Assignment #2: Chapter 8 Questions 12, 22, 46, 60, 102

Question 8.12: The reading on a voltage meter connected to a test circuit is uniformly distributed over the interval $(\theta, \theta+1)$, where θ is the true but unknown voltage of the circuit. Suppose that $Y_1, \dots, Y_n$ denote a random sample of such samplings.

a. Show that $\bar{Y}$ is a biased estimator of θ and compute the bias.
Since Y is from a uniform distribution over $(\theta, \theta+1)$, then

$$\mathrm{E}[Y_i] = \frac{1}{2}(\theta + \theta + 1) = \frac{2\theta + 1}{2} = \theta + \frac{1}{2}$$

Furthermore, for $\bar{Y} = \sum_i^n Y_i/n$,

$$\mathrm{E}[\bar{Y}] = \mathrm{E}\left[\frac{\sum_i^n Y_i}{n}\right] = \frac{\mathrm{E}[\sum_i^n Y_i]}{n} = \frac{\sum_i^n \mathrm{E}[Y_i]}{n} = \frac{n\left(\theta + \frac{1}{2}\right)}{n} = \theta + \frac{1}{2}$$

Now,

$$\mathrm{E}[\bar{Y}] = \theta + \frac{1}{2} \neq \theta = \mathrm{E}[\theta]$$

Therefore $\hat{\theta} = \bar{Y}$ is a biased estimator of θ. The bias is

$$\mathrm{Bias}[\hat{\theta}] = \mathrm{E}[\hat{\theta}] - \theta = \theta + \frac{1}{2} - \theta = \frac{1}{2}$$

b. Find a function of $\bar{Y}$ that is an unbiased estimator of θ.
According to the above result, an appropriate unbiased estimator of θ is

$$\hat{\theta} = \bar{Y} - \frac{1}{2}$$

c. Find $\mathrm{MSE}[\bar{Y}]$ when $\bar{Y}$ is used as an estimator of θ.
First note that if $\mathrm{Var}[Y_i] = \frac{1}{12}$, then

$$\mathrm{Var}[\bar{Y}] = \mathrm{Var}\left[\frac{\sum_i^n Y_i}{n}\right] = \frac{n \cdot \mathrm{Var}[Y_i]}{n^2} = \frac{1}{12n}$$

When $\hat{\theta} = \bar{Y}$ is used as an estimator of θ, then

$$\mathrm{MSE}[\bar{Y}] = \mathrm{E}[(\hat{\theta} - \theta)^2] = \mathrm{Var}[\hat{\theta}] + \mathrm{Bias}[\hat{\theta}]^2 = \frac{1}{12n} + \left(\frac{1}{2}\right)^2 = \frac{1}{12n} + \frac{1}{4}$$

Question 8.22: An increase in the rate of consumer savings frequently is tied to a lack of confidence in the economy and is said to be an indicator of a recessional tendency in the economy. A random sampling of $n = 200$ savings accounts in a local community showed the mean increase in savings account values to be 7.2% over the past 12 months, with standard deviation 5.6%. Estimate the mean percentage increase in savings account values over the past 12 months for depositors in the community. Place a bound on the error of estimation.

The mean percentage increase in savings account values over the past 12 months is 7.2%. Then the standard error of estimation is

$$\frac{\sigma}{\sqrt{n}} = \frac{5.6}{\sqrt{200}} = 0.395\%$$

and the bound on the error of estimation is

$$2 \cdot 0.395 = 0.791\%$$

Question 8.46: Suppose that Y is a single observation from an exponential distribution with mean θ.

a. Use the method of moments-generating functions to show that $\frac{2Y}{\theta}$ is a pivotal quantity and has a χ^2 distribution with 2 degrees of freedom.
The moments-generating function for the exponential distribution is

$$m_Y(t) = \frac{1}{1 - \theta t}$$

Then

$$m_{\frac{2Y}{\theta}}(t) = \mathrm{E}[e^{t\frac{2Y}{\theta}}] = \mathrm{E}[e^{\left(\frac{2t}{\theta}\right)Y}] = m_Y\left(\frac{2t}{\theta}\right) = \frac{1}{1-2t}$$

The moments-generating function for the χ^2 distribution, with v degrees of freedom, is

$$m_{\chi^2}(t) = \frac{1}{(1-2t)^{\frac{v}{2}}}$$

It can be seen that $m_{\frac{2Y}{\theta}}(t)$ and $m_{\chi^2}(t)$ resemble each other; the moments-generating function found is the one for the χ^2 distribution with $v = 2$ degrees of freedom. Therefore $\frac{2Y}{\theta} \sim \chi^2(2)$. Furthermore, it does not depend on θ and so $\frac{2Y}{\theta}$ is a pivotal quantity.

b. Use the pivotal quantity $\frac{2Y}{\theta}$ to derive a 90% confidence interval for θ.
Start with

$$P\left(\chi^2_{0.05}(2) \leq \frac{2Y}{\theta} \leq \chi^2_{0.95}(2)\right) = 0.90$$

Note that

$$\chi^2_{0.05}(2) = \texttt{qchisq(0.05, 2)} = 0.102$$
$$\chi^2_{0.95}(2) = \texttt{qchisq(0.95, 2)} = 5.991$$

So

$$P(0.102 \leq \frac{2Y}{\theta} \leq 5.991) = 0.90$$

Manipulate this so that the parameter is simply θ.

$$P(0.102 \leq \frac{2Y}{\theta} \leq 5.991) = 0.90$$
$$P\left(\frac{0.102}{2Y} \leq \frac{1}{\theta} \leq \frac{5.991}{2Y}\right) = 0.90$$
$$P\left(\frac{2Y}{0.102} \geq \theta \geq \frac{2Y}{5.991}\right) = 0.90$$

Hence the 90% confidence interval for θ is

$$\left(\frac{2Y}{5.991}, \frac{2Y}{0.102}\right) \to \left(\frac{Y}{2.996}, \frac{Y}{0.051}\right)$$

c. Compare the above interval with the interval obtained in example 8.4.
In example 8.4, it was found that using a single observation Y from an exponential distribution with mean θ, the confidence interval for θ with a confidence level of 0.90 was

$$\left(\frac{Y}{2.996}, \frac{Y}{0.051}\right)$$

This is the exact same result found above.

Question 8.60: What *is* the normal body temperature for healthy humans? A random sample of 130 healthy human body temperatures provided by Allen Shoemaker yielded 98.25% degrees and standard deviation 0.73 degrees.

a. Give a 99% confidence interval for the average body temperature of healthy people.
The 99% confidence interval can be given by $\hat{\theta} \pm z_{\frac{\alpha}{2}}\sigma_{\hat{\theta}}$. Now,

$$z_{\frac{\alpha}{2}} = z_{0.005} = \texttt{qnorm(0.005)} = 2.576$$

Then

$$\hat{\theta} \pm z_{\frac{\alpha}{2}}\sigma_{\hat{\theta}}$$
$$98.25 \pm \left(2.576 \cdot \frac{0.73}{\sqrt{130}}\right)$$
$$98.25 \pm 0.165$$

b. Does the confidence interval obtained in part (a) contain the value 98.6 degrees, the accepted average temperature cited by physicians and others? What conclusions can be drawn?
The confidence interval calculated does not contain the value 98.6 because the lower bound is $98.25 - 0.165 = 98.085$. This means that the accepted average temperature that physicians believe to be is inaccurate and has changed.

Question 8.102: The ages of a random sample of five university professors are 39, 54, 61, 72 and 59. Using this information, find a 99% confidence interval for the population standard deviation of the ages of all professions at the university, assuming that the ages of university professors are normally distributed.
To find the 99% confidence interval for the population standard deviation, find

$$\left(\frac{(n-1)S^2}{\chi^2_{\frac{\alpha}{2}}}, \frac{(n-1)S^2}{\chi^2_{1-\frac{\alpha}{2}}} \right)$$

Now

$$\begin{aligned} n-1 &= \texttt{5-1} = 4 \\ S^2 &= \texttt{sd(c(39, 54, 61, 72, 59))}^2 = 144.5 \\ \chi^2_{0.995} &= \texttt{qchisq(0.005, 4)} = 0.206 \\ \chi^2_{0.005} &= \texttt{qchisq(0.995, 4)} = 14.860 \end{aligned}$$

Therefore the 99% confidence interval for the population standard deviation is

$$\left(\frac{4 \cdot 144.5}{14.860}, \frac{4 \cdot 144.5}{0.206} \right) \to (38.895, 2792.418)$$