Math 310: Real Analysis Study Guide

Upper Bound: Suppose $E \subset \mathbb{R}$ and $w \in E$. w is an upper bound of E if $x \leq w$ for all $x \in E$.

Lower Bound: Suppose $E \subset \mathbb{R}$ and $w \in E$. w is a lower bound of E if $x \geq w$ for all $x \in E$.

Bounded Above: A subset is said to be bounded above it has at least one upper bound.

Bounded Below: A subset is said to be bounded below if it has at least one lower bound.

Least Upper Bound (LUB): A real number w is said to be the least upper bound of a subset $E \subset \mathbb{R}$ if

1. w is an upper bound of E
2. for all $y < w$, there exists $x \in E$ with $x > y$

Greatest Lower Bound (GLB): A real number w is said to be the greatest lower bound of a subset $E \subset \mathbb{R}$ if

1. w is a lower bound of E
2. for all $y > w$, there exists $x \in E$ with $x < y$

Sequence: A sequence in $\mathbb{R}$ is a function defined on $\mathbb{N}$ with values in $\mathbb{R}$ (denoted as: $\{x_n\} = \{x_1, x_2, \dots\}$)

Subsequence: A subsequence of (x_n) is a sequence (y_k) where for some sequence $\{n_k\}$ in $\mathbb{N}$ with $n_1 < n_2 < \dots$, there is $\lim y_k = x_{n_k}$

Limit of a Sequence: $\lim(x_n) = x$ if for any $\varepsilon > 0$, there exists $\mathbb{N} \in \mathbb{R}$ such that $|x_n - x| < \varepsilon$ for all $n \geq \mathbb{N}$

Convergent: (x_n) is convergent if $\lim x_n = x$

Cauchy Sequence: (x_n) is said to be a Cauchy sequence if for any $\varepsilon > 0$, there exists $\mathbb{N}$ such that $|x_n - x_k| < \varepsilon$ for all $n, k \geq \mathbb{N}$.

Limit of a Function: If for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ for all x with $|x - a| < \delta$ then $\lim_{x \to a} f(x) = L$.

Continuity (sequential): A function $f : E \to \mathbb{R}$ is continuous for all $x \in E$ if (x_n) is a sequence in E with $\lim x_n = x$ then $\lim f(x_n) = f(x)$.

Continuity ($\varepsilon - \delta$): A function $f : E \to \mathbb{R}$ is continuous if for every $x \in E$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(u) - f(x)| < \varepsilon$ for all $u \in E$ with $|u - x| < \delta$.

Intermediate Value Theorem: Let I be any interval in $\mathbb{R}$. Suppose $f : I \to \mathbb{R}$, f continuous, $a, b \in I$ and $a < b$. If y is in between $f(a)$ and $f(b)$, then there is at least one $c \in I$ with $f(c) = y$.

Function Bounded Above: $f : E \to \mathbb{R}$ is bounded above on E if there exists M such that $f(x) \leq M$ for all $x \in E$.

Function Bounded Below: $f : E \to \mathbb{R}$ is bounded below on E if there exists m such that $f(x) \geq m$ for all $x \in E$.

Extreme Value Theorem: Suppose f is continuous on $[a, b]$. Then f has a min and max on $[a, b]$.

Derivative of f: $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Local Maximum: f has a local maximum at $x = a$ if there is some $\delta > 0$ such that $f(x) \leq f(a)$ for all x with $|x - a| < \delta$.

Local Minimum: f has a local minimum at $x = a$ if there is some $\delta > 0$ such that $f(x) \geq f(a)$ for all x with $|x - a| < \delta$.

Rolle's Theorem: If $f'(x)$ exists on $[a, b]$, f continuous on $[a, b]$ and $f(a) = f(b)$, then $f'(c) = 0$ for some c in $[a, b]$.

Mean Value Theorem: Suppose f is "nice" on $[a, b]$. Then there exists a c with $\frac{f(b) - f(a)}{b - a} = f'(c)$

Continuity: A function $f : E \to \mathbb{R}$ is continuous if for all $x \in E$ and all $\varepsilon > 0$, there exists $\delta(x, \varepsilon)$ with $|f(u) - f(x)| < \varepsilon$ for all $u \in E$ with $|x - u| < \delta$.

Uniform Continuity: A function $f : E \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$ there exists (a single) $\delta(\varepsilon)$ such that $|f(u) - f(x)| < \varepsilon$ for all $x, u \in E$ with $|x - u| < \delta$. (δ does not depend on x.)

Partition: A partition of the interval $[a, b]$ with $a < b$ is a finite subset $P \subset [a, b]$ such that $a \in P$ and $b \in P$ and is written as: $P = \{x_0, x_1, \ldots, x_n\}$ where $a = x_0 < x_1 < \cdots < x_n = b$

Lower Sum: $L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k$ where $m_k = \text{glb}\{f(x) : x \in I_k\}$

Upper Sum: $U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$ where $M_k = \text{lub}\{f(x) : x \in I_k\}$

Lower Integral: $\underline{\int} = \underline{\int_a^b} f(x)\, dx = \text{glb}\{U(f, p) : p \in P\}$ (bounded above by $M(b - a)$)

Upper Integral: $\overline{\int} = \overline{\int_a^b} f(x)\, dx = \text{lub}\{L(f, p) : p \in P\}$ (bounded below by $-M(b - a)$)

Riemann Integrable: A continuous function f is Riemann integrable if $U(f, P) - L(f, P) < \varepsilon$.

Integrable: If $\underline{\int} f = \overline{\int} f$, then f is integrable on $[a, b]$

$f_n \to f$: For every $x \in E$ and $\varepsilon > 0$, there exists $N(\varepsilon, x)$ such that $|f_n(x) - f(x)| < \varepsilon$, all $n \geq N$, then f_n converges to f.

$f_n \rightrightarrows f$: If for every $\varepsilon > 0$, there exists (a single) $N(\varepsilon)$ such that for every $x \in E$, $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ then f_n converges uniformly to f.

Sum of Infinite Series: $\sum_{k=1}^{\infty} a_k = S$ if $\lim S_n = S$ where S_n are partial sums and $S_n = \lim_{k=1}^{n} a_k$.