmatrix} \frac{(\rho_2(t) - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\frac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

$$YY := \text{Radau}\left(\frac{y}{P_0}, 0, 5, 10000, D_2\right)$$

c)

$$\rho_3(t) := 1.0 \cdot \beta \cdot t$$

$$D_3(t,y) := \begin{bmatrix} \frac{(\rho_3(t) - \beta)}{\Lambda} y_0 + \lambda \cdot y_1 \\ \left(\frac{\beta}{\Lambda}\right) \cdot y_0 - \lambda \cdot y_1 \end{bmatrix}$$

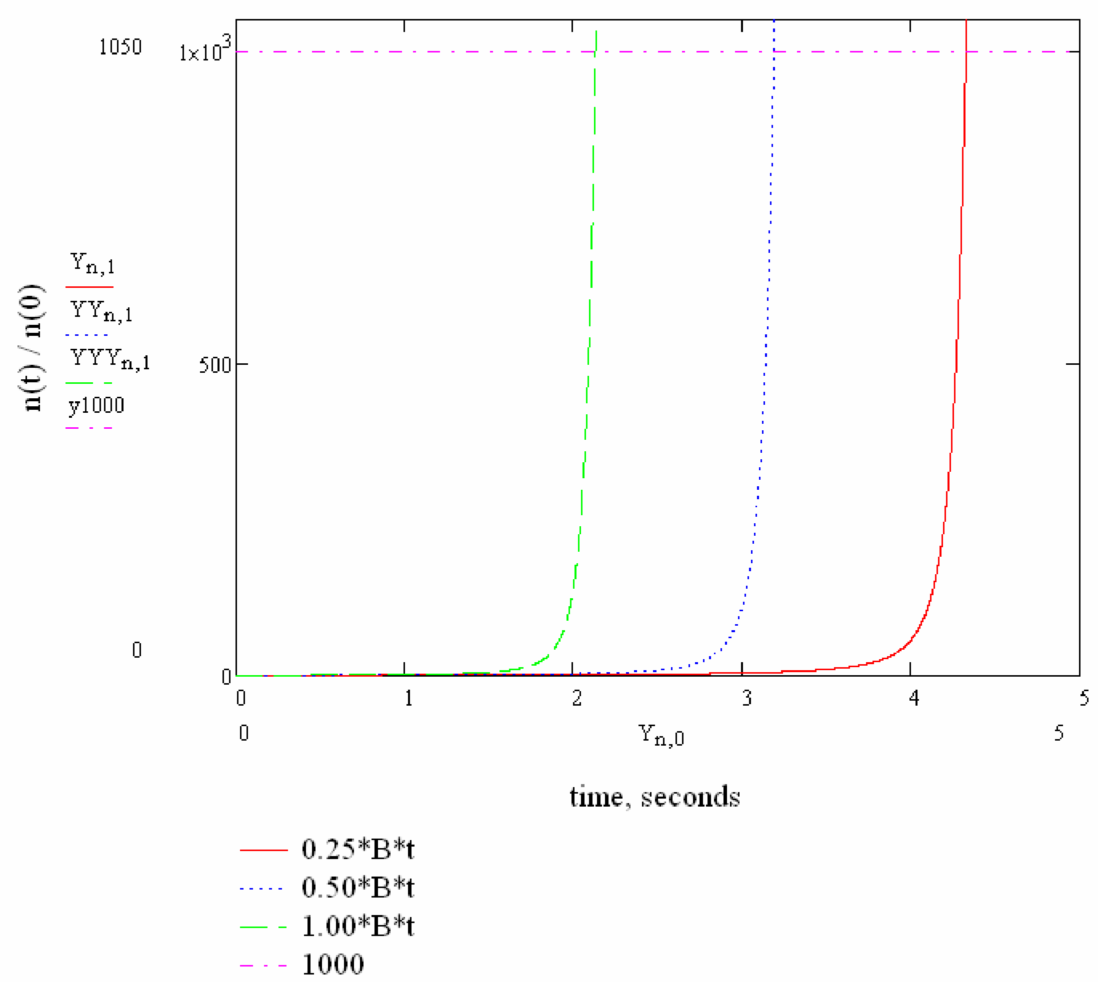
$$n := 0, 1 .. 1000\text{C}$$

$$y1000 := 100\text{C}$$

$$YYY := \text{Radau}\left(\frac{y}{P_0}, 0, 4, 10000, D_3\right)$$

Using the plots of the functions or inserting into the functions, the times at which neutron population reach $n(t)/n(0) = 1{,}000$ are obtained. Values listed are obtained from the graph below.

Y(1000)	=	2.1294
YY(1000)	=	3.1892
YYY(1000)	=	4.3267



2) Value of $n(t)/n(0)$ at the point in time when the system reaches prompt critical

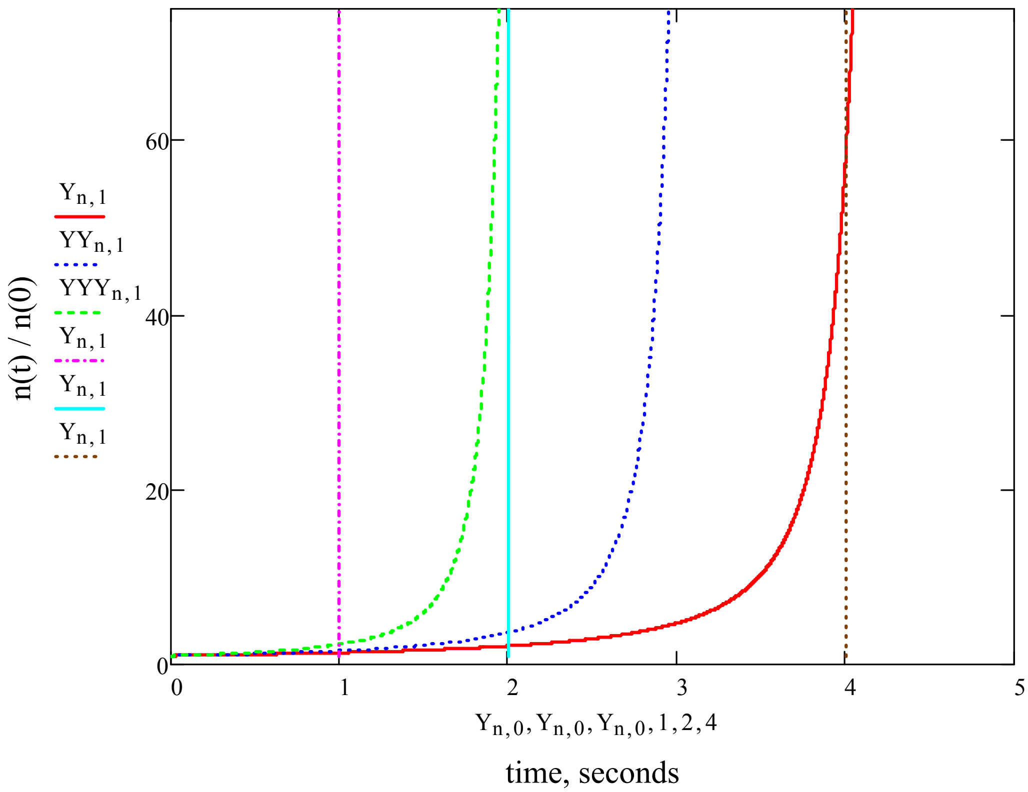
$$\rho' := \frac{\beta}{0.25\cdot\beta} = \frac{1}{0.25} = 4$$

For

0.25βt	→	t_{prompt} = 4 seconds
0.50βt	→	t_{prompt} = 2 seconds
1.00βt	→	t_{prompt} = 1 seconds

Value of n(t)/n(0) at prompt critical are obtained from the graph below.

0.25βt	→	n(t)/n(0) ≈ 58
0.50βt	→	n(t)/n(0) ≈ 4
1.00βt	→	n(t)/n(0) ≈ 2



0.25*B*t
0.50*B*t
1.00*B*t
1 second
2 second
4 second

[5.19] Suppose that a critical reactor is operating at a steady state level with a neutron population of n_o. You are to add reactivity such that the neutron population will increase linearly with time: $n(t) = n_o(1+\vartheta t)$ where ϑ is a constant. Using the one-delayed-group kinetics equations from problem [10] :

a. Determine the time dependent reactivity $\rho(t)$ that you should add to the reactor to achieve the linear increase.
b. Sketch $\rho(t)$ from part a. , assuming $\lambda\Lambda/\beta \ll 1$

Part a: Place $n(t) = n_o(1+\vartheta t)$ and $\frac{d}{dt}n(t) = n_o\vartheta$ into the equations for $n(t)$ and $C(t)$:

$$n_o\vartheta = \frac{[\rho(t)-\beta]}{\Lambda} n_o(1+\vartheta t) + \lambda C(t)$$

and

$$\frac{d}{dt}C(t) = \frac{\beta}{\Lambda} n_o(1+\vartheta t) - \lambda C(t).$$

Apply an integrating factor of $\exp(\lambda t)$ to the equation for C:

$$\frac{d}{dt}[C(t)\exp(\lambda t)] = \frac{\beta}{\Lambda} n_o(1+\vartheta t)\exp(\lambda t)$$

and integrate from 0 to t:

$$C(t)\exp(\lambda t) - C(0) = \frac{\beta}{\Lambda}\int_0^t n_o(1+\vartheta t')\exp(\lambda t')dt'$$

$$= \frac{\beta}{\Lambda\lambda} n_o[\exp(\lambda t)-1] + \frac{\beta}{\Lambda\lambda} n_o\vartheta t\exp(\lambda t) - \frac{\beta}{\Lambda\lambda^2} n_o\vartheta[\exp(\lambda t)-1]$$

Noting that for a reactor that is initially critical $C(0) = \frac{\beta}{\lambda\Lambda} n_o$. Thus solving for C:

$$C(t) = \frac{\beta}{\Lambda\lambda} n_o \exp(-\lambda t)$$
$$+ \frac{\beta}{\Lambda\lambda} n_o[1-\exp(-\lambda t)] + \frac{\beta}{\Lambda\lambda} n_o\vartheta t - \frac{\beta}{\Lambda\lambda^2} n_o\vartheta[1-\exp(-\lambda t)]$$

or

$$C(t) = \frac{\beta}{\Lambda\lambda} n_o\left\{1+\vartheta t - \frac{\vartheta}{\lambda}[1-\exp(-\lambda t)]\right\}$$

Inserting this into the equation for $n(t)$:

$$n_o\vartheta = \frac{[\rho(t)-\beta]}{\Lambda} n_o(1+\vartheta t) + n_o\frac{\beta}{\Lambda}\left\{1+\vartheta t - \frac{\vartheta}{\lambda}[1-\exp(-\lambda t)]\right\}\otimes$$

or

$$\vartheta = \frac{\rho(t)}{\Lambda}(1+\vartheta t) - \frac{\beta}{\Lambda}\frac{\vartheta}{\lambda}[1-\exp(-\lambda t)]$$

Solve for the reactivity:

$$\rho(t) = \left\{ \Lambda + \frac{\beta}{\lambda}[1 - \exp(-\lambda t)] \right\} \frac{\vartheta}{(1 + \vartheta t)}.$$

Part b:

$$\rho(t) = \frac{\beta}{\lambda} \left\{ \frac{\Lambda\lambda}{\beta} + 1 - \exp(-\lambda t) \right\} \frac{\vartheta}{(1 + \vartheta t)}$$

Since $\lambda\Lambda / \beta \ll 1$

$$\rho(t) \approx \frac{\beta}{\lambda}[1 - \exp(-\lambda t)] \frac{\vartheta}{(1 + \vartheta t)}$$

Chapter 6 Problems:

[6.1] Consider in plane geometry a slab of nonfissionable material with properties D and L and a thickness of a. s'' neutrons/cm^2/s enter the slab from the left. What fraction of the neutrons will

a. penetrate the slab
b. be reflected back from the left hand surface
c. be absorbed in the slab

We must first solve for the neutron distribution within the slab. It takes the form of Eq.(6.20), with two arbitrary constants to be determined from the boundary conditions. If we include the extrapolation to the right of the slab within a, the boundary conditions are

$$J^{+}(0) = s'' \qquad \text{and} \quad \phi(a) = 0 \, .$$

Applying the second condition, we have

$$0 = C_1 \exp\left(a / L\right) + C_2 \exp\left(-a / L\right)$$

Thus $\quad C_1 = -C_2 \exp\left(-2a / L\right)$ and

$$\phi(x) = C_2 \left\{ \exp\left(-a / L\right) - \exp\left[(x - 2a) / L\right] \right\}$$

It is convenient to rewrite the flux in terms of the hyperbolic sin (i.e.sinh):

$$\phi(x) = C \sinh[(a - x) / L] \qquad (*)$$

which is also seen to meet the boundary condition at $x = a$. We next apply the first boundary condition

$$J^{+}(0) = \tfrac{1}{4}\phi(0) - \tfrac{1}{2} D \frac{d}{dx}\phi(x)\bigg|_{x=0} = s''$$

Inserting Eq. (*)

$$\tfrac{1}{4} C \sinh(a / L) + \tfrac{1}{2} DC \frac{1}{L} \cosh(a / L) = s''$$

Solving for C:

$$C = \left\{ \sinh(a / L) + 2DL^{-1} \cosh(a / L) \right\}^{-1} 4s''$$

and thus:

$$\phi(x) = \left\{ \sinh(a / L) + 2DL^{-1} \cosh(a / L) \right\}^{-1} 4s'' \sinh[(a - x) / L]$$

Part a: Since there is vacuum on the right, no neutron enter from the right and thus:

$$J(a) / s'' = -\frac{D}{s''} \frac{d}{dx}\phi(x)\bigg|_{x=a}$$

$$= \frac{4D}{L} \left\{ \sinh(a / L) + 2DL^{-1} \cosh(a / L) \right\}^{-1} \cosh[(a - x) / L] \ \Big|_{x=a}$$

$$= \frac{4DL^{-1}}{\sinh(a / L) + 2DL^{-1} \cosh(a / L)}$$

Part b. This is just the albedo or

$$\frac{1}{s''}J^{-}(0)=\frac{1}{4s''}\phi(0)+\frac{1}{2s''}D\frac{d}{dx}\phi(x)\bigg|_{x=0}$$

$$=\frac{\sinh(a/L)-2DL^{-1}\cosh(a/L)}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

Part c. The fraction absorbed is

$$\frac{1}{s''}\int_0^a \Sigma_a\phi(x)dx=\frac{D}{s''L^2}\int_0^a \phi(x)dx$$

$$=\frac{4D}{L^2}\frac{\int_0^a \sinh[(a-x)/L]dx}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

$$=\frac{4D}{L}\frac{\cosh(a/L)-1}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

We can obtain the same result by taking one minus the fractions of neutrons escaping from the two slab surfaces:

$$1-\frac{4DL^{-1}}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}-\frac{\sinh(a/L)-2DL^{-1}\cosh(a/L)}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

$$=\frac{\sinh(a/L)+2DL^{-1}\cosh(a/L)-4DL^{-1}-\sinh(a/L)+2DL^{-1}\cosh(a/L)}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

$$=\frac{4D}{L}\frac{\cosh(a/L)-1}{\sinh(a/L)+2DL^{-1}\cosh(a/L)}$$

[6.2] Consider an infinite slab of nonfissionable material occupying the region $-a \le x \le a$ contains a uniform source s_o''' and has properties D_1 and Σ_{a1}. A second nonfissionable material, with D_2 and Σ_{a2} contains no source and occupies the remaining regions $-\infty \le x \le -a$ and $a \le x \le \infty$. Find the flux distribution for $0 \le x \le \infty$.

For a uniform source, Eq. (6.27) gives the problem solution: ($L_1 = \sqrt{D_1 / \Sigma_{a1}}$)

$$\phi(x) = C_1 \exp\left(x / L_1\right) + C_2 \exp\left(-x / L_1\right) + \frac{1}{\Sigma_{a1}} s_0''' \qquad 0 \le x \le a$$

while in the source free region Eq. (6.21) is the solution: ($L_1 = \sqrt{D_1 / \Sigma_{a1}}$)

$$\phi(x) = C_1' \exp\left(x / L_2\right) + C_2' \exp\left(-x / L_2\right) \qquad a \le x \le \infty$$

We must now employ boundary and interface conditions to determine the four arbitary constants. The problem is symmetric about x = 0 and thus we have a reflected boudary condition at the origin:

$$0 = \frac{d}{dx}\phi(x)\bigg|_{x=0} = \frac{C_1}{L_1} - \frac{C_2}{L_1}$$

Thus $C_1 = C_2$ and

$$\begin{aligned} \phi(x) &= C_1[\exp\left(x / L_1\right) + \exp\left(-x / L_1\right)] + \frac{1}{\Sigma_{a1}} s_0''' \\ &= C \cosh\left(x / L_1\right) + \frac{1}{\Sigma_{a1}} s_0''' \end{aligned} \qquad 0 \le x \le a$$

where $C = 2C_1$. At $x = \infty$, we must have $\phi(\infty) = 0$. Therefore $C' = 0$ and hence

$$\phi(x) = C_2' \exp\left(-x / L_2\right) \qquad a \le x \le \infty$$

Now we have the interface condition that the flux must be continuous $\phi(a_-) = \phi(a_+)$. Thus

$$C \cosh\left(a / L_1\right) + \frac{1}{\Sigma_{a1}} s_0''' = C_2' \exp\left(-a / L_2\right)$$

and the current must be continuous: $D_1 \frac{d}{dx}\phi(x)\Big|_{x=a_-} = D_2 \frac{d}{dx}\phi(x)\Big|_{x=a_+}$ Thus

$$D_1 L_1^{-1} C \sinh\left(a / L_1\right) = -D_2 L_2^{-1} C_2' \exp\left(-a / L_2\right)$$

We next solve these two equations for C and C_2': We obtain:

$$C = -\left[\cosh(a / L_1) + \frac{D_1 L_1^{-1}}{D_2 L_2^{-1}} \sinh\left(a / L_1\right)\right]^{-1} \frac{s_o'''}{\Sigma_{a1}}$$

and

$$C_2' = \left[\cosh(a / L_1) + \frac{D_1 L_1^{-1}}{D_2 L_2^{-1}} \sinh\left(a / L_1\right)\right]^{-1} \frac{D_1 L_1^{-1}}{D_2 L_2^{-1}} \exp(a / L_2) \sinh(a / L_1) \frac{s_o'''}{\Sigma_{a1}}$$

Thus for the flux, we obtain

$$\phi(x)=\left\{1-\frac{\cosh\left(x/L_1\right)}{\cosh(a/L_1)+\frac{D_1L_1^{-1}}{D_2L_2^{-1}}\sinh\left(a/L_1\right)}\right\}\frac{1}{\Sigma_{a1}}s_0''' \qquad 0\le x\le a$$

and

$$\phi(x)=\frac{\sinh(a/L_1)\exp[-(x-a)/L_2]}{\cosh(a/L_1)+\frac{D_1L_1^{-1}}{D_2L_2^{-1}}\sinh\left(a/L_1\right)}\frac{D_1L_1^{-1}}{D_2L_2^{-1}}\frac{s_o'''}{\Sigma_{a1}} \qquad a\le x\le\infty$$

[6.3] Determine the fraction of neutron that penetrate a one meter thick slab of graphite for which D = 0.84 cm and $\Sigma_a = 2.1 \times 10^{-4}$ cm^{-1}. Evaluate the albedo of the slab.

Eq. (6.20) is solution to Eq. (6.16) Let a = 100 cm $\phi(a) = 0$ boundary condition gives $0 = C_1 \exp(a/L) + C_2 \exp(-a/L)$ or $C_1 = -C_2 \exp(-2a/L)$ Thus

$\phi(x) = C_2 \left[\exp(-x/L) - \exp((x-2a)/L)\right]$ Now, let $s'' = J_x^+(0) = \tfrac{1}{4}\phi(0) - \tfrac{1}{2} D \frac{d}{dx}\phi(x)\Big|_0$

be the number of neutrons entering from the left. Thus

$$s'' = \tfrac{1}{4} C_2 [1 - \exp(-2a/L)] - \tfrac{1}{2} D C_2 \left[-\frac{1}{L} - \frac{1}{L}\exp(-2a/L)\right]$$

$$C_2 = \left\{(1 + 2DL^{-1}) - (1 - 2DL^{-1})\exp(-2a/L)\right\}^{-1} 4s''$$

$$\phi(x) = \frac{\left[\exp(-x/L) - \exp((x-2a)/L)\right]}{(1 + 2DL^{-1}) - (1 - 2DL^{-1})\exp(-2a/L)} 4s''$$

However, we don't need to determine $\phi(x)$ or even C_2 since they cancel out of the answers. The fraction penetrating is

$$\frac{J_x^+(a)}{J_x^+(0)} = \frac{\tfrac{1}{4}\phi(a) - \tfrac{1}{2} D \frac{d}{dx}\phi(x)\Big|_a}{\tfrac{1}{4}\phi(0) - \tfrac{1}{2} D \frac{d}{dx}\phi(x)\Big|_0}$$

$$= \frac{\tfrac{1}{4} C_2 [\exp(-a/L) - \exp(-a/L)] - \tfrac{1}{2} D C_2 \left[-\frac{1}{L}\exp(-a/L) - \frac{1}{L}\exp(-a/L)\right]}{\tfrac{1}{4} C_2 [1 - \exp(-2a/L)] - \tfrac{1}{2} D C_2 \left[-\frac{1}{L} - \frac{1}{L}\exp(-2a/L)\right]}$$

$$\frac{J_x^+(a)}{J_x^+(0)} = \frac{4DL^{-1}\exp(-a/L)}{1 - \exp(-2a/L) + 2DL^{-1}\left[1 + \exp(-2a/L)\right]}$$

The albedo is $\dfrac{J_x^-(0)}{J_x^+(0)} = \dfrac{\tfrac{1}{4}\phi(0) + \tfrac{1}{2} D \frac{d}{dx}\phi(x)\Big|_0}{\tfrac{1}{4}\phi(0) - \tfrac{1}{2} D \frac{d}{dx}\phi(x)\Big|_0}$

$$= \frac{\tfrac{1}{4} C_2 [1 - \exp(-2a/L)] + \tfrac{1}{2} D C_2 \left[-\frac{1}{L} - \frac{1}{L}\exp(-2a/L)\right]}{\tfrac{1}{4} C_2 [1 - \exp(-2a/L)] - \tfrac{1}{2} D C_2 \left[-\frac{1}{L} - \frac{1}{L}\exp(-2a/L)\right]}$$

$$\frac{J_x^-(0)}{J_x^+(0)} = \frac{[1 - \exp(-2a/L)] - 2DL^{-1}\left[1 + \exp(-2a/L)\right]}{[1 - \exp(-2a/L)] + 2DL^{-1}\left[1 + \exp(-2a/L)\right]}$$

Evaluations: $L = \sqrt{D/\Sigma_a} = \sqrt{0.84/2.1\cdot 10^{-4}}$ = 63.25 cm, $2DL^{-1}$=2*0.84/63.25 = 0.0266

a/L = 1.581 exp(-a/L) = 0.2058 exp(-2a/L) =0.0423

$$\frac{J_x^+(a)}{J_x^+(0)} = \frac{2\cdot 0.0266\cdot 0.2058}{1-0.0423+0.0266\left[1+0.0423\right]} = 0.011$$

$$\frac{J_x^-(0)}{J_x^+(0)} = \frac{1-0.0423-0.0266\left[1+0.0423\right]}{1-0.0423+0.0266\left[1+0.0423\right]} = 0.944$$

[6.4] In plane geometry, thermal neutrons enter a nonmultiplying slab of infinite thickness from the left. The properties of the composite slab are D_1 and Σ_{a1} for $0 \le x \le a$ and D_2 and Σ_{a2} for $a \le x \le \infty$

a. Show that with $\Upsilon_i \equiv D_i / L_i$ the albedo may be expressed as

$$\alpha = \frac{(1-2\Upsilon_1) + \dfrac{\Upsilon_2 - \Upsilon_1}{\Upsilon_2 + \Upsilon_1}(1+2\Upsilon_1)\exp(-2a/L_1)}{(1+2\Upsilon_1) + \dfrac{\Upsilon_2 - \Upsilon_1}{\Upsilon_2 + \Upsilon_1}(1-2\Upsilon_1)\exp(-2a/L_1)}$$

b. Evaluate the albedo for a = 10 cm when material 1 is graphite and material 2 is water

c. Repeat part b when materials 1 and 2 are switched.

Part a:

In slab geometry the solution in a source free region is given by Eq. (6.20), In region one, with arbitrary constants A & B we have:

$$\phi_1(x) = A\exp(x/L_1) + B\exp(-x/L_1)$$

Since region 2 extends to infinity, and must satisfy $\phi_2(\infty) = 0$, we have only one arbitrary constant (C):

$$\phi_2(x) = C\exp(-x/L_2)$$

Taking derivatives:

$$\frac{d}{dx}\phi_1(x) = \frac{A}{L_1}\exp(x/L_1) - \frac{B}{L_1}\exp(-x/L_1)$$

and

$$\frac{d}{dx}\phi_2(x) = -\frac{C}{L_2}\exp(-x/L_2)$$

At $x = a$ we apply the flux and current continuity conditions, Eqs. (6.42) and (6.43), to these equations:

$$A\exp(a/L_1) + B\exp(-a/L_1) = C\exp(-a/L_2)$$

and

$$-D_1\frac{A}{L_1}\exp(a/L_1) + D_1\frac{B}{L_1}\exp(-a/L_1) = D_2\frac{C}{L_2}\exp(-a/L_2)$$

We may eliminate C by dividing the second by the first:

$$\frac{-D_1\dfrac{A}{L_1}\exp(a/L_1) + D_1\dfrac{B}{L_1}\exp(-a/L_1)}{A\exp(a/L_1) + B\exp(-a/L_1)} = \frac{D_2}{L_2}$$

Let $\theta \equiv \dfrac{D_2 L_1}{D_1 L_2}$. Then we can solve for A in terms of B as

$$A = B\left(\frac{\theta - 1}{\theta + 1}\right)\exp(-2a/L_1)$$ allowing us to write the flux in region 1 as

`

$$\phi_1(x) = B\left\{\exp(-x/L_1) + \left(\frac{\theta-1}{\theta+1}\right)\exp[(x-2a)/L_1]\right\}$$

To calculate the albedo, we must first determine $\phi_1(0)$ and $J_1(0)$:

$$\phi_1(0) = B\left\{1 + \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\}$$

and

$$J_1(0) = -D_1\frac{d}{dx}\phi_1(x)\bigg|_{x=0} = D_1 B\left\{\frac{1}{L_1} - \frac{1}{L_1}\left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\}$$

From Eq. (6.33)

$$J^{\mp}(0) = \phi_1(0) \pm D_1\frac{d}{dx}\phi_1(x)\bigg|_{x=0}$$

$$= \frac{B}{4}\left\{1 + \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\} \mp \frac{D_1 B}{2L_1}\left\{1 - \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\}$$

Thus from Eq. (6.40) The albedo is

$$\alpha = \frac{\frac{1}{4}\left\{1 + \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\} - \frac{D_1}{2L_1}\left\{1 - \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\}}{\frac{1}{4}\left\{1 + \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\} + \frac{D_1}{2L_1}\left\{1 - \left(\frac{\theta-1}{\theta+1}\right)\exp(-2a/L_1)\right\}}$$

Letting $\Upsilon_i \equiv D_i/L_i$ and therefore $\left(\frac{\theta-1}{\theta+1}\right) = \frac{\left(\frac{D_2 L_1}{D_1 L_2} - 1\right)}{\left(\frac{D_2 L_1}{D_1 L_2} + 1\right)} = \frac{\Upsilon_2 - \Upsilon_1}{\Upsilon_2 + \Upsilon_1}$

$$\alpha = \frac{\left\{1 + \left(\frac{\Upsilon_2-\Upsilon_1}{\Upsilon_2+\Upsilon_1}\right)\exp(-2a/L_1)\right\} - 2\Upsilon_1\left\{1 - \left(\frac{\Upsilon_2-\Upsilon_1}{\Upsilon_2+\Upsilon_1}\right)\exp(-2a/L_1)\right\}}{\left\{1 + \left(\frac{\Upsilon_2-\Upsilon_1}{\Upsilon_2+\Upsilon_1}\right)\exp(-2a/L_1)\right\} + 2\Upsilon_1\left\{1 - \left(\frac{\Upsilon_2-\Upsilon_1}{\Upsilon_2+\Upsilon_1}\right)\exp(-2a/L_1)\right\}}$$

which reduces to the specified albedo given above.

[6.5] Note that if $2D/L > 1$, Eq. (6.41) yields a negative albedo! Explain in terms of cross sections why diffusion theory is not valid under these circumstances.

First express the inequality in terms of cross sections ($D = 1/(3\Sigma)$ and $L = \sqrt{D/\Sigma_a} = 1/\sqrt{3\Sigma\Sigma_a}$). Thus

$$1 < 2D/L = \frac{2\sqrt{3\Sigma\Sigma_a}}{3\Sigma} = 2\sqrt{\tfrac{1}{3}\Sigma_a/\Sigma}$$

but $\Sigma_a = (1-c)\Sigma$ where c is the ratio of scattering to total cross section. Thus

$$2\sqrt{\tfrac{1}{3}\Sigma_a/\Sigma} = \sqrt{\tfrac{4}{3}(1-c)} > 1 \quad \text{or} \quad 1-c > \tfrac{3}{4} \quad \text{or} \quad c < \tfrac{1}{4}.$$

This means that for the albedo to be negative the ratio of scattering to total cross section must be less than ¼. In media where ¾ of the collisions result in absorption, diffusion theory is not valid.

[6.6] Neutrons impinge uniformly over the surface of a sphere made of graphite that has a diameter of 1.0 m. For the graphite D = 0.84 cm and Σ_a = 2.1 x 10^{-4} cm^{-1}.

a. Determine the albedo of the graphite sphere.
b. Determine the fraction of the impinging neutrons that are absorbed in the sphere.

Part a: In source free spherical geometry the neutron flux is given by Eq. (6.51). Since in this problem there is no source at the origin, the flux $\phi(0)$ must be finite. But from Eq. (6.51) this condition can only hold if $C_2 = -C_1$. Thus Eq. (6.51) reduces to

$$\phi(r) = C\frac{\sinh(r/L)}{r}$$

where $C = 2C_1$. Suppose that there are s'' neutrons/s/cm^2 impinging on the surface of the sphere. Then the incoming partial current will be

$$J^-(R) = \tfrac{1}{4}\phi(R) + \tfrac{1}{2}D\frac{d}{dr}\phi(r)\Big|_{r=R} = s''$$

Inserting the flux distribution:

$$\frac{1}{4}C\frac{\sinh(R/L)}{R} + \frac{1}{2}DC\frac{\cosh(R/L)}{LR} - \frac{1}{2}DC\frac{\sinh(R/L)}{R^2} = s''$$

Solving for C:

$$C = \left[(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)\right]^{-1} 4Rs''$$

Yielding a flux distribution of

$$\phi(r) = \frac{\sinh(r/L)}{(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)}\left(\frac{R}{r}\right)4s''$$

Part b: The fraction absorbed will be just one minus the fraction reflected. The fraction reflected is just the albedo, or in this case

$$\alpha = J^+(R)/J^-(R)$$

but we already know that $J^-(R) = s''$. Thus

$$\alpha = \frac{1}{s''}\left[\tfrac{1}{4}\phi(R) - \tfrac{1}{2}D\frac{d}{dr}\phi(r)\Big|_{r=R}\right]$$

We first take the flux derivative

$$\frac{d}{dr}\phi(r) = \frac{L^{-1}\cosh(r/L) - r^{-1}\sinh(r/L)}{(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)}\left(\frac{R}{r}\right)4s'' :$$

and insert the flux and its derivative to determine α

$$\alpha = \frac{\sinh(R/L)}{(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)}$$
$$-2D\frac{L^{-1}\cosh(R/L) - R^{-1}\sinh(r/L)}{(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)}$$

Simplifying:

$$\alpha = \frac{(1+2DR^{-1})\sinh(R/L) - 2DL^{-1}\cosh(R/L)}{(1-2DR^{-1})\sinh(R/L) + 2DL^{-1}\cosh(R/L)}$$

$$\alpha = \frac{\dfrac{(1+2DR^{-1})}{2DL^{-1}}\tanh(R/L) - 1}{\dfrac{(1-2DR^{-1})}{2DL^{-1}}\tanh(R/L) + 1}$$

For graphite D = 0.84 cm and Σ_a = 2.1 x 10^{-4} cm^{-1}.

$L = \sqrt{D/\Sigma_a} = \sqrt{0.84/0.00021} = 63.2$ cm and R=100 cm

Thus $2DL^{-1} = 2\cdot 0.84/63.2 = 0.0266$, $2DR^{-1} = 2\cdot 0.84/100 = 0.0168$

$$\alpha = \frac{\dfrac{(1+0.0168)}{0.0266}\tanh(1.58) - 1}{\dfrac{(1-0.0168)}{0.0266}\tanh(1.58) + 1} = \frac{38.2\cdot 0.919 - 1}{37.0\cdot 0.919 + 1} = 0.970$$

Because graphite has a very small absorption cross section, 97% of the neutrons are reflected from the sphere, while only

$1-\alpha = 1-0.97 = 0.03$ or 3% are absorbed in the slab.

Note that if the sphere had a very large radius ($R \to \infty$), then

$$\alpha = \frac{\dfrac{1}{2DL^{-1}} - 1}{\dfrac{1}{2DL^{-1}} + 1} = \frac{\dfrac{1}{0.0266} - 1}{\dfrac{1}{0.0266} + 1} = 0.948$$

and only about 5% of the incident neutrons would be absorbed in the graphite

[6.7] Verify Eq. (6.55)

In spherical geometry: $dV = 4\pi r^2 dr$. Thus

$$\int_{all\ space} \Sigma_a \phi(r) dV = 4\pi \int_0^\infty \Sigma_a \phi(r) r^2 dr$$

Inserting Eq. (6.54):

$$\int_{all\ space} \Sigma_a \phi(r) dV = \cancel{4\pi} \int_0^\infty \Sigma_a \frac{s_p}{\cancel{4\pi} D \cancel{r}} \exp(-r/L) r^{\cancel{2}} dr$$

$$= \frac{s_p}{L^2} \int_0^\infty r \exp(-r/L) dr = s_p \int_0^\infty z \exp(-z) dz = s_p$$

where in the last two integrals we made the substitutions $L^2 = D/\Sigma_a$ and $z = r/L$, and evaluated the integral in *x* using the formula in Appendix A.

[6.8] In the text the spherical geometry condition $0 < \phi(0) < \infty$ is employed to reduce Eq. (6.60) to (6.63). Show that the condition $\lim_{r\to 0} 4\pi r^2 J_r(r) = 0$ produces the same result.

Beginning with Eq. (6.51) we have for the current

$$J(r) = -D\frac{d}{dr}\phi(r) = -D\left\{C_1\left(\frac{1}{rL} - \frac{1}{r^2}\right)\exp(r/L) - C_2\left(\frac{1}{rL} + \frac{1}{r^2}\right)\exp(-r/L)\right\}$$

$$0 = \lim_{r\to 0} 4\pi r^2 J(r) = \lim_{r\to 0} 4\pi D\left\{-C_1\left(\frac{r}{L} - 1\right)\exp(r/L) + C_2\left(\frac{r}{L} + 1\right)\exp(-r/L)\right\}$$

$$0 = 4\pi D\left\{-C_1\left(\frac{0}{L} - 1\right)\exp(0/L) + C_2\left(\frac{0}{L} + 1\right)\exp(-0/L)\right\} = 4\pi D(C_1 + C_2)$$

Thus we must have $C_1 = -C_2$ and then 6.51 reduces to

$\phi(r) = \sinh(r/L)\dfrac{C_1}{r}$ but for small r, $\sinh(r/L) \approx r/L$ Thus we have

$\phi(0) = C_1/L$ which is finite.

Adding a particular solution has no effect on the proof.

[6.9] The point source of strength s_p is located at the center of a sphere of nonmultiplying medium with properties D and Σ_a, and an extrapolated radius $\tilde{R}$, in an infinite vacuum:

a. Find the flux distribution in the sphere
b. Determine the fraction of neutrons that escape from the sphere without being absorbed if $\tilde{R} = L$.

Part a: Within the sphere (except at the origin) the flux is given by Eq. (6.51). To obtain the two arbitrary constants, we first set the flux to zero at the extrapolated boundary

$$\phi(\tilde{R}) = 0$$

Thus $C_1 = -\exp(-2\tilde{R}/L)C_2$ and

$$\phi(r) = \left\{\exp[-r/L] - \exp[(r-2\tilde{R})/L]\right\}\frac{C_2}{r}$$

This expression can also be written in terms of hyperbolic functions as

$$\phi(r) = \sinh[(\tilde{R}-r)/L]\frac{C}{r}$$

where C is an arbitrary constant. We determine C by the condition of Eq. (6.53). The current is

$$J(r) = -D\frac{d}{dr}\phi(r) = \left\{\frac{D}{Lr}\cosh[(\tilde{R}-r)/L] + \frac{D}{r^2}\sinh[(\tilde{R}-r)/L]\right\}C$$

$$s_p = \lim_{r\to 0} 4\pi r^2 J(r) = \lim_{r\to 0} 4\pi\left\{\frac{Dr}{L}\cosh[(\tilde{R}-r)/L] + D\sinh[(\tilde{R}-r)/L]\right\}C$$

$$= 4\pi\left\{\frac{D\cdot 0}{L}\cosh[\tilde{R}/L] + D\sinh[\tilde{R}/L]\right\}C = 4\pi D\sinh(\tilde{R}/L)C$$

Thus $C = \dfrac{s_p}{4\pi D\sinh(\tilde{R}/L)}$ and

$$\phi(r) = \frac{\sinh[(\tilde{R}-r)/L]}{\sinh(\tilde{R}/L)}\frac{s_p}{4\pi Dr}$$

Part b. The fraction escaping will be

$$\frac{4\pi\tilde{R}^2 J(\tilde{R})}{s_p} = \frac{4\pi\tilde{R}^2}{s_p}\left\{\frac{D}{L\tilde{R}}\cosh[0] + \frac{D}{\tilde{R}^2}\sinh[0]\right\}\frac{s_p}{4\pi D\sinh(\tilde{R}/L)}$$

$$= \frac{\tilde{R}/L}{\sinh(\tilde{R}/L)}$$

Alternately, we may write the fraction leaking as one minus the fraction absorbed in the sphere:

$$1-\frac{1}{s_p}\int_0^\infty \Sigma_a\phi(r)4\pi r^2 dr = 1-\int_0^\infty \frac{\Sigma_a}{D}\frac{\sinh[(\tilde{R}-r)/L]}{\sinh(\tilde{R}/L)} r dr$$

$$= 1-\frac{1}{\sinh(\tilde{R}/L)}\int_0^\infty \sinh[(\tilde{R}-r)/L]\frac{r}{L^2}dr$$

Let $x=(\tilde{R}-r)/L$ then

$$\int_0^{\tilde{R}} \sinh[(\tilde{R}-r)/L]\frac{r}{L^2}dr = \int_0^{\tilde{R}/L} \sinh(x)\left(\tilde{R}/L-x\right)dx$$

$$= (\tilde{R}/L)\int_0^{\tilde{R}/L} \sinh(x)dx - \int_0^{\tilde{R}/L} x\sinh(x)dx$$

[6.10] A thin spherical shell or radius R emits s''_{pl} neutrons/cm^2/s in an infinite nonmultiplying medium with properties D and Σ_a:

a. Determine the flux $\phi(r)$ for $0 \le r \le \infty$
b. Determine the flux ratio $\phi(0)/\phi(R)$.

Part a. Equation (6.51) applies on both sides of the spherical shell. Inside the shell we must have $C_1 = -C_2$ and therefore the solution may be written as

$$\phi(r) = \frac{\sinh(r/L)}{r}C \qquad\qquad 0 \le r < R$$

Outside the shell we must have $C_1 = 0$ so that the solution vanishes at infinity. Thus we may write

$$\phi(r) = \frac{\exp(-r/L)}{r}C' \qquad\qquad R < r \le \infty$$

We next employ the interface conditions, Eq. (6.42) and (6.44) to determine the remaining two constants: From Eq. (6.42);

$$\sinh(R/L)C = \exp(-R/L)C'$$

or

$$C' = \exp(R/L)\sinh(R/L)C$$

To employ Eq. (6.44) we must first calculate the currents:

$$J(r) = -D\frac{d}{dr}\phi(r) = -D\left[\frac{\cosh(r/L)}{Lr} - \frac{\sinh(r/L)}{r^2}\right]C \qquad\qquad 0 \le r < R$$

and

$$J(r) = -D\frac{d}{dr}\phi(r) = D\left[\frac{\exp(-r/L)}{Lr} + \frac{\exp(-r/L)}{r^2}\right]C' \qquad\qquad R < r \le \infty$$

For this spherical geometry problem, Eq. (6.44) becomes

$$- -D\frac{d}{dr}\phi(r)\bigg|_{R^-} + s'' = -D\frac{d}{dr}\phi(r)\bigg|_{R^+}$$

Thus

$$-D\left[\frac{\cosh(R/L)}{LR} - \frac{\sinh(R/L)}{R^2}\right]C + s'' = D\left[\frac{\exp(-R/L)}{LR} + \frac{\exp(-R/L)}{R^2}\right]C'$$

Substituting in C':

$$-D\left[RL^{-1}\cosh(R/L) - \cancel{\sinh(R/L)}\right]C + R^2 s'' = D(RL^{-1} + \cancel{1})\sinh(R/L)C$$

$$C = \left[\sinh(R/L) + \cosh(R/L)\right]^{-1}\frac{RLs''}{D} = \exp(-R/L)\frac{RLs''}{D}$$

Thus

$$\phi(r) = \left(\frac{R}{r}\right)\sinh(r/L)\exp(-R/L)\frac{Ls''}{D} \qquad\qquad 0 \le r < R$$

and

$$\phi(r) = \left(\frac{R}{r}\right)\exp(-r/L)\sinh(R/L)\frac{Ls''}{D} \qquad\qquad R < r \le \infty$$

Part b. Since for small r, $\sinh(r/L) \approx r/L$

$$\phi(0) = \exp(-R/L)\frac{Rs''}{D}$$

$$\phi(R) = \sinh(R/L)\exp(-R/L)\frac{Ls''}{D}$$

Thus

$$\frac{\phi(0)}{\phi(R)} = \frac{(R/L)}{\sinh(R/L)}$$

[6.11] Show that the ratio of uncollided to total flux from a point source in an infinite medium is

$$\frac{\phi_u(r)}{\phi(r)} = \sqrt{\tfrac{1}{3}(1-c)}\,\frac{1}{(r/L)}\exp\left[-\alpha(r/L)\right],$$

where $\alpha = \left[3(1-c)\right]^{-1/2} - 1$. Then plot the curve for $1/2 < r/L < 3$ using thermal neutron cross sections for

a. water,
b. heavy water,
c. graphite,
d. a one-to-one volume mixture of natural uranium and water.

Divide Eq. (2.9) by Eq. (6.54):

$$\frac{\phi_u(r)}{\phi(r)} = \frac{s_p \exp(-\Sigma r)/(4\pi r^2)}{s_p \exp(-r/L)/(4\pi D r)} = \frac{D}{r}\exp[-(\Sigma - 1/L)r] \qquad (*)$$

From Eq. (6.75): $L = \dfrac{\lambda}{\sqrt{3(1-c)}} = \dfrac{1}{\Sigma\sqrt{3(1-c)}}$ Thus $\Sigma = \dfrac{1}{L\sqrt{3(1-c)}}$

Likewise $D = \dfrac{1}{3\Sigma}$ and $\Sigma^{-1} = L\sqrt{3(1-c)}$ thus $D = L\sqrt{\tfrac{1}{3}(1-c)}$

Finally $\Sigma - 1/L = \dfrac{1}{L\sqrt{3(1-c)}} - \dfrac{1}{L} = \left[\dfrac{1}{\sqrt{3(1-c)}} - 1\right]\dfrac{1}{L} = \alpha/L$

Inserting D and $\Sigma - 1/L$ into Eq. (*) gives the specified equation.

[6.12] A semi-infinite multiplying medium having the properties D, Σ_a and $k_\infty < 1$ occupies the space $0 \le x \le \infty$, while $-\infty \le x < 0$ is a vacuum. A source embedded in the medium emits neutrons at a rate of $S_o'''\exp(-\alpha x)$ neutrons/cm^3 s^{-1}. Neglecting the extrapolation distance:

a. Show that the distribution of neutrons is

$$\phi(x) = \left[\alpha^2 - (1-k_\infty)/L^2\right]^{-1} \frac{S_o'''}{D}\left[\exp(-\sqrt{1-k_\infty}\,x/L) - \exp(-\alpha x)\right], \quad 0 \le x \le \infty$$

b. Determine the ratio of the number of neutrons escaping across the boundary at $x = 0$ to the number of source neutrons produced.

We solve the equation

$$-\frac{d^2}{dx^2}\phi(x) + \frac{1}{L^2}(1-k_\infty)\phi(x) = \frac{1}{D}S_o'''\exp(-\alpha x)$$

We need to sum of general and particular solutions as in Eq. (6.25) For the particular solution try $\phi_p(x) = A\exp(-\alpha x)$ where A is a constant. Plugging into the above equation we have $\left[-\alpha^2 + \frac{1}{L^2}(1-k_\infty)\right]A\exp(-\alpha x) = \frac{1}{D}S_o'''\exp(-\alpha x)$ or

$$A = \left[-\alpha^2 + \frac{1}{L^2}(1-k_\infty)\right]^{-1}\frac{1}{D}S_o'''$$ For the general solution we try the form

$\phi_g(x) = C\exp(\kappa x)$ and find $\kappa = \pm L^{-1}\sqrt{1-k_\infty}$ Hence

$$\phi_g(x) = C_1\exp(L^{-1}\sqrt{1-k_\infty}\,x) + C_2\exp(-L^{-1}\sqrt{1-k_\infty}\,x)$$

Adding general and particular solutions:

$$\phi(x) = C_1\exp(L^{-1}\sqrt{1-k_\infty}\,x) + C_2\exp(-L^{-1}\sqrt{1-k_\infty}\,x)$$
$$+\left[-\alpha^2 + L^{-2}(1-k_\infty)\right]^{-1}\frac{1}{D}S_o'''\exp(-\alpha x)$$

We next apply the boundary conditions $\phi(0) = 0$ and $\phi(\infty) = 0$ To satisfy $\phi(\infty) = 0$ we must have C_1=0. Then to satisfy $\phi(0) = 0$ we have

$$0 = C_2 + \left[-\alpha^2 + L^{-2}(1-k_\infty)\right]^{-1}\frac{1}{D}S_o'''$$

Substituting C_2 into the solution above, we obtain

$$\phi(x) = \left[\alpha^2 - (1-k_\infty)/L^2\right]^{-1}S_o'''\left[\exp(-\sqrt{1-k_\infty}\,x/L) - \exp(-\alpha x)\right]$$

[6.13] Verify Eqs. (6.67) through (6.69)

We may accomplish this either by solving (6.65) and (6.66) and determining C, or by showing the (6.67) and (6.68) meet the boundary and interface conditions with C given by Eq. (6.69). We use the second method. Since for small x $\sinh(x) \approx x + \tfrac{1}{6}x^3$, we have

$$\phi(0) = \left[1 - C\frac{R}{L\sinh(R/L)}\right]\frac{s_o'''}{\Sigma_a}$$

and the boundary condition as zero is met. Likewise from Eq. (6.68) $\phi(\infty) = 0$ so that the boundary condition at infinitiy is met for flux continuity at the interface, we obtain from Eqs. (6.67) and (6.68) evaluated at *R*:

$$\left[1 - C\frac{R\sinh(R/L)}{R\sinh(R/L)}\right]\frac{s_o'''}{\Sigma_a} = (1-C)\frac{s_o'''}{\Sigma_a}\frac{R}{R}\exp\left[-(R-R)/L\right]$$

or cancelling terms: $1 - C = 1 - C$ To determine the value of C we employ continuity of current. Thus taking the derivatives of Eqs. (6.67) and (6.68) and evaluating them at *R*:

$$-DCR\left[-\frac{\sinh(R/L)}{R^2} + \frac{\cosh(R/L)}{RL}\right]\frac{1}{\sinh(R/L)}\frac{s_o'''}{\Sigma_a}$$
$$= \hat{D}(1-C)\frac{s_o'''}{\Sigma_a}R\left[-\frac{1}{R^2} - \frac{1}{RL}\right]\exp\left[-(R-R)/L\right]$$

or

$$DC\left[1 - \frac{R}{L}\coth(R/L)\right]\frac{s_o'''}{\Sigma_a} = -\hat{D}(1-C)\frac{s_o'''}{\Sigma_a}\left(1 + \frac{R}{L}\right)$$

Solve for *C*:

$$C = \left\{1 + \frac{1}{1+(R/L)}\frac{D}{\hat{D}}\left[(R/L)\coth(R/L) - 1\right]\right\}^{-1}$$

[6.14] Show that Eqs. (6.95) and (6.103) agree in the limit of $k_\infty \to 1$.

In Eq. (6.95) let $\sqrt{1-k_\infty} = y$ and note from Appendix A that $\sinh(x) \approx x + \tfrac{1}{6}x^3$ for small x

$$\phi(r) = \frac{s'''_o}{y^2\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{\sinh\left(L^{-1}y\,r\right)}{\sinh\left(L^{-1}y\,\tilde{R}\right)}\right]$$

Thus taken together, we have

$$\phi(r) = \frac{s'''_o}{y^2\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{L^{-1}y\,r + \tfrac{1}{6}\left(L^{-1}y\,r\right)^3}{L^{-1}y\,\tilde{R} + \tfrac{1}{6}\left(L^{-1}y\,\tilde{R}\right)^3}\right] = \frac{s'''_o}{y^2\Sigma_a}\left[1 - \cancel{\frac{\tilde{R}}{r}}\,\cancel{\frac{r}{\tilde{R}}}\,\frac{\left[1 + \tfrac{1}{6}\left(L^{-1}y\,r\right)^2\right]}{\left[1 + \tfrac{1}{6}\left(L^{-1}y\,\tilde{R}\right)^2\right]}\right]$$

but for small X & Y $\quad \dfrac{(1+X)}{(1+Y)} \approx (1+X)(1-Y) \approx 1 + X - Y$. Thus

$$\frac{\left[1 + \tfrac{1}{6}\left(L^{-1}y\,r\right)^2\right]}{\left[1 + \tfrac{1}{6}\left(L^{-1}y\,\tilde{R}\right)^2\right]} \approx 1 + \tfrac{1}{6}\left(L^{-1}y\,r\right)^2 - \tfrac{1}{6}\left(L^{-1}y\,\tilde{R}\right)^2 \quad \text{and therefore}$$

$$\phi(r) = \frac{s'''_o}{6\Sigma_a L^2}(\tilde{R}^2 - r^2) \quad \text{or}$$

$$\phi(r) = \frac{s'''_o}{6D}(\tilde{R}^2 - r^2)$$

In Eq.(6.103) let $\sqrt{k_\infty - 1} = z$ and note from Appendix A that $\sin(x) \approx x - \tfrac{1}{6}x^3$ for small x

$$\phi(r) = \frac{s'''_o}{z^2\Sigma_a}\left[\frac{\tilde{R}}{r}\frac{\sin\left(L^{-1}z\,r\right)}{\sin\left(L^{-1}z\,\tilde{R}\right)} - 1\right]$$

Thus taken together, we have

$$\phi(r) = \frac{s'''_o}{z^2\Sigma_a}\left[\frac{\tilde{R}}{r}\frac{L^{-1}z\,r - \tfrac{1}{6}\left(L^{-1}z\,r\right)^3}{L^{-1}z\,\tilde{R} - \tfrac{1}{6}\left(L^{-1}z\,\tilde{R}\right)^3} - 1\right] = \frac{s'''_o}{z^2\Sigma_a}\left[\cancel{\frac{\tilde{R}}{r}}\,\cancel{\frac{r}{\tilde{R}}}\,\frac{\left[1 - \tfrac{1}{6}\left(L^{-1}z\,r\right)^2\right]}{\left[1 - \tfrac{1}{6}\left(L^{-1}z\,\tilde{R}\right)^2\right]} - 1\right]$$

but for small X & Y $\quad \dfrac{(1-X)}{(1-Y)} \approx (1-X)(1+Y) \approx 1 - X + Y$. Thus

$$\frac{\left[1 - \tfrac{1}{6}\left(L^{-1}z\,r\right)^2\right]}{\left[1 - \tfrac{1}{6}\left(L^{-1}z\,\tilde{R}\right)^2\right]} \approx 1 - \tfrac{1}{6}\left(L^{-1}z\,r\right)^2 + \tfrac{1}{6}\left(L^{-1}z\,\tilde{R}\right)^2 \quad \text{and therefore}$$

$$\phi(r) = \frac{s'''_o}{6\Sigma_a L^2}(\tilde{R}^2 - r^2) \quad \text{or}$$

$$\phi(r) = \frac{s_o'''}{6D}(\tilde{R}^2 - r^2)$$

Hence, both equations reduce to the same result as $k_\infty \rightarrow 1$.

[6.15] Suppose that the material in problem [6.9] is fissionable with $k_\infty < 1$. Find the flux distribution in the sphere

: Within the sphere (except at the origin) the flux is given by Eq. (6.92), but with the distributed source set equal to zero:

$$\phi(r) = \frac{C_1}{r}\exp\left(L^{-1}\sqrt{1-k_\infty}\, r\right) + \frac{C_2}{r}\exp\left(-L^{-1}\sqrt{1-k_\infty}\, r\right)$$

To obtain the two arbitrary constants, we first set the flux to zero at the extrapolated boundary

$$\phi(\tilde{R}) = 0$$

Thus $C_1 = -\exp(-2L^{-1}\sqrt{1-k_\infty}\,\tilde{R})C_2$ and

$$\phi(r) = \left\{\exp[-L^{-1}\sqrt{1-k_\infty}\, r] - \exp[L^{-1}\sqrt{1-k_\infty}\,(r-2\tilde{R})]\right\}\frac{C_2}{r}$$

This expression can also be written in terms of hyperbolic functions as

$$\phi(r) = \sinh[L^{-1}\sqrt{1-k_\infty}\,(\tilde{R}-r)]\frac{C}{r}$$

where C is an arbitrary constant. We determine C by the condition of Eq. (6.53). The current is

$$J(r) = -D\frac{d}{dr}\phi(r) = \left\{\frac{D}{Lr}\sqrt{1-k_\infty}\cosh[L^{-1}\sqrt{1-k_\infty}\,(\tilde{R}-r)] + \frac{D}{r^2}\sinh[L^{-1}\sqrt{1-k_\infty}\,(\tilde{R}-r)]\right\}C$$

$$s_p = \lim_{r\to 0} 4\pi r^2 J(r) = \lim_{r\to 0} 4\pi\left\{\frac{Dr}{L}\sqrt{1-k_\infty}\cosh[L^{-1}\sqrt{1-k_\infty}\,(\tilde{R}-r)] + D\sinh[L^{-1}\sqrt{1-k_\infty}\,(\tilde{R}-r)]\right\}C$$

$$= 4\pi\left\{\frac{D\cdot 0}{L}\sqrt{1-k_\infty}\cosh[L^{-1}\sqrt{1-k_\infty}\,\tilde{R}] + D\sinh[L^{-1}\sqrt{1-k_\infty}\,\tilde{R}]\right\}C = 4\pi D\sinh(\sqrt{1-k_\infty}\,\tilde{R}/L)C$$

Thus $C = \dfrac{s_p}{4\pi D\sinh(\sqrt{1-k_\infty}\,\tilde{R}/L)}$ and

$$\phi(r) = \frac{\sinh[\sqrt{1-k_\infty}\,(\tilde{R}-r)/L]}{\sinh(\sqrt{1-k_\infty}\tilde{R}/L)}\frac{s_p}{4\pi Dr}$$

[6.16] Suppose the material in [6.9] is fissionable with $k_\infty > 1$:

a. Find the flux distribution in the sphere.
b. Show that the criticality condition is the same as Eq. (6.105)

: Within the sphere (except at the origin) the flux is given by Eq. (6.101), but with the distributed source set equal to zero:

$$\phi(r) = \frac{C_1}{r}\sin\left(L^{-1}\sqrt{k_\infty - 1}\, r\right) + \frac{C_2}{r}\cos\left(L^{-1}\sqrt{k_\infty - 1}\, r\right).$$

To obtain the two arbitrary constants, we first set the flux to zero at the extrapolated boundary

$$\phi(\tilde{R}) = 0$$

Thus

$$0 = \frac{C_1}{\tilde{R}}\sin\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right) + \frac{C_2}{\tilde{R}}\cos\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)$$

or

$$C_2 = -C_1 \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)$$

Hence

$$\phi(r) = \frac{C_1}{r}\left[\sin\left(L^{-1}\sqrt{k_\infty - 1}\, r\right) - \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)\cos\left(L^{-1}\sqrt{k_\infty - 1}\, r\right)\right]$$

where C_1 is an arbitrary constant. We determine C_1 by the condition of Eq. (6.53). The current is

$$J(r) = -D\frac{d}{dr}\phi(r) = D\frac{C_1}{r^2}\left[\sin\left(L^{-1}\sqrt{k_\infty - 1}\, r\right) - \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)\cos\left(L^{-1}\sqrt{k_\infty - 1}\, r\right)\right]$$
$$-D\frac{C_1 L^{-1}\sqrt{k_\infty - 1}}{r}\left[\cos\left(L^{-1}\sqrt{k_\infty - 1}\, r\right) + \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)\sin\left(L^{-1}\sqrt{k_\infty - 1}\, r\right)\right]$$

Noting that for small x, $\sin(x) \approx x$ and $\cos(x) \approx 1$

$$s_p = \lim_{r\to 0} 4\pi r^2 J(r) = \lim_{r\to 0} 4\pi r^2 \left\{ D\frac{C_1}{r^2}\left[L^{-1}\sqrt{k_\infty - 1}\, r - \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)\right]\right.$$
$$\left. -D\frac{C_1 L^{-1}\sqrt{k_\infty - 1}}{r}\left[1 + \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)L^{-1}\sqrt{k_\infty - 1}\, r\right]\right\}$$

$$s_p = \lim_{r\to 0} 4\pi r^2 J(r) = \lim_{r\to 0} 4\pi \left\{ DC_1\left[L^{-1}\sqrt{k_\infty - 1}\, r - \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)\right]\right.$$
$$\left. -DC_1 L^{-1}\sqrt{k_\infty - 1}\left[r + \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)L^{-1}\sqrt{k_\infty - 1}\, r^2\right]\right\}$$

$$s_p = \lim_{r\to 0} 4\pi r^2 J(r) = -4\pi D C_1 \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)$$

$$C_1 = -\frac{s_p}{4\pi D \tan\left(L^{-1}\sqrt{k_\infty - 1}\,\tilde{R}\right)}$$

Thus

$$\phi(r)=\frac{1}{4\pi Dr}\left[\cos\left(L^{-1}\sqrt{k_\infty-1}\,r\right)-\frac{\sin\left(L^{-1}\sqrt{k_\infty-1}\,r\right)}{\tan\left(L^{-1}\sqrt{k_\infty-1}\,\tilde{R}\right)}\right]s_p$$

Part b. The equation becomes singular when the denominator of the second term vanishes. Since $\tan(\pi)=0$, the criticality equation is $L^{-1}\sqrt{k_\infty-1}\,\tilde{R}=\pi$ which is the same as Eq. (6.104)

[6.17] Equations (6.95) and (6.103) give the flux distributions for a subcritical sphere with a uniform source for $k_\infty < 1$ and $k_\infty > 1$, respectively. Find the equivalent expression for $k_\infty = 1$.

Begin be setting $k_\infty = 1$ in Eq. (6.82). We then have

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\phi(r) = \frac{s_o'''}{D}$$

This equation can be integrated directly. Multiply by $-r^2 dr$ and integrate

$$\int d\left[r^2\frac{d}{dr}\phi(r)\right] = -\int \frac{s_o'''}{D}r^2 dr$$

$$r^2\frac{d}{dr}\phi(r) = -\frac{s_o'''}{D}\frac{r^3}{3} + C_1$$

Multiply by dr/r^2 and integrate

$$\int d\phi(r) = -\frac{s_o'''}{D}\int \frac{r}{3}dr + C_1\int \frac{dr}{r^2}$$

$$\phi(r) = -\frac{s_o'''}{D}\frac{r^2}{6} - C_1\frac{1}{r} + C_2$$

We now use the boundary conditions to determine the two constants. Since this is a distributed source the flux must be finite at the origin. Thus we must take $C_1 = 0$. The flux must vanish at the extrapolated boundary $\tilde{R}$. Therefore

$$\phi(\tilde{R}) = 0 = -\frac{s_o'''}{D}\frac{\tilde{R}^2}{6} + C_2$$

Thus the flux distribution is

$$\phi(r) = \frac{s_o'''}{6D}\left(\tilde{R}^2 - r^2\right)$$

[6.18]* Using Eqs. (6.95) and (6.103)

a. Find expressions for the flux $\phi(0)$ at the center of the subcritical sphere
b. Using your results from a. make a. plot $\phi(0)$ for $0 \le k_\infty < 1.154$ with $\tilde{R}/L = 8$.
c. Using your results from a. make a plot $\phi(0)$ for $0 < \tilde{R}/L < 8$, with $k_\infty = 1.154$
d. Compare the two curves and discuss their significance.

Part a: Note from appendix A that for small x: $\sinh(x) \approx x$ and $\sin(x) \approx x$

From Eq. (6.95)

$$\phi(0) = \lim_{r\to 0} \frac{s_o'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{\sinh\left(L^{-1}\sqrt{1-k_\infty}\, r\right)}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\, \tilde{R}\right)}\right]$$

$$= \lim_{r\to 0} \frac{s_o'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{\tilde{R}}{r}\frac{L^{-1}\sqrt{1-k_\infty}\, r}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\, \tilde{R}\right)}\right]$$

$$\phi(0) = \frac{s_o'''}{(1-k_\infty)\Sigma_a}\left[1 - \frac{(\tilde{R}/L)\sqrt{1-k_\infty}}{\sinh\left(L^{-1}\sqrt{1-k_\infty}\, \tilde{R}/L\right)}\right]$$

From Eq. (6.103):

$$\phi(0) = \lim_{r\to 0} \frac{s_o'''}{(k_\infty - 1)\Sigma_a}\left[\frac{\tilde{R}}{r}\frac{\sin\left(L^{-1}\sqrt{k_\infty - 1}\, r\right)}{\sin\left(L^{-1}\sqrt{k_\infty - 1}\, \tilde{R}\right)} - 1\right]$$

$$\lim_{r\to 0} \frac{s_o'''}{(k_\infty - 1)\Sigma_a}\left[\frac{\tilde{R}}{r}\frac{L^{-1}\sqrt{k_\infty - 1}\, r}{\sin\left(L^{-1}\sqrt{k_\infty - 1}\, \tilde{R}\right)} - 1\right]$$

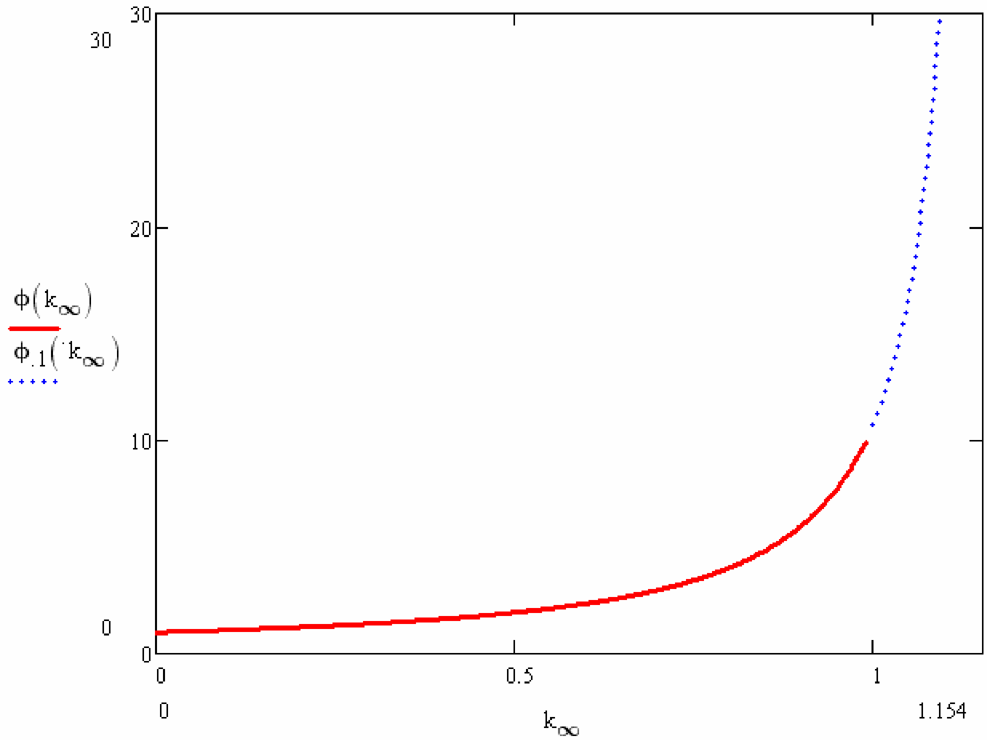
Thus

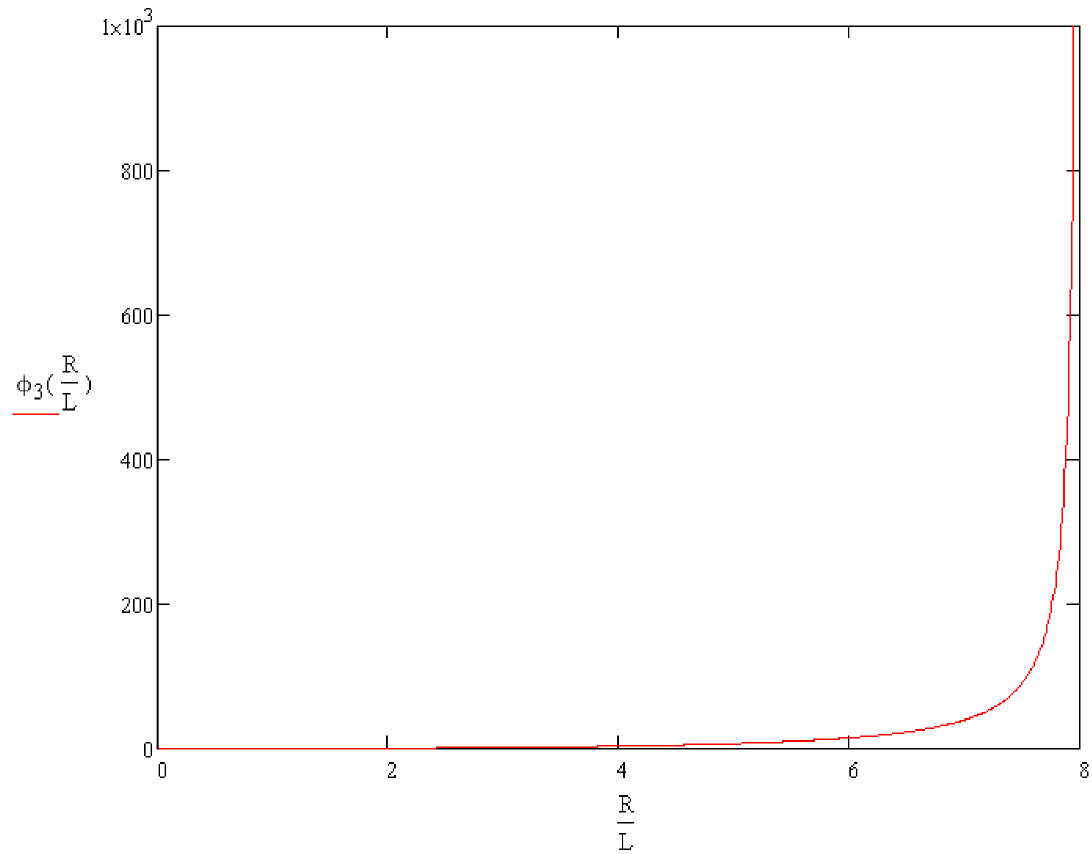
$$\phi(0) = \frac{s_o'''}{(k_\infty - 1)\Sigma_a}\left[\frac{(\tilde{R}/L)\sqrt{k_\infty - 1}}{\sin\left(\sqrt{k_\infty - 1}\, \tilde{R}/L\right)} - 1\right]$$

Note from Problem 17, that when $k_\infty = 1$, $\phi(0) = \dfrac{s_o'''}{6\Sigma_a}\tilde{R}/L$

Parts b & c: See plots, which are normalized to $\phi(0) = \dfrac{s_o'''}{\Sigma_a}$

Part d. Part b approaches criticality by increasing k_∞, while Part c approaches criticality by increasing the core radius. In both cases the flux becomes infinite as criticality is reached, since there is no longer a time-independent solution. Both approaches yield the same criticality condition.


30
30
20
φ(k∞)
φ.1(k∞)
10
0
0
0
0.5
1
0
1.154
k∞


1×10^3
800
600
φ3(R/L)
400
200
0
0
2
4
6
8
R/L

Chapter 7 Problems:

[7.1] The material composition for core of a large reactor yields $k_\infty = 1.02$ and M = 25 cm.

a. Calculate the critical volume for a bare cylinder with a height to diameter ratio of one.
b. Calculate the critical volume of a bare sphere.

Which of the two volumes did you expect to be larger? Why?

Let $M = B$ for convenience

Setting $k = 1$, $k_\infty = 1.02$ and L = 4.5 cm. in Eq. (7.6), we have $1 = \dfrac{1.02}{1 + 4.5^2 B^2}$ or

$B^2 = (1.02 - 1)/4.5^2 = 9.88 \cdot 10^{-4}\ cm^{-2}$

For a cylinder with a height to diameter ratio of one: $2R=D=H$: and thus

$$B^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2.405}{R}\right)^2 = \left(\frac{\pi}{D}\right)^2 + \left(\frac{4.810}{D}\right)^2 = \frac{33.0}{D^2} \quad \text{or}$$

$$D = \sqrt{33/B^2} = \sqrt{33/(9.88 \cdot 10^{-4})} = 1.828 \cdot 10^2 cm = 1.827 m$$

For the Cylindrical volume $V_c = \frac{1}{4}\pi D^2 H = \frac{1}{4}\pi D^3 = \frac{1}{4}\pi 1.827^3 = 4.79 m^3$

For the sphere $B^2 = (\pi / R)^2 \quad R = \sqrt{\pi^2 / B^2} = \sqrt{\pi^2/(9.88 \cdot 10^{-4})} = 0.9995 \cdot 10^2 cm = 0.999 m$

For the spherical volume $V_s = \frac{4}{3}\pi R^3 = \frac{4}{3}\pi 0.0.999^3 = 4.17 m^3$

Since the sphere is the most compact shape, you would expect it to have a smaller leakage than any other object of the same volume.

[7.2] Determine the height to diameter ratio of a bare cylindrical reactor that will lead to the smallest critical mass.

Since the core is uniform we equivalently minimize the volume. Conversely, we can ask: For a given volume what value of the height to diameter ratio yields the minimum buckling, and therefore the minimum leakage. Let x be the height to diameter ratio

$$x = H / D$$

Then the volume is

$$V = \frac{\pi D^2}{4} H = \frac{1}{4} x \pi D^3$$

and the bucking is

$$B^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 = \left(\frac{4.810}{D}\right)^2 + \left(\frac{\pi}{xD}\right)^2$$

or

$$B^2 = \frac{1}{D^2}\left[4.810^2 + (\pi / x)^2\right]$$

We next express the diameter in terms of the volume,

$$D = \left(\frac{4V}{\pi x}\right)^{1/3}$$

and eliminate it from the expression for the buckling:

$$B^2 = \left(\frac{\pi x}{4V}\right)^{2/3}\left[4.810^2 + (\pi / x)^2\right] = \left(\frac{\pi}{4V}\right)^{2/3}\left[4.810^2 x^{2/3} + \pi^2 x^{-4/3}\right]$$

To find the minimum buckling value, with V held constant, take its derivative with respect to x and set it equal to zero:

$$\frac{d}{dx} B^2 = \frac{d}{dx}\left(\frac{\pi}{4V}\right)^{2/3}\left[4.810^2 x^{2/3} + \pi^2 x^{-4/3}\right]$$

$$= \left(\frac{\pi}{4V}\right)^{2/3}\left[4.810^2 \tfrac{2}{3} x^{-1/3} - \pi^2 \tfrac{4}{3} x^{-7/3}\right] = 0$$

Thus $4.810^2 \tfrac{2}{3} x^{-1/3} - \pi^2 \tfrac{4}{3} x^{-7/3} = 0$ or

$$x^2 = 2\pi^2 / 4.810^2$$

or

$$x = \sqrt{2}\pi / 4.810 = 0.923$$

[7.3] Critical assemblies for studying the properties of fast reactors are sometimes built in halves as shown in the figure. The two halves are maintained in subcritical states by separating them with sufficient distance that neutronic coupling between the two is negligible; they are then brought together for form a critical assembly. Suppose the core composition under investigation has an infinite medium multiplication of 1.36, and a migration length of 18.0 cm. The assembly is configured with a height to diameter ratio of one ($H = D$). Neglecting extrapolation distances:

a. Determine the dimensions required to make the assembly exactly critical when the two halves are brought into contact.
b. Determine the value of k for each of the halves when they are isolated from each other.

(place problem 7.3 figure here)

Part a. For a cylinder with a height to diameter ratio of one: $2R=D=H$: and thus

$$B^2 = \left(\frac{\pi}{H}\right)^2 + \left(\frac{2.405}{R}\right)^2 = \left(\frac{\pi}{D}\right)^2 + \left(\frac{4.810}{D}\right)^2 = \frac{33.0}{D^2}$$

But since we must have

$$1 = \frac{k_\infty}{1+M^2B^2} = \frac{1.36}{1+18.0^2 \cdot 33.0/D^2}$$

$$D^2 = \frac{18^2 \cdot 33}{1.36-1} = 29,700 \text{ cm}^2$$

$$D = H = 172 \text{ cm} = 1.72 \text{ m}$$

Part b. The height is now cut in half $H' = H/2 = 172/2 = 86$ cm

$$k = \frac{k_\infty}{1+M^2B^2} = \frac{k_\infty}{1+M^2\left[\left(\frac{4.81}{D}\right)^2 + \left(\frac{\pi}{H'}\right)^2\right]}$$

$$k = \frac{1.36}{1+18^2\left[\left(\frac{4.81}{172}\right)^2 + \left(\frac{\pi}{86}\right)^2\right]} = 0.8068$$

[7.4] A sodium-cooled fast reactor is to be built with the composition given in problem [4.3] with a height to diameter ratio of 0.8. If the reactor, which is bare, is to have a value of *k* = 1.005 when no control rods are inserted, what should the value of the reactor's diameter be?

A sodium-cooled fast reactor is to be built with the composition given in Problem [4.3], for which $k_\infty = 1.342$. The core is to be a bare, uniform cylinder with a height to diameter ratio of one.

a. Determine the diffusion length of the core material.
b. If the core is to have a multiplication of *k* = 1.005 (*i.e.*with no control rods inserted) determine the core diameter.
c. What is the nonleakage probability for the core.

Here are All of the Calculations from a spread sheet:

	PuO2	UO2	Na	Fe
rho	11	11	0.97	7.87
nu	2.98	2.47		
A	271	270	23	55.85
micro f	1.95	0.05		
mico a	2.4	0.404	0.0018	0.0087
micro t	8.6	8.2	3.7	3.6
* e-24	0.024447601	0.024538148	0.025401348	0.084871996
macro f	0.047672823	0.001226907	0	0
nu* macro f	0.142065012	0.003030461	0	0
macro a	0.058674244	0.009913412	4.57224E-05	0.000738386
macro t	0.210249373	0.201212815	0.093984987	0.305539187
Vi/V	0.045	0.255	0.5	0.2

volume ave:	nu sig f	sig f	sig a	sig t
	0.007165693	0.002458138	0.005338799	0.16887082

k	1.342191858
D	1.973895386
L	19.22827221

a. Therefore the diffusion length is 19.2 cm = 0.192 m
b. For a cylindrical core with *H*=*Dia* the buckling is

$$B^2 = [(2\cdot 2.405)^2 + \pi^2]/Dia^2 = 33.0/Dia^2$$

$$k = k_\infty/(1+L^2B^2) \quad \text{Thus} \quad B^2 = L^{-2}(k_\infty/k - 1) \quad \text{or}$$

$$Dia^2 = 33.0L^2(k_\infty/k-1)^{-1}$$

$$Dia = \sqrt{\frac{33.0}{(k_\infty/k-1)}}\,L = \sqrt{\frac{33.0}{(1.342/1.05-1)}}\,19.2$$

c. For the nonleakage probability $P_{NL} = 1/(1+L^2B^2) = k/k_\infty = 1.05/1.342 = 0.782$

Thus *Dia* = 209 cm = 2.09 m

[7.5] A cylindrical tank is constructed for storage of liquids containing fissionable material. The tank diameter of 0.90 m., and it is surrounded by non-reflecting neutron absorber. Material with $k_\infty = 1.16$ and M = 7.0 cm. is poured into the tank. Neglecting extrapolation lengths:

a. To what height can the tank be filled before it becomes critical?
b. Estimate the maximum value k_∞ permissible (M remaining unchanged) if it must be guaranteed that criticality will not be reached no matter to what height the tank is filled.
c. It is decided to reduce the diameter of the tank so that the material in part a. can never reach criticality. Estimate the reduced diameter.

In this solution , we express migration lengths and dimensions in meters. Since the reactor is uniform and cylindrical, Eqs. (7.6) and (7.20) are applicable (with L replaced by M):

$$B_r^2 + B_z^2 = (k / k_\infty - 1) / M^2$$

Part a:

$$\begin{aligned} B_z^2 &= (k_\infty / k - 1) / M^2 - B_r^2 = (k_\infty / k - 1) / M^2 - (2.405 / R)^2 \\ &= (1.16 / 1.0 - 1) / 0.07^2 - (2.405 / 0.45)^2 = 4.09 \end{aligned} \quad \text{(with } R\text{=0.9/2=0.45)}$$

$$B_z = \pi / H \ \text{ or } \ H = \pi / B_z = 3.141 / \sqrt{4.09} = 1.55\, m$$

Part b. An infinite height implies that the axial bucking is equal to zero. Thus Eq. (7.6) becomes:

$$k_\infty = 1 + M^2 (2405 / R)^2 = 1 + 0.07^2 (2405 / 0.45)^2 = 1.140$$

Part c. Again the axial bucking is equal to zero.

$$(2.405 / R)^2 = (k_\infty / k - 1) / M^2 = (1.16 / 1.0 - 1) / .07^2 == 32.65$$

Thus

$$R = 2.405 / \sqrt{32.65} = 0.421 \quad m \quad \text{or a diameter of 0.842 m}$$

[7.6] Consider a critical reactor that is a cube with extrapolated side length *a*:

a. With the origin at the center, apply separation of variables in three-dimensional Cartesian geometry to show that the flux distribution is

$$\phi(x,y,z) = C\cos(\pi x/a)\cos(\pi y/a)\cos(\pi z/a)$$

b. Find C in terms of the reactor power, volume and $\gamma\Sigma_f$.

c. Determine the reactor's buckling

d. Suppose that a = 2.0 m and M = 20 cm. Determine the value of k_∞ required to obtain criticality (i.e. k = 1.0).

Part a. We begin by writing out Eq. (7.7) in Cartesian coordinates

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi(x,y,z)+B^2\phi(x,y,z)=0$$

Plug in $\phi(x,y,z)$ ::

$$C\cos(\pi y/a)\cos(\pi z/a)\frac{d^2}{dx^2}\cos(\pi x/a)+C\cos(\pi x/a)\cos(\pi z/a)\frac{d^2}{dy^2}\cos(\pi y/a)$$

$$+C\cos(\pi x/a)\cos(\pi y/a)\frac{d^2}{dz^2}\cos(\pi z/a)+B^2C\cos(\pi x/a)\cos(\pi y/a)\cos(\pi z/a)=0$$

and divide by $\phi(x,y,z)$:

$$\frac{\frac{d^2}{dx^2}\cos(\pi x/a)}{\cos(\pi x/a)}+\frac{\frac{d^2}{dy^2}\cos(\pi y/a)}{\cos(\pi y/a)}+\frac{\frac{d^2}{dz^2}\cos(\pi z/a)}{\cos(\pi z/a)}+B^2=0$$

Perform the derivatives:

$$\frac{d^2}{dx^2}\cos(\pi x/a)=-(\pi/a)\frac{d}{dx}\sin(\pi x/a)=-(\pi/a)^2\cos(\pi x/a)$$

and similarly for y and z. Combine equations:

$$-(\pi/a)^2-(\pi/a)^2-(\pi/a)^2+B^2=0$$

Thus the flux distribution is correct, since it satisfies the diffusion equation provided the bucking is set equal to $B^2=3(\pi/a)^2$; the flux positive within the reactor, and meets the boundary conditions because $\cos(\pi x/a)\ \big|_{x=\pm a/2}=\cos(\pm\pi/2)=0$ and similarly for y and z.

Part b. From Eq. (7.22)

$$P=\gamma\int\Sigma_f\phi dV=\gamma\Sigma_f\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\int_{-a/2}^{a/2}\phi(x,y,z)dxdydx$$

$$=\gamma\Sigma_f C\int_{-a/2}^{a/2}\cos(\pi x/a)dx\int_{-a/2}^{a/2}\cos(\pi y/a)dy\int_{-a/2}^{a/2}\cos(\pi z/a)dz$$

$$\int_{-a/2}^{a/2}\cos(\pi x/a)dx=(a/\pi)\sin(\pi x/a)\Big|_{-a/2}^{+a/2}=2a/\pi$$

and similarly for y and z. Thus

$$P=\gamma\Sigma_f C(2a/\pi)^3$$

Noting that the volume is $V = a^3$:

$$C = \frac{P}{\gamma\Sigma_f}\frac{\pi^3}{8a^3} = 3.86\frac{P}{\gamma\Sigma_f V}$$

Part c. In part a we determined the buckling to be

$$B^2 = 3(\pi / a)^2$$

Part d. We utilize $1 = \dfrac{k_\infty}{1+M^2B^2}$, Thus

$$k_\infty = 1 + M^2B^2 = 1 + M^2 3(\pi / a)^2$$

$$= 1 + 20^2 \cdot 3(\pi / 200)^2 = 1.296$$

[7.7] A cylindrical fast reactor has a volume of 14 m^3 and a migration length of 20 cm. For height to diameter ratios between 0.5 and 2.0 make plots on the same graph of

a. the nonleakage probability P_{NL}

b. the value of k_∞ required for the reactor to be critical

Part a: To determine the nonleakage probability, first we determine the buckling

$$B^2 = \frac{(4.81)^2}{D^2} + \frac{\pi^2}{H^2} = \left[(4.81)^2 \left(\frac{H}{D} \right)^2 + \pi^2 \right] H^{-2}$$

We next use the volume relationship to find *H* in terms of *H/D* :

$$\pi D^2 H / 4 = V \quad \text{or} \quad H^3 = \frac{4}{\pi} V \left(\frac{H}{D} \right)^2 \quad \text{or} \quad H^{-2} = \left(\frac{4}{\pi} V \right)^{-2/3} \left(\frac{H}{D} \right)^{-4/3}$$

Hence (with M = 0.20 m)

$$B^2 = \left[(4.81)^2 \left(\frac{H}{D} \right)^2 + \pi^2 \right] \left(\frac{4}{\pi} V \right)^{-2/3} \left(\frac{H}{D} \right)^{-4/3}$$

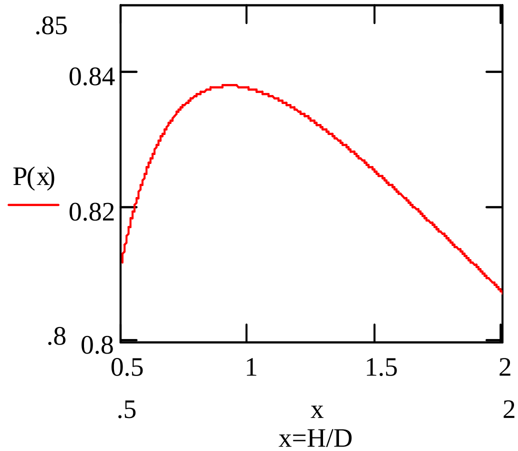
$$= \left(\frac{4}{\pi} V \right)^{-2/3} \left[(4.81)^2 \left(\frac{H}{D} \right)^{2/3} + \pi^2 \left(\frac{H}{D} \right)^{-4/3} \right] = 0.147 \left[23.1 \left(\frac{H}{D} \right)^{2/3} + 9.87 \left(\frac{H}{D} \right)^{-4/3} \right]$$

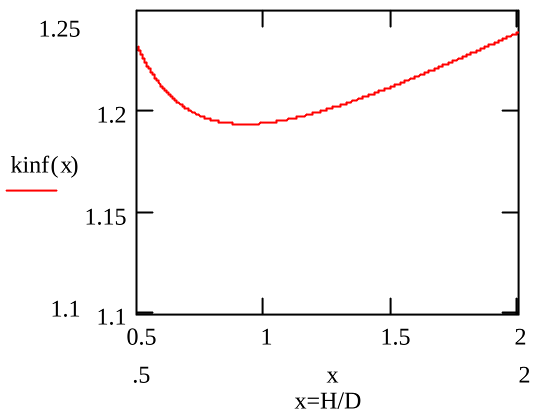
$$P_{NL} = \frac{1}{1 + M^2 B^2} = \frac{1}{1 + 0.20^2 \cdot 0.147 \left[23.1 \left(\frac{H}{D} \right)^{2/3} + 9.87 \left(\frac{H}{D} \right)^{-4/3} \right]}$$

$$P_{NL} = \frac{1}{1 + 0.136 \left(H/D \right)^{2/3} + 0.0580 \left(H/D \right)^{-4/3}}$$ See plot on next page

Part b: $1 = k_\infty P_{NL}$. Thus

$k_\infty = 1 / P_{NL}$ see plot on next page


.85
0.84
P(x)
0.82
.8
0.8
0.5
1
1.5
2
.5
x
2
x=H/D


1.25
1.2
kinf(x)
1.15
1.1
1.1
0.5
1
1.5
2
.5
x
2
x=H/D

[7.8] A critical bare cylindrical reactor has a height to diameter ratio of one, and a migration length of 7.5 cm. The core volume is 15 m^3. To simplify analysis, an engineer replaced the cylinder with a sphere of the same volume.

a. What is the sign and magnitude of the error in the multiplication caused by this simplification?

b.. If the reactor has a larger volume of 30 m^3, will the error be larger or smaller than in part a? Justify your result.

Part a. Since $V = \pi D^3/4$ the diameter of the cylinder will be

$$D = (4V/\pi)^{1/3} = (4\cdot 15/\pi)^{1/3} = 2.673\text{ m} \;,$$

With a height to diameter ratio of one, the bucking is

$$B^2 = 33/D^2 = 33/2.67^2 = 4.33\text{ m}^{-2}$$

The nonleakage probability is

$$P_{NL} = \frac{1}{1+M^2B^2} = \frac{1}{1+0.075^2 4.33} = 0.976$$

For a sphere $V = 4\pi R^3/3$, and the radius will be

$$R = (3V/4\pi)^{1/3} = (3\cdot 15/4\pi)^{1/3} = 1.53\text{ m}$$

The bucking is

$$B^2 = (\pi/R)^2 = (\pi/1.53)^2 = 4.22\text{ m}^{-2}$$

The nonleakage probability is

$$P_{NL} = \frac{1}{1+M^2B^2} = \frac{1}{1+0.075^2 4.22} = 0.977$$

Since $k = k_\infty P_{NL}$

$$\frac{k_{sp}-k_{cy}}{k_{cy}} = \frac{P_{sp}}{P_{cy}} - 1 = \frac{0.977}{0.976} - 1 = 0.0010 \qquad \text{or } 0.10\%$$

Part b For a larger core, the effect should be even less: Taking V = 30 m:

$$D = (4V/\pi)^{1/3} = (4\cdot 30/\pi)^{1/3} = 3.367\text{ m} \;,$$

With a height to diameter ratio of one, the bucking is

$$B^2 = 33/D^2 = 33/3.367^2 = 2.91\text{ m}^{-2}$$

The nonleakage probability is

$$P_{NL} = \frac{1}{1+M^2B^2} = \frac{1}{1+0.075^2 2.91} = 0.984$$

For a sphere the radius will be

$$R = (3V/4\pi)^{1/3} = (3\cdot 30/4\pi)^{1/3} = 1.93\text{ m}$$

The bucking is

$$B^2 = (\pi/R)^2 = (\pi/1.93)^2 = 2.65\text{ m}^{-2}$$

The nonleakage probability is

$$P_{NL} = \frac{1}{1+M^2B^2} = \frac{1}{1+0.075^2 2.65} = 0.985$$

Since $k = k_\infty P_{NL}$

$$\frac{k_{sp} - k_{cy}}{k_{cy}} = \frac{P_{sp}}{P_{cy}} - 1 = \frac{0.985}{0.984} - 1 = 0.0010 \qquad \text{or } 0.10\%$$

Or again about one tenth of one percent. However, for these small leakages, calculations of more than three place accuracy are required to assess the small changes in the multiplication. For smaller cores the differences will be larger.

[7.9] Show that $C_1 = 0$ in Eq. (7.16) from the boundary conditions $\chi(\pm\tilde{H}/2) = 0$ without employing the symmetry condition $\chi(z) = \chi(-z)$

Set $\chi(\tilde{H}/2) = 0$ and $\chi(-\tilde{H}/2) = 0$ Eq. (7.16) then gives, respectively,

$$0 = C_1 \sin(B_z\tilde{H}/2) + C_2 \cos(B_z\tilde{H}/2)$$

and

$$0 = C_1 \sin(-B_z\tilde{H}/2) + C_2 \cos(-B_z\tilde{H}/2)$$

But since $\sin(-x) = -\sin(x)$ and $\cos(-x) = \cos(x)$ we may rewrite the last equation as

$$0 = -C_1 \sin(B_z\tilde{H}/2) + C_2 \cos(B_z\tilde{H}/2)$$

Thus subtracting the last from the first equation:

$$0 = 2C_1 \sin(B_z\tilde{H}/2) \quad \text{or} \quad C_1 = 0\ C_1 = 0.$$

[7.10] Express C_1''' and C_2''' in Eqs. (7.61) in terms of C_1'' and C_2'' in Eq. (7.60).

Write out sinh and cosh in Eq.(7.61):

$$\varsigma(z) = C_1'''\tfrac{1}{2}[\exp(\alpha z) - \exp(-\alpha z)] + C_2'''\tfrac{1}{2}[\exp(\alpha z) + \exp(-\alpha z)]$$

$$= \tfrac{1}{2}\left(C_2''' + C_1'''\right)\exp(\alpha z) + \tfrac{1}{2}\left(C_2''' - C_1'''\right)\exp(-\alpha z)$$

Comparing this expression to Eq. (7.60) we see that

$$C_1'' = \tfrac{1}{2}\left(C_2''' + C_1'''\right) \qquad \text{and} \qquad C_2'' = \tfrac{1}{2}\left(C_2''' - C_1'''\right)$$

[7.11] Apply the boundary condition $\zeta(H'/2+a)=0$ to determine C_2''' in terms of C_1''' in Eq. (7.61). Then determine C' in Eq. (7.62) in terms of C_1''' and C_2'''.

In Eq. (7.61),

$$\zeta(H'/2+a)=0=C_1'''\sinh[\alpha(H'/2+a)]+C_2'''\cosh[\alpha(H'/2+a)]$$

Thus

$$C_2'''=-C_1'''\tanh[\alpha(H'/2+a)]$$

Next equate Eqs. (7.61) and (7.62):

$$C_1'''\sinh(\alpha z)+C_2'''\cosh(\alpha z)=C'\sinh[\alpha(H'/2+a-z)]$$

One way to proceed would be to apply hyperbolic identities. Instead, we convert everything to exponentials;

$$\begin{aligned}
&\tfrac{1}{2}(C_2'''+C_1''')\exp(\alpha z)+\tfrac{1}{2}(C_2'''-C_1''')\exp(-\alpha z)\\
&\quad =C'\tfrac{1}{2}\left\{\exp[\alpha(H'/2+a-z)]-\exp[-\alpha(H'/2+a-z)]\right\}\\
&\quad =C'\tfrac{1}{2}\left\{\exp[\alpha(H'/2+a)]\exp(-\alpha z)-\exp[-\alpha(H'/2+a)]\exp(\alpha z)\right\}
\end{aligned}$$

Regrouping terms:

$$\begin{aligned}
&\left\{(C_2'''+C_1''')+C'\exp[-\alpha(H'/2+a)]\right\}\exp(\alpha z)\\
&+\left\{(C_2'''-C_1''')-C'\exp[\alpha(H'/2+a)]\right\}\exp(-\alpha z)=0
\end{aligned}$$

Since $\exp(\pm\alpha z)$ are independent of one another, the two coefficients must be equal to zero: Hence

$$C'=-(C_2'''+C_1''')\exp[\alpha(H'/2+a)]$$

and

$$C'=(C_2'''-C_1''')\exp[-\alpha(H'/2+a)]$$

But these must be equal. Define $\beta\equiv\alpha(H'/2+a)$ If we eliminate C_2'''

$$\begin{aligned}
C'&=-(C_2'''+C_1''')\exp(\beta)=C_1'''(\tanh(\beta)-1)\exp(\beta)\\
&=C_1'''\frac{[\sinh(\beta)-\cosh(\beta)]\exp(\beta)}{\cosh(\beta)}\\
&=C_1'''\frac{\tfrac{1}{2}[\cancel{\exp(\beta)}-\exp(-\beta)-\cancel{\exp(\beta)}-\exp(-\beta)]\exp(\beta)}{\cosh(\beta)}=-\frac{C_1'''}{\cosh(\beta)}
\end{aligned}$$

and

$$\begin{aligned}
C'&=(C_2'''-C_1''')\exp(-\beta)=-C_1'''(\tanh(\beta)+1)\exp(-\beta)\\
&=-C_1'''\frac{[\sinh(\beta)+\cosh(\beta)]\exp(-\beta)}{\cosh(\beta)}\\
&=-C_1'''\frac{\tfrac{1}{2}[\exp(\beta)-\cancel{\exp(-\beta)}+\exp(\beta)+\cancel{\exp(-\beta)}]\exp(-\beta)}{\cosh(\beta)}=-\frac{C_1'''}{\cosh(\beta)}
\end{aligned}$$

Thus by two different routs we have arrived at the relationship

$$C'=-\frac{C_1'''}{\cosh[\alpha(H'/2+a)]}$$

[7.12] Consider the situation when the spherical system discussed in Chapter 6.7 is critical. Determine the ratio of maximum to average flux in the sphere.

The flux distribution is given by Eq. (6.107), which when combined with Eq. (6.104) yields

$$\phi(r) = \frac{C_1}{r}\sin(\pi r / R)$$

The maximum flux is at the center of the sphere. For small r, $\sin(\pi r) \approx \pi r$. Thus

$$\phi(0) = \pi C_1 / R$$

The volume-averaged flux is

$$\overline{\phi} = \frac{1}{V}\int \phi(r) dV = \frac{3}{4\pi R^3}\int_0^R \phi(r) 4\pi r^2 dr = C_1 \frac{3}{R^3}\int_0^R r\sin(\pi r) dr$$

We must integrate by parts (or use an integral table):

$$d\left[r\cos(\pi r)\right] = \cos(\pi r)dr - \pi r\sin(\pi r)dr$$

Hence

$$\left[r\cos(\pi r / R)\right]\Big|_o^R = \int_0^R \cos(\pi r / R)dr - \left(\pi / R\right)\int_0^R r\sin(\pi r / R)dr$$

and

$$\int_0^R r\sin(\pi r / R)dr = R^2 / \pi$$

The volume average is then

$$\overline{\phi} = C_1 \frac{3}{\pi R}$$

And the maximum to average ratio is

$$\phi(0) / \overline{\phi} = \pi^2 / 3 = 3.29$$

[7.13] A spherical reactor of radius *R* is surrounded by a reflector that extends to $r = \infty$. *L* and *D* are the same for core and reflector. Find the criticality equation relating k_∞, *R*, *L* and *D*.

The flux distribution within the core is given by Eq. (6.107), but with k_∞

$$\phi(r) = \frac{C_1}{r}\sin(L^{-1}\sqrt{k_\infty / k - 1}r) \qquad\qquad 0 \le r \le R$$

In the reflector, it is given by Eq.(6.51), but with C_1 =0 in order to meet the boundary condition $\phi(\infty) = 0$. Hence

$$\phi(r) = \frac{C_2}{r}\exp(-r/L) \qquad\qquad R \le r \le \infty$$

We next employ the interface conditions of Eqs. (6.42) and (6.43) at $r = R$:

$$\frac{C_1}{R}\sin(L^{-1}\sqrt{k_\infty - 1}\,R) = \frac{C_2}{R}\exp(-R/L)$$

and

$$-D\frac{C_1}{R^2}\sin(L^{-1}\sqrt{k_\infty - 1}\,R) + D\frac{C_1\sqrt{k_\infty - 1}}{R\ L}\cos(L^{-1}\sqrt{k_\infty - 1}\,R)$$
$$= -D\frac{C_2}{R^2}\exp(-R/L) - D\frac{C_2}{RL}\exp(-R/L)$$

Multiply the second equation by R^2/D and then divide the second by the first to eliminate the constants: Simplifying, we obtain:

$$\sqrt{k_\infty - 1}\cot(\sqrt{k_\infty - 1}\,R/L) + 1 = 0$$

which is the transcendental criticality equation.

[7.14] A spherical reactor is constructed with an internal reflector with parameters and D and Σ_a^r and extending $0 \le r \le R$. The annular core, with parameters D, Σ_a and k_∞ (>1), extends $R \le r \le 2R$.

a. Find the criticality condition (neglecting the extrapolation distance).
b. Sketch the flux distribution for $0 \le r \le 2R$.

Part a: Within the internal reflector the flux is given by Eq. (6.51) but with $C_2 = -C_1$ so that the flux is finite at the origin. Thus with $C = C_1/2$

$$\phi(r) = \frac{C_1}{r}\left[\exp(r/L) - \exp(-r/L)\right] = \frac{C}{r}\sinh(r/L) \qquad 0 \le r \le R$$

With the source term set equal to zero Eq. (6.102) specifies the flux in the core

$$\phi(r) = \frac{C_1}{r}\sin(\sqrt{k_\infty - 1}\, r/L) + \frac{C_2}{r}\cos(\sqrt{k_\infty - 1}\, r/L) \qquad R \le r \le 2R$$

To meet the boundary condition $\phi(2R) = 0$ we have

$$C_2 = -C_1 \tan(\sqrt{k_\infty - 1}\, 2R/L)$$

Thus

$$\phi(r) = \frac{C_1}{r}\left[\sin(\sqrt{k_\infty - 1}\, r/L) - \tan(\sqrt{k_\infty - 1}\, 2R/L)\cos(\sqrt{k_\infty - 1}\, r/L)\right]$$

which using trigonometric identities may also be written as

$$\phi(r) = \frac{C'}{r}\sin[\sqrt{k_\infty - 1}\,(2R - r)/L] \qquad R \le r \le 2R$$

We next apply the interace conditions given by Eqs. (6.42) and (6.43) at $r=R$:

$$\frac{C}{R}\sinh(R/L) = \frac{C'}{R}\sin[\sqrt{k_\infty - 1}\, R/L]$$

and

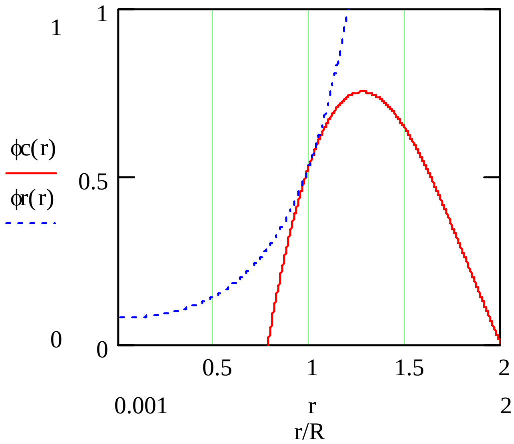
$$-D\frac{C}{R^2}\sinh(R/L) + D\frac{C}{RL}\cosh(R/L)$$
$$= -D\frac{C'}{R^2}\sin[\sqrt{k_\infty - 1}\, R/L] - D\frac{C'}{RL}\sqrt{k_\infty - 1}\cos[\sqrt{k_\infty - 1}\, R/L]$$

Divide the second by the first equation, and simplify

$$\coth(R/L) = -\sqrt{k_\infty - 1}\cot[\sqrt{k_\infty - 1}\, R/L]$$

which is the criticality equation.

Part b: For the plot we take R/L=4, which yields k_∞ =1.413, where the blue line is applicable in the center reflector region (r/R<1)and the red line in the annular core region(1<r/R<2).


1
1
φc(r)
φr(r)
0.5
0
0
0.5
1
1.5
2
0.001
r
2
r/R

[7.15] Show from Eq. (7.71) that the reflector savings is approximated by $\delta_z \approx \hat{M}D/\hat{D}$ for a thick reflector.

First plug Eq.(7.71) into $\delta_z = \frac{1}{2}(\tilde{H} - H')$ to obtain

$$\delta_z = \tfrac{1}{2}\tilde{H}\left\{1 - \frac{2}{\pi}\arctan\left(\alpha\frac{\tilde{H}\hat{D}}{\pi D}\right)\right\}$$

which we may rewrite as

$$\tan\left[\left(1 - 2\delta_z/\tilde{H}\right)\pi/2\right] = \alpha\frac{\tilde{H}\hat{D}}{\pi D}$$

From Eq. (7.58) we note that

$$\alpha^2 = \left(\frac{2.405}{R}\right)^2 + \left(\frac{1}{\hat{M}}\right)^2$$

If we assume that both the radius R and height of the reactor are much larger than $\hat{M}$, then $\alpha \approx \hat{M}^{-1}$ and

$$\tan\left[\left(1 - 2\delta_z/\tilde{H}\right)\pi/2\right] = \frac{\hat{D}}{\pi D}\left(\frac{\tilde{H}}{\hat{M}}\right)$$

Since we expect $\delta_z/\tilde{H}$ to be small the argument on the left will approach $\pi/2$, for which the tangent becomes infinite. Thus the left side of the equation should be large. This is consistent, since if $\hat{M}/\hat{H}$ is small, then the right side of the equation is also large. We expand sine and cosine about $\pi/2$ to obtain

`
$$\tan\left[\left(1 - 2\delta_z/\tilde{H}\right)\pi/2\right] = \frac{\sin\left[\left(1 - 2\delta_z/\tilde{H}\right)\pi/2\right]}{\cos\left[\left(1 - 2\delta_z/\tilde{H}\right)\pi/2\right]} \approx \frac{1}{\delta_z\pi/\tilde{H}}$$

combining equations, we obtain

$$\frac{1}{\delta_z\pi/\tilde{H}} = \frac{\hat{D}}{\pi D}\left(\frac{\tilde{H}}{\hat{M}}\right) \quad \text{or} \quad \delta_z = \frac{D}{\hat{D}}\hat{M}$$

[7.16] An infinite slab reactor (extending to infinity in the *y* and *z* directions) has a thickness of $2a$ with vacuum on either side. The properties for material #1 occupying $0 \le x \le a$ are $k_\infty^1 = k_\infty$, $D_1 = D$, $\Sigma_a^1 = \Sigma_a$ and for material #2 occupying $a \le x \le 2a$ are $k_\infty^2 = 0$, $D_2 = D$, $\Sigma_a^2 = 0$. Neglecting extrapolation distances:

a. Find a criticality equation relating a, k_∞, *D*, and Σ_a.
b. Sketch the flux between 0 and 2*a*.

Part a. In the core region we reduce Eq. (7,7) to slab geometry with $B^2 = (k_\infty - 1)/L^2$, and $L^2 = D/\Sigma_a$:

$$\frac{d^2}{dx^2}\phi(x) + B^2\phi(x) = 0 \qquad\qquad 0 \le x \le a$$

Which has a solution

$$\phi(x) = C_1' \sin(Bx) + C_2' \cos(Bx)$$

In the reflector region the solution is given by Eq. (6.20)

$$\phi(x) = C_1 \exp(x/L) + C_2 \exp(-x/L) \qquad\qquad a \le x \le 2a$$

We next apply boundary and interface conditions. On the left we have $\phi(0) = 0$. Thus $C_2' = 0$ and

$$\phi(x) = C_1' \sin(Bx) \qquad\qquad 0 \le x \le a\,.$$

On the right $\phi(2a) = 0$ and hence $C_2 - C_1 \exp(4a/L)$ and

$$\phi(x) = C_1\left[\exp(x/L) - \exp(4a/L)\exp(-x/L)\right] \qquad\qquad a \le x \le 2a$$

Which may also be written in terms of hyperbolic functions:

$$\phi(x) = C \sinh[(2a - x)/L] \qquad\qquad a \le x \le 2a$$

We next apply continuity of flux and current [Eqs (6.42) and (6.43)] at $x = a$:

$$C_1' \sin(Ba) = C \sinh(a/L)$$

and

$$DC_1' B \cos(Ba) = -DCL^{-1} \cosh(a/L)$$

dividing the first by the second, we obtain

$$B^{-1} \tan(Ba) = -L \tanh(a/L)$$

or substituting for *B*:

$$\tan(\sqrt{k_\infty - 1}\, a/L) = -\sqrt{k_\infty - 1}\, \tanh(a/L)$$

[7.17] Apply the interface conditions at the tip of the control rod bank to show that Eqs. (7.97) and (7.98) yield the criticality condition given by Eqs. (7.99) and (7.100).

First consider the $\alpha^2 > \beta^2$ case. Inserting Eqs. (7.97) and (7.98) into (7.95) and (7.96), gives

$$C\sin[\alpha(\tilde{H}-x)] = C'\sin\left(\sqrt{\alpha^2-\beta^2}\,x\right)$$

and

$$C\alpha\cos[\alpha(\tilde{H}-x)] = -C'\sqrt{\alpha^2-\beta^2}\cos\left(\sqrt{\alpha^2-\beta^2}\,x\right)$$

Dividing the second by the first yields

$$\alpha\cot[\alpha(1-x)] = -\sqrt{\alpha^2-\beta^2}\cot\left(\sqrt{\alpha^2-\beta^2}\,x\right) \qquad \alpha^2 > \beta^2$$

Which is Eq. (7.99). Now consider the $\alpha^2 < \beta^2$ case. Inserting Eqs. (7.97) and (7.98) into (7.95) and (7.96),

$$C\sin[\alpha(\tilde{H}-x)] = C'\sinh\left(\sqrt{\beta^2-\alpha^2}\,x\right)$$

and

$$C\alpha\cos[\alpha(\tilde{H}-x)] = -C'\sqrt{\beta^2-\alpha^2}\cosh\left(\sqrt{\beta^2-\alpha^2}\,x\right)$$

Again, dividing the second by the first yields Eq. (7.100)

$$\alpha\cot[\alpha(1-x)] = -\sqrt{\beta^2-\alpha^2}\coth\left(\sqrt{\beta^2-\alpha^2}\,x\right) \qquad \alpha^2 < \beta^2 .$$

[7.18] Beginning with Eq. (7.98) prove that with the rod bank fully inserted that

a. $\alpha_r^2 - \beta_r^2 = B_z^2$

b. $k_r = (1-\rho_b)k_u$

Part a: From Eq. 97.98) with the rods fully inserted we must have

$\chi_r(0) = C'\sin\left[\sqrt{\alpha_r^2 - \beta_r^2}\tilde{H}\right] = 0$ Thus for the flux to be positive everywhere within the core: $\sqrt{\alpha_r^2 - \beta_r^2}\tilde{H} = \pi$ Thus $\alpha_r^2 - \beta_r^2 = (\pi/\tilde{H})^2 = B_z^2$

Part b: Solve for $\alpha_r^2 = \beta_r^2 + B_z^2$ and substitute into Eq. (7.93)

$$\beta_r^2 + B_z^2 = \frac{1}{M^2}\left(\frac{k_\infty}{k} - 1\right) - B_r^2$$

Then eliminating β_r^2 between this expression and Eq. (7.94):

$$\frac{1}{M^2}\frac{k_\infty}{k}\rho_b + B_z^2 = \frac{1}{M^2}\left(\frac{k_\infty}{k} - 1\right) - B_r^2$$

Thus

$$k_r = (1-\rho_b)\frac{k_\infty}{1 + M^2(B_r^2 + B_z^2)} = (1-\rho_b)k_u$$

Chapter 8 Problems:

[8.1] The leakage probability of a power reactor is 0.065. As a fist approximation to a new reactor an engineer estimates that the same power density can be achieved if the power is to be increased by 20%. Assuming the height to diameter ratio of the cylindrical core remains the same:

a. What will the leakage probability be in the new reactor with the power increased by 20%?
b. If k_∞ is proportional to the fuel enrichment, by what percent will the enrichment of the core need to be changed to accommodate the 20% increase in power.

Let V and V' be the old and new volumes, and assume a height to diameter ratio $\alpha = H/D$. Then $V = \frac{1}{4}\pi\alpha D^3$ and $V' = \frac{1}{4}\pi\alpha D'^3$ Thus $V'/V = (D'/D)^3$

$D' = (V'/V)^{1/3} D = (1.2)^{1/3} D = 1.0627D$

or $\left(D'/D\right)^2 = (1.2)^{2/3} = 1.129$

$$B^2 = (2\cdot 2.405/D)^2 + (\pi/\alpha D)^2 = \beta/D^2 \quad \text{where} \quad \beta = (2\cdot 2.405)^2 + (\pi/\alpha)^2$$

$(B'/B)^2 = (D/D')^2$ =1/1.129=0.8855

$P_{NL} = 1 - P_L = 1 - 0.065 = 0.935$

$P_{NL} = 1/(1 + L^2B^2)$ => $B^2 = L^{-2}(1/P_{NL} - 1)$

$(B'/B)^2 = (1/P'_{NL} - 1)/(1/P_{NL} - 1)$

$P'_{NL} = [1 + (1/P_{NL} - 1)(B'/B)^2]^{-1} = [1 + (1/0.935 - 1)0.8855]^{-1} = 09420$

$P'_L = 0.05799 \qquad 1 = k'_\infty P'_{NL} = k_\infty P_{NL}$

$$\frac{k'_\infty - k_\infty}{k_\infty} = \frac{k'_\infty}{k_\infty} - 1 = \frac{P_{NL}}{P'_{NL}} - 1 = \frac{0.9350}{0.9420} - 1$$

$$\frac{k'_\infty - k_\infty}{k_\infty} = -0.0074 = -0.74\%$$

[8.2] A sodium-cooled fast reactor lattice is designed to have a migration length 20 cm, and a maximum power density 500 W/cm^3, Three bare cylindrical cores with height-to-diameter ratios of one are to be built, with power rating of 300 MW(t), 1000 MW(t) and 3000 MW(t). For each of the three cores determine

a. the core height H,
b. the buckling B^2
c. the nonleakage probability P_{NL}.

Take all dimensions is m and power in MW

Part a. For a bare uniform core, if follows from Eq. (8.22) that $P'''_{\max} = 3.63\bar{P}''' = 3.63P/V$

Thus, noting that 500 W/cm^3=500MW/m^3

$$V = 3.63P / P'''_{\max} = 3.63P/500 = .00726P$$

But for a cylinder with a height to diameter ratio of one: $V = \pi H^3/4$ or

$$H = (4V/\pi)^{1/3} = (4\cdot 0.00726P/\pi)^{1/3} = 0.210P^{1/3}$$

Part b. For a height to diameter ratio of one:

$$B^2 = (2.405\cdot 2/H)^2 + (\pi/H)^2 = 33.0H^{-2}$$

$$B^2 = 33.0(0.210P^{1/3})^{-2} = 478P^{-2/3}$$

Part c. $P_{NL} = 1/(1+M^2B^2) = 1/(1+0.20^2\cdot 478P^{-2/3})$

$$P_{NL} = 1/(1+19.12P^{-2/3})$$

Thus for the three reactors

Power P	300 MW	1000MW	3000MW
Height H	1.41 m	2.1 m	3.03 m
Buckling B^2	10.67 m^{-2}	4.78 m^{-2}	2.30 m^{-2}
Nonleakage P_{NL}	0.701	0.839	0.916

[8.3] Consider a nonuniform cylindrical reactor with core radius *R* and height *H*. With control rods partially inserted, the power density distribution is approximated by

$$P'''(r,z) = A\left[1-(r/R)^4\right]\left[\cos(\pi z/H) - 0.25\sin(2\pi z/H)\right]$$

(The (*r*,*z*)origin is at the center of the reactor)

a. Find *A* in terms of the reactor power *P*.
b. Determine $f_r(r)$ and $f_z(z)$.
c. Determine, F_r, F_z and F_q (assuming $F_l = 1.1$).
d. Plot $f_r(r)$ and $f_z(z)$.

First rewrite the power density equation as

$$P'''(r,z) = \frac{P}{V} C_r\left[1-(r/R)^4\right] C_z\left[\cos(\pi z/H) - 0.25\sin(2\pi z/H)\right]$$

Then $A = (P/V)C_r C_z$ and take

$$f_r(r) = C_r\left[1-(r/R)^4\right] \text{ and } f_z(z) = C_z\left[\cos(\pi z/H) - 0.25\sin(2\pi z/H)\right]$$

Next determine C_r and C_z from Eqs. (8.6) and (8.7):

$$1 = C_r\frac{2}{R^2}\int_0^R \left[1-(r/R)^4\right] r dr = C_r\frac{2}{R^2}\left(\frac{r^2}{2} - \frac{r^6}{6R^4}\right)\Bigg|_0^R = \frac{2}{3}C_r$$

$$1 = C_z\frac{1}{H}\int_{-H/2}^{H/2} \left[\cos(\pi z/H) - 0.25\sin(2\pi z/H)\right] dz$$

$$= C_z\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \left[\cos(\zeta) - 0.25\sin(2\zeta)\right] dz = C_z\frac{1}{\pi}(2 - 0.25\cdot 0) = \frac{2}{\pi}C_z$$

Thus $C_r = 1.5 \quad C_z = 1.571$

$$f_r(r) = 1.50\left[1-(r/R)^4\right]$$

$$f_z(z) = 1.571\left[\cos(\pi z/H) - 0.25\sin(2\pi z/H)\right]$$

$$A = 2.356P/V$$

For the peaking factors

$F_r = 1.50$ However, to determine F_r we must first determine the value of z for which $f_z(z)$ is a maximum, by setting its derivative equal to zero:

$$0 = \frac{d}{dz}f_z(z) = 1.571\left[-\frac{\pi}{H}\sin(\pi z/H) - 0.25\frac{2\pi}{H}\cos(2\pi z/H)\right]$$

or $\sin(\pi z/H) + 0.5\cos(2\pi z/H) = 0$ which has a solution of $\pi z/H = -0.3747$

$$F_z = f_z(z)\big|_{\max} = 1.571\left[\cos(-0.3747) - 0.25\sin(-2\cdot 0.3747)\right]$$

$$= 1.571\cdot\left(0.9603 + 0.25\cdot 0.681\right)$$

$$F_q = F_r F_z F_l = 1.50\cdot 3.60\cdot 1.1 = 5.94$$

[8.4] Consider a nonuniform cylindrical reactor with core radius *R* and height *H*. Two zones of fuel are employed, with a higher enrichment at the radial periphery to decrease the radial peaking factor. As a result, the power density if given by

$$P'''(r,z)=\begin{cases} A\left[1-(r/R)^4\right]\cos(\pi z/H) & for\ 0\le r<\tfrac{3}{4}R \\ 1.7A\left[1-(r/R)^4\right]\cos(\pi z/H) & for\ \tfrac{3}{4}R\le r\le R \end{cases}$$

(The (*r*,*z*)origin is at the center of the reactor)

a. Find *A* in terms of the reactor power *P*.
b. Determine $f_r(r)$ and $f_z(z)$.
c. Determine, F_r and F_z.

Part a: From Eqs. (8.1) and (8.6) we may write $P=\int P'''(\vec{r})dV$ which for a cylindrical reactor is

$$P=\int_{-H/2}^{H/2}\int_0^R \phi(r,z)2\pi r dr dz$$

$$P=\int_{-H/2}^{H/2}\int_0^{3R/4} A\left[1-(r/R)^4\right]\cos(\pi z/H)2\pi r dr dz$$

$$+\int_{-H/2}^{H/2}\int_{3R/4}^{R} 1.7A\left[1-(r/R)^4\right]\cos(\pi z/H)2\pi r dr dz$$

$$P=2\pi A\int_{-H/2}^{H/2}\cos(\pi z/H)dz\left\{\int_0^{3R/4}\left[1-(r/R)^4\right]rdr+1.7\int_{3R/4}^{R}\left[1-(r/R)^4\right]rdr\right\}$$

$$P=4HA\left\{\frac{1}{2}\left(\frac{3}{4}\right)^2R^2-\frac{1}{6}\left(\frac{3}{4}\right)^6R^2+1.7\left[\frac{1}{2}R^2-\frac{1}{2}\left(\frac{3}{4}\right)^2R^2-\frac{1}{6}R^2+\frac{1}{6}\left(\frac{3}{4}\right)^6R^2\right]\right\}$$

$$P=4HA\left\{O.2516R^2+1.7\left[0.08175R^2\right]\right\}=1.562AHR^2=1.562AV/\pi$$

$$A=\frac{\pi}{1.562}\frac{P}{V}=2.01\frac{P}{V}$$

Part b Rewrite the power density as

$$P'''(r,z)=\frac{P}{V}C_z\cos(\pi z/H)C_r\begin{cases}\left[1-(r/R)^4\right] & for\ 0\le r<\tfrac{3}{4}R \\ 1.7\left[1-(r/R)^4\right] & for\ \tfrac{3}{4}R\le r\le R\end{cases}$$

Then $A=(P/V)C_rC_z$. and

$$f_z(z)=C_z\cos(\pi z/H)$$

$$f_r(r)=C_r\begin{cases}\left[1-(r/R)^4\right] & for\ 0\le r<\tfrac{3}{4}R \\ 1.7\left[1-(r/R)^4\right] & for\ \tfrac{3}{4}R\le r\le R\end{cases}$$

From Eq.(8.10)

$$1=\frac{1}{H}\int_{-H/2}^{H/2} f_z(z)dz = C_z\frac{1}{H}\int_{-H/2}^{H/2} \cos(\pi z/H)dz = C_z 2/\pi$$

Hence,

$$f_z(z)=\frac{\pi}{2}\cos(\pi z/H)=1.57\cos(\pi z/H)$$

We could perform the integral of Eq. (8.9) to determine C_r. However, we already know that $A=\frac{\pi}{1.562}\frac{P}{V}$ and that $A=(P/V)C_rC_z$ Thus

$$C_r=\frac{\pi}{1.562}\frac{1}{C_z}=\frac{\pi}{1.562}\frac{2}{\pi}=1.28$$

Thus

$$f_r(r)=1.28\begin{cases}\left[1-(r/R)^4\right] & \text{for } 0\le r<\tfrac{3}{4}R\\ 1.7\left[1-(r/R)^4\right] & \text{for } \tfrac{3}{4}R\le r\le R\end{cases}$$

Part c: F_r and F_z are just the maximums of $f_r(r)$ and $f_z(r)$. Note that for $f_r(r)$ the maximum occurs not at r =0, but at $r=3R/4$. Thus

$$F_r=1.28\cdot1.7[1-(3/4)^4]=1.28\cdot1.162=1.49$$

The axial peak does occur at z=0, thus

$$F_z=1.57$$

[8.5] Beginning with the heat balance $W_{ch}c_p dT_c(r,z) = q'(r,z)dz$ show that if the power distribution is axially symmetric, $q'(r,-z) = q'(r,z)$, then Eq. (8.39) for the average coolant temperature is exact.

Integrating the above equation from the bottom of the core

$$T_c(r,z) = T_i + \frac{1}{W_{ch}c_p}\int_{-H/2}^{z} q'(r,z')dz'$$

Let

$$\hat{T}_c(r) = \frac{1}{H}\int_{-H/2}^{H/2} T_c(r,z)dz$$

represent the axial average. Then combining equations we have

$$\hat{T}_c(r) = T_i + \frac{1}{H}\int_{-H/2}^{H/2}\left(\frac{1}{W_{ch}c_p}\int_{-H/2}^{z} q'(r,z')dz'\right)dz$$

changing the order of integration

$$\hat{T}_c(r) = T_i + \frac{1}{HW_{ch}c_p}\int_{-H/2}^{H/2} q'(r,z')\left(\int_{z'}^{H/2} dz\right)dz'$$

$$= T_i + \frac{1}{HW_{ch}c_p}\int_{-H/2}^{H/2} q'(r,z')\left(H/2 - z'\right)dz'$$

$$= T_i + \frac{1}{W_{ch}c_p}\left[\frac{1}{2}\int_{-H/2}^{H/2} q'(r,z')dz' - \frac{1}{H}\int_{-H/2}^{H/2} q'(r,z')z'dz'\right]$$

With the change in variables $z'' = -z'$

$$\int_{-H/2}^{H/2} q'(r,z')z'dz' = \int_{-H/2}^{0} q'(r,z')z'dz' + \int_{0}^{H/2} q'(r,z')z'dz$$

$$= -\int_{0}^{H/2} q'(r,-z'')z''dz'' + \int_{0}^{H/2} q'(r,z')z'dz$$

Ths if $q'(r,-z) = q'(r,z)$,

$$\int_{-H/2}^{H/2} q'(r,z')z'dz' = 0$$

and

$$\hat{T}_c(r) = T_i + \frac{1}{2W_{ch}c_p}\int_{-H/2}^{H/2} q'(r,z')dz'$$

From Eq. (8.24) we have

$$\hat{T}_c(r) = T_i + \frac{A_{cell}}{2W_{ch}c_p}\int_{-H/2}^{H/2} P'''(r,z')dz'$$

Averaging over the radial direction:

$$\bar{T}_c \equiv \frac{2}{R^2}\int_0^R \hat{T}_c(r)rdr = T_i + \frac{A_{cell}H}{2W_{ch}c_p}\left[\frac{2}{R^2}\int_0^R \frac{1}{H}\int_{-H/2}^{H/2} P'''(r,z')dz'rdr\right]$$

$$= T_i + \frac{A_{cell}}{2W_{ch}c_p}\frac{HP}{V} = T_i + \frac{A_{cell}}{2W_{ch}c_p}\frac{P}{\pi R^2}$$

but $NA_{cell} = \pi R^2$, where N is the number of cooling channels, and so $W = NW_{ch}$ and

$$\frac{A_{cell}}{W_{ch}\pi R^2} = \frac{1}{W_{ch}N} = \frac{1}{W}$$

Hence

$$\overline{T}_c = T_i + \frac{1}{2Wc_p}P\,.$$

combined with $\overline{T}_o - T_i = \frac{1}{Wc_p}P$ yields Eq. (2.39):

$$\overline{T}_c = \tfrac{1}{2}(\overline{T}_o + T_i)$$

[8.6] You are to design a 3000 MW(t) pressurized water reactor. The reactor is a uniform bare cylinder with a height-to-diameter ratio of one. The coolant to fuel volume ratio is two to one in a square lattice. The volumes occupied by control and structural materials, as well as the extrapolation distances, can be neglected. The core inlet temperature is 290 ^{o}C. The reactor must operate under three thermal constraints:

1. maximum power density = 250 W/cm^3
2. maximum cladding surface heat flux = 125 W/cm^2
3. maximum core outlet temperature = 330 ^{o}C

Determine:

a. the reactor dimensions and volume
b. the fuel element diameter and lattice pitch
c. the approximate number of fuel elements
d. the mass flow rate and average coolant velocity

Part a. The peak to average power density in a uniform cylindrical reactor is 3.63 Thus

$$P'''_{\max} = 3.63\overline{P}''' = 3.63P/V$$

or

$$V = 3.63P/P'''_{\max} = 3.63 \cdot 3000\text{MW}/250(\text{MW/m}^3)$$

$$V = 43.6 \text{ m}^3$$

$$V = \pi\frac{D^2}{4}H = \pi\frac{D^3}{4}$$

$$D = H = (4V/\pi)^{1/3} = (4\cdot 43.6/\pi)^{1/3} = 3.81\,\text{m}$$

Part b. We also Have for the cladding heat flux $q''_{\max} = 3.63\overline{q}''$ or

$$\overline{q}'' = q''_{\max}/3.63 = 125/3.63 = 34.4 \text{ W/cm}$$

but if N is the number of rods and a is their radius, the total heat transfer surface is $2\pi aHN$, and

$$2\pi aHN\overline{q}'' = P \quad \text{or}$$

$$aN = \frac{P}{2\pi H\overline{q}''} = \frac{3\cdot 10^9\,\text{W}}{2\pi\cdot 381\text{cm}\cdot 34.4\text{W/cm}} = 3.64\cdot 10^4 \text{ cm}$$

However, if p is the lattice pitch, then

$$Np^2 = \pi D^2/4 = \pi D^2/4 = 11.4 \text{ m}^2 = 11.4\cdot 10^4 \text{ cm}^2$$

Divide this equation by aN:

$$\frac{p^2}{a} = \frac{Np^2}{aN} = \frac{11.4\cdot 10^4}{3.64\cdot 10^4} = 3.1\,\text{cm}$$

Also

$$\frac{p^2 - \pi a^2}{\pi a^2} = \frac{1}{\pi}(p/a)^2 - 1 = 2.0 \quad \text{(moderator to fuel volume ratio)}$$

$$p/a = \sqrt{3\pi} = 3.07$$

Thus

$$\frac{p^2}{a} = p\left(\frac{p}{a}\right) \quad \text{or} \qquad 3.1 = p\cdot 3.07 \text{ or } p = 1.01 \text{ cm for the lattice pitch}$$

The fuel diameter $d = 2a = 2(a/p)\,p = \dfrac{2 \cdot 1.01}{3.07} = 0.658$ cm

Part c.

$$N = 11.4 \cdot 10^4 \,/\, p^2 = 11.4 \cdot 10^4 \,/ 1.01^2 = 112 \cdot 10^3 \text{ fuel elements}$$

Part d: Since for a uniform cylindrical core $F_r = 2.32$ and from Eq.(8.42)

$$W = \frac{F_r P}{c_p (T_0\big|_{\max} - T_i)}$$

Take $c_p = 6.4 \cdot 10^3$ J/kg ^{o}C (see part e of the PWR example in section 8.3)

$$W = \frac{2.32 \cdot 3 \cdot 10^9 \text{ J/s}}{6.4 \cdot 10^3 \text{ (J/kg}\,^{o}\text{C)}(330\,^{o}\text{C} - 290\,^{o}\text{C})} = 27.2 \cdot 10^3 \text{kg/s}$$

Taking the density as 0.676 10^{-3} kg/cm^3 (see part e of the PWR example)

$$\bar{v} = \frac{W}{\rho N (p^2 - \pi a^2)} = \frac{27.2 \cdot 10^3}{0.676 \cdot 10^{-3} \cdot 112 \cdot 10^3 (1.01^2 - \pi \cdot 0.329^2)}$$

$\bar{v}$ = 528 cm/s = 5.28 m/s

[8.7]* A uniform cylindrical reactor core has a height-to-diameter ratio of one. The reactor is reflected both radially and axially with radial and axial reflector savings each being equal to *M*, the migration length of the core composition:

a. Show that the power peaking factor (with $F_l = 1.0$) is given by

$$F_q = \frac{1.889\left(1+R/M\right)^{-2}\left(R/M\right)^2}{J_1\left[2.405\left(1+R/M\right)^{-1}R/M\right]\sin\left[\left(\pi/2\right)\left(1+R/M\right)^{-1}R/M\right]}$$

b. Plot F_q versus *R/M* between *R/M*= 5 and 50 as well as the results for the same reactor without the reflector.

Part a: For a reflector savings of *M*, in Eq. (7.21) we add *M* to the radius of the reactor and 2*M* to the height. Thus Eq. (7.21) is replaced by

$$\phi(r,z) = CJ_0(2.405r/(R+M))\cos(\pi z/(2R+2M))$$

where *H=2R*. Correspondingly, Eqs. (8.16) and (8.17) become

$$C_r \frac{2}{R^2}\int_0^R J_0(2.405r/(R+M))rdr = 1$$

and

$$C_z \frac{1}{2R}\int_{-R}^{R} \cos(\pi z/(2R+2M))dz = 1$$

where integrals are taken over the core dimensions, and not the extrapolated dimensions. next solve for C_r and C_z. Making use of the Bessel integral B-7 in Appendix B and that fact that $J_1(0) = 0$

$$1 = C_r \frac{2}{R^2}\int_0^R J_0(2.405r/(R+M))rdr = C_r \frac{2}{R^2}\left(\frac{R+M}{2.405}\right)RJ_1\left(2.405\frac{R}{R+M}\right)$$

or

$$C_r = \frac{2.405}{2}\frac{(R/M)(1+R/M)^{-1}}{J_1\left(2.405(R/M)(1+R/M)^{-1})\right)}.$$

Using the trigonometric integral in Appendix A, along with sin(0) = 0:

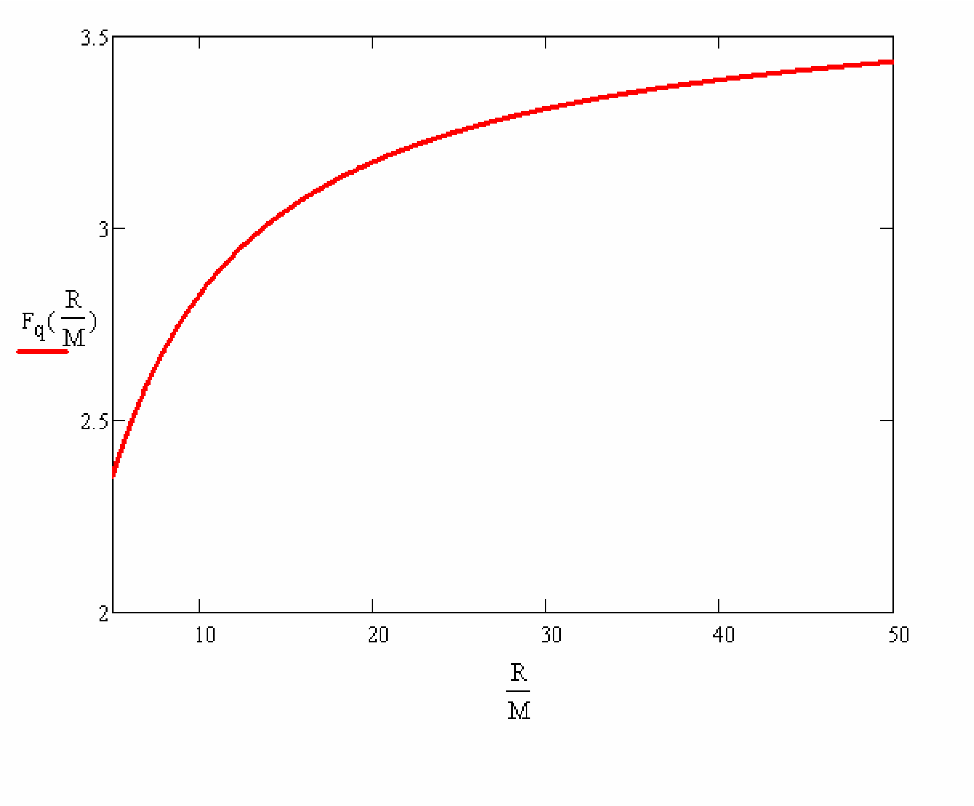
$$1 = C_z \frac{1}{2R}2\int_0^R \cos(\pi z/(2R+2M))dz = C_z \frac{1}{R}\frac{(2R+2M)}{\pi}\sin\left(\frac{\pi R}{2(R+M)}\right)$$

or

$$C_z = \frac{(\pi/2)(R/M)(1+R/M)^{-1}}{\sin\left((\pi/2)(R/M)(1+R/M)^{-1}\right)}.$$

Then since the maximum values of both J_0 and cos are equal to one, $F_q = C_r C_z$ and the expression for F_q given above results.

Part B:



Without the reflector the result is 3.63 as given in equation (7.30), regardless of the value of R/M.

[8.8] Suppose the reflected reactor in problem [8.7] is a sodium-cooled fast reactor with M = 18.0 cm and a power of 2000 MW(t). If the thermal design limits the maximum allowable power density to 450W/cm^3,

a. What is (1) the minimum value that the core radius can have, (2) the corresponding value of the core volume, and (3) the required value of k_∞ to maintain criticality?
b. Suppose that to increase the thermal safety margins, it is decided to reduce the maximum permissible power density by 10%. What are the percentage changes for the reflected reactor in radius, volume, and k_∞? (Assume that M remains the same.)

Part a. From Eq. (8.3) we may write

$$V = F_q P / P'''_{\max}$$

and with a height to diameter ratio of one $V = 2\pi R^3$. Since F_q is given in terms of the ratio $x \equiv R/M$, we rewrite the volume equation as

$$x^3 = \frac{P}{2\pi M^3 P'''_{\max}} F_q(x) = \frac{2000 \cdot 10^6}{2\pi 18^3 \cdot 450} F_q(x) = 121.3 F_q(x)$$

Plugging in the peaking factor and simplifying

$$x = \frac{229.1(1+x)^{-2}}{J_1\left[2.405(1+x)^{-1}x\right]\sin\left[(\pi/2)(1+x)^{-1}x\right]}$$

We must solve this transcendental equation for *x*. The value of F_q must be in the range of $1 < F_q < 3.63$, since 3.63 is the value derived in Chapter 7 for a bare reactor. Since $x^3 = 121.3 F_q(x)$ or $x = \left(121.3 F_q(x)\right)^{1/3}$, we look for a value in the range

$$4.95 < x < 7.60$$

The transcendental equation is solved using MathCad by plotting the left and right hand sides, as shown in the figure. The solution is x = 6.78

Therefore, R = xM = (6.78)(18.0 cm) = 122 cm.

Part b. The earlier equation becomes

$$x^3 = \frac{P}{2\pi M^3 P'''_{\max} \cdot (0.90)} F_q(x) = \frac{121.3}{0.90} F_q(x)$$

which reduces to

$$0.90x = \frac{229.1(1+x)^{-2}}{J_1\left[2.405(1+x)^{-1}x\right]\sin\left[(\pi/2)(1+x)^{-1}x\right]}$$

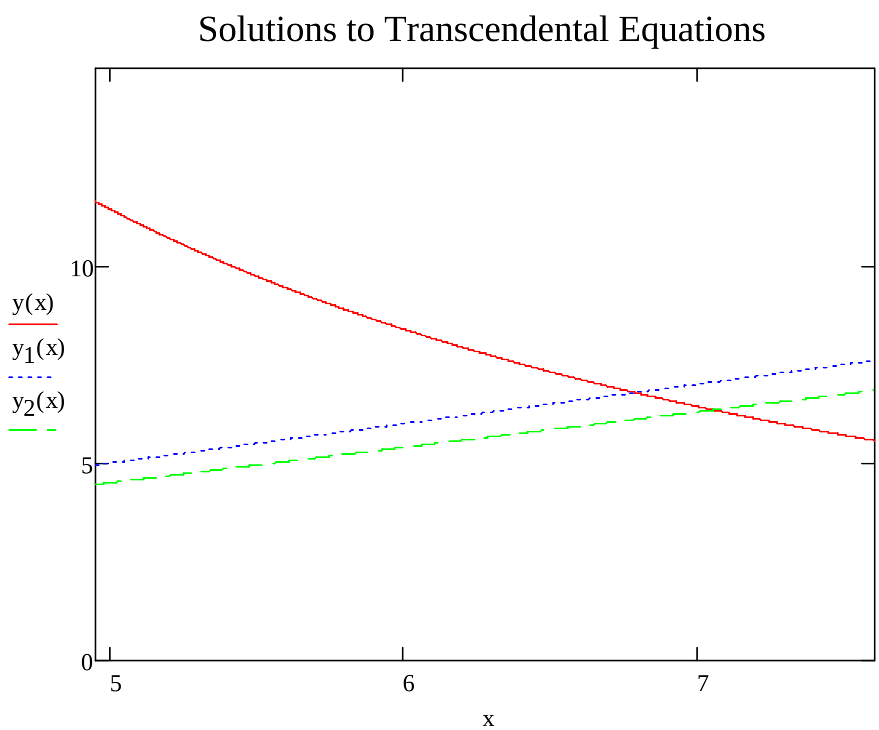
We can obtain the solution off of the same graph by replacing the left hand side with 0.90*x*. The solution is x = 7.05.

Therefore, R = xM = (7.05)(18.0 cm) = 127 cm.

$$y(x) := \frac{229.1\cdot(1+x)^{-2}}{J1\left[2.405(1+x)^{-1}x\right]\cdot\sin\left[\frac{\pi}{2}\cdot(1+x)^{-1}\cdot x\right]}$$

$$y_1(x) := x$$

$$y_2(x) := 0.90\cdot x$$



[8.9] Repeat problem [8] in the absence of the reflector.

Part a.   From Eq. (8.3) we may write

$$V = F_q P / P'''_{max}$$

and with a height to diameter ratio of one $V = 2\pi R^3$. For a bare cylindrical F_q = 3.63 (see Eq. (7.30)) From the above equation we may write

$$R^3 = \frac{P}{2\pi P'''_{max}} F_q = \frac{2000 \cdot 10^6}{2\pi 450} 3.63 = 2.57$$

The radius is

$$R = 1.37 \text{ m}$$

The core volume is

$$V = 2\pi R^3 = 16.1 \text{ m}^3$$

$$1 = \frac{k_\infty}{1 + M^2 B^2}$$

$$k_\infty = 1 + M^2 B^2 = 1 + M^2 33.0 / D^2 = 1 + 0.18^2 \cdot 33.0 / (2 \cdot 1.37)^2$$

$$k_\infty = 1.142$$

Part b. The earlier equation becomes

$$R^3 = \frac{P}{2\pi P'''_{max} \cdot (0.90)} F_q = \frac{1}{0.90} 2.57 = 2.86$$

The radius is

$$R = 1.42 \text{ m}$$

% change = 100(1.42-1.37)/1.37= 3.65 %

The core volume is

$$V = 2\pi R^3 = 18.0 \text{ m}^3$$

% change = 100(18.0-16.1)/16.1= 11.8 %

$$k_\infty = 1 + M^2 B^2 = 1 + M^2 33.0 / D^2 = 1 + 0.18^2 \cdot 33.0 / (2 \cdot 1.42)^2$$

$$k_\infty = 1.133$$

% change = 100(1.133-1.142)/1.142= -0.79 %

[8.10] Consider the PWR design at the end of Section 8.3. Suppose that by varying the enrichment in the fuel assemblies and distributing the control poisons in a non-uniform pattern the designers are able to reduce the radial and axial peaking factors to $F_r = 1.30$ and $F_z = 1.46$. Redesign the reactor by solving parts c through g of the pressurized water reactor example using these peaking factors.

With the new peaking factors stipulated, the solution follows the example in the text fairly closely. For the core volume we evaluate Eq. (8.26) at the point of the maximum linear heat rate to obtain $V = A_{cell} P F_r F_z / q'|_{\max}$ where $A_{cell} = p^2$. With the new peaking factors:

$$V = p^2 P F_r F_z / q'|_{\max} = 1.536^2 \cdot 3000 \cdot 10^6 \cdot 1.30 \cdot 1.46 / 400$$
$$= 3.36 \cdot 10^7 \text{ cm}^3 = 33.6 \text{ m}^3$$

For a height to diameter ratio of one, $V = \pi (H/2)^2 H$. Hence

(c) $H = (4V/\pi)^{1/3} = (4 \cdot 3.36 \cdot 10^7 / \pi)^{1/3} = 350$ cm = 3.50 m

The core averaged power density is just

(d) $\overline{P}''' = P/V = 3000 \cdot 10^6 / 3.36 \cdot 10^7 = 89.3$ W/cm^3 = 89.3 MW/m^3

The number of fuel elements is determined by dividing the core cross sectional area by the area of a lattice cell;

(e) $$N = \frac{\pi R^2}{A_{cell}} = \frac{\pi (H/2)^2}{p^2} = \frac{\pi (350/2)^2}{1.536^2} = 40,780$$

Since the maximum outlet temperature is limited to 330 ^{o}C we determine the mass flow rate from Eq. (8.42); taking $c_p = 6.4 \cdot 10^3$ J/kg^{o}C as the specific heat of water at the operating coolant temperature yields:

(f) $$W = \frac{1}{c_p} \frac{P F_r}{(T_0|_{\max} - T_i)} = \frac{1}{6.4 \cdot 10^3} \frac{3000 \cdot 10^6 \cdot 1.30}{(330 - 290)} = 15.2 \cdot 10^3 \text{ kg/s} = 15.2 \cdot 10^6 \text{ gm/s}$$

We determine the mean coolant velocity from $W = \rho A_{flow} \bar{\text{v}}$ where $A_{flow} = N \cdot (p^2 - \pi a^2)$ and we take the density of the pressurized water at 300 ^{o}C as 0.676 gm/cm^3. Thus

(g) $$\bar{\text{v}} = \frac{W}{\rho N (p^2 - \pi a^2)} = \frac{15.2 \cdot 10^6}{0.676 \cdot 40,780 \cdot (1.536^2 - \pi 0.509^2)} = 357 \text{ cm/s} = 3.57 \text{ m/s}$$

Note that using the same lattice, (i.e. fuel diameter and pitch)the core dimensions, the number of fuel pins, and the mass flow rate all decrease dramatically, but the average flow speed changes very little. If the size of the core had been kept the same, the power of the reactor could have been increased substantially.

[8.11] An unachievable ideal would be a reactor with a perfectly flat flux distributions: $F_r = 1.00$ and $F_z = 1.00$ Repeat problem [8.10] for such an idealized reactor.

With the idealized peaking factors stipulated, the solution follows the example in the text fairly closely. For the core volume we evaluate Eq. (8.26) at the point of the maximum linear heat rate to obtain $V = A_{cell} P F_r F_z / q'\big|_{\max}$ where $A_{cell} = p^2$. With the new peaking factors:

$$V = p^2 P F_r F_z / q'\big|_{\max} = 1.536^2 \cdot 3000 \cdot 10^6 \cdot 1.00 \cdot 1.00 / 400$$

$$= 1.77 \cdot 10^7 \text{ cm}^3 = 17.7 \text{ m}^3$$

For a height to diameter ratio of one, $V = \pi (H/2)^2 H$. Hence

(c) $H = (4V/\pi)^{1/3} = (4 \cdot 1.77 \cdot 10^7 / \pi)^{1/3} = 282$ cm = 2.82 m

The core averaged power density is just

(d) $\overline{P'''} = P/V = 3000 \cdot 10^6 / 1.77 \cdot 10^7 = 169$ W/cm^3 = 169 MW/m^3

The number of fuel elements is determined by dividing the core cross sectional area by the area of a lattice cell;

(e) $N = \dfrac{\pi R^2}{A_{cell}} = \dfrac{\pi (H/2)^2}{p^2} = \dfrac{\pi (282/2)^2}{1.536^2} = 26,474$

Since the maximum outlet temperature is limited to 330 ^{o}C we determine the mass flow rate from Eq. (8.42); taking $c_p = 6.4 \cdot 10^3$ J/kg^{o}C as the specific heat of water at the operating coolant temperature yields:

(f) $W = \dfrac{1}{c_p} \dfrac{P F_r}{(T_0\big|_{\max} - T_i)} = \dfrac{1}{6.4 \cdot 10^3} \dfrac{3000 \cdot 10^6 \cdot 1.00}{(330 - 290)} = 11.7 \cdot 10^3$ kg/s = $11.7 \cdot 10^6$ gm/s

We determine the mean coolant velocity from $W = \rho A_{flow} \bar{v}$ where $A_{flow} = N \cdot (p^2 - \pi a^2)$ and we take the density of the pressurized water at 300 ^{o}C as 0.676 gm/cm^3. Thus

(g) $\bar{v} = \dfrac{W}{\rho N (p^2 - \pi a^2)} = \dfrac{11.7 \cdot 10^6}{0.676 \cdot 26,474 \cdot (1.536^2 - \pi 0.509^2)} = 423$ cm/s = 4.23 m/s

[8.12] Suppose that the designers of the pressurized water reactor treated in Section 8.3 conclude that the thermal-hydraulic design must have larger safety margins by reducing flow velocity by 10% and the maximum coolant temperature by 5 ^{o}C. The reactor physicists are asked to accommodate these changes by reducing the radial peaking factor. What percentage reduction would be required?

Rewrite Eq. (8.42) as

$$F_r = Wc_p(T_o|_{\max} - T_i)/P = \bar{\mathrm{v}}\rho A_{flow}c_p(T_o|_{\max} - T_i)/P$$

designating the modified values with primes:

$$\frac{F_r'}{F_r} = \frac{\bar{\mathrm{v}}'}{\bar{\mathrm{v}}}\frac{(T_o'|_{\max} - T_i)}{(T_o|_{\max} - T_i)} = 0.9 \cdot \frac{325-290}{330-290} = 0.787$$

From the inlet and outlet temperatures are from section 8.3 , and $\bar{\mathrm{v}}'/\bar{\mathrm{v}}=0.90$. Thus the radial peaking factor must be reduced by 21.3%

[8.13] A reactor initially operating at a power P_o is put on a period T such that the power can be approximated as $P(t) = P_o \exp(t/T)$. Assuming that the coolant temperature is maintained at its initial value $T_c(0)$, solve Eq. (8.49) and show that the fuel temperature will be:

$$T_f(t) = T_c(0) + \frac{P_o R_f}{1+\tau/T}\left[\exp(t/T) + (\tau/T)\exp(-t/\tau)\right].$$

Rewrite Eq. (8.50) as

$$\frac{d}{dt}\theta(t) = \frac{1}{M_f c_f} P_o \exp(t/T) - \frac{1}{\tau}\theta(t)$$

where $\theta(t) = T_f(t) - T_c(0)$ since T_c remains constant. Apply an integrating factor $\exp(t/\tau)$:

$$\left[\frac{d}{dt}\theta(t) + \frac{1}{\tau}\theta(t)\right]\exp(t/\tau) = \frac{1}{M_f c_f} P_o \exp(t/T)\exp(t/\tau)$$

which is the same as

$$\frac{d}{dt}\left[\theta(t)\exp(t/\tau)\right] = \frac{1}{M_f c_f} P_o \exp\left[(1/T + 1/\tau)t\right]$$

Integrate between 0 and t:

$$\theta(t)\exp(t/\tau) - \theta(0) = \frac{1}{M_f c_f} P_o \int_0^t \exp\left[(1/T + 1/\tau)t'\right]dt'$$

$$= \frac{1}{M_f c_f} P_o \left\{\frac{\exp\left[(1/T + 1/\tau)t'\right] - 1}{(1/T + 1/\tau)}\right\}$$

Solving for $\theta(t)$ we have

$$\theta(t) = \theta(0)\exp(-t/\tau) + \frac{1}{M_f c_f} P_o \left\{\frac{\exp(t/T) - \exp(-t/\tau)}{(1/T + 1/\tau)}\right\}$$

Noting that the initial condition, Eq. (8.32) gives

$$\theta(0) = T_f(0) - T_c(0) = R_f P_o = \frac{\tau}{M_f c_f} P_o, \text{ since } \tau = M_f c_f R_f,$$

and that $\theta(t) = T_f(t) - T_c(0)$ we obtain

$$T_f(t) = T_c(0) + \frac{\tau}{M_f c_f} P_o \exp(-t/\tau) + \frac{1}{M_f c_f} P_o \left\{\frac{\exp(t/T) - \exp(-t/\tau)}{(1/T + 1/\tau)}\right\}$$

Then eliminating $M_f c_f$, we may reduce this equation to the form given in the problem

[8.14] In the prismatic block form of the graphite moderated gas-cooled reactor the heat passes through the moderator before reaching the coolant. Figures 4.1d and 4.2c show such a configuration. Develop a set of three coupled differential equations in forms similar to Eqs. (8.50) and (8.58) that describe the transient heat transfer in such a reactor. Assume that the heat transfer between fuel and moderator and between moderator and coolant is described by $P(t)=[\overline{T}_f(t)-\overline{T}_m(t)]/R_1$ and $P(t)=[\overline{T}_m(t)-\overline{T}_c(t)]/R_2$ respectively. Assume that W is the mass flow rate, and that the masses, specific heats and densities of fuel, moderator and coolant are given by M_i, c_i and ρ_i with $i = f, m, c$.

For the fuel region the equation is similar to Eq. (8.50)

$$M_f c_f \frac{d}{dt}\overline{T}_f(t)=P(t)-\frac{1}{R_1}\left[\overline{T}_f(t)-\overline{T}_m(t)\right]$$

For the moderator region heat transfer both into the moderator and out of it into the coolant must be represented.

$$M_m c_m \frac{d}{dt}\overline{T}_m(t)=\frac{1}{R_1}\left[\overline{T}_f(t)-\overline{T}_m(t)\right]-\frac{1}{R_2}\left[\overline{T}_m(t)-\overline{T}_c(t)\right]$$

The coolant equation looks similar to Eq. (8.58):

$$M_c c_c \frac{d}{dt}\overline{T}_c(t)=\frac{1}{R_2}\left[\overline{T}_m(t)-\overline{T}_c(t)\right]-2Wc_c\left[\overline{T}_c(t)-T_i\right]$$

In prism block reactors, the mass of the gas coolant is so small that it often can be ignored be setting $M_c c_c$ to zero in the last equation. Solving for the coolant temperature, we then have

$$\overline{T}_c(t)=\frac{1}{1+2R_2Wc_c}\left[2R_2Wc_c\overline{T}_m(t)+T_i\right]$$

Finally we note that the moderator mass in a graphite block reactor is very large. As a result the core heat capacity is large providing a slow response of moderator and coolant transients to changes in power level. The slow moderator adiabatic heat up rate, which we define as follows, indicates this:

$$\frac{d}{dt}\overline{T}_m(t)=\frac{P}{M_m c_m}$$

[8.15]* A sodium-cooled fast reactor has the following characteristics:

P = 2400 MW(t) $\quad$ W = 14,000 kg/s

τ = 4.0 s $\quad$ c_p = 1,250 J/(kg ^{0}C)

$M_f c_f$ = 13.5 x 10^6 J/^{0}C $\quad$ $M_c c_p$ = 1.90 x 10^6 J/^{0}C

T_i = 360 ^{0}C

Suppose the reactor undergoes a sudden trip, which may be approximated by setting the power equal to zero. Assuming the inlet temperature remains at its initial value, find the core outlet temperature and plot your results.

Using equations [8.49] and [8.57], initial conditions y, and the stiff differential equation solver Radau (Mathcad), a vector Y was obtained. The first data column of Y after the time step column is the fuel temperature and the second column is average coolant temperature.

To find the core outlet temperature a third column needs to be added algebraically using the second column (average coolant temperature) and the inlet temperature (T_i)

$$R_f := \frac{\tau}{M_{fCf}} = 2.963 \times 10^{-7}$$

$$T_{c0} := \frac{P}{2 \cdot W \cdot c_p} + T_i = 428.571$$

$$T_{f0} := \left(R_f + \frac{1}{2 \cdot W \cdot c_p} \right) \cdot P + T_i = 1.14 \times 10^3$$

$$y := \begin{pmatrix} T_{f0} \\ T_{c0} \end{pmatrix}$$

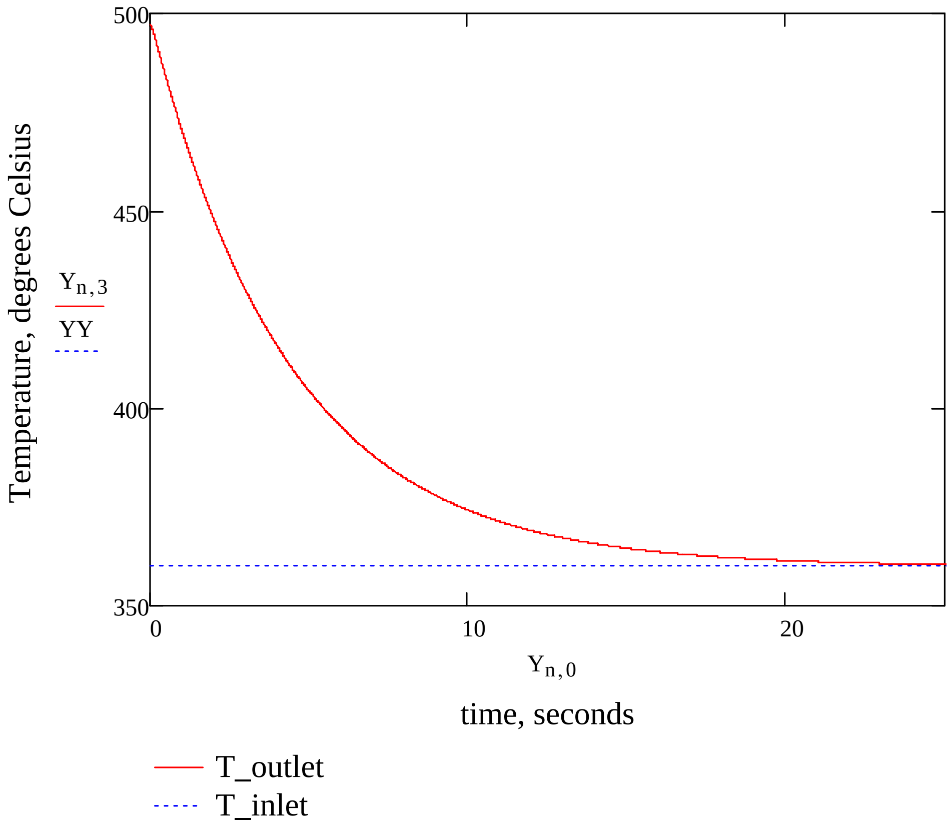
$$D_1(t, y) := \begin{bmatrix} \frac{1}{M_{fCf}} \cdot 0 - \frac{1}{\tau}\left(y_0 - T_i\right) \\ \left[\frac{1}{R_f} \cdot \left(y_0 - y_1\right) - 2 \cdot W \cdot c_p \cdot \left(y_1 - T_i\right) \right] \cdot \frac{1}{M_{ccp}} \end{bmatrix}$$

$$Y := \text{Radau}\left(y, 0, 100, 10000, D_1\right)$$

$$n := 0, 1 .. 10000$$

$$YY := 360$$

$$Y_{n,3} := \left(2 \cdot Y_{n,2}\right) - T_i$$


500
450
400
350
0
10
20
Temperature, degrees Celsius
Yn,3
YY
Yn,0
time, seconds
T_outlet
T_inlet

[8.16]* Assume that the reactor in the preceding problem suffers a control failure and undergoes a power transient $P(t) = P(0)[1+0.25t]$ where t is in seconds. The shut down system trips the reactor if the outlet temperature rises by more than 40 $^0\mathrm{C}$:

a. Determine the coolant outlet temperature transient and plot your result until the temperature rises by 40 $^0\mathrm{C}$.
b. At what time does the reactor shutdown system terminate the transient?

a)
Using equations [8.49] and [8.57], initial conditions y, and the stiff differential equation solver Radau (Mathcad), a vector Y was obtained. The first data column of Y after the time step column is the fuel temperature and the second column is average coolant temperature.

To find the core outlet temperature a third column needs to be added algebraically using the second column (average coolant temperature) and the inlet temperature (T_i)

$$R_f := \frac{\tau}{M_{fCf}} = 2.963 \times 10^{-7}$$

$$T_{c0} := \frac{P}{2 \cdot W \cdot c_p} + T_i = 428.571$$

$$P(t) := P \cdot (1 + 0.25 \cdot t)$$

$$T_{f0} := \left(R_f + \frac{1}{2 \cdot W \cdot c_p}\right) \cdot P + T_i = 1.14 \times 10^3$$

$$y := \begin{pmatrix} T_{f0} \\ T_{c0} \end{pmatrix}$$

$$D_1(t, y) := \begin{bmatrix} \dfrac{P(t)}{M_{fCf}} - \dfrac{1}{\tau}\left(y_0 - T_i\right) \\ \left[\dfrac{1}{R_f} \cdot \left(y_0 - y_1\right) - 2 \cdot W \cdot c_p \cdot \left(y_1 - T_i\right)\right] \cdot \dfrac{1}{M_{ccp}} \end{bmatrix}$$

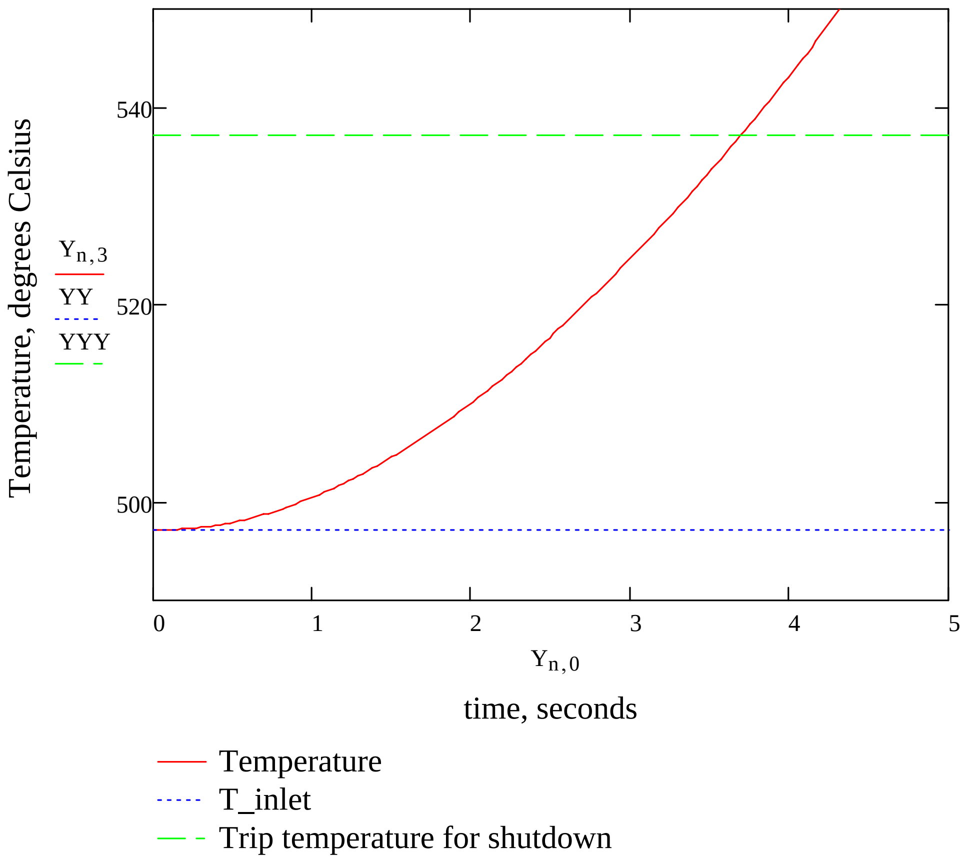
$$Y := \mathrm{Radau}\left(y, 0, 300, 10000, D_1\right)$$

$$Y_{n,3} := \left(2 \cdot Y_{n,2}\right) - T_i$$

$$YY := Y_{0,3}$$

$$n := 0, 1 .. 10000$$

$$YYY := Y_{0,3} + 40$$



b)

The reactor terminates the transient temperature when the temperature has risen by 40 degrees. This occurs at approximately 3.7 seconds as is seen on the above graph.

Chapter 9 Problems:

[9.1] Table 4.2 displays the four factors for a pressurized water reactor at 300^0 K . Assume the reactor is fueled with UO_2 fuel pins with diameters of 1.1cm and a density of 11.0 gm/cm^3:

a. Determine the fuel temperature coefficient and plot its value between 300 and $1{,}000^0$ K.
b. Estimate the coolant temperature coefficient, assuming that the water has a coefficient of thermal expansion given by $\beta_m = 0.004\ {}^o\mathrm{K}^{-1}$.
c.

Part a: The fuel temperature coefficient is given by Eqs. (9.16)

$$\alpha_f = -\frac{\tilde{\gamma}}{2\sqrt{\overline{T}_f}} \ln\left[1/\, p(T_o)\right]$$

where the temperature is absolute. For the specified fuel pin, and data from Table 4.2, Eq. (9.14) reduces to

$$\tilde{\gamma} = C_1 + C_2(4/\rho D) = 61\cdot 10^{-4} + 47\cdot \frac{4}{11.0\cdot 1.1} = 76.5\cdot 10^{-4}$$

Then taking the resonance escape probability as 0.63 from Table 4.2, we obtain

$$\alpha_f = -\frac{76.5\cdot 10^{-4}\tilde{\gamma}}{2\sqrt{\overline{T}_f}} \ln\left[1/0.63\right] = \frac{0.00177}{\sqrt{\overline{T}_f}} \quad /^oC$$

The plot follows

Part b From Eq. (9.24) and the data from Table 4.2:

$$\frac{1}{k_\infty}\frac{\partial k_\infty}{\partial T_m} = -\beta_m\left[\ln(1/\,p) - (1-\,f\,)\right] = -.004[\ln(1/.63) - (1-.94)] = 1.61\cdot 10^{-3}\ /^o\ \mathrm{C}$$

[9.2] Consider a hypothetical reactor in which all of the materials have the same volumetric coefficient of thermal expansion. Thus all of the nuclide densities decrease according to the same ratio: $N'/N = const. < 1$

a. Show that the expansion with increased temperature has no effect on k_∞.

b. Using the facts that the core mass, *NV*, remains constant, and that $M \propto N^{-1}$, show from Eq. (9.4) that the reactivity change is from expansion is negative, with a value of $\dfrac{dk}{k} = -\dfrac{4}{3} P_L \dfrac{dV}{V}$.

Part a; Since k_∞ is defined as a ratio of cross section, as for example in Eq. (4.1), if all of the densities decrease by the same ratio, the effect will cancel between numerator and denominator

Part b: Since k_∞ does not change Eq. (9.4) reduces to

$$\frac{dk}{k} = -\frac{M^2B^2}{1+M^2B^2}\left(\frac{dM^2}{M^2} + \frac{dB^2}{B^2}\right) = -P_L\left(\frac{dM^2}{M^2} + \frac{dB^2}{B^2}\right)$$

Since $M \propto N^{-1}$, we may write $M^2 = C/N^2 = CV^2/(NV)^2$ where *C* is a constant. Since the mass, *NV*, remains constant we may define a new constant $C_M = C/(NV)^2$ and write

$$M^2 = C_M V^2$$

The size of a reactor of a particular shape, say a sphere or cylinder, can be expressed in terms of a characteristic dimension, that we shall call *D*. The buckling , B, is proportional to D^{-1}, and the reactor volume, V, is proportional to $D^{3.}$ Thus we may write

$$B^2 = C'D^{-2} = C_B V^{-2/3}$$

where C' and C_B are constants. Taking differentials, we have

$$dM^2 = 2C_M V dV \qquad \text{and} \qquad \frac{dM^2}{M^2} = 2\frac{dV}{V}$$

$$dB^2 = -\frac{2}{3}C_B V^{-5/3} \qquad \text{and} \qquad \frac{dB^2}{B^2} = -\frac{2}{3}\frac{dV}{V}$$

Hence

$$\frac{dk}{k} = -P_L\left(2\frac{dV}{V} - \frac{2}{3}\frac{dV}{V}\right) = -\frac{2}{3}P_L\frac{dV}{V}$$

[9.3] Assume a pressurized water reactor has the parameters specified in the example at the end of Chapter 8.3. Assume the core has a thermal resistance or R_f = 0.50 ^{o}C/MW(t),

. If the reactor fuel and moderator temperature coefficients are $\alpha_f = -3.2 \cdot 10^{-5}\ {}^o\mathrm{C}^{-1}$ and $\alpha_m = -1.4 \cdot 10^{-5}\ {}^o\mathrm{C}^{-1}$

a. Determine the isothermal temperature coefficient.
b. Determine the power coefficient.

Part a: From Eq. (9.29) the isothermal temperature coefficient is just

$$\alpha_T = \alpha_f + \alpha_m = -3.2 \cdot 10^{-5} - 1.4 \cdot 10^{-5} = -4.6 \cdot 10^{-5}\ /^o\,\mathrm{C}$$

Part b: Following Eq. (9.33) the power coefficient is

$$\alpha_P = R_f \alpha_f + (2Wc_p)^{-1}(\alpha_f + \alpha_c)$$

Using data from Chapter 8 example

$$Wc_p = 27.2 \cdot 10^3\,\mathrm{kg/s} 6.4 \cdot 10^3\,\mathrm{J/kg} = 174 \cdot 10^6\,\mathrm{J/s} = 174\mathrm{MW}$$

Thus

$$\alpha_P = 0.50(-3.2 \cdot 10^{-5}) + (2 \cdot 174)^{-1}(-3.2 \cdot 10^{-5} - 1.4 \cdot 10^{-5})$$

$$= -(1.6 + 0.013) \cdot 10^{-5} = -1.61 \cdot 10^{-5}\ /\,\mathrm{MW}$$

Thus the fuel provides much more of the power coefficient than does the coolant.

[9.4] At full power 1000 MW(t) sodium-cooled fast reactor has coolant inlet and outlet temperatures of 350 and 500 ^oC, and an average fuel temperature of 1,150 ^oC. The fuel and coolant temperature coefficients are $\alpha_f = -1.8\times 10^{-5}\ /^0C$ and $\alpha_c = +0.45\times 10^{-5}\ /^0C$.

a. Estimate the core thermal resistance and the mass flow rate, taking for sodium c_p = 1,250 J/kg/ ^0C.
b. Estimate the temperature and power defects, assuming a "cold" temperature of 180 ^0C.

Part a: Take the average coolant temperature from Eq. (8.39) as

$$\bar{T}_c = \tfrac{1}{2}(T_i + \bar{T}_o) = \tfrac{1}{2}(350 + 500) = 425\ ^0C$$

From Eq. (8.32) the thermal resistance is

$$R_f = \frac{\bar{T}_f - \bar{T}_c}{P} = \frac{1150 - 425}{1000_c} = 0.725\ ^oC/MW$$

Form Eq. (8.37) the mass flow rate is

$$W = \frac{P}{c_p(\bar{T}_o - T_i)} = \frac{1000\cdot 10^6}{1250(500-350)} = 5,333\ \text{kg/s}$$

Part b: With constant temperature coefficients, for the temperature defect Eq. (9.34) simplifies to

$$D_T = \alpha_T(T_i - T_r) = (\alpha_f + \alpha_c)(T_i - T_r)$$

$$= (-1.8\cdot 10^{-5} + 0.45\cdot 10^{-5})(350 - 180) = -9.45\cdot 10^{-6}$$

For the power defect Eqs. (9.33) and (9.35) simplify to

$$D_P = \left[R_f\alpha_f + (2Wc_p)^{-1}(\alpha_f + \alpha_c)\right]P$$

$$= \left[0.725(-1.8\cdot 10^{-5}) + (2\cdot 5,333\cdot 1250\cdot 10^{-6})^{-1}(-1.8\cdot 10^{-5} + 0.45\cdot 10^{-5})\right]$$

$$= \left(-1.31\cdot 10^{-5} - 0.101\cdot 10^{-5}\right)1000 = -1.4\cdot 10^{-2}$$

[9.5] A 3000 MW(t) pressurized water reactor has the following specifications:; core thermal resistance 0.45 ^{o}C/MW(t), coolant flow, 68x10^6 kg/hr; coolant specific heat 6.4x10^3 J/kg ^{o}C . The fuel temperature coefficient is

$$\frac{1}{k}\frac{\partial k}{\partial \overline{T}_f} = -\frac{7.2\cdot 10^{-4}}{\sqrt{273+\overline{T}_f}} \quad (^o\mathrm{C})^{-1}$$

and the coolant temperature coefficient by

$$\frac{1}{k}\frac{\partial k}{\partial \overline{T}_c} = \left(30+1.5\overline{T}_c - 0.010\overline{T}_c^{\,2}\right)\cdot 10^{-6} \quad (^o\mathrm{C})^{-1}$$

a. Over what temperature range is the core overmoderated?
b. What is the value of the temperature defect? Assume room temperature of 35 ^{o}C and an operating coolant inlet temperature of 290 ^{o}C.
c. What is the value of the power defect?

Part a: As we learned in chapter 4 a liquid cooled and moderated reactor's temperature coefficient is positive if it is overmoderated. This is based on the isothermal temperature coefficient. Thus for this problem

$$\alpha_T = \alpha_f + \alpha_m = -\frac{7.2\cdot 10^{-4}}{\sqrt{273+T}} + \left(30+1.5T-0.010T^2\right)\cdot 10^{-6}$$

Determining the temperature at which this coefficient vanishes analytically, involves solving a quadric equation. Instead we simply plot the coefficient.

Part b. We apply Eq. (9.34) to α_T

$$D_T = \left\{ -\int_{35}^{290} \frac{720}{\sqrt{273+T}}dT + \int_{35}^{290} \left(30+1.5T-0.010T^2\right)dT \right\}\cdot 10^{-6}$$

taking $x=\sqrt{273+T}$ and hence $dT = 2xdx$ in the first integral then gives

$$D_T = \left\{ -720\cdot 2\int_{\sqrt{308}}^{\sqrt{563}} dx + \int_{35}^{290} \left(30+1.5T-0.010T^2\right)dT \right\}\cdot 10^{-6}$$

$$D_T = \left\{ -720\cdot 2\, x\Big|_{17.5}^{23.7} + \left(30T+\frac{1.5}{2}T^2 - \frac{0.010}{3}T^3\right)\Big|_{35}^{290} \right\}\cdot 10^{-6}$$

$$D_T = \left\{-89.28+7650+62156-81215\right\}\cdot 10^{-6} = -11.5\cdot 10^{-3}$$

Part c. The power defect is determined by substituting the temperature dependent coefficients into Eq. (9.35)

$$D_P = \int_{T_i}^{\overline{T}_f(P)} \alpha_f d\overline{T}_f + \int_{T_i}^{\overline{T}_c(P)} \alpha_c d\overline{T}_c$$

$$D_P = \left\{ -\int_{T_i}^{\overline{T}_f(P)} \frac{720}{\sqrt{273+\overline{T}_f}}d\overline{T}_f + \int_{T_i}^{\overline{T}_c(P)} \left(30+1.5\overline{T}_c - 0.010\overline{T}_c^{\,2}\right)d\overline{T}_c \right\}\cdot 10^{-6}$$

where T_i = T_i = 290 ^{o}C, R_f = 0.45 ^{o}C/MW(t), W= 68x10^6 kg/hr x3600^{-1} hr/s = 18.9x10^{3}kg/s c_p = 6.4x10^3 J/kg ^{o}C . Thus

$$\overline{T}_c = \frac{1}{2Wc_p} P + T_i$$

$$= \frac{1}{2 \cdot 18.8 \cdot 10^3 \cdot 6.4 \cdot 10^3 \cdot 10^{-6}} 3000 + 290 = 302.5\ ^o\text{C}$$

$$\overline{T}_f = R_f P + \overline{T}_c = 0.45 \cdot 3000 + 302.5 = 1,652.5\ ^o\text{C}$$

Thus

$$D_P = \left\{ -\int_{290}^{1652} \frac{720}{\sqrt{273 + \overline{T}_f}} d\overline{T}_f + \int_{290}^{302.5} \left(30 + 1.5\overline{T}_c - 0.010\overline{T}_c^{\,2} \right) d\overline{T}_c \right\} \cdot 10^{-6}$$

Evaluating the integrals:

$$D_P = \left\{ -720 \cdot 2\, x \Big|_{23.7}^{43.9} + \left(30T + \frac{1.5}{2} T^2 - \frac{0.010}{3} T^3 \right) \Big|_{290}^{302.5} \right\} \cdot 10^{-6}$$

$$D_P = \{-29,088 + 377 + 5,555 - 10,972\} \cdot 10^{-6} = -34.1 \cdot 10^{-3}$$

[9.6] A sodium-cooled fast reactor lattice is designed to have the following properties: migration length 18.0 cm. and a maximum power density, 450W/cm^3. Fractional sodium voiding results in the following reactivity effects:

$$\Delta k_\infty / k_\infty = +0.01, \qquad \Delta M / M = +0.01$$

Three bare cylindrical cores with height-to-diameter ratios of one are to be built with power ratings of 300 MW(t), 1,000 MW(t), and 3,000 MW(t).

a. Find *H*, the core height , B^2, the buckling, and k_∞ for each of these cores.
b. For each of the three cores, determine the reactivity change caused by the boiling.
c. Briefly interpret your results from part b.

Part A For a bare cylindrical reactor , from Eq. (8.3)

$$V = 3.63P / \bar{P}'''_{\max} = 3.63P / 450 = 8.07 \cdot 10^{-3} P \text{ (volume in m}^3\text{, power in MW)}$$

$$H = (4V/\pi)^{1/3} = 1.08V^{1/3} = 1.08(8.07 \cdot 10^{-3})^{1/3} P^{1/3}$$

=> $$H = 0.217P^{1/3}$$

$$B^2 = \left[(2 \cdot 2.405)^2 + \pi^2\right] H^{-2} = 33.0(0.217P^{1/3})^{-2}$$

=> $$B^2 = 701P^{-2/3}$$

$$k_\infty = 1 + M^2 B^2 = 1 + 0.18^2 \cdot (33/H^2)$$

$$= 1 + 1.07/H^2 = 1 + 1.07 \cdot (0.217P^{1/3})^{-2}$$

=> $$k_\infty = 1 + 22.7P^{-2/3}$$

See table for numerical results

Part b: From Eq. (9.4)

$$\frac{\Delta k}{k} = \frac{\Delta k_\infty}{k_\infty} - \frac{M^2 B^2}{1 + M^2 B^2}\left(\frac{\Delta M^2}{M^2}\right)$$

$$= \frac{\Delta k_\infty}{k_\infty} - \frac{22.7P^{-2/3}}{1 + 22.7P^{-2/3}}\left(\frac{\Delta M^2}{M^2}\right)$$

$$= \frac{\Delta k_\infty}{k_\infty} - \frac{1}{1 + 0.044P^{2/3}}\left(\frac{\Delta M^2}{M^2}\right)$$

=> $$\frac{\Delta k}{k} = 0.01 - \frac{0.01}{1 + 0.044P^{2/3}}$$

See table for results

Part c In a fast reactor sodium voiding causes light weight coolant to be lost, decreasing the slowing down of neutrons, and shifting the spectrum up in energy. At the same time the coolant loss increases the migration length, because neutrons travel further in the less dense core. This increases the leakage, causing the reactivity to decrease, as indicated by Eq. (9.4). Smaller cores have larger leakage probabilities, amplifying the negative effect. Thus as the core power therefore size increases, the leakage becomes smaller. The net effect is for the voiding reactivity effect to become increasingly positive for larger cores.

P MW	300	1,000	3,000
H m	1.45	2.17	3.13
B^2 m^{-2}	15.6	7.01	3.37
k_∞	1.51	1.23	1.11
$\Delta k / k$	$-1.36\text{x}10^{-3}$	$+0.0148\text{x}10^{-3}$	$+1.02\text{x}10^{-3}$

[9.7] For the reactor specified in problem [9.4] the power is maintained at 1,000 MW(t) while the following quasi-static changes are made

a. the inlet temperature is slowly decreased by 10 ^{0}C
b. The flow rate is slowly increased by 10%.

For each of these cases determine by how much the reactivity must be increased or decreased to keep the reactor running at constant power.

For the simple model of problem [9.4] the reactivity is found in Eq. (9.36) , with the absolute value signs removed to allow positive as well as negative temperature coefficients . Since the reactor must be maintained in a critical state for the power to be constant. We have $\rho(t) = 0$, and hence

$$\rho_i(t) + \alpha_f \left[\bar{T}_f(t) - \bar{T}_f(0)\right] + \alpha_c \left[\bar{T}_c(t) - \bar{T}_c(0)\right] = 0$$

where $\rho_i(t)$ is the control activity, which must be added or subtracted to keep the reactor at constant power Under quasi static conditions the fuel and moderator temperatures are determined by Eqs. (9.45) and (9.46), which we modify to allow slow changes in the flow rate and inlet temperature :

$$\bar{T}_c(t) = \frac{1}{2W(t)c_p} P + T_i(t) \,.$$

$$\bar{T}_f(t) = \left(R_f + \frac{1}{2W(t)c_p} \right) P + T_i(t) \,.$$

Hence

$$\left[\bar{T}_c(t) - \bar{T}_c(0)\right] = \left[\frac{1}{2W(t)c_p} - \frac{1}{2W(0)c_p} \right] P + \left[\bar{T}_i(t) - \bar{T}_c(0)_i\right]$$

and likewise

$$\left[\bar{T}_f(t) - \bar{T}_f(0)\right] = \left[\frac{1}{2W(t)c_p} - \frac{1}{2W(0)c_p} \right] P + \left[\bar{T}_i(t) - \bar{T}_c(0)_i\right]$$

The reactivity equation then becomes

$$\rho_i(t) = -\left(\alpha_f + \alpha_c\right) \left\{ \left[\frac{W(0)}{W(t)} - 1 \right] \frac{P}{2W(0)c_p} + \bar{T}_i(t) - \bar{T}_c(0) \right\}$$

Before proceeding we must calculate the flow rate: Form Eq. (8.37) the mass flow rate is

$$W(0) = \frac{P}{c_p \left[\bar{T}_o(0) - T(0_i)\right]} = \frac{1000 \cdot 10^6}{1250(500 - 350)} = 5{,}333 \text{ kg/s}$$

Plugging in the parameters from problem [9.4] into the reactivity equation:

$$\rho_i(t) = 1.35 \cdot 10^{-5} \left\{ \left[\frac{W(0)}{W(t)} - 1 \right] 225 + \bar{T}_i(t) - \bar{T}_i(0) \right\}$$

Part a. $\bar{T}_i(t) - \bar{T}_i(0)$ = - 5.0

$$\rho_i(t) = 1.35 \cdot 10^{-5} \left\{ \bar{T}_i(t) - \bar{T}_i(0) \right\} = -6.75 \cdot 10^{-5}$$

If the inlet temperature decreases, the feedback reactivity increases, thus the control system must subtract reactivity.

Part b: $W(0)/W(t) = 1/1.1 = 0.909$

$$\rho_i(t) = 1.35 \cdot 10^{-5}\left[\frac{W(0)}{W(t)} - 1\right]225 = -27.6 \cdot 10^{-5}$$

Increasing the flow rate, decreases fuel and coolant temperature. Thus the feedback reactivity positive, and the control system must subtract reactivity to maintain criticality.

[9.8] In the prismatic block form of the graphite moderated gas-cooled reactor the heat passes through the moderator before reaching the coolant. Figure 4.1d shows such a configuration. Assume that R_1 and R_2 are the thermal resistances between fuel and moderator, and between moderator and coolant respectively, and that W and c_p are the coolant mass flow rate and specific heat.

a. Develop a set of thee coupled equations similar to to those in Eqs. (8.40) and (8.41) to model the steady state heat transfer.
b. Determine the isothermal temperature coefficient in terms of the fuel, moderator and coolant temperature coefficients.
c. Determine the power coefficient in terms of the same temperature coefficients.

Part a. See solution to problem [8.14] for transient analysis. By setting the time derivatives to zero, we obtain the steady state heat transfer:

$$P = \frac{1}{R_1}\left[\overline{T}_f - \overline{T}_m\right]$$

$$\frac{1}{R_1}\left[\overline{T}_f - \overline{T}_m\right] = \frac{1}{R_2}\left[\overline{T}_m - \overline{T}_c\right]$$

$$\frac{1}{R_2}\left[\overline{T}_m - \overline{T}_c\right] = 2Wc_c\left[\overline{T}_c - T_i\right]$$

from which we may write the steady state model as:

$$\overline{T}_c = \frac{1}{2Wc_c}P + T_i$$

$$\overline{T}_m = \left(R_2 + \frac{1}{2Wc_p}\right)P + T_i$$

$$\overline{T}_f = \left(R_1 + R_2 + \frac{1}{2Wc_p}\right)P + T_i$$

Part b: Let α_f, α_m and α_c be the fuel, moderator and coolant temperature coefficients. The isothermal temperature coefficient is them simply:

$$\alpha_T = \alpha_f + \alpha_m + \alpha_c$$

Part c: The power coefficient is just a generalization of Eqs. (9.30) through (9.33) to include the third region:

$$\alpha_P = \alpha_f \frac{d}{dP}\overline{T}_f + \alpha_m \frac{d}{dP}\overline{T}_m + \alpha_c \frac{d}{dP}\overline{T}_c$$

assuming a constant inlet temperature. Find the power derivatives from the three final equations in part a. We then have

$$\alpha_P = \left(R_1 + R_2 + \frac{1}{2Wc_p}\right)\alpha_f + \left(R_2 + \frac{1}{2Wc_p}\right)\alpha_m + \frac{1}{2Wc_p}\alpha_c$$

[9.9] Suppose a power reactor has negative values of α_f and α_c, the fuel and coolant temperature coefficients. Using the thermal hydraulic model developed in Chapter 8:

a. Show that if very slow changes take place in the coolant inlet temperature and mass flow rate, but no control poisons are added or subtracted, the power will undergo a quasi-static change of

$$dP = \frac{\left|\alpha_f + \alpha_c\right|\left(\dfrac{P}{2W^2 c_p} dW - dT_i\right)}{\left|\left(R + \dfrac{1}{2Wc_p}\right)\alpha_f + \dfrac{1}{2Wc_p}\alpha_c\right|}$$

b. If the flow rate increases, does the power increase or decrease? Why?

c. If the inlet temperature increases, does the power increase or decrease? Why?

a. For negative temperature coefficients, we begin by writing Eq. (9.25) as

$$d\rho = -\left|\alpha_f\right| d\overline{T}_f - \left|\alpha_c\right| d\overline{T}_c$$

Use the quasi- steady state temperatures of Eqs. (8.40) and (8.41). Taking the differentials:

$$d\overline{T}_c = d\left(\frac{P}{2Wc_p}\right) + dT_i = \frac{1}{2Wc_p} dP - \frac{P}{2W^2 c_p} dW + dT_i$$

$$d\overline{T}_c = d\left(R + \frac{P}{2Wc_p}\right) + dT_i = \left(R + \frac{1}{2Wc_p}\right) dP - \frac{P}{2W^2 c_p} dW + dT_i$$

Combining equations:

$$d\rho = -\left|\alpha_f\right|\left\{\left(R + \frac{1}{2Wc_p}\right) dP - \frac{P}{2W^2 c_p} dW + dT_i\right\}_f - \left|\alpha_c\right|\left\{\frac{1}{2Wc_p} dP - \frac{P}{2W^2 c_p} dW + dT_i\right\}$$

For very slow changes, we assume that the feedback instantaneously compensates for the reactivity, thus $d\rho$ =0. and we have

$$d\rho = 0 = -\left\{\left|\alpha_f\right|\left(R + \frac{1}{2Wc_p}\right) + \left|\alpha_c\right|\frac{1}{2Wc_p}\right\} dP + \left(\left|\alpha_f\right| + \left|\alpha_c\right|\right)\left\{\frac{P}{2W^2 c_p} dW - dT_i\right\}$$

Solving for *P*, we obtain the desired equation.

b. The Power increases since increasing the flow rate decreases both fuel and moderator temperature, causing increased feedback reactivity.

c. The power decreases, since raising the inlet temperature causes increases in fuel and moderator temperature, causing negative reactivity from the feedback.

[9.10]* Apply appropriate software to Eqs. (9.37) through (9.42) for a uranium fueled reactor. Use the following parameters, which are typical of a large pressurized water reactor: $\Lambda = 50\times10^{-6}\,\text{s}$, $\alpha = -4.2\times10^{-5}\ /^{0}\text{C}$ $M_f c_f = 32\ \text{MW}/^{0}\text{C}$, $\tau = 4.5\,\text{s}$, With an initial steady state power of 10 MW, make a plot of the power for

a. step reactivity increase of 10 cents,
b. a step reactivity increase of 20 cents
c. a step reactivity decrease of. 10 cents,
d. a step reactivity decrease of 20 cents.

$\beta1 := 0.00021$

$\beta2 := 0.00142$

$\beta3 := 0.00128$

$\beta4 := 0.00257$

$\beta5 := 0.00075$

$\beta6 := 0.00027$

$c1 := \beta1\cdot\frac{P0}{(\lambda1\cdot\Lambda)}$ $\quad c2 := \beta2\cdot\frac{P0}{(\lambda2\cdot\Lambda)}$

$c3 := \beta3\cdot\frac{P0}{(\lambda3\cdot\Lambda)}$ $\quad c4 := \beta4\cdot\frac{P0}{(\lambda4\cdot\Lambda)}$

$c5 := \beta5\cdot\frac{P0}{(\lambda5\cdot\Lambda)}$ $\quad c6 := \beta6\cdot\frac{P0}{(\lambda6\cdot\Lambda)}$

Sample calculations shown for $0.10 below using the stiff differential equations solver Radau (Mathcad) to obtain the vector Y.

$Rf := \frac{\tau}{MfCf} = 0.141$

$Rf\cdot WCp = 8.438$

dollars:= 0.1

$\alpha1 = -4.2\times10^{-5}$

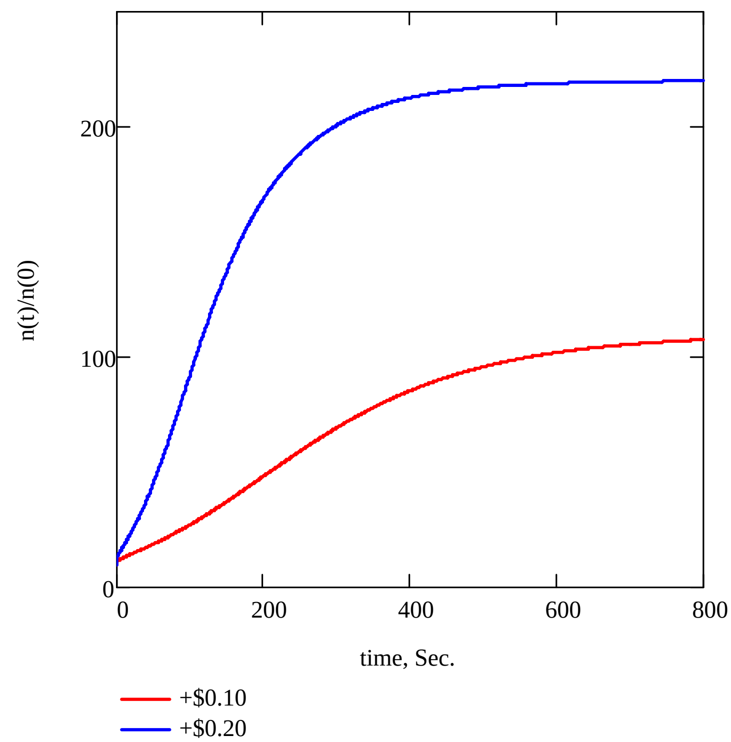
$Tf0 := Rf\cdot P0 + Ti = 551.406$

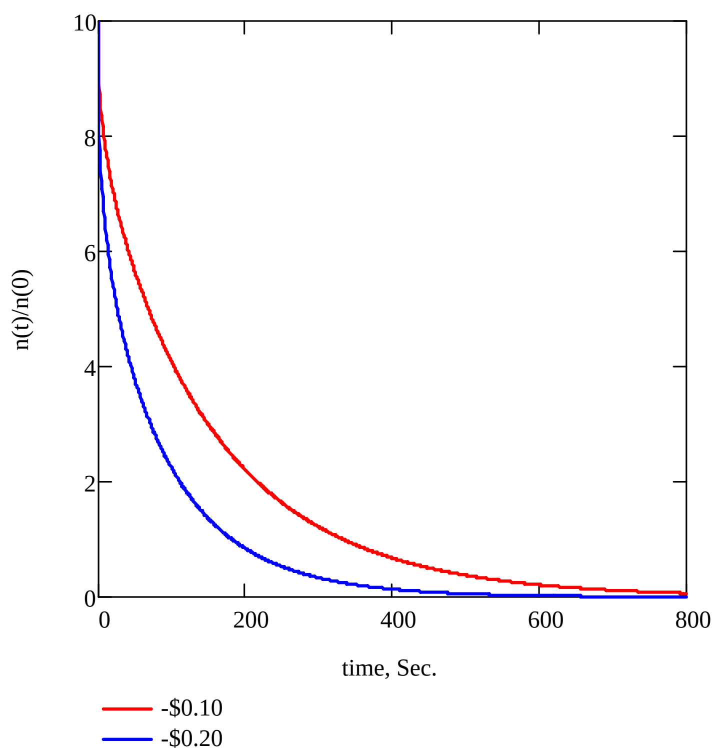
$$y := \begin{pmatrix} P0 \\ c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ Tf0 \end{pmatrix}$$

$$DD(t,y) := \begin{bmatrix} \frac{\left[dollars\cdot\beta + \alpha1\cdot(y_7 - Tf0) - \beta\right]}{\Lambda}\cdot y_0 + \lambda1\cdot y_1 + \lambda2\cdot y_2 + \lambda3\cdot y_3 + \lambda4\cdot y_4 + \lambda5\cdot y_5 + \lambda6\cdot y_6 \\ \left(\frac{\beta1}{\Lambda}\right)\cdot y_0 - \lambda1\cdot y_1 \\ \left(\frac{\beta2}{\Lambda}\right)\cdot y_0 - \lambda2\cdot y_2 \\ \left(\frac{\beta3}{\Lambda}\right)\cdot y_0 - \lambda3\cdot y_3 \\ \left(\frac{\beta4}{\Lambda}\right)\cdot y_0 - \lambda4\cdot y_4 \\ \left(\frac{\beta5}{\Lambda}\right)\cdot y_0 - \lambda5\cdot y_5 \\ \left(\frac{\beta6}{\Lambda}\right)\cdot y_0 - \lambda6\cdot y_6 \\ \frac{y_0}{MfCf} - \frac{(y_7 - Tf0)}{\tau} \end{bmatrix}$$

$Y := \mathrm{Radau}(y, 0, 800, 1000, DD)$

$n := 0 .. 1000$





[9.11]* Using the data and initial conditions from problem [10] apply appropriate software to

a. Determine what ramp rate of reactivity insertion will cause a power spike with a peak power that exceeds 100 MW(t)
b. Determine what ramp rate of reactivity insertion will cause a power spike with a peak that exceeds 1,000 MW(t)

$\beta1 := 0.00021$

$\beta2 := 0.00142$

$\beta3 := 0.00128$

$\beta4 := 0.00257$

$\beta5 := 0.00075$

$\beta6 := 0.00027$

$$c1 := \beta1 \cdot \frac{P0}{(\lambda1 \cdot \Lambda)} \qquad c2 := \beta2 \cdot \frac{P0}{(\lambda2 \cdot \Lambda)}$$

$$c3 := \beta3 \cdot \frac{P0}{(\lambda3 \cdot \Lambda)} \qquad c4 := \beta4 \cdot \frac{P0}{(\lambda4 \cdot \Lambda)}$$

$$c5 := \beta5 \cdot \frac{P0}{(\lambda5 \cdot \Lambda)} \qquad c6 := \beta6 \cdot \frac{P0}{(\lambda6 \cdot \Lambda)}$$

Sample calculations shown for $0.40 (5000 MW) below using the stiff differential equations solver Radau (Mathcad) to obtain the vector Y.

$$Rf := \frac{\tau}{MfCf} = 0.141 \qquad \alpha1 = -4.2 \times 10^{-5}$$

$$Rf \cdot WCp = 8.438 \qquad dollars := 0.40$$

$$y := \begin{pmatrix} P0 \\ c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ Tf0 \end{pmatrix} \qquad DD(t,y) := \begin{bmatrix} \frac{\left[dollars\beta \cdot t + \alpha1 \cdot (y_7 - Tf0) - \beta\right]}{\Lambda} \cdot y_0 + \lambda1 \cdot y_1 + \lambda2 \cdot y_2 + \lambda3 \cdot y_3 + \lambda4 \cdot y_4 + \lambda5 \cdot y_5 + \lambda6 \cdot y_6 \\ \left(\frac{\beta1}{\Lambda}\right) \cdot y_0 - \lambda1 \cdot y_1 \\ \left(\frac{\beta2}{\Lambda}\right) \cdot y_0 - \lambda2 \cdot y_2 \\ \left(\frac{\beta3}{\Lambda}\right) \cdot y_0 - \lambda3 \cdot y_3 \\ \left(\frac{\beta4}{\Lambda}\right) \cdot y_0 - \lambda4 \cdot y_4 \\ \left(\frac{\beta5}{\Lambda}\right) \cdot y_0 - \lambda5 \cdot y_5 \\ \left(\frac{\beta6}{\Lambda}\right) \cdot y_0 - \lambda6 \cdot y_6 \\ \frac{y_0}{MfCf} - \frac{(y_7 - Tf0)}{\tau} \end{bmatrix}$$

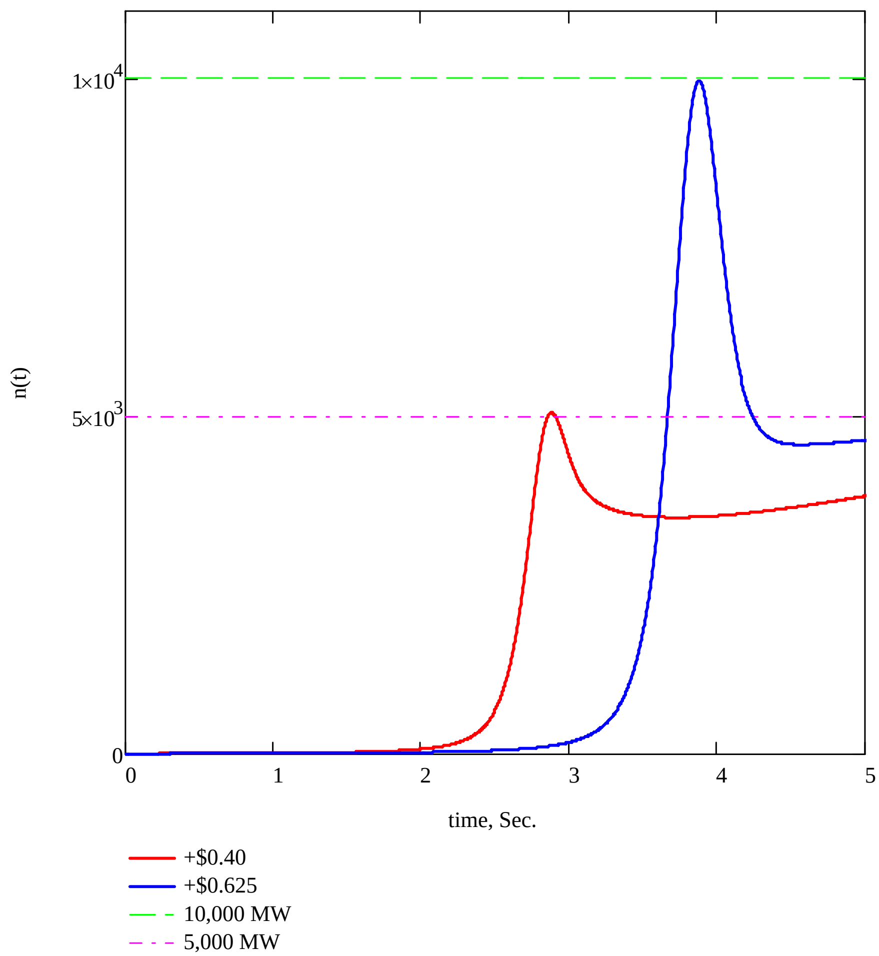
$n := 0..10000$

$Y := Radau(y, 0, 50, 10000, DD)$

$YY := 10000$

$YYY := 5000$

To obtain a power spike of 5,000 MW, a reactivity of $0.40 is used while to obtain a power spike of 10,000 MW, a reactivity of $0.625 is used.



[9.12]* Repeat problem [9.10] for a plutonium fueled sodium cooled fast reactor with the parameters: $\Lambda = 0.5\times10^{-6}$ s , $\alpha = -1.8\times10^{-5}$ /^{0}C $M_f c_f = 5.0\times10^6$ J/^{0}C , $\tau = 4.0$ s ,

$\beta1 := 0.00021$

$\beta2 := 0.00142$

$\beta3 := 0.00128$

$\beta4 := 0.00257$

$\beta5 := 0.00075$

$\beta6 := 0.00027$

$$c1 := \beta1\cdot\frac{P0}{(\lambda1\cdot\Lambda)} \qquad c2 := \beta2\cdot\frac{P0}{(\lambda2\cdot\Lambda)}$$

$$c3 := \beta3\cdot\frac{P0}{(\lambda3\cdot\Lambda)} \qquad c4 := \beta4\cdot\frac{P0}{(\lambda4\cdot\Lambda)}$$

$$c5 := \beta5\cdot\frac{P0}{(\lambda5\cdot\Lambda)} \qquad c6 := \beta6\cdot\frac{P0}{(\lambda6\cdot\Lambda)}$$

Sample calculations shown for $0.10 below using the stiff differential equations solver Radau (Mathcad) to obtain the vector Y.

$$Rf := \frac{\tau}{MfCf} = 0.8$$

$Rf\cdot WCp = 48$

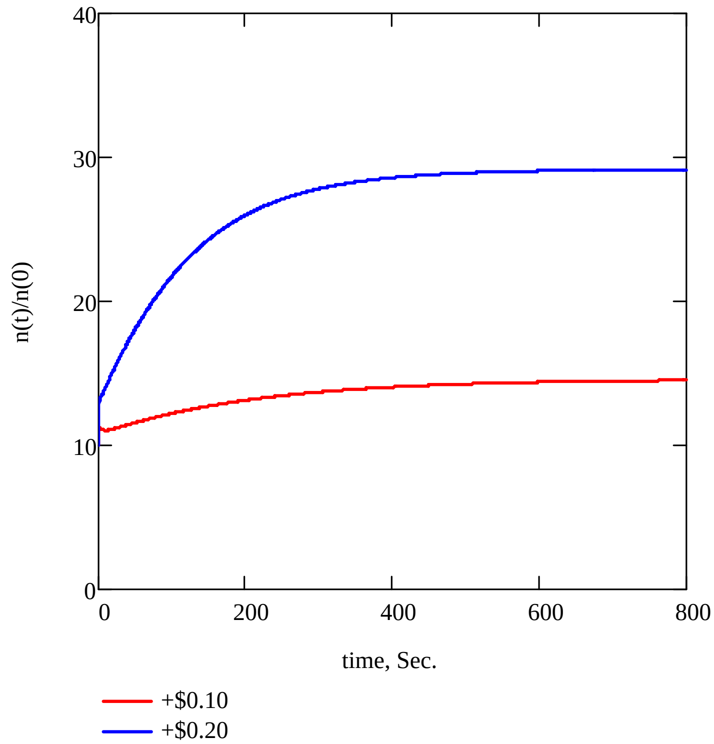
dollars := 0.1

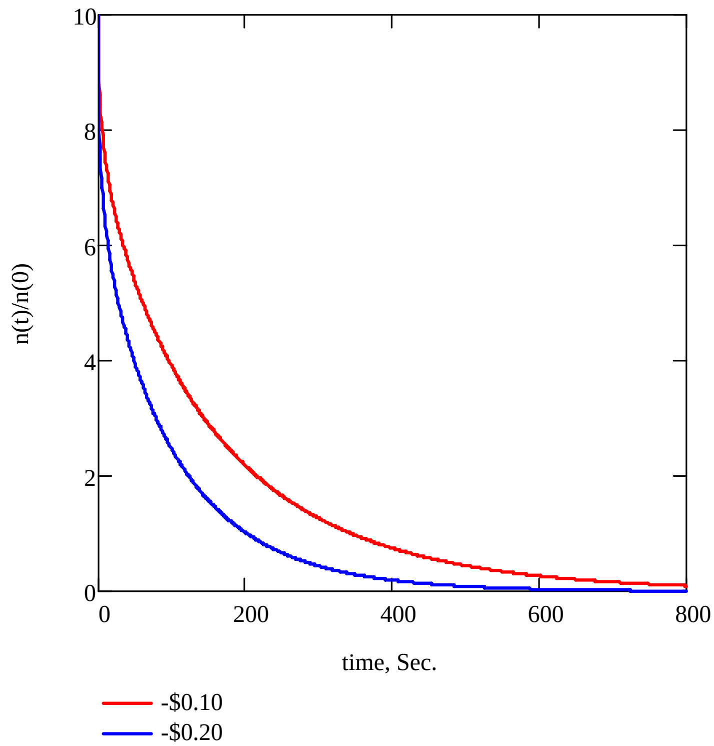
$\alpha1 = -1.8\times10^{-5}$

$$y := \begin{pmatrix} P0 \\ c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ Tf0 \end{pmatrix} \qquad DD(t,y) := \begin{bmatrix} \frac{\lfloor dollars\cdot\beta + \alpha1\cdot(y_7 - Tf0) - \beta \rfloor}{\Lambda}\cdot y_0 + \lambda1\cdot y_1 + \lambda2\cdot y_2 + \lambda3\cdot y_3 + \lambda4\cdot y_4 + \lambda5\cdot y_5 + \lambda6\cdot y_6 \\ \left(\frac{\beta1}{\Lambda}\right)\cdot y_0 - \lambda1\cdot y_1 \\ \left(\frac{\beta2}{\Lambda}\right)\cdot y_0 - \lambda2\cdot y_2 \\ \left(\frac{\beta3}{\Lambda}\right)\cdot y_0 - \lambda3\cdot y_3 \\ \left(\frac{\beta4}{\Lambda}\right)\cdot y_0 - \lambda4\cdot y_4 \\ \left(\frac{\beta5}{\Lambda}\right)\cdot y_0 - \lambda5\cdot y_5 \\ \left(\frac{\beta6}{\Lambda}\right)\cdot y_0 - \lambda6\cdot y_6 \\ \frac{y_0}{MfCf} - \frac{(y_7 - Tf0)}{\tau} \end{bmatrix}$$

$Y := Radau(y, 0, 800, 1000, DD)$

$n := 0 .. 1000$


40
30
20
10
0
n(t)/n(0)
0
200
400
600
800
time, Sec.
+$0.10
+$0.20


10
8
6
4
2
0
n(t)/n(0)
0
200
400
600
800
time, Sec.
-$0.10
-$0.20

[9.13]* A 2,400 MW(t) plutonium sodium-cooled fast reactor has the following characteristics:

W = 14,000 kg/s	τ = 4.0 s	c_p = 1,250 J/(kg ^{0}C)
$M_f c_f$ = 13.5 x 10^6 J/^{0}C	$M_c c_p$ = 1.90 x 10^6 J/^{0}C	T_i = 360 ^{0}C
$\Lambda = 0.5\times10^{-6}$s	$\alpha_f = -1.8\times10^{-5}$ /^{0}C	$\alpha_c = +0.45\times10^{-5}$ /^{0}C

The reactor undergoes a loss of flow transient with $W(t) = W(0)/(1+t/t_o)$ where $t_o = 5.0\,\mathrm{s}$. Employ appropriate software to Eqs. (8.36) through (9.40) to analyze the transient: Make plots of the reactor power, fuel and coolant outlet temperatures for 0 < t < 20 s (Hint, note that $\tilde{\tau}$ cannot be approximated by τ in this problem.)

$\beta1 := 0.00021$

$\beta2 := 0.00142$

$\beta3 := 0.00128$

$\beta4 := 0.00257$

$\beta5 := 0.00075$

$\beta6 := 0.00027$

$c1 := \beta1\cdot\frac{P0}{(\lambda1\cdot\Lambda)}$ $\quad c2 := \beta2\cdot\frac{P0}{(\lambda2\cdot\Lambda)}$

$c3 := \beta3\cdot\frac{P0}{(\lambda3\cdot\Lambda)}$ $\quad c4 := \beta4\cdot\frac{P0}{(\lambda4\cdot\Lambda)}$

$c5 := \beta5\cdot\frac{P0}{(\lambda5\cdot\Lambda)}$ $\quad c6 := \beta6\cdot\frac{P0}{(\lambda6\cdot\Lambda)}$

Sample calculations shown for \$0.10 below using the stiff differential equations solver Radau (Mathcad) to obtain the vector Y.

$\alpha_f := -1.8\cdot10^{-5}$

$\alpha_c := 0.45\cdot10^{-5}$

$Rf := \frac{\tau}{MfCf} = 0.296$

$Tf0 := Rf\cdot P0 + Ti = 1.071\times10^3$

$C := \left(2\cdot Rf\cdot W_0\cdot Cp\right)^{-1}$

$C = 0.096$

$$y := \begin{pmatrix} P0 \\ c1 \\ c2 \\ c3 \\ c4 \\ c5 \\ c6 \\ Tf0 \end{pmatrix}$$

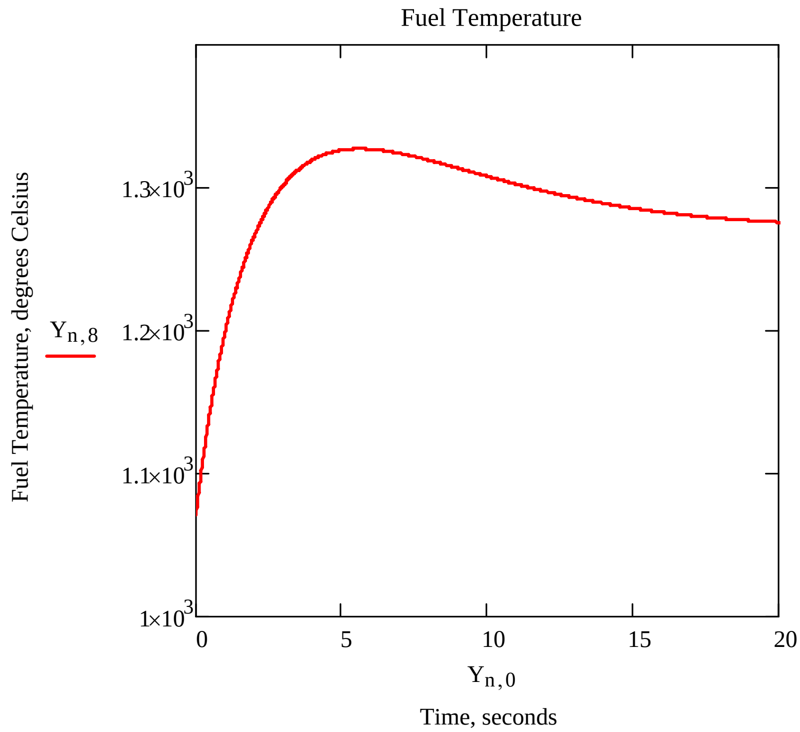
$$DD(t,y) := \begin{bmatrix} \dfrac{\left[\alpha_f\cdot(y_7 - Tf0) + \alpha_c\cdot C\cdot\left[\left(1+\dfrac{t}{t_0}\right)\cdot y_7 - Tf0\right] - \beta\right]}{\Lambda}\cdot y_0 + \lambda1\cdot y_1 + \lambda2\cdot y_2 + \lambda3\cdot y_3 + \lambda4\cdot y_4 + \lambda5\cdot y_5 + \lambda6\cdot y_6 \\ \left(\dfrac{\beta1}{\Lambda}\right)\cdot y_0 - \lambda1\cdot y_1 \\ \left(\dfrac{\beta2}{\Lambda}\right)\cdot y_0 - \lambda2\cdot y_2 \\ \left(\dfrac{\beta3}{\Lambda}\right)\cdot y_0 - \lambda3\cdot y_3 \\ \left(\dfrac{\beta4}{\Lambda}\right)\cdot y_0 - \lambda4\cdot y_4 \\ \left(\dfrac{\beta5}{\Lambda}\right)\cdot y_0 - \lambda5\cdot y_5 \\ \left(\dfrac{\beta6}{\Lambda}\right)\cdot y_0 - \lambda6\cdot y_6 \\ \dfrac{y0}{MfCf} - \dfrac{(y_7 - Tf0)\cdot\left[1 + C\cdot\left(1+\dfrac{t}{t_0}\right)\right]}{\tau} \end{bmatrix}$$

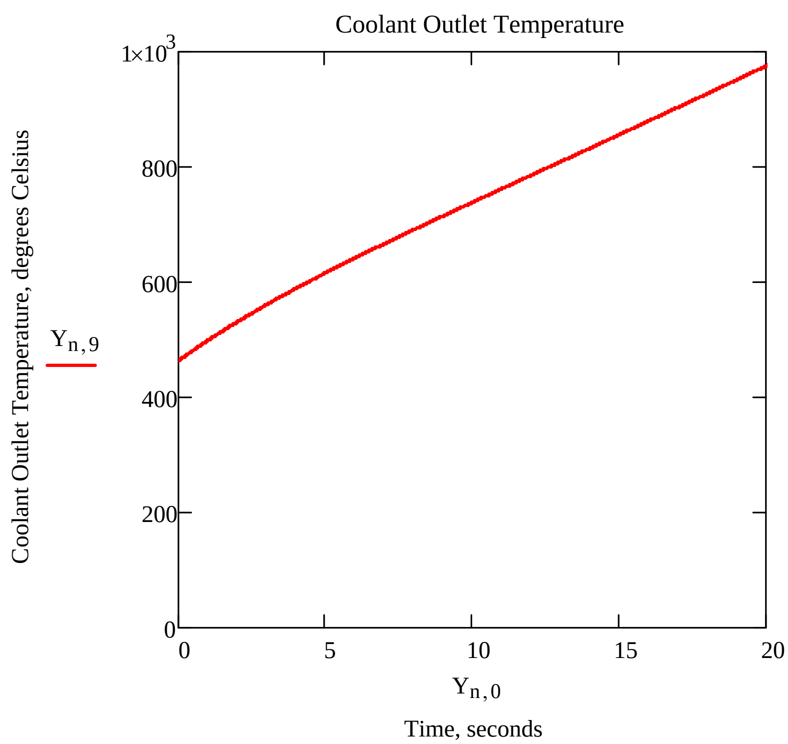
$Y := \mathrm{Radau}(y, 0, 100, 10000, DD)$

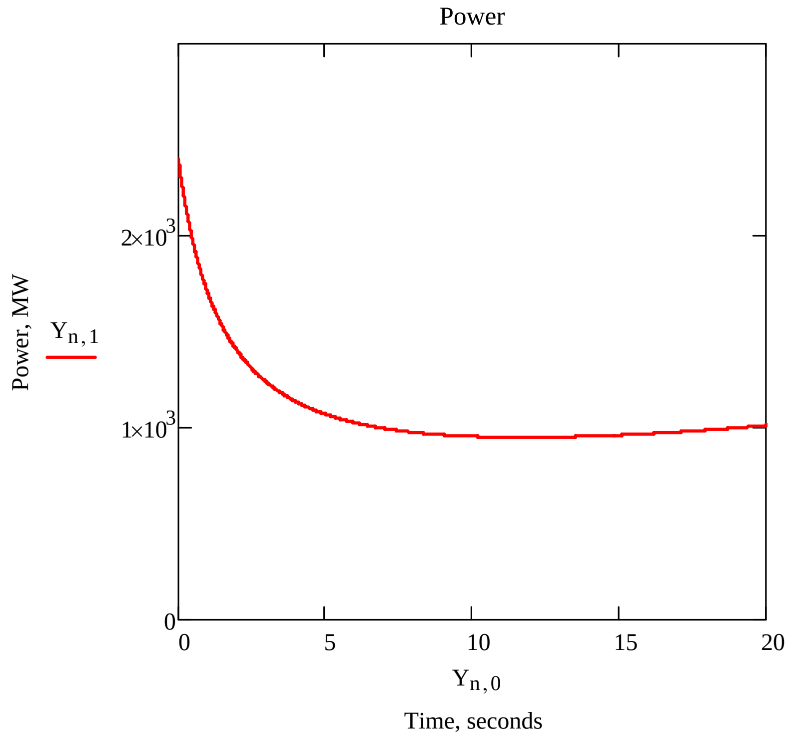
$n := 0 .. 10000$

Below is an algebraic equation to obtain the coolant outlet temperature using the fuel temperature and the inlet temperature.

$$Y_{n,9} := Ti + C\cdot\left(1 + \frac{Y_{n,0}}{t_0}\right)\cdot Y_{n,8}$$


Fuel Temperature
Fuel Temperature, degrees Celsius
$Y_{n,8}$
1.3×10^3
1.2×10^3
1.1×10^3
1×10^3
0
5
10
15
20
$Y_{n,0}$
Time, seconds


Coolant Outlet Temperature
Coolant Outlet Temperature, degrees Celsius
$Y_{n,9}$
1×10^3
800
600
400
200
0
0
5
10
15
20
$Y_{n,0}$
Time, seconds



Power
Power, MW
$Y_{n,1}$
2×10^3
1×10^3
0
0
5
10
15
20
$Y_{n,0}$
Time, seconds

Chapter 10 Problems:

[10.1] Prove that for a reactor operating at a very high flux level, the maximum xenon-135 concentration takes place at approximately 11.3 hours following shutdown.

The Xenon concentration following shutdown is given by Eq. (10.19):

$$X(t) = \overline{\Sigma}_f \phi \left[\frac{(\gamma_I + \gamma_X)}{\lambda_X + \sigma_{aX}\phi} e^{-\lambda_X t} + \frac{\gamma_I}{\lambda_I - \lambda_X}\left(e^{-\lambda_X t} - e^{-\lambda_I t}\right)\right]$$

To find the maximum we set its derivative equal to zero:

$$\frac{d}{dt} X(t) = \overline{\Sigma}_f \phi \left[-\frac{(\gamma_I + \gamma_X)\lambda_X}{\lambda_X + \sigma_{aX}\phi} e^{-\lambda_X t} + \frac{\gamma_I}{\lambda_I - \lambda_X}\left(-\lambda_x e^{-\lambda_X t} + \lambda_I e^{-\lambda_I t}\right)\right] = 0$$

For a very large value of the flux, the ϕ in the denominator causes the first term to vanish. Hence

$$\left(-\lambda_x e^{-\lambda_x t} + \lambda_I e^{-\lambda_I t}\right) = 0$$

or

$$t = \frac{1}{\lambda_I - \lambda_X} \ln(\lambda_I / \lambda_X)$$

From Eq. (10.11):

$$\lambda_I = 0.693 / t_{1/2I} = 0.693/6.7 = 0.1034 \text{ hr}^{-1}$$

$$\lambda_X = 0.693 / t_{1/2X} = 0.693/9.2 = 0.0753 \text{ hr}^{-1}$$

$$t = \frac{1}{0.1034 - 0.0753} \ln(0.1034/0.0753) = 11.3 \text{ hr}$$

[10.2] Make a logarithmic plot of the effective half-life of xenon-135 over the flux range of $10^{10} \le \phi \le 10^{15}\,\text{n/cm}^2\text{/s}$.

The effective half-life of xenon is given by $t'_{1/2} = 0.693/\lambda'$. From Eq. (10.9)

$$\lambda'_X = \lambda_X + \sigma_{aX}\phi$$

$$\lambda_X = 0.693/t_{1/2X} = 0.693/9.2 = 0.0753\,\text{hr}^{-1}$$

$$= 0.0753\,\text{hr}^{-1}/3600\,\text{s}\,\text{hr}^{-1} = 2.09\cdot 10^{-5}\ \text{s}^{-1}$$

and $\sigma_{aX} = 2.65\cdot 10^{6}$ b

$$\lambda'_X = 2.09\cdot 10^{-5} + 2.65\cdot 10^{-18}\phi\ \ \text{s}^{-1}$$

$$t'_{1/2X} = 0.693/(2.09\cdot 10^{-5} + 2.65\cdot 10^{-18}\phi)\ \text{s}$$

$$t'_{1/2X} = 1/(3.02\cdot 10^{-5} + 3.82\cdot 10^{-18}\phi)\ \text{s}$$

$$t'_{1/2X} = 1/(10.87\cdot 10^{-2} + 13.7\cdot 10^{-15}\phi)\ \text{hr}$$

See plot below

[10.3] A thermal reactor fueled with uranium has been operating at constant power for several days. Make a plot of the ratio of concentration of xenon-135 to uranium-235 atoms in the reactor versus its average flux. Determine the maximum value that this ratio can take.

To find the ratio *X/I*, divide Eq. (10.15) by (10.14)

$$X/I = \frac{\lambda_I}{\lambda_X + \sigma_{aX}\phi}\frac{(\gamma_I + \gamma_X)}{\gamma_I}$$

From Eq. (10.11):

$$\lambda_I = 0.693/t_{1/2I} = 0.693/6.7 = 0.1034\ \text{hr}^{-1} = 2.87\text{x}10^{-5}\ \text{s}^{-1}$$

$$\lambda_X = 0.693/t_{1/2X} = 0.693/9.2 = 0.0753\ \text{hr}^{-1} = 2.09\text{x}10^{-5}\ \text{s}^{-1}$$

$$\sigma_{aX} = 2.65\cdot 10^6\ \text{b}$$

From Table 10.1 for uranium-235 $\gamma_I = 0.0639$, $\gamma_X = 0.00237$

$$X/I = \frac{2.87\cdot 10^{-5}}{2.09\cdot 10^{-5} + 2.65\cdot 10^{-18}\phi}\frac{(0.0639 + 0.00237)}{0.0369}$$

$$X = \frac{(\gamma_I + \gamma_X)}{\lambda_X + \sigma_{aX}\phi}\Sigma_f\phi = \frac{(0.0639 + 0.00237)}{2.09\cdot 10^{-5} + 2.65\cdot 10^{-18}\phi}0.203\phi$$

$$X/I = \frac{1}{0.701 + 0.889\cdot 10^{-13}\phi}$$

See plot below.

[10.4] A pressurized water reactor at full power has an average power density of $\bar{P}''' = 80\,\mathrm{MW/m^3}$ and a peaking factor of $F_q = 2.0$. After the reactor has operated for several days:

a. What is the average xenon concentration?
b. What is the maximum xenon concentration?
c. What is the average samarium concentration?
d. What is the maximum samarium concentration?

(assume a fission cross section of $\bar{\Sigma}_f = 0.203$ cm^{-1}.)

First find the average flux $\bar{P}''' = \gamma\Sigma_f\bar{\phi}$ Therefore

$$\bar{\phi} = \bar{P}'''/(\gamma\Sigma_f) = 80/(3.1\cdot10^{-11}0.203) = 12.7\cdot10^{12}\ \mathrm{n/cm^2s}$$

$$\phi_{\max} = F_q\bar{P}'''/(\gamma\Sigma_f) = 2\cdot80/(3.1\cdot10^{-11}0.203) = 25.4\cdot10^{12}\ \mathrm{n/cm^2s}$$

From Eq. (10.11):

$$\lambda_I = 0.693/t_{1/2I} = 0.693/6.7 = 0.1034\ \mathrm{hr^{-1}} = 2.87\mathrm{x}10^{-5}\ \mathrm{s^{-1}}$$

$$\lambda_X = 0.693/t_{1/2X} = 0.693/9.2 = 0.0753\ \mathrm{hr^{-1}} = 2.09\mathrm{x}10^{-5}\ \mathrm{s^{-1}}$$

$$\sigma_{aX} = 2.65\cdot10^6\ \mathrm{b}$$

From Table 10.1 for uranium-235 $\gamma_I = 0.0639$, $\gamma_X = 0.00237$

Form Eq.(10.15):

$$X = \frac{(\gamma_I+\gamma_X)}{\lambda_X+\sigma_{aX}\phi}\Sigma_f\phi = \frac{(0.0639+0.00237)}{2.09\cdot10^{-5}+2.65\cdot10^{-18}\phi}0.203\phi$$

$$X = \frac{6.44\cdot10^2}{1.0+1.27\cdot10^{-13}\phi}\phi$$

Part a

$$\bar{X} = \frac{6.44\cdot10^2}{1.0+1.27\cdot10^{-13}\bar{\phi}}\bar{\phi} = \frac{6.44\cdot10^2}{1.0+1.27\cdot10^{-13}\cdot12.7\cdot10^{12}}12.7\cdot10^{12}$$

$$\bar{X} = 3\cdot13\cdot10^{15}\ \mathrm{cm^{-3}}$$

Part b

$$X_{\max} = \frac{6.44\cdot10^2}{1.0+1.27\cdot10^{-13}\phi_{\max}}\phi_{\max} = \frac{6.44\cdot10^2}{1.0+1.27\cdot10^{-13}\cdot25.4\cdot10^{12}}25.4\cdot10^{12}$$

$$X_{\max} = 3.87\cdot10^{15}\ \mathrm{cm^{-3}}$$

Note that although the flux doubles, the xenon concentration increases much less.

Parts. c and d: Setting the derivatives in Eqs. (10.21) and (10.22)

$$S = \gamma_P\Sigma_f/\sigma_{aS} = 0.0107\cdot0.203/(41\cdot10^3\cdot10^{-24})$$

$$S = 5.3\cdot10^{16}\ \mathrm{cm^{-3}}$$

Since the sumarium concentration is independent of the flux, it is the same throughout the reactor (so long as Σ_f is uniform).

[10.5] A reactor is started up and operated at full power. Solve Eqs. (10.12) and (10.13), and determine the iodine and xenon concentration as a function of time.

Eq. (10.12) for iodine is identical in form to Eq. (1.39), whose solution is worked out in Eq. (1.42). For iodine:

$$I(t) = \frac{\gamma_I \overline{\Sigma}_f \phi}{\lambda_I}\left[1 - \exp(-\lambda_I t)\right]$$

Inserting this equation into Eq. (10.13) yields

$$\frac{d}{dt}X(t) = (\gamma_X + \gamma_I)\overline{\Sigma}_f \phi - \gamma_I \overline{\Sigma}_f \phi \exp(-\lambda_I t) - \left(\lambda_X + \sigma_{aX}\phi\right) X(t)$$

Employing an integrating factor of $\exp[\left(\lambda_X + \sigma_{aX}\phi\right)t]$ we have

$$\frac{d}{dt}\left\{X(t)\exp[\left(\lambda_X + \sigma_{aX}\phi\right)t]\right\}$$
$$= \left[(\gamma_X + \gamma_I)\overline{\Sigma}_f \phi - \gamma_I \overline{\Sigma}_f \phi \exp(-\lambda_I t)\right]\exp[\left(\lambda_X + \sigma_{aX}\phi\right)t]$$

Integrating between 0 and t then yields (with $X(0) = 0$)

$$X(t)\exp[\left(\lambda_X + \sigma_{aX}\phi\right)t]$$
$$= \int_0^t \left\{(\gamma_X + \gamma_I)\overline{\Sigma}_f \phi \exp[\left(\lambda_X + \sigma_{aX}\phi\right)t'] - \gamma_I \overline{\Sigma}_f \phi \exp[\left(\lambda_X - \lambda_I + \sigma_{aX}\phi\right)t']\right\} dt'$$
$$= \frac{(\gamma_X + \gamma_I)\overline{\Sigma}_f \phi}{\lambda_X + \sigma_{aX}\phi}\left\{\exp[\left(\lambda_X + \sigma_{aX}\phi\right)t] - 1\right\} - \frac{\gamma_I \overline{\Sigma}_f \phi}{\lambda_X - \lambda_I + \sigma_{aX}\phi}\left\{\exp[\left(\lambda_X - \lambda_I + \sigma_{aX}\phi\right)t] - 1\right\}$$

Hence

$$X(t) = \frac{(\gamma_X + \gamma_I)\overline{\Sigma}_f \phi}{\lambda_X + \sigma_{aX}\phi}\left\{1 - \exp[-\left(\lambda_X + \sigma_{aX}\phi\right)t]\right\}$$
$$- \frac{\gamma_I \overline{\Sigma}_f \phi}{\lambda_X - \lambda_I + \sigma_{aX}\phi}\left\{\exp(-\lambda_I t) - \exp[-\left(\lambda_X + \sigma_{aX}\phi\right)t]\right\}$$

Note that as $t \to \infty$ this result reduces to Eq. (10.15) , as it should.

[10.6] Sumarium-157 is produced at a rate of 7.0x10^{-5} atoms/fission. It then undergoes decay: ${}^{157}_{62}Sm \xrightarrow[0.5\,\mathrm{min}]{\beta} {}^{157}_{63}Eu \xrightarrow[15.2\,\mathrm{hr}]{\beta} {}^{157}_{64}Gd$ While the absorption cross section of samarium and europium are negligible, the thermal absorption cross section of gadolinium is 240,000 b. Suppose that a reactor operates at a power density of 100 MW/m^3 and a flux level of 8.0x10^{12} n/cm^2/s.

a. Solve the decay equations for $G(t)$, the atom density of gadolinium, at a time t following reactor startup.
b. Evaluate $G(\infty)$.
c. If the reactor has been operated for several weeks and then is shut down, what is the concentration of gadolinium, after the reactor has been shut down for several weeks? (Assume that the energy produced per fission is 3.1x10^{-11} W s)

Part a. On a time scale of hours, assume that the Sm decays instantaneously. We then have

$$\frac{d}{dt}E = \gamma_s \Sigma_f \phi - \lambda E$$

with a solution of

$$E(t) = \frac{\gamma_s \Sigma_f \phi}{\lambda}\left[1 - \exp(-\lambda t)\right]$$

and

$$\frac{d}{dt}G = \lambda E - \sigma_{aG}\phi G$$

Plugging in E

$$\frac{d}{dt}G = \gamma_s \Sigma_f \phi\left[1 - \exp(-\lambda t)\right] - \sigma_{aG}\phi G$$

Using an integrating factor of $\exp(\sigma_{aG}\phi t)$, we have

$$\frac{d}{dt}\left[G\exp(\sigma_{aG}\phi t)\right] = \gamma_s \Sigma_f \phi\left[1 - \exp(-\lambda t)\right]\exp(\sigma_{aG}\phi t)$$

Integrating from 0 to t, with $G(0) = 0$

$$G(t)\exp(\sigma_{aG}\phi t) = \gamma_s \Sigma_f \phi \int_0^t \left[\exp(\sigma_{aG}\phi t') - \exp((\sigma_{aG}\phi - \lambda)t')\right]dt'$$

$$= \gamma_s \Sigma_f \phi \left\{ \frac{\exp(\sigma_{aG}\phi t) - 1}{\sigma_{aG}\phi} - \frac{\exp[(\sigma_{aG}\phi - \lambda)t] - 1}{\sigma_{aG}\phi - \lambda} \right\}$$

$$G(t) = \frac{\gamma_s \Sigma_f}{\sigma_{aG}} \left\{ 1 - \frac{1}{\sigma_{aG}\phi - \lambda}\left[\sigma_{aG}\phi \exp(-\lambda t) - \lambda \exp(-\sigma_{aG}\phi t)\right] \right\}$$

Part b:

$$G(\infty) = \frac{\gamma_s \Sigma_f}{\sigma_{aG}} = \frac{7.0\cdot 10^{-5}\Sigma_f}{240\cdot 10^3 \cdot 10^{-24}} = 2.9\cdot 10^{14}\Sigma_f$$

To determine the fission cross section we use

$$P''' = \gamma \Sigma_f \phi$$

or, noting that 1.0 MW/m^3 = 1. 0 W/cm^3

$$\Sigma_f = P''' / (\gamma\phi) = 100 / (3.1 \cdot 10^{-11} \cdot 8.0 \cdot 10^{12}) = 0.403\,\text{cm}^{-1}$$

Thus

$$G(\infty) = 11.7 \cdot 10^{12}\ \text{cm}^{-3}$$

Part c: Following shutdown the Eu decays at a rate,

$$E(t) = \frac{\gamma_s \Sigma_f \phi}{\lambda} \exp(-\lambda t)$$

However, no additional Gd is produced from Eu, since $\phi = 0$. Moreover, since Gd is stable its concentration remains at $G(\infty) = 11.7 \cdot 10^{12}\ \text{cm}^{-3}$

[10.7] Verify Eqs. (10.18) and (10.19).

Apply the integrating factor $\exp(\lambda_X t)$ to Eq. (10.17)

$$\left[\frac{d}{dt}X(t)+\lambda_X X(t)\right]\exp(\lambda_X t)=\frac{d}{dt}\left\{X(t)\exp(\lambda_X t)\right\}=\lambda_I I_o\exp(-\lambda_I t)\exp(\lambda_X t)$$

Integrate between 0 and t:

$$X(t)\exp(\lambda_X t)-X_o=\lambda_I I_o\int_0^t \exp[(\lambda_X-\lambda_I)t']dt'=\frac{\lambda_I}{\lambda_X-\lambda_I}I_o(e^{(\lambda_X-\lambda_I)t}-1)$$

Solving for X:

$$X(t)=X_o e^{-\lambda_X t}+\frac{\lambda_I}{\lambda_X-\lambda_I}I_o(e^{-\lambda_I t}-e^{-\lambda_X t})$$

or clearing minus signs

$$X(t)=X_o e^{-\lambda_X t}+\frac{\lambda_I}{\lambda_I-\lambda_X}I_o(e^{-\lambda_X t}-e^{-\lambda_I t})$$

Substituting in Eqs. (10.14) and (10.15) we have the second required equation:

$$X(t)=\overline{\Sigma}_f\phi\left[\frac{(\gamma_I+\gamma_X)}{\lambda_X+\sigma_{aX}\phi}e^{-\lambda_X t}+\frac{\gamma_I}{\lambda_I-\lambda_X}I_o(e^{-\lambda_X t}-e^{-\lambda_I t})\right]$$

[10.8] A reactor has operated for several weeks at constant power, reaching the equilibrium concentrations of iodine and xenon I_o and X_o given by Eqs. (10.14) and (10.15). At *t* = 0 the power is cut back, dropping the flux level from ϕ to $\tilde{\phi}$. Solve Eqs. (10.12) and (10.13) and show that the iodine and xenon concentrations following the power reduction are

$$I(t) = \frac{\gamma_I}{\lambda_I}\overline{\Sigma}_f\left[\tilde{\phi} + (\phi - \tilde{\phi})e^{-\lambda_I t}\right]$$

and

$$X(t) = \frac{(\gamma_I + \gamma_X)}{\lambda_X + \sigma_{aX}\phi}\overline{\Sigma}_f \phi e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t} + \frac{(\gamma_I + \gamma_X)}{\lambda_X + \sigma_{aX}\tilde{\phi}}\overline{\Sigma}_f\tilde{\phi}\left[1 - e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t}\right]$$
$$+ \frac{\gamma_I}{\lambda_X - \lambda_I + \sigma_{aX}\tilde{\phi}}\overline{\Sigma}_f(\phi - \tilde{\phi})\left[e^{-\lambda_I t} - e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t}\right]$$

Apply the integrating factor $\exp(\lambda_I t)$ to Eq. (10.12):

$$\frac{d}{dt}\left[I(t)\exp(\lambda_I t)\right] = \gamma_I \Sigma_f \tilde{\phi}\exp(\lambda_I t) \quad \text{and integrate between 0 and } t\text{:}$$

$$I(t)\exp(\lambda_I t) - I(0) = \gamma_I \Sigma_f \tilde{\phi}\int_0^t \exp(\lambda_I t')dt' = \gamma_I \Sigma_f \tilde{\phi}\frac{[\exp(\lambda_I t) - 1]}{\lambda_I}$$

Using Eq. (10.14) for the initial condition, we have

$$I(t) = \frac{\gamma_I \Sigma_f \phi}{\lambda_I}\exp(-\lambda_I t) + \frac{\gamma_I \Sigma_f \tilde{\phi}}{\lambda_I}[1 - \exp(-\lambda_I t)]$$

which reduces to the required equation.

Apply the integrating factor $\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t]$ to Eq. (10.13):

$$\frac{d}{dt}\left\{X(t)\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t]\right\} = \left[\gamma_X \Sigma_f \tilde{\phi} + \lambda_I I(t)\right]\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t]$$

Inserting $I(t)$:

$$\frac{d}{dt}\left\{X(t)\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t]\right\} = \left[(\gamma_X + \gamma_I)\Sigma_f \tilde{\phi} + \gamma_I \Sigma_f(\phi - \tilde{\phi})\exp(-\lambda_I t)\right]\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t]$$

Integrate between 0 and *t*:

$$X(t)\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t] - X(0)$$
$$= \int_0^t \left\{\left[(\gamma_X + \gamma_I)\Sigma_f \tilde{\phi} + \gamma_I \Sigma_f(\phi - \tilde{\phi})\exp(-\lambda_I t')\right]\exp[(\lambda_X + \sigma_{aX}\tilde{\phi})t']\right\}dt'$$
$$= \frac{(\gamma_X + \gamma_I)\Sigma_f \tilde{\phi}(e^{(\lambda_X + \sigma_{aX}\tilde{\phi})t} - 1)}{\lambda_X + \sigma_{aX}\tilde{\phi}} + \frac{\gamma_I \Sigma_f(\phi - \tilde{\phi})(e^{-(\lambda_X - \lambda_I + \sigma_{aX}\tilde{\phi})t} - 1)}{\lambda_X - \lambda_I + \sigma_{aX}\tilde{\phi}}$$

Replace $X(t)$ with Eq. (10.15) and solve:

$$X(t) = \frac{(\gamma_X + \gamma_I)\Sigma_f \phi}{\lambda_X + \sigma_{aX}\phi} e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t}$$

$$= \frac{(\gamma_X + \gamma_I)\Sigma_f \tilde{\phi}}{\lambda_X + \sigma_{aX}\tilde{\phi}}[1 - e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t}] + \frac{\gamma_I \Sigma_f (\phi - \tilde{\phi})}{\lambda_X - \lambda_I + \sigma_{aX}\tilde{\phi}}(e^{-\lambda_I t} - e^{-(\lambda_X + \sigma_{aX}\tilde{\phi})t})$$

Which is the required equation.

[10.9] Under load following conditions a reactor operates each day at full power for 12 hours, followed by a shutdown of 12 hours. Calculate the iodine concentration, $I(t)$, over a 24 hour time span. Use periodic boundary conditions $I(24hr) = I(0)$.

Apply the integrating factor $\exp(\lambda_I t)$ to Eq. (10.12)

$$\left[\frac{d}{dt}I(t)+\lambda_I I(t)\right]\exp(\lambda_I t)=\frac{d}{dt}\left\{I(t)\exp(\lambda_I t)\right\}=\gamma_I\Sigma_f\phi\exp(\lambda_I t)$$

Integrate between 0 and t:

$$I(t)\exp(\lambda_I t)-I(0)=\int_0^t \gamma_I\Sigma_f\phi\exp(\lambda_I t')dt'=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}[\exp(\lambda_I t)-1]$$

Thus

$$I(t)=I(0)e^{-\lambda_I t}+\frac{\gamma_I\Sigma_f\phi}{\lambda_I}(1-e^{-\lambda_I t})\quad \text{for } 0<t<12 \text{ hr.}$$

Let $t' = t - 12$ hr. Then while the reactor is shut down, no iodine is produced and thus

$$I(t)=I(12)e^{-\lambda_I t'}\qquad \text{for 12 hr.} < t < 24 \text{ hr.}$$

Now using the periodic boundary condition:

$$I(0)=I(24)=I(12)e^{-\lambda_I 12}=\left\{I(0)e^{-\lambda_I 12}+\frac{\gamma_I\Sigma_f\phi}{\lambda_I}(1-e^{-\lambda_I 12})\right\}e^{-\lambda_I 12}$$

Solve for $I(0)$:

$$I(0)=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\frac{(1-e^{-\lambda_I 12})}{(1-e^{-\lambda_I 24})}e^{-\lambda_I 12}=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\frac{1}{(e^{\lambda_I 12}+1)}$$

Thus

$$I(t)=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\frac{1}{(e^{\lambda_I 12}+1)}e^{-\lambda_I t}+\frac{\gamma_I\Sigma_f\phi}{\lambda_I}(1-e^{-\lambda_I t})\quad \text{for } 0<t<12 \text{ hr.}$$

$$I(t)=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\frac{e^{\lambda_I 12}}{(e^{\lambda_I 12}+1)}e^{-\lambda_I t'}\quad \text{for } 0<t'<12 \text{ hr., or}$$

$$I(t)=\frac{\gamma_I\Sigma_f\phi}{\lambda_I}\frac{e^{\lambda_I 24}}{(e^{\lambda_I 12}+1)}e^{-\lambda_I t}\quad \text{for 12 hr.} < t < 24 \text{ hr.}$$

[10.10] Under load following conditions a reactor operates each day at full power for 12 hours, followed by a shutdown of 12 hours. Calculate the promethium concentration, $P(t)$, over a 24 hour time span. Use periodic boundary conditions $P(24hr) = P(0)$.

Apply the integrating factor $\exp(\lambda_P t)$ to Eq. (10.12)

$$\left[\frac{d}{dt}P(t) + \lambda_P P(t)\right]\exp(\lambda_I t) = \frac{d}{dt}\left\{P(t)\exp(\lambda_P t)\right\} = \gamma_P \Sigma_f \phi \exp(\lambda_P t)$$

Integrate between 0 and t:

$$P(t)\exp(\lambda_P t) - P(0) = \int_0^t \gamma_P \Sigma_f \phi \exp(\lambda_P t')dt' = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}[\exp(\lambda_P t) - 1]$$

Thus

$$P(t) = P(0)e^{-\lambda_I t} + \frac{\gamma_P \Sigma_f \phi}{\lambda_P}(1 - e^{-\lambda_P t}) \quad \text{for } 0 < t < 12 \text{ hr.}$$

Let $t' = t - 12$ hr. Then while the reactor is shut down, no promethium is produced and thus

$$P(t) = P(12)e^{-\lambda_P t'} \qquad \text{for } 12 \text{ hr.} < t < 24 \text{ hr.}$$

Now using the periodic boundary condition:

$$P(0) = P(24) = P(12)e^{-\lambda_P 12} = \left\{P(0)e^{-\lambda_I 12} + \frac{\gamma_P \Sigma_f \phi}{\lambda_P}(1 - e^{-\lambda_P 12})\right\}e^{-\lambda_P 12}$$

Solve for $P(0)$:

$$P(0) = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}\frac{(1 - e^{-\lambda_P 12})}{(1 - e^{-\lambda_P 24})}e^{-\lambda_P 12} = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}\frac{1}{(e^{\lambda_P 12} + 1)}$$

Thus

$$P(t) = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}\frac{1}{(e^{\lambda_P 12} + 1)}e^{-\lambda_P t} + \frac{\gamma_P \Sigma_f \phi}{\lambda_P}(1 - e^{-\lambda_P t}) \quad \text{for } 0 < t < 12 \text{ hr.}$$

$$P(t) = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}\frac{e^{\lambda_P 12}}{(e^{\lambda_P 12} + 1)}e^{-\lambda_P t'} \quad \text{for } 0 < t' < 12 \text{ hr., or}$$

$$P(t) = \frac{\gamma_P \Sigma_f \phi}{\lambda_P}\frac{e^{\lambda_P 24}}{(e^{\lambda_P 12} + 1)}e^{-\lambda_P t} \quad \text{for } 12 \text{ hr.} < t < 24 \text{ hr.}$$

[10.11] Taking into account neutron capture in plutonium-239 and –240:

a. Write a rate equation for the concentration of plutonium-240
b. Solve the equation from part a. using Eq. (10.37) for the concentration of plutonium-239
c. Using the data in Table 3.2 show that your result behaves similarly to the plutonium-240 plot in Fig. 10.3.

(This problem may be solved in a simplified form wich assumes that the flux is time independent. We treat the more general case here)

Part a: Since plutonium-240 is a fissile material, we can ignore its fission cross section:

$$\frac{d}{dt}N^{40}(t) = \sigma_\gamma^{49}\phi(t)N^{49}(t) - \sigma_\gamma^{40}\phi(t)N^{40}(t)$$

Part b. Multiply by an integrating factor $\exp\left[\sigma_\gamma^{40}\int_0^t \phi(t')dt'\right]$ (See appendix A for details)

Since

$$\frac{d}{dt}\left\{N^{40}(t)\exp\left[\sigma_\gamma^{40}\int_0^t \phi(t')dt'\right]\right\} = \left[\frac{d}{dt}N^{40}(t) + \sigma_\gamma^{40}\phi(t)N^{40}(t)\right]\exp\left[\sigma_\gamma^{40}\int_0^t \phi(t')dt'\right]$$

we have

$$\frac{d}{dt}\left\{N^{40}(t)\exp\left[\sigma_\gamma^{40}\int_0^t \phi(t')dt'\right]\right\} = \sigma_\gamma^{49}\phi(t)N^{49}(t)\exp\left[\sigma_\gamma^{40}\int_0^t \phi(t')dt'\right]$$

Integrating between 0 and *t*, with $N^{40}(0) = 0$

$$N^{40}(t)e^{\sigma_\gamma^{40}\Phi(t)} = \sigma_\gamma^{49}\int_0^t \phi(t')N^{49}(t')e^{\sigma_\gamma^{40}\Phi(t')}dt'$$

where

$$\Phi(t) \equiv \int_0^t \phi(t')dt'$$

Inserting Eq. (10.37), and solving for $N^{40}(t)$:

$$N^{40}(t) = \frac{\sigma_\gamma^{49}\sigma_\gamma^{28}}{\sigma_a^{49}}N^{28}(0)e^{-\sigma_\gamma^{40}\Phi(t)}\int_0^t \phi(t')\left[1 - e^{-\sigma_a^{49}\Phi(t')}\right]e^{\sigma_\gamma^{40}\Phi(t')}dt'$$

We next note that

$$d\Phi(t) = \phi(t)dt$$

so that we may rewrite the equation as

$$N^{40}(t) = \frac{\sigma_\gamma^{49}\sigma_\gamma^{28}}{\sigma_a^{49}}N^{28}(0)e^{-\sigma_\gamma^{40}\Phi(t)}\int_0^{\Phi(t)} \left[e^{\sigma_\gamma^{40}\Phi} - e^{(\sigma_\gamma^{40} - \sigma_a^{49})\Phi}\right]d\Phi$$

$$= \frac{\sigma_\gamma^{49}\sigma_\gamma^{28}}{\sigma_a^{49}}N^{28}(0)e^{-\sigma_\gamma^{40}\Phi(t)}\left[\frac{e^{\sigma_\gamma^{40}\Phi(t)} - 1}{\sigma_\gamma^{40}} - \frac{e^{(\sigma_\gamma^{40} - \sigma_a^{49})\Phi(t)} - 1}{(\sigma_\gamma^{40} - \sigma_a^{49})}\right]$$

$$= \frac{\sigma_\gamma^{49}\sigma_\gamma^{28}}{\sigma_a^{49}} N^{28}(0) \left[\frac{1-e^{-\sigma_\gamma^{40}\Phi(t)}}{\sigma_\gamma^{40}} - \frac{e^{-\sigma_a^{49}\Phi(t)} - e^{-\sigma_\gamma^{40}\Phi(t)}}{(\sigma_\gamma^{40} - \sigma_a^{49})} \right]$$

$$N^{40}(t) = \sigma_\gamma^{49}\sigma_\gamma^{28} N^{28}(0) \left[\frac{1}{\sigma_a^{49}\sigma_\gamma^{40}} - \frac{e^{-\sigma_a^{49}\Phi(t)}}{\sigma_a^{49}(\sigma_\gamma^{40} - \sigma_a^{49})} + \frac{e^{-\sigma_\gamma^{40}\Phi(t)}}{\sigma_\gamma^{40}(\sigma_\gamma^{40} - \sigma_a^{49})} \right]$$

[10.12] Neptunium-238 has a thermal absorption cross section of 33 b, which we have neglected in deriving Eq. (10.30). In a reactor operating at a flux level of $\phi = 5\times 10^{14}$ n/cm^2/s what fraction of the neptunium will capture a neutron instead of decaying to plutonium-239?

From Eq. (10.29) $t_{1/2}^{39} = 2.36 \text{ d}\cdot(3600\cdot 24 \text{ s/d}) = 204\cdot 10^3 \text{ s}$

The capture rate in neptunium is $\sigma_a^{39}\phi N^{39}(t)$ while the decay rate is $\lambda^{39}N^{39}(t)$ Thus the fraction captured is

$$\frac{\sigma_a^{39}\phi N^{39}(t)}{\sigma_a^{39}\phi N^{39}(t) + \lambda^{39}N^{39}(t)} = \frac{1}{1+\lambda^{39}/(\sigma_a^{39}\phi)}$$

$$\lambda^{39}/(\sigma_a^{39}\phi) = 0.693/(t_{1/2}^{39}\sigma_a^{39}\phi) = \frac{0.693}{204\cdot 10^3\cdot 33\cdot 10^{-24}\cdot 5\cdot 10^{14}} = 206$$

$$\frac{1}{1+\lambda^{39}/(\sigma_a^{39}\phi)} = \frac{1}{1+206} = 0.00483 = 0.483\%$$

[10.13] Consider uranium fuel in a thermal reactor with an initial enrichment of enrichment of 4%:

a. What is the conversion ratio (CR) at the beginning of life.
b. After 50 % of the uranium-235 has been burned, what is the conversion ratio
c. After 50 % of the uranium-235 has been burned, what fraction of the power is being produced from plutonium-239

Hint: Make use of the approximate equations (10.31) and (10.37)

We begin with Eq. (10.38):

$$CR(t) = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(t) + \sigma_a^{49} N^{49}(t)}$$

Part a. At the beginning of life: $N^{49}(0) = 0$ The enrichment is

$$\tilde{e} = \frac{N^{25}(0)}{N^{25}(0) + N^{28}(0)}$$

or

$$\frac{N^{28}(0)}{N^{25}(0)} = \frac{1}{\tilde{e}} - 1 = \frac{1}{0.04} - 1 = 24$$

Using thermal cross section data from Table 3.2:

$$CR(0) = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(0)} = \frac{2.42}{591} 24.0 = 0.098$$

Part b: Substituting Eqs. (10.31) and (10.37) into the conversion ratio:

$$CR(t) = \frac{\sigma_\gamma^{28} N^{28}(0)}{\sigma_a^{25} N^{25}(0) \exp[-\sigma_a^{25}\Phi(t)] + \sigma_\gamma^{28} N^{28}(0)\left\{1 - \exp[-\sigma_a^{49}\Phi(t)]\right\}}$$

$$CR(t) = \frac{1}{\exp[-\sigma_a^{25}\Phi(t)] + CR(0)\left\{1 - \exp[-\sigma_a^{49}\Phi(t)]\right\}} CR(0)$$

For 50% burnup, from Eq. (10.31):

$$N^{25}(t) / N^{25}(0) = \exp[-\sigma_a^{25}\Phi(t)] = 0.50$$

Thus

$$\Phi(t) = -\frac{1}{\sigma_a^{25}} \ln(0.50) = -\frac{1}{591 \cdot 10^{-24}} \ln(0.50) = 1.13 \cdot 10^{21} \text{ n/cm}^2$$

$$\sigma_a^{25}\Phi(t) = -\ln(0.50) = 0.693$$

and

$$\sigma_a^{49}\Phi(t) = -\frac{\sigma_a^{49}}{\sigma_a^{25}}\left[\ln(0.50)\right] = \frac{973}{591} 0.693 = 1.14$$

Thus

$$CR(t) = \frac{1}{\exp(-0.5) + 0.098\{1 - \exp(-1.14)\}} 0.098 = 0.129$$

Part c.

$$\frac{P'''^{49}}{P'''} = \frac{P'''^{49}}{P'''^{25} + P'''^{49}} = \frac{\gamma \sigma_f^{49} N^{49}(t)\phi}{\gamma \sigma_f^{25} N^{25}(t)\phi + \gamma \sigma_f^{49} N^{49}(t)\phi}$$

Using Eqs. (10.31) and (10.37)

$$\frac{P'''^{49}}{P'''} = \frac{\sigma_f^{49}(\sigma_\gamma^{28}/\sigma_a^{49}) N^{28}(0)\{1 - \exp[-\sigma_a^{49}\Phi(t)]\}}{\sigma_f^{25} N^{25}(0)\exp[-\sigma_a^{25}\Phi(t)] + \sigma_f^{49}(\sigma_\gamma^{28}/\sigma_a^{49}) N^{28}(0)\{1 - \exp[-\sigma_a^{49}\Phi(t)]\}}$$

or

$$\frac{P'''^{49}}{P'''} = \frac{(\sigma_f^{49}/\sigma_f^{25})(\sigma_\gamma^{28}/\sigma_a^{49})[N^{28}(0)/N^{25}(0)]\{1 - \exp[-\sigma_a^{49}\Phi(t)]\}}{\exp[-\sigma_a^{25}\Phi(t)] + (\sigma_f^{49}/\sigma_f^{25})(\sigma_\gamma^{28}/\sigma_a^{49})[N^{28}(0)/N^{25}(0)]\{1 - \exp[-\sigma_a^{49}\Phi(t)]\}}$$

From Table 3.2

$$(\sigma_f^{49}/\sigma_f^{25})(\sigma_\gamma^{28}/\sigma_a^{49}) = (698/505)(2.42/973) = 3.44 \cdot 10^{-3}$$

and using the results from part b:

$$\frac{P'''^{49}}{P'''} = \frac{3.44 \cdot 10^{-3} \cdot 24\{1 - 0.320\}}{0.50 + 3.44 \cdot 10^{-3} \cdot 24\{1 - 0.320\}} = 0.101 \quad \text{or} \quad 10.1\ \%$$

[10.14] Thorium-232 is a fertile material the may be transmuted to fissile uranium-233 through the following reaction ${}^{232}_{90}Th \xrightarrow{n} {}^{233}_{90}Th \xrightarrow[22\,\text{min}]{\beta} {}^{233}_{91}Pa \xrightarrow[27.4\,\textbf{days}]{\beta} {}^{233}_{92}U$ where the half lives are indicated. Assume that a fresh core is put into operation containing only thorium-232 and uranium-235. Thereafter neutron capture in thorium takes place at a constant rate of $\Sigma_a^{th}\overline{\phi}$;

a. Assuming that the half-life ${}^{233}_{90}Th$ can be ignored, write down and solve the differential equation for the concentration of ${}^{233}_{91}Pa$.

b. Write down and solve the differential equation for the concentration of ${}^{233}_{92}U$

c. Plot your results for a. and b. for 0 < t < 6 months, normalized to $\Sigma_a^{th}\overline{\phi}$.

Part a. Assume that $N^{02}(t) \approx N^{02}(0)$

$$\frac{d}{dt}N^{13}(t) = \sigma_\gamma^{02}N^{02}(0)\phi - \lambda^{13}N^{13}(t)$$

Using an integrating factor of $\exp(\lambda^{13}t)$, and $N^{13}(0) = 0$, we obtain

$$N^{13}(t) = \frac{\sigma_\gamma^{02}N^{02}(0)\phi}{\lambda^{13}}\left[1 - \exp(-\lambda^{13}t)\right]$$

Part b:

$$\frac{d}{dt}N^{23}(t) = \lambda^{13}N^{13}(t) - \sigma_a^{23}\phi(t)N^{23}(t)$$

Substituting in $N^{13}(t)$, we obtain

$$\frac{d}{dt}N^{23}(t) = \sigma_\gamma^{02}N^{02}(0)\phi\left[1 - \exp(-\lambda^{13}t)\right] - \sigma_a^{23}\phi(t)N^{23}(t)$$

Employing an integrating factor of $\exp(\sigma_a^{23}\phi t)$ and $N^{23}(t)$ =0, we have,

$$N^{23}(t) = \exp(-\sigma_a^{23}\phi t)\sigma_\gamma^{02}N^{02}(0)\phi\int_0^t \left\{\exp(\sigma_a^{23}\phi t') - \exp[(\sigma_a^{23}\phi - \lambda^{13})t']\right\}dt'$$

$$N^{23}(t) = \exp(-\sigma_a^{23}\phi t)\sigma_\gamma^{02}N^{02}(0)\phi\left\{\frac{\exp(\sigma_a^{23}\phi t) - 1}{\sigma_a^{23}\phi} - \frac{\exp[(\sigma_a^{23}\phi - \lambda^{13})t] - 1}{\sigma_a^{23}\phi - \lambda^{13}}\right\}$$

$$N^{23}(t) = (\sigma_\gamma^{02}/\sigma_a^{23})N^{02}(0)\left\{1 - \frac{\sigma_a^{23}\phi}{\sigma_a^{23}\phi - \lambda^{13}}\exp(-\lambda^{13}t) + \frac{\lambda^{13}}{\sigma_a^{23}\phi - \lambda^{13}}\exp(-\sigma_a^{23}\phi t)\right\}$$