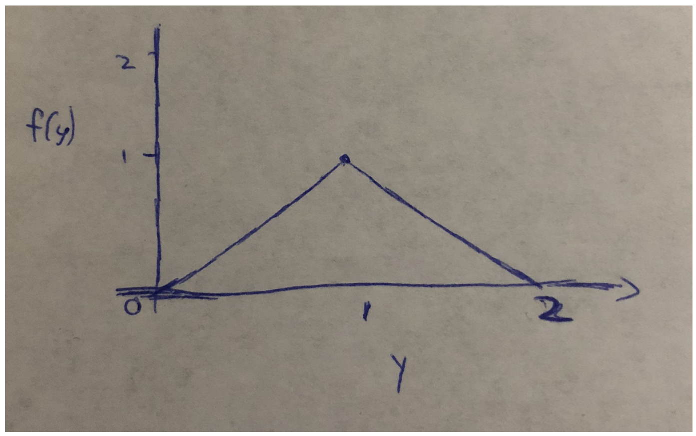
Assignment #3: Chapter 4 Questions 14, 32, 138, 160, 174

Question 4.14: A gas station operates two pumps, each of which can pump up to 10,000 gallons of gas in a month. The total amount of gas pumped at the station in a month is a random variable Y (measured in 10,000 gallons) with a probability density function given by

$$f(y) = \begin{cases} y & \text{if } 0 < y < 1 \\ 2 - y & \text{if } 1 \leq y < 2 \\ 0 & \text{elsewhere} \end{cases}$$

a. Graph $f(y)$.



b. Find $F(y)$ and graph it.
For $0 < y < 1$,

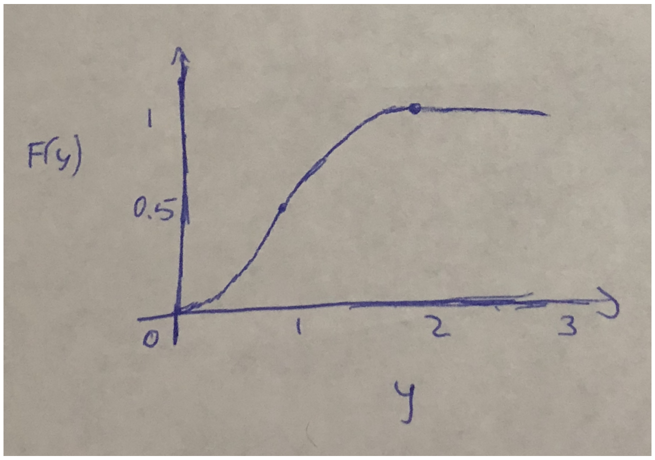
$$F(y) = \int_0^y t\, dt = \frac{y^2}{2}$$

For $1 \leq y < 2$,

$$F(y) = \frac{y^2}{2} + \int_1^y 2 - t\, dt = 2y - \frac{y^2}{2} - 1$$

For $y > 2$, $F(y) = 1$. Therefore

$$F(y) = \begin{cases} 0 & \text{if } y \leq 0 \\ \frac{y^2}{2} & \text{if } 0 < y < 1 \\ 2y - \frac{y^2}{2} - 1 & \text{if } 1 \leq y < 2 \\ 1 & \text{if } y \geq 2 \end{cases}$$



c. Find the probability that the station will pump between 8000 and 12,000 gallons in a particular month.

$$\begin{aligned}\mathbb{P}(0.8 < T < 1.2) &= F(1.2) - F(0.8) \\ &= \left(2(1.2) - \frac{1.2^2}{2} - 1\right) - \left(\frac{0.8^2}{2}\right) \\ &= 0.68 - 0.32 \\ &= 0.36\end{aligned}$$

d. Given that the station pumped more than 10,000 gallons in a particular month, find the probability that the station pumped more than 15,000 gallons during the month.

$$\begin{aligned}\mathbb{P}(Y > 1.5 \mid Y > 1) &= \frac{\mathbb{P}(Y > 1.5 \text{ and } Y > 1)}{\mathbb{P}(Y > 1)} \\ &= \frac{\mathbb{P}(Y > 1.5)}{\mathbb{P}(Y > 1)} \\ &= \frac{1 - F(1.5)}{1 - F(1)} \\ &= \frac{1 - [(2)(1.5) - (\frac{1.5^2}{2}) - 1]}{1 - [2(1) - (\frac{1^2}{2}) - 1]} \\ &= \frac{1 - 0.875}{1 - 0.5} \\ &= 0.25\end{aligned}$$

Question 4.32: Weekly CPU time used by an accounting firm has probability density function (measured in hours) given by

$$f(y) = \begin{cases} \frac{3}{64}y^2(4-y) & \text{if } 0 \leq y \leq 4 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the expected value and variance of weekly CPU time.

$$\begin{aligned}
\mathrm{E}[Y] &= \int_0^4 \frac{3}{64} y \cdot y^2(4-y)\,dy \\
&= \frac{3}{64}\int_0^4 y^3(4-y)\,dy \\
&= \frac{3}{64}\left(y^4 - \frac{y^5}{5}\right)\Big|_{y=0}^{y=4} \\
&= 2.4 \\
\mathrm{E}[Y^2] &= \int_0^4 \frac{3}{64} y^4(4-y)\,dy \\
&= \frac{3}{64}\left(\frac{4y^5}{5} - \frac{y^6}{6}\right)\Big|_{y=0}^{y=4} \\
&= 6.4
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}[Y] &= E[Y^2] - (E[Y])^2 \\
&= 6.4 - (2.4^2) \\
&= 0.64
\end{aligned}$$

b. The CPU time costs the firm \$200 per hour. Find the expected value and variance of the weekly cost for CPU time.

$$\begin{aligned}
\mathrm{E}[C] &= \mathrm{E}[200Y] = 200\mathrm{E}[Y] = 200 \cdot 2.4 = \$480 \\
\mathrm{Var}[C] &= \mathrm{Var}[200Y] = 200^2 \cdot 0.64 = \$25600
\end{aligned}$$

c. Would you expect the weekly cost to exceed \$600 very often? Why?

$$\mathbb{P}(200Y > 600) = \mathbb{P}(Y > 3) = \frac{3}{64}\int_3^4 y^2(4-y)\,dy = 0.26$$

The weekly cost will exceed \$600 about 26% of the time. Therefore it is expected to often exceed \$600.

Question 4.138: The moment generating function for $Y - \mu$, where Y is normally distributed with mean μ and variance σ^2 is

$$m(t) = e^{\frac{t^2}{2}\sigma^2}$$

a. Find the moment-generating function for Y.
Let $Z = Y - \mu$. Then $Y = Z + \mu$. Then

$$\begin{aligned}
m_Y(t) &= \mathrm{E}[e^{t(Z+\mu)}] \\
&= \mathrm{E}[e^{tZ}e^{t\mu}] \\
&= \mathrm{E}[e^{tZ}]\mathrm{E}[e^{t\mu}] \\
&= m(t) \cdot e^{t\mu} \\
&= e^{\frac{t^2}{2}\sigma^2} \cdot e^{t\mu} \\
&= e^{\frac{t^2}{2}\sigma^2 + t\mu}
\end{aligned}$$

b. Differentiate the above moment-generating function to show that $E(Y) = \mu$ and $V(Y) = \sigma^2$.

$$\begin{aligned}
m_Y'(t) &= e^{\frac{t^2}{2}\sigma^2 + t\mu} \cdot (t\sigma^2 + \mu) \\
m_Y'(0) &= e^{\frac{0^2}{2}\sigma^2 + (0)\mu} \cdot (0\sigma^2 + \mu) = 1 \cdot \mu \\
&= \mu = \mathrm{E}[Y] \\
m_Y''(t) &= (e^{\frac{t^2}{2}\sigma^2 + t\mu})(\sigma^2) + (e^{\frac{t^2}{2}\sigma^2 + t\mu})(t\sigma^2 + \mu)^2 \\
m_Y''(0) &= (e^{\frac{0^2}{2}\sigma^2 + (0)\mu})(\sigma^2) + (e^{\frac{0^2}{2}\sigma^2 + (0)\mu})((0)\sigma^2 + \mu)^2 \\
&= (1)(\sigma^2) + (1)(\mu)^2 = \sigma^2 + \mu^2 = \mathrm{E}[Y^2] \\
\mathrm{Var}[Y] &= \mathrm{E}[Y^2] - (\mathrm{E}[Y])^2 = \sigma^2 + \mu^2 - (\mu)^2 \\
&= \sigma^2
\end{aligned}$$

Question 4.160: Let the density function of a random variable Y be given by

$$f(y) = \begin{cases} \frac{2}{\pi(1+y^2)} & \text{if } -1 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

a. Find the distribution function.
For $y < -1$, $F(y) = 0$. For $-1 \leq y \leq 1$,

$$F(y) = \int_{-1}^{y} \frac{2}{\pi(1+y^2)}\, dy = \frac{2}{\pi}\tan^{-1} y + \frac{1}{2}$$

For $y > 1$, $F(y) = 1$. Thus

$$F(y) = \begin{cases} 0 & \text{if } y < -1 \\ \frac{2}{\pi}\tan^{-1} y + \frac{1}{2} & \text{if } -1 \leq y \leq 1 \\ 1 & \text{if } y > 1 \end{cases}$$

b. Find $E(Y)$.

$$\mathrm{E}[Y] = \int_{-1}^{1} \frac{2y}{\pi(1+y^2)}\, dy$$

Let $u = 1 + y^2$. Then $du = 2y\, dy$. Then

$$\mathrm{E}[Y] = \int_{u=2}^{u=2} \frac{1}{\pi u}\, du = 0$$

Question 4.174: The time (in hours) a manager takes to interview a job applicant has an exponential distribution with $\beta = \frac{1}{2}$. The applicants are scheduled at quarter-hour intervals, beginning at 8 : 00 AM, and the applicants arrive exactly on time. When the applicant with an 8 : 15 AM appointment arrives at the manager's office, what is the probability that he will have to wait before seeing the manager?
The applicant will only have to wait if the 8 : 00AM appointment is longer than 15 minutes.

$$\mathbb{P}(Y > 0.25) = \int_{0.25}^{\infty} 2e^{-2y}\, dy = e^{-0.5}$$