Assignment #3: Chapter 9 Questions 20, 46, 56, 70, 82

Question 9.20: If Y has a binomial distribution with n trials and success probability p, show that $\frac{Y}{n}$ is a consistent estimator of p.
If Y has a binomial distribution as stated, then

$$\begin{aligned} \mathrm{E}[Y] &= np \\ \frac{\mathrm{E}[Y]}{n} &= \frac{np}{n} \\ \mathrm{E}\left[\frac{Y}{n}\right] &= p \end{aligned}$$

Furthermore,

$$\begin{aligned} \mathrm{Var}[Y] &= npq \\ \frac{\mathrm{Var}[Y]}{n^2} &= \frac{npq}{n^2} \\ \mathrm{Var}\left[\frac{Y}{n}\right] &= \frac{pq}{n} \end{aligned}$$

Finally,

$$\lim_{n\to\infty} \mathrm{Var}\left[\frac{Y}{n}\right] = \lim_{n\to\infty} \frac{pq}{n} = 0$$

Since the variance of the estimator goes to 0 as $n \to \infty$, it shows that $\frac{Y}{n}$ is a consistent estimator of p.

Question 9.46: If $Y_1, Y_2, \dots, Y_n$ denote a random sample from an exponential distribution with mean β, show that $f(\mathcal{Y} \mid \beta)$ is in the exponential family and that $\bar{Y}$ is sufficient for β.

Let $Y \sim \mathrm{Exp}(\beta)$. Then the likelihood function $L(\theta)$ of the sample is the joint distribution

$$\begin{aligned} L(y_1, y_2, \dots, y_n \mid \theta) &= f(\mathcal{Y} \mid \theta) = f(y_1, y_2, \dots, y_n \mid \theta) \\ &= f(y_1 \mid \theta) \times f(y_2 \mid \theta) \times \cdots \times f(y_n \mid \theta) \\ &= \left(\frac{1}{\beta} e^{-\frac{y_1}{\beta}}\right) \times \left(\frac{1}{\beta} e^{-\frac{y_2}{\beta}}\right) \times \cdots \times \left(\frac{1}{\beta} e^{-\frac{y_n}{\beta}}\right) \\ &= \frac{1}{\beta^n} e^{-\frac{y_1}{\beta}} e^{-\frac{y_2}{\beta}} \dots e^{-\frac{y_n}{\beta}} = \frac{1}{\beta^n} e^{-\frac{1}{\beta^n}(y_1+y_2+\cdots+y_n)} \\ &= \frac{1}{\beta^n} e^{-\frac{n\bar{y}}{\beta^n}} \end{aligned}$$

$f(\mathcal{Y} \mid \beta)$ is in the exponential family where the parameter β is now β^n and the realization is $n\bar{y}$. Furthermore, since $f(\mathcal{Y} \mid \beta)$ can be broken into $g(\bar{y}, \beta)$ and $h(y_1, \dots, y_n)$, where

$$g(\bar{y}, \beta) = \frac{1}{\beta^n} e^{-\frac{n\bar{y}}{\beta^n}} \text{ and } h(y_1, \dots, y_n) = 1$$

$\bar{Y}$ is sufficient for β.

Question 9.56: Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from a normal distribution with mean μ and variance σ^2. If μ is known and σ^2 is unknown, then $\sum_{i=1}^n (Y_i - \mu)^2$ is sufficient for σ^2. Find an MVUE of σ^2.

If $\sum_{i=1}^n (Y_i - \mu)^2$ is sufficient for σ^2, then by the Rao-Blackwell theorem, there exists $\hat{\sigma}^{2*} = \mathrm{E}\left[\hat{\sigma^2} \mid \sum_{i=1}^n (Y_i - \mu)^2\right]$, where $\hat{\sigma^2}$ is an unbiased estimator for σ^2, such that

$$\mathrm{E}\left[\hat{\theta}^*\right] = \sigma^2 \text{ and } \mathrm{Var}\left[\hat{\theta}^*\right] \leq \mathrm{Var}\left[\hat{\theta}\right]$$

First find the sufficient estimator for σ^2.

The likelihood function is

$$\begin{aligned} L(y_1, y_2, \dots, y_n \mid \sigma^2) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_1-\mu)^2}{2\sigma^2}} \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_2-\mu)^2}{2\sigma^2}} \times \cdots \times \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_n-\mu)^2}{2\sigma^2}} \\ &= \frac{1}{(2\pi)^{\frac{n}{2}} (\sigma^2)^{\frac{n}{2}}} e^{-\frac{1}{2\sigma^2}((y_1-\mu)^2 + (y_2-\mu)^2 + \cdots + (y_n-\mu)^2)} \\ &= \underbrace{\frac{1}{(\sigma^2)^{-\frac{n}{2}}} e^{-\frac{\sum_{i=1}^n (y_i-\mu)^2}{2\sigma^2}}}_{g\left(\sum_{i=1}^n (y_i-\mu)^2, \sigma^2\right)} \times \underbrace{\frac{1}{(2\pi)^{-\frac{n}{2}}}}_{h(y_1, y_2, \dots, y_n)} \end{aligned}$$

The sufficient estimator for σ^2 is $\hat{\theta} = \sum_{i=1}^n (y_i - \mu)^2$. Now find $\hat{\theta}^*$ such that

$$\mathrm{E}\left[\hat{\theta}^*\right] = \theta = \sigma^2 \text{ and } \mathrm{Var}\left[\hat{\theta}^*\right] \leq \mathrm{Var}\left[\sum_{i=1}^n (y_i - \mu)^2\right]$$

A simple elegant solution would be the sample variance, a function of the sufficient statistic itself,

$$\hat{\theta}^* = \frac{1}{n} \sum_{i=1}^n (y_i - \mu)^2$$

since its expected value is indeed the population variance and the variance of this quantity would be $\frac{1}{n^2}$ times smaller than the variance of the sufficient statistic $\hat{\theta}$. Therefore this quantity is the MVUE of σ^2.

Question 9.70: Suppose that $Y_1, Y_2, \dots, Y_n$ constitute a random sample from a Poisson distribution with mean λ. Find the method of moments estimator of λ.
If $Y \sim \text{Poisson}(\lambda)$, then $\text{E}[Y] = \lambda$. Note that the first moment of a random variable taken about the origin is

$$\mu' = \text{E}[Y]$$

which in this case is

$$\mu' = \lambda$$

Now, the corresponding first sample moment of any random variable is the average

$$\mu' = m' = \frac{1}{n}\sum_{i=1}^n Y_i$$

Thus the method of moments estimator of λ can be found by equating the two.

$$\begin{aligned} \mu' &= m' \\ \lambda &= \frac{1}{n}\sum_{i=1}^n Y_i \\ &= \bar{Y} \end{aligned}$$

Thus the method of moments estimator of λ is

$$\hat{\lambda} = \bar{Y}$$

Question 9.82: Let $Y_1, Y_2, \dots, Y_n$ denote a random sample from the density function given by

$$f(y \mid \theta) = \begin{cases} \left(\frac{1}{\theta}\right) r y^{r-1} e^{-y^r/\theta} & \text{if } \theta > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

where r is a known positive constant.

a. Find a sufficient statistic for θ.
First find the likelihood function $L(y_1, \dots, y_n \mid \theta)$.

$$\begin{aligned} L(y_1, \dots, y_n \mid \theta) &= f(y_1, \dots, y_n \mid \theta) \\ &= \left(\frac{1}{\theta} r y_1^{r-1} e^{-\frac{y_1^r}{\theta}}\right) \times \left(\frac{1}{\theta} r y_2^{r-1} e^{-\frac{y_2^r}{\theta}}\right) \times \cdots \times \left(\frac{1}{\theta} r y_n^{r-1} e^{-\frac{y_n^r}{\theta}}\right) \\ &= \frac{1}{\theta^n} r^n (y_1 y_2 \dots y_n)^{r-1} e^{-\frac{y_1^r + y_2^r + \cdots + y_n^r}{\theta}} \end{aligned}$$

This can be broken into $g(\sum_i y_i^r, \theta)$ and $h(y_1, \dots, y_n)$ where

$$g(\sum_i y_i^r, \theta) = \frac{1}{\theta^n} r^n e^{-\frac{\sum_i y_i^r}{\theta}} \text{ and } h(y_1, \dots, y_n) = (y_1 y_2 \dots y_n)^{r-1}$$

Therefore a sufficient statistic for θ is

$$\hat{\theta} = \sum_i y_i^r$$

b. Find the MLE of θ.
To find the MLE, first take the natural log of the likelihood function.

$$\ln L(y_1,\dots,y_n \mid \theta) = -n\ln\theta + n\ln r + (r-1)\ln(y_1y_2\dots y_n) - \frac{y_1^r + y_2^r + \cdots + y_n^r}{\theta}$$

Differentiate this with respect to θ and equate it to 0.

$$-\frac{n}{\theta} + \frac{y_1^r + y_2^r + \cdots + y_n^r}{\theta^2} = 0$$

Solve for θ.

$$\begin{aligned} \frac{n}{\theta} &= \frac{y_1^r + y_2^r + \cdots + y_n^r}{\theta^2} \\ n\theta^2 &= \theta(y_1^r + y_2^r + \cdots + y_n^r) \\ \theta &= \frac{y_1^r + y_2^r + \cdots + y_n^r}{n} \end{aligned}$$

Thus the MLE of θ is

$$\hat{\theta} = \frac{y_1^r + y_2^r + \cdots + y_n^r}{n}$$

c. Is the estimator in part (b) an MVUE for θ?
In part(a), it was found that $\sum_i y_i^r$ was a sufficient statistic for θ. Furthermore,

$$\hat{\theta} = \frac{y_1^r + y_2^r + \cdots + y_n^r}{n} = \frac{\sum_i y_i^r}{n}$$

is a function of the sufficient statistic. This shows that the estimator in part (b) is an MVUE for θ.