Approximating the Sine

February 23, 2018

1 Project: Approximate the sine in six ways.

Find a cubic polynomial u that best approximates $\sin(t)$ in the sense that $\|u(t) - \sin(t)\|$ is minimized. The definition of the norm $\| \quad \|$, however, depends on a choice of inner product. Here are six different ways to define inner products on the vector space of polynomials of degree less than or equal to 3.

$$\langle f, g \rangle_6 = f(3)g(3) + f'(3)g'(3) + f''(3)g''(3) + f'''(3)g'''(3)$$
$$\langle f, g \rangle_5 = \int_0^6 f(t)g(t)dt$$
$$\langle f, g \rangle_3 = f(0)g(0) + f(2)g(2) + f(4)g(4) + f(6)g(6)$$
$$\langle f, g \rangle_2 = f(1)g(1) + f'(1)g'(1) + f(5)g(5) + f'(5)g'(5)$$
$$\langle f, g \rangle_4 = f(0)g(0) + f(3)g(3) + f'(3)g'(3) + f(6)g(6)$$
$$\langle f, g \rangle_1 = f(0)g(0) + f'(0)g'(0) + f(6)g(6) + f'(6)g'(6)$$

```
In [1]: import scipy.integrate as integ
        from sympy import *
        import math
```

```
In [4]: import numpy as np
```

```
In [5]: x = Symbol('x')
```

$$\langle f, g \rangle_1 = f(0)g(0) + f'(0)g'(0) + f(6)g(6) + f'(6)g'(6)$$

```
In [6]: def ip1(f,g):

            h = f*g
            diffF = lambdify(x,diff(f))
            diffG = lambdify(x,diff(g))

            return (N(h.subs(x,0)) + diffF(0)*diffG(0) +
                    N(h.subs(x,6)) + diffF(6)*diffG(6))
```

$$\langle f, g \rangle_2 = f(1)g(1) + f'(1)g'(1) + f(5)g(5) + f'(5)g'(5)$$

```
In [7]: def ip2(f,g):

            h = f*g
            diffF = lambdify(x,diff(f))
            diffG = lambdify(x,diff(g))

            return (N(h.subs(x,1)) + (diffF(1)*diffG(1)) +
                    N(h.subs(x,5)) + (diffF(5)*diffG(5)))
```

$$\langle f,g\rangle_3 = f(0)g(0) + f(2)g(2) + f(4)g(4) + f(6)g(6)$$

```
In [8]: def ip3(f,g):

            h = f*g
            diffF = lambdify(x,diff(f))
            diffG = lambdify(x,diff(g))

            return (N(h.subs(x,0)) + diffF(2)*diffG(2) +
                    N(h.subs(x,4)) + N(h.subs(x,6)))
```

$$\langle f,g\rangle_4 = f(0)g(0) + f(3)g(3) + f'(3)g'(3) + f(6)g(6)$$

```
In [9]: def ip4(f,g):

            h = f*g
            diffF = lambdify(x,diff(f))
            diffG = lambdify(x,diff(g))

            return (N(h.subs(x,0)) + (N(h.subs(x,3))) +
                    (diffF(3)*diffG(3)) + (N(h.subs(x,6))))
```

$$\langle f,g\rangle_5 = \int_0^6 f(t)g(t)dt$$

```
In [10]: def ip5(f,g):

             h = f*g

             return integ.quad(lambda t: h.subs(x,t),0,6)[0]
```

$$\langle f,g\rangle_6 = f(3)g(3) + f'(3)g'(3) + f''(3)g''(3) + f'''(3)g'''(3)$$

```
In [11]: def ip6(f,g):

             h = f*g
             diffF = lambdify(x,diff(f))
```

```python
    diffFTwo = lambdify(x,diff(f,x,x))
    diffFThree = lambdify(x,diff(f,x,x,x))
    diffG = lambdify(x,diff(g))
    diffGTwo = lambdify(x,diff(g,x,x))
    diffGThree = lambdify(x,diff(g,x,x,x))

    return (N(h.subs(x,3)) + (diffF(3)*diffG(3)) +
            (diffFTwo(3)*diffGTwo(3)) + (diffFThree(3)*diffGThree(3)))
```

In [13]:
```python
k=3 # max degree
myEOne=[] # initialize the output with zeros, for ip1
myETwo=[] # for ip2
myEThree=[] #for ip3
myEFour=[] #for ip4
myEFive=[] #for ip5
mySix=[] #for ip6


for i in range(k+1):
    alpha = x**i
    beta = x**i
    gamma = x**i
    delta = x**i
    epsilon = x**i
    zeta = x**i
    for j in range(i):
        alpha+=-ip1(myEOne[j],x**i)*myEOne[j]
        beta+=-ip2(myETwo[j],x**i)*myETwo[j]
        gamma+=-ip3(myEThree[j],x**i)*myEThree[j]
        delta+=-ip4(myEFour[j],x**i)*myEFour[j]
        epsilon+=-ip5(myEFive[j],x**i)*myEFive[j]
        zeta+=-ip6(myESix[j],x**i)*myESix[j]

    myEOne.append(alpha/(sqrt(ip1(alpha,alpha))))
    myETwo.append(beta/(sqrt(ip2(beta,beta))))
    myEThree.append(gamma/(sqrt(ip3(gamma,gamma))))
    myEFour.append(delta/(sqrt(ip4(delta,delta))))
    myEFive.append(epsilon/(sqrt(ip5(epsilon,epsilon))))
    myESix.append(zeta/(sqrt(ip6(zeta,zeta))))
```

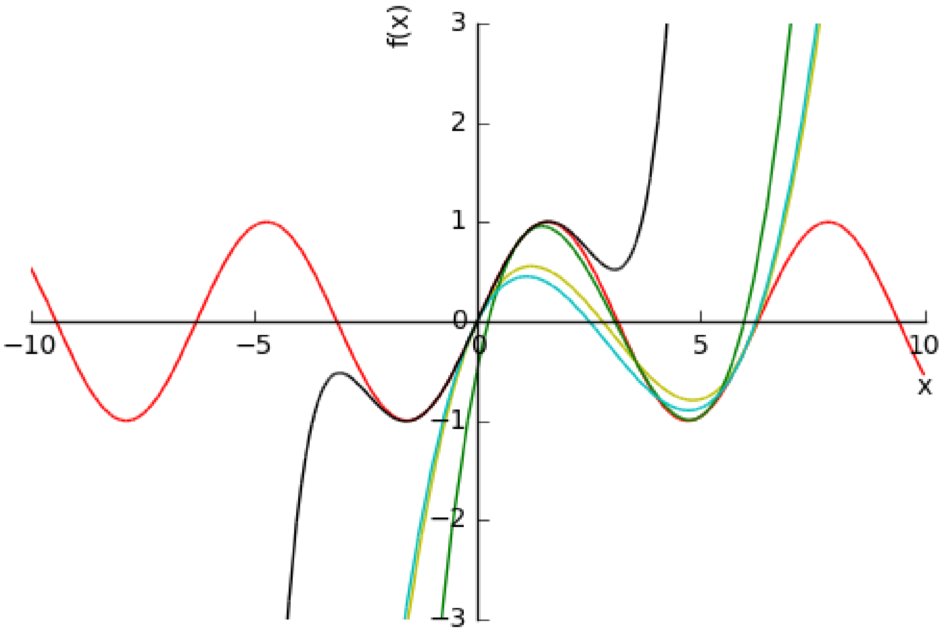
In [14]:
```python
u1=0.0
u2=0.0
u3=0.0
u4=0.0
u5=0.0
u6=0.0

for p in myEOne:
```

```
        u1+= ip1(sin(x),p) * p
    for q in myETwo:
        u2+= ip2(sin(x),q) * q
    for r in myEThree:
        u3+= ip3(sin(x),r) * r
    for s in myEFour:
        u4+= ip4(sin(x),s) * s
    for t in myEFive:
        u5+= ip5(sin(x),t) * t
    for v in myESix:
        u6+= ip6(sin(x),v) * v
    print(u1)
    print(u2)
    print(u3)
    print(u4)
    print(u5)
    print(u6)
```

```
0.0570363551680554*x**3 - 0.516646339291636*x**2 + 0.999999999999996*x - 6.66133814775094e-16
0.107760132566681*x**3 - 1.00192120815074*x**2 + 2.22086432446957*x - 0.48523226407762
0.0567365531800259*x**3 - 0.49604984473668*x**2 + 0.887213904239324*x + 1.0623446566882e-14
0.104824805211183*x**3 - 0.974626331029463*x**2 + 2.02749574887437*x - 1.22124532708767e-15
0.0975657436612037*x**3 - 0.913211464612763*x**2 + 1.97592056204524*x - 0.231135982929297
0.164998749433408*x**3 - 1.5555487489306*x**2 + 3.88833376228116*x - 1.9789087731102
```

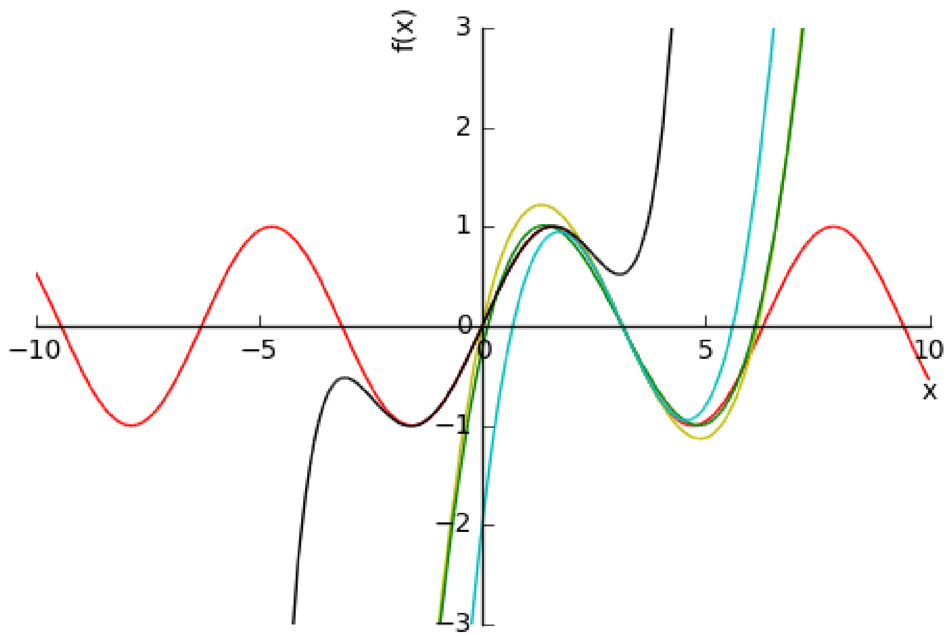
In [15]:
```
myPlot = plot(sin(x), u1,u2,u3,
              x-x**3 /6.+x**5 / 120.0,(x,-10,10), ylim=[-3,3], show=False)
myPlot[0].line_color = 'r' #sin(x)
myPlot[1].line_color = 'y' #from ip1
myPlot[2].line_color = 'g' #from ip2
myPlot[3].line_color = 'c' #from ip3
myPlot[4].line_color = 'k' #from taylor series
myPlot.show()
```

Out[15]:



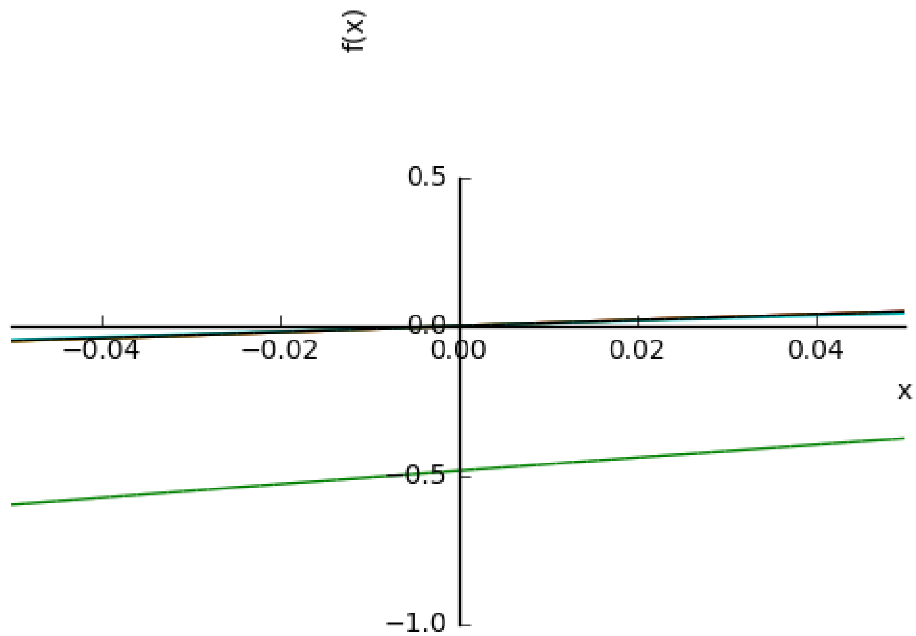

```
In [16]: myPlot2 = plot(sin(x), u4,u5,u6,
                        x-x**3 /6.+x**5 / 120.0,(x,-10,10), ylim=[-3,3], show=False)
         myPlot2[0].line_color = 'r' #sin(x)
         myPlot2[1].line_color = 'y' #from ip4
         myPlot2[2].line_color = 'g' #from ip5
         myPlot2[3].line_color = 'c' #from ip6
         myPlot2[4].line_color = 'k' #from taylor series
         myPlot2.show()
```

Out[16]:




```
In [17]: myplot = plot(sin(x), u1,u2,u3,
                        x-x**3 /6.+x**5 / 120.0, (x,-0.05,0.05), ylim=[-1,1], show=False)
         myplot[0].line_color = 'r' #sin(x)
         myplot[1].line_color = 'y' #from ip1
         myplot[2].line_color = 'g' #from ip2
         myplot[3].line_color = 'c' #from ip3
         myplot[4].line_color = 'k' #from taylor series
         myplot.show()
```

Out[17]:


f(x)
0.5
0.0
−0.5
−1.0
−0.04
−0.02
0.00
0.02
0.04
x