Sine Approximation to Neural Network

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1 Problem:

Use the following neural network topology to approximate the sine function. Here, approximate means minimize the distance between your approximation u and the function sin using the norm defined by

$$\|f-g\| = \frac{1}{2}\sum_{i=0}^{99}(f(x_i)-g(x_i))^2$$

where $x_0, x_1, \ldots, x_{99}$ are one hundred points evenly spaced throughout the interval $[0, 2\pi]$.

$$\begin{aligned}\nabla(\frac{1}{2}\|\mu(x)-\sin(x)\|) = \langle g,h\rangle * \langle &\theta_{0,0}+x\theta_{1,0}-\cos(x), \theta_0-\cos(x), x\alpha_0-\cos(x),\\ &\theta_{0,1}+x\theta_{1,1}-\cos(x), \alpha_1-\cos(x), x\alpha_1-\cos(x),\\ &\theta_{0,2}+x\theta_{1,2}-\cos(x), \alpha_2-\cos(x), x\alpha_2-\cos(x),\\ &\theta_{0,3}+x\theta_{1,3}-\cos(x), \alpha_3-\cos(x), x\alpha_3-\cos(x)\rangle\end{aligned}$$

```
In [1]: import math
        from sympy import *
        import numpy as np
```

```
In [2]: import matplotlib
        import matplotlib.pyplot as plt
```

```
In [3]: %matplotlib inline
```

$$C(\alpha^0,\alpha^1,\alpha^2,\alpha^3,\theta_0^0,\theta_0^1,\theta_0^2,\theta_0^3,\theta_1^0,\theta_1^1,\theta_1^2,\theta_1^3,\theta_2^0,\theta_2^1,\theta_2^2,\theta_2^3) = \frac{1}{2}\sum_{x=0}^{100}(\mu(x)-\sin(x))^2$$

$$\mu(x) = \alpha^0\sigma(\theta_0^0+\theta_1^0x)+\alpha^1\sigma(\theta_0^1+\theta_1^1x)+\alpha^2\sigma(\theta_0^2+\theta_1^2x)+\alpha^3\sigma(\theta_0^3+\theta_1^3x)$$

where

$$\sigma(x) = \frac{1}{1+e^{-x}}$$

Partial Derivatives:

$$\frac{\partial C}{\partial \alpha^0} = (\mu(x) - \sin(x))\sigma(\theta_0^0 + \theta_1^0 x) \quad \frac{\partial C}{\partial \theta_0^0} = \frac{\alpha^0(\mu(x) - \sin(x))e^{\theta_0^0 + x\theta_1^0}}{(1 + e^{\theta_0^0 + x\theta_1^0})^2} \quad \frac{\partial C}{\partial \theta_1^0} = \frac{x\alpha^0(\mu(x) - \sin(x))e^{\theta_0^0 + x\theta_1^0}}{(1 + e^{\theta_0^0 + x\theta_1^0})^2}$$

$$\frac{\partial C}{\partial \alpha^1} = (\mu(x) - \sin(x))\sigma(\theta_0^1 + \theta_1^1 x) \quad \frac{\partial C}{\partial \theta_0^1} = \frac{\alpha^0(\mu(x) - \sin(x))e^{\theta_0^1 + x\theta_1^1}}{(1 + e^{\theta_0^1 + x\theta_1^1})^2} \quad \frac{\partial C}{\partial \theta_1^1} = \frac{x\alpha^0(\mu(x) - \sin(x))e^{\theta_0^1 + x\theta_1^1}}{(1 + e^{\theta_0^1 + x\theta_1^1})^2}$$

$$\frac{\partial C}{\partial \alpha^2} = (\mu(x) - \sin(x))\sigma(\theta_0^2 + \theta_1^2 x) \quad \frac{\partial C}{\partial \theta_0^2} = \frac{\alpha^0(\mu(x) - \sin(x))e^{\theta_0^2 + x\theta_1^2}}{(1 + e^{\theta_0^2 + x\theta_1^2})^2} \quad \frac{\partial C}{\partial \theta_1^2} = \frac{x\alpha^0(\mu(x) - \sin(x))e^{\theta_0^2 + x\theta_1^2}}{(1 + e^{\theta_0^2 + x\theta_1^2})^2}$$

$$\frac{\partial C}{\partial \alpha^3} = (\mu(x) - \sin(x))\sigma(\theta_0^3 + \theta_1^3 x) \quad \frac{\partial C}{\partial \theta_0^3} = \frac{\alpha^0(\mu(x) - \sin(x))e^{\theta_0^3 + x\theta_1^3}}{(1 + e^{\theta_0^3 + x\theta_1^3})^2} \quad \frac{\partial C}{\partial \theta_1^3} = \frac{x\alpha^0(\mu(x) - \sin(x))e^{\theta_0^3 + x\theta_1^3}}{(1 + e^{\theta_0^3 + x\theta_1^3})^2}$$

Let's define two exponential function to tidy up our code:

$$\text{expOne}(x, y, z) = e^{(y+zx)}$$

$$\text{expTwo}(x, y, z) = e^{-(y+zx)}$$

Therefore our sigmoid function is:

$$\sigma(x, y, z) = \frac{1}{1 + \text{expTwo}(x, y, z)}$$

```
In [4]: def expOne(x,y,z):
            return (math.exp(y + (z*x)))
```

```
In [5]: def expTwo(x,y,z):
            return (math.exp(-1 * (y + z*x)))
```

```
In [6]: def sigmoid(x,y,z):
            return (1 + expTwo(x,y,z))**-1
```

The 12 variables and its gradient will be stored as follows:

$$\begin{bmatrix} \alpha^0 & \alpha^1 & \alpha^2 & \alpha^3 \\ \theta_0^0 & \theta_0^1 & \theta_0^2 & \theta_0^3 \\ \theta_1^0 & \theta_1^1 & \theta_1^2 & \theta_1^3 \end{bmatrix} \quad \begin{bmatrix} \frac{\partial C}{\partial \alpha^0} & \frac{\partial C}{\partial \alpha^1} & \frac{\partial C}{\partial \alpha^2} & \frac{\partial C}{\partial \alpha^3} \\ \frac{\partial C}{\partial \theta_0^0} & \frac{\partial C}{\partial \theta_0^1} & \frac{\partial C}{\partial \theta_0^2} & \frac{\partial C}{\partial \theta_0^3} \\ \frac{\partial C}{\partial \theta_1^0} & \frac{\partial C}{\partial \theta_1^1} & \frac{\partial C}{\partial \theta_1^2} & \frac{\partial C}{\partial \theta_1^3} \end{bmatrix}$$

```
In [7]: var = np.array([[0.0,0.0,0.0,0.0],[0.0,0.0,0.0,0.0],[0.0,0.0,0.0,0.0]])
```

```
In [8]: def mu(x,var_arr):

            y = 0
            for i in range(4):
                y += var_arr[0][i] * sigmoid(x, var_arr[1][i], var_arr[2][i])
            return y
```

In [9]:
```
def gradient(var_arr):

    grad_arr = np.zeros((3,4))
    interval = np.linspace(0,2*np.pi, 100)

    for x in interval:
        value = mu(x, var_arr) - np.sin(x)
        for i in range(4):
            grad_arr[0][i] += (value * sigmoid(x,var_arr[1][i],var_arr[2][i]))
            temp = expOne(x,var_arr[1][i],var_arr[2][i])
            tempTwo = (value * temp * var_arr[0][i]) / ((1 + temp)**2)
            grad_arr[1][i] += tempTwo
            grad_arr[2][i] += x * tempTwo

    return grad_arr
```

In [10]:
```
def cost(var_arr):

     output = 0.0
     interval = np.linspace(0,2*np.pi, 100)

     for i in interval:
         output += (mu(i,var_arr) - np.sin(i))**2
     return 0.5*output
```

In [11]:
```
a_old = np.random.rand(3,4)
h = 0.001
max_steps = 1000
tolerance = 0.0001

print("Initial Cost: ", cost(a_old))
print("The initial randomized guesses for the constants of the neural network are: ")
for a in range(4):
    print("alpha_",a, " = ", a_old[0][a])
for b in range(4):
    print("theta_0^",b, " = ", a_old[1][b])
for c in range(4):
    print("theta_1^",c, " = ", a_old[2][c])

a_new = a_old - h * gradient(a_old)

for i in range(max_steps):
    if(cost(a_old) - cost(a_new) <= tolerance):
        print("Tolerance has been reached.")
        break
    if(cost(a_old) < cost(a_new)):
        a_new = a_old
        break
```

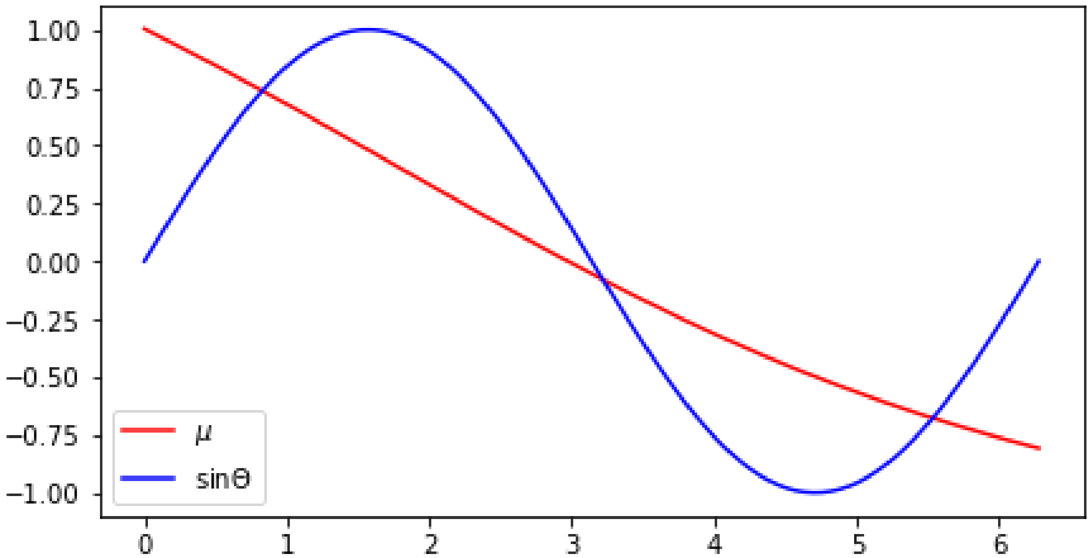
```
        a_old = a_new
        direction = -1 * gradient(a_old)
        a_new = a_old + (h * direction)

    print("Final Cost: ", cost(a_new))
    print("The constants for the neural network are: ")
    for j in range(4):
        print("alpha_",j, " = ", a_new[0][j])
    for k in range(4):
        print("theta_0^",k, " = ", a_new[1][k])
    for l in range(4):
        print("theta_1^",l, " = ", a_new[2][l])
```

```
Initial Cost:  152.853831491
The initial randomized guesses for the constants of the neural network are:
alpha_ 0  =  0.0392138049872
alpha_ 1  =  0.935069395134
alpha_ 2  =  0.801182370268
alpha_ 3  =  0.390018826213
theta_0^ 0  =  0.0188524897848
theta_0^ 1  =  0.197230919153
theta_0^ 2  =  0.634132031231
theta_0^ 3  =  0.312030611177
theta_1^ 0  =  0.993622193055
theta_1^ 1  =  0.0370337938615
theta_1^ 2  =  0.296799684981
theta_1^ 3  =  0.500139221653
Final Cost:  9.58989159705
The constants for the neural network are:
alpha_ 0  =  -0.799642639001
alpha_ 1  =  1.38572323448
alpha_ 2  =  0.876066461282
alpha_ 3  =  -0.466035521721
theta_0^ 0  =  -0.296995959199
theta_0^ 1  =  0.849889195078
theta_0^ 2  =  0.956045320777
theta_0^ 3  =  0.2279725582
theta_1^ 0  =  0.310373091075
theta_1^ 1  =  -0.451726089816
theta_1^ 2  =  -0.484716188143
theta_1^ 3  =  0.249518501377
```

In [12]:
```
x_0 = np.linspace(0, 2 * np.pi,100)
y_0 = []
y_1 = []
for p in x_0:
    y_0.append(mu(p,a_new))
    y_1.append(np.sin(p))
```

In [16]:
```
fig = plt.figure()
axes = fig.add_axes([0.1, 0.1, 0.9, 0.7])
axes.plot(x_0,y_0,label ="$\mu$",c = "red");
axes.plot(x_0,y_1, label = "$\sin\Theta$", c = "blue");
axes.legend(loc = 3);
```



A function of 12 variables that are related to each other by a neural network is approximated to the sin function here much better than before (a straight line at $y = 0$). The endpoints are not as close to the sin function as hoped for but a better algorithm can fix it, perhaps using stochastic gradient descent. If the max steps is changed to 10000, the function becomes much much closer to the sin function, even at the endpoints, but at the expense of time.