SET THEORY AND ALGEBRA

SETS

1. SETS AND SUBSETS

Set: well-defined unorderd collection of distinct elements

Ex: $A = \{1,2,3,4\}$

$S = \{$set of all students in a class$\}$

Nullset: Set with No elements is called Nullset, denoted as ϕ (or) $\{\}$

subset: If every ele of A is also an element of 'B' then A is subset of B.

$A = \{1,2,3\}$ $B = \{1,2\}$ Here $(B \subseteq A)$

Note: For every set 'A', 'A' and 'ϕ' are called Trivial subsets of 'A'.

Proper subset: Any subset of 'A' which is not a trivial subset is called proper subset of 'A'.

Note: If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

2. POWER SET

Denoted by $p(A)$

⇒ If 'A' is finite set then set of all finite subsets of 'A' is called power set of 'A'. It is denoted by $p(A)$.

Ex: $A = \{a,b\}$

subsets of $A = \phi, \{a\}, \{b\}, \{a,b\}$

$p(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

⇒ If a set contains 'n' elements $(|A| = n)$ then $\underbrace{|p(A)|}_{\text{contains}} = 2^n$ elements.

3. COMPLIMENT AND DIFFERENCE

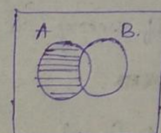
Universal set: set of all objects under discussion. denoted by 'U'

Compliment of a set: If 'A' is any set, then complement of 'A' denoted by $\overline{A}$ or A^c is called

$$A^c = \{x / x \notin A \text{ and } x \in U\}$$

Difference: If 'A' and 'B' are two sets then

$$A - B = \{x / x \in A \text{ and } x \notin B\}$$



$A = \{1,2,3,4\}$

$B = \{3,4,5,6\}$

$A - B = \{1,2\}$

4. UNION, INTERSECTION AND SYMMETRICAL DIFFERENCE

Set Intersection: $A \cap B = \{x / x \in A \text{ and } x \in B\}$

Set union: $A \cup B = \{x / x \in A \text{ OR } x \in B\}$

Note: If $A \cap B = \phi$, then 'A' and 'B' are disjoint sets.

Symmetric difference or Boolean sum: $A \triangle B / A \oplus B = \{x / x \in A \text{ or } x \in B$ but $x \notin A \cap B\}$.

$$\boxed{\begin{aligned} A \triangle B &= (A-B) \cup (B-A) \\ A \triangle B &= (A \cup B) - (A \cap B) \end{aligned}}$$

5. LAWS OF SETS

Commutative laws: (i) $A \cup B = B \cup A$

$A \cap B = B \cap A$

$A \oplus B = B \oplus A$

Associative laws:

1) $(A \cup B) \cup C = A \cup (B \cup C)$

2) $(A \cap B) \cap C = A \cap (B \cap C)$

3) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$

Distributive la

1) $A \cup (B \cap C) = (A \cup$

2) $A \cap (B \cup C) = (A \cap$

Modular laws

1) $(A \cup B) \cap C = A \cup ($

2) $(A \cap B) \cup C = A \cap (B$

3) $A \cup \phi = A$ 5)

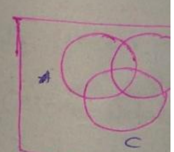
4) $A \cap \phi = \phi$ 6)

6. Example

Which of the

a) $A - (A - B) = B$

b) $A - (A - B) = ($



Use this diagra
3 sets

iii) $A - (A - B) = ($

$= \{2,3\} - \{2\} =$

(24)

Distributive laws:

1) $A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

2) $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

Modular laws

1) $(A\cup B)\cap C = A\cup(B\cap C)$ iff $A\subseteq C$

2) $(A\cap B)\cup C = A\cap(B\cup C)$ iff $C\subseteq A$

3) $A\cup\phi = A$ 5) $A\cup U = U$, $A\cap U = A$

4) $A\cap\phi = \phi$ 6) $A\cup A^c = U$ $A\cap A^c = \phi$

Demorgan law

1) $(A\cup B)^c = A^c\cap B^c$

2) $(A\cap B)^c = A^c\cap B^c$

3) $A-(B\cup C) = (A-B)\cup(A-C)$

4) $A-(B\cap C) = (A-B)\cap(A-C)$

Idempotent law

1) $A\cup A = A$

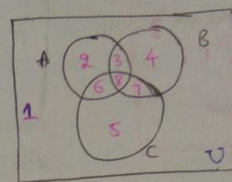
2) $B\cap B = B$

Absorption law

1) $A\cup(A\cap B) = A$

2) $A\cap(A\cup B) = A$

(3)



$A\cup(B\cap C) = \{2,3,6,8\}\cup\{8,7\}$
$= \{2,3,6,7,8\}$

$(A\cup B)\cap(A\cup C) = \{2,3,4,6,7,8\}\cap\{2,3,6,8,7,5\}$
$= \{2,3,6,8,7\}$

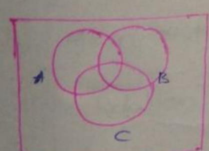
6. EXAMPLE 1

Which of the following is not TRUE?

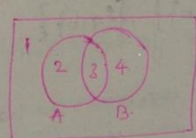
a) $A-(A-B) = B$ (c) $(A\cap B)\cup(A\cap B^c) = A$

b) $A-(A-B) = (A\cap B)$ (d) $B\cap(A\cup B) = A$

No. of Regions $= 2^n$
$n =$ no. of sets.



Use this diagram for 3 sets



for two sets.

i) $A\ \{2,3\} - \{2\} = \{3\}$
$\times$ $B = \{3,4\}$

ii) $(A\cap B)\cup(A\cap B^c) = A$
$= \{3\}\cup\{\{2,3\}\cap\{1,2\}\}$
$= \{3\}\cup\{2\} = \{2,3\}$
$= A$.

iii) $A-(A-B) = (A\cap B)$
$= \{2,3\}-\{2\} = \{3\} = A\cap B$.

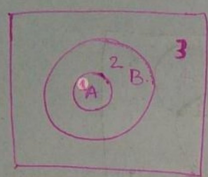
(iv) $B\cap(A\cup B) = A$
$= \{3,4\}\cap\{2,3,4\}$
$= \{3,4\} \neq A$.
$=$ False

7. EXAMPLE-2

Which of the following is not True?

a) If $A \subset B$, then $B^c \subseteq A^c$ ✓
(c) $A \cap p(A) = \phi$ ✓

b) $A \cap p(A) = A$ ✗
(d) $p(A) \cap p(p(A)) = \{\phi\}$ ✓



1> $B^c = \{3\}$, $A^c = \{2,3\}$ $\longrightarrow$ $\boxed{B^c \subset A^c \text{ (TRUE)}}$

2> $A \cap p(A) = \phi$

These are elements ← $A = \{a, b\}$

These are sets ← $p(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$A \cap p(A) = \{\phi\}$ ✗

∴ so $\boxed{A \cap p(A) = \phi}$

3> $p(A) \cap p(p(A)) = \{\phi\}$

$p(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

$p(p(A)) = \{\phi, \{\{a\}\}, \{\{b\}\}, \{\{a,b\}\}, \ldots\}$

8. EXAMPLE-3

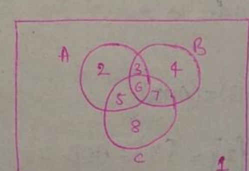
Which of the following is NOT TRUE?

a) $(A-B)-C = (A-C)-B$ ✓ = TRUE

b) $(A-B)-C = (A-C)-(B-C)$ ✓ = TRUE

c) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$ ✗ = FALSE

d) $A - (B \cup C) = (A-B) \cap (A-C)$ = TRUE



1) $(A-B) = \{2,3,5,6\} - \{3,6,7,4\} = \{2,5\} - \{5,6,7,4\}$
$= \{2\}$

$(A-C)-B = \{2,3\} - \{3,6,7,4\} = \{2\}$

(8) $A \oplus (B \cup C) = (A \oplus B) \cup (A \oplus C)$

$\{2,3,5,6\} \oplus \{3,4,6,7,5,8\}$

present here but not here (∪) and present here but not here

$= \{2\} \{2,4,7,8\}$

NOW, $(A \oplus B) = (2,5,4,7) \cup (2,3,7,8) = (2,3,4,7,8) \neq \{2,4,7,8\}$

2. RELATIO…

1. INTRODU…

Cartesian Pro…

$A \times B = \{(1, \ldots$

⇒ cartesian …

⇒ Here $|A\times$ …

⇒ In general …

⇒ we take …

Relation:

The Relation …

⇒ If $|A| = m$ …

(Since Rel…

contains …

over th…

be $2^{m \times n}$ …

Ex-1 Let R= …

Ex-2: Let R= …

Ex-3: Let A= …

R: $\{(x,y)$ …

2. RELATIONS

1. INTRODUCTION TO RELATIONS

Cartesian Product: $A = \{1,2,3\}$ $B = \{a,b\}$

$A \times B = \{(1,a)(1,b)\ (2,a)(2,b)\ (3,a)(3,b)\}$

↳ Each ele is called Orderd pair.

⇒ cartesian product = cross product

⇒ Here $|A \times B| = 3 \times 2 = 6$.

⇒ In general if $|A| = m$, $|B| = n$ then $(A \times B)$ contains $(m \times n)$ orderd pairs

⇒ we take a Relation from cartesian product.

Relation:

The Relation is a subset of cartesian product

For above example $R = \{(1,a)(1,b)\}$

$1Ra \Rightarrow (1,a) \in R$

$R' = \{(1,a)(2,a)(3,a)\}$

$R' = \{1R'a,\ 2R'a,\ 3R'a\}$

⇒ If $\boxed{|A| = m \text{ and } |B| = n \text{ then no. of Relations that can be formed} = 2^{m \times n}}$

(Since Relation is subset of cartesian product, and cross product contains $(m \times n)$ elements. Now, the no. of subsets that are possible over the cartesian product set is $2^{m \times n}$, so the no. of relations will be $2^{m \times n}$.)

Ex-1 Let $R = \{(x,y) \in Z \times Z : x < y\} = \{(1,2)(2,3)(4,5) \ldots\}$

Ex-2: Let $R = \{(x,y) \in Z \times Z : (x+y) \text{ is even}\} = \{(1,3)(1,5)(2,4)(6,4) \ldots\}$

Ex-3: Let $A = \{1,2,3,4\}$ $B = \{1,2\}$

$R : \{(x,y) \in A \times B : x+y = 3\} = \{(1,2)(2,1)\}$

2. REFLEXIVE RELATION

Reflexive Relation: A Relation 'R' on set 'A' is said to be Reflexive if $(xRx)\ \forall x \in A$

$A = \{1,2,3\}$

$A \times A = \{(1,1)(1,2)(1,3)$
$(2,1)(2,2)(2,3)$
$(3,1)(3,2)(3,3)\}$

$R = \{(1,1)(2,2)(3,3)\}$ Here 'R' is Reflexive Relation

$R_1 = \{(1,1)(2,2)\}$ is not Reflexive because it doesnot contain $(3,3)$ ($\forall x \in A$ xRx should be in R)

$R_2 = \{(1,1)(2,2)(3,3)(1,2)\}$ = Reflexive Relation

∴ Let $A = \{(1,2 \ldots n)\} \rightarrow n$ elements

$R_n = \{(1,1)(2,2) \ldots (n,n)\} \rightarrow$ Then the smallest Relation which is reflexive contains 'n' elements.

Note: If 'R' is Reflexive then any superset of 'R' is Reflexive

The largest Reflexive Relation on 'A' is "$A \times A$"

The smallest Reflexive Relation on 'A' contains 'n' elements

If $|A| = n$ then largest Reflexive Relation $= n \times n = n^2$ elements.
$|A| = n$ then smallest Reflexive Relation $= n$ elements.

3. EXAMPLE 1 ON REFLEXIVE RELATIONS

If $A = \{1,2,3 \ldots n\}$ then the no.of reflexive relations possible on 'A'?

$A = \{1,2\}$

$A \times A = \{(1,1)(1,2)(2,1)(2,2)\}$ ⟹ we have 2 choices we may or may not include it.

we have one choice

$R_1 = \{(1,1)(2,2)\}$ ✓

$R_2 = \{(1,1)(2,2)(1,2)\}$ ✓

$R_3 = \{(1,1)(2,2)(2,1)\}$ ✓

$R_4 = \{(1,1)(2,2)(2,1)(1,2)\} = A \times A$ ✓

$A = \{1,2,3\}$

$R_1 = \{(1,1)(2, \ldots$
2 2
2 2

∴ If $|A| = n$

$|A| = n$

The No.of Rel…

Non-…

4. EXAMPLE 2 …

Now whenever … an element from … element 2 times.

Ex:

1> The relation …

2> The relation … Real number…

3> The Relation … collection of …

4> The Relation …

5> $R: \{(x,y) \in$ …

$A = \{1,2,3\}$

$R_1 = \{(1,1)\ (2,2)\ (3,3)\}$ → 1 way

$\underline{2}\ \underline{2}\ \underline{2}$
$\underline{2}\ \underline{2}\ \underline{2}$ $= 2^6$ combinations

∴ The no. of Reflexive relations $= 64$.

∴ If $|A| = n$ then the no. of Reflexive Relations on $A = 2^{n^2-n}$

$$\boxed{|A| = n \Rightarrow \text{No. of Reflexive Relations on } A = 2^{n(n-1)}}$$

The No. of Relations that are not Reflexive = Total Relations − Reflexive Relations

$$= 2^{n\times n} - 2^{n^2-n}$$

$$\boxed{\text{Non-Reflexive Relations from A to A} = 2^{n^2} - 2^{n^2-n}}$$

4. EXAMPLE 2 ON REFLEXIVE RELATIONS

Now whenever they ask if a Relation is Reflexive then try to take an element from the relation and apply the condition between the same element 2 times.

Ex:

1) The relation $\leq$ is reflexive on any set of Real numbers $(x \leq x)$
2) The relation 'is a divisor of' is reflexive on a set of non-Zero Real numbers (x/x) x is divisible by 'x' = TRUE ⇒ Relation is Reflexive
3) The Relation is a subset of denoted by '$\subseteq$' is reflexive on any collection collection of sets $(A \subseteq A) \Rightarrow$ TRUE ⇒ Reflexive
4) The Relation 'is parallel to' is reflexive on a set of all lines $(L \parallel L)$ = TRUE
5) $R: \{(x,y) \in Z\times Z : x-y \text{ is even integer}\}$ $(x-x) = 0$ = even = TRUE ⇒ Reflexive

5. EXAMPLE 3 ON REFLEXIVE RELATIONS

Which of the following is False?

a) If R_1 is Reflexive then every superset of R_1 is Reflexive = TRUE

b) If R_1 is Reflexive, then subset of R_1 is reflexive = FALSE

c) If R_1, R_2 are Reflexive then $R_1 \cap R_2$ is Reflexive = TRUE

d) If $R_1 R_2$ are Reflexive then $R_1 \cup R_2$ is Reflexive = TRUE

(A)

$R = \{(1), 2, 3\}$

$R_1 = \{(1,1)(2,2)(3,3)\}$.

$R_1 = \{(1,1)(2,2)(3,3)(1,2)(1,3)(2,3)(3,1)\}$ are reflexive

(B) $R = \{1,2,3\}$.

$R_1 = \{(1,1)(2,2)(3,3)\}$

$R_2 = \{(1,1)\}$ → Not Reflexive

(C)

$R_1 = \{(1,1)(2,2)(3,3)\}$

$R_2 = \{(1,1)(2,2)(3,3)(1,2)(2,3)\}$

$R_1 \cap R_2 = \{(1,1)(2,2)(3,3)\}$

= Reflexive.

(D) $R_1 = \{(1,1)(2,2)(3,3)\}$.

$R_2 = \{(1,1)(2,2)(3,3)(1,2)(2,3)\}$

$R_1 \cup R_2 = \{(1,1)(2,2)(3,3)(1,2)(2,3)\}$

= Reflexive

6. IRREFLEXIVE RELATIONS

Irreflexive Relations:

The Relation 'R' on a set 'A' is called irreflexive if 'x' is not related to 'x' i.e $x \not R x \; \forall x \in A$ i.e the orderd pair $(x,x) \notin R \; \forall x \in A$

Ex:

If a Relation is Not reflexive then we think that it is irreflexive but this is wrong assumption.

Ex: $A = \{1,2,3\}$

$R_1 = \{(1,1)(2,2)(3,3)\}$ Reflexive, NOT Irreflexive

$R_2 = \{\}$ NOT Reflexive, Irreflexive

$R_3 = \{(1,1)\}$ NOT Reflexive, Not Irreflexive

$R_4 = \{(1,2)(2,1)\}$ NOT Reflexive, Irreflexive

There doesnot exist a relation which is both Reflexive and Irreflexive

⇒ There will be some relations which are neither reflexive n& Irreflexive

⇒ The min. set that is Irreflexive is { } and the min cardinality of a Irreflexive Relation is 'Zero'.

Now, $R_5 = A \times A$

$= \{(1,1)\ (1,2)\ (1,3)$
$(2,1)\ (2,2)\ (2,3)$
$(3,1)\ (3,2)\ (3,3)\}$

Now, Remove the diagnol elements because they make the Relation Reflexive.

Now $\{(A \times A) - (\text{diagnol elements})\} = \{(1,2)(1,3)\ (2,1)(2,3)\ (3,1)(3,2)\}$ This is the largest Irreflexive Relations

$\boxed{\text{If } |A| = n \text{ the then the cardinality of largest Irreflexive Relation will be } (n^2 - n)}$

$\{$Total no. of elements in $A \times A = (n \times n) = n^2\} - \{$Diagnol elements$\}$

7. EXAMPLE 1 ON IRREFLEXIVE RELATIONS

Now, I want to find no. of Irreflexive Relations on 'A' given $|A| = n$

⇒ then $\boxed{|A \times A| \text{ contains } n^2 \text{ elements}}$

let $A = \{1,2,3\}$

$A \times A = \{(1,1)\ (1,2)\ (1,3)$
$(2,1)\ (2,2)\ (2,3)$
$(3,1)\ (3,2)\ (3,3)\}$

⇒ 0 ways (they should not be included)

2 2 2 2 2 2

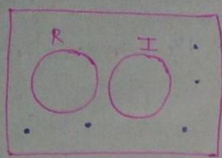
No. of Irreflexive Relations $= 2^6 = 64$

$\therefore$ In general $A \times A = \{\overbrace{\underbrace{(1,1)\ (2,2)(3,3) \cdots (n,n)}_{\text{'n' ele}},\ \underbrace{(1,2)\ (1,3) \cdots}_{n^2-n \text{ ele}}}^{n^2 \text{ elements}}\}$

2 0

$\therefore$ $\boxed{\text{The total no. of Irreflexive relations that can be formed over a set of 'n' elements is } = 2^{n^2-n}}$

Now, it is clear that



R = Reflexive Relations.

I = Irreflexive Relations

Now, No. of Relations which are either reflexive or irreflexive is $= 2^{n^2-n} + 2^{n^2-n}$

$= 2\left(2^{n^2-n}\right)$

$\therefore$ $\boxed{\text{No. of Relations that are either Reflexive or Irreflexive } 2^{n^2-n+1}}$

Now, No. of Relations that are neither Reflexive nor Irreflexive = (Total) − (No. of Reflexive + Irreflexive)

$= 2^{n^2} - \left(2^{n^2-n} + 2^{n^2-n}\right)$

$\therefore$ $\boxed{\text{No. of Relations that are Neither reflexive nor Irreflexive} = 2^{n^2} - \left(2^{n^2-n+1}\right)}$

8. EXAMPLE 2 ON IRREFLEXIVE RELATIONS

→ The Relation '<' on set of all Real numbers is Irreflexive = TRUE

→ The Relation '⊂' (→ proper subset) on set of all sets is "irreflexive" = $A \subset A$ = TRUE

→ The Relation ⊥ on set of all lines is "Irreflexive" = $x \perp x$ (A line x, x will be parllel not ⊥r ∴ Two ele (x,x) will never belong to this relation.

⇒ Fore the above examples choose a value and check the condition if it is valid then the Relation is Not IRReflexive if it fails then it is Irreflexive.

9. EXAMPLE 3 ON IRREFLEXIVE RELATION

→ which of the following is false?

→ Every subset of Irreflexive relation is irreflexive

→ Every superset of Irreflexive relation is Irreflexive

→ If R_1 is Irreflexive, R_2 is irreflexive then $R_1 \cap R_2$ is irreflexive

→ If $R_1 R_2$ are irreflexive then $R_1 \cup R_2$ is Irreflexive.

Ans

A) $A = \{1,2,3\}$

$R = \{(1,2)(1,3)$
$(2,3)(2,1)$
$(3,1)(3,2)\}$

R_1 = Subset of R

$= \{(1,2)(1,3)\}$ = IRREFLEXIVE

TRUE

(B)

$A = \{1,2,3\}$

$R = \{(1,2)(1,3)(2,3)$
$(2,1)(3,1)(3,2)$
$(1,1)(2,2)\}$

Super set of IR-Relation

= NOT IRREFLEXIVE

(C)

$A = \{(1), 2, 3\}$

$R_1 = \{(1,2)(1,3)\}$ $R_2 = \{(2,1)(2,3)\}$

$\Rightarrow R_1 \cap R_2 = \{ \} = \phi$

= IRREFLEXIVE

(D) TRUE (use same example)

10. EXAMPLE ON CLOSURE PROPERITIES

State TRUE/ FALSE

a) The set of all reflexive relations set closed under the operation set union.

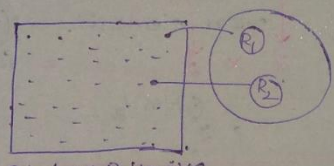
b) " " " " " " " " " " Intersection

c) X " " " " " " " " " " difference

d) The set of all irreflexive Relations are closed under the operation set union

e) " " " " " " " " " " Intersection

f) " " " " " " " " " " difference



set of all Reflexive Relations

11. SYMETRIC RELATIONS

A Relation 'R' on a set 'A' is said to be symetric if (xRy) then (yRx) $\forall x,y \in A$ i,e if the orderd pair $(x,y) \in R$ then $(y,x) \in R$ $\forall x,y \in A$.

$A = \{1,2,3\}$

$R_1 = \{(1,2)(2,1)\}$ → Symetric ✓
Reflexive ×
Irreflexive ✓

$R_2 = \{(1,1)\}$ – Symetric
Not Reflexive
Not Irreflexive

$R_3 = \{(2,3)\}$ → Not symetric
Not Reflexive
Irreflexive

$R_4 = \{(3,1)(1,1)\}$ – Not Symetric
– Not Reflexive
– Not Irreflexive

$R_5 = \{\}$ - symetric

The cardinality of the smallest symetric Relation is Zero

$R_6 = A \times A$ is symetric $\therefore$ The largest cardinality is "n^2" elements and

The largest symetric relation that is defined on a set 'A' is "$A \times A$"

12. NUMBER OF SYMETRIC RELATIONS

$A = n$ then how many Relations are possible that are symetric on $A \times A$

$|A| = n$ then $|A \times A| = n^2$

$A = \{1,2,3\}$

$$A \times A = \{(1,1)(2,2)(3,3)\boxed{(1,2)(2,1)}\boxed{(1,3)(3,1)}\boxed{(2,3)(3,2)}\}$$

$$2 \quad 2 \quad 2 \quad 2 \quad 2 \quad 2 = 2^6$$

$|A| = n = \{1,2,3 --- n\}$.

$$(A \times A) = \overbrace{\{\underbrace{(1,1)(2,2)(3,3)---(n,n)}_{n \text{ elements}}, \underbrace{\boxed{(1,2)\ (2,1)}\ \boxed{(2,3)(3,2)}}_{n^2-n \text{ elements}}\}}^{n^2 \text{ elements}}$$

$= n^2$ ele

$$2 \times 2 \times 2 \times \quad \times 2 = 2^n$$

$$2 \times 2 \ ... = 2^{(n^2-n/2)} = 2^{(n^2-n)/2}.$$

$\rightarrow$ No. of pairs that will be formed.

$$\boxed{\therefore \text{No. of symetric Relations} = 2^n \times 2^{(n^2-n)/2}. = 2^{n+\left(\frac{n^2-n}{2}\right)}}$$

13. EXAMPLE 1 ON SYMETRIC RELATIONS

$|A| = n$, now I want to find the relation between the no. of Symetric Relations and Reflexive Relations.

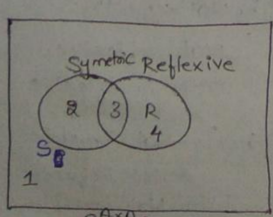
Let $A = \{1,2,3\}$

$R_1 = \{(1,2)(2,1)\}$

$R_2 = \{(1,1)(2,2)(3,3)\}$

$R_3 = \{(1,1)(2,2)(3,3)(1,2)\}$

$R_4 = \{(1,2)\}$



$\underbrace{2^{A \times A}}$ } $\rightarrow$ subset of powerset (Total no. of relations).

(12)

W.K.T the no. of symetric Relations possible are $2^n \times 2^{(n^2-n)/2} = 2^{\frac{n(n+1)}{2}}$

(13)

Now, set of all Relations that are both Symetric and Reflexive = Region 3 $= 2^{n(n-1)/2}$

$$AXA = \{\underbrace{\overbrace{(1,1)(2,2)(3,3)---(n,n)\boxed{(1,2)(2,1)}----}^{n^2 \text{ elements}}}_{}\}$$

1 way because the Relation should be reflexive (n elements).

n^2-n ele

$2^{(n^2-n)/2}$

$\therefore$ $\boxed{|S \cap R| = 2^{\frac{n(n-1)}{2}}}$ ← Symetric and Reflexive

Now, $|S-R| = |n(S)| - n(S \cap R)|$

$= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

$\therefore$ $|S-R|$ = Symetric but not Reflexive $= 2^{\frac{n(n+1)}{2}} - 2^{\frac{n(n-1)}{2}}$

Now, $|R-S| = n|R| - n|S \cap R|$

$\boxed{|R-S| = 2^{n(n-1)} - 2^{\frac{n(n-1)}{2}}}$

Now the Relations that are not Symetric and Reflexive $= n|\overline{S \cup R}|$

$= 2^{n^2} - (n(S) + n(R) - n(S \cap R))$

$= 2^{n^2} - \left(2^{\frac{n(n+1)}{2}} + 2^{n(n-1)} - \left(2^{\frac{n(n-1)}{2}}\right)\right)$

$\therefore$ The relations that are not both Reflexive and Symetric is $= 2^{n^2} - \left(2^{\frac{n(n+1)}{2}} + 2^{n^2-n} - \left(2^{\frac{n(n-1)}{2}}\right)\right)$

15. EXAMPLE 3 ON SYMETRIC RELATIONS

State which of the following are TRUE and False?

a) Every subset of symentric Relation is symetric (F)

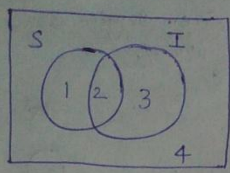
b) " superset " " " " (FALSE)

c) If R_1 and R_2 are symetric then $R_1 \cap R_2$ is symetric (TRUE)

d) " " " " " " " $R_1 \cup R_2$ " " (TRUE)

e) " " " " " " " $R_1 - R_2$ " " (TRUE).

16. RELATION BETWEEN SYMETRIC AND IRREFLEXIVE RELATION



$R_1 = \{(1,1)\} \rightarrow$ Symetric ✓ Not Irreflexive

$R_2 = \{(1,2)(2,1)\} \rightarrow$ Symetric ✓ Irreflexive ✓

$R_3 = \{(1,2)\} \rightarrow$ Symetric × Irreflexive ✓

$R_4 = \{(1,2)(1,1)\} \rightarrow$ Symetric × Irreflexive ×

WKT $n(S) = 2^{\frac{n(n+1)}{2}}$

$n(I) = 2^{n^2-n}$

Now, $n|S \cap I|$ = Both Symetric and Irreflexive

$$A \times A = \{\underbrace{(1,1)(2,2) \; --- \; (n,n)}_{\times} \boxed{(1,2)(2,1)} \; --- \}.$$

$$2 \quad 2 \quad 2 \; -- \; \frac{(n^2-n)}{2} \text{ times}$$

$$\therefore \boxed{n|S \cap I| = 2^{(n^2-n)/2}}$$

$n(S \cup I) = n(S) + n(I) - n(S \cap I) \rightarrow$ Either Symetric or Reflexive

$\therefore$ The no. of Relations that are Either Symetric or Irreflexive $= \left(2^{\frac{n(n+1)}{2}} + 2^{n^2-n}\right) - \left(2^{(n^2-n)/2}\right)$

$n(S-I) = n(S) - n(S \cap I)$

$\Rightarrow$ The no. of Relations that are symetric but not Irreflexive $= 2^{\frac{n(n+1)}{2}} - 2^{(n^2-n)/2}$

$n(I-S) = n(I) - n(S \cap I)$

$\Rightarrow$ The no of Relations that are but not symetric is $= 2^{n^2-n} - 2^{(n^2-n)/2}$

$n(\overline{I \cup S})$ = Relations that are neither Irreflexive nor Symetric $= n(U) - n(I \cup S) = 2^{n^2} - n(I \cup S)$

17. ANTISY

(14) A Relation

$\forall x, y \in A$.

$A = \{1, 2,$

$\therefore$ The min ca

Now, $A \times A = \{(1,1)$

$(1,2)$

$(2,3$

$(3,1$

Now, if $A \times A =$

$\therefore$ The max ca

17. ANTISYMETRIC RELATION

A Relation 'R' is said to be Antisymetric if (xRy and yRx) then $x=y$ $\forall x,y \in A$.

$A = \{1,2,3\}$

$R_1 = \{(1,2)\ (2,1)\}$ → NOT Antisymetric because we have if $(x,y) \in R$ then $(y,x) \in R$.

$R_2 = \{(1,1)\}$ → Antisymetric (This is the only exception (the pairs of type (x,x) are allowed).

symetric ✓ (we cant say symetric & Antisymetric are compliment to eachother).

$R_3 = \{(1,2)\ (1,1)\}$ ⇒ Not symetric but it is Antisymetric

$R_4 = \{(1,2)(2,1)(2,3)\}$ ⇒ Not Antisymetric. $(x,y) \in R$ & $(y,x) \in R$.
Not Symetric $(3,2)$ is not present.

$R_5 = \{(1,1)\ (2,2)(3,3)\}$ → Both Symetric and Antisymetric (Exceptional case).

$R_6 = \{\ \}$ – Antisymetric ✓

∴ $\boxed{\text{The min cardinality of Antisymetric Relation} = 0}$

$R_7 = \{(1,1)(2,2)(3,3)(1,2)(2,3)(3,1)\}$ = largest Antisymetric Relation

Now, $A \times A = \{(1,1)\ (2,2)(3,3)$
$(1,2)\ (2,1)$
$(2,3)\ (3,2)$
$(3,1)\ (1,3)\}$

Now, if $A \times A = \{\overbrace{\underbrace{(1,1)\ (2,2) \ldots (n,n)}_{n \text{ elements}}\ \underbrace{(1,2)\ (2,1)\ (3,1)(1,3) \ldots}_{n^2-n \text{ elements}}}^{n^2 \text{ elements}}\}$.

n elements → choose All

n^2-n elements: choose $\frac{n^2-n}{2}$ ele

∴ The max cardinality of Antisymetric Relation $= \left(n + \frac{n^2-n}{2}\right)$

18. NUMBER OF ANTISYMETRIC RELATIONS.

$|A| = n \quad |A \times A| = n^2$

$A = \{1,2,3\} \quad A\times A = \{(1,1)\ (2,2)\ (3,3)$ — $2 \times 2 \times 2$

$(1,2)\ (2,1)\ (2,3)$ → 3

$(3,2)\ (1,3)\ (3,1)\}$ → 3

→ case 1: Include (1,2)

2: Include (2,1)

3: Include None

$= 2^3 \times 3^3$ Relations

$= 2^{\text{diagnol ele}} \times 3^{\text{Nondiag pairs}}$

$\therefore$ No. of Antisymetric Relations possible on A where $|A| = n$ is $= \left(2^n \times 3^{\frac{n^2-n}{2}}\right)$

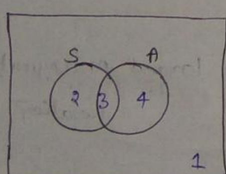
⇓

$A \times A = \{\underbrace{(1,1)\ (2,2)\ (3,3)\ \ldots\ (n,n)}_{n}\ \underbrace{(1,2)\ (2,1)}_{3}\ \underbrace{(3,2)\ (2,3)}_{3}\ \ldots\}$

$3 \ldots \frac{n^2-n}{2}$ times

$\Rightarrow 2^n \times 3^{\frac{n^2-n}{2}}$ Relations

19. RELATION BETWEEN SYMETRIC AND ANTISYMETRIC RELATIONS



$|U| = 2^{n^2}$

$n(S) = 2^{n(n+1)/2}$

$n(A) = 2^n \times 3^{\frac{n^2-n}{2}}$

$R_1 = \{(1,2)\ (2,1)\ (2,3)\} \rightarrow$ Antisymetric ✗ Symetric ✗

$R_2 = \{(1,2)\ (2,1)\} \rightarrow$ S ✓ AS ✗

$R_3 = \{\ \} \rightarrow$ AS ✓ S ✓

$R_4 = \{(1,1)\} \rightarrow$ AS ✓ S ✓

$R_5 = \{(2,1)\} \rightarrow$ AS ✓ S ✗

$A \times A = \{\underbrace{(1,1)\ (2,2)\ \ldots\ (n,n)}_{\text{Choose All}}\ (1,2)\ (2,1)\ \ldots\}$.

$= 2^n$

$\therefore$ The no. of Relations that will be Both symetric and Anti symetric $= 2^n$ → only diagonal elements

(6) Now, the no. of Relations that are either Symetric / Antisymetric are

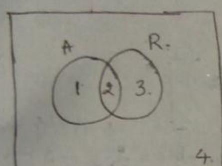
$$n(S \cup A) = n(S) + n(A) - n(S \cap A)$$ (A)

$$\boxed{n(S \cup A) = \left(2^{\frac{n(n+1)}{2}}\right) + \left(2^{n} \times 3^{(n^2-n)/2}\right) - \left(2^{n}\right)}$$

Similarly $n(S-A)$ and $n(A-S)$ can be found

$$\boxed{\begin{aligned} n(S-A) &= n(S) - n(S \cap A) \\ n(A-S) &= n(A) - n(S \cap A) \end{aligned}}$$

20. RELATION BETWEEN REFLEXIVE AND ANTISYMETRIC RELATIONS



$R_1 = \{(1,2)\}$

$R_2 = \{(1,1)\ (2,2)\ ---\ (n,n)\}$

$R_3 = \{(1,1)\ (2,2)\ --\ (n,n)\ (1,2)(2,1)\}$.

$R_4 = \{(2,1)\ (1,2)\}$

$n(A) = 2^{n}\ 3^{n(n-1)/2}$

$n(R) = 2^{n(n-1)}$

$n(u) = 2^{n^2}$

$A \times A = \{\underbrace{(1,1)\ (2,2)\ ---\ (n,n)}\ \boxed{(1,2)(2,1)}\ ---\ \}$.

$3 \times 3\ ---$

$3^{n(n-1)/2}$

Both Reflexive & Antisymetric $= 1 \times 3^{n(n-1)/2} = \boxed{3^{n(n-1)/2} = n(A \cap R)}$

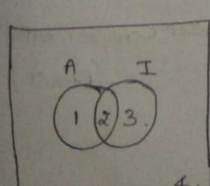
$n(A \cup R) = n(A) + n(R) - n(A \cap R) =$ Both Antisymetric and Reflexive

$n(A-R) = n(A) - n(A \cap R) =$ Antisymetric but not Reflexive

$n(R-A) = n(R) - n(A \cap R) =$ Reflexive but not Antisymetric

$n(\overline{A \cup R}) = n(U) - n(R \cup A) =$ Not Reflexive and not Antisymetric

21. RELATION BETWEEN IRREFLEXIVE AND ANTISYMETRIC RELATIONS



$R_1 = \{(1,1)\} \Rightarrow A$ ✓ IR ✗

$R_2 = \{(1,2)\} \Rightarrow A$ ✓ IR ✓

$R_3 = \{(1,2)(2,1)\} \Rightarrow A$ ✗ IR ✓

$R_4 = \{(1,1)(1,2)(2,1)\}$. A ✗ IR ✗

$n(U) = 2^{n^2}$

$n(A) = 2^{n}\ 3^{n(n-1)/2}$

$n(I) = 2^{n(n-1)}$.

$A \times A = \{(1,1)\ (2,2)\ \ldots\ (n,n)\ (1,2)\ (2,1)\ (2,3)\ (3,2)\ \ldots\}$.

The Relations that are Both Irreflexive and Antisymetric

$A \times A = \{\underbrace{(1,1)\ (2,2)\ \ldots\ (n,n)}_{\text{If i Include this they become reflexive so dont Include them}}\ \underbrace{\boxed{(1,2)\ (2,1)}\ \boxed{(2,3)\ (3,2)}\ \ldots}_{\text{3 choices}\quad \text{3 choices}\quad 3^{(n^2-n)/2}}\}$.

If i Include this
they become reflexive
so dont Include them

3choices 3 choices

$3^{(n^2-n)/2}$

$\boxed{\therefore \text{ The no. of Relations that are Both Antisymetric and Irreflexive is } = 3^{(n^2-n)/2}}$

Now, $n(A-I) = n(A) - n(A \cap I) = \left[\left(2^n\, 3^{n(n-1)/2} - 3^{(n^2-n)/2}\right)\right]$

$n(I-A) = n(I) - n(A \cap I)$ = Irreflexive but not Antisymetric $= 2^{n(n-1)} - 3^{(n^2-n)/2}$

$n(A \cup I) = n(A) + n(I) - n(A \cap I)$ = No. of Antisymetric but not Reflexive $= \left(2^n\, 3^{n(n-1)/2}\right) + 2^{n^2-n} - 3^{(n^2-n)/2}$

$n(\overline{A \cup I}) = n(U) - n(A \cup I)$

$2^{n^2} - \left(3^{(n^2-n)/2}\right)$

22. ANTISYMETRIC PROPERITIES.

State TRUE / FALSE

a) Every subset of Antisymetric relation is Antisymetric (TRUE)

b) " Superset " " " " " " " (FALSE)

c) Anti symetric relations are closed under set union. (FALSE)

d) " " " " " " Intersection (TRUE).

e) " " " " " " difference (TRUE)

f) " " " " " " Set Complementation - (FALSE)

23. EXAMPLES ON ANTI SYMETRIC RELATION

1) The Relation '≤' is Antisymmetric on any set of Real numbers

2) The Relation '<' is Antisymmetric on any set of Real numbers

3) The Relation "is a divisor" of denoted as '/' is an antisymetric on any set of +ve Real numbers.

4) The Relation '⊆' (set inclusion) is Anti symetric on any collection of sets

1) $x \le y$ then (y,x) will not be present and ($y \le x$ is false) ∴ The relation is Antisymetric

2) $x < y$ the ($y < x$) is false, so (y,x) will not be present in Relation therefore the relation is Anti symetric.

3) x/y (i mean if y is divisible by x)
↳ x' is divisor of 'y' then 'y' wont be divisor of x)
2/4 (4 is ÷ble by 2) then (4/2) (2 wont be ÷ble by 5).

4) Antisymetric.

24. ASYMETRIC RELATIONS

A Relation 'R' on set 'A' is said to be Assymetric if $(x R y)$ then $(y \not R x)$ $\forall\, x,y \in A$

$A = \{1,2,3\}$

$R_1 = \{(1,2)\}$ Assymetric Relation (because (x,y) is present but (y,x) is not present).

$R_2 = \{(1,2)\ (2,2)\}$ Not Assymetric (because we included x,x)
Anti symetric (✓) because (x,y) is present and (y,x) is not present and $(2,2)$ is allowed case

Note:. Diagnol elements they can be present in Antisymetric but not in Asymetric

$R_3 = \{\ \}$ Asymetric, Antisymetric, symmetric

∴ The min. cardinality of smallest Asymetric Relation = '0'.

$R_4 = \{(1,2)(2,1)\}$ → Antisymmetric ✗ Asymetric ✗ Symetric ✓

$A \times A = \{(1,1)(2,2)(3,3)(1,3)(3,1)$
$(1,2)(2,1)$
$(2,3)(3,2)\}$.

$R_5 = \{(1,3)\ (1,2)\ (2,3)\}$ — This is the largest asymetric Relation.

Now, $A \times A = \{(1,1)\ (2,2) - - - (n,n)\ \boxed{(1,2)\ (2,1)}\ \boxed{(3,2)\ (2,3)} - - \}$

They violate the Assymetric property.

Include one of $(1,2)$ & $(2,1)$

∴ $\boxed{\text{The cardinality of the largest Asymetric Relation that is possible with a set with 'n' elements is } = \left(\frac{n^2-n}{2}\right)}$

25. NO. OF ASYMERIC RELATIONS

$|A| = n \quad |A \times A| = n \times n$

Now, $|A \times A| = \{\underbrace{(1,1)\ (2,2) - - - (n,n)}_{\text{should not Include it}}\ \boxed{(1,2)\ (2,1)}\ \boxed{(3,2)\ (2,3)}\ \boxed{(1,3)\ (3,1)} - - - \}$

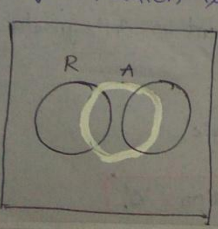
$\boxed{(1,2)\ (2,1)} \to 3C1$, $\boxed{(3,2)\ (2,3)} \to 3C1$, $\boxed{(1,3)\ (3,1)} \to 3C1$

$= 3 \times 3 \times 3 - - \left(\frac{n^2-n}{2}\right)$ times.

∴ $\boxed{\text{The no. of Assymetric Relations} = 3^{\left(\frac{n^2-n}{2}\right)}}$

26. REFLEXIVE AND ASSYMETRIC RELATIONS

→ If a Relation is Reflexive then it cannot be Assymetric



$n(A) = 3^{\frac{n(n-1)}{2}}$

$n(R) = 2^{n(n-1)}$

$n(A \cap R) = 0$

$n(A - R) = n(A)$

$n(R - A) = n(R)$

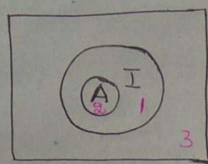
$n(\overline{R \cup A}) = n(U) - n(R \cup A)$

$n(A \cup R) = n(A) \cup n(R) = n(A) + n(R)$.

27. IRREFLEXIVE AND ASYMETRIC RELATIONS

⇒ Every Assymetric Relation is Irreflexive but

⇒ Every Irreflexive Relation is not Assymetric



$n(I) = 2^{n(n-1)}$

$n(A) = 3^{n(n-1)/2}$

$n(I \cup A) = n(I)$

$n(I \cap A) = n(A)$

$n(I - A) = n(I) - n(A)$

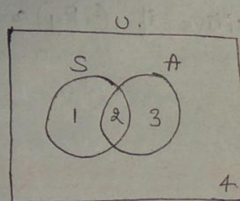
$n(A - I) = 0.$

$n(\overline{I \cup A}) =$

$= n(U) - n(I \cup A)$

$= n(U) - n(I)$

28. SYMETRIC AND ASYMETRIC RELATIONS



$R = \{(1,1)\}$ S ✓ AS ×

$R_2 = \{\ \}$ S ✓ AS ✓

$R_3 = \{(1,2)\}$ S × AS ✓

$R_4 = \{(1,1)(1,2)\}$ S × AS ×

$n(S) = 2^{\frac{n(n-1)}{2}}$

$n(A) = 3^{\frac{n(n-1)}{2}}$

$n(S \cap A) = 1$

Now, $n(S) \in n(S) - n(A \cap S)$ $n(S \cap A) = 1$

$n(S \cup A) = n(S) + n(A) - n(S \cap A)$

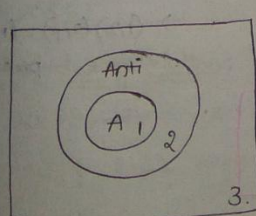
$n(\overline{S \cup A}) = n(U) - n(S \cup A)$

$n(S - A) = \{n(S) - n(S \cap A)\}$

$n(A - S) = \{n(A) - n(S \cap A)\}.$

29. ANTISYMETRIC AND ASYMETRIC RELATIONS.

Every Asymetric Relation is Antisymetric



$R_1 = \{(1,1)(1,2)\}$ Antisymetric ✓ Asymetric ×

$R_2 = \{(1,2)(2,1)\}$ Anti Symetric × Asymetric ×

$R_3 = \{(1,2)\}$ Anti Symetric ✓ Asymetric ✓

Now, $= n(A \cup Anti) = n(Anti)$

$= n(\overline{A \cup Anti}) = n(U) - n(Anti)$

$= n(Anti - Asy) = n(Anti) - n(Asy).$

$= n(Asy) - Anti) = 0.$

30. PROPERITIES OF ASYMETRIC RELATIONS

Assymetric Relations are closed under

a) subset operation (TRUE).

b) Superset operation (FALSE)

c) union " " (FALSE)

d) Intersection " (TRUE)

e) Set difference (TRUE)

f) Complementation (FALSE)

31. TRASITIVE RELATIONS

A Relation R on set A is said to be transitive if (xRy) and (yRz) then (xRz) $\forall x,y,z \in A$

$R_1 = \{\}$ – Transitive

$R_2 = \{(1,1)\}$ – Transitive

$R_3 = \{(a,b)\ (c,d)\} \rightarrow$ Transitive (we dont have $(x,y)(y,z)$ pair)

$R_4 = \{(x,y)\ (y,z)\} \rightarrow$ NOT Transitive [(x,z) is absent]

$R_5 = \{(x,y)\ (y,z)\ (x,z)\}$ = Transitive

$R_6 = \{(1,2)\ (2,2)\}$ – Transitive

$R_7 = \{(1,2)\ (2,1)\ (1,1)\}$ – Transitive $\Rightarrow (2,1)(1,1)$
(2,1) is present
$\Rightarrow (1,2)(2,1) = (1,1)$ present

$R_8 = A \times A$ = Transitive

The largest relation which is Transitive = $A \times A$
The smallest relation which is Transitive = $\{\}$.

32. EQUIVAL

A Relation 'R
if 'R' is 1)

Ex: $A = \{a,b,c$

$R_1 = \{(a,a)(b,b)($

$R_2 = \{(a,a)\ (b,b)$

$R_5 = \{(a,a)\ (b,$

The smallest
$\Rightarrow$ co

$\Rightarrow$ The largest
and it is

33. EXAMPLES

Which of the
of all Real nu

a) $R_1 = \{(a,b)/ a-$

b) $R_2 = \{(a,b)/a$

c) $R_3 = \{(a,b)/$

d) $R_4 = \{(a,b)/$

34. POSET

Partial Ordering R

A relation 'R'
order) if 'R'

Partially Orderd
A set 'A' with
Orderd set (po

22

32. EQUIVALENCE RELATION

- A Relation 'R' on a set 'A' is said to be Equivalence Relation on 'A' if 'R' is 1) Reflexive 2) Symmetric 3) Transitive

23

Ex: $A = \{a, b, c\}$

$R_1 = \{(a,a)(b,b)(c,c)\}$ $R_3 = \{(a,a)(b,b)(c,c)(b,c)(c,b)\}$

$R_2 = \{(a,a)(b,b)(c,c)(a,b)(b,a)\}$ $R_4 = \{(a,a)(b,b)(c,c)(a,c),(c,a)\}$.

$R_5 = \{(a,a)(b,b)(c,c)(a,b)(b,c)(a,c)(c,a)(c,b)(b,a)\}$

→ The smallest Equivalence set/Relation on set A contains 'n' elements.
⇒ contains only diagonal elements.

⇒ The largest Equivalence Relation on set A contains 'n^2' elements. and it is '$A \times A$'.

33. EXAMPLES OF EQUIVALENCE RELATIONS.

Which of the following is not an Equivalence relation on a set of all Real numbers?

a) $R_1 = \{(a,b) / a-b \text{ is a integer}\}$ — R ✓ All diagonal ele diff = 0 (integer); S ✓ If (a-b) is integer (b-a) is also integer; T ✓

b) $R_2 = \{(a,b) / a-b \text{ is } \div \text{le by } 5\}$ — R ✓ S ✓ T ✓

~~c)~~ $R_3 = \{(a,b) / a-b \text{ is odd no}\}$ → Not Reflexive (diagonal ele diff = 0 = Even no.

d) $R_4 = \{(a,b) / a-b \text{ is an even no.}\}$ → R ✓ (All diagonal ele are present)
S (Symmetric)
T (Transitive) } = Equivalence Relation.

34. POSET

Partial Ordering Relation:

A relation 'R' on set A is said to be partial ordering relation (partial order) if 'R' is Reflexive, Antisymmetric, and Transitive

Partially Ordered Set

A set 'A' with a partial order 'R' defined on 'A' is called partially Ordered set (poset) and it is denoted by [A; R]

$A = \{1,2,3\}$

$R_1 = \{(1,1)(2,2)(3,3)\}$ – Reflexive ✓, Transitive ✓, Symetric ✓, Antisymetric ✓

$\Rightarrow$ The Relation is Equivalence and partial order Relation.

$\Rightarrow$ This is the smallest Relation which is both partial ordering and Equivalence Relation.

$R_2 = \{(1,1)(2,2)\} \rightarrow$ cannot be Equivalent and partial ordering (Because this doesnot contain $(3,3)$)

$R_3 = \{(1,1)(2,2)(3,3)(1,2)\} \Rightarrow$ Not Equivalent Relation.

Reflexive ✓ Antisymetric ✓ Transitive ✓ } partial ordering Relation.

Now, let us consider set of all Real numbers 'R'. Now, let the Relation be "$\leq$" then 'R' is Reflexive, Antisymetric, Transitive. Therefore $[R; \leq]$ is called the partial order set / poset.

This Relation ($\leq$) is the partial orderd set on Relation 'R'.

Now, $[S; \subseteq]$ is also a poset –
$[R; /]$ is also a poset –
} Necessary condition is the Relation should be Reflexive, Antisymetric, Transitive.

35. TOS

Totally orderd set (Linearly orderd set or chain)

A poset $[A, R]$ is called a "Totally Orderd set" if every pair of elements in 'A' are comparable, i,e aRb or bRa $\forall\, a, b \in A$

Ex: Find whether the following are totally orderd sets or not

1) If 'A' is any set of real no's then poset $[A, \leq]$ is TOS
2) If $A = \{1,2,3,4 \text{---} 10\}$ then the poset $[A, \leq]$ is a TOS.
3) If $A = \{1,2,6,30,60,300\}$ then $[A; /]$ is TOS [2 ÷ 6, 6 ÷ 300, 60 ---].
4) If $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$ then $[S, \subseteq]$ is Not Tos.
5) If $S = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$ then $[S, \subseteq]$ is TOS.

1> Now, if i take any two Real no's, i can put the relation '≤' in btw them, $\left\{\begin{array}{l}\text{if i take } 1,2 \text{ i can say } (1 \le 2) \\ \text{if i take } 4,2 \text{ i can say } (2 \le 4)\end{array}\right\}$ ∴ Every pair of Real no's is Comparable

⇒ This is "TOS."

4>. $\{a\} \nsubseteq \{b\}$ ∴ This is not a "Total orderd set."

5> $\{a\} \subseteq \{a,b\}$. $\phi \subseteq \{a\}, \{a,b\}, \{a,b,c\}$

$\{a,b\} \subseteq \{a,b,c\}$.

36. GATE QUESTION-1

Let $A = \{a,b,c\}$ which of the following is Not TRUE (FALSE)?

a) $R_1 = \{(a,a)(c,c)\}$ is symetric, Antisymetric, Transitive (TRUE)

b) $R_2 = \{(a,b),(b,a)(a,c)\}$ is symetric, Antisymetric (FALSE) ((c,a) is not present)

c) $R_3 = \{(a,b)(b,a)(c,c)\}$ is symetric but not Antisymetric (TRUE)

d) $R_4 = \{(a,b)(b,c)(c,c)\}$ is Antisymetric but not symetric. (TRUE)

37. GATE QUESTION 2

Let $A = \{a,b,c,d\}$ and a relation on set A is defined as $R = \{(a,a)(b,a)(b,b)(b,c)(b,d)(c,a)(c,b)(c,c)(c,d)\}$ which of the following is TRUE?

a) R is Equivalence Relation = NOT EQUIVALENCE (NOT REFLEXIVE (d,d)X).

b) R is irreflexive or Antisymetric relation FALSE (NOT IRREFLEXIVE)

c) R is symetric or Asymetric Relation ( )(FALSE) {(a,b)} is absent

d) R is transitive ~~(FALSE)~~ (TRUE) ~~(b,a)(a,a) ⇒ (b,a) (not present in R)~~

38. GATE. QUESTION 3

Let A = set of all Real numbers, $R = \{(a,b) / b = a^k \text{ for some integer } k\}$

i.e $aRb \Leftrightarrow b = a^k$

a) R is Equivalence Relation $\left\{\begin{array}{l} R ✓ \ (a,a) = a = a^1 \\ S \times \\ T \end{array}\right\}$ ∴ Not Equivalent

b) 'R' is partial order ✓ $\left\{\begin{array}{l} R ✓ \\ \text{Anti} ✓ \\ T ✓ \end{array}\right\}$ ∴ The Given Relation is partial order.

c) 'R' is reflexive and symetric but not Transitive

d) 'R' is total order.

$R = \{(2,2)(2,4)(2,8)(2,16)\cdots (1,1)(3,3)(3,9)(3,27)\cdots\}$.

NOT Symetric = (8,2) is not present

$\Rightarrow 2 = 8^{1/3} \rightarrow$ but k is Integer

39. GATE QUESTION 4

Which of the following is Not TRUE (FALSE)?

a) If a Relation 'R' on set A is symetric and Transitive then 'R' is Reflexive

b) If a relation 'R' on set 'A' is irreflexive and transitive then 'R' is Antisymetric

c) If 'R' and 's' are Antisymetric on 'A' then $(R \cup S)$, $(R \cap S)$ are also Antisymetric

d) If R, S are Transitive then

a) A = {1,2,3}   A = {} → Not Reflexive, but not (Symetric & Transitive)

R = {(1,1)} – Transitive, symetric, not Reflexive ⇒ op1 : FALSE.

b) R = TRUE . (C) FALSE   (D) TRUE

41. GATE QUESTION 6

The no. of equivalence relations on set {1,2,3,4} is

a) 15   b) 16   c) 24   d) 4

Just Remember if a set has

3 elements → No. of Equivalence Relations = 5

4 elements → No. of Equivalence Relations = 15

R = {

SUMMARY ON RELATIONS

27

1> Reflexive Relation: All the diagnol ele should definitely be present

2> Irreflexive Relation: No diagonal ele should be present if you find atleast one diagnol ele then it is NOT IRREFLEXIVE

3> Symetric Relation: If (x,y) is present then only check for (y,x) (All the symetric pairs need not be present).

$\{\}$ - symetric $\begin{Bmatrix} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2 \end{Bmatrix}$.

4> Antisymetric Relation: Symetric pairs should not be present but diagnol pairs are allowed (Exception case)

$\{\}$ - Anti symetric $\begin{Bmatrix} \text{Min cardinality} = 0. \\ \text{Max cardinality} = n + \frac{n^2-n}{2} \end{Bmatrix}$.

5> Asymetric Relation: → symetric pairs should not be present, No exception on diagnol elements also.

$\{\}$ - Asymetric $\begin{Bmatrix} \text{Min cardinality} = 0 \\ \text{Max cardinality} = \frac{n^2-n}{2} \end{Bmatrix}$.

6> Transitive Relation: If (a,b) is present and (b,c) is present then only check for (a,c) if (a,c) is presents then Transitive else Not Transitive.

$\Rightarrow$ Iff, $(a,b)\,(b,c) \Rightarrow (a,c)$. | $R = \{abc\}$ | $R_1 = \{(ab)\,(bc)\}$ = Not Transitive

$\Rightarrow \{\}$ = Transitive $\begin{Bmatrix} \text{Min cardinality} = 0 \\ \text{Max cardinality} = n^2\,(A \times A) \end{Bmatrix}$.

7> Equivalence Relation: → Reflexive → Symetric → Transitive } (TRS).

8> POSET : Reflexive, Antisymetric, Transitive (RAT).

9> TOS. : Every pair should be comparable on the Relation given.

3. PARTIAL ORDERS AND LATTICES

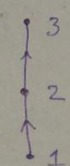
1. POSET DIAGRAM (HASSE DIAGRAM)

Let $[A:R]$ be a poset. The poset diagram is as follows

1) There is a vertex corresponding to each element of 'A'.
2) An edge between the elements 'a' and 'b' is not present in the diagram if there exists an element $x \in A$ such that (aRx) and (xRb)
3) An edge b/w the elements 'a' and 'b' is present iff aRb and there is no element $x \in A$ such that (aRx) and (xRb).

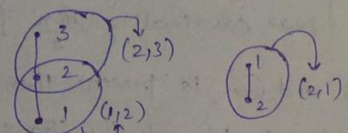
Ex: $A = (1,2,3)$

$R = \le = \{(1,2)(1,3)(2,3)$
$(1,1)(2,2)(3,3)\}$.



(1,3) is not needed because (1,2)(2,3) ⇒ (1,3), anyway (1,2) (2,3) edges are present

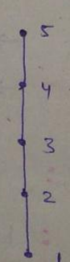
⇒ The general convention that we assume is bottom to top and we dont use arrows also.



2. EXAMPLES ON POSET DIAGRAMS

$A = \{1,2,3,4,5\}$, $R = \le$ } poset $= [A; \le]$

$R = \{ (1,1)(2,2)(3,3)(4,4)(5,5)$
$(1,2)(1,3)(1,4)(1,5)(2,3)$
$(2,4)(2,5)(3,4)(3,5)$
$(4,5)\}$

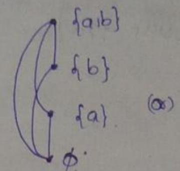


⇒ Here the diagram looks like chain

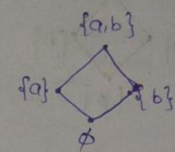
⇒ The relation in TOR (Total order Relation)

② $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

poset: $[S; \subseteq]$

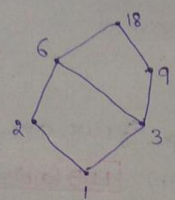


(or)



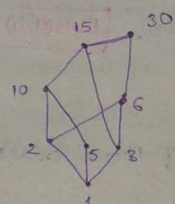
③ $A = \{1,2,3,9,6,18\}$.

poset: $[A, /]$

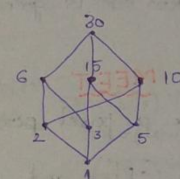


④ $A = \{1,2,3,5,6,10,15,30\}$.

poset: $[A, /]$

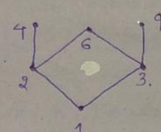


(or)



⑤ $A = \{1,2,3,4,6,9\}$.

poset $[A, /]$

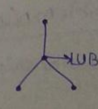


3. LUB

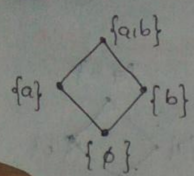
Least upper Bound (LUB or Join or Supremum)

Let $[A; R]$ be a poset. for $a, b \in A$, if there exists an element $c \in A$ such that i) aRc and bRc,

ii) if there exists any other element 'd' such that (aRd) and (bRd) then (cRd), then 'c' is the LUB of 'a' and 'b'.

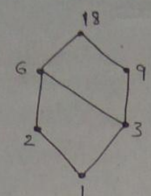


Ex:



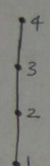
LUB = {a,b}.

[D_{18}, /]



LUB = {18}

⇒ In case of divisor operation $\boxed{LUB(a,b) = lcm(a,b)}$.



[A; ≤]

In case of ≤ Relationship if we are comparing (a,b) then the LUB will be max(a,b). $\boxed{LUB(a,b) = max(a,b)}$

⇒ In this diagram LUB = {4}.

⇒ In case of set Inclusion Relation (subset or equal to Relation) then union of two sets is going to be (LUB). $\boxed{LUB(a,b) = \{a\} \cup \{b\}}$

4. GLB OR MEET

Let [A; R] be a poset for a,b∈A if there exists an element c, such that

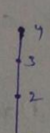
i) (cRa) and (cRb) and

ii) If there exists any other element 'd' such that (dRa) and (dRb) then (dRc) then 'c' is called the GLB of 'a' and 'b'.

(c,a)
(c,b)

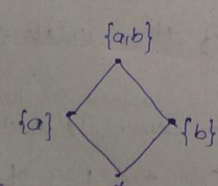
(d,a)
(d,b)

(c,d) → c is called the Greatest lower Bound.



$\boxed{GLB(a,b) = min(a,b)}$

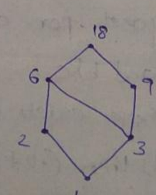
≤ Relationship



$\boxed{GLB(a,b) = a \cap b}$

Set Inclusion

Subset / Equalto



$\boxed{GLB(a,b) = GCD(a,b)}$

STANDARD EXAMPLES.

1) $[A; \leq]$ LUB $(a,b) = \max(a,b)$
GLB $(a,b) = \min(a,b)$ A = set of Real numbers

2) $[A; /]$ LUB$(a,b) =$ LCM(a,b)
GLB$(a,b) =$ GCD(a,b) A = set of Real no's.

3) $[S; \subseteq]$ LUB = union
GLB = Intersection. S = set of all sets.

For a given pair of elements the LUB, GLB may or may not exists.

6. LATTICE

<u>Join semi lattice</u>: If LUB exists for every pair of elements in poset

<u>Meet semi lattice</u> If GLB exists for every pair of elements in poset.

<u>lattice</u>: If Both LUB and GLB " " " " " " " "

Ex: $A = \{1,2,3 --- 10\}$ $[A, /]$ is meet semi lattice (GLB exists for every pair.) [For $(3,4)$ there is no LUB $= (\text{Lcm}(3,4) = 12)$ and 12 is not in set, so not LUB $\Rightarrow$ Not JSL)].

$S = \{\{a\}, \{b\}, \{a,b\}\}$ then $[S, \subseteq]$ is join semi lattice [GLB does not exist for $\{a\}\{b\}$ (Intersection of $\{a\}\{b\} = \phi$ (not in set).

$A = \{1,2,3,4\}$ $[A, \leq]$ is a lattice.

4
3
2
1

$\boxed{\begin{array}{l} \text{GLB} = (\max(a,b)) \\ \text{LUB} = \min(a,b) \end{array}}$

$\therefore$ A TOS (Total order set) is always a lattice.

7. LATTICE EXAMPLES.

→ If 'A' is set of all +ve integers, then poset $[A; /]$ is a lattice [$(a,b) \Rightarrow$ Lcm = +ve no; $(a,b) \Rightarrow$ GCD = +ve no; $\therefore$ LUB, GLB exists]

→ If 'n' is a positive integer then D_n = set of all +ve divisors of 'n'.

Ex: $D_6 = \{1,2,3,6\}$.

$D_{12} = \{1,2,3,4,6,12\}$

$D_{30} = \{1,2,3,5,6,10,15,30\}$.

Every $[D_n, /]$ is a lattice.

→ If $p(A)$ denotes powerset of A then $[p(A); \subseteq]$ is a lattice [$A = \{a,b\}$, $p(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$.]

8. PROPERITIES OF LATTICE

The following properity holds good in a lattice for any 3 ele $a, b, c \in A$

i) Commutative law: $a \vee b = b \vee a$
$a \wedge b = b \wedge a$

ii) Assocative law:
$(a \vee b) \vee c = a \vee (b \vee c)$
$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

iii) Idempotent law: $a \vee a = a$
$a \wedge a = a$

(iv) Absaption: $a \vee (a \wedge b) = a$
$a \wedge (a \vee b) = a$

v) Note: In a lattice $(a \vee b) = b$ iff $(a \wedge b) = a, \forall a, b \in L$ $\left\{ \begin{array}{l} \vee = LUB \\ \wedge = GLB \end{array} \right\}$

9. DISTRIBUTIVE LATTICE AND SUBLATTICE

The lattice on which the distributive properity holds is called Distributive lattice, and the distributive properities are

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\left\{ \begin{array}{l} \vee = LUB \\ \wedge = GLB \end{array} \right\}$

Sublattice:

Let L be a lattice $[L, \wedge, \vee]$. A subset 'M' of 'L' is called a sublattice of 'L' iff

i) M is a lattice i.e $[M, \vee, \wedge]$

ii) For any pair of elements $a, b \in M$ the LUB and GLB are same in M and L

To find whether a lattice is distributive there are 2 ways

1) Take all possible Triplets and check distributive properities (Headache)

2) Use the process of / Concept of sublattices

$A \rightarrow A \times A \rightarrow R \rightarrow R, A, T \rightarrow poset \rightarrow [A; R] \rightarrow lattice \xrightarrow{D} Distributive\ lattice$

$\downarrow$

M (sublattice)

10. BOUNDED

→ Let 'L' be a

such that (a

→ similarly if

'O' is called

→ In a lattice

Bounded la

L

Up

Lo

⇒ Every Finite

$\{\{1,2,3,4\}, \leq\}$

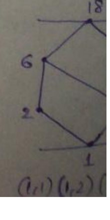
⇒ For a lattice

$[I, \leq]$

Set of Integers

⇒ $[D_{18}; /]$

$D_{18} = \{1, 2, 3, 6$



(1,1) (1,2)

10. BOUNDED LATTICE

→ Let 'L' be a lattice with respect to R, if there exists an element $I \in L$ such that $(aRI)\ \forall a \in L$, then I is called "Upper bound of Lattice L".

→ Similarly if there exists an element $O \in L$, such that $(ORa)\ \forall a \in L$, then 'O' is called "Lower bound of Lattice L".

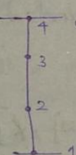
→ In a lattice if upper bound and lower bound exists then it is called Bounded lattice.

LUB, GLB } → Calculated on pair of elements.

Upper Bound, Lower Bound } → Calculated on entire lattice not on pair of elements.

→ Every finite lattice is Bounded. [Ex]

$\{\{1,2,3,4\}, \leq\}$

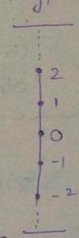


upper bound (4 is related to all other ele like (4,4) (3,4) (2,4) (1,4)

1 (lower bound because 1 is related to all other elements (1,1) (1,2) (1,3) (1,4)

→ For a lattice there may/may not be upper bound and lower bound.

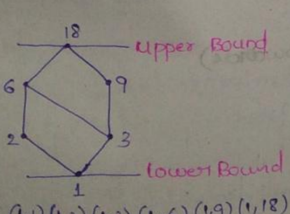
$[I, \leq]$ ↓ set of Integers.



No upperbound and lowerbound.

→ $[D_{18}; /]$

$D_{18} = \{1, 2, 3, 6, 9, 18\}$.



(1,1) (1,2) (1,3) (1,6) (1,9) (1,18)

11. PROPERITIES OF BOUNDED LATTICE

In a Bounded lattice, the following properities holds good. (34)

1> LUB of a and I i.e $a \vee I = I$ $\{(a,I)\}$
2> GLB of a and I i.e $a \wedge I = a$ $\{(a,I)\}$
3> LUB of a and O i.e $a \vee O = a$ $\{(O,a)\}$
4> GLB of 'a' and 'O' i.e $a \wedge O = a$ $\{(O,a)\}$

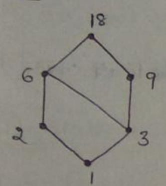
I - Upperbound of lattice
O = Lower bound of lattice

12. COMPLIMENT OF AN ELEMENT

Let 'L' be an bounded lattice, for any element $a \in L$, if there exists an element $b \in L$ such that $(a \vee b) = I$ and $(a \wedge b) = O$, then 'b' is called 'Compliment of a' written as $\bar{a}$. and 'a', 'b' are compliments of each other.

⇒ "Compliment" is only possible for "Bounded lattice."

Ex:

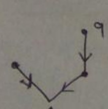


(2,9) are compliment to each other

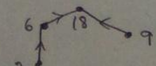
⇒ $(2 \vee 9)$ = Join operation = 18 (UB)
$(2 \wedge 9)$ = meet operation = 1 (LB)

$2 = \bar{9}$ or $9 = \bar{2}$

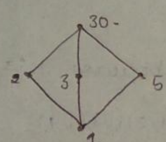
(2,9) are compliment



⇒ meet operation (come downwards).



= Join operation (go upwards)



$[\{1,2,3,5,30\}; /]$

$(2 \vee 3) = 30$ → upperbound (Join operation)
$(2 \wedge 3) = 1$ (Meet operation) → lower bound

⇒ (2,3) are compliment of each other ⇒ $\bar{2} = 3$ or $\bar{3} = 2$

⇒ (2,3) are not Compliment to each other because

$(2 \vee 3) = 6$ (Not UB) → X
$(2 \wedge 3) = 1$ (LB)

we should get both UB & LB.

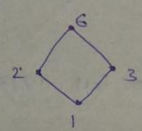
13. COMPLEMENTED LATTICE

⇒ If every element of a lattice has compliment then it is called Complimented lattice.

⇒ In a complemented lattice, each element has atleast one Compliment

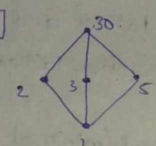
⇒ In a Distributive lattice, compliment of an element if exists, is unique i,e each element has atmost one complement.

[{1,2,3,6} : /]



⇒ Complimented lattice every ele has Compliment.

[{1,2,3,5,30}; /]



⇒ Compliment lattice

⇒ In a Distributive lattice each ele has (0 compliment & 1 compliment).

14. BOOLEAN ALGEBRA

Boolean Algebra

A lattice 'L' is called Boolean Algebra if it is distributive and complemented

⇒ In Boolean Algebra every element has atmost one Complement. i,e unique complement.

15. MAXIMAL AND MINIMAL ELEMENTS.

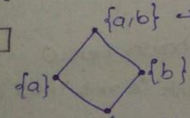
Maximal element : If in a poset, an element is not related to any other element, then it is called maximal element.

Minimal element: If in a poset, an ele is related to an element, then it is called minimal element.

Ex: A = {a,b}.

(ϕ, {a}) → ϕ is Related to {a}, Not {a} is related to ϕ.

[P(A); ⊆]

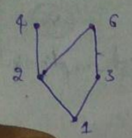


{a,b} → maximal ele

bcoz {{a,b}, —} {(a,b)} is not related to any other element

Ø → Minimal element

{—, Ø} → No ele is related to 'Ø' so Ø is the minimal ele.

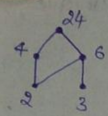


$[\{1,2,3,4,6\}; /]$

Maximal elements = 4, 6.

Minimal elements = 1

This is not a lattice because {4, 6} donot have LUB. and it is Meet semi lattice.



Maximal elements = 24.

Minimal elements = 2, 3.

Join semi lattice

* The upper Bound and lower bound are unique but maximal/minimal ele need not be unique.

⇒ Maximal, Minimal are applicable to both latices and posets But UB, LB can be /must be applied only for Bounded lattices.

⇒ In a poset if you have more than one maximal/minimal ele then It wont be lattice.

16. EXAMPLE 1

The poset $[\{2,3,4,9,12,18\}; /]$ is

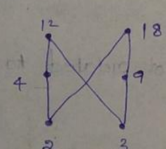
a) Join semi latice but not meet semilattice

b) Meet semi latice but not Join SL

c) A lattice

d) Neither join nor meet Semi lattice.

⇒



Maximal ele = 12, 18

Minimal ele = 2, 3

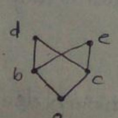
∴ This is not lattice

And there is no GLB for 2, 3 and LUB for 12, 18

∴ They arent Join and meet SL

17. EXAMPLE-2

The poset diagram of a poset $p = \{a, b, c, d, e\}$ is shown below.



which of the following statements is not TRUE (FALSE)?

a) 'p' is not lattice (True)

b) The subset {abcd} is lattice (T)

c) The subset {b, c, d, e} is lattice (F)

d) The subset {a, b, c, e} of 'p' is lattice (T)

a) TRUE becua

b) TRUE all

c) FALSE [d,e

18. EXAMPLE-

The Hasse dia

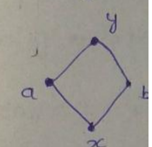
which of the foll

a) {x, a, b, y}

c) {x, a, e, y}

(e) {x, c, d, y}.

a) {x, a, b, y}



⇒ lattice becuas

every pair of elem

LUB, GLB Exists.

⇒ For sublattice f

LUB of every pa

the options, Here,

GLB and LUB a

but In original di

LUB (a, b) = C (≠

Cannot be the sa

a) TRUE becuase (d,e) donot have leastupper bound.

b) TRUE all pairs have LUB, GLB

FALSE [(d,e) has no LUB and (b,c) donot have GLB)

18. EXAMPLE-3

The Hasse diagram of a lattice $L = \{x, a, b, c, d, e, y\}$ is shown below which of the following subsets of 'L' are sublattices of L?

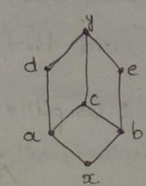
a) $\{x, a, b, y\}$ (b) $\{x, a, c, y\}$

c) $\{x, a, e, y\}$ (d) $\{x, d, e, y\}$

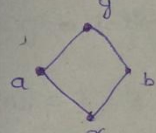
(e) $\{x, c, d, y\}$.

GLB = క్రింద

LUB = పైన



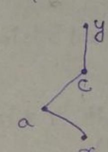
a) $\{x, a, b, y\}$



⇒ lattice becuase for every pair of elements both LUB, GLB Exists.

⇒ For sublattice find the GLB, LUB of every pair of ele in the options, Here, for (a,b) the GLB and LUB are x, y respectively but In original diagram the LUB $(a,b) = c\ (\neq y)$ ∴ This Cannot be the sublattice of 'L'.

(b) $\{x, a, c, y\}$.



⇒ Lattice bcoz LUB, GLB Exists for every pair

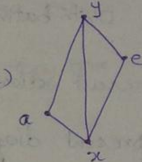
⇒ For (a,x) = GLB = x
LUB = a.

For (a,c) = GLB = c
LUB = a

For (c,y) = GLB = c
LUB = y

These are same in Both diagrams ∴ It is sublattice

(c)

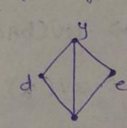


⇒ lattice For (a,e) GLB = x, LUB = y } Same in original diagram

For (a,y) GLB = a
LUB = y.

∴ This is the sublattice

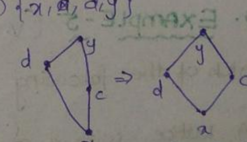
(d) $\{x, d, e, y\}$.



⇒ lattice

⇒ Sublattice also.

(e) $\{x, c, d, y\}$



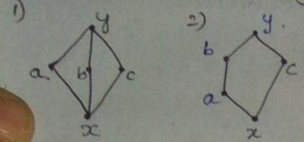
⇒ GLB(d,c) = x
GLB(d,c) = a (In original diagram)

∴ Not a sub lattice

19. EXAMPLE-4 (V. Imp Question)

Which of the following latices is not distributive

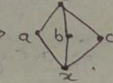
1) 2)



⇒ If a lattice is Distributive then we should have atmost one complement for each element so in the first diagram 'a' has two Complements 'b' and 'c' and therefore it cannot be distributive

⇒ In the 2nd Graph 'c' has 2 compliments 'a' and 'b'. so the 2nd diagram is also not Distributive

let L_1^* represent diag 1 ⟶



L_2^* represent diag 2.

GLB ⟵ $\wedge$ = see down

LUB. ⟵ $\vee$ = see up.

⇒ In L_1^* ⇒ $a\vee(b\wedge c) \overset{?}{=} (a\vee b)\wedge(a\vee c)$

⇒ $a\vee(x)$ | $(y)\wedge(y)$

⇒ a | $= y$

⇒ $(a\neq y)$ ∴ The distributive property doesnot hold true

⇒ In L_2^* ⇒ $a\vee(b\wedge c) \overset{?}{=} (a\vee b)\wedge(a\vee c)$

⇒ $a\vee(x)$ | ⇒ $(b)\wedge(y)$

⇒ a | ⇒ b

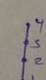
$(a\neq b)$ ⇒ The distributive property doesnot hold true for this lattice

20. EXAMPLE-5 (V. Imp Question)

which of the following statements are not true

a) A lattice with 4 or fewer elements is distributive (TRUE)

b) Every totally orderd set is a distributive lattice (TRUE) ⟶



c) Every sublattice of a distributive lattice is also distributive (TRUE)

No Isomorphic Structures exist similar to L_1^* or L_2^*.

d) Every distributive lattice is a bounded lattice

⇒ we have already seen L_1^*, L_2^* ~~as sublattice and they~~ are not distributive lattices. so If a lattice contains L_1^* or L_2^* as sublattice

then that

d). NO Relati

21. EXAMPL

Which of th

a) [P(A), ⊆]

b) [D_{81} ; /]

c) [R; ≤] R

d) [{1, 2, 3, 5, 3

⇒ To check wh

L_2^* are not

the given lat

a) [P(A), ⊆]

union and

we know that

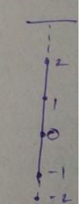
and Intersect

Distributive

b) D_{81} = [{1, 3, 9,

For every val

c) [R; ≤]



d) [{1, 2, 3, 5, 30} ;

~~[D_{30} ; /]~~

then that lattice won't be ~~sublattice~~ distributive lattice.

d) No Relation btn Distributive and Bounded lattices (FALSE)

31. EXAMPLE-6

Which of the following is not distributive lattice?

a) $[P(A), \subseteq]$ where $A = \{a, b, c, d\}$.

b) $[D_{81} ; /]$

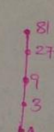
c) $[R; \leq]$ R is set of Real numbers

d) $[\{1, 2, 3, 5, 30\}; /]$

⇒ To check whether a lattice is distributive or not we check if L_1^* and L_2^* are ~~not~~ sublattices of given lattice, if they are ~~sublattices~~ then the given lattice is not distributive.

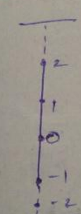
a) $[P(A), \subseteq]$ $A = \{a, b, c, d\}$ w.k.T the join of two elements is nothing but union and meet of two elements is nothing but Intersection. and we know that on a set of all sets Union is distributive over Intersection and Intersection is distributive over union. Therefore the option a is.
Distributive

b) $D_{81} = [\{1, 3, 9, 27, 81\}; /]$



$\boxed{\text{For every value of 'n' } [D_n ; /] \text{ is always Distributive}}$

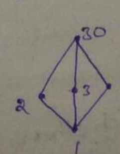
c) $[R; \leq]$



⇒ This is a Total order Relation and w.k.T. Every Total order Relation is $\boxed{\text{Distributive.}}$

d) $[\{1, 2, 3, 5, 30\}; /]$

$[\cancel{D_{30}} \cancel{/}]$



= Isomorphic to L_1^* so this lattice is Not distributive. and (2,3)(2,5) are Compliment. '2' has two Compliments.

22. EXAMPLE-7

For the lattice $[D_{18}; /]$ which of the following is not TRUE.

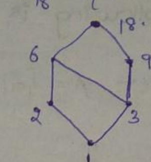
a) The compliment of $1 = 18$

b) The compliment of $2 = 9$

c) The compliment of $3 = 6$ $(3 = \bar{2})$

d) The compliment of '6' doesnot exists

$D_{18} = \{1,2,3,6,9,18\}$

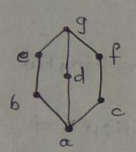


$(6 \vee 9) = 18$ (LUB)
$(6 \wedge 9) = 3$ (Not GLB) } 6,9 are not compliment

{GLB = 18 In the lattice}

23. EXAMPLE-8

For the lattice given below, how many complements does the ele 'e' have?



(e,a)
(e,b)
(e,c)
(e,d)
(e,f)
(e,g) } These are the combinations possible

Join = $\wedge$ = ~~upper~~ lower bound
meet = $\vee$ = ~~lower~~ upper bound

Now, The upper bound of lattice = g
lower bound of lattice = a

Now, $(e \wedge a) = a$, $(e \vee a) = e$ } (e,a) Not Compliment

$(e \wedge b) = b$, $(e \vee b) = e$ } Not compliment.

$(e \wedge d) = a$ - LB of lattice, $(e \vee d) = g$ = UB of lattice } (e,d) are compliment.

$(e \wedge f) = a$, $(e \vee f) = g$ } (e,f) are compliment

Similarly (e,c) are compliment to each other.

4. GROUPS.

1. ALGEBRAIC STRUCTURES

A Non-empty set 'S' is called an Algebraic structure with respect to Binary operation $*$ if $(a*b) \in S \ \forall\ a,b \in S$ i.e $*$ is closure operation on 'S'

1) $S = \{1, -1\}$.

   $* \rightarrow \times$ (multiplication)

   $(S,*)$ is Algebric structure becuase $(1)\times(-1) = -1$ (present in 'S')

   $-1 \times -1 = 1 \ (\in S)$

   $1 \times 1 = 1 \ (\in S)$

   $\therefore$ 'S' is called the Algebraic Structure.

2) $S = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

   $* = \cup$ (union operation)

   Now $(S,*)$ is A.S. becuase

$\phi \cup \{a\} = \{a\} \in S.$	$\{a,b\} \cup \{a\} = \{a,b\} \in S.$
$\phi \cup \{b\} = \{b\} \in S.$	$\vdots$
$\{a\} \cup \{a\} = \{a\} \in S.$	
$\phi \cup \phi = \phi \in S.$	

3) $A = \{1,2,3\}$

   $R =$ set of all Reflexive Relations $= \{(1,1)(2,2)(3,3)\}$.

   $(R, \cup) =$ A.S.
   $(R, \cap) =$ A.S.

   The set of all Reflexive Relations are closed under union/Intersection and therefore set of all Reflexive Relations w.r.t to Intersection/union is an "ABELIAN STRUCTURE"

4) $(R, +) \Rightarrow$ "ABELIAN STRUCTURE" (R = Real no's)

5) $(N, *) \Rightarrow$ "ABELIAN STRUCTURE" (N = Natural numbers)

6) $(S = \{1,2,3\}$ $\Rightarrow$ (NOT Algebraic STURCUTURE)

   $* \rightarrow$ multiplication

   Now, $2*3 = 6$ (Not in S). so

7) $[Z, /] \Rightarrow$ Not Algebraic structure.

   $2/3 = 0.6 \neq$ Integer

2. SEMI GROUP

An Algebraic structure $(S,*)$ is called a Semigroup if $(a*b)*c = a*(b*c)$ $\forall a,b,c \in S$ i,e $*$ is Associative on 'S'.

Ex:

Naturalno = Natural No.

1) $(N,+)$ = Algebraic structure (✓) → $a+(b+c)$ $(a+b)+c = a+(b+c)$

2) $(N,*)$ = (Algebraic Structure (✓) → $(a\times b)\times c = a\times(b\times c)$ ∴ SEMI GROUP

3) $(Z,-)$ = Z = {Integers set} AS(✓) ⟷ $(a-b)-c = a-(b-c)$ (F) ∴ NOT SEMI GROUP

1 2 3 1 2 3

4) $(Q^*,+)$ = Q = {Rational No's} not having '0'. AS(✗) $a+(-a)=0$. (−4) ≠ (2)

Not Rational.

5) $(Q^*,*)$ = SemiGroup

6) $(P(A),\cup)$ = Semi Group

7) $(P(A),\cap)$ = Semi Group

3. MONOID

A semigroup $(S,*)$ is called Monoid if there exists an element $e \in S$ such that $(a*e) = (e*a) = a$ $\forall a \in S$. ($*$ = operation defined)

The element 'e' is called Identity element of 'S' w.r.t. $*$.

⇒ If a semi Group contains Identity element(s) then it is a "Monoid".

Ex:

$(N,*)$ = semigroup (✓) $a*e = a$ Now $\boxed{e=1}$ ∴ $(N,*)$ is Monoid

and '1' is the identity element of N w.r.t. $*$

$(N,+)$ = AS (✓) SG (✓) $a+e = a$ ⇒ $\boxed{e=0 \notin N}$ ∴ This is not Monoid.

$(Z,+)$ = AS(✓) SG(✓) $a+e=a$ ⇒ $\boxed{e=0}$ $\in Z$ ∴ Monoid

$(P(A),\cup)$ = AS(✓) SG (✓) $X \cup e = X$ $\boxed{e=\phi \in P(A)}$ ∴ Monoid.

Algebraic Structures (closure property)

↓

Semi Groups (Associative property)

↓

Monoid (Identity elements)

4. GROUP

A monoid (S,*

each element

(b*a)=e then

Ex

(Z,+) = Monoid

(Q, ·) · = (Multip

(Q*, ·) ⇒ Q* = (set o

no's

· = Multi

(P(A), ∪) = X ∪ {φ}

5. ABELIAN GRO

→ In a Group (

i> The Identity

ii> The Inverse o

iii> The Inverse o

iv) Cancellation la

v) $(a*b)^{-1} = b^{-1}*a^{-1}$

6. EXAMPLE 1

Abelian Group (Com

A group (G,*) is

Ex:

(Z,+) = Group(✓) a

(R*, ·) = Group(✓)

(M,*) = M = (set of all

4. GROUP

A monoid $(S,*)$ with identity element 'e' is called a group if to each element $a \in S$, there exists an element $b \in S$, such that $(a*b) = (b*a) = e$ then 'b' is called Inverse of an element a, denoted by a^{-1}.

Ex

$(Z,+)$ = Monoid(✓) $= a+b = b+a = 0$ (Identity element). | For every ele there is Inverse

$= \boxed{a=-b}$ and $(-b) \in Z$ ∴ Group

$(Q,\cdot)$ $\cdot$ = (Multiplication) $= a*b = 1$ ∴ This is a Not Group

$\Rightarrow \boxed{b = (1/a)} \notin Q)$ if $(a=0)$

$(Q^*,\cdot) \Rightarrow Q^*$ = (set of all Rational no's without '0')

$\cdot$ = Multiplication

$a*b=1$

$\Rightarrow \boxed{a = 1/b}$ $\boxed{b = 1/a} \in Q^*$

$\Rightarrow$ This is Group because the set Q^* doesnot has '0'.

$(p(A),\cup) = X \cup \{\phi\} = \{\phi\} \Rightarrow$ identity element $\}$ = Group.

↳ $\boxed{\phi \in p(A)}$

5. ABELIAN GROUP.

→ In a Group $(G,*)$ the following properties must hold good

i> The Identity element of 'G' is unique

ii> The Inverse of any element in 'G' is unique

iii> The Inverse of identity element 'e' is 'e' itself

iv) Cancellation laws $(a*b) = (a*c) \Rightarrow b=c$

$(a*c) = (b*c) \Rightarrow a=b$

v) $(a*b)^{-1} = b^{-1} * a^{-1} \ \forall\, a,b \in G$

6. EXAMPLE 1

Abelian Group (Commutative Group)

A group $(G,*)$ is said to be Abelian if $(a*b) = (b*a) \ \forall\, a,b \in G$.

Ex:

$(Z,+)$ = Group(✓) $a+b = b+a \in Z \Rightarrow$ Abelian Group

$(R^*,\cdot)$ = Group(✓) $a*b = b*a \in R^* \Rightarrow$ Abelian Group.

$(M,*)$ = m = (set of all non singular matrices) $\Rightarrow$ ($*$ = matrix multiplication) = NOT Abelian Group

7. EXAMPLE 2

Which of the following is/are True?

1) In a group $(G,*)$ with an identity element 'e' if $a*a=a$ then $a=e$

2) In a Group $(G,*)$ if $x^{-1}=x$ $\forall x \in G$ then 'G' is Abelian Group.

3) " " " " " $(a*b)^2 = a^2 * b^2$ $\forall a,b \in G$ then 'G' is Abelian Group.

⇒ Use the formulas in 5th video. Now, $a*a=a$

1) $a*\cancel{a} = \cancel{a}*e$ (I can write 'a' as $a*e$ because 'e' is identity element)

$\boxed{a=e}$ TRUE.

2) $x^{-1}=x$ | $(a*b)^{-1} = b^{-1}*a^{-1}$

$x^{-1} = x*e$ | $\boxed{(a*b) = (b*a)}$ ⇒ Given $x^{-1}=x$ so $(a*b)^{-1}=(a*b)$

$b^{-1}=b$

$a^{-1}=a$

3) $a^2 = a*a$ write the operation specified.

⇒ $(a*b)^2 = a^2 * b^2$

$= \cancel{(a*b)}*\cancel{(a*b)}$ $\underbrace{(a*b)*(a*b)}$ $= a*a*b*b.$ | $\cancel{a}*(b*a)*\cancel{b} = \cancel{a}*a*b*\cancel{b}$

$= (a*a)*(b*b) = (a*a)*(b*b)$ $=$ LHS $=$ RHS. | $(b*a)=(a*b)$

8. EXAMPLE - 3

If $A = \{1,3,5,7,9---\infty\}$ and $B=\{2,4,6,8---\infty\}$ which of the following is semigroup?

a) $(A,+)$ = Algebraic structure (✗)

b) $(A,\cdot)$ = AS(✓) SG(✓) Monoid (✓) $(e=1)$ $a*\{x\}=1$ $x=1/a \notin A$ ∴ Not Abelian.

c) $(B,+)$ = AS(✓) SG(✓)

d) $(B,\cdot)$ = AS(✓) SG(✓)

9. EXAMPL

Let $A=\{1,2,3,$

$\forall a,b \in A$ whi

a) $(A,*)$ is Se

b) $(A,*)$ is n

c) $(A,*)$ is G

d) $(A,*)$ is no

10. EXAMPLE

Let $A=\{x/0<$

is

a) A semigroup

b) A monoid

c) A group

d) Not a semi

$(A,*)$ ⇒ AS

11. EXAMPLE-

Let 'A' is set

$(a*b)=\min(a,b$

$(Z,\cdot)$ ⇒

9. EXAMPLE-4

Let $A = \{1,2,3,4 --- \infty\}$ and a Binary Operation $*$ is defined by $a*b = a^b$ $\forall a,b \in A$ which of the following is true?

a) $(A,*)$ is Semi Group but not monoid   AS(✓) SG(✗)

b) $(A,*)$ is monoid but not group

c) $(A,*)$ is Group

~~d)~~ $(A,*)$ is not semi Group.

$a*(b*c) = (a*b)*c$

$1 \quad 2 \quad 3 \;|\; 1 \quad 2 \quad 3$

$= 1*(2^3)$ | $(1^2)*3$

$= 8$ | $= (1*3) = 1^3$

$= 1$

$(8 \neq 1)$

so not $\underline{SG}$

10. EXAMPLE-5

Let $A = \{x / 0 < x \leq 1$ and x is a Real number$\}$ then 'A'. w.r.t to multiplication is

$x \in (0,1]$

a) A semigroup but not monoid

~~b)~~ A monoid but not Group

c) A group

d) Not a semigroup

$(A,*) \Rightarrow$ AS(✓). semiGroup. (✓)

$(a*b)*c = a*(b*c)$

$0.1 \quad 0.1 = \quad 0.1 \quad 0.1$

$= (0.1)^2 = (0.1)^2$

Monoid (✓)

$\boxed{e=1}$ Identity element

Group.

$a * \{x\} = e$

$\Rightarrow \boxed{x = 1/a \notin A}$

$\frac{1}{0.1} = \underline{10 \notin A}$ $\underline{\text{Group}(\times)}$.

11. EXAMPLE-6

Let 'A' is set of all integers and Binary operation $*$ is defined by $(a*b) = \min(a,b)$, then $(A,*)$ is $\underline{\text{Semigroup}}$

$(z,\cdot) \Rightarrow$ Alegebraic Structure(✓)

semiGroup (✓)

$(a*b)*c = a*(b*c)$

$1 \quad 2 \quad 3 \;|\; 1 \quad 2 \quad 3.$

$= 1*3$ | $1*(2)$

$= 1$ | 1

Monoid

$\boxed{a * \{e\} = a}$

$\boxed{e \neq 1}$ identity element

$a * e = a.$

$a * e = \min(a,e)$

$101 \quad 100 = 100 \neq 101$

Group.

$a*b = a \Rightarrow a * ? = e$

$\Rightarrow a * ? = 1$

$? = 1/a \notin Z$

SemiGroup.   Monoid

$\therefore$ $\boxed{\text{It is } \text{~~Monoid~~} \text{ but not } \text{~~Group~~}}$

16. FINITE GROUPS.

A Group with finite no. of elements is called finite Groups.

$O(G) \rightarrow$ order of finite Group. (No. of ele in the Group)

Ex:

1) $(\{0\}, +) = (0+0) = 0$, Identity element = '0'. (Monoid) Group (✓)

$\Rightarrow$ $\boxed{\text{whenever a group is having only one element then that ele will be the Identity element. (V.V. Imp)}}$

2) $(\{1\}, *) = 1*1 = 1 \in$ set (AS) ✓.

$(1*1)*1 = 1*(1*1)$ (SG) ✓

$a*e = a = 1*e = 1 \Rightarrow \boxed{e=1} \in$ set (monoid) ✓

$a*\{x\} = e \Rightarrow x = 1/a \quad x = 1/1 \Rightarrow \boxed{x = 1 = e}$ $\therefore$ (Group)

3) $(\{1, -1\}, *)$ = Group (✓)

= Composition table

	1	-1
1	1	-1
-1	-1	1

$\in$ (set) $\therefore$ (AS) ✓

= (SG) ✓

= Monoid ✓ (e = 1 Identity element)

= Group.

4) $\{1, \omega, \omega^2\}, * \} =$

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

AS (✓)

Identify ele = 1 (mon) ✓ (SG) ✓

$\omega^4 = \omega^3 \times \omega$

$= 1 \times \omega$

$\boxed{\omega^4 = \omega}$

For all the ele we are able to find inverses $\therefore$ Group

	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

	1
1	1

1 is the Inverse of 1

	ω	ω^2
ω		1

ω^2 is the Inverse of ω

	ω
ω^2	1

ω is the Inverse of ω^2

EXAMPLES ON FINITE GROUPS

1) If $G = \{1,3,5,7\}$ is a group w.r.t $\times_8$ (Multiplication modulo 8), which of the following is not true?

a) The inv of 1 is 1
b) The Inv of 3 is 3
(c) The inverse of 5 is 7
(d) The inverse of 7 is 7.

Now, $1\times1 = 1$ (1 when divided by 8 gives '1' as Remainder)

$3\times3 = 9$ ($9 \div 8 = 1$ (belong to G) and 1 is identity element)

$5\times5 = 25$ ($25 \div 8 = 1$ (identity element)

$7\times7 = 49$ ($49 \div 8 = 1$ (identity element)

∴ Each element is Inverse of itself

(a)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

How did i write this is $(7\times5)\div8 = 35 \div 8 = 3$.

1 is the Inverse of 1
3 " " " 3
5 " " " 5
7 " " " 7

2) which of the following is a Group?

a) $\{1,2,3,4\}$ w.r.t $\times_6$
b) $\{0,1,2,3,4,5\}$ w.r.t $\times_6$
c) $\{1,2,3,4,5,6\}$ w.r.t $\times_7$
d) $\{1,2,3,4,5,6\}$ w.r.t $+_7$

we know that

$\oplus_m$ contains $0,1,2 \ldots (m-1)$ elements

$\otimes_m$ contains 'm' elements.

a) $\{1,2,3,4,5\}$ w.r.t $\times_6$

$= (2\times3) \div (6)$

$\Rightarrow 0 \notin$ set

$\Rightarrow$ (AS) ✗.

(b) $\{0,1,2,3,4,5\}$ w.r.t $\times_6$

$\Rightarrow$ No Inverse to '0' and we cannot get Identity element so doesnot form Group.

(c) $\underbrace{\{1,2,3,4,5,6\}}_{S_7}$ w.r.t $\times_7$

S_7 (set of all no's less than 7 and are relatively prime to 7).

∴ Group

(d) $\{1,2,3,4,5,6\}$ w.r.t $\oplus_7$

(0, 6) must be present

'0' is missing.

Monoid property fails

18. EXAMPLES ON FINITE GROUPS

17

1) $(\{0,1,2 \ldots (m-1)\}, \oplus_m)$ Addition modulo 'm' (This will always be Group)

$a \oplus_m b$ — If $(a+b) < m \Rightarrow (a+b)$ is the result
— If $(a+b) > m \Rightarrow (a+b) \% m$ is the result

$(\{0,1,2\}, \oplus_3) \Rightarrow$ This is a group

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$(0,1,2) \in$ set $\therefore$ AS ✓ (SG) ✓ (M) ✓ (G) ✓

⇒ For getting the identity element check the row containing $\{0,1,2\}$ (in the same order) then that element will be Identity element here '0' row has $\{0,1,2\}$ ∴ '0' is the identity element

⇒ Here In this case check for 0's

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

row 1: ⇒ 2 is the Inverse of 1.

row 2: ↳ 1 is the Inverse of 2.

$a \times a^{-1} = e$

$a \times a^{-1} = 0$

$\boxed{a^{-1} = 0}$

2) $(S_m, \otimes_m) \Rightarrow \boxed{a \otimes_m b = (a \times b) \% m}$

↓

Set of all no's that are less than 'm' and relatively prime to 'm'.

$S_{10} = \{1, 3, 7, 9\}$ ∴ $S_{10} = \{1, 3, 7, 9\}$.

Two no's are said to be Relatively prime if $GCD(a,b) = 1$

Now, $\{S_{10}, \otimes_{10}\}$ = Group.

$= \{S_{10}, \otimes_{10}\} = \{\{1,3,7,9\}, \otimes_{10}\}$ - Group.

19. ORDER

Order of an

Let $(G, *)$

smallest po

Ex:

1) $(\{1,-1\},$

Now,

⇒ order of

2) $(\{1, \omega, \omega^2\},$

$1' = 1$

$\omega^3 = 1$

∴ $\boxed{O(\omega}$

⇒ $\boxed{\text{Order o}}$

3) $(\{0,1,2\}, \oplus$

Now, $0' =$

$1 \oplus_3 1$

$= \oplus_3$

In a G

19. ORDER

Order of an element of a group

Let $(G,*)$ be a Group and $a \in G$, then order of element 'a' is the smallest positive integer 'n' such that a^n is identity element.

Ex:

1) $(\{1,-1\}, *)$ Identity element $= 1$

Now, $(-1)^2 = 1$ = identity element $\therefore$ order of $(-1) = 2$

$(-1)^4 = 1$ '4 cannot be order we should always take least no).

⇒ order of Identity element is always one.

2) $(\{1,\omega,\omega^2\}, *)$ Identity element $= 1$

$1^1 = 1$ ⇒ order of $1 = 1$

$\omega^3 = 1$ ⇒ order of $\omega = 3$

~~$\omega^2 \times \omega \neq 1 \Rightarrow$ order of $\omega^2 = 1$~~ Now, $(\omega^2)^3 = \omega^6 = (\omega^3)^2 = 1$

$\therefore \boxed{O(\omega^2) = 3 \quad O(\omega) = 3 \quad O(1) = 1}$

⇒ **Order of any element divides the order of group (Finite Group)**

3) $(\{0,1,2\}, \oplus_3)$ Identity element $= 0$.

Now, $0^1 = 0$. $\therefore O(0) = 1$

$1 \oplus_3 1 \oplus_3 1 = 0$ ⇒ $\boxed{O(1) = 3}$ ⇒ $\boxed{1^3 = 0}$

$2 \oplus_3 2 \oplus_3 2 = 0$ ⇒ $\boxed{O(2) = 3}$ ⇒ $\boxed{2^3 = 0}$

In a Group $O(a)$ and $O(a^{-1})$ are always same/Equal

20. EXAMPLES ON ORDER

State True / False

S1) In a group $(Z, +)$ the order of any element except '0' doesnot exist

S2) In the group $(Q^*, \cdot)$ where Q^* is set of all Non-zero rational numbers i.e $Q^* = Q - \{0\}$, the order of any element except 1 doesnot exist

S1) TRUE, because order of '0' is 1 (because '0' is Identity element) and for other ele order doesnot exist

S2) 1 is the Identity element so $O(1) = 1$

Now, $(-1)^2 = 1$ $\therefore \boxed{O(-1) = 2}$ $\therefore$ S2 is FALSE.

21. SUBGROUPS.

SUBGROUPS.

Let $(G, *)$ be a group. A subset 'H' of 'G' is called a SubGroup of 'G' if $(H, *)$ is a group.

Ex:

let $(G, *)$ be a group with Identity element 'e', then $\{e\}$ and 'G' are the trivial Sub Groups of 'G'. Any subgroup which is not a trivial subgroup is called proper SubGroup.

$G = (\{1, -1, i, -i\}, *)$ then $H = (\{1, -1\} *)$ is a proper subgroup.

⇒ Every Group is going to have atleast two subgroups 1> Itself 2> set containing Identity elements. (These are called Trivial subGroups)

The SubGroups other than Trivial SubGroups are called proper sub Grp.

22. THEOREMS ON SUB GROUPS.

Th 1: Let 'H' be non empty subset of a group $(G,*)$ 'H' is a sub group of 'G' iff $a*b^{-1} \in H \ \forall a,b \in H$

Th 2: Let 'H' be non empty finite subset of a group $(G,*)$ 'H' is a subgroup of 'G' iff $(a*b) \in H \ \forall a,b \in H$

Th 3: Lagranges Theorem :-

If 'H' is a subgroup of finite group $(G,*)$ then $O(H)$ is the divisor of $O(G)$. The converse of the above theorem need not be True.

$\left.\begin{array}{l} O(G) = m \\ O(H) = n \end{array}\right\} \Rightarrow$ m is divisable with/by 'n'. (This is what this theorem says). Ex: $\left.\begin{array}{l} O(G) = 10 \\ O(H) = 3 \end{array}\right\}$ 'H' cannot be subgroup of 'G' becuase '3' does not divide 10.

23. EXAMPLES ON SUBGROUPS - 1

Let $G = (\{0,1,2,3,4,5\}, \oplus_6)$ is a group. which of the following is/are subgroups of G?

a) $H_1 = \{1,3\} \Rightarrow$ Now, $\{\{1,3\} \oplus_6\}$ $(1+3) \oplus_6 = 4 \bmod 6 = 4 \notin H_1$

b) $H_2 = \{1,5\} \Rightarrow$ Now, $(\{1,5\}, \oplus_6) = (1+5) \bmod 6 = 0 \notin H_2$

c) $H_3 = \{1,3\} \Rightarrow$ Now, $(\{1,3\}, \oplus_6) = (4) \bmod 6 = 4 \notin H_3$.

d) $H_4 = \{0,2,4\} \Rightarrow$ Now, $(\{0,2,4\}, \oplus_6) =$ $\begin{array}{ll} 2 \bmod 6 = 2 & (0+2) \bmod 6 = 2 \\ 0 \bmod 6 = 0 & (0+4) \bmod 6 = 4 \\ 4 \bmod 6 = 4 & (2+4) \bmod 6 = 0 \end{array} \Bigg\} \in H_4$

$\Rightarrow$ SubGroup

e) $H_5 = \{0,2,3,5\}$.

Now $(\{0,2,3,5\}, \oplus_6)$

$= 0 \bmod 6 = 0$

$= 2 \bmod 6 = 3 \bmod 6$

$= 5 \bmod 6 = 2,3,5$ respectively

$\Rightarrow 0,2,3,5 \in H_5$

$(0,2) \bmod 6 = 2$

$(0,3) \bmod 6 = 3$

$(0+5) \bmod 6 = 5$

$(2+3) \bmod 6 = 5$

$(2+5) \bmod 6 = 1 \notin H_5$

$\therefore$ This wont be subGroup.

(OR) "PROVE BY CONSTRUCTING COMPOSITION TABLE"

24. EXAMPLES ON SUB GROUPS -- 2

$G = (\{1,2,3,4,5,6\}, \otimes_7)$ which of the following are subgroups of 'G'.

a) $H_1 = \{1,6\}$ c) $H_3 = \{1,3,5\}$

b) $H_2 = \{1,2,4\}$ d) $H_4 = \{1,2,3,5\}$.

H_1:

	1	6
1	1	6
6	6	1

H_1 is sub group

H_2:

	1	2	4
1	1	2	4
2	2	4	1
4	4	1	2 $\rightarrow \in H_2$

H_2 is subGroup.

H_3:

	1	3	5
1	1	3	5
3	3	2	1
5	5	1	4

$2 \notin H_3$.

$\therefore H_3$ is not sub Group.

H_4:

	1	2	3	5
1	1	2	3	5
2	2	4	6	3
3	3	6	2	1
5	5	3	1	4

$6 \notin H_4$ $\therefore$ H_4 is not subGroup.

25. EXAMPLES ON SUB GROUPS -- 3

Let $(G,*)$ be a group of order p where 'p' is prime no, then the no. of proper subgroups of 'G' is ___ ?

Sol: Given the order of 'G' has prime number $= p$ (1 and 'p' are only factors)

$\therefore$ The subgroups of G has the order '1 and 'p'.

$\Rightarrow$ so 2 subgroups are possible that are the total set $(G,*)$ and the Identity element set say $(e,*)$

$\longrightarrow$ But these are Trivial subgroups.

$\therefore$ The total no. of subgroups = Total − Trival subGroups

$= 2 - (2)$

$\boxed{\text{SubGrps} = 0}$

26. EXAM

Which of th

a) The union

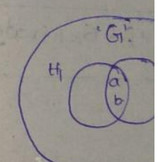
b) The Intersec

c) The union

d) Every subg

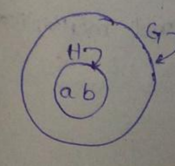
a) $(S_8, \otimes_8)$

b) TRUE. (same



c) FALSE (Refe

d)



26. EXAMPLES ON SUBGRAPHS -- 4

Which of the following are not true? (FALSE)

a) The union of any two subgroups of 'G' is also a subgroup of 'G'.

b) The Intersection " " " " " " " " "

c) The union of 2 subgroups H_1 and H_2 of group $(G,*)$ is also a subgroup of 'G'.

d) Every subgroup of an Abelian group is also Abelian group.

a) $(S_8, \otimes_8) = (\{1,3,5,7\}, \otimes_8)$

= subgroups $H_1 = \{1,3\}$ $H_2 = \{1,5\}$

	1	3
1	1	3
3	3	1

	1	5
1	1	5
5	5	1

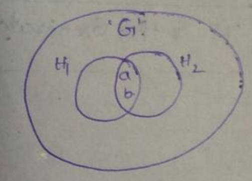
= union of H_1 and H_2 = $\{1,3\} \cup \{1,5\} = \{1,3,5\}$

	1	3	5
1	1	3	5
3	3	1	7
5	5	7	4

$\notin (H_1 \cup H_2)$

$\therefore$ $\boxed{\text{a is false}}$

b) TRUE. (same above example) $\{1,3\} \cap \{1,5\} = \{1\}$

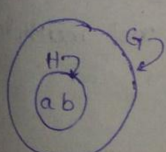


	1
1	1

$1 \rightarrow \in H_1 \cap H_2$ $\therefore$ SubGroup.

c) FALSE (Refer (a))

d)



Given 'G' is Abelian Group and hence 'H' which is subset of Abelian Group is again Abelian Group.

$\boxed{\text{Every subset of Abelian Group is Abelian.}}$

27. CYCLIC GROUPS

A group $(G,*)$ is called a cyclic Group if there exists an element $a \in G$ such that every element of 'G' can be written as a^n for some integer n. Then 'a' is called generating element/generator.

1) $G = (\{1,-1\},*)$ Now $(-1)^1 = (-1)^1 = -1$, $(-1)^2 = 1$ } The elements can be generated using (-1) as $(-1)^1$ and $(-1)^2 = 1$.

$\therefore$ $\boxed{(-1) \text{ is the GENERATOR}}$

2) $G = (\{1,\omega,\omega^2\},*)$ Now, $\omega^1 = \omega$, $\omega^2 = \omega^2$, $\omega^3 = 1$ } all the elements can be generated using "ω"

$\therefore$ $\boxed{\text{Generator} = \omega}$

3) $G = (\{1,-1,i,-i\},*)$ Now, $i = i^1 = i$

$i^2 = -1$

$(i^2)^2 = i^4 = 1$ $\quad\therefore$ $\boxed{\text{'i' is the Generator}}$

$i^3 = -i$

4) $G = (\{0,1,2,3\},\oplus_4)$ Now, $1^1 = 1$

$1^2 = 1+1 = 2$

$1^3 = 1+1+1 = 3$

$1^4 = 0$ $(4 \bmod 4 = 0)$ } Operation defined here is addition not multiplication

Now, $3^1 = 3$

$3^2 = 6 \bmod 4 = 2$

$3^3 = 9 \bmod 4 = 1$

$3^4 = 12 \bmod 4 = 0$ } '3' is also a Generator

28. EXAMPLES

→ If $(G,*)$ ...

i) a^{-1} is als...

ii) The Ord...

Ex: $(\{0,1,2,3\},$ ...

$1^1 = 1$ $\quad 1^2 =$ ...

	0	1
0	0	1
1	1	2
2	2	3
3	3	4

$3^1 = 3$

$3^2 = 2$

$3^3 = 1$

$3^4 = 0$

$\therefore$ $O(1) = 4$

$O(3) = 4$

Ex $(S_5, 5) =$ ...

$2^1 = 2$, $2^2 = 4$ ...

$3^1 = 3$, $3^2 = 4$, ...

$O(3) = 4$

$O(2) = 4$

EXAMPLES ON CYCLIC GROUPS

→ If $(G,*)$ is a cyclic group with generator 'a' then

i) a^{-1} is also a generator

ii) The order of the Generator $= O(G)$ = Order of the Group.

Ex: $(\{0,1,2,3\}, \oplus_4)$

$1^1 = 1 \quad 1^2 = 2,\ 1^3 = 3 \quad 1^4 = 4 \bmod 4 = 0.$ ∴ 1 is the generator

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	4	1	2

Now, the Inverse of $a*a^{-1} = e$ (Identity ele)

Here 0, is the Identity element $\Rightarrow \cancel{1*a^{-1}=0}$

$\Rightarrow \cancel{a^{-1}=0}$

∴ check in 1 row where '0' is present

∴ under '3' numbered column.

∴ $\boxed{\text{Inverse of 1 is '3'}}$

$$\left.\begin{array}{l} 3^1 = 3 \\ 3^2 = 2 \\ 3^3 = 1 \\ 3^4 = 0 \end{array}\right\} \therefore \text{3 is also a Gerator.}$$

∴ $\boxed{\text{1st point is satisfied}}$

∴ $O(1) = 4$ (because $1^4 = 0$ (identity element)

$O(3) = 4$ ($3^4 = 0$ (Identity element).

Ex: $(S_5, 5) = (\{1,2,3,4\}, \otimes_5)$

$$\left.\begin{array}{l} 2^1 = 2,\ 2^2 = 4,\ 2^3 = 3,\ 2^4 = 1 \\ 3^1 = 3,\ 3^2 = 4,\ 3^3 = 2,\ 3^4 = 1 \end{array}\right\} \text{2 and 3 are generators}$$

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Now, the identity element is '1' Now

'3' is the Inverse of '2'.

$$\left.\begin{array}{l} O(3) = 4 \\ O(2) = 4. \end{array}\right\} = O(4). = O(5).$$

29. THEOREM ON CYCLIC GROUPS

Let $(G,*)$ be a cyclic group of order 'n' with generator 'a' then

1) The no. of generators in $G = \boxed{\phi(n)}$ ← Eulers function of 'n'.

2) a^m is also generator of G if $GCD(m,n) = 1$

$\boxed{\phi(n) = \text{The no. of numbers that are less than 'n' and relatively prime to n.}}$

Ex

Let $(G,*)$ be a cyclic group of order 8 with generator 'a'

(i) No. of generators in $G = ?$ 4

(ii) which of the following is not a generator of G ?

(a) a^2 (b) a^3 (c) a^5 (d) a^7

Now, Here $n=8 \Rightarrow$ The set $\phi(n) = \{1,3,5,7\}$

$\therefore$ No. of generators in $\boxed{G = 4}$

Now, $n = 8$. Now, $\because$ a^m is also generator if $GCD(m,n)=1$

$\Rightarrow a^2 = GCD(2,8) \neq 1$ — NOT Generator

$\Rightarrow a^3 = GCD(3,8) = 1$ — Generator

$\Rightarrow a^5 = GCD(5,8) = 1$ — Generator

$\Rightarrow a^7 = GCD(7,8) = 1$ — Generator.

The problem with this method is as the value of 'n' increases then it is difficult to find the prime no's less than 'n'. So the product rule has been defined.

If $n = p \times q$ (If 'n' can be written as product of two distinct prime numbers 'p' and 'q' then)

$\boxed{\phi(n) = \phi(p)\,\phi(q)}$

↓

The Advantage with this method is if 'p' is a prime number then $\boxed{\phi(p) = (p-1)}$ $[\phi(7) = 6,\ \phi(11) = 10]$

Now, $\phi(77) = \phi(7) \times \phi(11)$

$= 6 \times 10 = 60$ //.

Now, In the above procedure Both 'p' and 'q' should be distinct if they are same prime numbers then the formula is

$$\boxed{\phi(p^n) = p^n - p^{n-1}}$$

'p' should be prime

$\phi(25) = \phi(5^2) = 5^2 - 5$

$\boxed{\phi(25) = 20}$

Now, $\phi(84) = 2\times2\times3\times7$

$\phi(84) = \phi(2^2 \times 3\times7)$

$= \phi(2^2)\times\phi(3)\times\phi(7)$

$= (2^2-2)\times(2)\times(6)$ $\left[\begin{array}{l}\phi(3) = (3-1) = 2 \\ \phi(7) = (7-1) = 6\end{array}\right.$ $\phi(2^2) = 2^2 - 2^1 = 2$

$= 2\times2\times6$

$\boxed{\phi(84) = 24}$

EXAMPLES ON CYCLIC GROUPS:

1) $G_1 = (\{1,2,3,4,5,6\}, \otimes_7)$ Find all the generators of G_1, G_2, G_3.

2) $G_2 = (\{0,1,2,3,4\}, \oplus_5)$

3) G_3 $(\{1,3,5,7\}, \otimes_8)$

Now, $G_1 = (\{1,2,3,4,5,6\}, \otimes_7)$

No. of generators $= \phi(6) = \{1,5\} = 2$ generators $\Rightarrow a^1, a^5$ are generators

Now, 1 cannot be generator because it is identity element.

2 is not generator $2^1 = 2$ $2^4 = 2$; $2^2 = 4$ $2^5 = 4$; $2^3 = 1$ $2^6 = 1$ $\Big\}$ 4, 5, 6 are not generated

3 is generator.

$3^1 = 3$ $3^5 = 3^2\times3^3$

$3^2 = 2$ $= 2\times6$

$3^3 = 6$ $= 5$ (12 mod 7)

$3^4 = 3^2\cdot3^2 = 4$ $3^6 = 3^3\times3^3$

$= 6\times6$

$= 1$ (36 mod 7)

The 2 generators are 3^1 and 3^5

$= 3$ and 5

Now,

(2) $G_2 = (\{0,1,2,3,4\}, \oplus_5)$ $o(G) = (5)$

No. of generators = $\phi(5) = \{4\} = \{0,1,2,3,4\} \Rightarrow a^0, a^1, a^2, a^3, a^4$ are generators

Now, 0 is the identity element $\Rightarrow$ it cannot be generator

$1^1 = 1$ $1^2 = 2$ $1^3 = 3$ $1^4 = 4$ $1^5 = 0$. 1 is generator.

$= 1^1 \quad 1^2 \quad 1^3 \quad 1^4$ are generators.

1	2	3	4

are generators

(3) $G_3 = (\{1,3,5,7\}, \otimes_8)$

Now, $\phi(4) = \{1,3\}$. $\therefore$ a^1, a^3 are the 2 generators.

Now 1 is identity element

$3^1 = 3$ $3^2 = 1$ $3^3 = 3$ $3^4 = 1$ $\therefore$ 3 is not generator

$5^1 = 5$ $5^2 = 1$ $5^3 = 5$ $5^4 =$ not generator

$7^1 = 7$ $7^2 = 1$ not generator.

$\therefore$ There are no generators for this group $\Rightarrow$ The group is not cyclic group.

31. SOME POINTS ON CYCLIC GROUPS.

For cyclic groups, the following properties hold good. g = generator (assume)

1) Every cyclic group is an Abelian Group $[a * b = g^n \times g^m = g^{m+n} = g^m \times g^n = b * a]$
2) Every Group of prime order is cyclic and so every group of prime order is Abelian Group.
3) Every subgroup of a cyclic group is also cyclic, but the generator of the subgroup need not be same as that of cyclic group.

   Ex: $G = \{1, -1, i, -i\}$; $H = \{1, -1\}$ so, H is subgroup of G.

   Generators of G are $i, -i$, Generator of $H = -1$

4) Let $(G, *)$ be a group of even order, then there exists atleast one element $a \in G$ $(a \neq e)$ such that $a^{-1} = a$.

5. FUNCTIONS

INTRODUCTION TO FUNCTIONS

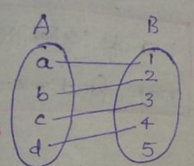
A Relation 'f' from 'set A' to a 'set B' is called a function if to each element $a \in A$, we can assign an unique element of 'B'. It is denoted as $f: A \to B$. A is domain and 'B' is co-Domain.

Range: Range of function is $\{y / y \in B \text{ and } (x,y) \in f\}$ so range of $f \subseteq B$.

A function $f: A \to A$ is called a function on 'A'.

$f: (A \to B)$

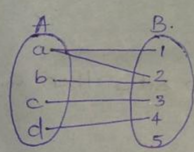
Range = {1,2,3,4}



{(a,1)(b,2)(c,3)(d,4)}

Image of a = 1
b = 2
c = 3
d = 4

$f: (A \to B)$

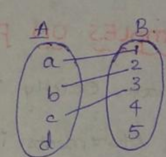


R

Not a function

{(a,1)(a,2)}

Same ele is mapped to two elements.



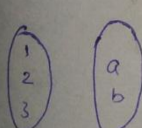
Not function

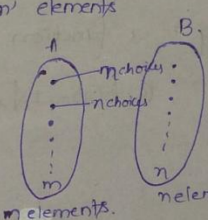
'd' is not mapped.

COUNTING THE FUNCTIONS

$f: A \to B$ Assume 'A' has 'm' elements

'B' has 'n' elements





1 has 2 choices (1,a) (1,b)

2 has 2 choices (2,a)(2,b)

3 has 2 choices (3,a)(3,b)

$n \times n \times n \times \dots$ (m) times

$= n^m$

$\therefore$ $\boxed{\text{The no. of functions from } A \to B = n^m}$

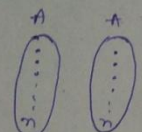
$= (\text{No. of ele in B})^{\text{No. of ele in A}}$

Now,

The total no. of Relations from $A \to B$ which are not functions is

The no. of Relations that are not functions $= \boxed{2^{mn} - (n)^m}$

Now, if we are defining the function on the same set $f: A \to A$ then, The total no. of Relations that are not functions are



No. of Relations $= 2^{n^2}$

No. of functions $= n^n$

$\therefore$ No. of Relations that are not functions $= 2^{n^2} - n^n$

3. EXAMPLES ON FUNCTIONS

If there are exactly 81 functions from set A to set B then which of the following statements is not True?

a) $|A| = 4$ $|B| = 3$

b) $|A| = 2$ $|B| = 9$

c) $|A| = 1$ $|B| = 81$

d) $|A| = 9$ $|B| = 9$.

No. of functions from $A \to B = (\text{No. of ele in B})^{\text{No. of ele in A}}$

option 1 $\Rightarrow 3^4 = 81$ ✓ function

2 $\Rightarrow 9^2 = 81$ ✓

3 $\Rightarrow 81^1 = 81$ ✓

4 $\Rightarrow 9^9 \neq 81$. $\therefore$ option '4' is false.

Which of the following is a function if domain is set of all real no's.

a) $f(x) = 1/x$ (b) $f(x) = \sqrt{x}$ (c) $h(x) = \pm\sqrt{x^2+1}$ d) $\phi(x) = |x|$

Domain = All Real no's.

a) $f(x) = 1/x$ if $x = 0$?? then the image will not be present for '0' so this is not a function.

b) $f(x) = \sqrt{x}$: for $-ve$ no's square root is not possible (Not function)

c) Not function because for every element there are two values $+x$ and $-x$.

d) function.

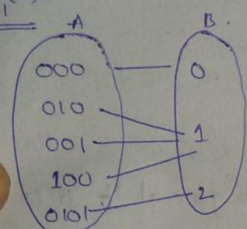
→ Consider the following relations from set of all bit strings to set of all integers.

$f_1(S)$ = The no. of 1's in the bit string 'S'.

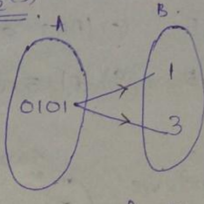
$f_2(S)$ = The position of a 0-bit in a bit string 'S'.

which of the above relations are functions.

$f_1(S)$



$f_2(S)$



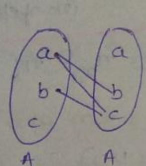
['0' is present at position '1' and '3' so it will be mapped to 2 elements in B.

Not, a function

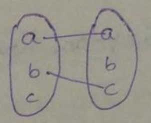
4. EXAMPLES ON FUNCTIONS

Which of the following relations on set A is a function? $A = \{a, b, c\}$

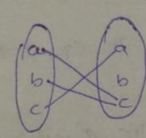
a) $R_1 = \{(a,b)(b,c)(a,c)\}$ (b) $R_2 = \{(a,a)(b,c)\}$ c) $R_3 = \{(a,c)(b,c)(c,a)\}$



NOT function



Not function



Function.

b) State True / False ?

S1) There exists equalence Relation which is function. TRUE $\{(1,1)(2,2)(3,3)\}$.

S2) The functions $f(x) = x$, $g(x) = \sqrt{x^2}$ are identical FALSE

S3) $f(x) = \log_e x^2$ and $g(x) = 2\log_e x$ are identical. FALSE

S4) The domain of $f(x) = \frac{1}{\sqrt{|x| - x}}$ is $(-\alpha, 0)$. $\Rightarrow |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$ TRUE.

if $x > 0 \Rightarrow f(x) = 1/0$ ∴ $f(x)$ is not defined for +ve no's

∴ Domain = $(-\alpha, 0)$

S2

$f(x) = x$	$g(x) = \sqrt{x^2}$
$f(1) = 1$	$g(1) = 1$
$f(-1) = -1$	$g(-1) = 1$

$(-1 \neq 1)$ ∴ NOT IDENTICAL

S3

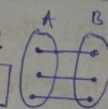
$f(x) = \log_e x^2 \Rightarrow f(-1) = \log_e (-1)^2 = \log_e 1 = 0$

$g(x) = 2\log_e x = \downarrow 2\log_e(-1)$ = does not exist.

$g(-1) = \uparrow$

5. ONE-ONE FUNCTIONS (INJECTION)

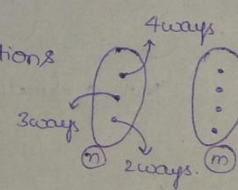
A function 'f' from a set 'A' to set 'B' is said tobe one-to-one if no two elements in 'A' are mapped to same element in 'B'. $\boxed{|B| \geq |A|}$



→ If there are exactly 120 one-to one functions possible from A to B then which of the following is not true?

a) $|A| = 5$ and $|B| = 5$ (c) $|A| = 3$ and $|B| = 6$
b) $|A| = 4$ and $|B| = 5$ (d) $|A| = 5$ and $|B| = 4$

No. of one-to one functions



∴ In general if 'A' is having 'n' elements and B is having 'm' elements then the no. of one-one function from $A \to B = (m)(m-1)(m-2)(m-3)\cdots(m-n+1)$

$$= {}^mP_n = \boxed{(\text{No of ele in B})_{P_{(\text{no. of ele in A})}}}$$

if m, n are equal then the no. of one-one functions $= {}^nP_n = n!$

Now, In the Question (No. of one-one functions $= {}^mP_n = 120$)

⇒ op1 : $|A| = 5$ $|B| = 5$ $= {}^5P_5 = 5! = 120$. TRUE

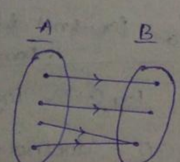
⇒ op2 : $|A| = 4$ $|B| = 5$ $= {}^5P_4 = 5 \times 4 \times 3 \times 2 = 120$. = TRUE

⇒ op3 : $|A| = 3$ and $|B| = 6 \Rightarrow {}^6P_3 = 6 \times 5 \times 4 = 120$.

⇒ op4 : $|A| = 5$ $|B| = 4$ (${}^4P_5$) — Not possible

6. ONTO FUNCTIONS

A function $f : A \to B$ is said tobe onto if each element of 'B' is mapped by atleast one element of 'A'. i.e Range of $f = B$.



onto function

The condition for a function tobe onto is $\boxed{|B| \leq |A|}$ ⇒ If this doesnt satisfy then func is not onto.

↳ we cannot say a func is onto if it satisfy the above condition.

If $|A| = |B|$ then the no. of onto functions $= n!$

7. EXAMPLES ON ONTO FUNCTIONS - 1

If $|A| = m$ and $|B| = n$, $(m>n)$ then the no. of onto fundions possible from A to B is.

$$\boxed{n^m - n_{c_1}(n-1)^m + n_{c_2}(n-2)^m - n_{c_3}(n-3)^m + \cdots + (-1)^n n_{c_{n-1}}(1)^m}$$

Ex: If $|A| = 6$, $|B| = 3$ then the no. of onto functions from A to B is —?

$|A| = 6 \Rightarrow m = 6$

$|B| = 3 \quad n = 3.$

$$\Rightarrow 3^6 - 3_{c_1}(2)^6 + 3_{c_2}(1)^6 - 3_{c_3}(0)^6 = 729 - 3\times64 + 3\times1 = 540$$

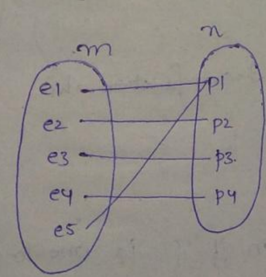
Ex: If $|A| = n$, $|B| = 2$ $(n>2)$ then the no. of onto functions from A to B?

$$\Rightarrow 2^n - 2_{c_1}(1)^n = \boxed{2^n - 2.}$$

8. EXAMPLES ON ONTO FUNCTIONS - 2

In how many ways we can assign 5 employees to 4 projects, so that every employee is assigned to only one project and every project is assigned to atleast one employee.?

Sol:



$m = 5 \quad n = 4.$

Required ways: $4^5 - 4_{c_1}(3)^5 + 4_{c_2}(2)^5 - 4_{c_3}(1)^5 + 0$

$= 1024 - (243\times4) + 6(32) + (-4) = 240$

$\boxed{240 \text{ ways.}}$

9. EXAMPLES ON ONTO FUNCTIONS - 3

Consider the following functions on set of all integers $f(x) = x^2, g(x) = x^3$ and $h(x) = \lceil x/2 \rceil$ which of the following is TRUE?

S1) f_1 is one-one (FALSE)   S4) g is onto (FALSE)

S2) f is onto (FALSE)   S5) h is one-one (TRUE)

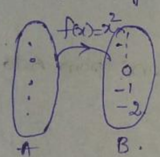
S3) g is one-one (TRUE)   S6) 'h' is -onto

1>

$f(x) = x^2$

$\left.\begin{array}{l} f(1) = 1 \\ f(-1) = 1 \end{array}\right\}$ Two elements are mapped to same element so NOT ONE-ONE

2> Set of Integers $= \{-\infty - - - 0 - - - \infty\}$.

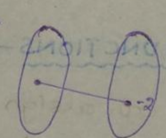


square of a number cannot be Negative but so Negative nos in B cannot be mapped with any element in 'A' not one-one.

3> $g(x) = x^3$

~~Not~~ one-one function

4>



Not pre Image for (-2) ∴ Not onto

5> $h(1) = \lceil 1/2 \rceil$

$h(1) = 1$
$h(2) = 1$ — Not one-one function. and onto function also.

10. BIJECTION

A function $f: A \to B$ is called a Bijection if 'f' is one-to-one as well as onto.

→ If 'A' and 'B' are finite sets then Bijection from 'A' to 'B' is possible if $|A| = |B|$

→ If $|A| = |B| = n$ then No. of Bijections possible from A to B is $n!$

one-one $f: A \to B$.
$|A| \le |B|$

onto function $f: A \to B$.
$|A| \ge |B|$

$\underbrace{|A| = |B|}$ Bijection function condition.

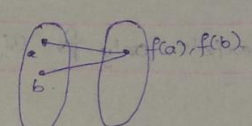
11. EXAMPLE ON BIJECTION

Let $A = R - \{3\}$ and $B = R - \{1\}$. A function $f: A \to B$ is defined by $f(x) = \frac{x-2}{x-3}$ which of the following is true?

a) f is one-one but not onto(X) (c) f is bijection (X)

b) f is onto but not one-one (d) f is neither one-one or onto.

1) Let $f(a) = f(b)$



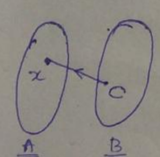
$= \frac{a-2}{a-3} = \frac{b-2}{b-3}$

$= (a-2)(b-3) = (b-2)(a-3)$

$= \cancel{ab} - 3a - 2b + \cancel{6} = \cancel{ab} - 3b - 2a + \cancel{6}$

$\Rightarrow \boxed{a=b}$ ∴ Not one to one

2)



$f(x) = c$

$\Rightarrow \frac{x-2}{x-3} = c$

$\Rightarrow x - 2 = cx - 3c$

$\Rightarrow x(1+c) = -3c + 2$

$\Rightarrow \boxed{x = \frac{2-3c}{1-c}}$ for every element 'c' we can find an element 'x' in 'A'.

12. INVERSE OF A FUNCTION

Let $f: A \to B$, if the Inverse relation $f^{-1}: B \to A$ is a function then it is called Inverse of function 'f' and it is ~~call~~ denoted by $f^{-1}: B \to A$

<u>Theorem</u>: Inverse of $f: A \to B$ exists iff 'f' is a bijection

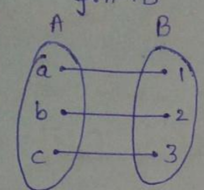
Q) which of the following functions have inverse defined on their Ranges?

a) $f(x) = x^2$ b) $f(x) = x^3$.

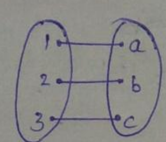
Sol

$f : A \to B$

function.

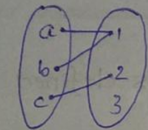


$f^{-1} : B \to A$



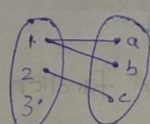
function $\Rightarrow f^{-1}$ exists.

$f : A \to B$



function

$f^{-1} : B \to A$



Not function

For a function to have Inverse it should be one-one function and onto function.

$\therefore$ $\boxed{\text{Inverse exists only for Bijection functions.}}$

a) $f(x) = x^2$

$\left.\begin{array}{l} f(1) = 1 \\ f(-1) = 1 \end{array}\right\}$ Not one-one $\Rightarrow$ Not Bijection.

b) $f(x) = x^3$. $\to$ one-one, onto, Bijection $\Rightarrow$ Inverse exists.