Assignment #4: Chapter 5 Questions 14, 26, 52, 82, 96

Question 5.14: Suppose that the random variable Y_1 and Y_2 have join probability density function $f(y_1, y_2)$ given by

$$f(y_1,y_2) = \begin{cases} 6y_1^2y_2 & \text{if } 0 \leq y_1 \leq y_1 + y_2 \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

a. Verify that this is a valid joint density function.
Note that y_1 depends on y_2 and y_2 is bounded by $2 - y_1$ and y_1. Thus

$$\begin{aligned} \int_0^1 \int_{y_1}^{2-y_1} 6y_1y_2 \, dy_2 \, dy_1 &= \int_0^1 3y_1^2y_2^2 \Big|_{y_2=y_1}^{y_2=2-y_1} \\ &= \int_0^1 3y_1^2[(2-y_1)^2 - y_1^2] \, dy_1 \\ &= \int_0^1 3y_1^2(4 - 4y_1 + y_1^2 - y_1^2) \, dy_1 \\ &= \int_0^1 3y_1^2(4 - 4y_1) \, dy_1 \\ &= \int_0^1 12y_1^2 - 12y_1^3 \, dy_1 \\ &= 4y_1^3 - 3y_1^4 \Big|_{y_1=0}^{y_1=1} \\ &= (4-3) - (0-0) \\ &= 1 \end{aligned}$$

b. What is the probability that $Y_1 + Y_2$ is less than 1?
Since $Y_1 + Y_2 < 1$, that means the maximum either can be is $\frac{1}{2}$.

$$\begin{aligned} \mathbb{P}(Y_1 + Y_2 < 1) = \mathbb{P}(Y_2 < 1 - Y_1) &= \int_0^{\frac{1}{2}} \int_{y_1}^{1-y_1} 6y_1^2y_2 \, dy_2 \, dy_1 \\ &= \int_0^{\frac{1}{2}} 3y_1^2y_2^2 \Big|_{y_2=y_1}^{y_2=1-y_1} dy_1 = \int_0^{\frac{1}{2}} 3y_1^2[(1-y_1)^2 - y_1^2] \, dy_1 \\ &= \int_0^{\frac{1}{2}} 3y_1^2(1 - 2y_1 + y_1^2 - y_1^2) \, dy_1 \\ &= \int_0^{\frac{1}{2}} 3y_1^2(1 - 2y_1) \, dy_1 \\ &= \int_0^{\frac{1}{2}} 3y_1^2 - 6y_1^3 \, dy_1 \\ &= y_1^3 - \frac{6}{4}y_1^4 \Big|_{y_1=0}^{y_1=\frac{1}{2}} \\ &= \frac{1}{8} - \left(\frac{6}{4} \cdot \frac{1}{16}\right) = \frac{4}{32} - \frac{3}{32} = \frac{1}{32} \end{aligned}$$

Question 5.26: Let

$$f(y_1, y_2) = \begin{cases} 4y_1y_2 & \text{if } 0 \leq y_1 \leq 1,\ 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

be a valid probability density function. Find:

a. the marginal density functions for Y_1 and Y_2.
For Y_1, $0 \leq y_1 \leq 1$:

$$f_{Y_1}(y_1) = \int_0^1 4y_1y_2\,dy_2 = 2y_1y_2^2\Big|_{y_2=0}^{y_2=1} = 2y_1$$

For Y_2, $0 \leq y_2 \leq 1$:

$$f_{Y_2}(y_2) = \int_0^1 4y_1y_2\,dy_1 = 2y_1^2y_2\Big|_{y_1=0}^{y_1=1} = 2y_2$$

Therefore

$$f_{Y_1}(y_1) = \begin{cases} 2y_1 & \text{if } 0 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}, \quad f_{Y_2}(y_2) = \begin{cases} 2y_2 & \text{if } 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

b. $\mathbb{P}\left(Y_1 \leq \frac{1}{2} \mid Y_2 \geq \frac{3}{4}\right)$.

$$\begin{aligned} \mathbb{P}\left(Y_1 \leq \frac{1}{2} \mid Y_2 \geq \frac{3}{4}\right) &= \frac{\mathbb{P}(Y_1 \leq \frac{1}{2}) \text{ and } \mathbb{P}(Y_2 \geq \frac{3}{4})}{\mathbb{P}(Y_2 \geq \frac{3}{4})} = \frac{\int_0^{\frac{1}{2}} \int_{\frac{3}{4}}^1 4y_1y_2\,dy_1\,dy_2}{\int_{\frac{3}{4}}^1 2y_2\,dy_2} \\ &= \frac{\int_0^{\frac{1}{2}} 2y_1\,dy_1 \cdot \int_{\frac{3}{4}}^1 2y_2\,dy_2}{\int_{\frac{3}{4}}^1 2y_2\,dy_2} \\ &= \int_0^{\frac{1}{2}} 2y_1\,dy_1 \\ &= y_1^2\Big|_{y_1=0}^{y_1=\frac{1}{2}} \\ &= \frac{1}{4} - 0 = \frac{1}{4} \end{aligned}$$

c. the conditional density function of Y_1 given $Y_2 = y_2$.
For $0 \leq y_1 \leq 1$,

$$f(Y_1 \mid Y_2 = y_2) = \frac{f(Y_1, Y_2 = y_2)}{f(Y_2 = y_2)} = \frac{4y_1y_2}{2y_2} = 2y_1$$

So

$$f(Y_1 \mid Y_2 = y_2) = \begin{cases} 2y_1 & \text{if } 0 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

d. the conditional density function of Y_2 given $Y_1 = y_1$.
For $0 \leq y_2 \leq 1$,

$$f(Y_2 \mid Y_1 = y_1) = \frac{f(Y_2, Y_1 = y_1)}{f(Y_1 = y_1)} = \frac{4y_1y_2}{2y_1} = 2y_2$$

So

$$f(Y_2 \mid Y_1 = y_1) = \begin{cases} 2y_2 & \text{if } 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

e. $\mathbb{P}\left(Y_1 \leq \frac{3}{4} \mid Y_2 = \frac{1}{2}\right)$.

$$\begin{aligned}\mathbb{P}\left(Y_1 \leq \frac{3}{4} \mid Y_2 = \frac{1}{2}\right) &= \frac{\mathbb{P}(Y_1 \leq \frac{3}{4}) \text{ and } \mathbb{P}(Y_2 = \frac{1}{2})}{\mathbb{P}(Y_2 = \frac{1}{2})} \\ &= \frac{\int_0^{\frac{3}{4}} 2y_1 \, dy_1 \cdot f_{Y_2}(\frac{1}{2})}{f_{Y_2}(\frac{1}{2})} \\ &= \int_0^{\frac{3}{4}} 2y_1 \, dy_1 \\ &= y_1^2 \Big|_{y_1=0}^{y_1=\frac{3}{4}} \\ &= \frac{9}{16} - 0 \\ &= \frac{9}{16}\end{aligned}$$

Question 5.52: Let

$$f(y_1, y_2) = \begin{cases} 4y_1y_2 & \text{if } 0 \leq y_1 \leq 1, \ 0 \leq y_2 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

be a valid joint probability density function. Are Y_1 and Y_2 independent?
If $f(y_1, y_2)$ can be factored into $g(y_1)h(y_1)$ for all values of y_1 and y_2, then we can say that Y_1 and Y_2 are independent.

$$f(y_1, y_2) = 4y_1y_2 = (2 \cdot y_1) \cdot (2 \cdot y_2) = g(y_1)h(y_2)$$

Clearly $f(y_1, y_2)$ can be factored into functions of separate variables in the range where $0 \leq y_1 \leq 1$ and $0 \leq y_2 \leq 1$. In addition, the ranges of y_1 and y_2 do not depend on each other. Outside of this range, probability is zero.Therefore Y_1 and Y_2 are independent.

Question 5.82: The joint density function for Y_1, the weight in tons of a bulk item stocked by a supplier, and Y_2, the weight of the item sold by the supplier, is

$$f(y_1, y_2) = \begin{cases} \frac{1}{y_1} & \text{if } 0 \leq y_2 \leq y_1 \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

In this case, the random variable $Y_1 - Y_2$ measures the amount of stock remaining at the end of the week, a quantity of great importance to the supplier. Find $\text{E}[Y_1 - Y_2]$.

$$\begin{aligned}
\text{E}[Y_1 - Y_2] &= \text{E}[Y_1] - \text{E}[Y_2] \\
\text{E}[Y_1] &= \int_0^1 \int_{y_2}^1 y_1 \cdot \frac{1}{y_1} \, dy_1 \, dy_2 = \int_0^1 \int_{y_2}^1 dy_1 \, dy_2 = \int_0^1 y_1 \Big|_{y_1=y_2}^{y_1=1} dy_2 \\
&= \int_0^1 1 - y_2 \, dy_2 = y_2 - \frac{y_2^2}{2} \Big|_{y_2=0}^{y_2=1} = 1 - \frac{1}{2} - (0-0) = \frac{1}{2} \\
\text{E}[Y_2] &= \int_0^1 \int_{y_2}^1 y_2 \cdot \frac{1}{y_1} \, dy_1 \, dy_2 = \int_0^1 \frac{y_2}{y_1} \, dy_1 \, dy_2 \\
&= \int_0^1 y_2 \ln y_1 \Big|_{y_1=y_2}^{y_1=1} dy_2 = \int_0^1 y_2(0 - \ln y_2) \, dy_2 = \int_0^1 -y_2 \ln y_2 \, dy_2 \\
&= \frac{y_2^2}{4} - \frac{y_2^2}{2} \ln y_2 \Big|_{y_2=0}^{y_2=1} = (\frac{1}{4} - 0) - (0 - 0) = \frac{1}{4} \\
\text{E}[Y_1 - Y_2] &= \text{E}[Y_1] - \text{E}[Y_2] \\
&= \frac{1}{2} - \frac{1}{4} = \frac{1}{4}
\end{aligned}$$

Question 5.96: Suppose that the random variables Y_1 and Y_2 have means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Use the basic definition of the covariance of two random variables to establish that

a. $\text{Cov}[Y_1, Y_2] = \text{Cov}[Y_2, Y_1]$.

$$\text{Cov}[Y_1, Y_2] = \text{E}[(Y_1 - \mu_1)(Y_2 - \mu_2)] = \text{E}[(Y_2 - \mu_2)(Y_1 - \mu_1)] = \text{Cov}[Y_2, Y_1]$$

b. $\text{Cov}[Y_1, Y_1] = \text{Var}[Y_1] = \sigma_1^2$. That is, the covariance of a random variable and itself is just the variance of the random variable.

$$\text{Cov}[Y_1, Y_1] = \text{E}[(Y_1 - \mu_1)(Y_1 - \mu_1)] = \text{E}[(Y_1 - \mu_1)^2] = \text{Var}[Y_1] = \sigma_1^2$$