Assignment #2: Chapter 3 Questions 48, 130, 152, 184, 196

Question 3.48: Two teams A and B play a series of games until one team wins four gams. Assume that the games are played independently and that the probability that A wins any game is p. Compute the probability that the series lasts exactly five games.

To last 5 games, a single team must win thrice and then lose, followed by a win. Thus

$$\mathbb{P}(5 \text{ games}) = \mathbb{P}(\text{win 3 of 4 games})\mathbb{P}(\text{win 5th game}) = \binom{4}{3}p^3(1-p)p = 4p^4(1-p)$$

Question 3.130: A parking lot has two entrances. Cars arrive at entrance I according to a Poisson distribution at an average of three per hour and at entrance II according to a Poisson distribution at an average of four per hour. What is the probability that a total of three cars will arrive at the parking lot in a given hour? Assume that the numbers of cars arriving at the two entrances are independent.

Let $X_1 \sim \text{Poisson}(3)$ and $X_2 \sim \text{Poisson}(4)$. Then

$$\begin{aligned}
\mathbb{P}(3 \text{ cars}) &= \mathbb{P}(X_1 = 0, X_2 = 3) + \mathbb{P}(X_1 = 1, X_2 = 2) \\
&+ \mathbb{P}(X_1 = 2, X_2 = 1) + \mathbb{P}(X_1 = 3, X_2 = 0) \\
&= \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 3) + \mathbb{P}(X_1 = 1)\mathbb{P}(X_2 = 2) \\
&+ \mathbb{P}(X_1 = 2)\mathbb{P}(X_2 = 1) + \mathbb{P}(X_1 = 3)\mathbb{P}(X_2 = 0) \\
&= \left(\frac{3^0e^{-3}}{0!}\right)\left(\frac{4^3e^{-4}}{3!}\right) + \left(\frac{3^1e^{-3}}{1!}\right)\left(\frac{4^2e^{-4}}{2!}\right) \\
&+ \left(\frac{3^2e^{-3}}{2!}\right)\left(\frac{4^1e^{-4}}{1!}\right) + \left(\frac{3^3e^{-3}}{3!}\right)\left(\frac{4^0e^{-4}}{0!}\right) \\
&= 0.0521
\end{aligned}$$

Question 3.152: If Y has moment-generating function $m(t) = e^{6(e^t-1)}$, what is $\mathbb{P}(|Y-\mu| \leq 2\sigma)$?

This is the moment-generation function for Poisson($\lambda = 6$). For the Poisson distribution, $\mu = \sigma^2 = 6$ and $\sigma = \sqrt{6}$. Then

$$\begin{aligned}
\mathbb{P}(|Y-\mu| \leq 2\sigma) &= \mathbb{P}(\mu - 2\sigma \leq Y \leq \mu + 2\sigma) \\
&= \mathbb{P}(6 - 2\sqrt{6} \leq Y \leq 6 + 2\sqrt{6}) \\
&= \mathbb{P}(1.1 \leq Y \leq 10.9) \\
&= \mathbb{P}(2 \leq Y \leq 10) \\
&= \sum_{i=2}^{10} \frac{6^ie^{-6}}{i!} \\
&= 0.940
\end{aligned}$$

Question 3.184: A city commissioner claims that 80% of the people living in the city favor garbage collection by contract to a private company over collection by city employees. To test the commissioner's claim, 25 city residents are randomly selected, yielding 22 who prefer contracting to a private company.

a. If the commissioner's claim is correct, what is the probability that the sample would contain at least 22 who prefer contracting to a private company?
   This is $X \sim \text{Binomial}(25, 0.8)$.
   $$\mathbb{P}(X \geq 22) = \sum_{i=22}^{25} \binom{25}{i} \times 0.8^i \times 0.2^{25-i} = 0.234$$

b. If the commissioner's claim is correct, what is the probability that exactly 22 would prefer contracting to a private company?
   $$\mathbb{P}(X = 22) = \binom{25}{22} 0.8^{22} 0.2^3 = 0.135$$

c. Based on observing 22 in a sample of size 25 who prefer contracting to a private company, what do you conclude about the commissioner's claim that 80% of city residents prefer contracting to a private company?

   The commissioner's claim is correct.

Question 3.196: The number of imperfections in the weave of a certain textile has a Poisson distribution with a mean of 4 per square yard. The cost of repairing the imperfections in the weave is \$10 per imperfection. Find the mean and standard deviation of the repair cost for an 8-square-yard bolt of the textile.
If the textile is 8-square-yard long, then the number of imperfections in the weave has a Poisson distribution with a mean of $4 \times 8 = 32$. Hence the mean and standard deviation of the repair cost is

$$\begin{aligned} \text{E}[\text{cost}] &= \text{E}[10X] = 10\text{E}[X] = 10 \cdot 32 = \$320 \\ \sigma_{\text{cost}} &= \sqrt{\text{Var}[\text{cost}]} = \sqrt{\text{Var}[10X]} = \sqrt{10^2 \text{Var}[X]} = \sqrt{100 \times 32} = \$56.568 \end{aligned}$$