Assignment #5: Chapter 6 Questions 2, 10, 32, 40, 88

Question 6.2: Let Y be a random variable with a density function given by

$$f(y) = \begin{cases} \frac{3}{2}y^2 & \text{if } -1 \leq y \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

The CDF of $f(y)$ is

$$F_Y(y) = \int_{-1}^{y} \frac{3}{2}t^2\,dt = \frac{t^3}{2}\bigg|_{t=-1}^{t=y} = \frac{1}{2}(y^3 - 1)$$

in the range of $-1 \leq y \leq 1$, 0 otherwise.

a. Find the density function of $U_1 = 3Y$.

$$\begin{aligned} F_{U_1}(u) = \mathbb{P}(U_1 \leq u) &= \mathbb{P}(3Y \leq u) \\ &= \mathbb{P}(Y \leq \frac{u}{3}) \\ &= F_Y(\frac{u}{3}) \\ &= \frac{1}{2}(\frac{u^3}{27} - 1) \end{aligned}$$

Then

$$f_{U_1}(u) = \frac{d}{du}F_{U_1}(u) = \frac{d}{du}\frac{1}{2}(\frac{u^3}{27} - 1) = \frac{1}{2} \cdot \frac{3u^2}{27} = \frac{u^2}{18}$$

where $-1 \leq \frac{u}{3} \leq 1$ or $-3 \leq u \leq 3$.

b. Find the density function of $U_2 = 3 - Y$.

$$\begin{aligned} F_{U_2}(u) = \mathbb{P}(U_2 \leq u) &= \mathbb{P}(3 - y \leq u) \\ &= \mathbb{P}(Y \geq 3 - u) \\ &= 1 - \mathbb{P}(Y \leq 3 - u) = 1 - F_Y(3 - u) \\ &= 1 - \frac{1}{2}((3 - u)^3 - 1) \end{aligned}$$

Then

$$f_{U_2}(u) = \frac{d}{du}F_{U_2}(u) = \frac{d}{du}[1 - \frac{1}{2}((3 - u)^3 - 1)] = -\frac{1}{2} \cdot 3(3 - u)^2 \cdot -1 = \frac{3}{2}(3 - u)^2$$

where $-1 \leq 3 - u \leq 1$ or $2 \leq u \leq 4$.

c. Find the density function of $U_3 = Y^2$.

$$\begin{aligned} F_{U_3}(u) = \mathbb{P}(U_3 \leq u) &= \mathbb{P}(Y^2 \leq u) \\ &= \mathbb{P}(-\sqrt{u} \leq U \leq \sqrt{u}) \\ &= F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) \\ &= \frac{1}{2}(u^{\frac{3}{2}} - 1) - \frac{1}{2}(-u^{\frac{3}{2}} - 1) \\ &= u^{\frac{3}{2}} \end{aligned}$$

Then

$$f_{U_3}(u) = \frac{d}{du}F_{U_3}(u) = \frac{d}{du}u^{\frac{3}{2}} = \frac{3}{2}\sqrt{u}$$

where $-1 \leq \sqrt{u} \leq 1$ or $0 \leq u \leq 1$.

Question 6.10: The total time from arrival to completion of service at a fast-food outlet, Y_1, and the time spent waiting in line before arriving at the service window, Y_2 has the joint density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1} & \text{if } 0 \leq y_2 \leq y_1 < \infty \\ 0 & \text{elsewhere} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window. Find

a. the probability density function for U.

$$\begin{aligned} F_U(u) = \mathbb{P}(U \leq u) &= \mathbb{P}(Y_1 - Y_2 \leq u) \\ &= \mathbb{P}(Y_1 \leq u + Y_2) \\ &= \int_0^\infty \int_{y_2}^{y_2+u} e^{-y_1}\, dy_1\, dy_2 \\ &= \int_0^\infty -e^{-y_1}\Big|_{y_1=y_2}^{y_1=y_2+u} dy_2 \\ &= \int_0^\infty -e^{-y_2-u} + e^{-y_2}\, dy_2 \\ &= (e^{-u-y_2} - e^{-y_2})\Big|_{y_2=0}^{y_2=\infty} \\ &= (0-0) - (e^{-u} - 1) \\ &= 1 - e^{-u} \end{aligned}$$

Then

$$f_U(u) = \frac{d}{du}F_U(u) = \frac{d}{du}1 - e^{-u} = e^{-u}$$

where $u \geq 0$. This is the Exponential distribution with parameter $\lambda = 1$.(Will simplify results for next part.)

b. E$[U]$ and Var$[U]$.
Since $U \sim \text{Exp}(\lambda = 1)$,

$$\text{E}[U] = \frac{1}{\lambda} = 1$$

and

$$\text{Var}[U] = \frac{1}{\lambda^2} = 1$$

Question 6.32: Consider a random variable Y that has a uniform distribution on the interval $[1,5]$. The cost of delay is given by $U = 2Y^2 + 3$. Use the method of transformations to derive the density function of U.
Let $Y \sim U(1,5)$. Then $f_Y(y) = \frac{1}{5-1} = \frac{1}{4}$ on the interval of $1 \leq y \leq 5$. Its CDF is $F_Y(y) = \frac{y}{4}$. Then

$$\begin{aligned} U &= 2Y^2 + 3 \\ Y^2 &= \frac{U-3}{2} \\ Y &= \sqrt{\frac{U-3}{2}} = h^{-1}(u) \end{aligned}$$

Now

$$\left|\frac{dh^{-1}}{du}\right| = \frac{1}{2} \cdot \left(\frac{U-3}{2}\right)^{-\frac{1}{2}} \cdot \frac{1}{2} = \frac{1}{4}\sqrt{\frac{2}{u-3}}$$

Then

$$f_U(u) = f_Y[h^{-1}(u)]\left|\frac{dh^{-1}}{du}\right| = \frac{1}{4} \cdot \frac{1}{4}\sqrt{\frac{2}{u-3}} = \frac{1}{16}\sqrt{\frac{2}{u-3}}$$

where $1 \leq y \leq 5$ or $5 \leq u \leq 53$.

Question 6.40: Suppose that Y_1 and Y_2 are independent, standard normal random variables. Find the density function of $U = Y_1^2 + Y_2^2$.
If $Y_1 \sim \mathrm{N}(0,1)$ and $Y_2 \sim \mathrm{N}(0,1)$, then $Y_1^2 \sim \chi^2(v=1)$ and $Y_2^2 \sim \chi^2(v=1)$. Then if $U = Y_1^2 + Y_2^2$, we can use the moment generating functions of Y_1^2 and Y_2^2 to derive the density function of U. Now,

$$\begin{aligned} m_{Y_1}(t) &= \frac{1}{\sqrt{1-2t}} \\ m_{Y_2} &= \frac{1}{\sqrt{1-2t}} \\ m_U(u) &= m_{Y_1}(t) \cdot m_{Y_2}(t) = \frac{1}{1-2t} \\ &= (1-2t)^{-1} \end{aligned}$$

This is the moment generating function for a χ^2 distribution with degrees of freedom $v = 2$. Its probability density function is

$$f_U(u) = \frac{u^{\frac{2}{2}-1}e^{-\frac{u}{2}}}{\Gamma(\frac{2}{2})2^{\frac{2}{2}}} = \frac{1}{2}e^{-\frac{u}{2}}$$

Question 6.88: Suppose that the length of time Y it takes a worker to complete a certain task has the probability density function given by

$$f(y) = \begin{cases} e^{-(y-\theta)} & \text{if } y > \theta \\ 0 & \text{elsewhere} \end{cases}$$

where θ is a positive constant that represents the minimum time until task completion. Let $Y_1, \dots, Y_n$ denote a random sample of completion times from this distribution. Find

a. the density function for $Y_{(1)} = \min(Y_1, \dots, Y_n)$.
   The CDF of Y is

$$\begin{aligned} F_Y(y) &= \int_0^y e^{-x-\theta}\,dx \\ &= -e^{-(x-\theta)}\Big|_{x=\theta}^{x=y} \\ &= 1 - e^{-(y-\theta)} \end{aligned}$$

   Then the CDF of $Y_{(1)}$ is

$$\begin{aligned} F_{Y_{(1)}}(y) &= 1 - (1-F(y))^n \\ &= 1 - (e^{-(y-\theta)})^n \\ &= 1 - e^{-n(y-\theta)} \end{aligned}$$

   Thus

$$\begin{aligned} f_{Y_{(1)}}(y) &= \frac{d}{dy}[1 - e^{-n(y-\theta)}] \\ &= ne^{-n(y-\theta)} \end{aligned}$$

   where $y \geq \theta$.

b. $\text{E}[Y_{(1)}]$.

$$\text{E}[Y_{(1)}] = \int_0^\infty yne^{-n(y-\theta)}\,dy$$

   Let $x = y - \theta$, then

$$\begin{aligned} \text{E}[Y_{(1)}] &= \int_0^\infty (x+\theta)ne^{-nx}\,dx \\ &= \int_0^\infty xne^{-nx}\,dx + \theta\int_0^\infty ne^{-nx}\,dx \\ &= \left[\left(-\frac{1}{n}e^{-nx}(2n+1)\right) + \theta\left(-e^{-nx}\right)\right]\Bigg|_{x=0}^{x=\infty} \\ &= (0+0) - (-\frac{1}{n} - \theta) \\ &= (0+0) - (-1(\frac{1}{n} + \theta)) \\ &= \frac{1}{n} + \theta \end{aligned}$$