Math 628: Functions of Complex Variables

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1 Lecture 1

Let $a + bi$ where $a, b \in \mathbb{R}$ and $i^2 + 1 = 0$.
Let $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$. Then

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$$

a is the real part ($a = \text{Re}\{z\}$) and b is the imaginary part ($b = \text{Im}\{z\}$).

$$z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$$

$$z_1 z_2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1)i$$

Let $z = a + bi$, its complex conjugate is $\overline{z} = a - bi$.
Modulus: $|z| = \sqrt{a^2 + b^2}$, $|z|^2 = a^2 + b^2$

$$z \cong z = a^2 + b^2 = |z|^2$$

$$\frac{1}{3+4i} = \frac{1}{3+4i} \cdot \frac{3-4i}{3-4i} = \frac{3-4i}{25} = \frac{3}{25} + \frac{-4}{25}i$$

Note: $0 = 0 + 0i$
For $a, b \neq 0$,

$$\frac{1}{z} = \frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

$\frac{1}{z}$ is well defined if and only if $z \neq 0$ ($a, b \neq 0$).

$$z \cdot \frac{1}{z} = (a+bi)(\frac{a-bi}{a^2+b^2}) = \frac{a^2+b^2}{a^2+b^2} = 1$$

$$\frac{z_1}{z_2} = \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + \frac{a_2 b_1 - a_1 b_2}{a_2^2 + b_2^2} i$$

Let $z = a + bi$ and $\overline{z} = a - bi$. Then $z + \overline{z} = 2a$

$$\text{Re}\{z\} = a = \frac{1}{2}(z + \overline{z})$$

Furthermore, $z - \overline{z} = 2bi$

$$\text{Im}\{z\} = b = \frac{1}{2i}(z - \overline{z})$$

$$a^2 \leq a^2 + b^2 \to a \leq \sqrt{a^2 + b^2}$$

$$\text{Re}\{z\} \leq |z| \quad \text{Im}\{z\} \leq |z|$$

Note that if $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$,

$$|z_1 z_2| = |z_1||z_2|$$

$$\overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1)i$$

$$\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2} = (a_1 - b_1 i)(a_2 - b_2 i) = (a_1 a_2 - b_1 b_2) - (a_1 b_2 + a_2 b_1)i$$

$$(\overline{z_1})(\overline{z_2}) = (a_1 - b_1 i)(a_2 - b_2 i)$$

Similarly, $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

$$\begin{aligned} |z_1 z_2|^2 &= (z_1 z_2)|(z_1 z_2)| \\ &= z_1 z_2 |z_1||z_2| \\ &= z_1 |z_1| z_2 |z_2| \\ &= |z_1|^2 |z_2|^2 \end{aligned}$$

$$|z_1 z_2|^2 = |z_1||z_2|$$

Note:

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i \to \overline{z_1 + z_2} = (a_1 + a_2) - (b_1 + b_2)i$$

$$\overline{z_1} + \overline{z_2} = (a_1 - b_1 i) + (a_2 - b_2 i) = (a_1 + a_2) - (b_1 + b_2)i$$

Therefore

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$

Note: $\overline{(\overline{z})} = z$ and $|z| = |\overline{z}|$.
Preface: $\text{Re}\{z\} = \frac{1}{2}(z + \overline{z}) \to 2\,\text{Re}\{z_1 \overline{z_2}\} = z_1 \overline{z_2} + \overline{(z_1 \overline{z_2})} = z_1 \overline{z_2} + \overline{z_1} z_2$

$$\begin{aligned} |z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} \\ &= (z_1 + z_2)(\overline{z_1} + \overline{z_2}) \\ &= z_1 \overline{z_1} + z_2 \overline{z_2} + z_1 \overline{z_2} + \overline{z_1} z_2 = |z_1|^2 + |z_2|^2 + 2\,\text{Re}\{z_1 \overline{z_2}\} \\ &\leq |z_1|^2 + |z_2|^2 + 2|z_1 \overline{z_2}| \end{aligned}$$

Hence
$$|z_1+z_2|^2 \leq |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
Furthermore,
$$|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2 \to |z_1+z_2| \leq |z_1|+|z_2|$$
Prove: $|z_1+z_2|^2 = |z_1-z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.
$$\begin{aligned} |z_1+z_2|^2 + |z_1-z_2|^2 &= (z_1+z_2)\overline{(z_1+z_2)} + (z_1-z_2)\overline{(z_1-z_2)} \\ &= (z_1+z_2)(\overline{z_1}+\overline{z_2}) + (z_1-z_2)(\overline{z_1}-\overline{z_2}) \\ &= z_1\overline{z_1} + z_2\overline{z_2} + \overline{z_2}z_1 + z_1\overline{z_1} + z_2\overline{z_2} - z_1\overline{z_2} - z_2\overline{z_1} \\ &= |z_1|^2 + |z_1|^2 + |z_2|^2 + |z_2|^2 \\ &= 2(|z_1|^2 + |z_2|^2) \end{aligned}$$

Suppose $|z_1| < 1$ and $|z_2| < 1$. Prove $\left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right| < 1$ and $\left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right| = 1$ if either $|z_1| = 1$ or $|z_2| = 1$.

$$\begin{aligned} \left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right|^2 &< 1 \\ |z_1-z_2|^2 &< |1-z_1\overline{z_2}|^2 \\ 0 &< |1-z_1\overline{z_2}|^2 - |z_1-z_2|^2 \\ &= (1-z_1\overline{z_2})(1-\overline{z_1}z_2) - (z_1-z_2)(\overline{z_1}-\overline{z_2}) \\ &= 1 - z_1\overline{z_2} - \overline{z_1}z_2 + z_1\overline{z_1}z_2\overline{z_2} - z_1\overline{z_1} - z_2\overline{z_2} + z_1\overline{z_2} + \overline{z_1}z_2 \\ &= 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2 \\ &= (1-|z_1|^2)(1-|z_2|^2) \end{aligned}$$
$$0 < (1-|z_1|^2)(1-|z_2|^2)$$
because both $|z_1| < 1$ and $|z_2| < 1$

If either $|z_1| = 1$ or $|z_2| = 1$, then
$$(1-|z_1|^2)(1-|z_2|^2) = 0 \to \left|\frac{z_1-z_2}{1-z_1\overline{z_2}}\right| = 1$$

2 Lecture 2

Prove that $||z_1| - |z_2|| \leq |z_1 - z_2|$.
$$\begin{aligned} |z_1| &= |z_1 - z_2 + z_2| \leq |z_1-z_2| + |z_2| \to |z_1| - |z_2| \leq |z_1 - z_2| \\ |z_2| &= |z_2 - z_1 + z_1| \leq |z_2-z_1| + |z_1| \to |z_2| - |z_1| \leq |z_1 - z_2| \end{aligned}$$
$$||z_1| - |z_2|| \leq |z_1 - z_2|$$

Let X be a nonempty set. A map $d : X \times X \to \mathbb{R}$ is called a metric on X if

1. $d(x, y) \geq 0 \; \forall x, y \in X$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x) \forall x, y \in \mathbb{R}$
4. $d(x, z) \leq d(x, y) + d(y, z), x, y, z \in X$

If so, then (X, d) is called a metric space.
Let $\mathbb{C}$ be the set of all complex numbers. Define $d(z_1, z_2) = |z_1 - z_2|$ where $z_1, z_2 \in \mathbb{C}$.

1. $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \geq 0$ and $|z_1 - z_2| = 0 \iff z_1 - z_2 = 0 \iff z_1 = z_2$
2. $|z_1 - z_2| = |z_2 - z_1|$
3. $|z_1 - z_3| = |z_1 - z_2 + z_2 - z_3| \leq |z_1 - z_2| + |z_2 - z_3|$ Hence $d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3)$

Therefore $(\mathbb{C}, |\cdot|)$ is a metric space.

A complex number is an ordered pair of real numbers $z = (a, b)$ where $a = \text{Re}\{z\}$ and $b = \text{Im}\{z\}$. We say $(a, 0)$ is purely real and $(0, b)$ is purely imaginary. Note that $i = (0, 1)$.

Let $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$. Then

$$z_1 + z_2 = (a_1 + a_2, b_1 + b_2)$$

For each $z = (a, b)$, $\exists - z = (-a, -b)$ such that $z + (-z) = 0$.
Note: $0 = (0, 0)$ and $1 = (1, 0)$.
$\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 \in \mathbb{C}$.
$\forall z_1, z_2, z_3 \in \mathbb{C}, (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.
$\forall z_1, z_2 \in \mathbb{C}, z_1 + z_2 = z_2 + z_1$.
$\exists 0 \in \mathbb{C}$ such that $z1 = 1z = z \forall z \in \mathbb{C}$.
For each $z \in \mathbb{C}$ such that $z \neq 0, \exists z^{-1} \in \mathbb{C}$ such that $zz^{-1} = 1$.

If $z \neq 0$ then $(a, b) \neq 0$ and so $a \neq 0$ and $b \neq 0$.
If $z = (a, b)$ where $z \neq 0$, then $z^{-1} = \frac{1}{z} = \left(\frac{a}{a^2+b^2}, -\frac{b}{a^2+b^2}\right)$. Therefore $zz^{-1} = (1, 0)$.
$(C/\{0\}, \cdot)$ is an abelian group.

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

The set of all complex numbers $(\mathbb{C}, +, \cdot)$ is a field.

We write $z = (a, b)$ as $z = a + bi$ where $i^2 = -1$.
There exists a 1-1 correspondence between all points on the plane and the set of all complex numbers (seen as ordered pairs of real numbers).
By $\mathbb{C}$, we denote the complex plane where the real axis is horizontal and the imaginary

axis is vertical. By Δ, we denote the open unit disc $= \{z \in \mathbb{C} | |z| < 1\}$. By $\hat{\mathbb{C}}$, we denote $\mathbb{C} \bigcup \{\infty\}$, a Riemann sphere.
Note that $\mathcal{U}$ is the upper half plane $= z \in \mathbb{C} : \text{Im}\{z\} > 0$.
Associated to each complex number $z = (a, b)$ there exists a complex conjugate $\overline{z} = (a, -b)$ and its modulus $|z| = \sqrt{a^2 + b^2}$.

Describe the set of points:

1. $|z + 2| = |z - 1|$

$$\begin{aligned} |z+2|^2 &= |z-1|^2 \\ z &= x + yi \\ |(x+z) + yi|^2 &= |(x-1) + yi|^2 \\ (x+2)^2 + y^2 &= (x-1)^2 + y^2 \\ (x+2)^2 &= (x-1)^2 \\ x &= -\frac{1}{2} \end{aligned}$$

2. $|z - 1| = \text{Re}\{z\} + 1$

$$\begin{aligned} \sqrt{(x-1)^2 + y^2} &= x + 1 \\ (x-1)^2 + y^2 &= (x+1)^2 \\ y^2 &= 4x \end{aligned}$$

3. $\text{Re}\{z\} \geq 4$, this is $x \geq 4$

4. $|z - i| < 2$, this is a open disc of radius 2

5. $|z - 1| = |z + i|$

$$\begin{aligned} (x-1)^2 + y^2 &= x^2 + (y+1)^2 \\ y &= -x \end{aligned}$$

6. $|z| \geq 6$, this is the region outside of an open disc of radius 6

7. $|z| = a$, a circle of radius a and centered at the origin

8. $|z| < a$, an open disk of radius a

9. $|z| \leq a$, a closed disk of radius a

10. $|z| = \text{Re}\{z\} + 2$

$$\begin{aligned} \sqrt{x^2 + y^2} &= x^2 + 2 \\ x^2 + y^2 &= (x^2 + 2)^2 \\ y^2 &= 4x + 4 \end{aligned}$$

11. $|z - 1 + i| = 3$, this is a circle with center $(1, -1)$ and radius 3

Let $z = (x, y)$ be a point in a plane with length r and angle θ to the real axis. Then

$$\begin{aligned} r &= |z| = \sqrt{x^2 + y^2} \\ \cos\theta &= \frac{x}{r} \to x = r\cos\theta \\ \sin\theta &= \frac{y}{r} \to y = r\sin\theta \\ z &= x + yi = r(\cos\theta + i\sin\theta) \end{aligned}$$

Let a unit surface be represented as follows: $\hat{S} = \{x \in \mathbb{C} : |z| = 1\} = \cos\theta + i\sin\theta$.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = x + yi = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

3 Lecture 3

Let $\frac{x-yi}{x+yi} = a + bi$. Prove that $a^2 + b^2 = 1$.
Let $z = x + yi$ and $\alpha = a + bi$.

$$\begin{aligned} \frac{\overline{z}}{z} &= \alpha \\ \overline{\alpha} &= \overline{\left(\frac{\overline{z}}{z}\right)} \\ &= \frac{z}{\overline{z}} \\ \alpha\overline{\alpha} &= \frac{\overline{z}}{z} \cdot \frac{z}{\overline{z}} \\ &= 1 \\ |\alpha|^2 &= 1 \\ a^2 + b^2 &= 1 \end{aligned}$$

Let $z = a + bi$. Define $\psi(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

- $\psi(z + w) = \psi(z) + \psi(w)$
  Let $w = x + yi$ and $z = a + bi$.

$$\begin{aligned} \psi(w) &= \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\ \psi(z + w) &= \psi((a + x) + (b + y)i) \\ &= \begin{bmatrix} a + x & -b - y \\ b + y & a + x \end{bmatrix} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\ &= \psi(z) + \psi(w) \end{aligned}$$

- $\psi(zw) = \psi(z)\psi(w)$

$$\begin{aligned} zw &= (ax - by) + (bx + ay)i \\ \psi(zw) &= \begin{bmatrix} ax - by & -bx - ay \\ bx + ay & ax - by \end{bmatrix} \\ \psi(z)\psi(w) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix} \\ &= \begin{bmatrix} ax - by & -bx - ay \\ bx + ay & ax - by \end{bmatrix} \\ &= \psi(zw) \end{aligned}$$

- $\psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\psi(\lambda z) = \lambda \psi(z)$ if λ is real

$$\begin{aligned} \lambda z &= \lambda a + \lambda b i \\ \psi(\lambda z) &= \begin{bmatrix} \lambda a & -\lambda b \\ \lambda b & \lambda a \end{bmatrix} \\ &= \lambda \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ &= \lambda \psi(z) \end{aligned}$$

- $\psi(\overline{z}) = (\psi(z))^T$

$$\begin{aligned} \psi(z) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ \overline{z} &= a - bi \\ \psi(\overline{z}) &= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ &= (\psi(z))^T \end{aligned}$$

- $\psi\left(\frac{1}{z}\right) = (\psi(z))^{-1}$

$$\begin{aligned} z &= a + bi \\ \frac{1}{z} &= \frac{a - bi}{a^2 + b^2} \\ \psi\left(\frac{1}{z}\right) &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ \psi(z) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ (\psi(z))^{-1} &= \frac{1}{a^2 + b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ &= \psi\left(\frac{1}{z}\right) \text{ if } z \neq 0 \end{aligned}$$

- z is real $\iff \psi(z) = (\psi(z))^T$

$$\begin{aligned} \psi(z) &= (\psi(z))^T \\ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} &= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ -b &= b \\ b &= 0 \\ z & \text{ is real} \end{aligned}$$

- $|z| = 1 \iff \psi(z)$ is orthogonal. (Matrix A is orthogonal if $A^T = A^{-1} \iff AA^T = AA^{-1} = I$)

$$\begin{aligned} z &= a + bi \\ |z| &= a^2 + b^2 = 1 \\ \psi(z) &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \\ & \text{If } \psi(z) \text{ is orthogonal} \\ (\psi(z))^{-1} &= (\psi(z))^T \\ \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} &= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ a^2 + b^2 &= 1 \\ |z| &= 1 \end{aligned}$$

Let $\varphi : \mathbb{C} \to \Lambda$ where $\Lambda = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} : a, b \in \mathbb{R} \right\}$ and $\psi(z) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

- $\psi(z + w) = \psi(z) + \psi(w)$
- $\psi(zq) = \psi(z)\psi(w)$
- $\psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- $\psi(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- $\psi(z^{-1}) = (\psi(z))^{-1}$ if $z \neq 0$

Let $r = 1$ ($|z| = 1$).

$$\begin{aligned} (\cos\theta + i\sin\theta)^2 &= (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta) \\ &= \cos 2\theta + i \sin 2\theta \\ (\cos\theta + i\sin\theta)^3 &= (\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta) \\ &= (\cos 2\theta + i\sin 2\theta)(\cos\theta + i\sin\theta) \\ &= (\cos 2\theta\cos\theta - \sin 2\theta\sin\theta) + i(\sin 2\theta\cos\theta + \cos 2\theta\sin\theta) \\ &= \cos(2\theta + \theta) + i\sin(2\theta + \theta) \\ &= \cos 3\theta + i\sin 3\theta \end{aligned}$$

De Moivre's Theorem:
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$
where n is a postive integer.
Suppose n is a positive integer.
$$\begin{aligned}(\cos\theta + i\sin\theta)^{-n} &= \frac{1}{(\cos\theta + i\sin\theta)^n} \\ &= \frac{1}{\cos n\theta + i\sin n\theta} \\ &= \cos n\theta - i\sin n\theta \\ &= \cos(-n\theta) + i\sin(-n\theta)\end{aligned}$$
Hence,
$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \forall n \in \mathcal{Z}$$
Let n be a positive integer. The set of all values of $(\cos\theta + i\sin\theta)^{\frac{1}{n}}$ is
$$\left\{\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right\} \text{ where } k = 0, 1, 2, \dots, n-1$$
Let $z^n = 1$ where n is a positive integer.
$$1 = \cos 0 + i\sin 0 \quad (\theta = 0)$$
All roots of $z^n = 1$ are given by
$$\cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right) \text{ where } k = 0, 1, 2, \dots, n-1$$
When $k = 0$, $\cos 0 + i\sin 0 = 1$.
When $k = 1$, let $w = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$.
When $k = 2$,
$$\cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right) = w^2$$
Hence, all n^{th} (distinct) roots of $z^n = 1$ are given by $1, w, w^2, \dots, w^{n-1}$ where $w = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right)$. Thus the n^{th} roots of unity form a geometric series.

Solve $z^8 = 1$.

$$\begin{aligned}
w &= \cos\Big(\frac{2\pi}{8}\Big) + i\sin\Big(\frac{2\pi}{8}\Big) = \cos\Big(\frac{\pi}{4}\Big) + i\sin\Big(\frac{\pi}{4}\Big) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\
w^2 &= \cos\Big(\frac{4\pi}{8}\Big) + i\sin\Big(\frac{4\pi}{8}\Big) = \cos\Big(\frac{\pi}{2}\Big) + i\sin\Big(\frac{\pi}{2}\Big) = i \\
w^3 &= \cos\Big(\frac{6\pi}{8}\Big) + i\sin\Big(\frac{6\pi}{8}\Big) = \cos\Big(\frac{3\pi}{4}\Big) + i\sin\Big(\frac{3\pi}{4}\Big) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\
w^4 &= \cos\Big(\pi\Big) + i\sin\Big(\pi\Big) = -1 \\
w^5 &= \cos\Big(\frac{10\pi}{8}\Big) + i\sin\Big(\frac{10\pi}{8}\Big) = \cos\Big(\frac{5\pi}{4}\Big) + i\sin\Big(\frac{5\pi}{4}\Big) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \\
w^6 &= \cos\Big(\frac{12\pi}{8}\Big) + i\sin\Big(\frac{12\pi}{8}\Big) = \cos\Big(\frac{3\pi}{2}\Big) + i\sin\Big(\frac{3\pi}{2}\Big) = -i \\
w^7 &= \cos\Big(\frac{14\pi}{8}\Big) + i\sin\Big(\frac{14\pi}{8}\Big) = \cos\Big(\frac{7\pi}{4}\Big) + i\sin\Big(\frac{7\pi}{4}\Big) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i
\end{aligned}$$

Let $z = r(\cos\theta + i\sin\theta)$. Then

$$z^n = r^n(\cos n\theta + i\sin n\theta)\ \forall n \in \mathcal{Z}$$

and

$$z^{\frac{m}{n}} = r^{\frac{m}{n}}\Big(\cos\Big(\frac{\theta + 2\pi k}{n}\Big) + i\sin\Big(\frac{\theta+2\pi k}{n}\Big)\Big)^m \text{ where } k = 0, 1, 2, \dots, n-1$$

4 Lecture 4

Let $z = x + yi = r(\cos\theta + i\sin\theta)$ where $\arg z = \theta + 2\pi n$. The principal argument is defined as follows

$$-\pi < \text{Arg } z \leq \pi$$

and $\arg z = \text{Arg } z + 2\pi n$, $n \in \mathcal{Z}$.

Express $-1 - i$ in terms of $\cos\theta$ and $\sin\theta$.

$$\begin{aligned}
-1 - i &= r\cos\theta + ir\sin\theta \\
r\cos\theta &= -1 \\
r\sin\theta &= -1 \\
r^2 &= 2 \to r = \sqrt{2} \\
\cos\theta &= -\frac{1}{\sqrt{2}} \\
\sin\theta &= -\frac{1}{\sqrt{2}} \\
\text{Arg } z &= -\frac{3\pi}{4} \\
z &= \sqrt{2}\Big(\cos\Big(-\frac{3\pi}{4}\Big) + i\sin\Big(-\frac{3\pi}{4}\Big)\Big)
\end{aligned}$$

Evaluate $(1-\sqrt{3}i)^{\frac{1}{2}}$.

$$\begin{aligned}
r\cos\theta &= 1 \\
r\sin\theta &= -\sqrt{3} \\
r^2 &= 4 \to r = 2 \\
\cos\theta &= \frac{1}{2} \\
\sin\theta &= -\frac{\sqrt{3}}{2} \\
\theta &= -\frac{\pi}{3} \\
z &= 2\Big(\cos\Big(-\frac{\pi}{3}\Big) + i\sin\Big(-\frac{\pi}{3}\Big)\Big) \\
z^{\frac{1}{2}} &= 2^{\frac{1}{2}}\Big(\cos\Big(\frac{-\frac{\pi}{3}+2\pi k}{2}\Big) + i\sin\Big(\frac{-\frac{\pi}{3}+2\pi k}{2}\Big)\Big) \quad k = 0, 1
\end{aligned}$$

For $k=0$, $\sqrt{2}\Big(\cos\Big(-\frac{\pi}{6}\Big) + i\sin\Big(-\frac{\pi}{6}\Big)\Big) = \sqrt{2}\Big(\frac{\sqrt{3}}{2} - \frac{1}{2}i\Big) = \frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

For $k=1$, $\sqrt{2}\Big(\cos\Big(\frac{5\pi}{6}\Big) + i\sin\Big(\frac{5\pi}{6}\Big)\Big) = -\frac{\sqrt{3}}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$

Evaluate $(-8-8\sqrt{3}i)^{\frac{1}{4}}$.

$$\begin{aligned}
r\cos\theta &= -8 \\
r\sin\theta &= -8\sqrt{3} \\
r^2 &= 64 + 64(3) = 256 \to r = 16 \\
\cos\theta &= -\frac{8}{16} = -\frac{1}{2} \\
\sin\theta &= -\frac{8}{16\sqrt{3}} = -\frac{1}{2\sqrt{3}} \\
\theta &= -\frac{2\pi}{3} \\
z &= 16\Big(\cos\Big(-\frac{2\pi}{3}\Big) + i\sin\Big(-\frac{2\pi}{3}\Big)\Big) \\
z^{\frac{1}{4}} &= 2\Big(\cos\Big(\frac{-\frac{2\pi}{3}+2\pi k}{4}\Big) + i\sin\Big(\frac{-\frac{2\pi}{3}+2\pi k}{4}\Big)\Big) \quad k = 0, 1, 2, 3
\end{aligned}$$

For $k=0$, $2\Big(\cos\Big(-\frac{\pi}{6}\Big) + i\sin\Big(-\frac{\pi}{6}\Big)\Big) = 2\Big(\frac{\sqrt{3}}{2} - \frac{1}{2}i\Big) = \sqrt{3} - i$

For $k=1$, $2\Big(\cos\Big(\pi\Big) + i\sin\Big(\pi\Big)3\Big) = 2\Big(\frac{1}{2} + \frac{\sqrt{3}}{2}\Big) = 1 + \sqrt{3}i$

For $k=2$, $2\Big(\cos\Big(\frac{5\pi}{6}\Big) + i\sin\Big(\frac{5\pi}{6}\Big)\Big) = 2\Big(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\Big) = -\sqrt{3} + i$

For $k=3$, $2\Big(\cos\Big(\frac{4\pi}{3}\Big) + i\sin\Big(\frac{4\pi}{3}\Big)\Big) = 2\Big(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\Big) = -1 - \sqrt{3}i$

Express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$ using De Moivre's Theorem.

$$\begin{aligned}
(\cos\theta + i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta - i\sin^3\theta + 3i\sin\theta\cos^2\theta - 3\cos\theta\sin^2\theta &= \cos 3\theta + i\sin 3\theta \\
(\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\sin\theta\cos^2\theta - \sin^3\theta) &= \cos 3\theta + i\sin 3\theta \\
\cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta \\
\sin 3\theta &= 3\sin\theta\cos^2\theta - \sin^3\theta
\end{aligned}$$

Let $w = f(z) = f(x+yi)$.
We say $\lim_{z\to z_0} f(z) = L$ if: Given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - L| < \varepsilon$ whenever $0 < |z - z_0| < \delta$.
Properties

- $\lim_{z\to z_0}[f(z) \pm g(z)] = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z)$
- $\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z)\lim_{z\to z_0} g(z)$
- $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{\lim_{z\to z_0} f(z)}{\lim_{z\to z_0} g(z)}$ provided $\lim_{z\to z_0} g(z) \neq 0$
- $\lim_{z\to z_0} \lambda g(z) = \lambda \lim_{z\to z_0} g(z)$

A function $w = f(z)$ is continuous at z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$. That is, given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $|z - z_0| < \delta$.
Lemma: Suppose f is continuous on a disk $D(a,r) = \left\{z : |z-a| < r\right\}$ and $f(a) \neq 0$ ($|f(a)| > 0$). Then there exists $\delta > 0$ such that $|f(z)| \neq 0$ for all $z \in D(a,\delta)$.
Proof: Choose $\varepsilon = \frac{1}{2}|f(a)|$¿ Then $\varepsilon > 0$. There exists $\delta > 0$ such that $|f(z) - f(a)| < \frac{1}{2}|f(a)|$ for all $|z-a| < \delta$. Then $||f(z)| - |f(a)|| \leq |f(z) - f(a)|$. So for all $|z-a| < \delta$, we have $||f(z)| - |f(a)|| < \frac{|f(a)|}{2}$. Therefore

$$-\frac{1}{2}|f(a)| < |f(z)| - |f(a)| < \frac{1}{2}|f(a)|$$

Hence for all $|z-a| < \delta$, $|f(z)| > \frac{1}{2}|f(a)| > 0$. Therefore there exists $B(a,\delta) = |z-a| < \delta$ such that $f(z) \neq 0$.

A sequence $z_n \to z_0$ means that given $\varepsilon > 0$, there exists a positive integer N such that $|z_n - z_0| < \varepsilon$ for all $n \geq N$. Then $\left\{z_n\right\}$ converges to z_0.
A sequence $\left\{z_n\right\}$ is said to be Cauchy if given $\varepsilon > 0$, there exists a positive integer N such that $|z_m - z_n| < \varepsilon$ for all $m, n > N$.
A sequence $\left\{z_n\right\} \in \mathbb{C}$ is convergence $\iff$ $\left\{z_n\right\}$ is Cauchy. In other words, $(C, |\cdot|)$ is a complete metric space.

5 Lecture 5

Definition 5.1. Let $\mathbb{C}$ be a complex plane and let $a \in \mathbb{C}$. If $\delta > 0$, then a neighborhood N or N_δ around a is defined as follows

$$N(a, \delta) = N_\delta(a) = \left\{ z : |z - a| < \delta \right\}$$

Definition 5.2. Let $G \subseteq \mathbb{C}$. A point $x_0 \in G$ is called an interior point if there exists $\delta > 0$ such that $N_\delta(x_0) \subseteq G$.

Definition 5.3. A set $G \subseteq \mathbb{C}$ is called an open set if each point of G is an interior point.

Note: $N_\delta(a)$ and $\mathbb{C}$ are open sets.

Definition 5.4. Let $F \in \mathbb{C}$ and $x_0 \in \mathbb{C}$. Then x_0 is a limit point of F if for every $\delta > 0$, $N_\delta(x_0) \bigcap F / \left\{ x_0 \right\} \neq 0$. In other words, every neighborhood of x_0 must contain a point in F distinct from x_0.

Definition 5.5. A set $F \subseteq \mathbb{C}$ is called a closed set if every limit point of F belongs to F.

Definition 5.6. Let $F \subseteq \mathbb{C}$ and $z_0 \in \mathbb{C}$. Then z_0 is called a boundary point of F is for every $\delta > 0$, $N_\delta(z_0) \bigcap \neq 0$ and $N_\delta(z_0) \bigcap F^C \neq 0$.

Definition 5.7. The set of all boundary points of F is called the boundary of F and is written as ∂F.

Facts:

- A set G is open $\iff$ G^c is closed.
- An arbitrary union of open sets is open. In other words, if $\left\{ G_i \right\}_{i \in I}$ each G_i open, then $\bigcup_i G_i$ is open.
- A finite intersection of open sets is open. In other words, if $G_1, \dots, G_n$ are open, then $\bigcap_i^n G_i$ is open.
- A finite union of closed sets is closed. In other words, if $F_1, \dots, F_n$ are closed, then $\bigcup_i^n F_i$ is closed.
- An arbitrary intersection of closed sets is closed. In other words, if $\left\{ F_i \right\}_{i \in I}$ each F_i closed, then $\bigcap_i F_i$ is closed.

Definition 5.8. Let $K \subseteq \mathbb{C}$. A family G of open sets, G_i, $G = \left\{ G_i \right\}$ is called an open covering of K if $K = \bigcup_i G_i$.

Definition 5.9. A set $K \subseteq \mathbb{C}$ is called compact if every open covering admits a finite subcovering. In other words, if $G = \left\{ G_i \right\}$ is any open covering of K, then there exists $G_1, \dots, G_n \in G$ such that $K = \bigcup_i^n G_i$.

Theorem 5.1. A set $K \subseteq \mathbb{C}$ in compact $\iff$ K is closed and bounded.

Definition 5.10. A set K is called bounded if there exists $R > 0$ such that $K \subseteq N(0, R)$, or $K \subseteq \Big\{z : |z| \leq R\Big\}$.

Definition 5.11. Let S be a bounded set of real numbers. Then

$$\sup S = \text{lub } S = \lambda$$

This means that $x \leq \lambda$ for all $x \in S$ and given any $\varepsilon > 0$, there exists $t \in S$ such that $t - \varepsilon < t < \lambda$.

Definition 5.12. Let S be a bounded set of real numbers. Then

$$\inf S = \text{glb } S = \eta$$

This means that $\eta \leq x$ for all $x \in S$ and given any $\varepsilon > 0$, there exists $p \in S$ such that $\eta < p < \eta + \varepsilon$.

Theorem 5.2. Let $K \subseteq \mathbb{C}$. If $f : K \to \mathbb{C}$ is continuous and K is compact, then there exists $R > 0$ such that $|f(z)| \leq R$ for all $z \in K$. Furthermore, there exists $z_1, z_2 \in K$ such that $|f(z_1)| = \sup_{z \in K} |f(z)|$ and $|f(z_2)| = \inf_{z \in K} |f(z)|$,

Definition 5.13. Let $F \subseteq \mathbb{C}$. Then the derived set F' (of F) is the set of all limit points of F.

Note: The closure of F is written as $\overline{F} = F \bigcup F'$.

Definition 5.14. A set F is dense in $\mathbb{C}$ if $\overline{F} = \emptyset$. In other words, given any $z \in \mathbb{C}$, every neighborhood $N_\delta(z)$ must intersect F.

Definition 5.15. Let X be a metric space and $K \subseteq X$. Let $x_0 \in K$. Then

$$d(x_0, K) = \inf \Big\{d(x_0, x) : x \in K\Big\}$$

and

$$\text{diam } K = \sup \Big\{d(x_1, x_2) : x_1, x_2 \in K\Big\}$$

Let X be a metric space and $F, K \subseteq X$ such that F is compact and K is closed. If $F \bigcap K = \emptyset$, prove that $d(F, K) > 0$.

Note: $d(F, K) = \inf \Big\{d(x, y) : x \in F, y \in K\Big\}$.
Let $K = \Big\{(x,) : x \in \mathbb{R}, y = 0\Big\}$ and $F = \Big\{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}, y = e^x\Big\}$. Then K, F are closed. K is not compact. Furthermore, $K \bigcap F = \emptyset$ but $d(K, F) \not> 0$.

Definition 5.16. Let $S \subseteq \mathbb{C}$ and $x_0 \in \overline{S}$. Then there exists a sequence $z_i \in S$ such that $z_n \to z0$.

Definition 5.17. Let X be a metric space. If $X = S_1 \bigcup S_2$ where $S_1, S_2 \neq \emptyset$, both S_1, S_2 are open and $S_1 \bigcap S_2 = \emptyset$, then X is not connected.

Fact: A metric space X is connected if otherwise. In other words, X is connected if there exists no separation of X.
Fact: Equivalently, X is connected $\iff$ the only subsets of X that are both open and closed are $\emptyset$ and X.
Fact: $S \subseteq \mathbb{R}^1$ is connected $\iff$ S is an interval.

Theorem 5.3. If $S \subseteq \mathbb{C}$ is connected, then given any two points $z_1, z_2 \in \mathbb{C}$, there exists a polygon joining z_1, z_2 that is contained in S.

Corollary: If $S \subseteq \mathbb{C}$ is connected and open, then any two points in S can be joined by a polygon whose segments are parallel to the real or imaginary axis.

Definition 5.18. If $K \subseteq \mathbb{C}$ is compact and $f : K \to \mathbb{C}$ is continuous, then $f(K)$ is compact.

Definition 5.19. If $K \subseteq \mathbb{C}$ is connected and $f : K \to \mathbb{C}$ is continuous, then $f(K)$ is connected.

Definition 5.20. A region $\Omega \subseteq \mathbb{C}$ is a connected open set. In other words, Ω is a region $\iff$ $\Omega \subseteq \mathbb{C}$, Ω is open, Ω is connected.

6 Lecture 6

Example Problems:

- $\Big\{z : 0 < |z| \leq 1\Big\}$: not open, not closed, not compact, connected
- $\Big\{z : 1 \leq \text{Re}\{z\} \leq 2\Big\}$: not open, closed, not compact, connected
- $\Big\{z : \text{Im}\{z\} > 2\Big\}$: open, not closed, not compact, connected
- $\Big\{z : 1 \leq z \leq 2\Big\}$: not open, closed, compact, connected
- $\Big\{z : -2 < \text{Re}\{z\} \leq 2\Big\}$: not open, not closed, not compact, connected
- $\Big\{z : |z| \leq 3 \text{ and } |\text{Re}\{z\}| \geq 1\Big\}$: not open, closed, compact, not connected
- $\Big\{z : |\text{Re}\{z\}| \geq 1\Big\}$: not open, closed, compact, not connected
- $\Big\{z : |z| \geq 5 \text{ and } |\text{Im}\{z\}| \geq 1\Big\}$: not open, closed, compact, not connected

Definition 6.1. Simply Connected Example: $\mathbb{C}/\Big\{z : \text{Re}\{z\} \leq 0 \text{ and } \text{Im}\{z\} = 0\Big\}$

Every simply connected region is homomorphic to $\Delta = \left\{z : |z| < 1\right\}$.

Let X be a metric space, $A \subset A$ and $x \in X$. Then define $d(x, A)$ as follows:

$$d(x, A) = \inf \left\{d(x, A) : a \in A\right\}$$

Properties

- $d(x, a) = d(x, \overline{A})$
  Pf: Let $A \subseteq \overline{A}$. then $d(x, \overline{A}) \leq d(x, A)$. Let $\varepsilon > 0$. There exists $y \in \overline{A}$ such that $d(x, \overline{A}) \geq d(x, y) - \frac{\varepsilon}{2}$ and there exists $a \in A$ such that $s(x, a) < \frac{\varepsilon}{2}$. Then $|d(x, y) - d(x, a)| \leq d(x, a) < \frac{\varepsilon}{2}$. In particular, $d(x, y) > d(x, a) - \frac{\varepsilon}{2}$. Therefore $d(x, \overline{A}) \geq d(x, a) - \varepsilon$. Hence $d(x, \overline{A}) \geq d(x, A) - \varepsilon$. But $\varepsilon > 0$ is arbitrary. Hence $d(x, \overline{A}) \geq d(x, A)$. Thens $d(x, A) = d(x, \overline{A})$.

- $d(x, A) = 0 \iff x \in \overline{A}$
  Pf: Forward, let $x \in \overline{A}$. Then $d(x, A) = d(x, \overline{A}) = 0$. Now suppose $d(x, A) = 0$. For any $x \in \overline{A}$, there exists a sequence $\left\{a_n\right\}$ in A such that $d(x, S) = \lim d(x, a_n)$. Since $d(x, A) = 0$, then $\lim d(x, a_n) = 0$. Therefore $x = \lim a_n$ and thus $x \in \overline{A}$.

- $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.
  Pf: Let $a \in A$. Then $d(x, a) \leq d(x, y) + d(y, a)$. This means that

  $$d(x, A) \leq \inf \left\{d(x, a) : a \in A\right\} \leq \inf \left\{d(x, y) + d(y, a)\right\} \leq d(x, y) + \inf \left\{d(y, a)\right\}$$

  Therefore

  $$d(x, A) \leq d(x, y) + d(y, A)$$

  So

  $$d(x, A) - d(y, A) \leq d(x, y)$$

  Hence

  $$|d(x, A) - d(y, A)| \leq d(x, y)$$

Let K be compact and $f : K \to \mathbb{R}$ be continuous. There exists m, M such that $m \leq |f(x)| M$ for all $x \in K$. Furthermore, there exists $a, b \in K$ such that $f(a) = m$ and $f(b) = M$.
Corollary: Let $A \subseteq K$. Let $f(x) = d(x, A)$ for all $x \in X$ be continuous. If $K \subseteq X$ and K is compact and $x \in X$, there exists $y \in K$ such that $d(x, y) = d(x, K)$.
Let $A, B \subseteq X$. Then

$$d(A, B) = \inf \left\{d(a, b) : a \in A, b \in B\right\}$$

Theorem 6.1. If A and B are disjoint sets in X with B closed and A compact, then $d(A, B) > 0$.

Proof. Define $f : X \to \mathbb{R}$ as $f(x) = d(x, B)$. Claim: $f(a) > 0$ for each $a \in A$ because $A \bigcap B = \emptyset$ and B closed. A is compact therefore there exists $a \in A$ such that $f(a) = \inf \Big\{ f(x) : x \in A \Big\}$. Therefore

$$0 < \inf \Big\{ f(x) : x \in A \Big\} = d(A, B)$$

□

Let Ω be a connected and open set. Let $G \subseteq \mathbb{C}$ be open. Then f is continuous on G if and only if whenever $z_n \to z_0$ in G, $f(z_n) \to f(z_0)$. By continuous at z_0, we mean that given $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(z) - f(z_0)| < \varepsilon$ for all $|z - z_0| < \delta$.
Let $z_n \to z_0$. Then given $\delta > 0$, there exists $N > 0$ such that $|z_n - z_0| < \delta$ for all $n \geq N$. Therefore for all $n \geq N$, $|f(z_n) - f(z_0)| < \varepsilon$ and thus $f(z_n) \to f(z_0)$.
Suppose $z_n \to z_0$. Let $\varepsilon > 0$. Then there exists $N > 0$ such that $|f(z_n) - f(z_0)| < \varepsilon$ for all $n \geq N$. For this, $\varepsilon > 0$, then there exists $M > 0$ such that $|z_n - z_0| < \varepsilon$ for all $n \geq M$. Choose $\tilde{M} > \max \Big\{ M, N \Big\}$. Then for $\varepsilon > 0$, there exists $\delta > 0$ ($\delta = \varepsilon$) such that $|f(z) - f(z_0)| < \varepsilon$ for all $|z - z_0| < \delta$. Then $|f(z_n) - f(z_0)| < \varepsilon$ and $|z_n - z_0| < \varepsilon$ for all $n \geq \tilde{M}$.

7 Lecture 7

Homomorphic/ Analytic Functions: Let G be a nonempty open set $\mathbb{C}$. Let $f : G \to \mathbb{C}$ and $z \in G$. We say that f has a derivative at z, written as $f'(z)$ if the following exists

$$\lim_{h \to \infty} \frac{f(z+h) - f(z)}{h} = f'(z)$$

We say that f is holomorphic in G if $f'(z)$ exists at each $z \in G$.
The set of all homomorphic functions in G is denoted by $\mathcal{O}(G)$. It is a ring with respect to $+$ and $\cdot$. In other words, if $f, g \in \mathcal{O}(G)$, then

- $f + g \in \mathcal{O}(G)$
- $f \cdot g \in \mathcal{O}(G)$
- $\lambda f \in \mathcal{O}(G)$ where λ is a constant
- $\frac{f}{g} \in \mathcal{O}(G)$ if $g \neq 0$

Let $\mathfrak{G}(G)$ denote the set of all continuous functions in G.
Lemma: If $f \in \mathcal{O}(G)$, then $f \in \mathfrak{G}$.
Proof: The following exists: $f'(z) = \lim_{h \to \infty} \frac{f(z+h) - f(z)}{h}$. So then,

$$\begin{aligned}
\lim_{h \to \infty} f(z+h) - f(z) &= \lim_{h \to \infty} \Big(\frac{f(z+h) - f(z)}{h} \Big) \cdot h \\
&= f'(z) \cdot \lim_{h \to \infty} h \\
&= 0 \\
f &\in \mathfrak{G}(G)
\end{aligned}$$

Cauchy-Riemann Equations: Let $w = f(z)$ where $z = x + iy$ and $w = u + iv$. So then $u + iv = f(x + iy)$. Let $z \in G$ where G is an open set in $\mathbb{C}$.

Theorem 7.1. If f is holomorphic in G, then the Cauchy Riemann equations hold in G; in other words, $u_x = v_y$ and $u_y = -v_x$.

Proof. Let $f \in \mathcal{O}(G)$. Then $f'(z)$ exists for all $z \in G$, or $f'(z) = \lim_{h\to\infty} \frac{f(z+h)-f(z)}{h}$ exists for each $z \in G$. This means, given $z \in G$, $f'(z)$ exists and the limit $(f'(z))$ is independent of how $h \to 0$. So we first let $h \to 0$ through purely real values:

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) + iv(x+h,y) - u(x,y) - iv(x,y)}{h} \\ &= \lim_{h\to 0} \frac{u(x+h,y) - u(x,y)}{h} + i \lim_{h\to 0} \frac{v(x+h,y) - v(x,y)}{h} \\ &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \end{aligned}$$

Now let $h \to 0$ through purely imaginary values, in other words, $ih \to 0$:

$$\begin{aligned} f'(z) &= \lim_{h\to 0} \frac{f(z+ih) - f(z)}{h} \\ &= \lim_{h\to 0} \frac{u(x,y+h) + iv(x,y+h) - u(x,y) - iv(x,y)}{ih} \\ &= \lim_{h\to 0} \frac{-iu(x,y+h) + v(x,y+h) + iu(x,y) - v(x,y)}{h} \\ &= \lim_{h\to 0} \frac{v(x,y+h) - v(x,y)}{h} - i \lim_{h\to 0} \frac{u(x,y+h) - u(x,y)}{h} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \end{aligned}$$

Since $f'(z)$ is independent of the way it tends to zero, we that have $f'(z) = u_x + iv_x = v_y - u_y$. Equating real and imaginary parts, we get

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}$$

□

Theorem 7.2. If $w = f(z)$ is holomorphic on G where $w = u + iv$ and $z = x + iy$, then $u_x = v_y$ and $u_y = -v_x$ for all $z = (x, y) \in G$. Furthermore, since $f'(z) = u_x + iv_x$ and $|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = u_x v_y - u_y v_x$,

$$|f'(z)|^2 = \det \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

Let Ω be a region and $f \subseteq \mathcal{O}(\Omega)$.

- If $f'(z) = 0$ for all $z \in \Omega$, then f is a constant.
  Proof: If $f'(z) = u_x + iv_x = v_y - iu_y = 0$, then $u_x = v_x = 0$ and $u_y = v_y = 0$. Consider $u(x, y)$. If $u_x = u_y = 0$, then $u(x, y) = k_1$, a constant. Consider $v(x, y)$. If $v_x = v_y = 0$, then $v(x, y) = k_2$, a constant. Hence $f'(z) = k_1 + ik_2$, which itself is a constant.

- If $|f(z)|$ is constant for all $z \in \Omega$, then f is constant in Ω.
  Proof: Let $f = u + iv$ and $|f|^2 = u^2 + v^2 =$ constant. Then the derivative with respect to x gives $2uu_x + 2vv_x = 0$ and the derivative with respect to y gives $2uu_y + 2vv_y = 0$. Multiply the first equation by v and the second equation by u to get

$$\begin{aligned} v(uu_x + vv_x) &= uvu_x + v^2 v_x = 0 \\ u(uu_y + vv_y) &= u^2 u_y + uvv_y = 0 \\ uvu_x + v^2 v_x &= u^2 u_y + uvv_y \\ uvu_x - v^2 u_y &= 0 \\ uvu_x + u^2 u_y &= 0 \end{aligned}$$

  Then $u_x(u^2 + v^2) = 0$ and so $u_y = 0$ and similarly, $u_x = 0$. By the C-R equations, $v_x = 0$ and $v_y = 0$. Thus we find that $u_x = u_y = 0$ and so $u(x, y)$ is constant and $v_x = v_y = 0$ and $v(x, y)$ is constant. Therefore $f = u + iv$ is a constant.

- If $\text{Re}\{f\}$ is a constant, then f is a constant.
  Proof: Let $f = u + iv$. Then $\text{Re}\{f\} = u$, a constant. Furthermore, $u_x = u_y = 0$. By C-R equations, $u_x = v_y = 0$ and $u_y = -v_x = 0$. So $u_x = u_y = v_x = v_y = 0$. Therefore f is a constant.

- If $\text{Im}\{f\}$ is a constant, then f is a constant.
  Proof: Let $f = u + iv$. Then $\text{Im}\{f\} = v$, a constant. Furthermore, $v_x = v_y = 0$. By C-R equations, $v_x = -u_y = 0$ and $v_y = u_x = 0$. So $u_x = u_y = v_x = v_y = 0$. Therefore f is a constant.

- If $\text{Arg}(f(x))$ is a constant, then f is a constant.
  Proof: Let $f = u + iv$. Then $\text{Arg}(f) = \theta$ is a constant. Hence $\tan\theta = \tan\frac{v}{u}$ is a constant. So we have $u = kv$ for some constant k. Then $u - kv = \text{Re}\{(1 + ki)f\}$. Check:

$$(1 + ki)(u + vi) = (u - kv) + (ku + v)i \rightarrow u - kv = \text{Re}\{(1 + ki)f\}$$

  Then $\text{Re}\{(1 + ki)f\} = 0$. Therefore $(1 + ki)f$ is a constant and so f is a constant.

- If $f \in \mathcal{O}(\Omega)$ nd $\overline{f} \in \mathcal{O}(\Omega)$, then f is a constant on Ω.
  Proof: Let $f = u + iv$ and $\overline{f} = u - iv = p + iq$. If $\overline{f} \in \mathcal{O}(\Omega)$, then if $p = u$ and $q = v$, $p_x = q_y$ and $p_y = -q_x$. Therefore since $p_x = q_y$, $u_x = -v_y$. Since $p_y = -q_x$, $u_y = v_x$. Henceforth, $u_x = v_y = -v_y$ and so $v_y = 0$. Also, $v_x = u_y = -v_x$ and so $v_x = 0$. Hence $v(x, y)$ is a constant. By the same logic, since $u_x = v_y = 0$ and $u_y = -v_x = 0$, $u(x, y)$ is constant. Thus f is a constant.

8 Lecture 8

Note that if f is continuous on $[a,b]$ and differentiable on (a,b), there exists $a<c<b$ such that
$$f'(v)=\frac{f(b)-f(a)}{b-a}\to f(a+h)-f(a)=hf'(a+t)$$
where $|t|<|h|$.

Theorem 8.1. Let $f=u(x,y)+iv(x,y)$ be holomorphic on an open set $G\subseteq\mathbb{C}$. Then the Cauchy-Riemann equations hold
$$u_x=v_y \text{ and } u_y=-v_x$$

Theorem 8.2. Let $u(x,y)$ and $v(x,y)$ have continuous first partial derivatives on a region Ω such that the Cauchy-Riemann equations are satisfied. Then the function $f(z)=u(x,y)+iv(x,y)$ is holomorphic in Ω.

Proof. To show that $\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}$ exists, let $z=x+yi$ and $h=s+ti$.
$$\frac{f(z+h)-f(z)}{h}=\frac{u(x+s,y+t)-u(x,y)+iv(x+s,y+t)-iv(x,y)}{s+ti}\quad(1)$$

Now
$$u(x+s,y+t)-u(x,y)=[u(x+s,y+t)-u(x,y+t)]+[u(x,y+t)-u(x,y)]$$

By the Mean Value Theorem, there exists s_1 and t_1 such that $|s_1|<|s|$ and $|t_1|<|s|$ so that
$$u(x+s,y+t)-u(x,y+t)=su_x(x+s_1,y+t)\quad(2a)$$
where $|s_1|<|s|$, and
$$u(x,y+t)-u(x,y)=tu_y(x,y+t_1)\quad(2b)$$
where $|t_1|<|t|$.
Define
$$\varphi(s,t)=[u(x+s,y+t)-u(x,y)]-[su_x(x,y)-tu_y(x,y)]$$

Then
$$\begin{aligned}\frac{\varphi(s,t)}{s+ti}&=\frac{su_x(x+s_1,y+t)+tu_y(x,y+t_1)-su_x(x,y)-tu_y(x,y)}{s+ti}\\&=\frac{s(u_x(x+s_1,y+t)+tu_y(x,y+t_1))}{s+ti}+\frac{t(u_y(x,y+t_1)-u_y(x,y))}{s+ti}\quad(3)\end{aligned}$$

Claim: $\lim_{s+ti\to 0}\frac{\varphi(s,t)}{s+ti}=0$ because $|s|\leq|s+ti|$, $|t|\leq|s+ti|$, $|s_1|\leq|s|$ and $|t_1|\leq|t|$ and u_x and u_y are continuous. Hence
$$u(x+s,y+t)-u(x,y)=su_x+tu_y+\varphi(s,t)$$

where

$$\lim_{s+ti} \frac{\varphi(s,t)}{s+ti} = 0 \quad (4)$$

Similarly,

$$v(x+s, y+t) - v(x,y) = sv_x + tv_y + \psi(s,t)$$

where

$$\lim_{s+ti} \frac{\psi(s,t)}{s+ti} = 0 \quad (5)$$

By (1), (4) and (5),

$$\begin{aligned}
\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} &= \lim_{s+ti\to 0} \frac{su_x + tu_y + \varphi(s,t)}{s+ti} + i \lim_{s+ti\to 0} \frac{sv_x + tv_y + \psi(s,t)}{s+ti} \\
&= \lim_{s+ti\to 0} \frac{su_x - tv_x + \varphi(s,t)}{s+ti} + i \lim_{s+ti\to 0} \frac{sv_x + tu_x + \psi(s,t)}{s+ti} \\
&= \lim_{s+ti\to 0} \frac{s(u_x + iv_x) + ti(u_x + iv_x)}{s+ti} + \lim_{s+ti\to 0} \frac{sv_x + tu_x + \psi(s,t)}{s+ti} \\
&= \lim_{s+ti\to 0} \frac{(s+ti)(u_x + iv_x) + ti(u_x + v_x)}{s+ti} + \lim_{s+ti\to 0} \frac{\varphi(s,t)}{s+ti} + \lim_{s+ti\to 0} \frac{\psi(s,t)}{s+ti} \\
&= \lim_{s+ti\to 0} \frac{(s+ti)(u_x + iv_x)}{s+ti} \\
&= u_x + iv_x \\
f'(z) &= u_x + iv_x
\end{aligned}$$

□

Summary of Theorem 1 and 2: Suppose $u(x,y)$ and $v(x,y)$ are 2 real-valued functions with continuous first partial derivatives on a region Ω, a connected open subset of the complex plane. Then the complex-valued function $f(z) = u(x,y) + iv(x,y)$ is holomorphic in Ω if and only if the Cauchy-Riemann equations hold in Ω:

$$u_x = v_y \text{ and } u_y = -v_x$$

Furthermore,

$$f'(z) = u_x + iv_x$$

9 Lecture 9

Let U be an open set in $\mathbb{C}$. Let $f \in \mathcal{O}(U)$ and $g \in \mathcal{O}(U)$. Then if $f+g \in \mathcal{O}(U)$, $fg \in \mathcal{O}(U)$ and $\lambda_1 f + \lambda_2 g \in \mathcal{O}(U)$ (where $\lambda_1, \lambda_2 \in \mathbb{C}$), then $\mathcal{O}(U)$ is a ring.

Theorem 9.1. If $f \in \mathcal{O}(U)$ and if $f(U) \in U$, $4g \in \mathcal{O}(U)$ and $h = g \cdot f$, then $h \in \mathcal{O}(U)$ and

$$h'(z) = g'(f(z))f(z) \ \forall z \in U$$

Proof. Fix $z_0 \in U$. Let $w = f(z)$ and so $w_0 = f(z_0)$. To show $h'(z_0) = g'(f(z_0)) \cdot f'(z_0)$, we have

$$f(z) - f(z_0) = (f'(z_0) + \varepsilon(z))(z - z_0)$$

where $\varepsilon(z) \to 0$ as $z \to z_0$ and

$$g(w) - g(w_0) = (g'(w_0) + \eta(f(w)))(w - w_0)$$

where $\eta(w) \to 0$ as $w \to w_0$. Then

$$\begin{aligned} g(f(z)) - f(f(z_0)) &= (g'(f(z_0)) + \eta(f(z)))(f(z) - f(z_0)) \\ h(z) - h(z_0) &= (g'(f(z_0)) + \eta(f(z)))(f(z) - f(z_0))) \\ &= (g'(f(z_0)) + \eta(f(z)))(f'(z_0) + \varepsilon(z))(z - z_0) \end{aligned}$$

So

$$\frac{h(z) - h(z_0)}{z - z_0} = (g'(f(z_0)) + \eta(f(z)))(f'(z_0) + \varepsilon(z))$$

for all $z \neq z_0$. Since $f \in \mathcal{O}(U)$, f is continuous on U. So as $z \to z_0$, we have $f(z) \to f(z_0)$. This means $w \to w_0$. So taking limits,

$$\begin{aligned} \lim_{z \to z_0} \frac{h(z) - h(z_0)}{z - z_0} &= g'(f(z_0)) \cdot f'(z_0) \\ h'(z_0) &= g'(f(z_0)) \cdot f'(z_0) \end{aligned}$$

and since $z_0 \in U$ is arbitrary in $\mathcal{O}(U)$,

$$h'(z) = g'(f(z)) \cdot f'(z)$$

for all $z \in U$. □

Let $u(x, y)$ be a real valued function on U, an open set in $\mathbb{C}$ such that $u(x, y)$ has continuous second partials and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \;\; \forall (x, y) \in U$$

then $u(x, y)$ is harmonic on U.
If $f \in \mathcal{O}(\Omega)$, then all of its higher-order derivatives exist and are holomorphic.

Suppose $f = u + iv$ is holomorphic in a region Ω. Claim: Both u and v are harmonic in Ω.

Proof. Let $f \in \mathcal{O}(\Omega)$, by the above property, u and v both have continuous second partials on Ω. Furthermore,

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\Big(\frac{\partial v}{\partial y}\Big) = \frac{\partial^2 v}{\partial x \partial y} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y}\Big(-\frac{\partial v}{\partial x}\Big) = -\frac{\partial^2 v}{\partial y \partial x} \\
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y2} &= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x} = 0
\end{aligned}$$

because the second partial derivatives of $u(x,y)$ are continous. Hence $u(x,y)$ is harmonic. Similarly,

$$\begin{aligned}
\frac{\partial^2 v}{\partial x^2} &= -\frac{\partial}{\partial x}\Big(\frac{\partial u}{\partial y}\Big) = -\frac{\partial^2 u}{\partial x \partial y} \\
\frac{\partial^2 v}{\partial y^2} &= \frac{\partial}{\partial y}\Big(\frac{\partial u}{\partial x}\Big) = \frac{\partial^2 u}{\partial y \partial x}
\end{aligned}$$

Hence

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and so $v(x,y)$ is harmonic. □

Theorem 9.2. The real and imaginary parts of a holomorphic function on a region are harmonic.

Suppose $u(x,y)$ is harmonic on an open set $U \subseteq \mathbb{C}$. If there exists a harmonic function $v(x,y) \in U$ such that $f(z) = u(x,y) + iv(x,y)$ is holomorphic on U, then $v(x,y)$ is a harmonic conjugate of $u(x,y)$.

Let $u(x,y) = x^3 - 3xy^2 + y$. Determine if $u(x,y)$ is harmonic and if so, find its harmonic conjugate.

$$\begin{aligned}
u_x &= 3x^2 - 3y^2 \\
u_{xx} &= 6x \\
u_y &= -6xy + 1 \\
u_{yy} &= -6x \\
u_{xx} + u_{yy} &= 0
\end{aligned}$$

Since $u(x,y)$ have continuous second partials, then $u(x,y)$ is harmonic on $\mathbb{C}$. Suppose $v(x,y)$ is its harmonic conjuhate. Then $f = u + iv$ is holomorphic. Then

$$u_x = v_y \text{ and } u_y = -v_x$$

This means

$$\begin{aligned} v_x &= -u_y = 6xy - 1 \\ \frac{\partial v}{\partial x} &= 6xy - 1 \\ v(x,y) &= 3x^2y - x + \varphi(y) \\ v_y &= 3x^2 + \varphi'(y) = 3x^2 - 3y^2 \\ \varphi'(y) &= -3y^2 \\ \varphi(y) &= -y^3 + k \\ v(x,y) &= 3x^2y - x - y^2 + k \end{aligned}$$

Let Ω be a region. Propositions:

1. Any two harmonic conjugates must differ by a constant.
   Proof: Let $u(x,y)$ be harmonic on Ω. Suppose $v(x,y)$ and $V(x,y)$ are two harmonic conjugates of $u(x,y)$. Then $u+iv$ and $u+iV$ are both holomorphic on Ω. By Cauchy-Riemann equations, this means

   $$\begin{aligned} u_x = v_y \text{and } u_y = -v_x \\ u_x = V_y \text{ and } u_y = -V_x \end{aligned}$$

   So $\frac{\partial V}{\partial x} = \frac{\partial v}{\partial x}$ and $\frac{\partial V}{\partial y} = \frac{\partial v}{\partial y}$. Therefore $V_x - v_x = 0$ and $V_y - v_y = 0$. Then $V(x,y) - v(x,y) =$ constant.

2. Suppose v is a harmonic conjugate of u in Ω. Then $-u$ is a harmonic conjugate of v in Ω.
   Proof: v is a harmonic conjugate of u in Ω. Then $f = u + iv$ is holomorphic in Ω. So $v - iu = -if$, which is also holomorphic in Ω. Therefore $-u$ is a harmonic conjugate of v.

3. If u is a harmonic conjugate of v and v is a harmonic conjugate of u, then both u and v must be constants.
   Proof: Let $f = u + iv$ be holomorphic in Ω. Then $g = v - iu$ is holomorphic in Ω. Then $-ig = u - iv$ is holomorphic; this is $\overline{f}$. Therefore f and $\overline{f}$ are both holomorphic in Ω. Then f is a constant and so u and v are constants.

Let Ω be a region. Suppose v is a harmonic conjugate of u in Ω. Show that uv is a harmonic function on Ω.

Proof. Let $f = u + iv$ be holomorphic in Ω. Then $g = v - iu$ is also holomorphic in Ω.

$$fg = (u+iv)(v-iu) = (uv+uv) + i(v^2 - u^2) = 2uv + i(v^2 - u^2)$$

Therefore $2uv$ is harmonic and so uv is harmonic.
Since real and imaginary parts of a holomorphic function for a region are harmonic, the real part of a holomorphic function is harmonic. □

10 Lecture 10

Let $z = x + iy$ and $\overline{z} = x - iy$. Then

$$\begin{aligned}
x &= \frac{1}{2}(x + \overline{z}) \\
iy &= \frac{1}{2}(z - \overline{z}) \\
y &= -\frac{i}{2}(z - \overline{z}) \\
\frac{\partial x}{\partial z} &= \frac{\partial x}{\partial \overline{z}} \\
&= \frac{1}{2} \\
\frac{\partial y}{\partial z} &= -\frac{i}{2} \\
\frac{\partial y}{\partial \overline{z}} &= \frac{i}{2}
\end{aligned}$$

Let $f(x, y)$ exist. Then

$$\begin{aligned}
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{i}{2}\frac{\partial f}{\partial y} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\Big) \\
\frac{\partial f}{\partial \overline{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{z}} = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y} = \frac{1}{2}\Big(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\Big)
\end{aligned}$$

Define the operators ∂ and $\overline{\partial}$ as follows:

$$\begin{aligned}
\partial &= \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big) \\
\overline{\partial} &= \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)
\end{aligned}$$

Let $f = u(x, y) + iv(x, y)$. Then

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{1}{2}\Bigg(\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big) - i\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\Big)\Bigg) \\
&= \frac{1}{2}\Big((u_x + v_x) - i(v_y - u_y)\Big) \\
\frac{\partial f}{\partial \overline{z}} &= \frac{1}{2}\Bigg(\Big(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\Big) + i\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\Big)\Bigg) \\
&= \frac{1}{2}\Big((u_x + v_x) + i(u_y + v_y)\Big)
\end{aligned}$$

Suppose f is holomorphic. Then $u_x = v_y$ and $u_y = -v_x$. Then

$$\frac{\partial f}{\partial z} = \frac{1}{2}(u_x + i2u_x) = u_x + iv$$

and

$$\frac{\partial f}{\partial \bar{z}} = 0$$

Summary: Suppose $f = u(x,y) + iv(x,y)$ where u and v have continuous first partials. Then f is holomorphic if and only if $u_x = v_y$ and $u_y = -v_x$. Equivalently, $\frac{\partial f}{\partial z} = f'(z)$ and $\underset{=}{\frac{\partial \bar{z}}{}} 0$. Thus $\frac{\partial}{\partial \bar{z}} = 0$ if and only if $u_x = v_y$ and $u_y = -v_x$. Hence $f(z)$ is a holomorphic function.

Properties:

1. ∂ and $\overline{\partial}$ are $\mathbb{C}$-linear maps for which product and quotient rules apply
2. $\overline{\partial} f = \overline{(\partial \overline{f})}$
3. $\overline{\partial f} = \overline{(\partial f)}$
4. Let $f \in \mathcal{O}(\Omega)$ and so $\overline{\partial} f = 0$ and $\partial f = f'$. Let $\overline{f} \in \mathcal{O}(\Omega)$ and so $\overline{\partial f} = 0$ and $\partial f = 0$ and $\overline{f}' = (\overline{\partial} f)$. Then $\partial f = \overline{\partial} f = 0$ and so f is a constant.

A series $\left\{ z_n \right\}$ is said to converge if and only if $\text{Re}\{z_n\}$ and $\text{Im}\{z_n\}$ converges.
A power series is of the format $\sum_{n=0}^\infty a_n z^n$ where $a_n \in \mathbb{C}$ and $n \geq 0$.

Lemma: There exists $0 \leq R \leq \infty$ such that if $z \in \mathbb{C}$ and $|z| < R$, then $\sum a_n z^n$ converges.
Lemma: If $\sum a_n z^n$ has a radius of convergence R, then so does the derived series $\sum_{n=1}^\infty n a_n z^{n-1}$.
Lemma: If $a, b \in \mathbb{C}$ and $|a| < \rho$, $|b| < \rho$, then

$$\left| b^k - a^k \right| \leq k\rho^{k-1} |b-a| \quad \forall k \geq 0$$

Proof:

$$\begin{aligned} b^k - a^k &= (b-a)(b^{k-1} + b^{k-2}a^2 + b^{k-3}a^3 + \cdots + a^{k-1}) \\ &= (b-a) \sum_{j=0}^{k-1} a^j b^{k-1-j} \\ \left| b^k - a^k \right| &\leq |b-a| \sum_{j=0}^{k-1} \rho^j \rho^{k-1-j} \\ \left| b^k - a^k \right| &\leq |b-a| \sum_{j=0}^{k-1} \rho^{k-1} \end{aligned}$$

So

$$\left| b^k - a^k \right| \leq |b-a| k p^{k-1}$$

Theorem 10.1. Let $\sum a_n z^n$ have a radius of convergence $R \geq 0$ and let $D(0,R) = \left\{ z \in \mathbb{C} : |z| < R \right\}$. Then the function $f(z) = \sum a_n z^n$ is holomorphic to $D(0,R)$ and for all $z \in D(0,R)$, $f'(z) = \sum n a_n z^{n-1}$.

Proof. Define $g(x) = \sum_{n=1}^\infty na_nz^{n-1}$ where $|z| < R$. Fix z_0 with $|z_0| < R$. Choose ρ such that $|z_0| < \rho < R$. Assume $z \neq z_0$ and $|z| < \rho$. Then

$$\frac{f(z)-f(z_0)}{z-z_0} - g(z_0) = \sum_{n=2}^\infty a_n\Big(\frac{z_n - z_0^n}{z-z_0} - nz_0^{n-1}\Big)$$

Consider:

$$\begin{aligned}\left|\frac{z^n - z_0^n}{z-z_0} - nz_0^{n-1}\right| &= \left|\sum_{k=0}^{n-1}(z^kz_0^{n-1-k} - z_0^{n-1})\right| \\ &\leq \sum_{k=0}^{n-1}|z_0|^{n-1-k}\big|z^k - z_0^k\big| \\ &\leq \sum_{k=0}^{n-1}\rho^{n-1-k}k\rho^{k-1}|z-z_0| \\ &= |z-z_0|\rho^{n-2}\sum_{n=0}^{k-1}k\end{aligned}$$

Hence

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| \leq |z-z_0|\sum_{n=2}^\infty |a_n|\rho^{n-2}\frac{n(n-1)}{2}$$

Claim: $\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| \to 0$ as $z \to z_0$. Proof: If $\sum_{n=0}^\infty a_nz^n$ converges in $|z| < R$, then $\sum_{n=1}^\infty na_nz^{n-1}$ converges in $|z| < R$. Therefore $\sum_{n=2}^\infty n(n-1)a_nz^{n-2}$ converges in $|z| < R$. Hence $\sum_{n=2}^\infty n(n-1)|a_n||z|^{n-2}$ converges in $|z| < R$. Thus $\sum_{n=2}^\infty n(n-1)|a_n|\rho^{n-2}$ converges in $|z| < R$.
Hence

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} = g(z_0)$$

or $f'(z_0) = g(z_0)$ and since z_0 is arbitrary in $D(0,R)$, we are done. □

11 Lecture 11

Let the following be Riemann surfaces:

- $\Delta = \Big\{z \in \mathbb{C} : |z| < 1\Big\}$
- $\mathcal{U} = \Big\{z \in \mathbb{C} : \text{Im}\{z\} > 0\Big\}$
- $\hat{\mathbb{C}} = \mathbb{C}\bigcup\Big\{\infty\Big\}$ - Riemann sphere

The Riemann sphere is a "one point" compactification:

$$\hat{\mathbb{C}} : \mathbb{C}\bigcup\Big\{\infty\Big\}$$

of $\mathbb{C}$. It is given the Hausdorff topology such that $V \subseteq \mathbb{C}$ is open if and only if

- $V \bigcap \mathbb{C}$ is open
- if $\infty \in V$, then $\hat{\mathbb{C}}\ V$ is compact in $\mathbb{C}$

Let S^2 be defined as follows:

$$S^2 = \left\{ \vec{x} \in \mathbb{R}^3 : \vec{x} = (x_1, x_2, x_3), x_1^2 + x_2^2 + x_3^2 = 1 \right\}$$

Theorem 11.1. The stereographic function $f : S^2 \to \hat{\mathbb{C}}$, defined by

$$f(\vec{x}) = \begin{cases} \infty & \text{if } \vec{x} = (0,0,1) \\ \frac{x_1 + ix_2}{1 - x^3} \in \mathbb{C} & \text{if } \vec{x} \neq (0,0,1) \end{cases}$$

is a homomorphism.

Proof. Consider $S^2\ \left\{(0,0,1)\right\}$. Function f is continuous on $S^2\ \left\{(0,0,1)\right\}$.

$$|f(\vec{x})|^2 = \frac{x_1^2}{(1-x_3)^2} + \frac{x_2^2}{(1-x_3)^2} = \frac{x_1^2 + x_2^2}{(1-x_3)^2} = \frac{1 - x_3^2}{(1-x_3)^2} = \frac{1+x_3}{1-x_3}$$

So $|f(\vec{x})| \to \infty$ as $\vec{x} \to (0,0,1)$. Here f is continuous on all of S^2. Let $f(\vec{x}) = z \in \mathbb{C}$. Then

$$|\vec{z}|^2 = |f(\vec{x})|^2 = \frac{1+x_3}{1-x_3}$$

Then

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Let $z = \frac{x_1 + ix_2}{1 - x_3}$ or $(1 - x_3)z = x_1 + ix_2$. Substitute $z = x + iy$. Then

$$(1 - x_3)(x + iy) = x_1 + ix_2$$
$$x(1 - x_3) + iy(1 - x_3) = x_1 + ix_2$$

Therefore

$$x = \frac{x_1}{1 - x_3} = \frac{x_1}{1 - \left(\frac{|z|^2 - 1}{|z|^2 + 1}\right)} = \frac{x_1(|z|^2 + 1)}{2}$$
$$iy = \frac{x_2}{1 - x_3}$$

Here

$$x_1 = \frac{2\operatorname{Re}\{z\}}{1 + |z|^2} \text{ and } x_2 = \frac{2\operatorname{Im}\{z\}}{1 + |z|^2}$$

Then

$$f^{-1}(z) = \begin{cases} (0,0,1) & \text{if } z = \infty \\ \left(\frac{2\operatorname{Re}\{z\}}{1+|z|^2}, \frac{2\operatorname{Im}\{z\}}{1+|z|^2}, \frac{|z|^2-1}{|z|^2+1}\right) & \text{if } z \in \mathbb{C} \end{cases}$$

Clearly f^{-1} is continuous on $\mathbb{C}$. If $|z| \to \infty$, then $\frac{|z|^2-1}{|z|^2+1} \to 1$ and so $f^{-1}(z) \to (0,0,1)$ as $z \to \infty$. Thus f^{-1} is continuous on all of $\hat{\mathbb{C}}$. $\square$

A Möbius transformation is is a map $\varphi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ given by

$$\varphi(z) = \frac{az+b}{cz+d}$$

where $z \in \hat{\mathbb{C}}$ and $ad - bc \neq 0$. If $c \neq 0$, $\varphi(\infty) = \frac{a}{c}$ and $\varphi(-\frac{d}{c}) = \infty$. If $c = 0$, $\varphi(\infty) = \infty$.

Lemma: Each Möbius transformation is continuous.

Proof. $\varphi|\mathbb{C} \left\{\varphi^{-1}(\infty)\right\}$ is homomorphic and hence continuous. If $c = 0$,

$$\varphi(z) = \frac{az}{d} + \frac{b}{d} = \alpha z + \beta$$

where $\alpha \neq 0$ and $|\varphi(z)| \geq |\alpha||z| - |\beta| \to \infty$ as $|z| \to \infty$. Therefore φ is everywhere continuous. If $c \neq 0$, then

$$\varphi(z) - \frac{a}{c} = \frac{az+b}{cz+d} - \frac{a}{c} = \frac{bc-ad}{c(cz+d)} \to$$

so $|z| \to \infty$. Therefore $\varphi(z) \to \frac{a}{c}$ as $|z| \to \infty$. So φ is continuous at ∞. Finally, as $z \to -\frac{d}{c}$, then $az + b \to \frac{bc-ad}{c} \neq 0$. So

$$\left|\frac{az+b}{cz+d}\right| \to \infty$$

and so φ is continuous at $-\frac{d}{c}$. □

Theorem 11.2. The set $\bigwedge$ of all Möbius transformation is a group of homeomorphisms of $\hat{\mathbb{C}}$ onto itself. Let general linear group $GL(2,\mathbb{C})$ be the group of all invertible 2×2 complex matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the map $\Phi : GL(2,\mathbb{C}) \to \bigwedge$ given by

$$\Phi\Big(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\Big) = \frac{az+b}{cz+d}$$

is a surjective homomorphism.

Proof. Let $\varphi_1(z) = \frac{az+b}{cz+d}$ and $\varphi_2(z) = \frac{\alpha z+\beta}{\gamma z+\delta}$. Then

$$\varphi_1 \circ \varphi_2 = \varphi_1(\varphi_2(z)) \in \bigwedge$$

If $\varphi_1 \in \bigwedge$ and $\varphi_2 \in \bigwedge$, then $\varphi_1 \circ \varphi_2 \in \bigwedge$.
If $\varphi_1, \varphi_2, \varphi_3 \in \bigwedge$, then

$$\varphi_1 \circ (\varphi_2 \circ \varphi_3) = (\varphi_1 \circ \varphi_2) \circ \varphi_3$$

$$\varphi(z) = z \in \bigwedge$$

If $\varphi(z) = w = \frac{az+b}{cz+d}$, then $wcz + ws = az + b$. This means $z(wc - a) = b - wd$. Hence

$$z = \frac{b - wd}{wc - a} = \frac{-dw + b}{cw - a}$$

Lastly, if $\varphi \in \bigwedge$ then $\varphi^{-1} \in \bigwedge$.

$$\varphi_{-1}(z) = \frac{-dz+b}{cz-a} = \frac{dz-b}{-cz+a} = \frac{dz-b}{a-cz}$$

Hence $\bigwedge$ is a group.

To show if $A, B \in GL(2,\mathbb{C})$, show that $\Phi(AB) = \Phi(A)\Phi(B)$.
Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Then

$$AB = \begin{bmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & d\beta + d\delta \end{bmatrix}$$

Then

$$\Phi(AB) = \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)}$$

Now

$$\Phi(A) = \frac{az+b}{cz+d} \text{ and } \Phi(B) = \frac{\alpha z + \beta}{\gamma z + \delta}$$

Then

$$\begin{aligned} \Phi(A) \circ \Phi(B) &= \varphi_1 \circ \varphi_2 \\ &= \varphi_1(\varphi_2(z)) \\ &= \frac{a\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + b}{c\left(\frac{\alpha z+\beta}{\gamma z+\delta}\right) + d} \\ &= \frac{(a\alpha + b\gamma)z + (a\beta + b\delta)}{(c\alpha + d\gamma)z + (c\beta + d\delta)} \\ &= \Phi(A)\Phi(B) \end{aligned}$$

Φ is obviously onto. For example, if $\Phi : GL(2,\mathbb{C}) \to \bigwedge$ and $\bigwedge = \frac{pz+q}{rz+s}$, then $GL(2,\mathbb{C}) = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Furthermore, the kernel of Φ is:

$$\text{Ker } \Phi = \left\{ A \in GL(2,\mathbb{C}) : \Phi(A) = \text{Id} \right\}$$

For Id to be in $\bigwedge$, it mush be the case that $\varphi(z) = \frac{az+b}{cz+d} = z$. This means $a = 1$, $b = 0$, $c = 0$ and $d = 1$. This forms the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. For 1 is arbitrary; all we need is $a = d$ and $b = c = 0$. Therefore $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, where $\lambda \in \mathbb{C} \left\{0\right\}$, will produce this result since if this is $G(2,\mathbb{C})$, then $\bigwedge = \frac{\lambda z}{\lambda} = z$. Hence

$$K = \text{Ker } \Phi = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

where $\lambda \in \mathbb{C} \left\{0\right\} = \mathbb{C}^*$. □

Composition of Transformations:

- Translation: $s(z) = z + a$
- Dilation: $s(z) = az$ where $a \in \mathbb{R}$ and $a > 0$
- Rotation: $s(z) = e^{i\theta} z$
- Inversion: $s(z) = \frac{1}{z}$

Proposition: If $S \in \bigwedge$, meaning if S is a Möbius transformation , then S is a composition of translations, dilations and inversions.

Proof. Step 1: Let $c = 0$. Define $S(z) = \left(\frac{a}{d}\right)z + \left(\frac{b}{d}\right)$. Then

$$\begin{aligned} S_1(z) &= \frac{a}{d}z \\ S_2(z) &= z + \frac{b}{d} \\ S &= S_2 \circ S_1 \end{aligned}$$

Step 2: If $c \neq 0$, then

$$\begin{aligned} S_3(z) &= \frac{bc - ad}{c^2} z \\ S_4(z) &= z + \frac{a}{c} \\ S &= S_4 \circ S_3 \circ S_2 \circ S_1 \end{aligned}$$

□

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Let $\varphi(z) = \frac{az+b}{cz+d}$ be a Möbius transformation and $\varphi(z) = z$, then

$$\begin{aligned} cz^2 + dz - az - b &= 0 \\ cz^2 + z(d - a) - b &= 0 \end{aligned}$$

which has at most 2 roots. Thus a Möbius transformation can have at most 2 fixed points unless $\varphi(z) = z$ for all $z \in \hat{\mathbb{C}}$.

Let z_1, z_2 and z_3 be distinct points in $\hat{\mathbb{C}}$ and w_1, w_2 and w_3 be distinct points in $\hat{\mathbb{C}}$. Suppose there exists two Möbius transformation T and S such that $T(z_i) = w_i$ and $S(T_i) = w_i$ for $i = 1, 2, 3$. Then

$$TS^{-1}(w_i) = w_i$$

for $i = 1, 2, 3$. Therefore

$$TS^{-1} = Id \text{ or } T = S$$

A Möbius transformation is uniquely determined by its action on 3 distinct points in $\hat{\mathbb{C}}$.
Cross Ratio:

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Suppose

$$S = [z, z_2, z_3, z_4] = \frac{(z - z_3)(z_2 - z_4)}{(z - z_4)(z_2 - z_3)}$$

This is a Möbius transformation if when $z = z_2$, then $S(z_2) = 1$, if when $z = z_3$, then $S(z_3) = 0$ and if when $z = z_4$, then $S(z_4) = \infty$. In other words, if $S(z_i) = w_i$, then z_2 and w_1 go to 1, z_3 and w_2 go to 0 and z_4 and w_3 go to ∞.

Important Proposition: The cross ratio is invariance under Möbius transformation . That is, if z_1, z_2, z_3, z_4 are distinct points in $\hat{\mathbb{C}}$, then

$$[z_1, z_2, z_3, z_4] = [T(z_1), T(z_2), T(z_3), T(z_4)]$$

where T is any Möbius transformation .

Proof. Let $S(z) = [z, z_2, z_3, z_4]$ and defined $M = ST^{-1}$. Let S map z_2 to 1, z_3 to 0 and z_4 to ∞. This means $MT(z_2) = 1$, $MT(z_3) = 0$ and $MT(z_4) = \infty$. Then

$$M(z) = [z, T(z_2), T(z_3), T(z_4)]$$

or in other words,

$$ST^{-1}(z) = [z, T(z_2), T(z_3), T(z_4)]$$

for all $z \in \mathbb{C}$. In particular, if $z = T(z_1)$, then

$$ST^{-1}(T(z_1)) = [T(z_1), T(z_2), T(z_3), T(z_4)]$$

Hence

$$S(z_1) = [T(z_1), T(z_2), T(z_3), T(z_4)]$$

and so

$$[z_1, z_2, z_3, z_4] = [T(z_1), T(z_2), T(z_3), T(z_4)]$$

□

Proposition: If z_1, z_2 and z_3 are distinct points in $\mathbb{C}$ and w_1, w_2 and w_3 are distinct points in $\mathbb{C}$, there exists a unique Möbius transformation such that $T(z_i) = w_i$ where $i = 1, 2, 3$.

Proof. Let $\varphi_1(z) = [z, z_1, z_2, z_3]$ and $\varphi_2(w) = [w, w_1, w_2, w_3]$. Then let z_1 and w_1 map to 1, z_2 and w_2 map to 0 and z_3 and w_3 map to ∞. Define $T = \varphi_2^{-1} \circ \varphi_1$. Then

$$\begin{aligned} T(z_1) &= \varphi_2^{-1}(\varphi_1(z_1)) = w_1 \\ T(z_2) &= \varphi_2^{-1}(\varphi_1(z_2)) = w_2 \\ T(z_3) &= \varphi_2^{-1}(\varphi_1(z_3)) = w_3 \end{aligned}$$

□

Let $w = \frac{az+b}{cz+d}$ be a Möbius transformation where $ad - bc \neq 0$. This means $cwz + dw - az - b = 0$ is of the form

$$Azw + Bz + Cw + D = 0$$

where $A = c$, $B = -a$, $C = d$ and $D = -b$ and so $AD - BC = -bc + ad \neq 0$.
Claim:

$$[w, w_1, w_2, w_3] = [z, z_1, z_2, z_3]$$

is the Möbius transformation such that $w(z_i) = w_i$ for $i = 1, 2, 3$.

Proof. Given the identity above,

$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$
$$(w-w_2)(w_1-w_3)(z-z_3)(z_1-z_2) = (w-w_3)(w_1-w_2)(z-z_2)(z_1-z_3)$$

If $z = z_2$, then $w = w_2$. If $z = z_3$, then $w = w_3$. If $z = z_1$,

$$\begin{aligned}
(w-w_1)(w_1-w_3)(z_1-z_3)(z_1-z_2) &= (w-w_3)(w_1-w_2)(z_1-z_2)(z_1-z_3) \\
(w-w_1)(w_1-w_3) &= (w-w_3)(w_1-w_2) \\
ww_1 - w_1w_2 - ww_3 + w_2w_3 &= ww_1 - w_1w_3 - ww_2 + w_2w_3 \\
-w_1w_2 - ww_3 &= -w_1w_3 - ww_2 \\
w(w_2 - w_3) &= w_1(w_2 - w_3) \\
w &= w_1
\end{aligned}$$

□

Find a Möbius transformation that maps $z_1 = 2$, $z_2 = i$, $z_3 = -2$ to $w_1 = 1$, $w_2 = i$, $w_3 = -1$.

$$[w, 1, i, -1] = [z, 2, i, -2]$$

This means

$$\begin{aligned}
\frac{(w-i)(2)}{(w+1)(1-i)} &= \frac{(z-1)(4)}{(z+2)(2-i)} \\
\frac{w-i}{(w+1)(1-i)} &= \frac{2(z-1)}{(z+2)(2-i)} \\
\frac{w-i}{w+1-iw-i} &= \frac{2z-2}{2z+4-iz-2i} \\
2wz + 4w - izw - 2wi - 2iz - 4i - 2 &= 2zw + 2z - 2izw - 2iz - 2iw - 2i - 2w - 2 \\
4w - izw - 4i - z &= 2z - 2izw - 2i - 2w \\
6w + izw &= 3z + 2i \\
w &= \frac{3z+2i}{iz+6}
\end{aligned}$$

Find a Möbius transformation that maps $z_1 = 1$, $z_2 = 0$, $z_3 = -1$ to $w_1 = i$, $w_2 = \infty$, $w_3 = 1$.

$$[w, w_1, w_2, w_3] = [z, z_1, z_2, z_3]$$
$$[w, i, \infty, 1] = [z, 1, 0, -1]$$

This means
$$\frac{(w-w_2)(w_1-w_3)}{(w-w_3)(w_1-w_2)} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$
If $w_2 = \infty$,
$$\frac{w_1-w_3}{w-w_3} = \frac{(z-z_2)(z_1-z_3)}{(z-z_3)(z_1-z_2)}$$
Then
$$\begin{aligned} \frac{i-1}{w-1} = \frac{z(2)}{(z+1)(1)} &= \frac{2z}{z+1} \\ iz - z + i - 1 &= 2zw - 2z \\ 2wz &= z + iz + i - 1 \\ w &= \frac{z(1+i)+i-1}{2z} \end{aligned}$$
A circle in $\hat{\mathbb{C}}$ is a (closed) subset of $\hat{\mathbb{C}}$ which is either a circle in $\mathbb{C}$ or else a set $L \bigcup \left\{\infty\right\}$ where L is a straight line in $\mathbb{C}$.
For example, $\hat{\mathbb{R}} : \mathbb{R} \bigcup \left\{\infty\right\}$ is a circle in $\hat{\mathbb{C}}$.

Lemma: If $\varphi \in \bigwedge$, then $\varphi^{-1}(\hat{\mathbb{R}})$ is a circle in $\hat{\mathbb{C}}$.

Proof. Let $\varphi(z) = \frac{az+b}{cz+d}$. For $z \in \mathbb{C}$, $\varphi(z) \in \hat{\mathbb{R}}$ if and only if $(az+b)(\overline{cz}+\overline{d}) = (\overline{az}+\overline{b})(cz+d)$. So $\mathbb{C} \bigcup \varphi^{-1}(\hat{\mathbb{R}})$ is the set of all $z \in \mathbb{C}$ such that
$$(a\overline{c} - \overline{a}c)|z|^2 + (a\overline{d} - \overline{b}c)z + (b\overline{c} - d\overline{a}) + (b\overline{d} - \overline{b}d) = 0$$
If $a\overline{c} - \overline{a}c \neq 0$, then this becomes
$$|(a\overline{c} - \overline{a}c)z - (\overline{a}d - b\overline{c})|^2 = |ad - bc|^2$$
in $\mathbb{C}$ which is a circle in $\mathbb{C}$.
If $a\overline{c} - \overline{a}c = 0$, then this defines a line in $\mathbb{C}$ and so $\varphi^{-1}(\hat{\mathbb{R}}) = L \bigcup \left\{\infty\right\}$. $\square$

Lemma: If C is a circle in $\hat{\mathbb{C}}$, there exists $\varphi \in \bigwedge$ such that $\varphi(C) = \hat{\mathbb{R}}$.

Proof. Choose α, β and γ distinct points on C and define
$$\varphi(z) = \frac{(z-\alpha)(\beta-\gamma)}{(z-\gamma)(\alpha-\beta)}$$
If $\varphi(\alpha) = 0$, $\varphi(\beta) = 1$ and $\varphi(\gamma) = \infty$, then $\varphi^{-1}(\hat{\mathbb{R}})$ is a circle in $\hat{\mathbb{C}}$ through α, β, γ and the only such circle is C. $\square$

Theorem 12.1. If $\varphi \in \bigwedge$ and C is a circle in $\hat{\mathbb{C}}$, then are $\varphi^{-1}(C)$ and $\varphi(C)$.

Proof. Choose $\psi \in \bigwedge$ such that $\psi^{-1}(\hat{\mathbb{R}}) = C$. Then
$$\varphi^{-1}(C) = \varphi^{-1}(\psi^{-1}(\hat{\mathbb{R}})) = (\psi \circ \varphi)^{-1}(\hat{\mathbb{R}})$$
which is a circle in $\hat{\mathbb{C}}$. If so, then $\varphi^{-1} \in \bigwedge$ and so $\varphi(C) = (\varphi^{-1})^{-1}(C)$ is also a circle in $\hat{\mathbb{C}}$. $\square$

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Let
$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y)$$
Let
$$f(z) = e^z = e^x \cos y + ie^x \sin y = u(x,y) + iv(x,y)$$
This means $u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$.
All first partials are continuous
$$\begin{aligned} u_x &= e^x \cos y = v_y \\ u_y &= -e^x \sin y = -v_x \end{aligned}$$
So the Cauchy-Riemann equations hold and hence $f(z) = e^z$ for all $z \in \mathbb{C}$ is holomorphic. Furthermore,
$$f'(z) = u_x + iv_x = e^x \cos y + 0e^x \sin y = e^x(\cos y + i\sin y) = e^z$$
Conclusion: The function $f(z) = e^z$ is holomorphic on $\mathbb{C}$ and
$$\frac{d}{dz}d^z = e^z \quad \forall z \in \mathbb{C}$$
A function holomorphic on the entire complex plane is called an entire function.
Note that
$$|z| = e^x = e^{\text{Re}\{z\}}$$
Write $|e^{2z+i}|$ in terms of x and y.
$$e^{2z+i} = e^{2x+2iy+i} = e^{2x} + e^{i(2y+1)} \to \left|e^{2z+i}\right| = e^{2x}$$
Write $\left|e^{iz^2}\right|$ in terms of x and y.
$$e^{iz^2} = e^{i(x^2-y^2+2ixy)} = e^{-2xy+i(x^2-y^2)} \to \left|e^{iz^2}\right| = e^{-2xy}$$
Show that $\left|e^{z^2}\right| \leq e^{|z|^2}$.
$$\begin{aligned} \left|e^{z^2}\right| &= e^{\text{Re}\left\{z^2\right\}} = e^{x^2-y^2} \\ e^{|z|^2} &= e^{x^2+y^2} \\ e^{x^2-y^2} &\leq e^{x^2+y^2} \\ \left|e^{z^2}\right| &\leq e^{|z|^2} \end{aligned}$$
Prove that $|e^{-2x}| \iff \text{Re}\{z\} > 0$.
$$\begin{aligned} \left|e^{-2z}\right| &= e^{\text{Re}\{-2z\}} \\ &= e^{-2\,\text{Re}\{z\}} \leq 1 \\ -2\,\text{Re}\{z\} &< 0 \\ \text{Re}\{z\} &> 0 \end{aligned}$$

Let $f(z) = u(x,y)+iv(x,y)$ be holomorphic on a region Ω. Define $U(x,y) = e^{u(x,y)}\cos v(x,y)$ and $V(x,y) = e^{u(x,y)}\sin v(x,y)$. Show that $U(x,y)$ and $V(x,y)$ are harmonic.
Define $F(z) = e^{f(z)}$ is which is holomorphic on Ω.

$$\begin{aligned} F(z) &= e^{f(z)} \\ &= e^{u(x,y)+iv(x,y)} \\ &= e^{u(x,y)}[\cos v(x,y) + i\sin v(x,y)] \\ &= e^{u(x,y)}\cos v(x,y) + ie^{u(x,y)}\sin v(x,y) \\ &= U(x,y) + iV(x,y) \end{aligned}$$

So $U(x,y) = \text{Re}\{F(z)\}$ and $V(x,y) = \text{Im}\{F(z)\}$ and so they are harmonic.

Define the following:

$$\begin{aligned} \sin z &= \frac{e^{iz} - e^{-iz}}{2i} \\ \cos z &= \frac{e^{iz} + e^{-iz}}{2} \\ \frac{d}{dz}\sin z &= \frac{ie^{iz} + ie^{iz}}{2} = \frac{e^{iz} + e^{-iz}}{2} = \cos z \\ \frac{d}{dz}\cos z &= \frac{ie^{iz} - ie^{iz}}{2} = \frac{-e^{iz} + e{-iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z \end{aligned}$$

Note that

$$\cos z + i\sin z = \frac{e^{iz} + e^{-iz}}{2} + i\frac{e^{iz} - e^{-iz}}{2i} = e^{iz}$$

For $n \in \mathbb{Z}$,

$$e^{z+2\pi ni} = e^z e^{2\pi ni} = e^z$$

Therefore, e^z is a periodic function with period $2\pi ni$.
Note the following:

$$\begin{aligned} \sin(z_1 + z_2) &= \sin z_1 \cos_2 + \sin z_2 \cos z_1 \\ \cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\ \sin^2 z + \cos^2 z &= 1 \end{aligned}$$

Hyperbolic functions:

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} \\ \cosh x &= \frac{e^x + e^{-x}}{2} \end{aligned}$$

Note the following:

$$\begin{aligned} \sin iy &= \frac{e^{-y} - e^{y}}{2i} = i\sinh y \\ \cos iy &= \cosh y \end{aligned}$$

If so, then

$$\sin z = \sin(x+iy) = \sin x \cos iy + \cos x \sin iy = \sin x \cosh y + i \cos x \sinh y$$

Furthermore, let

$$|\sin x|^2 = \sin^2 x \cosh^2 y + \cos^2 y \sinh^2 x$$

Suppose

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2 - e^{2x} - e^{-2x} + 2}{4} = 1$$

then

$$|\sin z|^2 = \sin^2 x(1 - \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y = \sin^2 x + \sinh^2 y$$

Similarly,

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Facts:

$$\begin{aligned} \frac{d}{dz}\sinh z &= \cosh z \\ \frac{d}{dz}\cosh z &= \sinh z \\ \sin iy &= i \sinh y \\ \cos iy &= \cosh y \\ \cosh^2 x - \sinh^2 x &= 1 \end{aligned}$$

Verify that $-i \sinh iz = \sin z$.

$$-i \sinh iz = -i\left(\frac{e^{iz} - e^{-iz}}{2}\right) = \left(\frac{e^{iz} - e^{-iz}}{2i}\right) = \sin z$$

Prove the following:

$$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$$

From the LHS:

$$\sinh(z_1 + z_2) = \frac{e^{z_1+z_2} - e^{-i(z_1+z_2)}}{2} = \frac{e^{z_1}e^{z_2} - e^{-z_1}e^{-z_2}}{2}$$

From the RHS:

$$\sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 = \frac{e^{z_2} - e^{-z_1}}{2}\frac{e^{z_2} + e^{-z_2}}{2} + \frac{e^{z_1} + e^{-z_1}}{2}\frac{e^{z_2} - e^{-z_2}}{2}$$

Prove that $\sinh z = \sinh x \cos y + i \cosh x \sin y$.

$$\begin{aligned} \sinh z &= \sinh(x+iy) \\ &= \sinh x \cosh iy + \cosh x \sinh iy \\ &= \sinh x \cos y + i \cosh x \sin y \end{aligned}$$

Note that

$$\begin{aligned} |\sinh z|^2 &= \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \\ &= \sinh^2 x(1 - \sin^2 y) + (1 + \sinh^2 x) \sin^2 y \\ &= \sinh^2 x + \sin^2 y \end{aligned}$$

Similarly,

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$

where

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Cauchy Riemann Equations in Polar Form: Let $z = x + iy$, $x = r\cos\theta$, and $y = r\sin\theta$. Let $w = f(z) = u(x, y) + iv(x, y)$. Then

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)u_x + \sin(\theta)u_y \\ \frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = -r\sin(\theta)u_x + r\cos(\theta)u_y \\ \frac{\partial v}{\partial r} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)v_x + \sin(\theta)v_y \\ \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin(\theta)v_x + r\cos(\theta)v_y \end{aligned}$$

The Cauchy Riemann Equations are as follows:

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x \\ ru_r = r\cos(\theta)u_x + r\sin(\theta)u_y &= r\cos(\theta)v_y - r\sin(\theta)v_x = v_\theta \\ u_\theta = -r\sin(\theta)u_x + r\cos(\theta)u_y &= -r\sin(\theta)v_y - r\cos(\theta)v_x = -rv_r \end{aligned}$$

Therefore the Cauchy Riemann Equations are:

$$ru_r = v_\theta \qquad -rv_r = u_\theta$$

Furthermore,

$$\begin{aligned} f'(z) &= u_r + iv_r \\ &= \cos(\theta)u_x + \sin(\theta)u_y + i(\cos(\theta)v_x + \sin(\theta)v_y) \\ &= u_x(\cos\theta + i\sin\theta) + iv_x(\cos\theta + i\sin\theta) \\ &= e^{-i\theta}(u_x + iv_x) \\ f'(z) &= e^{-i\theta}(u_r + iv_r) \end{aligned}$$

Let $f(z) = |z|$ be continuous. Show that $||z_n| - |z|| \leq |z_n - z|$ if $z_n \to z$ and $|z_n| \to |z|$.

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Let $z = re^{i\theta}$. Define $\Omega = \mathbb{C}/\left\{z : z = x + iy : x \leq 0, y = 0\right\}$.
Problem: Suppose $z_n, z \in \Omega$ where $z_n = r_n e^{i\theta_n}$ and $z = re^{i\theta}$ and $-\pi < \theta_n < \pi$ and $-\pi < \theta < \pi$. Provethat if $z_n \to z$, then $r_n \to r$ and $\theta_n \to \theta$.
Let Ω be a region. If there exists a function $f : \Omega \to \mathbb{C}$ such that f is continuous on Ω and $e^{f(z)} = z$ for all $z \in \Omega$, then f is called a branch of the logarithm $\log z$¿ Note that $0 \notin \Omega$.
Suppose f is a given branch and k is an integer. Let $g(z) = f(z) + 2\pi ki$. Then

$$e^{g(z)} = e^{f(z)}e^{2\pi ki} = e^{f(z)} = z$$

Therefore $g(z)$ is also a branch. Consequently, if f and g are branches of $\log z$, then

$$f(z) = g(z) + 2\pi ki$$

for some $k \in \mathbb{Z}$ where k depends on z.
Does the same k work for all $z \in \Omega$? Let $h(z) = \frac{f(z)-g(z)}{2\pi i}$. So h is continuous on Ω. Since Ω is connected and h is connected in connected on Ω, then $g(z)$ is connected and hence a point. Therefore there exists $k \in \mathbb{Z}$ such that

$$f(z) + 2\pi ki = g(z) \qquad \forall z \in \Omega$$

Proposition: If Ω is a region and f is a branch of $\log z$, then the totality of all branches of $\log z$ are

$$f(z) + 2\pi ki, \ k \in \mathbb{Z}$$

Now back to the problem. Let $\Omega = \mathbb{C} \left\{z : z = x + iy : x \leq 0, y = 0\right\}$. Seach $z \in \Omega$ can be written as $z = re^{i\theta}$ where $i\pi < \theta < \pi$. By the problem, $f(z) = \ln|r| + i\theta$ is a continuous function on Ω and

$$e^{f(z)} = e^{\ln|r|+i\theta} = e^{\ln r}e^{i\theta} = re^{i\theta} = z$$

Given a nonzero complex number z,

$$\log z = \ln r + i\theta$$

where $z = re^{i\theta}$ and $-\pi < \theta < \pi$. This is called the principal branch of $\log z$. The principal branch is written as $\log z$. So the general values of $\log z$ are:

$$\log(z) = \ln r + i\theta + 2n\pi i$$

where $n \in \mathbb{Z}$ and $-\pi < \theta < \pi$.

Note that

$$\log z = \ln r + i\theta$$

where $r = |z|$, $\theta = \arg z$ and $-\pi < \theta < \pi$.

If $z_n \to z$, to show that $f(z_n) \to f(z)$, show that $\ln|z_n| + i\theta_n \to \ln|z| + i\theta$.

Recall: Polar form of Cauchy Riemann Equations: If $f(z) = u(x,y) + iv(x,y)$ and $x = r\cos\theta$ and $y = r\sin\theta$ then

$$\begin{aligned} ru_r &= v_\theta \\ u_\theta &= -rv_r \\ f'(z) &= e^{i\theta}(u_r + iv_r) \end{aligned}$$

Consider $f(z) = \log z = \ln r + i\theta$ where $z = re^{i\theta}$ and $-\pi < \theta < \pi$. Then

$$\begin{aligned} u(r,\theta) &= \ln r \\ v(r,\theta) &= \theta \\ u_r &= \frac{1}{r} \\ v_\theta &= 1 \end{aligned}$$

Therefore $ru_r = v_\theta$ and if $u_\theta = 0$ and $v_r = 0$, then $u_\theta = -rv_r$. Furthermore,

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(\frac{1}{r}\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Conclusion: $\log z$ is a holomorphic function on $\Omega = \mathbb{C}/\Big\{z : z = x + iy : x \leq 0, y = 0\Big\}$ and $\frac{d}{dz}\log z = \frac{1}{z}$ for all $z \in \Omega$.

When $z \neq 0$ and $z \in \mathbb{C}$,

$$z^c = e^{c\log z}$$

This gives the values of the principal value of z^c.
Find the principal value of $(1+i)^{1+i}$.
Let $z = 1 + i$.

$$z^z = e^{(1+i)\log(1+i)}$$

Let $z = 1 + i = r(\cos\theta + i\sin\theta)$. Then $1 = r\cos\theta$ and $1 = r\sin\theta$. Since $r^2 = 2$, $r = \sqrt{2}$. Therefore $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$. So $\theta = \frac{\pi}{4}$. So

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

Then the principal branch is

$$\log(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\ln 2 + i\frac{\pi}{4}$$

Hence the principal value of $(1+i)^{1+i}$ is

$$\begin{aligned} e^{(1+i)(\ln\sqrt{2} + i\frac{\pi}{4})} &= e^{\ln\sqrt{2} - \frac{\pi}{4} + i\ln\sqrt{2} + i\frac{\pi}{4}} \\ &= e^{\ln\sqrt{2} - \frac{\pi}{4}}(\cos\Big(\ln\sqrt{2} + \frac{\pi}{4}\Big) + i\sin\Big(\ln\sqrt{2} + \frac{\pi}{4}\Big)) \end{aligned}$$

Find all values.

$$\log(1+i) = \ln\sqrt{2} + i\frac{\pi}{4} + 2n\pi i$$

Then

$$\begin{aligned} e^{(1+i)(\log(1+i))} &= e^{(1+i)[\ln\sqrt{2}+i(\frac{\pi}{4}+2n\pi)]} \\ &= e^{\ln\sqrt{2}-(\frac{\pi}{4}+2n\pi)}e^{i[\ln\sqrt{2}+\frac{\pi}{4}+2n\pi]} \\ &= e^{\ln\sqrt{2}-(\frac{\pi}{4}+2n\pi)}[\cos\Big(\ln\sqrt{2}+\frac{\pi}{4}+2n\pi\Big)+i\sin\Big(\ln\sqrt{2}+\frac{\pi}{4}+2n\pi\Big)] \end{aligned}$$

Find the principle value of i^i.
Let $z = i$ and $z^z = i^i = e^{i\log i}$. Then $z = i = r(\cos\theta + i\sin\theta)$. So $r\cos\theta = 0$ and $r\sin\theta = 1$. Since $-\pi < \theta < \pi$ and $r^2 = 1$ and so $r = 1$, $\cos\theta = 0$ and $\sin\theta = 1$ and hence $\theta = \frac{\pi}{2}$. So

$$i = e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$$

The principal branch is

$$\log i = \ln 1 + i\frac{\pi}{2} = i\frac{\pi}{2}$$

Therefore the principal value is

$$i^i = e^{i\log i} = e^{i(i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}$$

Show that the principal value of $\left[\frac{e}{2}(-1-\sqrt{3}i)\right]^{3\pi i}$ is $-e^{2\pi^2}$.

$$-\frac{e}{2} - \frac{\sqrt{3}}{2}ei = r(\cos\theta + i\sin\theta)$$

Therefore $-\frac{e}{2} = r\cos\theta$ and $-\frac{\sqrt{3}}{2}e = r\sin\theta$. Since $r^2 = e^2$ and so $r = 2$, then $\cos\theta = -\frac{1}{2}$ and $\sin\theta = -\frac{\sqrt{3}}{2}$. Hence $\theta = -\frac{2\pi}{3}$. The principal branch is

$$\log z = lne - i\frac{2\pi}{3} = 1 - i\frac{2\pi}{3}$$

and the principal value is

$$e^{3\pi i(1-\frac{2\pi}{3}i)} = e^{3\pi i}e^{2\pi^2} = e^{2\pi^2}(\cos 3\pi + i\sin 3\pi) = -e^{2\pi^2}$$

Find the principal value of $(1-i)^{4i}$.
Let $z = 1 - i = r(\cos\theta + i\sin\theta)$. Then $1 = r\cos\theta$ and $-1 = r\sin\theta$. Since $r^2 = 2$, then $r = \sqrt{2}$ and so $\cos\theta = \frac{1}{\sqrt{2}}$ and $\sin\theta = -\frac{1}{\sqrt{2}}$ and hence $\theta = -\frac{\pi}{4}$. The principal branch is

$$\log(1-i) = \ln\sqrt{2} - i\frac{\pi}{4}$$

The principal value is

$$\begin{aligned} e^{4i(\ln\sqrt{2}-i\frac{\pi}{4})} &= e^{\pi}e^{i4\ln\sqrt{2}} \\ &= e^{\pi i4\frac{1}{2}\ln 2} \\ &= e^{\pi}e^{i2\ln 2} \\ &= e^{\pi}(\cos 2\ln 2 + i\sin 2\ln 2) \end{aligned}$$

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Let $z_n = r_n e^{i\theta_n}$ and $z = re^{i\theta}$ where $-\pi < \theta_n < \pi$ and $-\pi < \theta < \pi$. Prove that if $z_n \to z$, then $\theta_n \to \theta$ and $r_n \to r$.

Proof. If $z_n \to z$, then $|z_n| \to |z|$ because

$$||z_n| - |z|| \leq |z_n - z| \to 0$$

and so $|z_n| \to |z|$. This means $r_n \to r$. If $z_n \to z$, then

$$r_n e^{i\theta_n} \to re^{i\theta}$$

Since $r_n \to r$, then

$$\frac{r_n e^{i\theta_n}}{r_n} \to \frac{re^{i\theta}}{r}$$
$$e^{i\theta_n} \to e^{i\theta}$$

Now if $\left\{\theta_n\right\}$ is a bounded sequence, then there exists a convergent subsequent $\theta_{n_j} \to \phi$. Then

$$e^{i\theta_{n_j}} \to e^{i\phi}$$
$$\text{Let } e^{i\phi} = e^{i\theta}$$
$$\text{Then } e^{i(\phi-\theta)} = 1$$

and so $\phi = \theta$. So $e^{i\theta_{n_j}} \to e^{i\theta}$. Claim: if $\left\{\theta_{n_k}\right\}$ is any subsequence of $\left\{\theta_n\right\}$, then $e^{i\theta_{n_k}} \to e^{i\theta}$. Suppose that $\theta_{n_k} \to \alpha$. Then $e^{i\theta_{n_k}} \to e^{i\alpha}$. Hence $e^{i\alpha} = e^{i\theta}$ or $\alpha = \theta$. Therefore $\theta_n \to \theta$. □

16 Midterm Practice Questions

Theorems:

1. Let f be holomorphic in a region Ω. Then
   - if $f'(z) = 0$ for all $z \in \Omega$, then f is constant in Ω.
   - if $|f(z)|$ is constant, then f is constant.
   - if $\text{Re}\{f(z)\}$ is constant, then f is constant.
   - if $\text{Im}\{f(z)\}$ is constant, then f is constant.
2. Let f be holomorphic in a region Ω. Then if $\overline{f}$ is holomorphic in Ω, then f is constant in Ω.
3. Define the cross ratio of four points: z_1, z_2, z_3, z_4 as follows

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

Let
$$\varphi(z) = [z, z_1, z_2, z_3] = \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)}$$
where $z_1 \to 1$, $z_2 \to 0$ and $z_3 \to \infty$. Prove that if T is a Möbius transformation and z_1, z_2, z_3, z_4 are distinct points in $\hat{\mathbb{C}}$, then
$$[z_1, z_2, z_3, z_4] = [T(z_1), T(z_2), T(z_3), T(z_4)]$$

Problems:

1. Suppose $u(x, y)$ is a harmonic function on G. Define $f = u_x - iu_y$. Show that f is holomorphic on G.
   Let $f = u_x - iu_y = U + iV$. Then $U = u_x = \frac{\partial u}{\partial x}$ and $V = -u_y = -\frac{\partial u}{\partial y}$. U and V have continuous first partials because $u(x, y)$ is harmonic and so its second partials are all continuous. Now,
$$U_x = \frac{\partial}{\partial x}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial x^2}$$
$$V_y = -\frac{\partial}{\partial y}(\frac{\partial u}{\partial y}) = -\frac{\partial^2 u}{\partial y^2}$$
$$U_y = \frac{\partial}{\partial y}(\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial y \partial x}$$
$$V_x = -\frac{\partial}{\partial x}(\frac{\partial u}{\partial y}) = -\frac{\partial u}{\partial x \partial y}$$
   Since $u(x, y)$ is harmonic, $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and so $u_y = -v_x$ and hence
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
   Thus f is holomorphic on G.

2. Show that $u(x, y) = x^3 - 3xy^2$ is harmonic on $\mathbb{C}$ and find the harmonic conjugates.
$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$
$$\frac{\partial^2 u}{\partial x^2} = 6x$$
$$\frac{\partial u}{\partial y} = -6xy$$
$$\frac{\partial^2 u}{\partial y^2} = -6x$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

Therefore $u(x,y) = x^3 - 3xy^2$ is harmonic. Furthermore, let $v(x,y)$ be a harmonic conjugate of u. Then $u + iv$ is holomorphic.

$$\begin{aligned}
u_x &= v_y \\
u_y &= -v_x \\
v_x &= -u_y = 6xy \\
v &= \int 6xy\,dx = 3x^2y + h(y) \\
v_y &= u_x = \frac{\partial v}{\partial y} \\
&= 3x^2 + h'(y) = 3x^2 - 3y^2 \\
h'(y) &= -3y^2 \\
h(y) &= \int -3y^2\,dy = -y^3 + k \\
v(x,y) &= 3x^2y - y^3 + k
\end{aligned}$$

3. Find a Möbius transformation such that $f(z_i) = w_i$ where

- $z_1 = -1$, $z_2 = 1$, $z_3 = 2$; $w_1 = 0$, $w_2 = -1$, $w_3 = -3$

$$\begin{aligned}
\frac{(w+1)(3)}{(w+3)(2)} &= \frac{(z-1)(-3)}{(z-2)(-2)} \\
\frac{w+1}{w+3} &= \frac{z-1}{2(z-2)} \\
2(z-2)(w+1) &= (w+3)(z-1) \\
2[zw - 2w + z - 2] &= wz + 3z - w - 3 \\
wz - 3w &= z + 1 \\
w &= \frac{z+1}{z-3}
\end{aligned}$$

- $z_1 = -1$, $z_2 = 1$, $z_3 = 2$; $w_1 = -3$, $w_2 = -1$, $w_3 - 0$

$$\begin{aligned}
\frac{(w+1)(-3)}{(w-0)(-2)} &= \frac{(z-1)(-3)}{(z-2)(-2)} \\
\frac{w+1}{w} &= \frac{z-1}{z-2} \\
(w+1)(z-2) &= w(z-1) \\
wz - 2w + z - 2 &= wz - w \\
w &= z - 2
\end{aligned}$$

- $z_1 = 0$, $z_2 = 1$, $z_3 = 2$; $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$

If $w_3 = \infty$,

$$\begin{aligned}
\frac{w - w_2}{w_1 - w_2} &= \frac{(z - z_2)(z_1 - z_3)}{(z - z_3)(z_1 - z_2)} \\
\frac{w - 1}{-1} &= \frac{(z - 1)(-2)}{(z - 2)(-1)} \\
(w - 1)(z - 2) &= -2(z - 1) \\
wz - 2w - z + 2 &= -2z + 2 \\
w(z + 2) &= -2 \\
w &= -\frac{z}{z - 2} = \frac{z}{2 - z}
\end{aligned}$$

- $z_1 = -i$, $z_2 = 0$, $z_3 = i$; $w_1 = -1$, $w_2 = i$, $w_3 = 1$

$$\begin{aligned}
\frac{(w - i)(-2)}{(w - 1)(-1 - i)} &= \frac{(z - 0)(-i - i)}{(z - i)(-i - 0)} \\
\frac{(w - i)(-2)}{(w - 1)(-1 - i)} &= \frac{2z}{z - i} \\
\frac{-(w - i)}{(w - 1)(-1 - i)} &= \frac{2}{z - i} \\
2(w - 1)(-1 - i) &= -(w - i)(z - i) \\
z(-w - iw + 1 + i) &= -zq - iqz + z + iz = -wz + iw + iz + 1 \\
w &= \frac{z - 1}{iz + 1}
\end{aligned}$$

- $z_1 = 1$, $z_2 = i$, $z_3 = -1$; $w_1 = 0$, $w_2 = 1$, $w_3 = \infty$

$$\begin{aligned}
\frac{w - 1}{-1} = 1 - w &= \frac{(z - i)(2)}{(z + 1)(1 - i)} = \frac{2z - 2i}{z + 1 - iz - i} \\
z + 1 - iz - i - wz - w + wiz + wi &= 2z - 2i \\
wi(z + 1) - w(z + 1) &= z - 1 + iz - 1 = (z - 1) = i(z - 1) \\
(wi - w)(z + 1) &= (z - 1)(1 + i) \\
w(i - 1)(z + 1) &= (z - 1)(1 + i) \\
w &= \frac{(z - 1)(1 + i)}{(z + 1)(i - 1)} \\
w &= \frac{z(1 + i) - (1 + i)}{z(-1 + i) - (1 - i)}
\end{aligned}$$

4. Find the principal values of

- $\log\big(1+\sqrt{3}i\big)$

$$\begin{aligned} 1+\sqrt{3}i &= r(\cos\theta + i\sin\theta) \\ r\cos\theta &= 1 \\ r\sin\theta &= \sqrt{3} \\ r^2 &= 4 \to r = 2 \\ \cos\theta &= \frac{1}{2} \\ \sin\theta &= \frac{\sqrt{3}}{2} \\ \theta &= \frac{\pi}{3} \\ \log\Big(1+\sqrt{3}i\Big) &= \ln 2 + i\frac{\pi}{3} + 2n\pi i \end{aligned}$$

- $(1-i)^{4i}$

$$\begin{aligned} (1-i)^{4i} &= e^{4i\log(1-i)} \\ 1-i &= r(\cos\theta + i\sin\theta) \\ r\cos\theta &= 1 \\ r\sin\theta &= -1 \\ r^2 &= 2 \to r = \sqrt{2} \\ \cos\theta &= \frac{1}{\sqrt{2}} \\ \sin\theta &= -\frac{1}{\sqrt{2}} \\ \theta &= -\frac{\pi}{4} \\ \log(1-i) &= \ln\sqrt{2} - \frac{\pi}{4} \\ (1-i)^{4i} &= e^{4i[\ln\sqrt{2} - i\frac{\pi}{4}]} \\ &= e^{\pi}e^{(4\ln\sqrt{2})i} \\ &= e^{\pi}e^{(2\ln 2)i} \\ &= e\pi(\cos 2\ln 2 + i\sin 2\ln 2) \end{aligned}$$

- $(1+i)^i$

$$\begin{aligned}
(1+i)^i &= e^{i\log(1+t)} \\
1+i &= r(\cos\theta + i\sin\theta) \\
r\cos\theta &= 1 \\
r\sin\theta &= 1 \\
r^2 &= 2 \to r = \sqrt{2} \\
\cos\theta &= \frac{1}{\sqrt{2}} \\
\sin\theta &= \frac{1}{\sqrt{2}} \\
\theta &= \frac{\pi}{4} \\
\log(1+i) &= \ln\sqrt{2} + i\frac{\pi}{4} \\
(1+i)^i &= e^{i(\ln\sqrt{2}+i\frac{\pi}{4})} \\
&= e^{-\frac{\pi}{4}}e^{(\ln\sqrt{2})i} \\
&= e^{-\frac{\pi}{4}}(\cos\ln\sqrt{2} + i\sin\ln\sqrt{2})
\end{aligned}$$

- $(1+i)^{1+i}$

$$\begin{aligned}
(1+i)^{1+i} &= e^{(1+i)\log(1+i)} \\
&= e^{(1+i)(\ln\sqrt{2}+i\frac{\pi}{4})} \\
&= e^{\ln\sqrt{2}-\frac{\pi}{4}}e^{i(\ln\sqrt{2}+\frac{\pi}{4})} \\
&= e^{\ln\sqrt{2}-\frac{\pi}{4}}(\cos\Big(\ln\sqrt{2}+\frac{\pi}{4}\Big) + i\sin\Big(\ln\sqrt{2}+\frac{\pi}{4}\Big))
\end{aligned}$$

5. Find all values of $(-8-8\sqrt{3}i)^{\frac{1}{4}}$.

$$[r(\cos\theta + i\sin\theta)]^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right] \text{ where } k = 0,1,2,\dots,n-1$$

$$\begin{aligned}
(-8-8\sqrt{3}i) &= r(\cos\theta + i\sin\theta) \\
r\cos\theta &= -8 \\
r\sin\theta &= -8\sqrt{3} \\
r^2 &= 256 \to r = 16 \\
\cos\theta &= -\frac{1}{2} \\
\sin\theta &= -\frac{\sqrt{3}}{2} \\
\theta &= -\frac{2\pi}{3} \\
(-8-8\sqrt{3}i) &= 16(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right))
\end{aligned}$$

$$[16(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right))]^{\frac{1}{4}} = 2[\cos\left(\frac{-\frac{2}{3}\pi + 2k\pi}{4}\right) + i\sin\left(\frac{-\frac{2\pi}{3} + 2k\pi}{4}\right)],$$

where $k = 0, 1, 2, 3$

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Let a curve be defined as: $\gamma : [0,1] \to \mathbb{C}$, a continuous function where $\gamma(0) =$ initial point and $\gamma(1) =$ terminal point.
Let a path be defined as: $\gamma : [,1] \to \mathbb{C}$ such that γ' is continuous and a closed path if $\gamma(0) = \gamma(1)$.
Let γ^* be the trace. Suppose f is a continuous complex-valued function on γ^*. Then

$$\int_\gamma f = \int_\gamma f(z)\,dz = \int_0^1 f(\gamma(t))\gamma'(t)\,dt$$

Suppose $\gamma : [0, 2\pi] \to \mathbb{C}$ and $\gamma(t) = e^{it}$ and $f(z) = \frac{1}{z}$, where $z \neq 0$. Then $\gamma'(t) = ie^{it}$ and $dz = ie^{it}\,dt$. Then

$$\int_\gamma \frac{dz}{z} = \int_0^{2\pi} \frac{ie^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i$$

Goal: Let f be holomorphic on a region that contains a disk $B(a,r) = \Big\{z : |z-a| < r\Big\}$.
Let γ be the boundary. Then

$$f(a) = \frac{2\pi i}{\int_\gamma} \frac{f(z)}{z-a}\,dz$$

Let Ω be simply connected and $f \in \mathcal{O}(\Omega)$ and γ_1 and γ_2 be two boundaries. Then

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

Let Ω be simply connected and $f \in \mathcal{O}(\Omega)$. If γ is a closed path in Ω, then

$$\int_\gamma f = 0$$

Cauchy's Integral Formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Let γ be square such that $x = \pm 2$ and $y = \pm 2$ and γ is traversing counter-clockwise. Calculate $\int_\gamma \frac{e^{-z}}{z - \pi\frac{i}{2}}\,dz$.
Note that $f(z) = e^{iz}$. Therefore

$$\int_\gamma \frac{f(z)}{z-a} = 2\pi i \cdot f(a)$$

$$* = 2\pi i \cdot f(\frac{\pi i}{2})$$

$$= 2\pi i \cdot e^{-\frac{\pi}{2}i}$$

$$= 2\pi i \cdot -1$$

$$= -2\pi i$$

Calculate $\int_\gamma \frac{\cos z}{z(z^2+8)}\,dz$.
Let $f(z) = \frac{\cos z}{z^2+8}$. Then

$$\int_\gamma \frac{f(z)}{z-0}\,dz = 2\pi i \cdot f(0)$$

$$= 2\pi i \cdot \frac{1}{8}$$

$$= \frac{\pi i}{4}$$

Let $\gamma : |z - i| = 2$. Calculate $\int_\gamma \frac{dz}{z^2+4}$.
Note first that

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$

$z - 2i$ is not on the boundary. Let $f(z) = \frac{1}{z+2i}$. Then

$$\int_\gamma \frac{f(z)}{z-2i}\,dz = 2\pi i \cdot f(2i) = 2\pi i(\frac{1}{4i}) = \frac{\pi}{2}$$

Calculate $\int_\gamma \frac{dz}{(z^2+4)^2}$.
Note that

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$$

Let $f(z) = \frac{1}{(z+2i)^2}$. Note that $f'(a) = \frac{1!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^2}\,dz$, from Cauchy's Integral Formula. Hence, we'll need $f'(z)$, which is $f'(z) = -\frac{z}{(z+2i)^2}$. Therefore

$$\begin{aligned}\int_\gamma \frac{dz}{(z^2+4)^2} &= \int_\gamma \frac{f(z)}{(z-2i)^2}\,dz \\ &= 2\pi i \cdot f'(2i) \\ &= 2\pi i \cdot (\frac{-2}{-64i}) \\ &= \frac{\pi}{16}\end{aligned}$$

Calculate $\int_\gamma \frac{e^z - e^{-z}}{z^4}\,dz$ where $\gamma : |z| = 4$.
Let $f(z) = e^z - e^{-z}$. Then $f'(z) = e^z + e^{-z}$, $f''(z) = e^z - e^{-z}$ and $f'''(z) = e^z + e^{-z}$. Therefore

$$\begin{aligned}\int_\gamma \frac{e^z - e^{-z}}{(z-0)^4}\,dz &= \int_\gamma \frac{f(z)}{(z-0)^4}\,dz \\ &= \frac{2\pi i}{3!} \cdot f'''(0) \\ &= \frac{\pi i}{3} \cdot (1+1) \\ &= \frac{2\pi i}{3}\end{aligned}$$

Calculate $\int_\gamma \frac{z^3+2z}{(z-2)^3}\,dz$ where $\gamma : |z| = 3$.
Let $f(z) = z^3 + 2z$. Then $f'(z) = 3z^2 + 2$ and $f''(z) = 6z$. Hence

$$\int_\gamma \frac{z^3+2z}{(z-2)^3}\,dz = \frac{2\pi i}{2!} \cdot f''(2) = \frac{2\pi i}{2}(12) = 12\pi i$$

18 Lecture 17

A curve in $\mathbb{C}$ is a continuous map γ of $[\alpha, \beta]$ into $\mathbb{C}$. The parameter interval is $[\alpha, \beta]$. Let $\gamma^* = \left\{\gamma(t) : \alpha \leq t \leq \beta\right\}$ where $\gamma(\alpha)$ is the initial point of γ and $\gamma(\beta)$ is the end point of γ. If $\gamma(\alpha) = \gamma(\beta)$ then γ is a closed curve.
A path is a piecewise C^1 curve, in other words, $\gamma : [\alpha, \beta] \to \mathbb{C}$ is continuous and there are infinitely many points. Let $\alpha = S_0 < S_1 < \cdots < S_n = \beta$ such that $\gamma[S_{j-1}, S_j]$ has a continuous derivative on the interval. However at the points $S_1, \dots, S_{n-1}$, the left and right derivatives of δ may differ. Now suppose that δ is a path and f is a continuous function on γ^*. Then

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt$$

Let φ be a continuous differentiable $1-1$ map of $[\alpha_1, \beta_1]$ onto $[\alpha, \beta]$ such that $\varphi(\alpha_1) = \alpha$ and $\varphi(\beta_1) = \beta$. Let $\gamma_1 = \gamma \circ \varphi$. Then γ_1 is a path with parameter intervals $[\alpha_1, \beta_1]$ and

$$\int_{\gamma_1} f(z)\,dz = \int_{\alpha_1}^{\beta_1} f(\gamma_1(t))\gamma_1'\,dt$$

But $\gamma_1'(t) = \gamma'(\varphi(t))\varphi'(t)$ and so

$$\int_{\gamma_1} f(z)\,dz = \int_{\alpha_1}^{\beta_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,dt = \int_{\alpha}^{\beta} f(\varphi(s))\gamma'(s)\,ds$$

Note that if $\gamma = \gamma_1 + \gamma_2$, then

$$\int_{\gamma_1+\gamma_2} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

Let $[0,1]$ be the parameter interval of γ. Consider $\varphi_1(t) = \varphi(1-t)$ where $0 \leq t \leq 1$ and φ_1 is the opposite path of φ.Then

$$\int_{\gamma} f(z)\,dz = \int_0^1 f(\varphi_1(t))\gamma_1'(t)\,dt = -\int_0^1 f(\gamma(1-t))\gamma'(1-t)\,dt = -\int_0^1 f(\gamma(s))\gamma'(s)\,ds = -\int_{\gamma} f(z)\,dz$$

Hence

$$\int_{\gamma_1} f(z)\,dz = -\int_{\gamma} f(z)\,dz$$

Suppose $\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt$. Suppose $|f(z)| \leq M$ for all $z \in \gamma$. Then

$$\begin{aligned}\left|\int_\gamma f(z)\,dz\right| &= \left|\int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt\right| \\ &\leq \int_\alpha^\beta |f(\gamma(t))||\gamma'(t)|\,dt \\ &\leq M\int_\alpha^\beta |\gamma'(t)|\,dt \\ &\leq ML(\gamma)\end{aligned}$$

where $L(\gamma)$ is the length of γ.

Recall: Cauchy's Integral Formula: Let $B(a,R) = \Big\{z : |z-a| < R\Big\}$. Then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$$

where $\gamma = \Big\{z : |z-a| = R\Big\}$.

Theorem 18.1. Cauchy's Estimate: Suppose $|f(z)| \leq M$ for all $z \in Ba, R$.

$$\left|f^{(n)}(a)\right| = \frac{n!}{2\pi i}\left|\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz\right| \leq \frac{n!}{2\pi}M \cdot \frac{2\pi R}{R^{n+1}}$$

Hence, if f is holomorphic on a region containing $B(a,R) = \Big\{z : |z-a| < R\Big\}$ and $|f(z)| \leq M$ on $B(a,R)$, then

$$\frac{\left|f^{(n)}(a)\right|}{n!} \leq \frac{M}{R^n}$$

Theorem 18.2. Liouville's Theorem: Every bounded entire function is a constant.

Proof. Let f be an entire function such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be an arbitrary point in $\mathbb{C}$ and consider a disk of radius R centered at z_0. By Cauchy's estimate, $|f'(z)| \leq \frac{M}{R}$. But $R > 0$ is arbitrary and hence $f'(z) = 0$. Since $z_0 \in \mathbb{C}$ is arbitrary, $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is constant. $\square$

A polynomial of degree $n \geq 0$ is of the form

$$f(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$$

where $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$.

Theorem 18.3. Fundamental Theorem of Algebra: If $p(z)$ is a nonconstant polynomial, then there exists a complex number z such that $p(z) = 0$.

Proof. Let

$$p(z) = z_n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 = z^n[1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}]$$

be a nonconstant polynomial. Then $\lim_{z\to\infty} p(z) = \infty$. Suppose there exists no $z \in \mathbb{C}$ such that $p(z) = 0$. Define $f(z) = \frac{1}{p(z)}$. Then f is an entire function. Furthermore, $\lim_{z\to\infty} f(z) = 0$. So there exists $N > 0$ such that $|f(z)| < 1$ for all $|z| > N$. Now consider the closed disk $\overline{B(0,N)} = \left\{z : |z| \leq N\right\}$ which is compact. Since f is holomorphic, and therefore continuous on $\overline{B(0,N)}$, it must be bounded on $\overline{B(0,N)}$. In other words, there exists $M > 0$ such that $|f(z)| \leq M$ for all z such that $|z| \leq N$. Thus f is a bounded entire function. By Liouville's theorem, f is a constant. Therefore $p(z)$ is a constant which contradicts that $p(z)$ is a nonconstant polynomial. Hence there exists $z \in \mathbb{C}$ such that $p(z) = 0$. $\square$

19 Lecture 18

Let X be a set and $A \subseteq X$. Then we say A is dense in X which means that $\overline{A} = X$. That means, given any point $x \in X$, any neighborhood $N(x)$ intersects A.

Consequences of Liouville's Theorem:

Theorem 19.1. The range of a nonconstant entire function is dense in the complex plane.

Proof. Let f be a nonconstant entire function. Suppose the range of f is not dense in $\mathbb{C}$. That means, there exists $z_0 \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - z_0| > \delta$. Let $g(z) = \frac{1}{f(z)-z_0}$. This is an entire function because $|f(z) - z_0| > \delta$. Furthermore

$$|g(z)| = \frac{1}{|f(z) - z_0|} < \frac{1}{\delta}$$

for all $z \in \mathbb{C}$. So then g is a bounded entire function. Hence by Liouville's theorem, g is constant. That means $f(z) - z_0$ is constant. But z_0 is constant as well and so $f(z)$ is constant. Contradiction. Hence the range of f must be dense in $\mathbb{C}$. $\square$

Suppose f is an entire function such that $\text{Re}\{f\}$ is bounded above. Prove that f is a constant.

Proof. Suppose f is an entire function such that $\text{Re}\{f\} \leq M$. Define $F = e^f$. F is an entire function and $|F| = \left|e^f\right| = e^{\text{Re}\{f\}} \leq e^M$. So F is a bounded entire function. By Liouville's theorem, F is a constant. That means $F'(z) = 0$ for all $z \in \mathbb{C}$. Then $e^{f(z)}f'(z) = 0$. Hence $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore F is constant. □

Suppose f is an entire function such that $\text{Im}\{f\}$ is bounded above. Prove that f is a constant.

Proof. Suppose f is an entire function such that $\text{Im}\{f\} \leq M$. Define $F = e^{-if}$. Then $|F| = \left|e^{-if}\right| = e^{\text{Im}\{f\}} \leq e^M$. So F is a bounded entire function. That means F is a constant. Then $F'(z) = 0$ for all $z \in \mathbb{C}$. Then $e^{-if}f'(z) = 0$. That is, $f'(z) = 0$ for all $z \in \mathbb{C}$ and so f is constant. □

Suppose that f is an entire function such that $\text{Re}\{f\}$ is bounded below. Prove that f is a constant.

Proof. Suppose f is an entire function such that $\text{Re}\{f\} \geq M$. That means, $M \leq \text{Re}\{f\} \leq |f|$. So $|f| \geq M$. Let $g(z) = \frac{1}{f(z)}$. Then g is an entire function and $|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{M}$. Hence g is a bounded entire function. Hence g is a constant and so f is a constant. □

Suppose f is an entire function such that $|f(z)| > 1$. Show that f is a constant.

Proof. Let $g(z) = \frac{1}{f(z)}$. Since $|f(z)| > 1$ for all $z \in \mathbb{C}$. g is an entire function. Furthermore, $|g(z)| = \frac{1}{|f(z)|} < 1$. So g is a bounded entire function. Hence g is a constant function and so f is a constant. □

Theorem 19.2. Let U be an open set in $\mathbb{C}$ and suppose $F \in \mathcal{O}(U)$ and F' is continuous in U. Then

$$\int_\gamma F'(z)\,dz = 0$$

for every closed path γ in U.

Proof. Let $[\alpha, \beta]$ be the parameter interval of γ.

$$\int_\gamma F'(z)\,dz = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)\,dt = F(\gamma(\beta)) - F(\gamma(\alpha)) = 0$$

since $\gamma(\alpha) = \gamma(\beta)$. □

Corollary: Since z^n is the derivative of $\frac{z^{n+1}}{n+1}$, for all integers $n \neq -1$, then

$$\int_\gamma z^n\,dz = 0$$

for any closed path γ if $n = 0, 1, 2, \dots$ and for those closed paths γ such that $0 \notin \gamma^*$ if $n = -2, -3, \dots$.
Proposition: If $\gamma : [0,1] \to \mathbb{C}$ is a closed smooth path and $a \notin \gamma$, then

$$\frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

is an integer.

Proof. Define $g : [0,1] \to \mathbb{C}$ as follows:

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - a} \, ds$$

Then $g(0) = 0$ and $g(1) = \int_\gamma \frac{dz}{z-a}$. In addition, $g'(t) = \frac{\gamma'(t)}{\gamma(t)-a}$ for $0 \leq t \leq 1$. Note

$$\begin{aligned}
\frac{d}{dt}(e^{-g(t)}(\gamma(t) - a)) &= -g'(t)e^{-g(t)}(\gamma(t) - a) + e^{-g(t)}\gamma'(t) \\
&= -g'(t)e^{-g(t)}(\gamma(t) - a) + e^{-g(t)}(\gamma(t) - a)g'(t) \\
&= 0
\end{aligned}$$

Hence $e^{-g(t)}(\gamma(t) - a)$ is a constant. Then

$$\begin{aligned}
e^{-g(0)}(\gamma(0) - a) &= e^{-g(1)}(\gamma(1) - a) \\
e^{-g(0)} &= e^{-g(1)} \\
1 &= e^{-g(1)} \\
&= \frac{1}{e^{g(1)}} \\
e^{g(1)} &= 1
\end{aligned}$$

Then $g(1) = 2\pi i k$ for some integer k and so

$$\frac{1}{2\pi i} g(1) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} = k$$

□

If γ is a closed path in $\mathbb{C}$ and $\alpha \notin \gamma$, then

$$\text{Ind}(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a}$$

is called the Index of a with respect to γ on the winding number of a with respect to γ.

20 Lecture 19

If $\left\{F_n\right\}$ is a sequenced compact set such that

$$F_n \geq F_{n+1}$$

for all $n \geq 1$ and $\lim_{n\to\infty} \text{diam}(F_n) = 0$, then

$$\bigcap_{n=1}^{\infty} F_n$$

contains exactly 1 point. (Note: $\text{diam}(S) = \sup_{x\in S, y\in S} d(x,y)$.)
For any $a,b,c \in \mathbb{C}$¡ the triangle whose vertices are a,b,c is $\Delta = \Delta(a,b,c)$. Let $\partial\Delta$ be the boundary of Δ. For any function f continuous on $\partial\Delta$,

$$\int_{\partial\Delta} f(z)\,dz = \int_{[a,b]} f(z)\,dz + \int_{[b,c]} f(z)\,dz + \int_{[c,a]} f(z)\,dz$$

Theorem 20.1. Local Cauchy Theorem: If Δ is a triangle contained in a region Ω and if $f \in O(\Omega)$ (f is holomorphic), then

$$\int_{\partial\Delta} f(z)\,dz = 0$$

Proof. Let a', b', c' be the midpoints of $[b,c]$, $[c,a]$ and $[a,b]$ respectively. Consider the four triangles

$$\Delta^1 = \left\{a, c', b'\right\}$$
$$\Delta^2 = \left\{b, a', c'\right\}$$
$$\Delta^3 = \left\{c, b', a'\right\}$$
$$\Delta^4 = \left\{a', b', c'\right\}$$

Put

$$J = \int_{\partial\Delta} f(z)\,dz = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z)\,dz$$

St least one of the triangles Δ^j must satisfy

$$\left|\int_{\partial\Delta^j} f(z)\,dz\right| \geq \frac{|J|}{4}$$

Choose one of them and call it Δ_i. Repeat this process to form a sequence of triangles Δ_1, Δ_2, ... such that $\Delta_{n+1} \subseteq \Delta$. The lengths of $\partial\Delta_n = \frac{L}{2^n}$ where L is the length of the boundary of Δ, or $\int_{\partial\Delta} |dz|$ and Δ_n has diam $= \frac{D}{2^n}$ where $D = \text{diam}(\Delta)$ and

$$\left|\int_{\partial\Delta_n} f(z)\,dz\right| \geq \frac{|J|}{4^n}$$

So $\bigcap_{n=1}^{\infty} \Delta_n = \Big\{ z_0 \Big\} \subseteq \Delta \subseteq \Omega$. Let $\varepsilon > 0$ be given. Choose $r > 0$ such that $B(z_0, r) \subseteq \Omega$. Note that

$$B(z_0, r) = \Big\{ z : |z - z_0| < r \Big\}$$

and

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon |z - z_0|$$

if $z \in B(z_0, r)$. Choose n so that $\Delta_n \subseteq B(z_0, r)$. Then

$$\begin{aligned}
\left| \int_{\partial \Delta_n} f(z)\, dz \right| &= \left| \int_{\partial \Delta_n} [f(z) - f(z_0) - f'(z)(z - z_0)]\, dz \right| \\
&\leq \int_{\partial \Delta_n} |f(z) - f(z_0) - f'(z_0)(z - z_0)|\, |d_z| \\
&\leq \varepsilon \int_{\partial \Delta_n} |z - z_0|\, |dz| \\
&\leq \varepsilon \cdot \text{diam}(\Delta_n) \int_{\partial \Delta_n} |dz| \\
&\leq \varepsilon \cdot \text{diam}(\Delta_n)(\text{length of } \partial \Delta_n) \\
&= \varepsilon \cdot \frac{D}{2^n} \cdot \frac{L}{2^n} \\
&= \frac{\varepsilon DL}{4^n}
\end{aligned}$$

So

$$|J| \leq 4^n \left| \int_{\partial \Delta_n} f(z)\, dz \right| \leq 4^n \cdot \frac{\varepsilon DL}{4^n} = \varepsilon DL$$

Hence $J = 0$. □

Theorem 20.2. Let $\Delta \subseteq \Omega$ and let p be a point in Ω. Let f be continuous in Ω and holomorphic in $\Omega / \Big\{ p \Big\}$. Then

$$\int_{\partial \Delta} f(z)\, dz = 0$$

Proof. There is nothing to prove if $p \in \Omega$ but $p \not\in \Delta$. Case 1: $\Delta = \Big\{ p, b, c \Big\}$ where p is a vertex. Let $\varepsilon > 0$ be given. Choose $x \in [p, b]$ and $y \in [p, c]$ so close to p such that

$$\left| \int_{\partial \Big\{ p,x,y \Big\}} f(z)\, dz \right| < \varepsilon$$

Now

$$\int_{\partial \Delta} f(z)\, dz = \int_{\partial \Big\{ p,x,y \Big\}} f(z)\, dz + \int_{\partial \Big\{ x,b,y \Big\}} f(z)\, dz + \int_{\partial \Big\{ b,c,y \Big\}} f(z)\, dz$$

$$* = \int_{\partial \Big\{ p,x,y \Big\}} f(z)\, dz$$

Case 2: If $p \in \Delta$ and p is not a vertex, then

$$\int_{\partial\Delta} f(z)\,dz = \int_{\partial\left\{a,b,c\right\}} f(z)\,dz + \int_{\partial\left\{a,b,p\right\}} f(z)\,dz + \int_{\partial\left\{b,c,p\right\}} f(z)\,dz = 0$$

□

21 Lecture 20

A set E is convex is it has the following geometric property: whenever $x \in E$, $y \in E$, and $0 < t < 1$, the point

$$z_t = (1-t)x + ty$$

also lies in E. As t runs from 0 to 1, one may visualized z_t as describing a straight line segment in V, from x to y. Convexity requires that E contains the segments between any two of its points.
Recall: If Ω is a region and $f \in O(\Omega)$ and f' is continuous in Ω, then

$$\int_\gamma f'(z)\,dz = 0$$

where γ is a closed path in Ω.
The region V is starlike with respect to the point z_0 if for every $z \in V$, the line segment $[z_0, z]$ is contained in V. The region V is starlike if it is starlike with respect to any point in V.

Theorem 21.1. Let V be a starlike region with respect to $z_0 \in V$. For any $p \in V$, if f is continuous in V and holomorphic in $V/\left\{p\right\}$, then

1. $\int_\gamma f(z)\,dz = 0$ for every closed path in V
2. $f = F'$ for some $F \in O(V)$

Proof. Define $F : V \to \mathbb{C}$ by $F(z) = \int_{[z_0,z]} f(G)\,dG$. Since V is starlike with respect to z_0, $\left\{z_0, z, z+h\right\} \subseteq V$ for all h sufficiently small. Then

$$F(z+h) - F(z) = \int_{[z_0,z+h]} f(G)\,dG - \int_{[z_0,z]} f(G)\,dG$$

But

$$\int_{[z_0,z]} f(G)\,dG + \int_{[z,z+h]} f(G)\,dG + \int_{[z+h,z_0]} f(G)\,dG = 0$$

So

$$F(z+h) - F(z) = \int_{[z,z+h]} f(G)\,dG$$

Now

$$\left|\frac{1}{h}(F(z+h)-F(z))-f(z)\right| = \left|\frac{1}{h}\int_{[z,z+h]} f(G)-f(z)\,,dG\right|$$

But

$$\left|\frac{1}{h}\int_{[z,z+h]} f(z)\,dG\right| = |f(z)|\frac{1}{|h|}\int_{[z,z+h]} |dG| = |f(z)|$$

So

$$\left|\frac{1}{h}\int_{[z,z+h]} f(G)-f(z)\,dG\right| \leq \frac{1}{|h|}\int_{[z,z+h]} |f(G)-f(z)|\,|dG| \to 0$$

as $h \to 0$. Hence

$$\lim_{n\to\infty} \frac{f(z+h)-f(z)}{h} = f(z)$$

So $F = O(V)$ and $F' = f$. Finally,

$$\int_\gamma F'(z)\,dz = 0$$

or

$$\int_\gamma f(z)\,dz = 0$$

□

Theorem 21.2. Cauchy's Integral Formula: Let z be a starlike region and $f \in O(V)$. If γ is a closed path in V and $z \in V/\left\{\gamma\right\}$, then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G-z}\,dG = f(z)\mathrm{Ind}(\gamma, z)$$

Proof. Fix $p \in V/\gamma$. Define $g : V \to \mathbb{C}$ by $g(G) = \begin{cases} \frac{f(G)-f(p)}{G-p} & \text{if } G \neq p \\ f'(p) & \text{if } G = p \end{cases}$. Apply the above theorem to g: $\int_\gamma g(G)\,dG = 0$. That is,

$$\frac{1}{2\pi i}\int_\gamma g(G)\,dG = 0$$

or

$$\frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G-p}\,dG = \frac{1}{2\pi i}f(p)\int_\gamma \frac{dG}{G-p} = f(p)\mathrm{Ind}(\gamma, p)$$

□

Special Case: If γ is a circle and $\mathrm{Ind}(\gamma, p) = 1$, then

$$f(p) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-p}\,dz$$

Corollary: Let $\Delta = \Big\{z : |z| < 1\Big\}$. If $f \in O(\Delta)$, then there exists a power series $\sum_{n=0}^{\infty} a_n z^n$ with radius of convergence ≥ 1 such that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for all $z \in \Delta$. Furthermore,

$$a_n = \frac{2\pi i}{\int}_{|G|=r} \frac{f(G)}{G^{n+1}}\, dG$$

if $0 < r < 1$.

Proof. Suppose $0 < r < 1$. Let $\gamma(t) = re^{2\pi i t}$ for $0 \leq t \leq 1$. If $|z| < r$, then $\text{Ind}(\gamma, z) = \text{Ind}(\gamma, 0) = 1$. Now

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G-z}\, dG = \frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G}(1-\frac{z}{G})^{-1}\, dG = \frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G}(\sum_{n=0}^{\infty}\frac{z^n}{G^n})\, dG = \sum_{n=0}^{\infty} a_n z^n$$

Hence

$$a_n = \frac{1}{2\pi i}\int_\gamma \frac{f(G)}{G^{n+1}}\, dG$$

This expression is valid for $|z| < r$. But $a_n = \frac{f^{(0)}}{n!}$. Hence

$$\int_\gamma \frac{f(G)}{G^{n+1}}\, dG = \frac{2\pi i}{n!} f^{(n)}(0)$$

Since $a_n = \frac{f^{(n)}(0)}{n!}$ is independent of r,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

for all $z \in \Delta$. □

Corollary: Let $D = D(a,r) = \Big\{z : |z-a| < r\Big\}$. If $f in O(D)$, then the Taylor series of f about a has radius of convergence $\geq r$ and converges to f in D.

Proof. Apply the above corollary to $g(G) = f(a + rG)$ where $G \in \Delta$. □

Corollary: If V is any region in $\mathbb{C}$ and $f \in O(V)$, then $f' \in O(V)$.
Remark: If $f \in O(V)$, then all higher derivatives of f are holomorphic in V.
Corollary: If $f \in O(\Delta)$ and $|f()| \leq M$ for all $z \in \Delta$, then

$$\left|\frac{f^{(n)}(0)}{n!}\right| \leq M$$

for all $n \geq 0$.

Proof. If $0 < r < 1$,

$$\left|\frac{f^{(n)}(0)}{n!}\right| = |a_n| = \left|\frac{1}{2\pi i}\int_{|G|=r}\frac{f(G)}{G^{r+1}}\, dG\right| \leq \frac{1}{2\pi}\frac{M}{r^{n+1}} \cdot 2\pi r \leq \frac{M}{r^n}$$

□

Corollary: Cauchy's Estimate: If $f \in O(D(a,r))$ and $|f(z)| \leq M$ for all $z \in D(a,r)$, then

$$\left| f^{(n)}(a) \right| \leq \frac{M}{r^n}$$

for all $n \geq 0$.

Proof. Use the above corollary to $g(G) = f(a + rG)$ for $G \in \Delta$ so that

$$g^{(n)}(G) = f^{(n)}(a + rG)r^n$$

□

Remark: Suppose f is an entire function. Then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z^1 + a_2 z^2 + \dots + a_n z^n + \dots$$

where

$$a_n = \frac{f^{(n)}(0)}{n!}$$

22 Lecture 21

Let f be holomorphic in a region Ω and $a \in \Omega$. There exists $R > 0$ such that

$$f(a) = \sum_{n=0}^{\infty} c_n (z - a)^n$$

where

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem 22.1. Let Ω be a region and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then the following are equivalent.

- $f \equiv 0$
- There exists a point $a \in \Omega$ such that $f^{(n)}(a) = 0$ for all $n \geq 0$.
- $\left\{ z \in \Omega : f(z) = 0 \right\}$ has a limit point in Ω.

Proof. For $1 \to 2$: If $f = 0$, then all $f^{(n)}(a) = 0$ for any $n \geq 0$ and $a \in \Omega$. For $2 \to 3$, it is obvious. For $3 \to 2$: Let $Z = \left\{ z \in \Omega : f(z) = 0 \right\}$. Let a be a limit point of Z and $a \in \Omega$. There exists $R > 0$ such that $B(a,R) = \left\{ z : |z - a| < R \right\} \subseteq \Omega$. Note that $f(a) = 0$ (by continuity of f). Suppose there exist an integer $n \geq 1$ such that $f(a) = f^1(a) = f^2(a) = \dots = f^{n-1}(a) = 0$, but $f^n(a) \neq 0$. Then

$$f(z) = \sum_{k=n}^{\infty} a_k (z - a)^k$$

for $|z-a| < R$. Let $g(z) = \sum_{k=n}^{\infty} a_k(z-a)^{k-n}$ be holomorphic in $B(a,R)$. Then $f(z) = (z-a)^n g(z)$. Note that $g(a) = a_n \neq 0$. This means there exists $r > 0$ such that $g(z) \neq 0$ for all $|z-a| < r$. Since a is a limit point of Z, the neighborhood $B(a,R)$ cannot contain a point $b \in Z$ ($b \neq a$). This means $f(b) = 0$, or $f(b) = (b-a)^n g(b)$. Then $g(b) = 0$. Contradiction.
For $2 \to 1$: Let $A = \left\{ z \in \Omega : f^{(n)}(z) = 0 \forall n \geq 0 \right\}$. Claim: $A \neq \emptyset$. True because $a \in A$. Claim: A is closed. Let $z \in \overline{A}$. So there exists $z_0 \in A$ such that $z_k \to z$. Since each $f^{(n)}$ is continuous, it follows that $f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$. So $z \in A$ and so A is closed. Claim: A is open. Let $a \in A$. There exists $R > 0$ such that $B(a,R) \subseteq \Omega$. Then $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ where $a_n = \frac{f^{(n)}(a)}{n!}$ for all $|z-a| < R$ in $B(a,R)$. But $f(z) = 0$ for each $n \geq 0$. So $f(z) = 0$ for all $z \in B(a,R)$. So $B(a,R) \subseteq A$ and so A is open. Finally, since $A \neq 0$ and is open and is closed and Ω is connected, $A = \Omega$. $\square$

Corollary: Suppose $f \in O(\Omega)$ and there exists $a \in \Omega$ such that $f(z) = 0$ for all $B(a,r) = \left\{ z : |z-a| < r \right\}$. Then $f(z) = 0$ for all $z \in \Omega$. Proof: True because $3 \to 1$.
Corollary: Suppose $f, g \in O(\Omega)$ and $a \in \Omega$ such that $f(z) = g(z)$ for all $z \in B(a,r) = \left\{ z : |z-a| < r \right\}$. Then $f(g) = g(z)$ for all $z \in \Omega$. Proof: Let $h(z) = f(z) - g(z)$. Then $h \in O(\Omega)$ and by the above corollary, $h(z) = 0$ for all $z \in \Omega$. So $f(z) = g(z)$ for all $z \in \Omega$.
Corollary: The zeros of a nonconstant holomorphic function on a region must be isolated. Proof: If $f \in O(\Omega)$ and if the zero set Z has a limit point in Ω, then $f \equiv 0$. This means that if $a \in \Omega$ such that $f(a) = 0$, there exists $R > 0$ such that $f(z) \neq 0$ for all $0 < |z-a| < R$.
Remark: A holomorphic function f is said to have a zero of order $n \geq 0$ if there exists a holomorphic function g and $a\delta > 0$ such that $f(z) = (z-a)^n g(z)$ where $g(z) \neq 0$ for all $z \in B(a,\delta) = \left\{ z : |z-a| < \delta \right\}$.
Let Ω be a region. Let $f, g \in O(\Omega)$ such that $f(z)g(z) = 0$. Show that either $f(z) = 0$ for all $z \in \Omega$ or $g(z) = 0$ for all $z \in \Omega$. Proof: Suppose $g(z) \neq 0$ for all $z \in \Omega$. This means there exists $a \in \Omega$ such that $g(a) \neq 0$. By the continuity of g, there exists $R > 0$ such that $g(z) \neq 0$ for all $z \in B(a,R) = \left\{ z : |z-a| < R \right\}$. This implies $f(z) = 0$ for all $z \in B(a,R)$. Hence by the Identity Theorem, $f(z) = 0$ for all $z \in \Omega$.

23 Lecture 22

Suppose f, g are holomorphic on a region Ω such that $\overline{f}g$ is holomorphic. Show that either f is a constant or $g(z) = 0$ for all $z \in \Omega$. Proof: Suppose $g(z) \neq 0$ for all$z \in \Omega$, meaning $g \not\equiv 0$, or there exists $a \in \Omega$ such that $g(a) \neq 0$. By the continuity of g, there exists a neighborhood $B(a,r) = \left\{ z : |z-a| < r \right\}$ such that $g(z) \neq 0$ for all $z \in B(a,r)$. Let $\overline{f}g = h$ given that $h \in O(\Omega)$. Then $\overline{f}(z) = \frac{h(z)}{g(z)}$ for all $z \in B(a,r)$ because $g(z) \neq 0$ for all $z \in B(a,r)$. Since h and g are both holomorphic and $g(z) \neq 0$ in $B(a,r)$, it follows that $\overline{f}$ is holomorphic in $B(a,r)$. Thus f and $\overline{f}$ are both holomorphic in $B(a,r)$ and so f is constant on $B(a,r)$. Hence by the Identity Theorem, f is constant on Ω.

Let $\Delta = \left\{z : |z| < 1\right\}$. Suppose $f \in O(\Delta)$ and $g \in O(\Delta)$ and neither f and g have a zero in Δ. If $\frac{f'}{f}(\frac{1}{n}) = \frac{g'}{g}(\frac{1}{n})$, where $n = 1, 2, 3, \dots$, find a simple relation between f and g. Proof: Define $h = \frac{f}{g}$. Since $f, g \in O(\Delta)$ and g has no zeros in Δ, $h \in O(\Delta)$. Then

$$h'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$$

for all $z \in \Delta$. By hypothesis, $h'(z) = 0$ for $z = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$. So the zero set of h is $Z = \left\{\frac{1}{n}\right\}_{n=2}^{\infty}$ which has a limit point 0 in Δ. Hence by the Identity Theorem, $h'(z) = 0$ for all $z \in \Omega$. This implies $h'(z) = \lambda$, a constant, for all $z \in \Omega$ and so $f(z) = \lambda g(z)$ for all $z \in \Delta$.

Let f be an entire function and suppose there exists a constant M and $R > 0$ and an integer $n \geq 1$ such that

$$|f(z)| \leq M|z|^n$$

for all $|z| > R$. Show that f is a polynomial of degree $\leq n$. Proof: Since f is an entire function,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

or

$$f(z) = f(0) + f'(0)z + \frac{f^2(0)}{2!}z^2 + \dots + \frac{f^n(0)}{n!}z^n + \dots$$

By Cauchy's estimate,

$$\frac{\left|f^{(k)}(0)\right|}{k!} \leq \frac{Mr^n}{r^k}$$

if $r > R$. So for all $k > n$,

$$\frac{\left|f^{(n)}(0)\right|}{k!} \leq \frac{M}{r^{k-n}}$$

where n is fixed and is true for all $k > 0$. Since $r > R$ is arbitrary, it follows that $f^{(k)}(0) = 0$ for all $k > n$. Hence by the expansion of $f(z)$, f is a polynomial of degree $\leq n$.

Let f be an entire function and $|f(z)| < 1 + |z|^{\frac{1}{2}}$ for all $z \in \mathbb{C}$. Show that f is a constant. Proof: If f is an entire function, then

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

or

$$f(z) = f(0) + f'(0)z + \frac{f^2(0)}{2!}z^2 + \dots + \frac{f^n(0)}{n!}z^n + \dots$$

for all $z \in \mathbb{C}$. Consider $|z| = R$. Then

$$|f(z)| < 1 + R^{\frac{1}{2}}$$

By Cauchy's estimate,

$$\frac{\left|f^{(n)}(0)\right|}{n!} \leq \frac{1 + R^{\frac{1}{2}}}{R^n}$$

Since $R > 0$ can be arbitrary, it follows that $f^{(n)}(0) = 0$ for all $n \geq 1$. Hence $f(z) = f(0)$ for all $z \in \mathbb{C}$ and so f is a constant.

24 Lecture 23

Let U be an open set. If $a \in U$ and $f \in O(U \backslash \left\{a\right\})$, then f is said to be an isolated singularity at the point a. If f can be so defined at a such that the external function is holomorphic in U, then the singularity is removable.

Theorem 24.1. Riemann's Removable Singularity Theorem: Suppose $f \in O(U \backslash \left\{a\right\})$ and f is bounded in $D'(a,r) = \left\{z : 0 < |z-a| < r\right\}$, for some $r > 0$¿ Then f has a removable singularity at a.

Proof. Define $h(a) = 0$ and $h(z) = (z-a)^2 f(z)$ in $U \backslash \left\{a\right\}$. Claim: $h \in O(U)$ and $h'(a) = 0$. Note that

$$h'(a) = \lim_{z \to z} \frac{h(z) - h(a)}{z - a} = \lim_{z \to a} \frac{(z-a)^2 f(z)}{z-a} = \lim_{z \to a} (z-a) f(a) = 0$$

because f is bounded in $D'(a,r)$. Hence $h \in O(U)$ and $h'(a) = 0$. Now,

$$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= c_0 + c_1(z-a) + c_2(z-a)^2 + \dots \\ h(a) &= c_0 = 0 \\ h'(z) &= \sum_{n=0}^{\infty} n c_n (z-a)^{n-1} \\ &= c_1 + 2(z-a)^n + \dots \end{aligned}$$

$$h'(a) = c_1 = 0$$

Hence

$$h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n$$

Therefore

$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z-a)^n$$

for all $z \in D(a,r)$. So $f \in O(D(a,r))$ and hence a is a removable singularity. □

Theorem 24.2. If $a \in U$ and $f \in O(U \backslash \left\{a\right\})$, then one of the following three cases must occur:

1. f has a removable singularity at a

2. there exists complex numbers $c_1, \dots, c_m$, where m is a positive integer and $c_m \neq 0$, such that $f(z) = \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$ has a removable singularity at a

3. if $R > 0$ and $D(a,R) \subseteq U$, then $f(D'(a,R))$ is dense in the complex plane

Remark: In case b, we say that f has a pole of order m at a. In case c, we say that f has an essential singularity at a. Case c means that for every complex number w, there exists a sequence such that $z_n \to a$ and $f(z_n) \to w$, as $n \to \infty$.
Conclusion: An isolated singularity is either a removable singularity, a pole, or an essential singularity.

Proof. Suppose (c) fails. Then there exists $R > 0$ and a complex number w such that $|f(z) - w| > \delta$ in $D'(a, R) = D'$. Let $g(z) = \frac{1}{f(z)-w}$ for $z \in D'$. Then $g \in O(D')$ and $|g| < \frac{1}{\delta}$. So by RRST, g extends to a holomorphic function in D.
Case 1: If $g(a) \neq 0$. then

$$f(z) = w + \frac{1}{g(z)}$$

and so $f(a) = w + \frac{1}{g(a)}$. Furthermore,

$$\lim_{z\to a} f(z) = w + \lim_{z\to a} \frac{1}{g(z)} = w + \frac{1}{g(a)}$$

This means f is continuous at a and so continuous on $D(a, R)$ and so there exists some $0 < \rho < R$ such that f is bounded in $D(a, \rho)$ where $f(a) = w + \frac{1}{g(a)}$. Then by RRST, $z = a$ is a removable singularity of f, which is (a).
Case 2: If $g(a) = 0$, suppose g has a zero of order $m \geq 1$ at $z = a$. Then $f(z) = (z-a)^m g_1(z)$, for all $z \in D$ where $g_1 \in O(D)$ and $g_1(a) \neq 0$. Next, observe that g_1 does not have any zero in D'. So g_1 has no zero in D. Let $h = \frac{1}{g_1}$ in D. Hence $h \in O(D)$ and h has no zero in D. So

$$f(z) - w + \frac{1}{(z-a)^m g_1(z)} = \frac{h(z)}{(z-a)^m}$$

or

$$f(z) = w + \frac{h(z)}{(z-a)^m}$$

where $z \in D'$. If

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n$$

for $z \in D$ and $b_0 \neq 0$, then

$$f(z) = w + \frac{b_) + b_1(z-a) + b_2(z-a)^2 + \cdots + b_m(z-a)^m + \dots}{(z-a)^m}$$

and so

$$f(a) = \frac{b_0}{(z-a)^m} + \frac{b_1}{(z-a)^{m-1}} + \cdots + (b_m + w) + \dots$$

, where $c_k = b_{m-k}$ for $k = 1, 2, \dots, m$. This is (b). ∎

25 Lecture 24

Let $D(a,r) = \Big\{z : |z-a| < r\Big\}$. Let f be holomorphic in $D(a,r)$. f is said o have a zero of order n at a if there exists a holomorphic function g in $D(a,r)$ such that $f(z) = (z-a)^n g(z)$ and $g(a) \neq 0$.
Let $D'(a,r) = \Big\{z : 0 < |z-a| < r\Big\}$. Let f be holomorphic in $D'(a,r)$. f is said to have a pole of order n at a if there exists a holomorphic function g in $D(a,r)$ such that $f(z) = \frac{g(z)}{(z-a)^n}$ and $g(a) \neq 0$.
Laurent Series: Suppose f is holomorphic in the annulus $R_1 < |z-a| < R_2$ and let γ be any positively correlated circle centered at z_0 lying in the annulus. Then $|z - z_0| = r$ where $R_1 < r < R_2$. For each $R_1 < z < R_2$,

$$f(z) = \sum_{n=0} a_n(z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where $R_1 < |z-z_0| < R_2$ and

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz \text{ where } n = 0,1,2,\dots \\ b_n &= \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-z_0)^{-n+1}}\,dz \text{ where } n = 1,2,3,\dots \end{aligned}$$

In other words,

$$f(z) = \dots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Note:

1. If $b_0 = 0$ for all $n \geq 1$, $z = z_0$ is a removable singularity
2. If $b_i = 0$ for all $i > n$, $z = z_0$ is a pole of order n (A pole of order 1 is called a simple pole)
3. If $b_n \neq 0$ for infinitely many n, $z = z_0$ is an essential singularity

Theorem 25.1. Suppose $z = z_0$ is a pole of order n. Then the residue of f at z_0 is b_1 and

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$

Suppose f has a pole of order 1. Then

$$f(z) = \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

Let $g(z) = f(z)(z-z_0)$. Then

$$g(z) = b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + a_2(z-z_0)^3 + \dots$$

Hence
$$f(z) = \frac{g(z)}{z - z_0}$$
and $g(z_0) = b_1$ and so
$$\operatorname*{Res}_{z=z=z_0} f(z) = g(z_0) = b_1$$
Suppose f has a pole of order 2. Then
$$f(z) = \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$
Let $g(z) = f(z)(z - z_0)^2$. Then
$$g(z) = b - 2 + b_1(z - z_0) + a_0(z - z_)^2 + \dots$$
Hence
$$f(z) = \frac{g(z)}{(z - z_0)^2}$$
and $g(z_0) = b_1$ and so
$$\operatorname*{Res}_{z=z=z_0} f(z) = g(z_0) = b_1$$
Suppose f has a pole of order 3. Then
$$f(z) = \frac{b_3}{(z - z_0)^3} + \frac{b_2}{(z - z_0)^2} + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + \dots$$
Let $g(z) = f(z)(z - z_0)^3$. Then
$$g(z) = b_3 + b_2(z - z_0) + b_1(z - z_0)^2 + a_0(z - z_0)^3 + \dots$$
Then $f(z) = \frac{g(z)}{(z-z_0)^3}$. Now,
$$g'(z) = b_2 + 2b_1(z - z_0) + 3a_0(z - z_0)^2 + \dots$$
and
$$g''(z) = 2b_1 + 6a_0(z - z_0) + \dots$$
Hence $g''(z_0) = 2b_1$ and so
$$b_1 = \frac{g''(z_0)}{2}$$
Then
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{g''(z_0)}{2}$$
Rule:
$$\operatorname*{Res}_{z=z_0} f(z) = \begin{cases} g(z_0) & \text{if } n = 1 \\ \frac{g^{(n-1)}(z_0)}{(n-1)!} & \text{if } n > 1 \end{cases}$$
$$f(z) = \frac{g(z)}{(z - z_0)^n}$$

where g is holomorphic and $g(z_0) \neq 0$.

Suppose $f(z) = \frac{z^3-2z}{(z-i)^3}$. This is

$$f(z) = \frac{g(z)}{(z-i)^3}$$

where $g(z) = z^3 - 2z$. Then $z = i$ is a pole of order 3 and

$$\operatorname*{Res}_{z=i} f(z) = \frac{g''(z)}{2!} = \frac{6i}{2} = 3i$$

since

$$\begin{aligned} g'(z) &= 3z^2 - 2 \\ g''(z) &= 6z \\ g''(i) &= 6i \end{aligned}$$

Suppose $f(z) = (\frac{z}{2z+1})^3$. This is equivalent to

$$f(z) = (\frac{z}{2(z+\frac{1}{2})})^3 = \frac{\frac{z^3}{8}}{(z-(-\frac{1}{2}))^3} = \frac{g(z)}{(z-(-\frac{1}{2}))^3}$$

Then $z = -\frac{1}{2}$ is a pole of order 3. Note that

$$\begin{aligned} g'(z) &= \frac{3}{8}z^2 \\ g''(z) &= \frac{6}{8}z = \frac{3}{4}z \\ g''(-\frac{1}{2}) &= \frac{3}{4}(-\frac{1}{2}) = -\frac{3}{8} \end{aligned}$$

Then

$$\operatorname*{Res}_{z=-\frac{1}{2}} f(z) = \frac{g''(-\frac{1}{2})}{2!} = \frac{-\frac{3}{8}}{2} = -\frac{3}{16}$$

26 Lecture 25

Laurent Series: Let $R_1 < |z - z_0| < R_2$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}} \, dz$$

for $n = 0, 1, 2, \dots$ and

$$b_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-z_0)^{-n+1}} \, dz$$

for $n = 1, 2, 3, \dots$. In other words,

$$f(z) = \dots + \frac{b_1}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

Then

$$\operatorname*{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

$z = z_0$ is a pole if $f(z) = \frac{g(z)}{(z-z_0)^n}$ where g is a holomorphic in a neighborhood of z_0 and $g(z_0) \neq 0$.
If $n = 1$, $\operatorname*{Res}_{z=z_0} f(z) = g(z_0)$. If $n \geq 2$, $\operatorname*{Res}_{z=z_0} f(z) = \frac{g^{(n-1)}(z_0)}{(n-1)!}$.

Theorem 26.1. Cauchy's Residue Theorem: Let f be holomorphic except for some poles at $z_1, \dots, z_m$. Then

$$\int_\gamma f(z)\, dz = \sum_{i=1}^n \int_{\gamma_i} f(z)\, dz = 2\pi i \cdot (\text{sum of the residuals})$$

Evaluate:

$$\int_\gamma \frac{3z^3 + 2}{(z-1)(z^2+9)}\, dz$$

where γ is the circle $|z| = 4$ and γ is taken counterclockwise.
First note that

$$f(z) = \frac{3z^3 + 2}{(z-1)(z-3i)(z+3i)}$$

That means the singularities are at $z = 1$, $z = 3i$ and $z = -3i$, all of which are inside γ.
At $z = 1$, $f(z) = \frac{g(z)}{z-1}$ where $g(z) = \frac{3z^3-2}{z^2+9}$. This function is holomorphic in a small neighborhood of $z = 1$. Then

$$\operatorname*{Res}_{z=1} f(z) = g(1) = \frac{3(1)^3 + 2}{1 + 9} = \frac{5}{10} = \frac{1}{2}$$

At $z = 3i$, $f(z) = \frac{\phi(z)}{z-3i}$ where $\phi(z) = \frac{3z^3+2}{(z-1)(z+3i)}$. This function is holomorphic in a small neighborhood of $z = 3i$. Thus

$$\operatorname*{Res}_{z=3i} f(z) = \frac{2 - 81i}{(-1+3i)(6i)} = \frac{81 - 2i}{6(-1+3i)} = \frac{(81-2i)(-1-3i)}{-6(10)} = \frac{-87 - 241i}{-60} = \frac{87 + 241i}{60}$$

At $z = -3i$, $f(z) = \frac{h(z)}{z+3i}$ where $h(z) = \frac{3z^3+2}{(z-1)(z-3i)}$. This function is holomorphic in a small neighborhood of $z = -3i$. Then

$$\operatorname*{Res}_{z=-3i} f(z) = \frac{2 + 81i}{(-1-3i)(-6i)} = \frac{-81 + 2i}{(-1-3i)6} = \frac{75 - 245i}{60}$$

Then

$$\int_\gamma \frac{3z^3 + 2}{(z-1)(z^2+9)}\, dz = 2\pi i (\frac{1}{2} + \frac{5}{4} + \frac{5}{4}) = 6\pi i$$

Evaluate
$$\int_\gamma \frac{dz}{z^3(z+4)}$$
where $\gamma : |z| = 2$ in the counterclockwise direction.
First, note that $f(z) = \frac{1}{z^3(z^2+4)}$. Inside γ, f has only one singularity, at $z = 0$. Now let $f(z) = \frac{g(z)}{z^3}$ where $g(z) = \frac{1}{z+4}$. This function is holomorphic in a small neighborhood of $z = 0$. Then
$$\operatorname*{Res}_{z=0} f(z) = \frac{g''(0)}{2!} = \frac{1}{32} \cdot \frac{1}{2} = \frac{1}{64}$$
Therefore
$$\int_\gamma \frac{dz}{z^3(z+4)} = 2\pi i \cdot \frac{1}{64} = \frac{\pi}{32} i$$
Evaluate
$$\int_\gamma \frac{\cosh \pi z}{z(z^2+1)}\, dz$$
where $\gamma : |z| = 2$ counterclockwise. Note that $\cosh z = \frac{e^z + e^{-z}}{2}$.
Let $f(z) = \frac{\cosh \pi z}{z(z^2+1)}$. f has singularities at $z = 0$, $z = i$ and $z = -i$.
At $z = 0$, $g(z) = \frac{e^{\pi z} + e^{-\pi z}}{2(z^2+1)}$. Then $f(z) = \frac{g(z)}{2}$ which is holomorphic in a small neighborhood of $z = 0$. Then
$$\operatorname*{Res}_{z=0} f(z) = g(0) = 1$$
At $z = i$, $\phi(z) = \frac{e^{\pi z} + e^{-\pi z}}{2z(z+i)}$. Then $f(z) = \frac{\phi(z)}{z-i}$ which is holomorphic in a neighborhood of $z = i$. Then
$$\operatorname*{Res}_{z=i} f(z) = \phi(i) = \frac{-1-1}{2i(2i)} = \frac{-2}{-4} = \frac{1}{2}$$
At $z = -i$, $h(z) = \frac{e^{\pi z} + e^{-\pi z}}{1z(z-i)}$. Then $f(z) = \frac{h(z)}{2+i}$ which is holomorphic in a small neighborhood of $z = -i$. Then
$$\operatorname*{Res}_{z=-i} f(z) = h(-i) = \frac{-1 + -1}{(-2i)(-2i)} = \frac{-2}{-4} = \frac{1}{2}$$
Hence
$$\int_\gamma \frac{\cosh \pi z}{z(z^2+1)}\, dz = 2\pi i(1 + \frac{1}{2} + \frac{1}{2}) = 4\pi i$$

27 Lecture 26

Theorems:

- Liouville's Theorem: Every bounded entire function is a constant.

  Proof. Let f be an entire function such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$ be an arbitrary point in $\mathbb{C}$ and consider a disk of radius R centered at z_0. By Cauchy's estimate, $|f'(z)| \leq \frac{M}{R}$. But $R > 0$ is arbitrary and hence $f'(z) = 0$. Since $z_0 \in \mathbb{C}$ is arbitrary, $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore f is constant. □

A polynomial of degree $n \geq 0$ is of the form

$$f(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0$$

where $a_0, a_1, \dots, a_{n-1} \in \mathbb{C}$.

- FTA (Fundamental Theorem of Algebra): If $p(z)$ is a nonconstant polynomial, then there exists a complex number z such that $p(z) = 0$.

  Proof. Let

  $$p(z) = z_n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0 = z^n[1 + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}]$$

  be a nonconstant polynomial. Then $\lim_{z\to\infty} p(z) = \infty$. Suppose there exists no $z \in \mathbb{C}$ such that $p(z) = 0$. Define $f(z) = \frac{1}{p(z)}$. Then f is an entire function. Furthermore, $\lim_{z\to\infty} f(z) = 0$. So there exists $N > 0$ such that $|f(z)| < 1$ for all $|z| > N$. Now consider the closed disk $\overline{B(0,N)} = \left\{z : |z| \leq N\right\}$ which is compact. Since f is holomorphic, and therefore continuous on $\overline{B(0,N)}$, it must be bounded on $\overline{B(0,N)}$. In other words, there exists $M > 0$ such that $|f(z)| \leq M$ for all z such that $|z| \leq N$. Thus f is a bounded entire function. By Louville's theorem, f is a constant. Therefore $p(z)$ is a constant which contradicts that $p(z)$ is a nonconstant polynomial. Hence there exists $z \in \mathbb{C}$ such that $p(z) = 0$. $\square$

- RRST (Riemann's Removable Singularity Theorem): Suppose $f \in O(U \setminus \left\{a\right\})$ and f is bounded in $D'(a,r) = \left\{z : 0 < |z-a| < r\right\}$, for some $r > 0$¿ Then f has a removable singularity at a.

  Proof. Define $h(a) = 0$ and $h(z) = (z-a)^2 f(z)$ in $U \setminus \left\{a\right\}$. Claim: $h \in O(U)$ and $h'(a) = 0$. Note that

  $$h'(a) = \lim_{z\to z} \frac{h(z) - h(a)}{z-a} = \lim_{z\to a} \frac{(z-a)^2 f(z)}{z-a} = \lim_{z\to a} (z-a)f(a) = 0$$

  because f is bounded in $D'(a,r)$. Hence $h \in O(U)$ and $h'(a) = 0$. Now,

  $$\begin{aligned} h(z) &= \sum_{n=0}^{\infty} c_n (z-a)^n \\ &= c_0 + c_1(z-a) + c_2(z-a)^2 + \dots \\ h(a) &= c_0 = 0 \\ h'(z) &= \sum_{n=0}^{\infty} nc_n (z-a)^{n-1} \\ &= c_1 + 2(z-a)^n + \dots \\ h'(a) &= c_1 = 0 \end{aligned}$$

Hence

$$h(z) = \sum_{n=2}^{\infty} c_n (z-a)^n$$

Therefore

$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z-a)^n$$

for all $z \in D(a,r)$. So $f \in O(D(a,r))$ and hence a is a removable singularity. □

Problems:

- f is an entire function such that $\text{Re}\{f\} \leq M$. Show that f is a constant.

  Proof. Suppose f is an entire function such that $\text{Re}\{f\} \leq M$. Define $F = e^f$. F is an entire function and $|F| = \left|e^f\right| = e^{\text{Re}\{f\}} \leq e^M$. So F is a bounded entire function. By Liouville's theorem, F is a constant. That means $F'(z) = 0$ for all $z \in \mathbb{C}$. Then $e^{f(z)} f'(z) = 0$. Hence $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore F is constant. □

- f is an entire function such that $\text{Im}\{f\} \leq M$. Show that f is a constant.

  Proof. Suppose f is an entire function such that $\text{Im}\{f\} \leq M$. Define $F = e^{-if}$. Then $|F| = \left|e^{-if}\right| = e^{\text{Im}\{f\}} \leq e^M$. So F is a bounded entire function. That means F is a constant. Then $F'(z) = 0$ for all $z \in \mathbb{C}$. Then $e^{-if} f'(z) = 0$. That is, $f'(z) = 0$ for all $z \in \mathbb{C}$ and so f is constant. □

- f is an entire function. Suppose there exists a constant M, $R \geq 0$ and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for all $|z| > R$. Show that f is a polynomial of degree $\leq n$.

  Proof. Since f is an entire function,

  $$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

  or

  $$f(z) = f(0) + f'(0)z + \frac{f^2(0)}{2!} z^2 + \cdots + \frac{f^n(0)}{n!} z^n + \dots$$

  By Cauchy's estimate,

  $$\frac{\left|f^{(k)}(0)\right|}{k!} \leq \frac{Mr^n}{r^k}$$

  if $r > R$. So for all $k > n$,

  $$\frac{\left|f^{(n)}(0)\right|}{k!} \leq \frac{M}{r^{k-n}}$$

  where n is fixed and is true for all $k > 0$. Since $r > R$ is arbitrary, it follows that $f^{(k)}(0) = 0$ for all $k > n$. Hence by the expansion of $f(z)$, f is a polynomial of degree $\leq n$. □

- Let Ω be a region and $f, g \in O(\Omega)$ such that $f(z)g(z) = 0$ for all $z \in \Omega$. Show that either $f(z)$ is a constant or $g(z) = 0$ for all $z \in \Omega$.

  Proof. Suppose $g(z) \neq 0$ for all $z \in \Omega$. This means there exists $a \in \Omega$ such that $g(a) \neq 0$. By the continuity of g, there exists $R > 0$ such that $g(z) \neq 0$ for all $z \in B(a, R) = \left\{z : |z - a| < R\right\}$. This implies $f(z) = 0$ for all $z \in B(a, R)$. Hence by the Identity Theorem, $f(z) = 0$ for all $z \in \Omega$. $\square$

- Let Ω be a region and $f, g \in O(\Omega)$ such that $\overline{f}g \in O(\Omega)$. Show that either $f(z)$ is a constant or $g(z) = 0$ for all $z \in \Omega$.

  Proof. Suppose $g(z) \neq 0$ for all$z \in \Omega$, meaning $g \not\equiv 0$, or there exists $a \in \Omega$ such that $g(a) \neq 0$. By the continuity of g, there exists a neighborhood $B(a, r) = \left\{z : |z - a| < r\right\}$ such that $g(z) \neq 0$ for all $z \in B(a, r)$. Let $\overline{f}g = h$ given that $h \in O(\Omega)$. Then $\overline{f}(z) = \frac{h(z)}{g(z)}$ for all $z \in B(a, r)$ because $g(z) \neq 0$ for all $z \in B(a, r)$. Since h and g are both holomorphic and $g(z) \neq 0$ in $B(a, r)$, it follows that $\overline{f}$ is holomorphic in $B(a, r)$. Thus f and $\overline{f}$ are both holomorphic in $B(a, r)$ and so f is constant on $B(a, r)$. Hence by the Identity Theorem, f is constant on Ω. $\square$

Note: Identity Theorem: Suppose $f, g \in O(\Omega)$ and $a \in \Omega$ such that $f(z) = g(z)$ for all $z \in B(a, r) = \left\{z : |z - a| < r\right\}$. Then $f(g) = g(z)$ for all $z \in \Omega$.

Cauchy's Integral Formula:

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z - a)^{n+1}}\, dz$$

- $\int_\gamma \frac{5z^2+2z+1}{(z-i)^3}\, dz$ in the region $\gamma : |z| = 2$

$$\begin{aligned}
\int_\gamma \frac{5z^2 + 2z + 1}{(z - i)^3}\, dz &= \int_\gamma \frac{f(z)}{(z - i)^3}\, dz \\
f(z) = 5z^2 + 2z + 1 \\
f'(z) &= 10z \\
f''(z) &= 10 \to f''(i) = 10 \\
\int_\gamma \frac{5z^2 + 2z + 1}{(z - i)^3}\, dz &= \frac{2\pi i}{2!} f''(i) \\
&= \frac{2\pi i}{2} \cdot 10 = 10\pi i
\end{aligned}$$

- $\int_\gamma \frac{e^{2z}-e^{-2z}}{z^5}\,dz$ in the region $\gamma : |z| = 4$

$$\int_\gamma \frac{e^{2z}-e^{-2z}}{z^5} = \int_\gamma \frac{f(z)}{z^5}\,dz$$
$$f(z) = e^{2z} - e^{-2z}$$
$$f'(z) = 2e^{2z} + 2e^{-2z}$$
$$f''(z) = 4e^{2z} - 4e^{-2z}$$
$$f'''(z) = 8e^{2z} + 8e^{-2z}$$
$$f^4(z) = 16e^{2z} - 16e^{-2z}$$
$$f^5(z) = 32e^{2z} + 32e^{-2z} \to f^5(0) = 64$$
$$\int_\gamma \frac{e^{2z}-e^{-2z}}{z^5} = \frac{2\pi i}{5!}\cdot 64 = \frac{128}{120}\pi i = \frac{16}{15}\pi i$$

Cauchy's Residue Formula:

$$\operatorname*{Res}_{z=z_0} f(z) = \begin{cases} g(z_0) & \text{if } n = 1 \\ \frac{g^{(n-1)}(z_0)}{(n-1)!} & \text{if } n \geq 2 \end{cases}$$

- $\int_\gamma \frac{1-2z}{z(z-1)(z-3)}$ where $\gamma : |z| = 2$.
  Inside γ, there are only two singularities, $z = 0$ and $z = 1$, both of order 1.
  At $z = 0$, $f(z) = \frac{g(z)}{z}$ where $g(z) = \frac{1-2z}{(z-1)(z-3)} = \frac{1-2z}{z^2-4z+3}$, which is holomorphic in a small neighborhood of $z = 0$. Then

$$\operatorname*{Res}_{z=0} = g(0) = \frac{1}{3}$$

  At $z = 1$, $f(z) = \frac{\phi(z)}{z-1}$ where $\phi(z) = \frac{1-2z}{z(z-3)}$ which is holomorphic in a small neighborhood of $z = 1$. Then

$$\operatorname*{Res}_{z=1} f(z) = \phi(1) = \frac{-1}{-2} = \frac{1}{2}$$

  Therefore

$$\int_\gamma \frac{1-2z}{z(z-1)(z-3)} = 2\pi i(\frac{1}{3} + \frac{1}{2}) = \frac{5}{3}\pi i$$

- $\int_\gamma \frac{e^z}{z(z-2)^3}\,dz$ where $\gamma : |z| = 3$.
  Inside γ, there are only two singularities, $z = 0$ and $z = 2$, of order 1 and 3 respectively.
  At $z = 0$, $f(z) = \frac{g(z)}{z}$ where $g(z) = \frac{e^z}{(z-2)^3}$ which is holomorphic in a small neighborhood of $z = 0$. Then

$$\operatorname*{Res}_{z=0} f(z) = g(0) = -\frac{1}{8}$$

  At $z = 2$, $f(z) = \frac{\phi(z)}{(z-2)^3}$ where $\phi(z) = \frac{e^z}{z}$ which is holomorphic in a small neighborhood of $z = 2$. Now

$$\phi(z) = \frac{e^z}{z}$$
$$\phi'(z) = \frac{ze^z - e^z}{z^2}$$
$$\phi''(z) = \frac{z^2(ze^z + e^z - e^z) - (ze^z - e^z)2z}{z^4}$$
$$\phi''(2) = \frac{4(2e^2) - 4(2e^2 - e^2)}{16} = \frac{4e^2}{16} = \frac{e^2}{4}$$

Therefore

$$\operatorname*{Res}_{z=2} f(z) = \frac{\phi''(2)}{2!} = \frac{e^2}{8}$$

Furthermore,

$$\int_\gamma \frac{e^z}{z(z-2)^3}\,dz = 2\pi i(-\frac{1}{8} + \frac{e^2}{8}) = (\frac{e^2-1}{4})\pi i$$

- $\int_\gamma \frac{\cos z}{z^2(z-\pi)^3}\,dz$ where $\gamma : |z| = 4$.
Inside γ, there are two singularities, $z = 0$ and $z = \pi$, of order 1 and 2 respectively. At $z = 0$, $f(z) = \frac{g(z)}{z^2}$ where $g(z) = \frac{\cos z}{(z-\pi)^3}$ which is holomorphic in a small neighborhood of $z = 0$. Now

$$g'(z) = \frac{-(\sin z)(z-\pi)^3 - 3(\cos z)(z-\pi)^2}{(z-\pi)^4}$$

and

$$g'(0) = \frac{-3\pi^2}{\pi^6} = -\frac{3}{\pi^4}$$

Therefore

$$\operatorname*{Res}_{z=0} f(z) = g'(0) = -\frac{3}{\pi^4}$$

At $z = \pi$, $f(z) = \frac{\phi(z)}{(z-\pi)^3}$ where $\phi(z) = \frac{\cos z}{z^2}$ which is holomorphic in a small neighborhood of $z = \pi$. Now

$$\phi(z) = \frac{\cos z}{z^2}$$
$$\phi'(z) = \frac{-z^2\sin z - 2z\cos z}{z^4}$$
$$\phi''(z) = \frac{z^4[(-z^2\cos z - 2z\sin z) - (-2z\sin z + 2\cos z)] + 4z^3(z^2\sin z + 2z\cos z)}{z^8}$$
$$\phi''(z) = \frac{\pi^6 + 2\pi^4 - 8\pi^4}{\pi^8} = \frac{\pi^6 - 6\pi^4}{\pi^8} = \frac{\pi^2 - 6}{\pi^4}$$

Therefore

$$\operatorname*{Res}_{z=\pi} f(z) = \frac{\phi''(\pi)}{2!} = \frac{\pi^2 - 6}{2\pi^4}$$

Furthermore,

$$\int_\gamma \frac{\cos z}{z^2(z-\pi)^3} = 2\pi i(\frac{-3}{\pi^4} + \frac{\pi^2 - 6}{2\pi^4}) = 2\pi i(\frac{1}{2\pi^2}) = \frac{1}{\pi}$$

Laurent Series: Use the fact that

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$$

for $|z| < 1$. Find the Laurent expansion of the following in the given region

- $f(z) = \frac{1}{z^2(1-z)}$

  1. $0 < |z| < 1$
$$\begin{aligned} f(z) &= \frac{1}{z^2}\frac{1}{1-z} \\ &= \frac{1}{z^2}(1 + z + z^2 + z^3 + \dots + z^n + \dots) \\ &= \frac{1}{z^2} + \frac{1}{z} + z + 1 + z^2 + \dots + z^{n-2} + \dots \\ &= \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} z^n \end{aligned}$$

  2. $1 < |z| < \infty$
$$\begin{aligned} f(z) &= \frac{1}{z^2(1-z)} \\ &= \frac{1}{z^2 - z^3} \\ &= \frac{1}{-z^3(1-\frac{1}{z})} \\ &= -\frac{1}{z^3}\frac{1}{1-\frac{1}{z}} \\ &= -\frac{1}{z^3}(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n} + \dots) \\ &= -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \dots \\ &= -\sum_{n=3}^{\infty} \frac{1}{z^n} \end{aligned}$$

- $f(z) = -\frac{1}{(z-1)(z-2)}$ Note first that $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$ by partial fraction decomposition.

1. $|z| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= -\frac{1}{1-z} + \frac{1}{2-z} \\ &= -\frac{1}{1-z} + \frac{1}{2(1-\frac{1}{z})} \\ &= -(1+z+z^2+\dots+z^n+\dots) + \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots+(\frac{z}{2})^n+\dots) \\ &= \sum_{n=0}^{\infty}(\frac{1}{2^{n+1}}-1)z^n \end{aligned}$$

2. $1 < |z| < 2$

$$\begin{aligned} f(z) &= \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-\frac{z}{2})} \\ &= \frac{1}{z}(1+\frac{1}{z}+\frac{1}{z^2}+\dots+\frac{1}{z^n}+\dots) + \frac{1}{2}(1+\frac{z}{2}+(\frac{z}{2})^2+\dots+(\frac{z}{2})^n+\dots) \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^n}{2^{n+1}} \end{aligned}$$

3. $|z| > 2$

$$\begin{aligned} f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= \frac{1}{z(1-\frac{1}{z})} - \frac{1}{z(1-\frac{2}{z})} \\ &= \frac{1}{z}(1+\frac{1}{z}+(\frac{1}{z})^2+\dots+(\frac{1}{z})^n+\dots) - \frac{1}{z}(1+\frac{2}{z}+(\frac{2}{z})^2+\dots+(\frac{2}{z})^n+\dots) \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}} \end{aligned}$$