FEM SO FAR

PROBLEM (BVP) $\quad -\frac{\partial}{\partial x} D \frac{\partial}{\partial x}\phi + \Sigma_a \phi = S$

BC:

Reflecting: $\left.\frac{\partial \phi}{\partial x}\right|_{x = x_L / x_R} = 0$ /

Vacuum: Do ~~angular flux~~ $\phi \rightarrow$ linear in angle

Define J^+ & J^-

finally $\left. D\frac{\partial \phi}{\partial x}\right|_{x_L} = \left.\frac{\phi}{2}\right|_{x_L}$ /

$\left. D\frac{\partial}{\partial x}\phi\right|_{x_R} = -\frac{\phi}{2}$ or $D\left|\frac{\partial \phi}{\partial x}\right|_{x_R} = \left.\frac{\phi}{2}\right|_{x_R}$ /

OR

$\left.\phi\right|_{x = x_L / x_R + \frac{2}{3}\lambda} = 0$ /

FEM:

① Divide domain into subdomains (elements)

$x=0$ ———— $x=L$

⇓

element: E_1 E_2 E_3

$x=0$ ———— $x=L$

position: x_1 x_2 x_3 x_4

node: 1 2 3 4

Element	nodes
E_1	1, 2
E_2	2, 3
E_3	3, 4

4 nodes ⇔ 3 ~~elements~~ elements.

② Write equations in weighted residual form, develop weak form.

- Do it over entire domain, choose functions with compact support (eg hat fn)
- or do it over elements
- either way is okay to think about it & go about doing FEM.

Strong: $-\partial_x \phi + \Sigma_a \phi = S$

WR:

$$\int_{x_i}^{x_{i+1}} dx \, \omega \left(-\frac{\partial}{\partial x} D \frac{\partial \phi}{\partial x} + \Sigma_a \phi - S \right) = 0$$

arbitrary weight function

minimize the residual against an arbitrary weight ω.

- Can think of this as a projection of residual on ~~w~~ space of w and ~~minimizing this projection~~ setting this projection to 0.

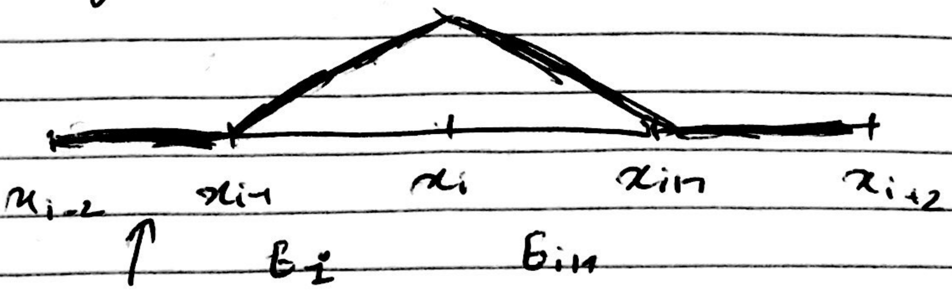
<u>weak form:</u> Integrate 1st term of weighted residual form by parts:

$$\boxed{D_i \int_{x_i}^{x_{i+1}} \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial \phi}{\partial x}\right) dx - \left[w D_i \frac{\partial \phi}{\partial x}\right]_{x_i}^{x_{i+1}} + \Sigma_{a_i} \int_{x_i}^{x_{i+1}} w \phi \, dx = \int_{x_i}^{x_{i+1}} w S \, dx.}$$

- weak form because $\frac{\partial^2 \phi}{\partial x^2}$ reduced to $\frac{\partial \phi}{\partial x}$
- ϕ no longer needs to have a 2nd derivative.

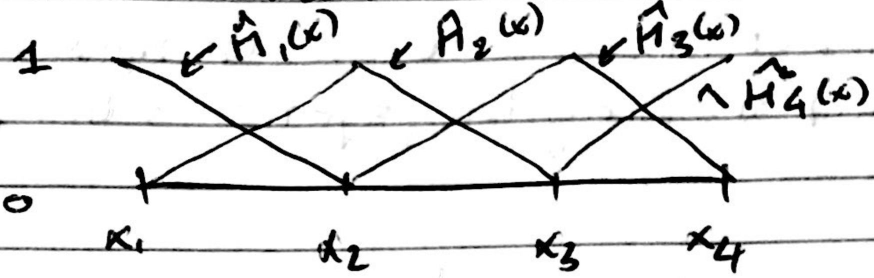
(3) Choose weight & trial function.

<u>hat function</u>



Compact support → means f^n non zero only ~~locally~~ locally around node.

Over entire domain this translates to



when we do weighted residual over entire domain

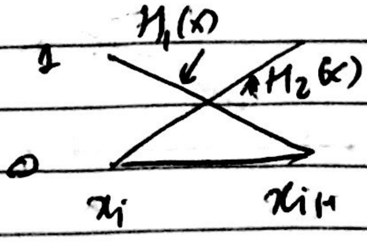
$$\int_{x_0}^{x_L} dx \; \omega(\cdots) = 0 \quad \& \text{ choose } \phi(x) = \sum_{k=1}^{N} H_k(x)\phi_k$$

k = # of nodes on domain.

& choose hat functions - or other functions with compact support we get discrete eqn.

We are essentially doing ~~the~~ something similar but we tweak things a little.

in context of elementwise procedure that we are doing here, for each element, ~~the~~ we have



$$H_1(x) = \frac{x_{i+1} - x}{x_{i+1} - x_i} = \frac{x_{i+1} - x}{h_i}$$

$$H_2(x) = \frac{x - x_i}{x_{i+1} - x_i} = \frac{x - x_i}{h_i}$$

Moreover, for each element,

$$\phi(x) = H_1(x)\phi_i + H_2(x)\phi_{i+1}$$ ← Trial function.

Two ~~diff~~ families of character for weight function:

Galerkin - wt. f^n has same form as trial f^n.

Petrov-Galerkin - wt. f^n has different form.

We do Galerkin, so: $\omega = H_1(x)\omega_1 + H_2(x)\omega_2$.

$$\phi = \begin{bmatrix} H_1(x) & H_2(x) \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_{i+1} \end{bmatrix} = [H](\tilde{\phi})$$

$$\omega = \omega^T = \begin{bmatrix} \omega_1 & \omega_2 \end{bmatrix} \begin{bmatrix} H_1(x) \\ H_2(x) \end{bmatrix} = [\tilde{\omega}]^T [H]^T$$

(4) Equation for each element:

$$\begin{bmatrix} D_i/h_i + \Sigma a_i h_i/3 & -D_i/h_i + \Sigma a_i h_i/6 \\ -D_i/h_i + \Sigma a_i h_i/6 & D_i/h_i + \Sigma a_i h_i/3 \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_{i+1} \end{bmatrix}$$

$$- \begin{bmatrix} -D_i \frac{d\phi}{dx}\Big|_{x_i} \\ D_i \frac{d\phi}{dx}\Big|_{x_{i+1}} \end{bmatrix} = S_i \begin{bmatrix} h_i/2 \\ h_i/2 \end{bmatrix} \quad //$$

← Boundary term

Boundary conditions:

Vacuum:

$$\begin{bmatrix} -D_i \frac{d\phi}{dx}\Big|_{x_i} \\ D_i \frac{d\phi}{dx}\Big|_{x_{i+1}} \end{bmatrix}$$

left: $J^+ = 0 \Rightarrow \frac{\phi(x)}{2} = D\frac{d\phi}{dx}\phi(x)$

then on left boundary, we have for the element.

$$\begin{bmatrix} -\phi(x)/2 \\ D_i \frac{d\phi}{dx}\Big|_{x_{i+1}} \end{bmatrix}$$

right: $J^- = 0 \Rightarrow \frac{\phi(x)}{2} = -D\frac{d\phi}{dx}$

then at right boundary, we have

$$\begin{bmatrix} -D_i \frac{d\phi}{dx}\Big|_{x_i} \\ -\frac{\phi(x)}{2} \end{bmatrix}$$

Reflecting: $\frac{d\phi}{dx} = 0.$

Left:

$$\begin{bmatrix} 0 \\ D_i \frac{d\phi}{dx}\Big|_{x_{i+1}} \end{bmatrix}$$

right:

$$\begin{bmatrix} -D_i \frac{d\phi}{dx}\Big|_{x_i} \\ 0 \end{bmatrix}$$

$\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4$

$E_1 \quad E_2 \quad E_3$

(5) Global matrix assembly:

we have considered 3 elements: E_1, E_2, E_3

$E_1 \rightarrow$ nodes 1, 2
$E_2 \rightarrow$ nodes 2, 3
$E_3 \rightarrow$ nodes 3, 4

} note that there are common nodes at interfaces of each element.

Matrix eqn. for element 2 (interior element)

$$\begin{bmatrix} D_i/h_i + \Sigma a_i h_i/3 & -D_i/h_i + \Sigma a_i h_i/6 \\ -D_i/h_i + \Sigma a_i h_i/6 & D_i/h_i + \Sigma a_i h_i/3 \end{bmatrix} \begin{bmatrix} \phi_i \\ \phi_{i+1} \end{bmatrix}$$

$$- \begin{bmatrix} -D_i \left.\dfrac{d\phi}{dx}\right|_{x_i} \\ D_i \left.\dfrac{d\phi}{dx}\right|_{x_{i+1}} \end{bmatrix} = S_i \begin{bmatrix} h_i/2 \\ h_i/2 \end{bmatrix}$$

$$\begin{bmatrix} D_2/h_2 + \Sigma a_2 h_2/3 & -D_2/h_2 + \Sigma a_2 h_2/6 \\ -D_2/h_2 + \Sigma a_2 h_2/6 & D_2/h_2 + \Sigma a_2 h_2/3 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix}$$

$$- \begin{bmatrix} -D_2 \left.\dfrac{d\phi}{dx}\right|_{x_2} \\ D_2 \left.\dfrac{d\phi}{dx}\right|_{x_3} \end{bmatrix} = S_2 \begin{bmatrix} h_2/2 \\ h_2/2 \end{bmatrix}$$

Consider vacuum boundaries on both left & right.

Matrix eqn. for left boundary:

$$\begin{bmatrix} D_1/h_1 + \Sigma a_1 h_1/3 & -D_1/h_1 + \Sigma a_1 h_1/6 \\ -D_1/h_1 + \Sigma a_1 h_1/6 & D_1/h_1 + \Sigma a_1 h_1/3 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}$$

$$- \begin{bmatrix} -\phi_1/2 \\ D_1 \frac{d\phi}{dx}\Big|_{x_2} \end{bmatrix} = S_1 \begin{bmatrix} h_1/2 \\ h_1/2 \end{bmatrix}$$

right boundary.

$$\begin{bmatrix} D_3/h_3 + \Sigma a_3 h_3/3 & -D_3/h_3 + \Sigma a_3 h_3/6 \\ -D_3/h_3 + \Sigma a_3 h_3/6 & D_3/h_3 + \Sigma a_3 h_3/6 \end{bmatrix} \begin{bmatrix} \phi_3 \\ \phi_4 \end{bmatrix}$$

$$- \begin{bmatrix} -D_i \frac{d\phi}{dx}\Big|_{x_3} \\ -\frac{\phi_4}{2} \end{bmatrix} = S_3 \begin{bmatrix} h_3/2 \\ h_3/2 \end{bmatrix}$$

Now we combine the 3 matrix equations.
Note that each equation is a 2×2 ~~eqn.~~ system.

Moreover note that there are 4 nodes - ~~4 ϕ unknown~~
$\Rightarrow$ 4 ϕ's are unknown $\rightarrow \phi_1, \phi_2, \phi_3, \phi_4$

In order to see how things are to be assembled, lets start look at the boundary terms in each equation

we have:

① E_1:

$$\begin{bmatrix} -\dfrac{\phi_1}{2} \\ D_1 \dfrac{d\phi}{dx}\Big|_{x_2} \end{bmatrix} \quad \text{at } x_2$$

E_2:

$$\begin{bmatrix} -D_2 \dfrac{d\phi}{dx}\Big|_{x_2} \\ D_2 \dfrac{d\phi}{dx}\Big|_{x_3} \end{bmatrix} \quad \text{at } x_3$$

E_3:

$$\begin{bmatrix} -D_3 \dfrac{d\phi}{dx}\Big|_{x_3} \\ -\dfrac{\phi_4}{2} \end{bmatrix}$$

According to interface condition, ϕ & J at the interface must be conserved.

$$\Rightarrow \quad \phi_{x^-} = \phi_{x^+}$$

~~Also~~, $J_{x_i^-} = \cancel{D\phi} - D_i \dfrac{d\phi}{dx}\Big|_{x_i} = D_{i+1} \dfrac{d\phi}{dx}\Big|_{x_i} = J_{x_i^+}$

Then when we combine these, values at common evaluation points get added (by superposition)

$\Rightarrow$ we have

by interface condition

$$\begin{bmatrix} -\phi_1/2 \\ D_1 \frac{d\phi}{dx}\Big|_{x_2} - D_2 \frac{d\phi}{dx}\Big|_{x_2} = 0 \\ D_2 \frac{d\phi}{dx}\Big|_{x_3} - D_3 \frac{d\phi}{dx}\Big|_{x_3} \\ \frac{\phi_4}{2} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -\phi_1/2 \\ 0 \\ 0 \\ -\phi_4/2 \end{bmatrix}$$

Similarly src term becomes

$$\begin{bmatrix} S_1 h_1/2 \\ S_1 h_1/2 + S_2 h_2/2 \\ S_2 h_2/2 + S_3 h_3/2 \\ S_3 h_3/2 \end{bmatrix}$$

finally ~~stiffness~~ global matrix for 2x2 matrices

$$
\begin{array}{c|cccc}
 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\
\hline
\phi_1 & \frac{D_1}{h_1}+\frac{\varepsilon a_1 h_1}{3} & \frac{-D_1}{h_1}+\frac{\varepsilon a_1 h_1}{6} & & \\
\phi_2 & \frac{-D_1}{h_1}+\frac{\varepsilon a_1 h_1}{6} & \frac{D_1}{h_1}+\frac{\varepsilon a_1 h_1}{3} & & \\
 & & \frac{D_2}{h_2}+\frac{\varepsilon a_2 h_2}{3} & \frac{-D_2}{h_2}+\frac{\varepsilon a_2 h_2}{6} & \\
\phi_3 & & \frac{-D_2}{h_2}+\frac{\varepsilon a_2 h_2}{6} & \frac{D_2}{h_2}+\frac{\varepsilon a_2 h_2}{3} & \\
 & & & \frac{D_3}{h_3}+\frac{\varepsilon a_3 h_3}{3} & \frac{-D_3}{h_3}+\frac{\varepsilon a_3 h_3}{6} \\
\phi_4 & & & \frac{-D_3}{h_3}+\frac{\varepsilon a_3 h_3}{6} & \frac{D_3}{h_3}+\frac{\varepsilon a_3 h_3}{3}
\end{array}
$$

$\leftarrow e^1$ for $e^1_{(1,1)}$, $e^2_{(2,2)}$, ϕ_2, ϕ_2 common so combine (add)

$\leftarrow e^2$

common ϕ_3, ϕ_3 so add.

e^3

$x^{1/2}$

$$
A = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\left[\begin{array}{c}\frac{D_1}{h_1}+\frac{\varepsilon a_1 h_1}{3}\\[4pt] \frac{-D_1}{h_1}+\frac{\varepsilon a_1 h_1}{6}\\[4pt] 0\\[4pt] \\ \end{array}\right.
&
\begin{array}{c}\frac{-D_1}{h_1}+\frac{\varepsilon a_1 h_1}{6}\\[4pt] \frac{D_1}{h_1}+\frac{D_2}{h_2}+\frac{\varepsilon a_1 h_1}{3}+\frac{\varepsilon a_2 h_2}{3}\\[4pt] -\frac{D_2}{h_2}+\frac{\varepsilon a_2 h_2}{6}\\[4pt] \\ \end{array}
&
\begin{array}{c}\\[4pt] \frac{-D_2}{h_2}+\frac{\varepsilon a_2 h_2}{6}\\[4pt] \frac{D_2}{h_2}+\frac{D_3}{h_3}+\frac{\varepsilon a_2 h_2}{3}+\frac{\varepsilon a_3 h_3}{3}\\[4pt] \frac{-D_3}{h_3}+\frac{\varepsilon a_3 h_3}{6}\end{array}
&
\left.\begin{array}{c}\\[4pt] \\[4pt] \frac{-D_3}{h_3}+\frac{\varepsilon a_3 h_3}{6}\\[4pt] \frac{D_3}{h_3}+\frac{\varepsilon a_3 h_3}{3}\end{array}\right]
\end{array}
$$

$+1/2$

Then the matrix system is

$\cancel{[A][\Phi]}$ $\cancel{=}$ $[\Phi]$

$$\begin{bmatrix} A & \\ & \end{bmatrix} \begin{bmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \end{bmatrix} = \begin{bmatrix} -\Phi_1/2 \\ 0 \\ 0 \\ -\Phi_4/2 \end{bmatrix} = [S]$$

↳ add $1/2$ to (1,1) term
& (4,4) term to incorporate
vacuum boundary.

Eventually, we have

$\Rightarrow [L][\Phi] = [S]$

Find stiffness matrix

or simply $L\Phi = S$

$\cancel{\Phi =}$

(6) Solve system

$\hat{L}\Phi = S$

$\Rightarrow \underline{\Phi = \hat{L}^{-1} S}$

in matlab
$\underline{\Phi = L \backslash S;}$

Gauss Jacobi

First thing we do is divide operator (matrix) $\hat{L}$ into upper, lower & diagonal components.

$$\hat{L} = \underset{\text{lower}}{L} + \underset{\text{upper}}{U} + \underset{\text{diag}}{D}$$

Now,

$$\hat{L}\phi = S$$

$$\Rightarrow (L+U+D)\phi = S.$$

$$\Rightarrow D\phi = -(L+U)\phi + S$$

Lets call $-(L+U) \rightarrow T$.

$$\Rightarrow D\phi = T\phi + S$$

$$\Rightarrow \phi = D^{-1}T\phi + D^{-1}S.$$

Obviously we don't know ϕ so we ~~try~~ make the equation iterative

$$\boxed{\phi^{n} = D^{-1}T\phi^{n-1} + D^{-1}S.}$$

Note that $T\phi^{n-1}$ is just a matrix vector product now. Algorithm:

1. $\hat{L} \rightarrow D+L+U$.
2. Guess ϕ so that $\phi^{n-1} \rightarrow$ same minimization. Usually choose close to expected sol^n ϕ_0
3. $\phi^{n} = D^{-1}T\phi^{n-1} + D^{-1}S$
4. find nor: $\|\phi^{n} - \phi^{n-1}\| / \|\phi^{n}\|$ & check for convergence

Gauss Seidel :

We have the same decomposition for $\hat{L}$

$$\hat{L} = L + U + D$$

$$\hat{L}\phi = S$$

then $(L+U+D)\phi = S$ call T

$\Rightarrow \quad (L+D)\phi = -U\phi + S$

$\Rightarrow \quad (L+D)\phi^{n} = T\phi^{n-1} + S$

$\Rightarrow \quad \phi^{n} = (L+D)^{-1} T\phi^{n-1} + (L+D)^{-1} S$

$\hookrightarrow \quad \boxed{\phi^{n} = (L+D)^{-1} T\phi^{n-1} + (L+D)^{-1} S}$