ALSM: Chapter 5

Matrix Approach to Simple Linear Regression Analysis

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```r
library(tidyverse)
library(latex2exp)
library(gridExtra)
library(wesanderson)
library(broom)
theme_set(theme_minimal())
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Problem 1:

For the matrices below, obtain (1) $\mathbf{A}+\mathbf{B}$, (2) $\mathbf{A}-\mathbf{B}$, (3) $\mathbf{AC}$, (4) $\mathbf{AB}'$, (5) $\mathbf{B}'\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 6 \\ 3 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 3 \\ 1 & 4 \\ 2 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 8 & 1 \\ 5 & 4 & 0 \end{bmatrix}$$

State the dimension of each resulting matrix.

Answer:

Problem 2:

For the matrices below, obtain (1) $\mathbf{A}+\mathbf{C}$, (2) $\mathbf{A}-\mathbf{C}$, (3) $\mathbf{B}'\mathbf{A}$, (4) $\mathbf{AC}'$, (5) $\mathbf{C}'\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 5 \\ 5 & 7 \\ 4 & 8 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 6 \\ 9 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 3 & 8 \\ 8 & 6 \\ 5 & 1 \\ 2 & 4 \end{bmatrix}$$

State the dimension of each resulting matrix.

Answer:

Problem 3:

Show how the following expressions are written in terms of matrices: (1) $Y_i - \hat{Y}_i = e_i$, (2) $\sum X_i e_i = 0$. Assume $i = 1, \dots, 4$.

Answer:

Problem 4:

Flavor deterioration, Results were obtained in a small-scale experiment to study the relation between $^\circ F$ of storage temperature (X) and number of weeks before flavor deterioration of a food product begins to occur (Y).

Assume that first-order regression model is applicable. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$, (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

Answer:

Problem 5:

Consumer finance. The data shows, for a consumer finance company operating in six cities, the number of competing loan companies operating in the city (X) and the number per thousand of the company's loans made in that city that are currently delinquent (Y).

Assume that first-order regression model is applicable. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$ (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

Answer:

Problem 6:

Refer to *Airfreight breakage* Problem 1.21. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$ (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

Answer:

Problem 7:

Refer to *Plastic hardness* Problem 1.22. Using matrix methods, find (1) $\mathbf{Y}'\mathbf{Y}$ (2) $\mathbf{X}'\mathbf{X}$, (3) $\mathbf{X}'\mathbf{Y}$.

Answer:

Problem 8:

Let $\mathbf{B}$ be defined as follows:

$$\mathbf{B} = \begin{bmatrix} 1 & 5 & 0 \\ 1 & 0 & 5 \\ 1 & 0 & 5 \end{bmatrix}$$

(a) Are the column vectors of $\mathbf{B}$ linearly dependent?

Answer:

(b) What is the rank of $\mathbf{B}$?

Answer:

(c) What must be the determinant of $\mathbf{B}$?

Answer:

Problem 9:

Let $\mathbf{A}$ be defined as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 8 \\ 0 & 3 & 1 \\ 0 & 5 & 5 \end{bmatrix}$$

(a) Are the column vectors of $\mathbf{A}$ linearly dependent?

Answer:

(b) Restate definition (5.20) in terms of row vectors. Are the row vectors of $\mathbf{A}$ linearly dependent?

Answer:

(c) What is the rank of $\mathbf{A}$?

Answer:

(d) Calculate the determinant of $\mathbf{A}$.

Answer:

Problem 10:

Find the inverse of each of the following matrices:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 4 & 3 & 2 \\ 6 & 5 & 10 \\ 10 & 1 & 6 \end{bmatrix}$$

Check that in each case, the resulting matrix is indeed the inverse.

Answer:

Problem 11:

Find the inverse of the following matrix:

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 3 \\ 4 & 0 & 5 \\ 1 & 9 & 6 \end{bmatrix}$$

Check that the resulting matrix is indeed the inverse.

Answer:

Problem 12:

Refer to *Flavor deterioration* Problem 5.4. Find $(\mathbf{X}'\mathbf{X})^{-1}$.

Answer:

Problem 13:

Refer to *Consumer finance* Problem 5.5. Find $(\mathbf{X}'\mathbf{X})^{-1}$.

Answer:

Problem 14:

Consider the simultaneous equations:

$$4y_1 + 7y_2 = 25$$
$$2y_1 + 3y_2 = 12$$

(a) Write these equations in matrix notation.

Answer:

(b) Using matrix methods, find the solutions for y_1 and y_2.

Answer:

Problem 15:

Consider the simultaneous equations:

$$5y_1 + 2y_2 = 8$$
$$23y_1 + 7y_2 = 28$$

(a) Write these equations in matrix notation.

Answer:

(b) Using matrix methods, find the solutions for y_1 and y_2.

Answer:

Problem 16:

Consider the estimated linear regression function in the form of (1.15). Write expressions in this form for the fitted values $\hat{Y}_i$ in matrix terms for $i = 1, \ldots, 5$.

Answer:

Problem 17:

Consider the following functions of the random variables Y_1, Y_2, and Y_3.

$$\begin{aligned}W_1 &= Y_1 + Y_2 + Y_3 \\ W_2 &= Y_1 - Y_2 \\ W_3 &= Y_1 - Y_2 - Y_3\end{aligned}$$

(a) State the above in matrix notation.

Answer:

(b) Find the expectation of the random vector $\mathbf{W}$.

Answer:

(c) Find the variance-covariance matrix of $\mathbf{W}$.

Answer:

Problem 18:

Consider the following functions of the random variables Y_1, Y_2, Y_3, and Y_4.

$$\begin{aligned}W_1 &= \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4) \\ W_2 &= \frac{1}{2}(Y_1 + Y_2) - \frac{1}{2}(Y_3 + Y_4)\end{aligned}$$

(a) State the above in matrix notation.

Answer:

(b) Find the expectation of the random vector $\mathbf{W}$.

Answer:

(c) Find the variance-covariance matrix of $\mathbf{W}$.

Answer:

Problem 19:

Find the matrix $\mathbf{A}$ of the quadratic form:

$$3Y_1^2 + 10Y_1Y_2 + 17Y_2^2$$

Answer:

Problem 20:

Find the matrix $\mathbf{A}$ of the quadratic form

$$7Y_1^2 - 8Y_1Y_2 + 8Y_2^2$$

Answer:

Problem 21:

For the matrix

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

find the quadratic form of the observations Y_1 and Y_2.

Answer:

Problem 22:

For the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 9 \end{bmatrix}$$

find the quadratic form of the observations Y_1, Y_2, and Y_3.

Answer:

Problem 23:

Refer to *Flavor deterioration* Problems 5.4 and 5.12.

(a) Using matrix methods, obtain the following: (1) vector of estimated regression coefficients, (2) vector of residuals, (3) SSR, (4) SSE, (5) estimated variance-covariance matrix of $\mathbf{b}$, (6) point estimate of $E[Y_h]$ when $X_h = -6$, (7) estimated variance of $\hat{Y}_h$ when $X_h = -6$.

Answer:

(b) What simplifications arose from the spacing of the X levels in the experiment?

Answer:

(c) Find the hat matrix $\mathbf{H}$.

Answer:

(d) Find $s^2[\mathbf{e}]$.

Answer:

Problem 24:

Refer to *Consumer finance* Problems 5.5 and 5.13.

(a) Using matrix methods, obtain the following: (1) vector of estimated regression coefficients, (2) vector of residuals, (3) SSR, (4) SSE, (5) estimated variance-covariance matrix of $\mathbf{b}$, (6) point estimate of $E[Y_h]$ when $X_h = 4$, (7) $s^2[\text{pred}]$ when $X_h = 4$.

Answer:

(b) From your estimated variance-covariance matrix in part (a5), obtain the following: (1) $s[b_0, b_1]$, (2) $s^2[b_0]$, (3) $s[b_1]$.

Answer:

(c) Find the hat matrix $\mathbf{H}$.

Answer:

(d) Find $s^2[\mathbf{e}]$.

Answer:

Problem 25:

Refer to *Airfreight breakage* Problems 1.21 and 5.6.

(a) Using matrix methods, obtain the following: (1) $(\mathbf{X}'\mathbf{X})^{-1}$, (2) $\mathbf{b}$, (3) $\mathbf{e}$, (4) $\mathbf{H}$, (5) SSE, (6) $s^2[\mathbf{b}]$, (7) $\hat{Y}_h$ when $X_h = 2$, (8) $s^2[\hat{Y}_h]$ when $X_h = 2$.

Answer:

(b) From part (a6), obtain the following: (1) $s^2[b_1]$, (2) $s[b_0, b_1]$, (3) $[b_0]$.

Answer:

(c) Find the matrix of the quadratic form for SSR.

Problem 26:

Refer to *Plastic hardness* Problems 1.22 and 5.7.

(a) Using matrix methods, obtain the following: (1) $(\mathbf{X}'\mathbf{X})^{-1}$, (2) $\mathbf{b}$, (3) $\hat{\mathbf{Y}}$, (4) $\mathbf{H}$, (5) SSE, (6) $s^2[\mathbf{b}]$, (7) $s^2[\text{pred}]$ when $X_h = 30$.

Answer:

(b) From part (a6), obtain the following: (1) $s^2[b_0]$, (2) $s[b_0, b_1]$, (3) $s[b_1]$.

Answer:

(c) Find the matrix of the quadratic form for SSE.

Problem 27:

Refer to regression-through-the-origin model (4.10). Set up the expectation vector for ε. Assume that $i = 1, \dots, 4$.

Answer:

Problem 28:

Consider model (4.10) for regression through the origin and the estimator b_1 given in (4.14). Obtain (4.14) by utilizing (5.60) with $\mathbf{X}$ suitably defined.

Answer:

Problem 29:

Consider the least squares estimator $\mathbf{b}$ given in (5.60). Using matrix methods, show that $\mathbf{b}$ is an unbiased estimator.

Answer:

Problem 30:

Show that $\hat{Y}_h$ in (5.96) can be expressed in matrix terms as $\mathbf{b}'\mathbf{X}_h$.

Answer:

Problem 31:

Obtain an expression for the variance-covariance matrix of the fitted values $\hat{Y}_i$, $i = 1, \ldots, n$, in terms of the hat matrix.

Answer: