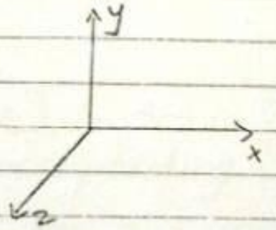
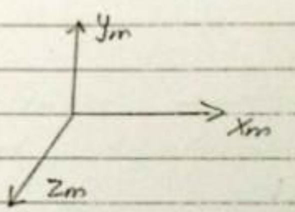
SYSTEM APPROACH	MEMBER APPROACH
1.) In this process whole structure is considered for analysis	1.) In this process memberwise analysis is carried out
2) Structural axis is considered for analysis 	2.) Member axis is considered for analysis 
3.) Number of diagrams are required for each degree of freedom	3.) No diagrams are required
4.) It is convenient for less number of members.	4.) It is convenient for more number of members for complicated geometry load configrations.

2. Explain the following matrices with illustrion:-

(i) $[S_J]$:- It is overall stiffness matrix by ascending contribution of member stiffness matrix $[S_M]$ of each member considering all joints unrestrained

$$[S_J] = \sum_{i=1}^{n} S_{Mi} = \sum_{i=1}^{n} [R_T^T]_i [S_M]_i [R_T]_i$$

i) [ARC] :-

→ It is the combined loads applied at supports where there is no DOF.

eg:- $[AC] = [\,10 \quad 4 \;|\; \underbrace{9 \quad 6 \quad 3}_{ARC} \quad 2\,]$

ii) [SRF] :- It is structure stiffness matrix corresponding to restrained displacement

$$[AR] = [ARC] + [SRF][DF]$$

iii) $[R_T]$:- R_T is a matrix of direction cosines which will be referred to as rotation matrix.

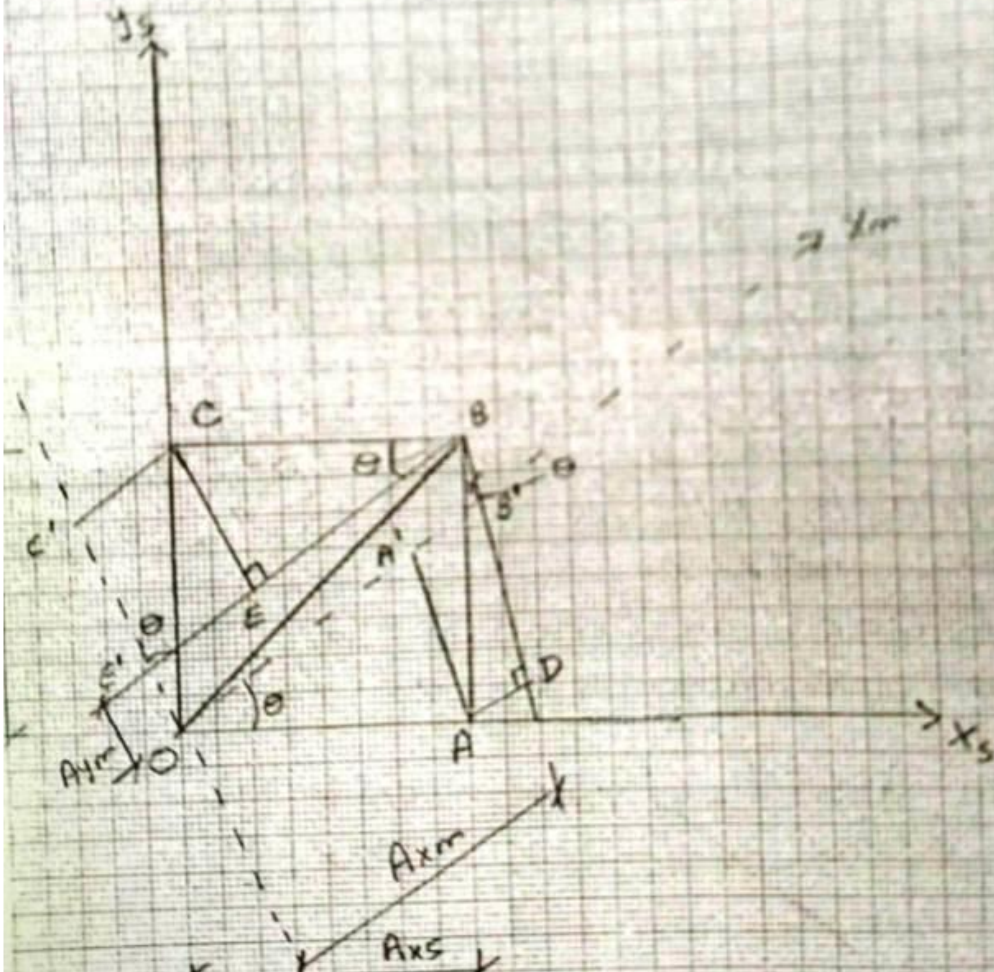
→ The rotation matrix in a two dimension problem is as follows:-

$$R_T = \begin{bmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{bmatrix} = \begin{bmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{bmatrix}$$

v) [SMS] :-

→ It is the member stiffness matrix for structure axis:-

$$[\overline{SMS}] = [R_T]^T [SM] [R_T]$$



TRUSS

1.1.

v) [AML]:-

⇒ The matrix of actions [AML] due to loads acting on the members when the joints of the structure are restrained against translation.

$$[AM] = [AML] + [SM][DM]$$

Year:

3. Derive the relationship between vectors A_{xs} & A_{xm} for

- Plane truss
- Plane frame
- Grid.

1) Plane truss:-

⇒ Consider a vector A which lies in x-y plane a shown in fig.

Two sets of orthogonal is also shown where X_m, Y_m = member oriented (local) x and y axis

Similarly X_s, Y_s = structure oriented (global) x & y axis

θ = angle between above two sets of orthogonal axis

From fig, $OB' = OA' + A'B'$

In $\triangle OA'A$, $OA' = OA\cos\theta$

$= A_{xs}\cos\theta$

& In $\triangle BAD$, $AD = AB\sin\theta$

But, $AD = A'B'$

$\therefore A'B' = A_{ys}\sin\theta$

$A_{xm} = A_{xs}\cos\theta + A_{ys}\sin\theta$ ---- (i)

Now, $OE' = OC' - E'C'$, $OE' = A_{ym}$

In $\triangle OCC'$, $OC' = A_{ys}\cos\theta$, ~~...~~ CE

In $\triangle CEB$, $CE = CB\sin\theta$ ⇒ $CE = A_{xs}\sin\theta$

$A_{ym} = A_{ys}\cos\theta - A_{xs}\sin\theta$ ----(ii)

$$\begin{bmatrix} A_{xm} \\ A_{ym} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} A_{xs} \\ A_{ys} \end{bmatrix}$$

$\therefore [A_m] = [R][A_s]$ ....(iii) where, $[R]$ = rotation matrix

$$= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$[A_s] = [R^{-1}][A_m]$ & $[R^{-1}] = R^T$

$A_s = R^T A_m$ ----(iv)

$[D_m] = [R][D_s]$

$[D_s] = [R^T][D_m]$

Plane frame & 3) Grid :-

⇒ Consider a vector A which lies in XY plane as shown in fig.

Three sets of orthogonal is also shown, where, X_m, Y_m, Z_m = member oriented (local) x, y & z axis

Similarly, X_s, Y_s, Z_s = Structure oriented (global) x, y & z axis

θ = angle between above two sets of orthogonal axis

From fig, $OB' = OA' + A'B'$

In $\Delta OA'A$, $OA' = OA\cos\theta = A_{xs}\cos\theta$

& In ΔBAD, $AD = AB\sin\theta$

But $AD = A'B'$

$\therefore A'B' = A_{ys}\sin\theta$

$\therefore A_{xm} = A_{xs}\cos\theta + A_{ys}\sin\theta$ — (i)

Q. List out the secondary effects & explain how to incorporate it in analysis by stiffness member approach.

→ The secondary effects are:-

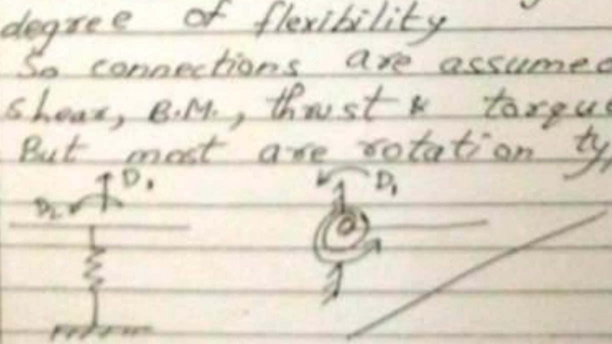
1) Spring stiffness
2) Rotation
3) Settlement
4) Temperature → Uniform, → Linear
5) Bar with excess length
6) Bar with a bend
7) Beam with initial curvature.

1) Spring stiffness:-

→ Joints of framed structure - either pinned or rigid.

→ However connections may have a significant degree of flexibility

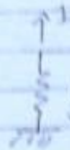
→ So connections are assumed as elastic type: shear, B.M., thrust & torque type.
But most are rotation type.



Linear Spring   Torsional Spring

→ Add the stiffness of spring in respective term of SM matrix

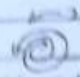
i) If linear (unit translation corresponding to 1)



Add spring stiffness in K_{11} i.e. $\frac{12EI}{L^3}$

$\therefore K_{11}$ + stiffness

ii) If torsional

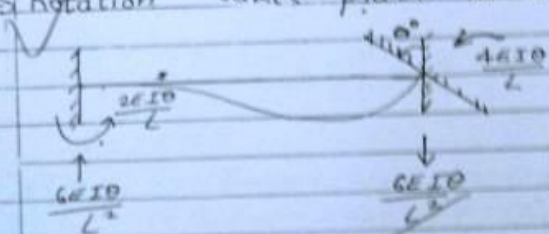


unit rotation correspond to 2

K_{22} + stiffness.

2.) Rotation

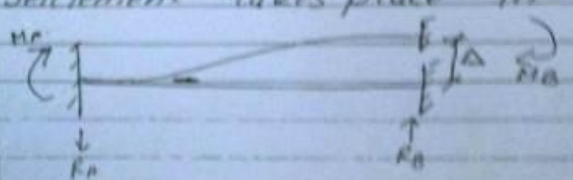
→ Rotation takes place in beam and frame.



→ Due to rotation changes takes place in load vector matrix.

3.) Settlement

→ Settlement takes place in beam and frame



where, $M_A = M_B = 6EI\Delta/L^2$

$R_A = R_B = \frac{12EI\Delta}{L^3}$

→ Because of settlement changes ~~is~~ takes place in load vector matrix.

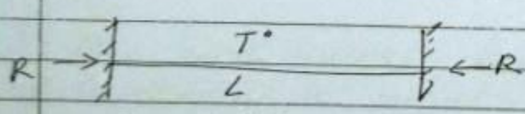
4) Temperature.

(i) Uniform

(ii) Linear

i) Uniform:-

→ Beam subjected to uniform temperature increase of $T°$.



$R = EA\alpha T$

where, T = Temperature increase

α = Coefficient of thermal expansion

A = c/s area

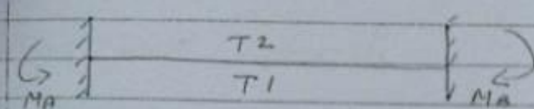
E = modulus of elasticity

→ Length of beam try to increase so reaction will be inward.

→ Changes will take place in load vector matrix.

ii) Linear:-

→ Beam subjected to linear temperature gradient.



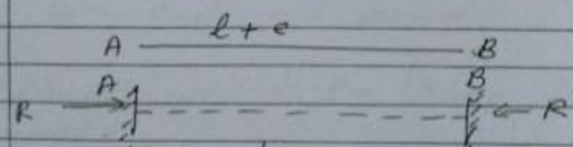
$$M_A = M_B = \frac{\alpha EI(T_1 - T_2)}{d}$$

where, I = moment of inertia
$T1$ = Temperature of bottom beam
$T2$ = Temperature at top of beam
d = depth of beam.

→ Here temperature of centroidal axis remains same, so no change in length - so FEA are moments only as a result of which changes takes place in load vector matrix.

E) Bar with excess length.

→ Prestrain is initial deformation of a member bar with excess length

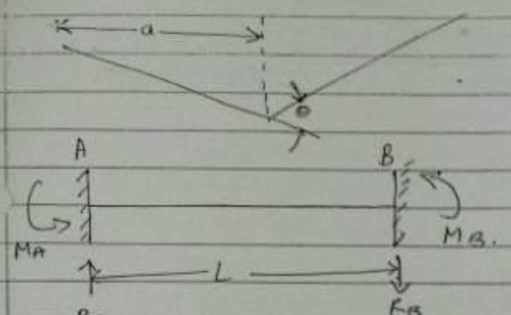


$$R = \frac{EAe}{L}$$

→ Members having initial length > distance between support by small amount e.

→ Resulting FEA are compressive forces R as a result of which changes will take place in load vector matrix.

6.) Bar with a bend.



where, $M_A = \frac{2EI\theta}{L^2}(2L - 3a)$

$M_B = \frac{2EI\theta}{L^2}(L - 3a)$

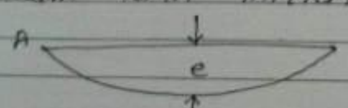
$R_A = +R_B = \frac{6EI\theta}{L^3}(L - 2a)$

∴ I = Moment of inertia
θ = Angle of bend.

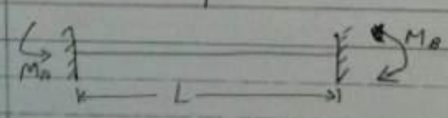
5/11/09

→ Changes will take place in load vector matrix.

7.) Beam with initial circular curvature.



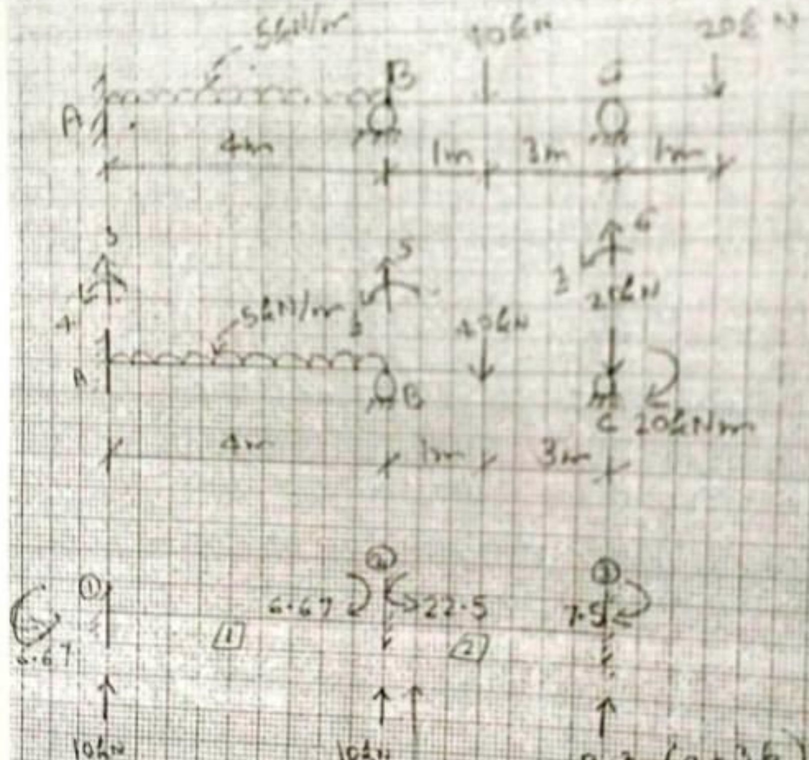
$M_A = +M_B = \frac{8EIe}{L^2}$



e = initial deflection at middle of bar

⇒ Changes will take place in load vector matrix.

3.9375 =



$$\frac{Pb^2(3a+b)}{L^3} = \frac{40\times3^2(3\times1+3)}{4^3} = 33.75$$

$$\frac{Pa^2(a+3b)}{L^3} = \frac{40\times1^2(1+3\times3)}{4^3} = 6.25$$

$$[\;\ldots\; -20 \quad 0 \quad 0 \quad 0 \quad -20\;]$$

$$+\,[-15.83 \quad 7.5 \quad -10 \quad -6.67 \quad -43.75 \quad -6.25]$$

$$= \Big[\underbrace{\overset{1}{-15.83} \quad \overset{2}{-12.5}}_{AFC} \;\Big|\; \underbrace{\overset{3}{-10} \quad \overset{4}{-6.67} \quad \overset{5}{-43.75} \quad \overset{6}{-26.25}}_{ARC}\Big]$$

(i) Primary unknowns:-
Displacements,

$$[D_F] = SPF^{-1} \times AFC$$

$$= \frac{1}{EI}\begin{bmatrix} 2 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -15.83 \\ -12.5 \end{bmatrix}$$

$$\begin{bmatrix} M_B \\ M_C \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} -5.47 \\ -9.76 \end{bmatrix}$$

(ii) Secondary unknowns:-

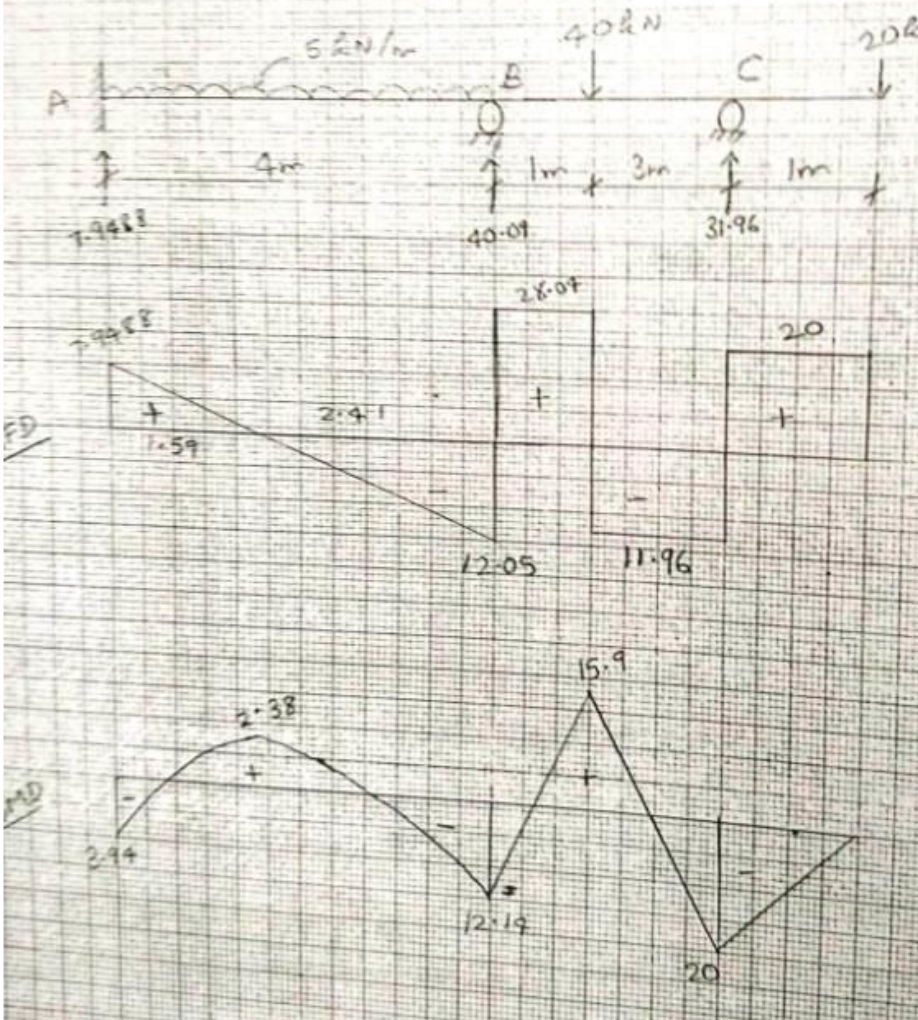
$$\rightarrow [AR] = -[ARC] + [SRF][DF]$$

$$= -\begin{bmatrix}-10\\-6.67\\-43.75\\-26.25\end{bmatrix} + \cancel{EI}\begin{bmatrix}0.375 & 0\\0.5 & 0\\0 & 0.375\\-0.375 & -0.375\end{bmatrix}\frac{1}{\cancel{EI}}\begin{bmatrix}-5.47\\-9.76\end{bmatrix}$$

$$\begin{bmatrix}R_A\\M_A\\R_B\\R_C\end{bmatrix} = \begin{bmatrix}7.9488\\3.94\\40.09\\31.96\end{bmatrix} kN$$

$$\rightarrow [AM]_1 = [AML]_1 + [SM]_1[DM]_1$$

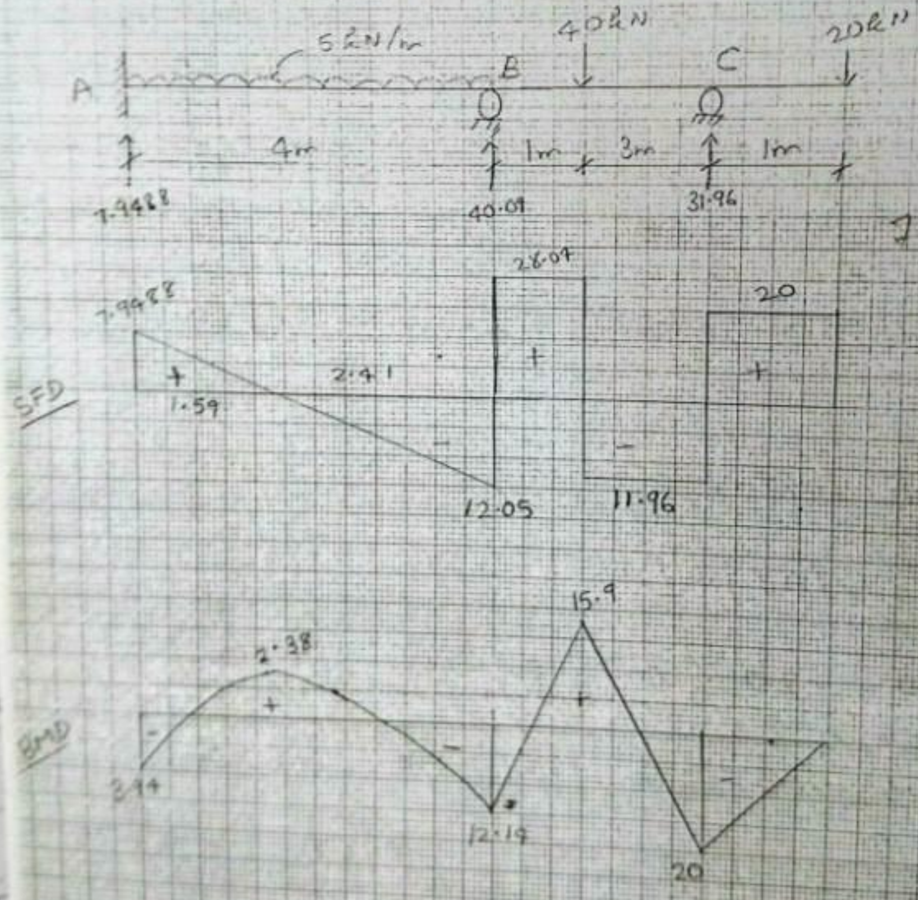
$$= \begin{bmatrix}10\\6.67\\\ldots\end{bmatrix} + \cancel{EI}\begin{array}{c}\begin{array}{cccc}3 & 4 & 5 & 1\end{array}\\\begin{bmatrix}0.1875 & 0.375 & -0.1875 & 0.375\\0.375 & 1 & -0.375 & 0.5\\\ldots\end{bmatrix}\end{array}\frac{1}{\ldots}\begin{bmatrix}0\\0\\\ldots\end{bmatrix}$$



$$\frac{7.94}{5} = 1.59$$

$$\frac{x}{7.9488} = \frac{4-x}{12.05}$$

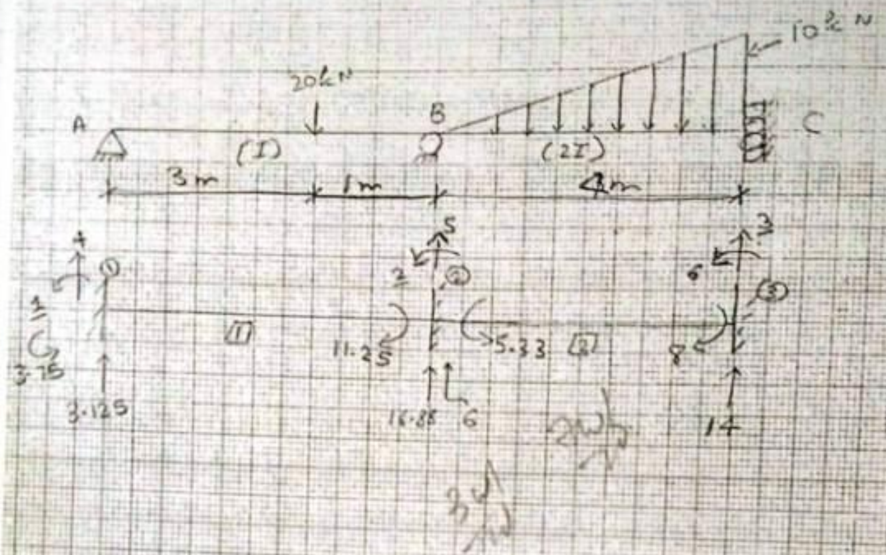
$$x = 1.59$$



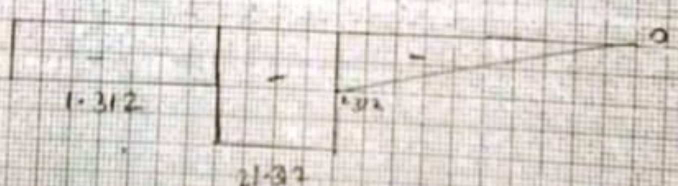
$$\frac{7.94 \times 4}{5} = 1.59$$

$$\frac{x}{7.9488} = \frac{4-x}{12.05}$$

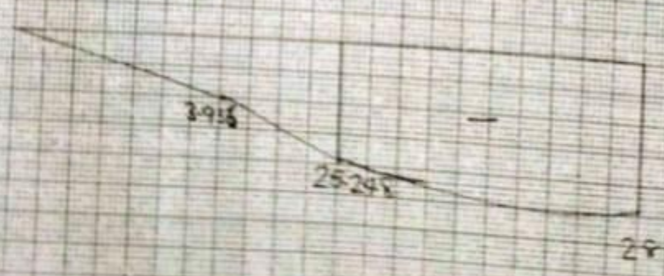
$$x = 1.59$$


10kN
20kN
A
B
C
(I)
(2I)
3m
1m
4m
3.75
3.125
11.25
16.88
6
5.33
8
14

SFD


1.312
21.312
0

BMD


3.936
25.248
28

AFC ARC

i) Primary unknowns:-
Displacements,

$[D_F] = S_{FF}^{-1} \times A_{FC}$

$$= \frac{1}{EI}\begin{bmatrix} 1 & 0.5 \\ 0.5 & 3 \end{bmatrix}^{-1} \times$$

$$= \frac{1}{EI}\begin{bmatrix} 1 & 0.5 & 0 \\ 0.5 & 3 & -0.75 \\ 0 & -0.75 & 0.375 \end{bmatrix}^{-1} \begin{bmatrix} -3.75 \\ 5.92 \\ -14 \end{bmatrix}$$

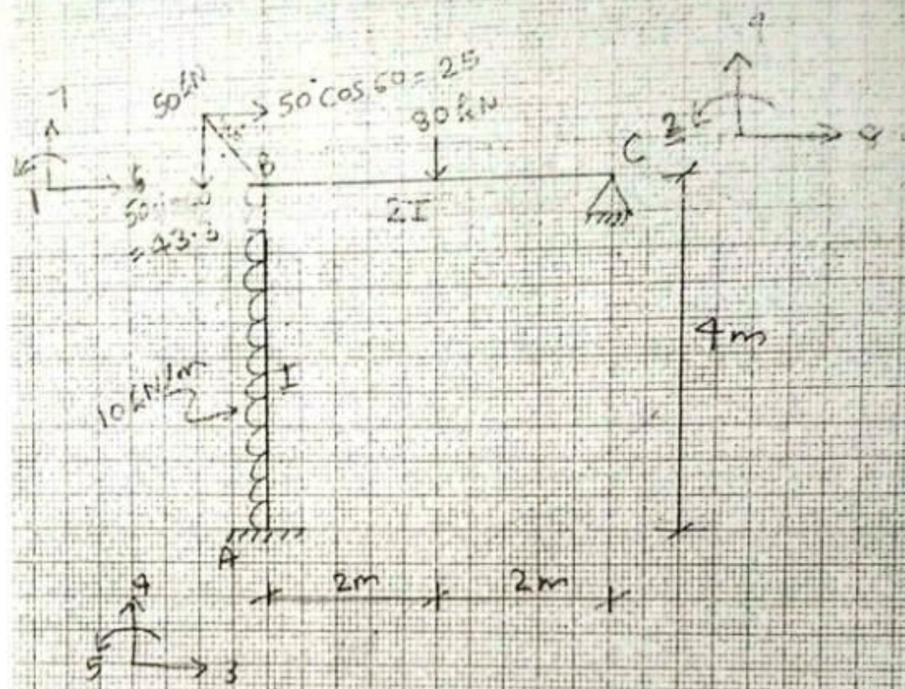
$$\begin{bmatrix} M_A \\ M_B \\ V_C \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 4.332 \\ -16.164 \\ -69.66 \end{bmatrix}$$

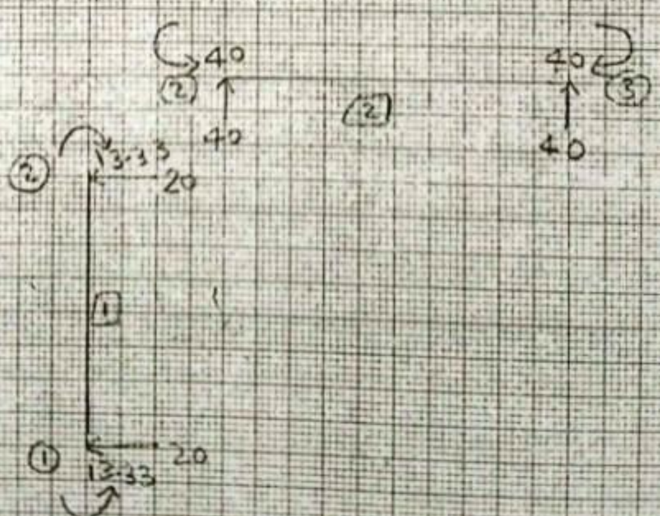
i) Secondary unknowns:-

$[A_R] = -[A_{RC}] + [S_{RF}][D_F]$

$$= -\begin{bmatrix} -3.125 \\ -22.88 \\ 8 \end{bmatrix} + EI\begin{bmatrix} 0.375 & 0.375 & 0 \\ -0.375 & 0.375 & -0.375 \\ 0 & 1 & -0.75 \end{bmatrix} \frac{1}{EI}\begin{bmatrix} 4.332 \\ -16.164 \\ -69.66 \end{bmatrix}$$

$$\begin{bmatrix} R_A \\ R_B \\ M_C \end{bmatrix} = \begin{bmatrix} -1.312 \\ 41.3165 \\ 28.081 \end{bmatrix}$$


50kN
50 cos 60 = 25
80kN
50 sin 60
= 43.3
B
C
2I
4m
10kN/m
I
A
2m
2m


40
40
40
40
3.33
20
20
13.33

	J	K	θ	C_x	C_y	EA (10^3)	EI (10^3)	L
1	1	2	90	0	1	40	20	4
2	2	3	0	1	0	80	40	4

$[AC] = [AJ] + [AE]$

$$
\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
$$

$$= [\,0 \quad 0 \;\vdots\; 0 \quad 0 \quad 0 \quad 25 \quad -43.3 \quad 0 \quad 0\,]$$

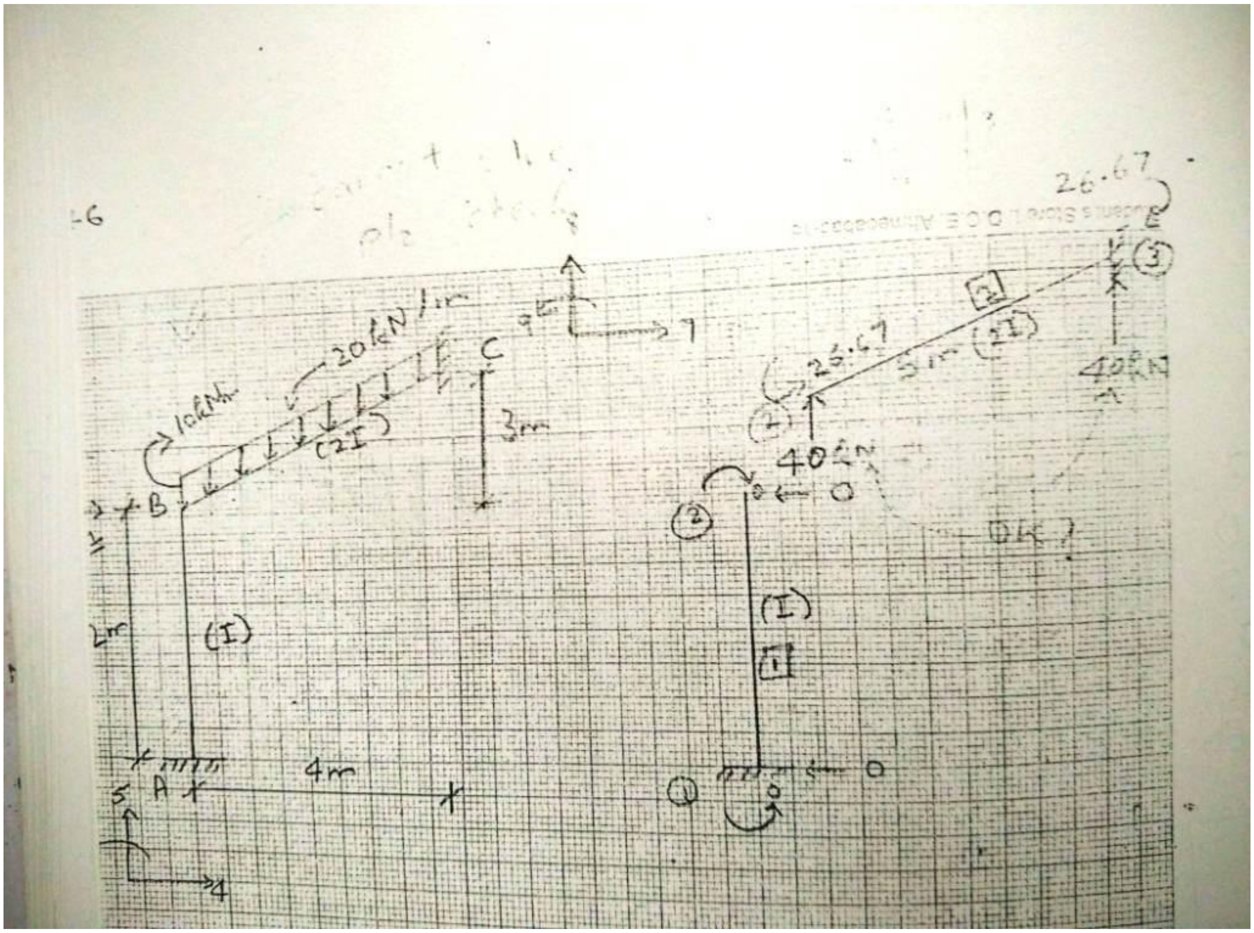
$$+ [\,-26.67 \quad 40 \;\vdots\; 20 \quad 0 \quad -13.33 \quad 20 \quad -40 \quad 0 \quad -40\,]$$

$$= [\,-26.67 \quad 40 \;\vdots\; 20 \quad 0 \quad -13.33 \quad 45 \quad -83.3 \quad 0 \quad -40\,]$$

AFC   ARL

$$
SM1 = 10^3 \begin{bmatrix}
EA/L & 0 & 0 & -EA/L & 0 & 0 \\
0 & 12EI/L^3 & 6EI/L^2 & 0 & -12EI/L^3 & 6EI/L^2 \\
0 & 6EI/L^2 & 4EI/L & 0 & -6EI/L^2 & 2EI/L \\
-EA/L & 0 & 0 & EA/L & 0 & 0 \\
0 & -12EI/L^3 & -6EI/L^2 & 0 & 12EI/L^3 & -6EI/L^2 \\
0 & 6EI/L^2 & 2EI/L & 0 & -6EI/L^2 & 4EI/L
\end{bmatrix}
$$

$SM1 = 10^3$ [10   0   0   10   0   0]



-6
20kN/m
10kNm
C
B
A
(2I)
(I)
3m
2m
4m
26.67
5m
(2I)
40kN
(I)
OK ?

$I = \frac{bd^3}{12}$

$A = (bd)$

Member	J	K	θ	C_x	C_y	EA (10^3)	EI (10^3)	L
1	1	2	90	0	1	40	20	2
2	2	3	36.87	0.8	0.6	80	40	5

$[AC] = [AJ] + [AE]$

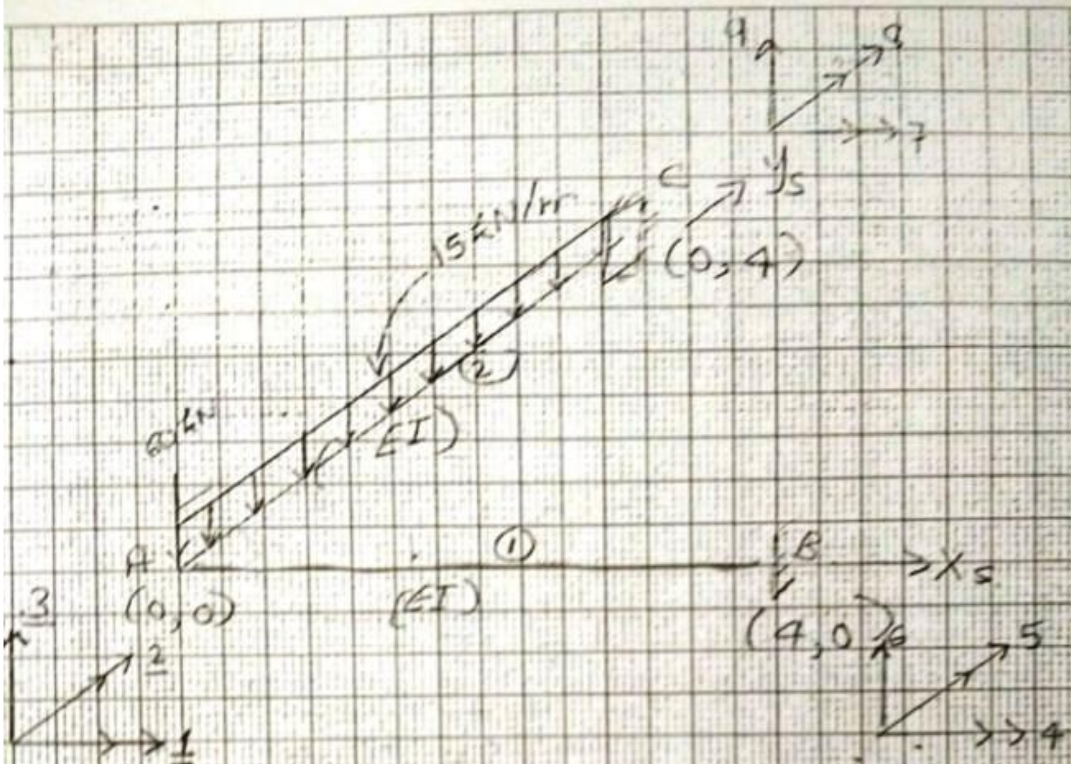
$$
\begin{array}{ccccccccc}
[0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0] \\
+ \; 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
= [0 & -40 & -26.67 & 0 & 0 & 0 & 0 & -40 & 26.67] \\
= [0 & -40 & -36.67 & 0 & 0 & 0 & 0 & -40 & 26.67]
\end{array}
$$

$$
[SM1] = 10^3 \begin{bmatrix} 20 & 0 & 0 & -20 & 0 & 0 \\ 0 & 30 & 30 & 0 & -30 & 30 \\ 0 & 30 & 40 & 0 & -30 & 20 \\ -20 & 0 & 0 & 20 & 0 & 0 \\ 0 & -30 & -30 & 0 & 30 & -30 \\ 0 & 30 & 20 & 0 & -30 & 40 \end{bmatrix}
$$

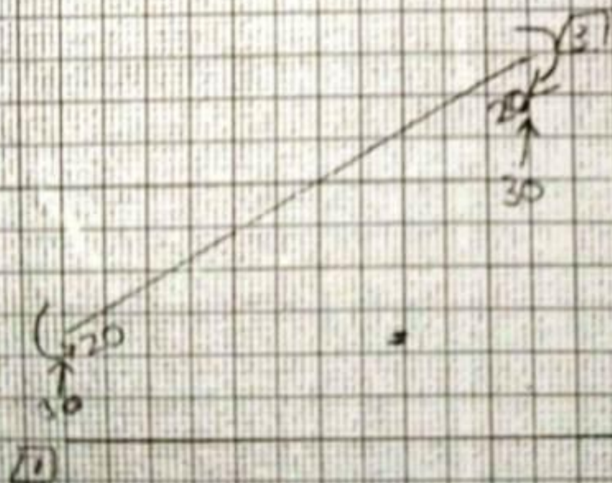
$$
[RT1] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad [RT_1]^T = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
[SM1][RT1] = 10^3 \begin{bmatrix} 0 & 20 & 0 & 0 & -20 & 0 \\ -30 & 0 & 30 & 30 & 0 & 30 \\ -30 & 0 & 40 & 30 & 0 & 20 \\ 0 & -20 & 0 & 0 & 20 & 0 \\ 30 & 0 & -30 & -30 & 0 & -30 \\ -30 & 0 & \cancel{40}\,20 & +30 & 0 & \cancel{20}\,40 \end{bmatrix}
$$

$$
\Rightarrow [SMS1] = 10^3 \begin{array}{c} \\ \end{array}
\begin{array}{cccccc|c}
4 & 5 & 6 & 1 & 2 & 3 & \\
\hline
30 & 0 & -30 & -30 & 0 & -30 & 4 \\
0 & 20 & 0 & 0 & -20 & 0 & 5 \\
-30 & 0 & 40 & 30 & 0 & 20 & 6 \\
-30 & 0 & 30 & 30 & 0 & 30 & 1 \\
0 & -20 & 0 & 0 & 20 & 0 & 2 \\
-30 & 0 & 20 & 30 & 0 & 40 & 3
\end{array}
$$



15 kN/m
C
(0, 4)
2
(EI)
A
(0, 0)
1
B
(4, 0)
(EI)
X_s
Y_s
1
2
3
4
5
6
7
8
9



3
30
20
10
1
2

Sr No.	Member	Joint J	Joint K	L	EI (10^3)	GJ (10^3)	θ	C_x	C_y
1)	1	1	2	4	20	16	0	1	0
2)	2	1	3	4	20	16	90	0	1

→ Load vector:-

$$[AC] = [AJ] + [AE]$$

$$= [0 \quad 0 \quad -60 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$+ [-20 \quad 0 \quad -30 \quad 0 \quad 0 \quad 0 \quad 20 \quad 0 \quad -30]$$

$$\begin{array}{c} \\ = \end{array} \begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underbrace{-20 \quad 0 \quad -90}_{AFC} & & & \underbrace{0 \quad 0 \quad 0 \quad 20 \quad 0 \quad -30}_{ARC} \end{array}$$

$$\left.\begin{array}{l} \to [SM1] \\ [SM1][RT1] \\ [SMS1] \end{array}\right\} = 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 20 & -7.5 & 0 & 10 & 7.5 \\ 0 & -7.5 & 3.75 & 0 & -7.5 & -3.75 \\ -4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 10 & -7.5 & 0 & 20 & 7.5 \\ 0 & 7.5 & -3.75 & 0 & 7.5 & 3.75 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$[SM1] = 10^3 \begin{bmatrix} 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 20 & -7.5 & 0 & 10 & 7.5 \\ 0 & -7.5 & 3.75 & 0 & -7.5 & -3.75 \\ -4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 10 & -7.5 & 0 & 20 & 7.5 \\ 0 & 7.5 & -3.75 & 0 & 7.5 & 3.75 \end{bmatrix}$$

$$\begin{bmatrix} & 7.5 & -3.75 \\ 10 & 0 & 7.5 \\ 0 & -4 & 0 \\ -7.5 & 0 & -3.75 \end{bmatrix} \begin{matrix} (5) \\ (6) \\ (7) \\ (8) \\ (9) \end{matrix}$$

(i) Primary unknowns:-

$[D_i] = [SFF]^{-1} \times [AFC]$

$$= \frac{1}{10^3} \begin{bmatrix} 24 & 0 & 7.5 \\ 0 & 24 & -7.5 \\ 7.5 & -7.5 & 7.5 \end{bmatrix}^{-1} \begin{bmatrix} -20 \\ 0 \\ -90 \end{bmatrix}$$

$$= \frac{1}{10^3} \begin{bmatrix} 8.472 \\ -9.31 \\ -29.78 \end{bmatrix}$$

(ii) Secondary unknowns:-

$[AMi] = [AMLi] + [SMi]\ [RTi][DMi]$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 & -4 & 0 & 0 \\ 0 & 20 & -7.5 & 0 & 10 & 7.5 \\ 0 & -7.5 & 3.75 & & & \end{bmatrix}$$

Secondary effects:-

⇒ Analyse the continuous & propped cantilever beam.

(i) $EI = 20 \times 10^3$ kN m

$EA = 10 \times 10^3$ kN

$\alpha = 12 \times 10^{-6}\ {}^\circ C^{-1}$

$d = 300$ mm

⇒ The beam is subjected to following effects. Downward settlement of support A & B by 1.5mm & 4.5mm respectively

⇒ Clockwise rotation of support B by 0.001 rad

⇒ Member BC is subjected to temperature rise (differential) 40°C at top & 60°C at bottom.

$[AC] = [AJ] + [AE]$

$$= \begin{array}{cccccc} ① & ② & ③ & ④ & ⑤ & ⑥ \\ \end{array}$$
$$\left[\begin{array}{cc|cccc} 109 & 171 & -60 & -70 & 225 & -165 \end{array}\right]$$

(A) Structural Data:-

Member No.	J end	K end	Length (m)	EI
1	①	②	2	1
2	②	③	2	1

Secondary effects:-

Analyse the continuous & propped cantilever beam.

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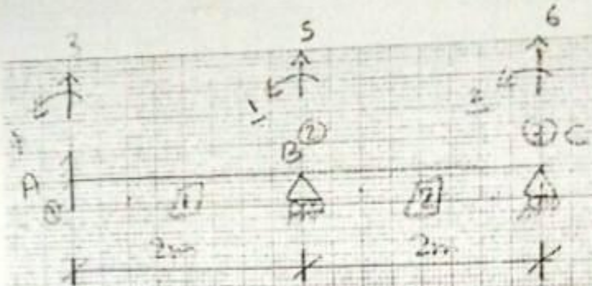
(A) Structural Data:-

Member No.	J end	K end	Length (m)	EI
1	①	②	2	1
2	②	③	2	1

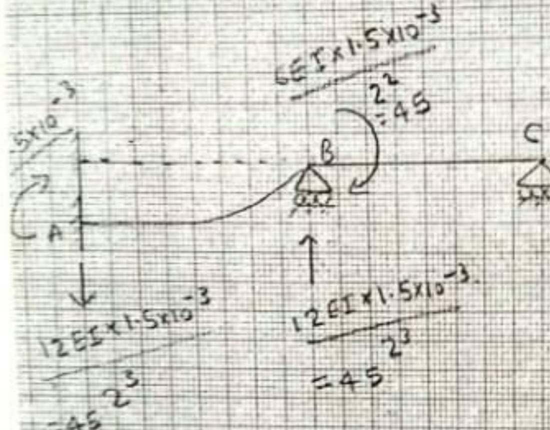


$EI = 20 \times 10^3$ kNm

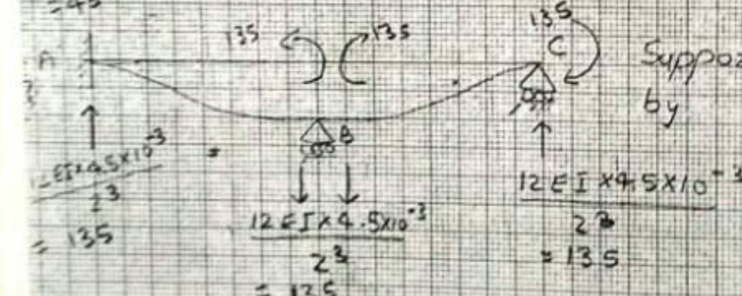
$EA = 10 \times 10^3$ kN

$\alpha = 12 \times 10^{-6}$ °C^{-1}

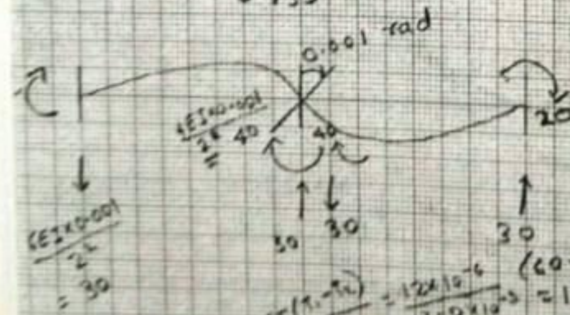
$d = 300$ mm



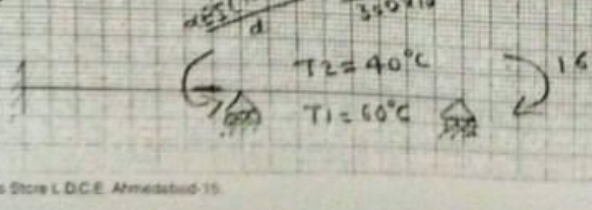
Support A settles by 1.5mm



Support B settles by 4.5mm



B rotates clockwise by 0.001 rad.



BC is subjected to temp rise

T1 = 60°C

T2 = 40°C

Stiffness matrices:-

$$[SM1] = EI \begin{array}{c} \\ \left[\begin{array}{cccc} 1.5 & 0.375 & -1.5 & 0.375 \\ 0.375 & 2 & -0.375 & 1 \\ -1.5 & -0.375 & 1.5 & -0.375 \\ 0.375 & 1 & -0.375 & 2 \end{array}\right] \end{array} \begin{array}{c} 3 \\ 4 \\ 1 \\ 2 \end{array}$$

(columns: 3, 4, 1, 2)

$$[SM2] = EI \left[\begin{array}{cccc} 1.5 & 0.375 & -1.5 & 0.375 \\ 0.375 & 2 & -0.375 & 1 \\ -1.5 & -0.375 & 1.5 & -0.375 \\ 0.375 & 1 & -0.375 & 2 \end{array}\right] \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array}$$

(columns: 5, 1, 6, 2)

$$SJ = EI \left[\begin{array}{cc} 3.5 & +0.625 \\ 0.625 & 4 \\ -1.5 & 0.375 \\ -0.375 & \text{[illegible]} \\ 0.375 & +0.375 \\ -0.375 & -0.375 \end{array} \quad \cdots \right] \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

(columns: 1, 2)

(i) Primary unknowns:-

$$D_F = [SFF]^{-1} \times [AFC] = \frac{1}{EI} \begin{bmatrix} 3.5 & 0.625 \\ 0.625 & 4 \end{bmatrix}^{-1} \times \begin{bmatrix} 109 \\ 171 \end{bmatrix}$$

$$D_F = \frac{1}{EI} \begin{bmatrix} 24.18 \\ 38.97 \end{bmatrix}$$

$$[AR] = -[ARC] + [SRF][DF]$$

$$= -\begin{bmatrix} -60 \\ -70 \\ 225 \\ -165 \end{bmatrix} + EI\begin{bmatrix} -1.5 & 0.375 \\ -0.375 & 1 \\ 0.375 & 0.375 \\ -0.375 & -0.375 \end{bmatrix} \frac{1}{EI}\begin{bmatrix} 24.18 \\ 38.97 \end{bmatrix}$$

$$\begin{bmatrix} V_A \\ M_A \\ V_B \\ V_C \end{bmatrix} = \begin{bmatrix} 38.34 \\ 100 \\ -201.31 \\ 141.32 \end{bmatrix}$$

The beam is subjected to following effects.
Member AB subjected to temperature rise from 30°C to 40°C (uniform)
Member AB is having initial sag of 5mm at centre of span.
curvature

$[AC] = [AJ] + [AE]$

$$= \begin{array}{cccc} ① & ② & ③ & ④ \\ \left[\, 60.27 \right. & -40 & -60.27 & \left. -40 \,\right] \end{array}$$

Structural Data:-

Member No.	J end	K end	Length (m)	EI
1	①	②	4	1

(i) Stiffness matrices:-

$$[SM1] = EI \begin{array}{c} \\ \\ \end{array} \begin{array}{cccc|c} & ② & ③ & ④ & ① \\ \end{array}$$

$$[SM1] = EI \left[\begin{array}{ccc|c} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ \hline 0.375 & 0.5 & -0.375 & 1 \end{array}\right] \begin{array}{c} ② \\ ③ \\ ④ \\ ① \end{array}$$

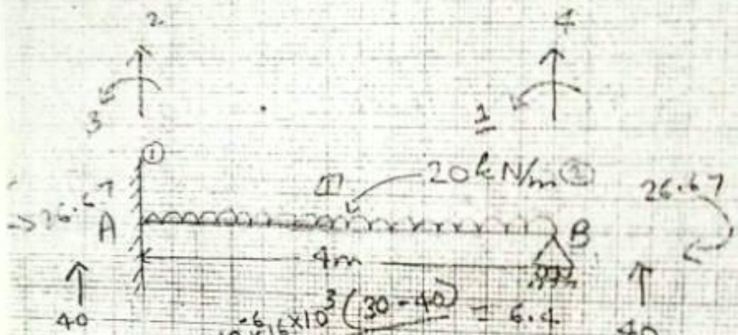
(ii) Primary unknowns:-

$[D_F] = [S_{FF}]^{-1} \times [A_{FC}] = \frac{1}{EI}[1]^{-1} \times 60.27$

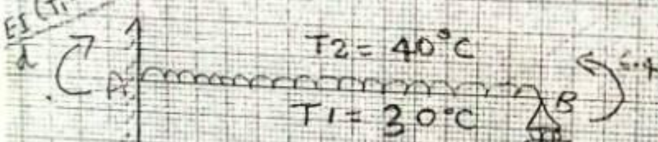
$$D_F = \frac{60.27}{EI}$$

Assume

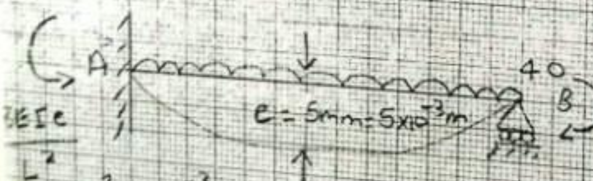
$d = 300\,mm$



$$\frac{EI(T_1-T_2)}{d} = \frac{12\times10^{-6}\times16\times10^{3}(30-40)}{300\times10^{-3}} = 6.4$$



Member AB is subjected to temp rise from 30°C to 40°C (Uniform



Member AB is having initial sag of 5mm at centre span

$$\frac{2EIe}{L^2}$$

$$\frac{2\times16\times10^{3}\times5\times10^{-3}}{4^2}$$

40

$\frac{2EIe}{L^2}$

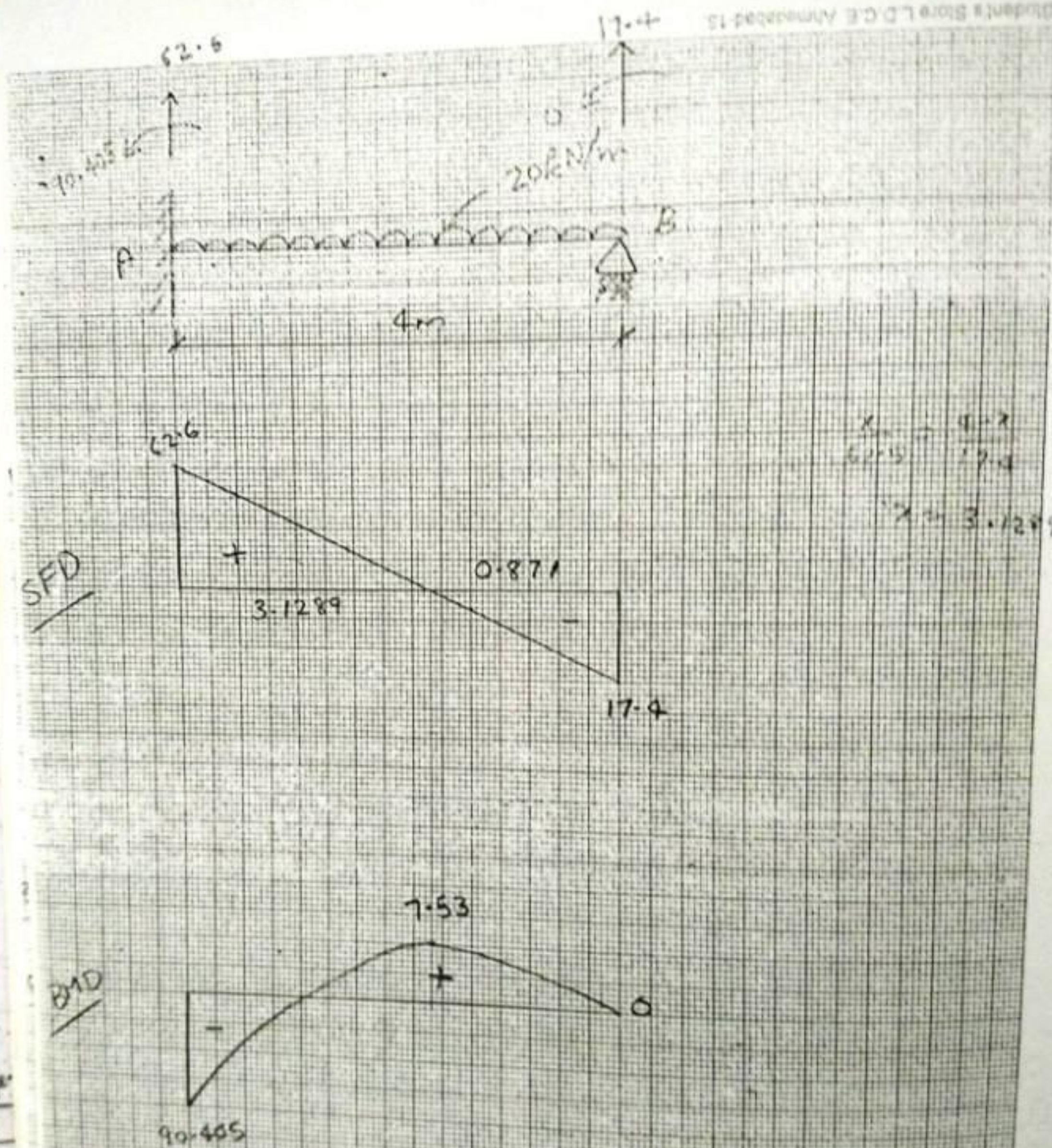
Secondary unknowns:-

$$[AM] = [AML] + [SM][DM]$$

$$\begin{bmatrix} 40 \\ 60.27 \\ 40 \\ -60.27 \end{bmatrix} = \begin{bmatrix} 60.27 \\ 40 \\ 40 \\ -60.27 \end{bmatrix} + \cancel{EI} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1 \end{bmatrix}$$

$$\frac{1}{\cancel{EI}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 60.27 \end{bmatrix}$$

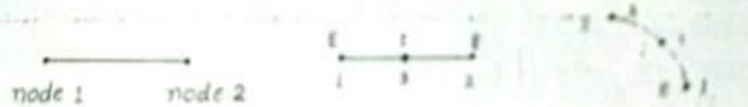
$$\begin{bmatrix} V_A \\ M_A \\ V_B \\ M_B \end{bmatrix} = \begin{bmatrix} 62.6 \\ 90.405 \\ 17.4 \\ 0 \end{bmatrix}$$


62.6
17.4
90.405
20kN/m
A
B
4m
62.6
SFD
+
0.871
3.1289
−
17.4
x/62.6 = (4−x)/17.4
x = 3.1289
BMD
7.53
+
−
0
90.405

K

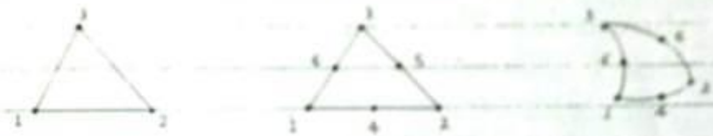
1. Define the term 'Finite element' and list atleast four common types of finite element.

Actually, any system is made up of infinite numbers of subrigion, but in FEM the finite number of subregions are considered which are called as Elements or Finite elements.



Basic Element   H-Element   Curved Element

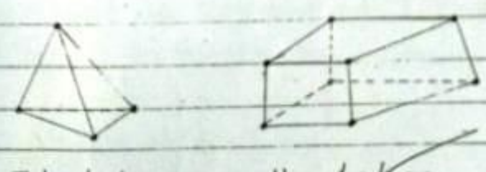
(i) 1-D Elements



Triangular Elements

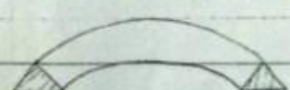
Quadrilateral Elements

(ii) 2-D Elements



Tetrahedron   Hexahedron

(iii) 3-D Elements



(iv) Axisymmetric Elements

2. What is finite element method ?
Explain briefly general steps for solution of problem by FEM.

① Finite Element Method:

(The finite element method is a numerical technique for obtaining approximate numerical solutions to boundary value problems to predict the response of physical systems subjected to external loads.)

General steps for solution of problem by FEM —

(1) Discretize the continuum
(2) Select element functions
(3) Calculate element properties
(4) Assemble element properties
(5) Introduce boundary conditions
(6) Calculate primary unknowns
(7) Calculate secondary unknowns

Step-1 Discretize the continuum

Discretization is the process of subdividing the given body into number of elements which result into an equivalent body of finite elements.

In the discretization we have to decide the number, size and type of the elements in such a way that the original body is simulated as closely as possible.

Step 2 Select element functions

Displacement functions

The basic theory of the FEM is going from a part to the whole. We subdivide the given problem and represent the solution within each element by a relatively simple function. We assume simple functions to approximate the displacement for each element. These functions are called displacement functions, displacement models, displacement fields. The unknown magnitudes of the displacement functions are the displacement at the nodal points.

A displacement function can be expressed as

- Trigonometric function
- Polynomial function

Generally polynomials are selected as displacement functions because they are easy to integrate and differentiate and thus simplify the mathematical manipulation. Further a polynomial of arbitrary order permits a recognizable approximation to the true solution and it is easy to vary degree of approximation.

One dimensional displacement function:

$$u = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + \dots + C_{m+1} x^n$$

Two dimensional displacement functions:

$$u = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2 + \dots + C_m y^n$$

$$v = C_{m+1} + C_{m+2} x + C_{m+3} y + \dots + C_{2m} y^n$$

Three dimensional displacement functions :

$u = C_1 + C_2 x + C_3 y + C_4 z + C_5 xy + C_6 yz + C_7 zx + \ldots + C_m z^n$

$v = C_{m+1} + C_{m+2} x + \ldots + C_{2m} z^n$

$w = C_{2m+1} + C_{2m+2} x + \ldots + C_{3m} z^n$

Step-3 Calculate element properties :

The element equilibrium equation

$$[K]\{q\} = \{F\}$$

$[K] = \int [B]^T [D] [B] \, dv$, stiffness matrix

$[B] = \frac{d}{dx} [N]$, strain-displacement matrix

$[D]$ = Constitutive matrix

Step-4 Assemble element properties

Assuming that we have found the necessary equilibrium equations describing the characteristics of each element of the system, the next step is to comb all these equations to form a complete structure.

The procedure for constructing the system equat from the element equations is the same regardless of t type of the problem being considered or the complexity of the system of elements.

The system assembly procedure is based on insistence of compatibility at the element nodes. By thi we mean that at nodes where elements are connected values of the unknown nodal variables are the same all elements connecting at that node. The consequences this rule is the basis for assembly process. When the displacements of surrounding elements are matched at the node the nodal stiffnesses and nodal loads for each of the elements sharing the node are added to obtain the net stiffness and the net load at the node

Step-5 Introduce boundary conditions:

The overall stiffness matrix which we get at assembly [K] is, infact, singular. Mathematically this is equivalent to saying that its determinant vanishes, i.e. it is equal to zero and therefore its inverse does not exist. This means that the associated set of simultaneous equations for the unknown displacements cannot be solved. The physical significance of this is that a loaded body or structure is free to experience unlimited rigid body motion unless some support constraints are imposed that will ensure the equilibrium of the loads. These constrains are the boundary conditions.

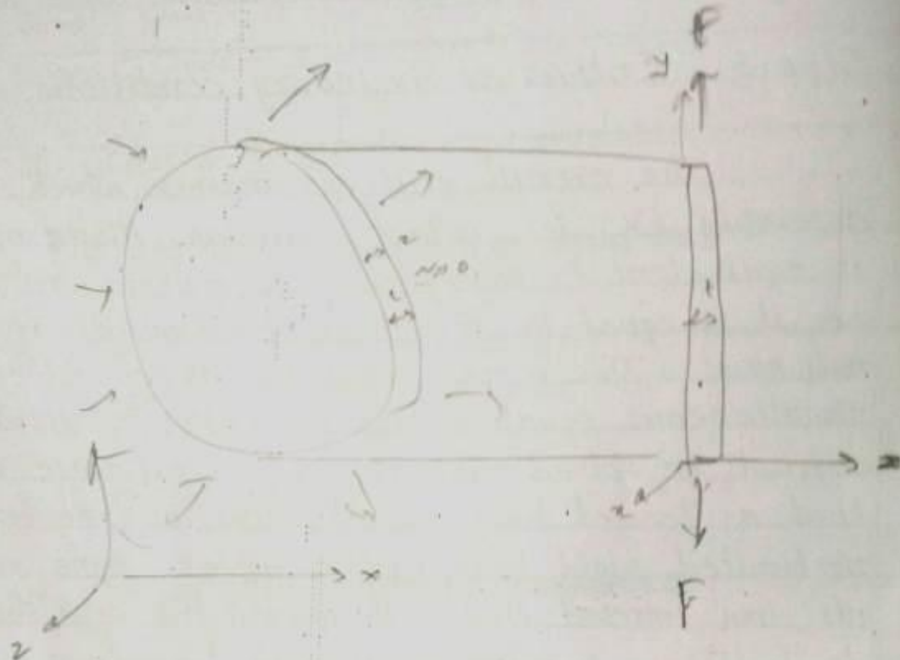
Step-6 Calculate primary unknowns:

Primary unknowns are displacements.

Step-7 Calculate secondary unknowns:

Strains, stresses, moments and shear forces are secondary unknowns.

Once the nodal displacements have been obtained the stresses within the element may be obtained quite simply.



$$\left.\begin{array}{l}\sigma_z = 0\\ \tau_{xz} = 0\\ \tau_{yz} = 0\end{array}\right\} \Rightarrow \begin{array}{l}\varepsilon_z = 0\\ \gamma_{xz} = 0\\ \gamma_{yz} = 0\end{array}$$

$$\left.\begin{array}{l}\varepsilon_z = 0\\ \gamma_{xz} = 0\\ \gamma_{yz} = 0\end{array}\right\} \Rightarrow \begin{array}{l}\sigma_z = 0\\ \tau_{xz} = 0\\ \tau_{yz} = 0\end{array}$$

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \frac{E}{(1-2\mu)(1+\mu)}\begin{bmatrix}1-\mu & \mu & 0\\ \mu & 1-\mu & 0\\ 0 & 0 & \frac{1-2\mu}{2}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}$$

3. Explain 'Plane stress' and 'Plane strain' conditions.

(Book - Bhavi Kutti)
of Zero - 3

Plane stress

A thin planer body subjected to in-plane loading on its edge surface is said to be in plane stress.

A ring press fitted on a shaft as shown in fig. is an example

Here stresses σ_z, τ_{xz} and τ_{yz} are set as zero

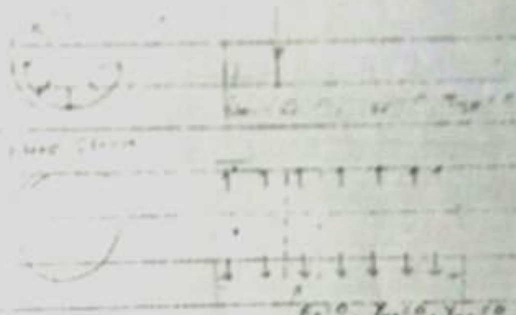
The Hooke's law relations then give us

$$\epsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E}$$

$$\epsilon_y = -\nu \frac{\sigma_x}{E} + \frac{\sigma_y}{E}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E} \tau_{xy}$$

$$\epsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$



The inverse relations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad \frac{E}{1-\mu^2}\begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix}$$

which is used as $\sigma = D\epsilon$

Plane strain

If a long body of uniform cross section is subjected to transverse loading along its length, a small thickness in the loaded area, as shown in fig, can be treated as subjected to plane strain.

problem

(3) The displacement model must include the rig

body displacements of the element.

(4) The displacement models must include the

constant strain states of the element.

5. Explain
in mod

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5. Explain the importance of 'Element Aspect Ratio' in modeling FEM problem.

A characteristic of the discretization that effects a finite element solution is the aspect ratio of the elements.

The element aspect ratio describes the shape of the element in the assemblage.

For two dimensional elements, this parameter is conveniently defined as the ratio of the largest dimension of the element to the smallest dimension.

The optimum aspect ratio at any location within the grid depends largely upon the difference in rate of change of displacements in different directions.

If the displacement vary at about the same rate in each direction, the closer the aspect ratio to unity, the better the quality of the solution.

In other words, generally we must avoid use of long narrow elements.

6. What is Pascal's Triangle?

For two dimensional displacement functions, usin[g] n^{th} order polynomial, we can write

$$u = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2 + \dots + C_m y^n$$

$$v = C_{m+1} + C_{m+2} x + C_{m+3} y + \dots + C_{2m} y^n$$

A convenient way to illustrate the terms in a complete two dimensional polynomial is by means of Pascal Triangle. If the terms are placed in a triangu[lar] array in ascending order, we obtain an arrangement similar to the Pascal Triangle.

	Name	No. of
1	Constant	1
$x \quad y$	Linear	3
$x^2 \quad xy \quad y^2$	Quadratic	6
$x^3 \quad x^2y \quad xy^2 \quad y^3$	Cubic	10
$x^4 \quad x^3y \quad x^2y^2 \quad xy^3 \quad y^4$	Quartic	15
$x^5 \quad x^4y \quad x^3y^2 \quad x^2y^3 \quad xy^4 \quad y^5$	Quintic	21
$x^6 \quad x^5y \quad x^4y^2 \quad x^3y^3 \quad x^2y^4 \quad xy^5 \quad y^6$	Hexadic	28
$x^7 \quad x^6y \quad x^5y^2 \quad x^4y^3 \quad x^3y^4 \quad x^2y^5 \quad xy^6 \quad y^7$	Septic	36

No. of terms $= \dfrac{(n+1)(n+2)}{2!}$

known
displac
the m
Obvious
wherei
finite
Two-d
$u = C_1$
$v = C_r$
where
in the

The coefficients of the polynomial, the C's are known as generalized coordinates or generalized displacement amplitudes.

Thus greater the number of terms in approximation the more closely the exact solution is represented. Obviously this statement does not apply to the case wherein the exact solution is a polynomial of some finite order.

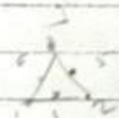
Two-dimensional displacement models

Using the n^{th} order polynomial, we can write

$$u = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2 + \dots + C_m y^n$$

$$v = C_{m+1} + C_{m+2} x + C_{m+3} y + \dots + C_{2m} y^n$$

where u & v denotes the components of displacements in the x and y directions, respectively.



$\Rightarrow$

6 Noded Triangular Element of displacement function

$$u = C_1 + C_2 x + C_3 y + C_4 x^2 + C_5 xy + C_6 y^2$$

$$v = C_7 + C_8 x + C_9 y + C_{10} x^2 + C_{11} xy + C_{12} y^2$$

1. Prepare a table showing
   1. Displacement field
   2. DOF per node
   3. Approximate function
   4. Shape function
   5. Strain displacement function

   for following elements
   1. Linear spring
   2. 1-D Torsional element
   3. Seepage problem
   4. Ideal fluid flow in pipe
   5. Steady state heat problem

Element	Displacement field	DOF per node	Constitutive relationship	Approximate function	Shape function	Strain-Displacement function
Linear Spring	Axial displacement	1	$T = K\delta$ $[k]\{q\} = \{f\}$	$u = a_1 + a_2 x$	$N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$	$[-1/L \;\; 1/L]$
1-D Torsional element	Torsional displacement	1	$T = GJ(d\phi/dx)$ $[k]\{\phi\} = \{T\}$ $k = GJ/L$	$\phi = a_1 + a_2 x$	$N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$	$[-1/L \;\; 1/L]$
Seepage problem	Hydraulic head	1	$V = -k(dh/dx)$ $[k]\{h\} = \{q\}$ $k = k_x A/L$	$h = a_1 + a_2 x$	$N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$	$[-1/L \;\; 1/L]$
Ideal fluid flow in pipe	Velocity potential	1	$V = d\phi/dx$	$\phi = a_1 + a_2 x$	$N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$	$[-1/L \;\; 1/L]$
Steady state heat problem	Temperature	1	$q = -k(dt/dx)$ $[k]\{T\} = \{q\}$ $k = k/L$	$t = a_1 + a_2 x$	$N_1 = 1 - \frac{x}{L}$ $N_2 = \frac{x}{L}$	$[-1/L \;\; 1/L]$

2. For the given spring element (linear)
1. Assemble stiffness matrix
2. Calculate displacement at node 2, 3, 4
3. Calculate reaction at node 1, 5
4. Calculate forces in each spring.

$$[k^1] = \begin{array}{cc} 1 & 2 \\ \left[\begin{matrix} 3 & -3 \\ -3 & 3 \end{matrix}\right] \end{array} \begin{array}{c} 1 \\ 2 \end{array} \qquad [k^2] = \begin{array}{cc} 2 & 3 \\ \left[\begin{matrix} 6 & -6 \\ -6 & 6 \end{matrix}\right] \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$$[k^3] = \begin{array}{cc} 3 & 4 \\ \left[\begin{matrix} 4 & -4 \\ -4 & 4 \end{matrix}\right] \end{array} \begin{array}{c} 3 \\ 4 \end{array} \qquad [k^4] = \begin{array}{cc} 4 & 5 \\ \left[\begin{matrix} 3 & -3 \\ -3 & 3 \end{matrix}\right] \end{array} \begin{array}{c} 4 \\ 5 \end{array}$$

$$[k]\{q\} = \{f\}$$

$$\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \left[\begin{matrix} 3 & -3 & 0 & 0 & 0 \\ -3 & 9 & -6 & 0 & 0 \\ 0 & -6 & 10 & -4 & 0 \\ 0 & 0 & -4 & 7 & -3 \\ 0 & 0 & 0 & -3 & 3 \end{matrix}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 = 4 \\ f_3 = 0 \\ f_4 = -6 \\ f_5 \end{Bmatrix}$$

$$\begin{bmatrix} 9 & -6 & 0 \\ -6 & 10 & -4 \\ 0 & -4 & 7 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 0 \\ -6 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 4/13 \\ -8/39 \\ -38/39 \end{Bmatrix} = \begin{Bmatrix} 0.308 \\ -0.205 \\ -0.974 \end{Bmatrix} m$$

$$V = k\left(\frac{h_1 - h_2}{L}\right)$$

Velocity in upper part

$$V^{1} = 1 \times \left(\frac{20 - 11}{30}\right)$$

$$= 0.3 \text{ cm/s}$$

1. Derive k matrix for two noded and three-
bar element.

Two noded bar element

Referring to figure, assume that the displace-
u varies linearly along the legth of the element.
Then we can assume

$$u = a_1 + a_2 x$$

where a_1 and a_2 are constants.

Since u_1 and u_2 are the displacements at the nodes 1 and 2, we get:

$$u_1 = a_1$$

$$u_2 = a_1 + a_2 \cdot l$$

$$\therefore u_2 = u_1 + a_2 \cdot l$$

$$\therefore a_2 = \frac{u_2 - u_1}{l}$$

$$\therefore u = u_1 + \frac{u_2 - u_1}{l} x$$

$$\therefore = u_1 + u_2 \frac{x}{l} - u_1 \frac{x}{l}$$

$$= \left(1 - \frac{x}{l}\right) u_1 + \left(\frac{x}{l}\right) u_2$$

$$u = N_1 u_1 + N_2 u_2$$

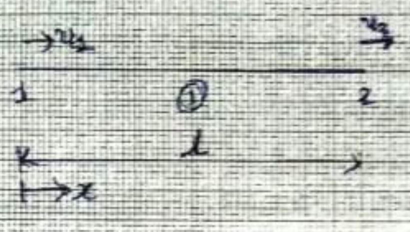
$$\therefore u = \underbrace{[N_1 \; N_2]}_{\text{shape function}} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore u = [N]\{q^e\}$$

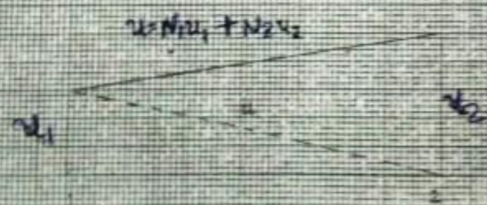
$N_1 = 1 - x/l$

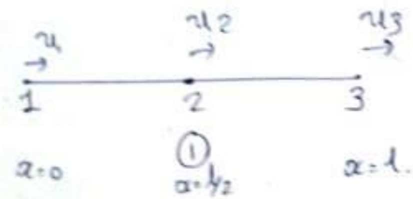
$N_2 = x/l$

$B = \frac{d}{dx}[N]$ $\quad$ $\sigma = E\epsilon$


1
2
l
x


N1
N2
1
2


u=N1u1 + N2u2



u1
u2
u3
1
2
3
1
x=0
x=1/2
x=1.

Now the strain in the element is given by

$$\epsilon_x = \frac{du}{dx}$$

$$= \frac{d}{dx}[N]\{q^e\}$$

$$[\epsilon_x] = [B]\{q^e\}$$

where $[B] = \frac{d}{dx}[N] = \frac{1}{l}[-1 \;\; 1]$

is the element strain-displacement matrix

By Hooke's law

$$\sigma = E\,\epsilon$$

$$\therefore [\sigma_x] = [D][\epsilon_x]$$

$$[\sigma_x] = [D][B]\{q^e\}$$

[D] is called constitutive matrix

For one-dimensional deformation, the strain energy is given by

$$U = \frac{1}{2}\int_V \sigma_x \, \epsilon_x \, dV$$

$$\therefore U = \frac{1}{2}\int_0^l [\sigma_x]^T [\epsilon_x] A\,dx$$

$$= \frac{1}{2}\int_0^l \left[[D][B]\{q^e\}\right]^T [B]\{q^e\} A\,dx$$

$$= \frac{1}{2}\int_0^l \{q^e\}^T [B]^T [D]^T [B]\{q^e\} A\,dx$$

$$\therefore U = 1/2 \{q^e\}^T [k^e]\{q^e\}$$

where

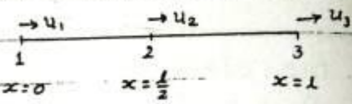
$$[k^e] = \int_0^l [B]^T [D][B] A\,dx \qquad (\because D = E)$$

$$= AE\int_0^l \frac{1}{l}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \frac{1}{l}[-1 \;\; 1]\,dx$$

$$\therefore [k^e] = \frac{AE}{l^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \int_0^l dx$$

$$\therefore [k^e] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Three noded bar element



$$u = a_1 + a_2 x + a_3 x^2$$

$$u(0) = u_1 = a_1$$

$$u(l/2) = u_2 = a_1 + a_2(l/2) + a_3(l/2)^2$$

$$u_2 = u_1 + a_2\, l/2 + a_3\, l^2/4$$

$$u(l) = u_3 = a_1 + a_2 l + a_3 l^2$$

$$u_3 = u_1 + a_2 l + a_3 l^2$$

eq. ② × 4 $\quad 4u_2 = 4u_1 + 2a_2 l + a_3 l^2$

eq. ③ × (−1) $\quad -u_3 = -u_1 - a_2 l - a_3 l^2$

$$\therefore 4u_2 - u_3 = 3u_1 + a_2 l$$

$$\therefore a_2 = \frac{1}{l}(-3u_1 + 4u_2 - u_3)$$

eq. ② × 2 $\quad 2u_2 = 2u_1 + a_2 l + a_3 l^2/2$

eq. ③ × (−1) $\quad -u_3 = -u_1 - a_2 l - a_3 l^2$

$$\therefore 2u_2 - u_3 = u_1 - a_3 l^2/2$$

$$\therefore a_3 = \frac{2}{l^2}(u_1 - 2u_2 + u_3)$$

$$\therefore u = u_1 + \frac{1}{l}(-3u_1 + 4u_2 - u_3)x + \frac{2}{l^2}(u_1 - 2u_2 + u_3)x$$

$$= u_1 - \frac{3x}{l}u_1 + \frac{4x}{l}u_2 - \frac{x}{l}u_3 + \frac{2x^2}{l^2}u_1 - \frac{4x^2}{l^2}u_2 + $$

$$u = (1 - 3x/l + 2x^2/l^2)u_1 + (4x/l - 4x^2/l^2)u_2 + (-x/l + 2x$$

$$u = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$\left.\begin{aligned} N_1 &= 1 - 3x/l + 2x^2/l^2 \\ N_2 &= 4x/l - 4x^2/l^2 \\ N_3 &= -x/l + 2x^2/l^2 \end{aligned}\right] \checkmark$$

$$[B] = \frac{d}{dx}[N]$$

$$= \frac{d}{dx}[N_1 \quad N_2 \quad N_3] = [N_1' \quad N_2' \quad N_3']$$

$$[B] = \left[(-3/l + 4x/l^2) \quad (4/l - 8x/l^2) \quad (-1/l + 4x/l^2)\right]$$

$$[k^e] = \int_0^l [B]^T [D][B] A \, dx$$

$$= \int_0^l \begin{bmatrix} (-3/l + 4x/l^2) \\ (4/l - 8x/l^2) \\ (-1/l + 4x/l^2) \end{bmatrix} E \left[(-3/l + 4x/l^2)\ (4/l - 8x/l^2)\ (-1/l + 4x/l^2)\right] A \, d$$

$$= \int_0^l \begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} E \begin{bmatrix} N_1' & N_2' & N_3' \end{bmatrix} A \, dx$$

$$= \int_0^l AE \begin{bmatrix} N_1'N_1' & N_1'N_2' & N_1'N_3' \\ N_2'N_1' & N_2'N_2' & N_2'N_3' \\ N_3'N_1' & N_3'N_2' & N_3'N_3' \end{bmatrix} dx = \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$$\int_0^l N_1'N_1' \, dx = \int_0^l (-3/l + 4x/l^2)(-3/l + 4x/l^2) \, dx$$

$$= \int_0^l 1/l^4 (-3l + 4x)(-3l + 4x) \, dx$$

$$= \int_0^l 1/l^4 (9l^2 - 24lx + 16x^2) \, dx$$

$$= \frac{1}{l^4}\left[9l^2x - 24l\frac{x^2}{2} + 16\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(9l^3 - 12l^3 + 16l^3/3\right)$$

$$\int_0^l N_1' N_1' \, dx = 7/3l$$

$$\int_0^l N_2' N_1' \, dx = \int_0^l \left(4/l - 8x/l^2\right)\left(-3/l + 4x/l^2\right) dx$$

$$= \int_0^l \frac{1}{l^4}(4l - 8x)(-3l + 4x)\, dx$$

$$= \int_0^l \frac{1}{l^4}\left(-12l^2 + 40lx - 32x^2\right) dx$$

$$= \frac{1}{l^4}\left[-12l^2x + 40l\frac{x^2}{2} - 32\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(-12l^3 + 20l^3 - 32l^3/3\right)$$

$$\int_0^l N_2' N_1' \, dx = -8/3l$$

$$\int_0^l N_3' N_1' \, dx = \int_0^l \left(-1/l + 4x/l^2\right)\left(-3/l + 4x/l^2\right) dx$$

$$= \int_0^l \frac{1}{l^4}(-l + 4x)(-3l + 4x)\, dx$$

$$= \int_0^l \frac{1}{l^4}\left(3l^2 - 16lx + 16x^2\right) dx$$

$$= \frac{1}{l^4}\left[3l^2x - 16l\frac{x^2}{2} + 16\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(3l^3 - 8l^3 + 16l^3/3\right)$$

$$\int_0^l N_3' N_1' \, dx = 1/3l$$

$$\int_0^l N_1' N_2' \, dx$$

$$\int_0^l N_2' N_2' \, dx$$

$$\int_0^l N_3' N_2' \, dx$$

$$\int_0^l N_1' N_3' \, dx$$

$$\int_0^l N_2' N_3' \, dx$$

$$= \frac{1}{l^4}\left[9l^2x - 24l\frac{x^2}{2} + 16\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(9l^3 - 12l^3 + 16\frac{l^3}{3}\right)$$

$$\int_0^l N_1' N_1' \, dx = \frac{7}{3l}$$

$$\int_0^l N_2' N_1' \, dx = \int_0^l \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right) dx$$

$$= \int_0^l \frac{1}{l^4}(4l - 8x)(-3l + 4x) \, dx$$

$$= \int_0^l \frac{1}{l^4}(-12l^2 + 40lx - 32x^2) \, dx$$

$$= \frac{1}{l^4}\left[-12l^2x + 40l\frac{x^2}{2} - 32\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(-12l^3 + 20l^3 - 32\frac{l^3}{3}\right)$$

$$\int_0^l N_2' N_1' \, dx = -\frac{8}{3l}$$

$$\int_0^l N_3' N_1' \, dx = \int_0^l \left(-\frac{1}{l} + \frac{4x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right) dx$$

$$= \int_0^l \frac{1}{l^4}(-l + 4x)(-3l + 4x) \, dx$$

$$= \int_0^l \frac{1}{l^4}(3l^2 - 16lx + 16x^2) \, dx$$

$$= \frac{1}{l^4}\left[3l^2x - 16l\frac{x^2}{2} + 16\frac{x^3}{3}\right]_0^l$$

$$= \frac{1}{l^4}\left(3l^3 - 8l^3 + 16\frac{l^3}{3}\right)$$

$$\int_0^l N_3' N_1' \, dx = \frac{1}{3l}$$

$$\int_0^l N_2' N_2' \, dx =$$

$$\int_0^l N_2 N_2 \, dx$$

$$\int_0^l N_3 N_2' \, dx$$

$$\int_0^l N_3' N_2' \, dx$$

$$\int_0^l N_3' N_3' \, dx$$

$$\int_0^l N_2' N_2'\, dx = \int_0^l (4/l - 8x/l^2)(4/l - 8x/l^2)\, dx$$

$$= \int_0^l 1/l^4 \,(4l - 8x)^2\, dx$$

$$= \int_0^l 1/l^4 \,(16l^2 - 64\,lx + 64x^2)\, dx$$

$$= 1/l^4 \left[16l^2x - 64\,lx^2/2 + 64x^3/3\right]_0^l$$

$$= 1/l^4 \,(16\,l^3 - 32\,l^3 + 64\,l^3/3)$$

$$\int_0^l N_2' N_2'\, dx = 16/3l$$

$$\int_0^l N_3' N_2'\, dx = \int_0^l (-1/l + 4x/l^2)(4/l - 8x/l^2)\, dx$$

$$= \int_0^l 1/l^4 \,(-l + 4x)(4l - 8x)\, dx$$

$$= \int_0^l 1/l^4 \,(-4l^2 + 24\,lx - 32x^2)\, dx$$

$$= 1/l^4 \left[-4l^2x + 24\,lx^2/2 - 32\,x^3/3\right]_0^l$$

$$= 1/l^4 \,(-4l^3 + 12\,l^3 - 32\,l^3/3)$$

$$\int_0^l N_3' N_2'\, dx = -8/3l$$

$$\int_0^l N_3' N_3'\, dx = \int_0^l (-1/l + 4x/l^2)(-1/l + 4x/l^2)\, dx$$

$$= \int_0^l 1/l^4 \,(-l + 4x)^2$$

$$= \int_0^l 1/l^4 \,(l^2 - 8lx + 16x^2)\, dx$$

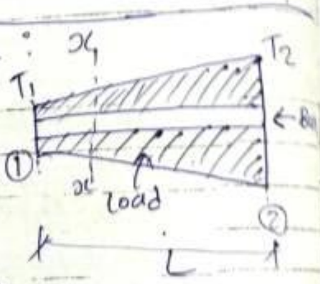
$$= \frac{1}{l^4}\left[l^2 x - 8 l \frac{x^2}{2} + 16 \frac{x^3}{3} \right]_0^l$$

$$= \frac{1}{l^4}\left(l^3 - 4 l^3 + \frac{16 l^3}{3} \right)$$

$$\int_0^l N_i' N_i' \, dx = \frac{7}{3l}$$

$$[k^e] = \frac{AE}{3l} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

two noded - UVL along length L :



→ Shape function $n[N] = \left[\left(1-\frac{x}{L}\right) \left(\frac{x}{L}\right) \right]$

→ loading = UVL along length having
@ section x-x

T_1 intensity @ node (1) &

T_2 intensity @ node (2).

$$\therefore \text{load} = T_1 + \left(\frac{T_2 - T_1}{L} \right) \cdot x$$

→ now, load vector $= \int_0^l \left([N]^T \cdot \underset{@sec. x-x}{\text{load}} \right) \times dx$

$$= \int_0^l \begin{bmatrix} 1 - x/L \\ x/L \end{bmatrix} \cdot \left(T_1 + \left(\frac{T_2 - T_1}{L} \right) \times x \right) \cdot dx$$

simplifying this we get,

load vector

$$[T^e] = \begin{bmatrix} T_1 * \left(\frac{l}{3}\right) + T_2 \left(\frac{l}{6}\right) \\ T_1 * \left(\frac{l}{6}\right) + T_2 \left(\frac{l}{3}\right) \end{bmatrix}$$

Derive load

subjected to

The element

$\int_0 u^T T \, dx$

where

$\{T^e\}$ is t

$\{T^e\} = \int [$

$\{T^e\} = \int \{$

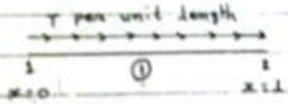
$\int N_1 \, dx =$

$\int N_2 \, dx$

$\therefore \{T^e\} =$

Derive load vector for two noded bar element subjected to uniformly distributed axial force.

The element traction force



$$\int_0^l u^T T\,dx = \int_0^l [[N]\{q^e\}]^T T\,dx$$

$$= \int_0^l [N]^T [q^e]^T T\,dx$$

$$= \int_0^l \{q^e\}^T [N]^T T\,dx$$

$$= \{q^e\}^T \{T^e\}$$

where

$\{T^e\}$ is the element traction force vector.

$$\therefore \{T^e\} = \int_0^l [N]^T T\,dx$$

$$\{T^e\} = \int_0^l \begin{Bmatrix} N_1 \\ N_2 \end{Bmatrix} T\,dx$$

$$\int_0^l N_1\,dx = \int_0^l (1 - x/l)\,dx$$

$$= \left[x - x^2/2l\right]_0^l = l/2$$

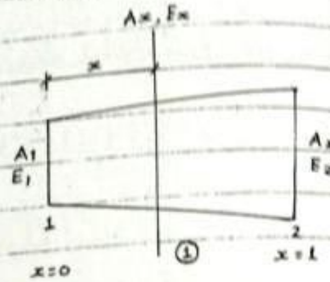
$$\int_0^l N_2\,dx = \int_0^l x/l\,dx$$

$$= \left[x^2/2l\right]_0^l = l/2$$

$$\therefore \{T^e\} = \frac{Tl}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

3. Derive k matrix for 2-noded bar element va[ry]
from A_1 to A_2 and modulus of elasticity [f]
E_1 to E_2.

Refferring to figure



$A_x = A_1 + (A_2 - A_1)\,x/l$

$E_x = E_1 + (E_2 - E_1)\,x/l$

$[k^e] = \int_l [B]^T [D] [B]\, A\, dx$

$= \int_0^l \frac{1}{l}\begin{bmatrix}-1\\1\end{bmatrix}\left(E_1 + (E_2 - E_1)x/l\right)\frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix}\left(A_1 + (A_2 - A_1\right.$

$= \int_0^l \frac{1}{l^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\left(A_1E_1 + \frac{A_2 - A_1}{l}E_1 x + A_1\frac{E_2 - E_1}{l}x + \frac{(A_2 - A_1}{}\right.$

$= \frac{1}{l^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\left[A_1E_1 x + \frac{A_2 - A_1}{l}E_1\frac{x^2}{2} + A_1\frac{E_2 - E_1}{l}\cdot\frac{x^2}{2} + \frac{(A_2 - A_1)(}{l^2}\right.$

$= \frac{1}{l^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\left(A_1E_1 l + \frac{A_2 - A_1}{2}E_1 l + A_1\frac{E_2 - E_1}{2}l + \frac{(A_2 - A_1)(E_2}{3}\right.$

$= \frac{1}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\left(A_1E_1 + \frac{A_2E_1}{2} - \frac{A_1E_1}{2} + \frac{A_1E_2}{2} - \frac{A_1E_1}{2}\right.$

$+ \frac{A_2E_2}{3} - \frac{A_2E_1}{3} - \frac{A_1E_2}{3} + \frac{A_1E_1}{3}$

$= \frac{1}{6l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\left(2A_1E_1 + 2A_2E_2 + A_1E_2 + A_2E_1\right)$

$[k^e] = \frac{1}{6l}\left(A_1(2E_1 + E_2) + A_2(E_1 + 2E_2)\right)\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

* E = const.

* E = const.

$$= \frac{E}{\cancel{?}} \int_0^l \frac{1}{l}\begin{bmatrix}-1\\1\end{bmatrix} \cdot \frac{1}{l}\begin{bmatrix}-1 & 1\end{bmatrix} \left(A_1 + (A_2 - A_1)\frac{x}{l}\right) dx$$

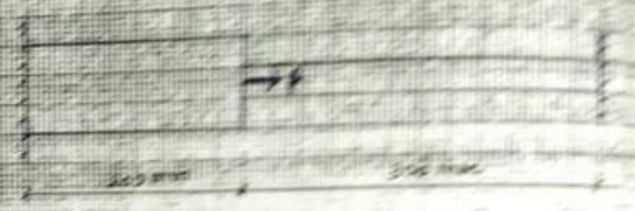
$$= \frac{E}{l^2} \int_0^l \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left(A_1 + (A_2 - A_1)\frac{x}{l}\right) dx$$

$$= \frac{E}{l^2} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left[A_1 l + (A_2 - A_1)\frac{l^2}{2l}\right]$$

$$= \frac{lE}{l^2} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left[A_1 + \frac{(A_2 - A_1)}{2}\right]$$

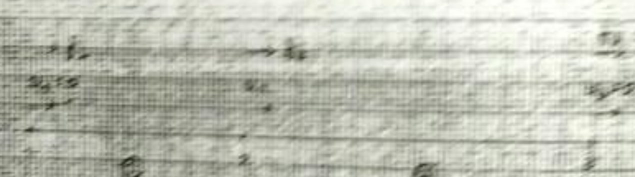
$$= \frac{E}{l} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left[\frac{A_1}{2} + \frac{A_2}{2}\right]$$

$$= \frac{E}{2l} \begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} \left[A_1 + A_2\right].$$



$A_1 = 2400\ mm^2$ $A_2 = 600\ mm^2$

$E_1 = 70 \times 10^3\ N/mm^2$ $E_2 = 200 \times 10^3\ N/mm^2$



Consider th

$P = 100 \times 10^3$

(i) Determ

(ii) Determ

(iii) Determi

$P = 100 \times 10^3$ N

$l_1 = 200$ mm

$A_1 = 2400$ m

$E_1 = 70 \times 10^3$

$= 70 \times 10^3$

$[k^1] = \frac{AE}{l}$ [

$[k^1] = 2400$

$[k^2] = 600 \times$

$[k]\{q\} = [F]$

10^6 [0.84 − ; −0.84 ; 0 −]

1.24×10^6

$u_2 = \frac{5}{62}$

Consider the bar shown in figure. An axial load $P = 100 \times 10^3$ N.

(i) Determine the nodal displacements.

(ii) Determine the stress in each material.

(iii) Determine the reaction forces.

$P = 100 \times 10^3$ N

$l_1 = 200$ mm $\qquad l_2 = 300$ mm

$A_1 = 2400$ mm^2 $\qquad A_2 = 600$ mm^2

$E_1 = 70 \times 10^9$ N/m^2 $\qquad E_2 = 200 \times 10^9$ N/m^2

$= 70 \times 10^3$ N/mm^2 $\qquad = 200 \times 10^3$ N/mm^2

$$[k^e] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^1] = \frac{2400 \times 70 \times 10^3}{200}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{cc} 1 & 2 \\ \end{array}$$
$$\begin{array}{c|c} \begin{bmatrix} 0.84 & -0.84 \\ -0.84 & 0.84 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array}$$

$$[k^2] = \frac{600 \times 200 \times 10^3}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^6 \begin{array}{cc} 2 & 3 \\ \end{array}$$
$$\begin{array}{c|c} \begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} & \begin{array}{c} 2 \\ 3 \end{array} \end{array}$$

$[k]\{q\} = \{F\}$

$$10^6 \begin{array}{ccc|c} 1 & 2 & 3 & \\ 0.84 & -0.84 & 0 & 1 \\ -0.84 & 1.24 & -0.4 & 2 \\ 0 & -0.4 & 0.4 & 3 \end{array} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 = 100 \times 10^3 \\ F_3 \end{Bmatrix}$$

$1.24 \times 10^6 \; u_2 = 100 \times 10^3$

$\therefore u_2 = \frac{5}{62}$ mm $= 0.0806$ mm

$-0.84 \times 10^6 \times 5/62 = f_1$

$\therefore f_1 = -67.74 \times 10^3$ N

$-0.4 \times 10^6 \times 5/62 = f_3$

$\therefore f_3 = -32.26 \times 10^3$ N

$$\sigma = EBq = E \cdot 1/l \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

element-1

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{200} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 5/62 \end{Bmatrix} =$$

$$= \frac{70 \times 10^3}{200} \times \frac{5}{62}$$

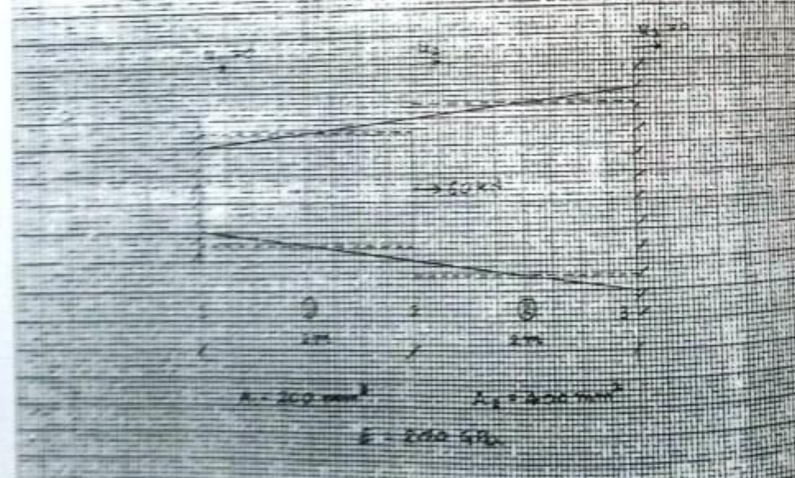
$$= 28.23 \text{ N/mm}^2$$

element-2

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{300} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 5/62 \\ 0 \end{Bmatrix}$$

$$= \frac{200 \times 10^3}{300} \times \left(-\frac{5}{62}\right)$$

$$= -53.76 \text{ N/mm}^2$$



(i) 2-noded - 2 elements

For a given bar, shou

displacements and el

(i) 2-noded - 2-el

(ii) 2-noded - 4-e

(iii) 3-noded - 2-e

Shape functions for

$[N] = [\,1-3s+2s^2$

where $s = x/l$

Comment on the res

(i) 2-noded - 2-eleme

$[k^1] = \dfrac{200 \times 2\times10^5}{2000}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$[k^2] = \dfrac{400 \times 2\times10^5}{2000}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$[k]\{q\} = \{f\}$

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ 10^4 & \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 = \\ u_2 \\ u_3 \end{Bmatrix}$$

$6\times10^4\ u_2 = 60\times10^3$

$\therefore u_2 = 1$ mm

$-2\times10^4 \times 1 = f_1$

$\therefore f_1 = -20\times10^3$ N

$-4\times10^4\times1 = f_3$

$\therefore f_3 = -40\times10^3$ N

For a given bar, shown in figure, Find nodal displacements and element stresses it.

(i) 2-noded - 2-elements

(ii) 2-noded - 4-elements

(iii) 3-noded - 2-elements

Shape functions for 3-noded bar

$[N] = [\ 1-3s+2s^2 \quad 4s(1-s) \quad s(2s-1)\ ]$

where $s = x/l$

Comment on the results.

(i) 2-noded - 2-elements

$$[k^1] = \frac{200 \times 2 \times 10^5}{2000}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ & \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \end{array}$$

$$[k^2] = \frac{400 \times 2 \times 10^5}{2000}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 10^4 \begin{array}{cc} & \begin{array}{cc} 2 & 3 \end{array} \\ & \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix} \end{array} \begin{array}{c} 2 \\ 3 \end{array}$$

$[k]\{q\} = \{f\}$

$$10^4 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 2 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 = 0 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 = 60 \times 10^3 \\ f_3 \end{Bmatrix}$$

$6 \times 10^4\ u_2 = 60 \times 10^3$

$\therefore u_2 = 1$ mm

$-2 \times 10^4 \times 1 = f_1$

$\therefore f_1 = -20 \times 10^3$ N

$-4 \times 10^4 \times 1 = f_3$

$\therefore f_3 = -40 \times 10^3$ N

$\sigma = E\, B\, q$

element - 1

$$\sigma_1 = 2\times10^5 \times \frac{1}{2000}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix}$$

$$= \frac{2\times10^5}{2000}\times 1$$

$$= 100\ \text{N/mm}^2$$

element - 2

$$\sigma_2 = 2\times10^5 \cdot \frac{1}{2000}\begin{bmatrix}-1 & 1\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix}$$

$$= \frac{2\times10^5}{2000}\times(-1)$$

$$= -100\ \text{N/mm}^2$$

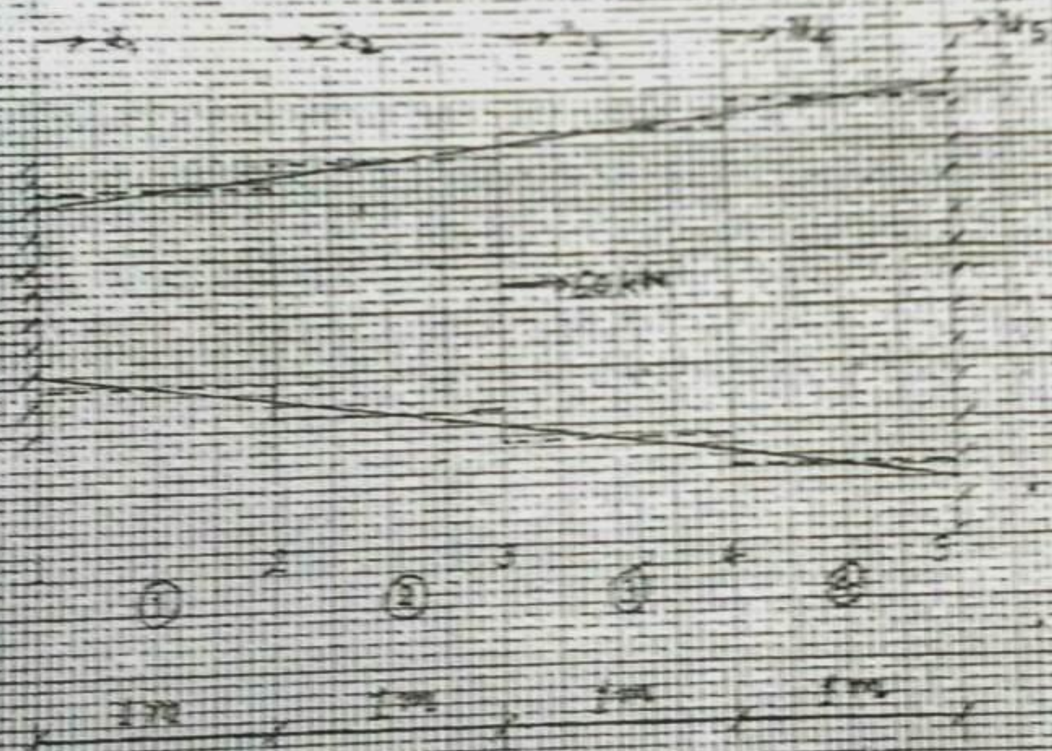
(ii) 2-noded - 4-elements

$$[k^1] = \frac{150\times2\times10^5}{1000}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^4\ \begin{array}{cc} 1 & 2 \\ \begin{bmatrix}3 & -3\\-3 & 3\end{bmatrix} \end{array}\begin{array}{c} 1\\2\end{array}$$

$$[k^2] = \frac{250\times2\times10^5}{1000}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^4\ \begin{array}{cc} 2 & 3 \\ \begin{bmatrix}5 & -5\\-5 & 5\end{bmatrix} \end{array}\begin{array}{c} 2\\3\end{array}$$

$$[k^3] = \frac{350\times2\times10^5}{1000}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^4\ \begin{array}{cc} 3 & 4 \\ \begin{bmatrix}7 & -7\\-7 & 7\end{bmatrix} \end{array}\begin{array}{c} 3\\4\end{array}$$

$$[k^4] = \frac{450\times2\times10^5}{1000}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = 10^4\ \begin{array}{cc} 4 & 5 \\ \begin{bmatrix}9 & -9\\-9 & 9\end{bmatrix} \end{array}\begin{array}{c} 4\\5\end{array}$$



$A_1 = 150\ mm^2$ $A_3 = 350\ mm^2$

$A_2 = 250\ mm^2$ $A_4 = 450\ mm^2$

$E = 200\ GPa$

(ii) 2-noded - 4 elements

$[k]\{q\} = \{f\}$

$$
10^4 \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 3 & -3 & 0 & 0 & 0 \\ -3 & 8 & -5 & 0 & 0 \\ 0 & -5 & 12 & -7 & 0 \\ 0 & 0 & -7 & 16 & -9 \\ 0 & 0 & 0 & -9 & 9 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 = 0 \\ f_3 = 60 \times 10^3 \\ f_4 = 0 \\ f_5 \end{Bmatrix}
$$

$$
10^4 \begin{bmatrix} 8 & -5 & 0 \\ -5 & 12 & -7 \\ 0 & -7 & 16 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix}
$$

$$
\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 20/31 \\ 32/31 \\ 14/31 \end{Bmatrix} = \begin{Bmatrix} 0.6452 \\ 1.0323 \\ 0.4516 \end{Bmatrix} \text{ mm}
$$

$\sigma = E\,B\,q$

element-1

$$
\sigma_1 = 2 \times 10^5 \times \frac{1}{1000} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 20/31 \end{Bmatrix} = 129.03 \text{ N/mm}^2
$$

element-2

$$
\sigma_2 = 2 \times 10^5 \times \frac{1}{1000} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 20/31 \\ 32/31 \end{Bmatrix} = 77.42 \text{ N/mm}^2
$$

element-3

$$
\sigma_3 = 2 \times 10^5 \times \frac{1}{1000} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 32/31 \\ 14/31 \end{Bmatrix} = -116.13 \text{ N/mm}^2
$$

element-4

$$
\sigma_4 = 2 \times 10^5 \times \frac{1}{1000} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{Bmatrix} 14/31 \\ 0 \end{Bmatrix} = -90.32 \text{ N/mm}^2
$$

3-noded - 2-elements

$$[k^1] = \frac{200 \times 2 \times 10^5}{3 \times 2000} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} = \frac{10^4}{3} \begin{array}{c} \begin{matrix} 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 14 & -16 & 2 \\ -16 & 32 & -16 \\ 2 & -16 & 14 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$[k^2] = \frac{400 \times 2 \times 10^5}{3 \times 2000} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} = \frac{10^4}{3} \begin{array}{c} \begin{matrix} 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 28 & -32 & 4 \\ -32 & 64 & -32 \\ 4 & -32 & 28 \end{bmatrix} \end{array} \begin{matrix} 3 \\ 4 \\ 5 \end{matrix}$$

$[k]\{q\} = \{f\}$

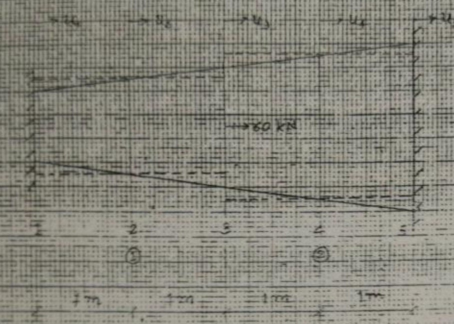
$$\frac{10^4}{3} \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{bmatrix} 14 & -16 & 2 & 0 & 0 \\ -16 & 32 & -16 & 0 & 0 \\ 2 & -16 & 42 & -32 & 4 \\ 0 & 0 & -32 & 64 & -32 \\ 0 & 0 & 4 & -32 & 28 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} \begin{Bmatrix} u_1 = 0 \\ u_2 \\ u_3 \\ u_4 \\ u_5 = 0 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 = 0 \\ f_3 = 60 \times 10^3 \\ f_4 = 0 \\ f_5 \end{Bmatrix}$$

$$\frac{10^4}{3} \begin{bmatrix} 32 & -16 & 0 \\ -16 & 42 & -32 \\ 0 & -32 & 64 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60 \times 10^3 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1 \\ 0.5 \end{Bmatrix} \text{ mm}$$

$$\frac{10^4}{3} \begin{bmatrix} -16 & 2 & 0 \\ 0 & 4 & -32 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 1 \\ 0.5 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_5 \end{Bmatrix}$$

$$\begin{Bmatrix} f_1 \\ f_5 \end{Bmatrix} = 10^3 \begin{Bmatrix} -20 \\ -40 \end{Bmatrix} \text{ N}$$



$A_1 = 200\ mm^2$ $\quad$ $A_2 = 400\ mm^2$

$E = 200\ GPa$

(iii) 3-noded = 2-elements

$$[B] = \frac{d}{dx}[N] = \frac{d}{dx}\cdot[N_1 \;\; N_2 \;\; N_3]$$

$$= [N_1' \;\; N_2' \;\; N_3']$$

$$N_1' = \frac{d}{dx}\left(1 - 3x/l + 2x^2/l^2\right) = -3/l + 4x/l^2 = 1/l\,(4x/l - 3)$$

$$N_2' = \frac{d}{dx}\left(4x/l - 4x^2/l^2\right) = 4/l - 8x/l^2 = 1/l\,(4 - 8x/l)$$

$$N_3' = \frac{d}{dx}\left(2x^2/l^2 - x/l\right) = 4x/l^2 - 1/l = 1/l\,(4x/l - 1)$$

	$x = 0$	$x = 1000$ mm	$x = 2000$ mm
N_1'	$-3/2000$	$-1/2000$	$1/2000$
N_2'	$4/2000$	0	$-4/2000$
N_3'	$-1/2000$	$1/2000$	$3/2000$

$\sigma = E B q$

element-1

node-1 $\quad \sigma_{11} = 2\times10^5 \times \frac{1}{2000}\begin{bmatrix}-3 & 4 & -1\end{bmatrix}\begin{Bmatrix}0\\0.5\\1\end{Bmatrix} = 100\ N/mm^2$

node-2 $\quad \sigma_{12} = 2\times10^5 \times \frac{1}{2000}\begin{bmatrix}-1 & 0 & 1\end{bmatrix}\begin{Bmatrix}0\\0.5\\1\end{Bmatrix} = 100\ N/mm^2$

node-3 $\quad \sigma_{13} = 2\times10^5 \times \frac{1}{2000}\begin{bmatrix}1 & -4 & 3\end{bmatrix}\begin{Bmatrix}0\\0.5\\1\end{Bmatrix} = 100\ N/mm^2$

element-2

node-3 $\quad \sigma_{23} = 2\times 10^5 \times \frac{1}{2000} \begin{bmatrix} -3 & 4 & -1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.5 \\ 0 \end{Bmatrix} = -100$ N/m

node-4 $\quad \sigma_{24} = 2\times 10^5 \times \frac{1}{2000} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.5 \\ 0 \end{Bmatrix} = -100$ N/m

node-5 $\quad \sigma_{25} = 2\times 10^5 \times \frac{1}{2000} \begin{bmatrix} 1 & -4 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 0.5 \\ 0 \end{Bmatrix} = -100$ N/m

Derive sh

having co

as constan

$u = a_1 + a$

$\{u\} = [1$

$\therefore \{u\} = [\phi]$

where

$[\phi] = [$

$u(0) = u_1 =$

$u(0.4l) = u_2 =$

$u(l) = u_3$

$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\{q\} = [A]$

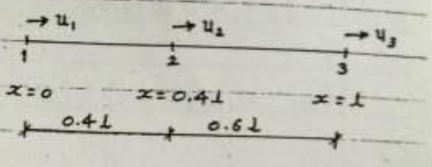
where

$[A] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore \{\alpha\} = [A]^{-1}$

$\therefore \{u\} = [\phi]$

Derive shape functions of 3-noded bar element, having coordinate 0, 0.4l, l. Assume axial rigidity as constant.



$$u = a_1 + a_2 x + a_3 x^2$$

$$\{u\} = [1 \quad x \quad x^2] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\therefore \{u\} = [\phi]\{\alpha\}$$

where

$$[\phi] = [1 \quad x \quad x^2] \qquad \{\alpha\} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$u(0) = u_1 = a_1$$
$$u(0.4l) = u_2 = a_1 + 0.4l\, a_2 + 0.16 l^2 a_3$$
$$u(l) = u_3 = a_1 + l\, a_2 + l^2 a_3$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.4l & 0.16l^2 \\ 1 & l & l^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{q\} = [A]\{\alpha\}$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.4l & 0.16l^2 \\ 1 & l & l^2 \end{bmatrix}$$

$$\therefore \{\alpha\} = [A]^{-1}\{q\}$$

$$\therefore \{u\} = [\phi][A]^{-1}\{q\}$$

$$\{u\} = [N]\{q\}$$

$$\therefore [N] = [\Phi][A]^{-1}$$

$$[N_1 \quad N_2 \quad N_3] = [1 \quad x \quad x^2] \frac{1}{0.24 l^3} \begin{bmatrix} 0.24 l^3 & 0 & 0 \\ -0.84 l^2 & l^2 & -0.16 l^2 \\ 0.6 l & -l & 0.4 l \end{bmatrix}$$

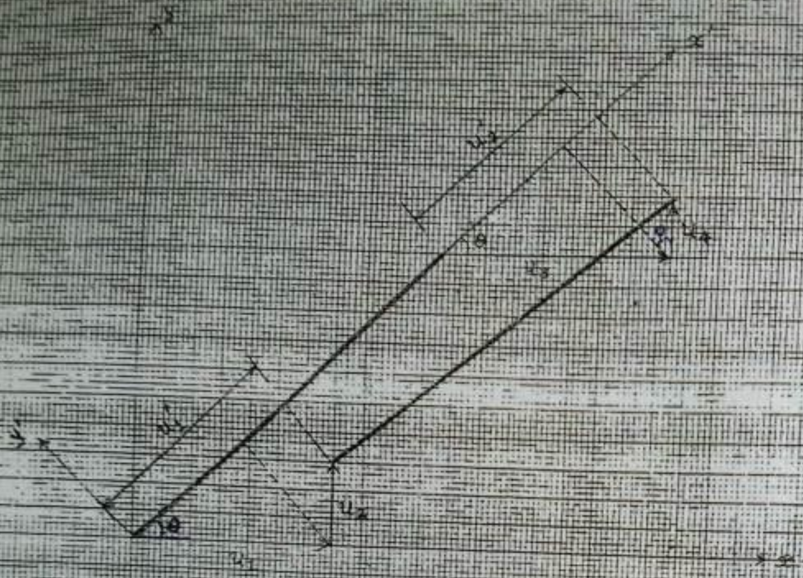
$$= [1 \quad x \quad x^2] \begin{bmatrix} 1 & 0 & 0 \\ -7/2l & 25/6l & -2/3l \\ 5/2l^2 & -25/6l^2 & 5/3l^2 \end{bmatrix}$$

$$N_3] = \left[1 - \frac{7x}{2l} + \frac{5x^2}{2l^2} \quad \frac{25x}{6l} - \frac{25x^2}{6l^2} \quad -\frac{2x}{3l} + \frac{5x^2}{3l^2}\right]$$

$$\therefore N_1 = 1 - \frac{7x}{2l} + \frac{5x^2}{2l^2}$$

$$N_2 = \frac{25x}{6l} - \frac{25x^2}{6l^2}$$

$$N_3 = -\frac{2x}{3l} + \frac{5x^2}{3l^2}$$



Derive

and th

ξ as v

Two nod

$u = a_1 +$

$u = [1$

$\xi = -1$

$\xi = 1$

$\therefore$

$\therefore \; u = [1$

$= \frac{1}{2} [$

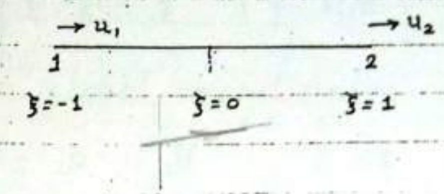
$u = [N_1$

$\therefore \; N_1 = \frac{1-\xi}{2}$

$N_2 = \frac{1+\xi}{2}$

3. Derive the expression for shape function for two node and three noded bar element taking natural coordinat ξ as varying from -1 to 1

Two noded bar element



$$u = a_1 + a_2 \xi$$

$$u = \begin{bmatrix} 1 & \xi \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$\xi = -1 \qquad u_1 = a_1 - a_2$

$\xi = 1 \qquad u_2 = a_1 + a_2$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

$$\therefore \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad u = \begin{bmatrix} 1 & \xi \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1-\xi & 1+\xi \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

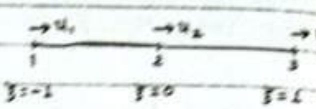
$$u = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$\therefore \quad N_1 = \frac{1-\xi}{2}$$

$$N_2 = \frac{1+\xi}{2}$$

Three noded bar element

$u = a_1 + a_2\xi + a_3\xi^2$



$$u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$\xi = -1 \qquad u_1 = a_1 - a_2 + a_3$

$\xi = 0 \qquad u_2 = a_1$

$\xi = 1 \qquad u_3 = a_1 + a_2 + a_3$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\therefore \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$\therefore u = \begin{bmatrix} 1 & \xi & \xi^2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$u = \begin{bmatrix} -\xi/2 + \xi^2/2 & 1-\xi^2 & \xi/2 + \xi^2/2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$\therefore N_1 = -\xi/2\,(1-\xi)$

$N_2 = (1+\xi)(1-\xi)$

$N_3 = \xi/2\,(1+\xi)$

Derive element stiffness matrix of truss element.
Also derive the expressions for nodal load vector in a two noded bar element due to body force and surface load.

Plane truss

Element stiffness matrix in global coordinates

A typical plane truss element is shown in local and global coordinate system in fig.

Let u_1' and u_2' be the displacements of nodes 1 and 2, respectively, in the local coordinate system.

$$q' = \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix}$$

The element displacement vector in the global coordinate system is denoted by

$$\{q\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

The relationship between q' and q is developed as follows:

$$u_1' = u_1 \cos\theta + u_2 \sin\theta$$
$$u_2' = u_3 \cos\theta + u_4 \sin\theta$$

The direction cosines l and m are introduced as

$l = \cos\theta$ and $m = \cos\phi = \sin\theta$

$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix}$$

$$\therefore \{q'\} = [L]\{q\}$$

where the transformation matrix L is given by

$$[L] = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

The truss element is a one-dimensional element viewed in the local coordinate system.

$$[k'] = \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The element strain energy in local coordinates is given by

$$U_e = \frac{1}{2}\{q'\}^T [k'] \{q'\}$$

$$= \frac{1}{2}\{[L]\{q\}\}^T [k'] \{[L]\{q\}\}$$

$$= \frac{1}{2}\{q\}^T [L]^T [k'] [L] \{q\}$$

$$U_e = \tfrac{1}{2}\{q\}^T [k] \{q\}$$

where

$$[k] = [L]^T [k'] [L]$$

$$= \begin{bmatrix} l & 0 \\ m & 0 \\ 0 & l \\ 0 & m \end{bmatrix} \frac{A_e E_e}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

$$[k] = \frac{A_e E_e}{l_e}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}$$

Two noded bar element

nodal load vector due to body force

The element body force

$$\int u^T f A\, dx = \int_0^l \{[N]\{q^e\}\}^T f A\, dx$$

$$= A f \int_0^l [N]^T \{q\}^T dx$$

$$= \{q^e\}^T \{f^e\}$$

where

$\{f^e\}$ is the element body force vector

$$\{f^e\} = A f \int_0^l [N]^T dx$$

$$\{f^e\} = A f \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$\int_0^l N_1\, dx = \int_0^l (1 - x/l)\, dx = \left[x - x^2/2l\right]_0^l = l/2$$

$$\int_0^l N_2\, dx = \int_0^l x/l\, dx = \left[x^2/2l\right]_0^l = l/2$$

$$\therefore \{f^e\} = \frac{A f l}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

f is the body force per unit volume.

nodal load vector due to surface load

The element surface force (traction force)

$$\int u^T T A\, dx = \int_0^l \{[N]\{q^e\}\}^T T A\, dx$$

$$= AT \int_0^l \{q^e\}^T [N]^T dx$$

$$= \{q^e\}^T \{T^e\}$$

where

$\{T^e\}$ is the element traction force vector

$$\{T^e\} = \int_0^l AT [N]^T dx$$

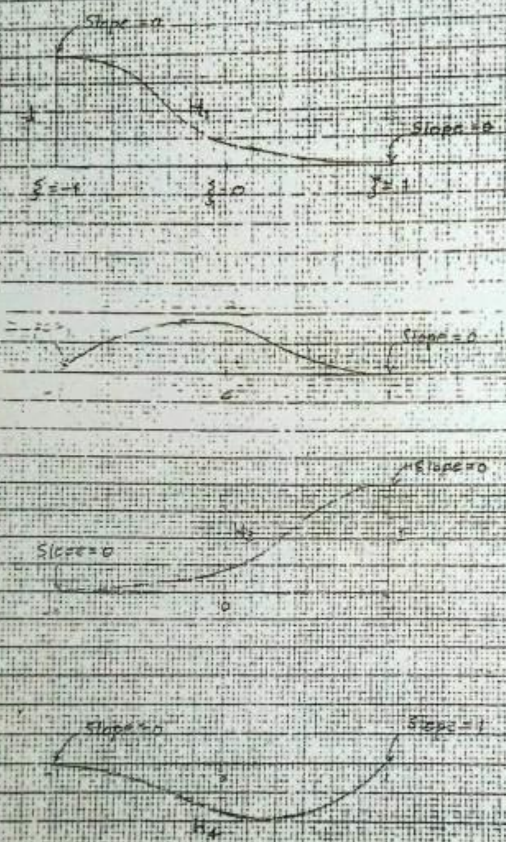
$$\{T^e\} = AT \int_0^l \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} dx$$

$$\int_0^l N_1\, dx = \int_0^l (1 - x/l)\, dx = \left[x - x^2/2l\right]_0^l = l/2$$

$$\int_0^l N_2\, dx = \int_0^l x/l\, dx = \left[x^2/2l\right]_0^l = l/2$$

$$\therefore \{T^e\} = \frac{ATl}{2} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

T is the surface traction per unit area



Hermite shape function

Derive & Draw the shape function and its variation for two noded beam element. Use natural coordinat

$$v = a_1 + a_2\xi + a_3\xi^2 + a_4\xi^3$$

$$v = \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$$v = [\phi]\{\alpha\}$$

$$v' = \frac{dv}{d\xi} = a_2 + 2a_3\xi + 3a_4\xi^2$$

$\xi = -1$ $\quad v_1 = a_1 - a_2 + a_3 - a_4$

$\quad\quad\quad v_1' = a_2 - 2a_3 + 3a_4$

$\xi = +1$ $\quad v_2 = a_1 + a_2 + a_3 + a_4$

$\quad\quad\quad v_2' = a_2 + 2a_3 + 3a_4$

$$\begin{Bmatrix} v_1 \\ v_1' \\ v_2 \\ v_2' \end{Bmatrix} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = [A]\{\alpha\}$$

$$[H] = [\phi][A]^{-1}$$

$$= \begin{bmatrix} 1 & \xi & \xi^2 & \xi^3 \end{bmatrix} \frac{1}{4} \begin{bmatrix} 2 & 1 & 2 & -1 \\ -3 & -1 & 3 & -1 \\ 0 & -1 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$$

$H_1 = \frac{1}{4}(2 - 3\xi + \xi^3)$

$H_2 = \frac{1}{4}(1 - \xi - \xi^2 + \xi^3)$

$H_3 = \frac{1}{4}(2 + 3\xi - \xi^3)$

$H_4 = \frac{1}{4}(-1 - \xi + \xi^2 + \xi^3)$ Hermite shape functions

$$v(\xi) = H_1 v_1 + H_2 \left(\frac{dv}{d\xi}\right)_1 + H_3 v_2 + H_4 \left(\frac{dv}{d\xi}\right)_2$$

$$x = \frac{1-\xi}{2} x_1 + \frac{1+\xi}{2} x_2$$

$$x = \frac{x_1 + x_2}{2} + \frac{x_2 - x_1}{2} \xi$$

$$x = \frac{x_1 + x_2}{2} + \frac{l}{2} \xi$$

$$dx = l/2 \; d\xi$$

$$\frac{dv}{d\xi} = \frac{dv}{dx} \frac{dx}{d\xi} = \frac{l}{2} \frac{dv}{dx} = \frac{l}{2} \theta$$

$$\therefore v(\xi) = H_1 v_1 + \frac{l}{2} H_2 \theta_1 + H_3 v_2 + \frac{l}{2} H_4 \theta_2$$

$\therefore$ Shape functions

$N_1 = H_1$

$N_2 = l/2 \; H_2$

$N_3 = H_3$

$N_4 = l/2 \; H_4$

2. Derive element stiffness matrix for beam element. Show sample calculation for k_{44}

At each node D.O.F = 2

vertical displacement along y-direction = v

angular displacement (rotation) about z-direction = θ

$v = a_1 + a_2 x + a_3 x^2 + a_4 x^3$

$= [1 \quad x \quad x^2 \quad x^3] \{a_1 \quad a_2 \quad a_3 \quad a_4\}^T$

$v = [\phi]\{\alpha\}$

$\theta = \frac{dv}{dx} = 0 + a_2 + 2a_3 x + 3a_4 x^2$

$$\begin{Bmatrix} v \\ \theta \end{Bmatrix} = \begin{bmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$v_1 = a_1$

$\theta_1 = a_2$

$v_2 = a_1 + a_2 l + a_3 l^2 + a_4 l^3$

$\theta_2 = 0 + a_2 + 2a_3 l + 3a_4 l^2$

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & l & l^2 & l^3 \\ 0 & 1 & 2l & 3l^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$$

$\{q\} = [A]\{\alpha\}$

$\therefore \{\alpha\} = [A]^{-1}\{q\}$

$$[N] = [\Phi][A]^{-1}$$

$$= \begin{bmatrix} 1 & x & x^2 & x^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/l^2 & -2/l & 3/l^2 & -1/l \\ 2/l^3 & 1/l^2 & -2/l^3 & 1/l^2 \end{bmatrix}$$

$$[N] = \left[\tfrac{1}{l^3}(l^3 - 3lx^2 + 2x^3),\ \tfrac{1}{l^2}(l^2x - 2lx^2 + x^3),\ \tfrac{1}{l^3}(3lx^2 - 2x^3),\ \tfrac{1}{l^2}(x^3 \ldots \right.$$

$$N_1 = \tfrac{1}{l^3}(l^3 - 3lx^2 + 2x^3)$$
$$N_2 = \tfrac{1}{l^2}(l^2x - 2lx^2 + x^3)$$
$$N_3 = \tfrac{1}{l^3}(3lx^2 - 2x^3)$$
$$N_4 = \tfrac{1}{l^2}(x^3 - lx^2)$$

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R} \qquad \varepsilon_x = \frac{\sigma_x}{E}$$

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} = -\frac{\sigma_x}{Ey} = -\frac{\varepsilon_x}{y}$$

$$\varepsilon_x = -y\frac{d^2v}{dx^2} = -y\frac{d^2[N]}{dx^2}\{q\} = -y[B'][q]$$

$$\sigma_x = E\,\varepsilon_x = -Ey[B']\{q\} = -y[D][B']\{q\}$$

$$U = \frac{1}{2}\int_v [\sigma_x]^T[\varepsilon_x]\,dv$$

$$= \frac{1}{2}\int_l\int_A \left[-y[D][B']\{q\}\right]^T\left[-y[B']\{q\}\right] dA\,dx$$

$$= \frac{1}{2}\int_l I_z\{q\}^T[B']^T[D][B']\{q\}\,dx \qquad \left(\because \int_A y^2\,dA = I_z\right)$$

$$U = \frac{1}{2}\{q\}^T[k]\{q\}$$

where $[k] = I_z \int_l [B']^T[D][B']\,dx$

$[k^e] = EI_z \int_0$

$[k^e] = EI_z \int_0$

where,

$N_1'' = 1/l^3\ (-6.$

$N_2'' = 1/l^2\ (-4.$

$N_3'' = 1/l^3\ (6l$

$N_4'' = 1/l^2\ (6$

$[k^e] = \frac{EI_z}{l^3}$

Sample calc

$k_{44} = EI_z \int_0^l$

$= \frac{EI}{l^4}\int_0^l ($

$= \frac{EI}{l^4}\ [$

$= \frac{EI}{l^4}\ ($

$\therefore k_{44} = \frac{4EI_z}{l}$

$$[k^e] = EI_z \int_0^l \begin{bmatrix} N_1'' \\ N_2'' \\ N_3'' \\ N_4'' \end{bmatrix} \begin{bmatrix} N_1'' & N_2'' & N_3'' & N_4'' \end{bmatrix} dx$$

$$[k^e] = EI_z \int_0^l \begin{bmatrix} N_1''N_1'' & N_1''N_2'' & N_1''N_3'' & N_1''N_4'' \\ N_2''N_1'' & N_2''N_2'' & N_2''N_3'' & N_2''N_4'' \\ N_3''N_1'' & N_3''N_2'' & N_3''N_3'' & N_3''N_4'' \\ N_4''N_1'' & N_4''N_2'' & N_4''N_3'' & N_4''N_4'' \end{bmatrix} dx$$

where,

$N_1'' = 1/l^3 \, (-6l + 12x)$

$N_2'' = 1/l^2 \, (-4l + 6x)$

$N_3'' = 1/l^3 \, (6l - 12x)$

$N_4'' = 1/l^2 \, (6x - 2l)$

$$[k^e] = \frac{EI_z}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$

Sample calculation for k_{44}

$$k_{44} = EI_z \int_0^l 1/l^2 (6x - 2l) \, 1/l^2 (6x - 2l) \, dx$$

$$= \frac{EI}{l^4} \int_0^l (36x^2 - 24lx + 4l^2) \, dx$$

$$= \frac{EI}{l^4} \left[36\frac{x^3}{3} - 24l\frac{x^2}{2} + 4l^2 x \right]_0^l$$

$$= \frac{EI}{l^4} \left(12l^3 - 12l^3 + 4l^3 \right)$$

$$\therefore k_{44} = \frac{4EI_z}{l}$$

3. Analyse the beam by FEM and determine the ex
reactions

$E = 2 \times 10^5 \ N/mm^2 \quad = 2 \times 10^8 \ kN/m^2$

$I = 5 \times 10^6 \ mm^4 \quad = 5 \times 10^{-6} \ m^4$

$\therefore EI = (2 \times 10^8)(5 \times 10^{-6}) = 10^3 \ kNm^2$

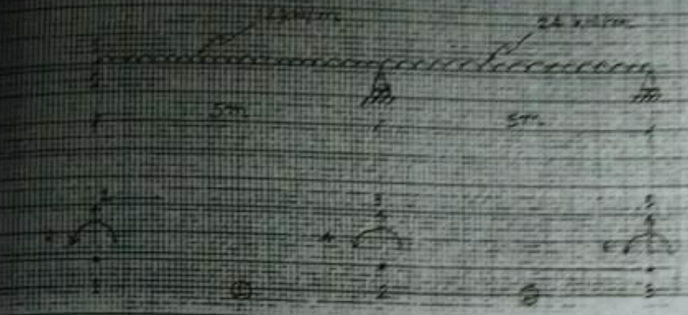
$$[k^e] = \frac{EI}{l^3} \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

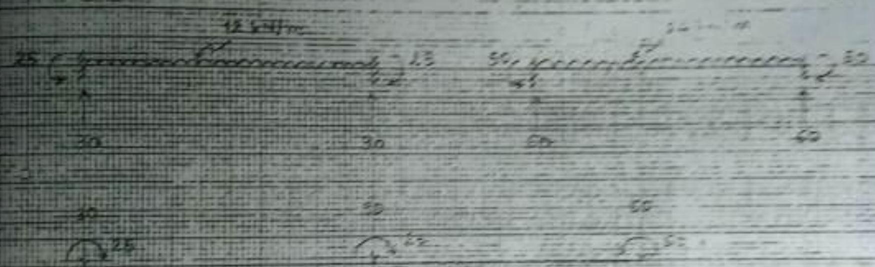
$$[k^1] = 8 \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array} \qquad [k^2] = 8 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \end{array}$$

$[k]\{q\} = \{f\}$

$$8 \begin{array}{cc} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 12 & 30 & -12 & 30 & 0 & 0 \\ 30 & 100 & -30 & 50 & 0 & 0 \\ -12 & -30 & 24 & 0 & -12 & 30 \\ 30 & 50 & 0 & 200 & -30 & 50 \\ 0 & 0 & -12 & -30 & 12 & -30 \\ 0 & 0 & 30 & 50 & -30 & 100 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array} \begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 \\ v_3 = 0 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -25 \\ -90 \\ -25 \\ -60 \\ 50 \end{Bmatrix}$$

$$8 \begin{bmatrix} 200 & 50 \\ 50 & 100 \end{bmatrix} \begin{bmatrix} \theta_2 \\ \theta_3 \end{bmatrix} = \begin{bmatrix} -25 \\ 50 \end{bmatrix}$$


5m
5m


25
30
30
60

$$\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -1/28 \\ 9/112 \end{Bmatrix} = \begin{Bmatrix} -0.0357 \\ 0.0804 \end{Bmatrix} \text{ rad}$$

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = 8 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & -30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1/28 \end{Bmatrix} + \begin{Bmatrix} 30 \\ 25 \\ 30 \\ -25 \end{Bmatrix} = \begin{Bmatrix} 21.43 \\ 10.71 \\ 38.57 \\ -53.57 \end{Bmatrix}$$

$$\begin{Bmatrix} V_2 \\ M_2 \\ V_3 \\ M_3 \end{Bmatrix} = 8 \begin{bmatrix} 12 & 30 & -12 & 30 \\ 30 & 100 & 30 & 50 \\ -12 & -30 & 12 & -30 \\ 30 & 50 & -30 & 100 \end{bmatrix} \begin{Bmatrix} 0 \\ -1/28 \\ 0 \\ 9/112 \end{Bmatrix} + \begin{Bmatrix} 60 \\ 50 \\ 60 \\ -50 \end{Bmatrix} = \begin{Bmatrix} 70.71 \\ 53.57 \\ 49.29 \\ 0 \end{Bmatrix}$$

4. Find the deflection at the load and the slopes at ends for the steel shaft.

$E = 200 \text{ GPa} = 2 \times 10^5 \text{ N/mm}^2 = 2 \times 10^8 \text{ kN/m}^2$

$I_1 = I_2 = 1.25 \times 10^5 \text{ mm}^4 = 1.25 \times 10^{-7} \text{ m}^4$

$I_3 = 4 \times 10^4 \text{ mm}^4 = 4 \times 10^{-8} \text{ m}^4$

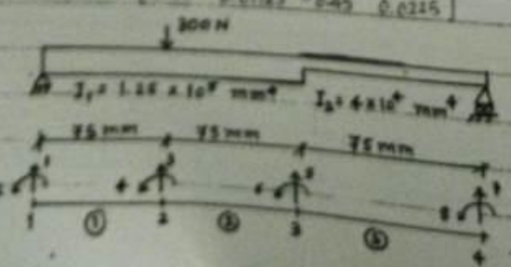
$EI_1 = EI_2 = 25$

$EI_3 = 8$

$$[k^1] = 59.26 \times 10^3 \begin{bmatrix} 12 & 0.45 & -12 & 0.45 \\ 0.45 & 0.0225 & -0.45 & 0.01125 \\ -12 & -0.45 & 12 & -0.45 \\ 0.45 & 0.01125 & -0.45 & 0.0225 \end{bmatrix} = 10^3 \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 711.1111 & 26.6667 & -711.1111 \\ 26.6667 & 1.3333 & -26.6667 \\ -711.1111 & -26.6667 & 711.1111 \\ 26.6667 & 0.6667 & -26.6667 \end{bmatrix} \end{array}$$

$$[k^2] = 10^3 \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 711.1111 & 26.6667 & -711.1111 & 26.6667 \\ 26.6667 & 1.3333 & -26.6667 & 0.6667 \\ -711.1111 & -26.6667 & 711.1111 & -26.6667 \\ 26.6667 & 0.6667 & -26.6667 & 1.3333 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$$[k^3] = 18.963 \times 10^3 \begin{bmatrix} 12 & 0.45 & -12 & 0.45 \\ 0.45 & 0.0225 & -0.45 & 0.01125 \\ -12 & -0.45 & 12 & -0.45 \\ 0.45 & 0.01125 & -0.45 & 0.0225 \end{bmatrix} = 10^3 \begin{array}{c} \begin{array}{ccc} 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 227.5556 & 8.5333 & -227.5556 \\ 8.5333 & 0.4267 & -8.5333 \\ -227.5556 & -8.5333 & 227.5556 \\ 8.5333 & 0.2133 & -8.5333 \end{bmatrix} \end{array}$$



$$[k] = 10^3 \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 711.1111 & 26.6667 & -711.1111 & 26.6667 & 0 & 0 & 0 & 0 \\ 26.6667 & 1.3333 & -26.6667 & 0.6667 & 0 & 0 & 0 & 0 \\ -711.1111 & -26.6667 & 1422.2222 & 0 & -711.1111 & 26.6667 & 0 & 0 \\ 26.6667 & 0.6667 & 0 & 2.6666 & -26.6667 & 0.6667 & 0 & 0 \\ 0 & 0 & -711.1111 & -26.6667 & 938.6667 & -18.1334 & -227.5556 & 8.5333 \\ 0 & 0 & 26.6667 & 0.6667 & -18.1334 & 1.76 & -8.5333 & 0.2133 \\ 0 & 0 & 0 & 0 & -227.5556 & -8.5333 & 227.5556 & -8.5333 \\ 0 & 0 & 0 & 0 & 8.5333 & 0.2133 & -8.5333 & 0.4267 \end{bmatrix} \end{array}$$

$$\{q\} = \begin{Bmatrix} v_1 = 0 \\ \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ v_4 = 0 \\ \theta_4 \end{Bmatrix} \qquad \{f\} = \begin{Bmatrix} 0 \\ 0 \\ -0.300 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

$$10^3 \begin{bmatrix} 1.3333 & -26.6667 & 0.6667 & 0 & 0 & 0 \\ -26.6667 & 1422.2222 & 0 & -711.1111 & 26.6667 & 0 \\ 0.6667 & 0 & 2.6666 & -26.6667 & 0.6667 & 0 \\ 0 & -711.1111 & -26.6667 & 938.6667 & -18.1334 & 8.5333 \\ 0 & 26.6667 & 0.6667 & -18.1334 & 1.76 & 0.2133 \\ 0 & 0 & 0 & 8.5333 & 0.2133 & 0.4267 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.3 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_1 \\ v_2 \\ \theta_2 \\ v_3 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} -4.2813 \times 10^{-5} \\ -2.6483 \times 10^{-6} \\ -2.0309 \times 10^{-5} \\ 2.7653 \times 10^{-6} \\ 1.3441 \times 10^{-5} \\ 4.8583 \times 10^{-5} \end{Bmatrix}$$

5. For the beam and loading as shown in fig. deter[mine]
(i) the slopes at 2 and 3
(ii) vertical deflection at midpoint of the distributed

$E = 200\ GPa = 2 \times 10^5\ N/mm^2 = 2 \times 10^8\ kN/m^2$

$I = 4 \times 10^6$

$EI = 800\ kN\,m^2$

$$[k'] = 800 \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & \\ -12 & & & \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ \\ 4 \end{array}$$

$$[illegible] \begin{array}{c} \begin{array}{cccc} 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ & 4 & -6 & 2 \\ & & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \end{array}$$

$[k]\{q\} = \{f\}$

$$800 \begin{bmatrix} 12 & 6 & -12 & 6 & 0 & 0 \\ 6 & 4 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} v_1 = 0 \\ \theta_1 = 0 \\ v_2 = 0 \\ \theta_2 \\ v_3 = 0 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -6 \\ -1 \\ -6 \\ 1 \end{Bmatrix}$$

$$800 \begin{bmatrix} 8 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$\begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} -2.6786 \times 10^{-4} \\ 4.4643 \times 10^{-4} \end{Bmatrix} \text{ rad.}$$

$v = [N]\{q\}$

$v = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2$

For the vertical deflection at mid point of the distributed loa

$v = N_1 v_2 + N_2 \theta_2 + N_3 v_3 + N_4 \theta_3$

$v_2 = 0 \;,\; v_3 = 0$

at $x = 0.5$ m , $l = 1$ m

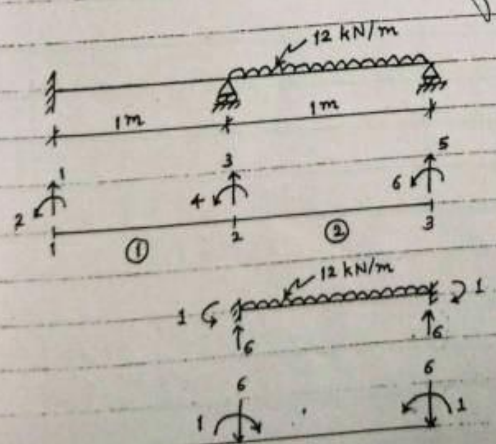
$N_2 = \frac{1}{l^2}(l^2 x - 2 l x^2 + x^3) = 0.125$

$N_4 = \frac{1}{l^2}(x^3 - l x^2) = -0.125$

$v = 0 + 0.125 \times (-2.6786 \times 10^{-4}) + 0 + (-0.125) \times (4.4643 \times 10^{-4})$

$\therefore v = 8.9286 \times 10^{-5}$ m

$\therefore v = 0.0893$ mm



For a given bar element determine the reaction effec and stresses if element 1 is subjected to temperature drop of 10°C

$E_1 = 70\text{ GPa} = 70\text{ kN/mm}^2$ $\quad$ $E_{2,3} = 100\text{ GPa} = 100\text{ kN/mm}^2$

$\alpha_1 = 23 \times 10^{-6}\ /°C$ $\quad$ $\alpha_{2,3} = 20 \times 10^{-6}\ /°C$

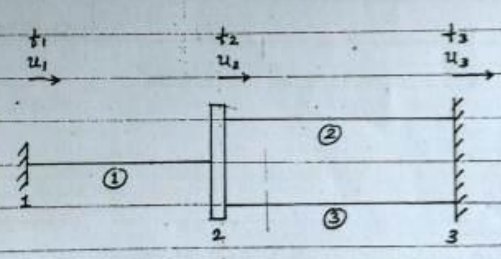
$A_1 = 12 \times 10^{-4}\ m^2 = 1200\ mm^2$ $\quad$ $A_{2,3} = 6 \times 10^{-4}\ m^2 = 600\ mm^2$

$l_1 = l_2 = l_3 = 2\ m = 2000\ mm$

$$[k^e] = \frac{AE}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k^1] = \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{bmatrix} 42 & -42 \\ -42 & 42 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \qquad [k^2] = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array} \qquad [k^3] = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \begin{bmatrix} 30 & -30 \\ -30 & 30 \end{bmatrix} \end{array} \begin{array}{c} \\ 2 \\ 3 \end{array}$$

$$[k] = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 42 & -42 & 0 \\ -42 & 102 & -60 \\ 0 & -60 & 60 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}$$



$$f_T = AE\alpha\,\Delta T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$$

$$f_{T1} = 1200 \times 70 \times 23 \times 10^{-6} \times (-10) \begin{Bmatrix} -1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 19.32 \\ -19.32 \end{Bmatrix} \begin{array}{c} 1 \\ 2 \end{array}$$

$$f_{T2} = f_{T3} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \begin{array}{c} 2 \\ 3 \end{array}$$

$$f_T = \begin{Bmatrix} 19.32 \\ -19.32 \\ 0 \end{Bmatrix}$$

$[k]\{q\} = \{f\}$

$$\begin{bmatrix} 42 & -42 & 0 \\ -42 & 102 & -60 \\ 0 & -60 & 60 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \end{Bmatrix}$$

$u_1 = 0 \qquad u_3 = 0 \qquad f_2 = -19.32$

$\therefore\ 102\ u_2 = f_2 = -19.32$

$\therefore\ u_2 = -0.1894\ mm$

$\sigma^e = E[B]\{q\} - E\alpha\Delta T$

$$\sigma^1 = 70 \times \begin{bmatrix} -1/2000 & 1/2000 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.1894 \end{Bmatrix} - 70 \times 23 \times 10^{-6} \times (-10)$$

$\therefore\ \sigma^1 = 9.471 \times 10^{-3}\ kN/mm^2$

$$\sigma^2 = \sigma^3 = 100 \times \begin{bmatrix} -1/2000 & 1/2000 \end{bmatrix} \begin{Bmatrix} -.1894 \\ 0 \end{Bmatrix}$$

$$= 9.47 \times 10^{-3}\ kN/mm^2$$

$$R_1 = \begin{bmatrix} 42 & -42 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.1894 \\ 0 \end{Bmatrix} - \{19.32\} = -11.3652\ kN$$

$$R_3 = \begin{bmatrix} 0 & -60 & 60 \end{bmatrix} \begin{Bmatrix} 0 \\ -0.1894 \\ 0 \end{Bmatrix} - \{0\} = 11.364\ kN$$

TUTORIAL - 6

Plane Stress And
Plane Strain Problems
[CST Elements]

Q. Derive the strain displacement matrix for 3-no[de] element using linear displacement function and [state] its name as CST element.

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

$$v = \alpha_4 + \alpha_5 x + \alpha_6 y$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$u_1 = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

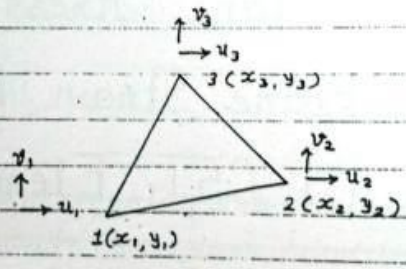
$$v_1 = \alpha_4 + \alpha_5 x_1 + \alpha_6 y_1$$

$$u_2 = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$v_2 = \alpha_4 + \alpha_5 x_2 + \alpha_6 y_2$$

$$u_3 = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

$$v_3 = \alpha_4 + \alpha_5 x_3 + \alpha_6 y_3$$



$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_1 & y_1 \\ 1 & x_2 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & y_2 \\ 1 & x_3 & y_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} \qquad \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{Bmatrix}$$

$$\{q\} = [A]\{\alpha\}$$

matrix for 3-n
ent function and

$$\{\alpha\} = [A]^{-1}\{q\}$$

$$u = [\phi]\{\alpha\}$$

$$\therefore u = [\phi][A]^{-1}\{q\} = [N]\{q\}$$

$$\therefore [N] = [\phi][A]^{-1}$$

$$[N_1 \;\; N_2 \;\; N_3] = [1 \;\; x \;\; y]\,\frac{1}{2\Delta}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$[N_1 \;\; N_2 \;\; N_3] = \frac{1}{2\Delta}\left[(a_1+b_1x+c_1y) \quad (a_2+b_2x+c_2y) \quad (a_3+b_3x+c_3y)\right]$$

$$2\Delta = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \qquad 2\Delta^T = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

(x_3, y_3)

v_2 u_2

$2(x_2, y_2)$

$a_1 = (x_2y_3 - x_3y_2)$ $\quad a_2 = (x_3y_1 - x_1y_3)$ $\quad a_3 = (x_1y_2 - x_2y_1)$

$b_1 = (y_2 - y_3)$ $\quad b_2 = (y_3 - y_1)$ $\quad b_3 = (y_1 - y_2)$

$c_1 = (x_3 - x_2)$ $\quad c_2 = (x_1 - x_3)$ $\quad c_3 = (x_2 - x_1)$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{Bmatrix} = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

α_4

α_5

$\{\varepsilon\} = [B]\{q\}$

$$\therefore\ [B] = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix}$$

Strain is constant i.e. independent of x and hence the element is known as constant strain triangle or CST.

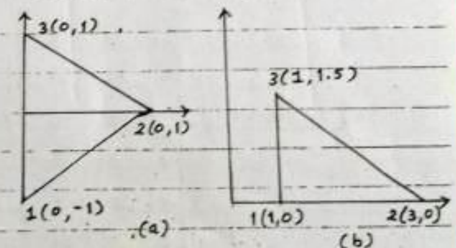
Evaluate the stiffness matrix for the elements shown in fig. The coordinates are in inches. Assume plane stress conditions. Take $E = 210\,GPa$, $\nu = 0.25$ and the thickness $t = 1\,in$.

$E = 210\,GPa = 210 \times 10^3\,N/mm^2 = 210\,kN/(1/25.4^2\,inch^2) = 135.4836 \times 10^3\,kN/in^2$

$\nu = 0.25$

$t = 1\,in$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = 144.51584 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$



$$[A] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$2\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 4 \qquad \therefore \Delta = 2$$

$$[A]^{-1} = \frac{1}{2\Delta}\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & -1 \\ -2 & 0 & 2 \end{bmatrix}$$

$$[B] = \frac{1}{2\Delta}\begin{bmatrix} b_1 & 0 & b_2 & 0 & b_3 & 0 \\ 0 & c_1 & 0 & c_2 & 0 & c_3 \\ c_1 & b_1 & c_2 & b_2 & c_3 & b_3 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

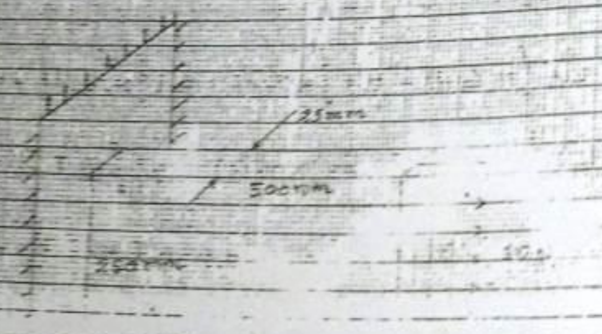
$$[k] = \Delta t \, [B]^T [D][B]$$

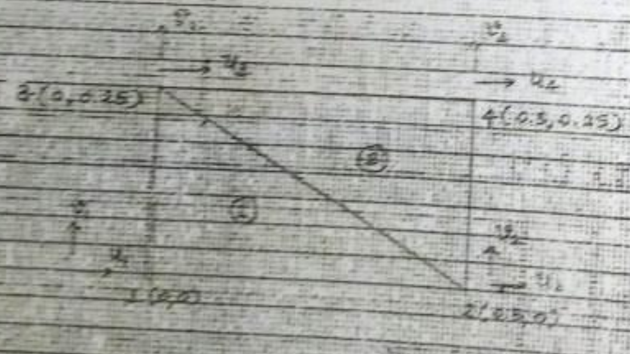
$$[D][B] = 144.51584 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1 & 0 & 2 & 2 & -1 \end{bmatrix}$$

$$[D][B] = 36.12896 \times 10^3 \begin{bmatrix} -1 & -0.5 & 2 & 0 & -1 & 0.5 \\ -0.25 & -2 & 0.5 & 0 & -0.25 & 2 \\ -0.75 & -0.375 & 0 & 0.75 & 0.75 & -0.375 \end{bmatrix}$$

$$[k] = 2 \times 1 \times \frac{1}{4} \begin{bmatrix} -1 & 0 & -2 \\ 0 & -2 & -1 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} 36.12896 \times 10^3 \begin{bmatrix} -1 & -0.5 & 2 & 0 & -1 & 0.5 \\ -0.25 & -2 & 0.5 & 0 & -0.25 & 2 \\ -0.75 & -0.375 & 0 & 0.75 & 0.75 & -0.375 \end{bmatrix}$$

$$[k] = 18.06448 \times 10^3 \begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 2.5 & 1.25 & -2 & -1.5 & -0.5 & 0.25 \\ 1.25 & 4.375 & -1 & -0.75 & -0.25 & -3.625 \\ -2 & -1 & 4 & 0 & -2 & 1 \\ -1.5 & -0.75 & 0 & 1.5 & 1.5 & -0.75 \\ -0.5 & -0.25 & -2 & 1.5 & 2.5 & -1.25 \\ 0.25 & -3.625 & 1 & -0.75 & -1.25 & 4.375 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

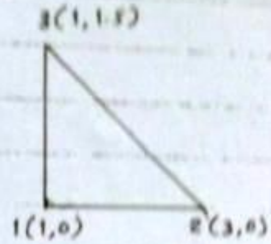

25mm
500mm
250mm


3 (0, 0.25)
4 (0.5, 0.25)
1 (0,0)
2 (0.5, 0)

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1.5 \end{bmatrix}$$

$2\Delta = 3$

$\therefore \Delta = 1.5$



$$[A]^{-1} = \frac{1}{3} \begin{bmatrix} 4.5 & -1.5 & 0 \\ -1.5 & 1.5 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$[B] = \frac{1}{3} \begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1.5 & 0 & 1.5 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 144.51584 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1.5 & 0 & 1.5 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 48.1719 \times 10^3 \begin{bmatrix} -1.5 & 1 & 0 & 0 & 0 & 0.5 \\ -0.375 & -1.625 & 0 & 0 & 0 & 2 \\ -0.75 & 0.5625 & 0 & 0.5625 & 0.75 & 0 \end{bmatrix}$$

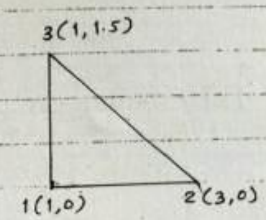
$$[k] = \Delta \; t \; [B]^T [D][B]$$

$$= 1.5 \times 1 \times \frac{1}{3} \begin{bmatrix} -1.5 & 0 & -2 \\ 0 & -2 & -1.5 \\ 1.5 & 0 & 0 \\ 0 & 0 & 1.5 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} 48.1719 \times 10^3 \begin{bmatrix} -1.5 & 1 & 0 & 0 & 0 & 0.5 \\ -0.375 & -1.625 & 0 & 0 & 0 & 2 \\ -0.75 & 0.5625 & 0 & 0.5625 & 0.75 & 0 \end{bmatrix}$$

$$[A] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1.5 \end{bmatrix}$$

$2\Delta = 3$

$\therefore \Delta = 1.5$



$$[A]^{-1} = \frac{1}{3}\begin{bmatrix} 4.5 & -1.5 & 0 \\ -1.5 & 1.5 & 0 \\ -2 & 0 & 2 \end{bmatrix}$$

$$[B] = \frac{1}{3}\begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1.5 & 0 & 1.5 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 144.51584 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{3}\begin{bmatrix} -1.5 & 0 & 1.5 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 2 \\ -2 & -1.5 & 0 & 1.5 & 2 & 0 \end{bmatrix}$$

$$[D][B] = 48.1719 \times 10^3 \begin{bmatrix} -1.5 & 1 & 0 & 0 & 0 & 0.5 \\ -0.375 & -1.625 & 0 & 0 & 0 & 2 \\ -0.75 & 0.5625 & 0 & 0.5625 & 0.75 & 0 \end{bmatrix}$$

$[k] = \Delta\ t\ [B]^T [D][B]$

$$= 1.5 \times 1 \times \frac{1}{3}\begin{bmatrix} -1.5 & 0 & -2 \\ 0 & -2 & -1.5 \\ 1.5 & 0 & 0 \\ 0 & 0 & 1.5 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} 48.1719 \times 10^3 \begin{bmatrix} -1.5 & 1 & 0 & 0 & 0 & 0.5 \\ -0.375 & -1.625 & 0 & 0 & 0 & 2 \\ -0.75 & 0.5625 & 0 & 0.5625 & 0.75 & 0 \end{bmatrix}$$

$$[k] = 24.086 \times 10^3 \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 3.75 & -2.625 & 0 & -1.125 & -1.5 & -0.75 \\ -2.625 & 5.5938 & 0 & 0.8438 & 1.125 & -3.25 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1.125 & 0.8438 & 0 & 0.8438 & 1.125 & 0 \\ -1.5 & 1.125 & 0 & 1.125 & 1.5 & 0 \\ -0.75 & -3.25 & 0 & 0 & 0 & 4 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}$$

...r the element

...e given as

$..._1 = 2.0$ mm

$..._1 = 1.0$ mm

Determine the

stresses σ_1 & ...

determine Von

Assume plane

Take $E = 210$ GPa

$$[A] = \begin{bmatrix} 1 & 0 & \dots \\ 1 & 10 & \dots \\ 1 & 5 & \dots \end{bmatrix}$$

$$[A]^{-1} = \frac{1}{100} \begin{bmatrix} 100 & \dots \\ -10 & \dots \\ 5 & \dots \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{\dots$$

$$[B]\{q\} = \frac{1}{100} \begin{bmatrix} -10 & \dots \\ 0 & \dots \\ -5 & \dots \end{bmatrix}$$

$$[B]\{q\} = \frac{1}{254\dots}$$

For the element shown in fig., the nodal displacement are given as

$u_1 = 2.0$ mm $\quad u_2 = 0.5$ mm $\quad u_3 = 3.0$ mm

$v_1 = 1.0$ mm $\quad v_2 = 0.0$ mm $\quad v_3 = 1.0$ mm

Determine the element stress σ_x, σ_y, τ_{xy}, principal stresses σ_1 & σ_2 and angle of principal plane θ. Also determine Von Mises stresses.

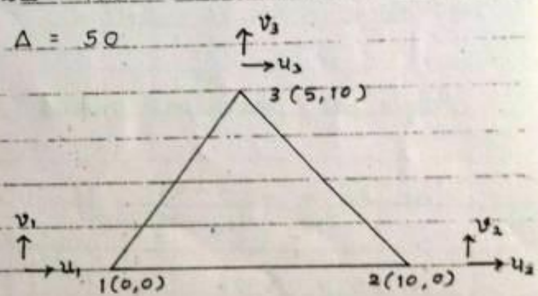
Assume plane stress condition.

Take $E = 210$ GPa, $\nu = 0.25$, thickness $t = 1$ in.

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 10 & 0 \\ 1 & 5 & 10 \end{bmatrix}$$

$2\Delta = 100$

$\Delta = 50$



(a)

$$[A]^{-1} = \frac{1}{100}\begin{bmatrix} 100 & 0 & 0 \\ -10 & 10 & 0 \\ 5 & -5 & 10 \end{bmatrix}$$

$\{\sigma\} = [D][B]\{q\}$

$$[B]\{q\} = \frac{1}{100}\begin{bmatrix} -10 & 0 & 10 & 0 & 0 & 0 \\ 0 & -5 & 0 & -5 & 0 & 10 \\ -5 & -10 & -5 & 10 & 10 & 0 \end{bmatrix} \frac{1}{25.4} \begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 0 \\ 3 \\ 1 \end{Bmatrix}$$

$$[B]\{q\} = \frac{1}{2540}\begin{Bmatrix} -15 \\ 5 \\ 7.5 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 144.51584 \times 10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{2540} \begin{Bmatrix} -15 \\ 5 \\ 7.5 \end{Bmatrix} = \begin{Bmatrix} -782.32 \\ 71.12 \\ 160.02 \end{Bmatrix}$$

Principal stresses

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma = -355.2 \pm 455.74$

$\therefore \sigma_1 = 100.14 \quad kN/in^2$

$\sigma_2 = -511.34 \quad kN/in^2$

Angle of principal plane

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$\tan 2\theta = 0.375$

$\therefore 2\theta_1 = 20.556^\circ$ and $2\theta_2 = 200.556^\circ$

$\therefore \theta_1 = 10.278^\circ \qquad \theta_2 = 100.278^\circ$

Von Mises stress

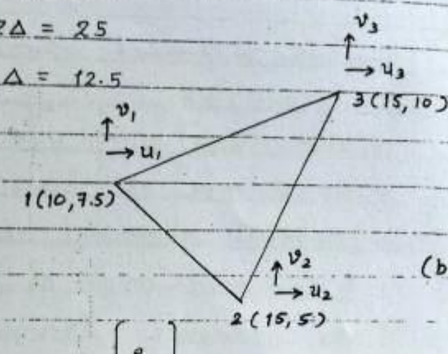
$$\sigma_{VM} = \frac{1}{\sqrt{2}} \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3^2) + (\sigma_3 - \sigma_1)^2}$$

$= 867.76 \; kN/in^2$

$$[A] = \begin{bmatrix} 1 & 10 & 7.5 \\ 1 & 15 & 5 \\ 1 & 15 & 10 \end{bmatrix}$$

$2\Delta = 25$

$\therefore \Delta = 12.5$



$$[A]^{-1} = \frac{1}{25}\begin{bmatrix} 75 & 12.5 & -62.5 \\ -5 & 2.5 & 2.5 \\ 0 & -5 & 5 \end{bmatrix}$$

$$[B]\{q\} = \frac{1}{25}\begin{bmatrix} -5 & 0 & 2.5 & 0 & 2.5 & 0 \\ 0 & 0 & 0 & -5 & 0 & 5 \\ 0 & -5 & -5 & 2.5 & 5 & 2.5 \end{bmatrix} \frac{1}{25.4}\begin{Bmatrix} 2 \\ 1 \\ 0.5 \\ 0 \\ 3 \\ 1 \end{Bmatrix} = \frac{1}{635}\begin{Bmatrix} -1.25 \\ 5 \\ 10 \end{Bmatrix}$$

$\{\sigma\} = [D][B]\{q\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 144.51584\times10^3 \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \frac{1}{635}\begin{Bmatrix} -1.25 \\ 5 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1066.8 \\ 853.44 \end{Bmatrix}$$

Principal stresses and Angle of principal plane

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$\sigma = 533.4 \pm 1006.42$

$\therefore \sigma_1 = 1539.82 \text{ kN/in}^2$ $\quad \sigma_2 = -953.02 \text{ kN/in}^2$

$$\tan 2\theta = -\frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$\tan 2\theta = 1.6$

$\therefore \theta_1 = 29^\circ$ and, $\theta_2 = 119^\circ$.

Von Mises stress

$$\sigma_{VM} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2} = 2178.71 \text{ kN/in}^2$$

1. For the elements shown in fig., the nodal displacements are given as:

$u_1 = 2.0$ mm $\quad u_2 = 0.5$ mm $\quad u_3 = 3.0$ mm

$v_1 = 1.0$ mm $\quad v_2 = 0.0$ mm $\quad v_3 = 1.0$ mm

Determine element stresses $\sigma_x, \sigma_y, \tau_{xy}$, principal stresses σ_1 & σ_2 and angle of principal plane. Also determine Von Mises stresses. Assume plane strain condition.

Take $E = 210$ GPa, $\nu = 0.25$, thickness $t = 1$ in.

(Refer Ex 3)

(a) $E = 210 \text{ GPa} = 135.4836 \times 10^3 \text{ kN/in}^2$

$$[D] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} = 216.77376 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{q\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 216.77376 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \frac{1}{2540} \begin{Bmatrix} -15 \\ 5 \\ 7.5 \end{Bmatrix} = \begin{Bmatrix} -853.44 \\ 0 \\ 160.02 \end{Bmatrix}$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma = -426.72 \pm 455.74$$

$\therefore \sigma_1 = 29.02 \text{ kN/in}^2 \qquad \sigma_2 = -882.46$

$$\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_x - \sigma_y} = 0.375$$

$\theta_1 = 10.278^\circ \qquad \theta_2 = 100.278^\circ$

$$\sigma_{vm} = 1/\sqrt{2}\ \sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$$

$$= 897.32 \text{ kN/in}^2$$

$$\{\sigma\} = [D][B]\{q\}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = 216.77376 \times 10^3 \begin{bmatrix} 0.75 & 0.25 & 0 \\ 0.25 & 0.75 & 0 \\ 0 & 0 & 0.25 \end{bmatrix} \frac{1}{635} \begin{Bmatrix} -1.25 \\ 5 \\ 10 \end{Bmatrix} = \begin{Bmatrix} 106.68 \\ 1173.48 \\ 853.44 \end{Bmatrix}$$

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma = 640.08 \pm 1006.42$$

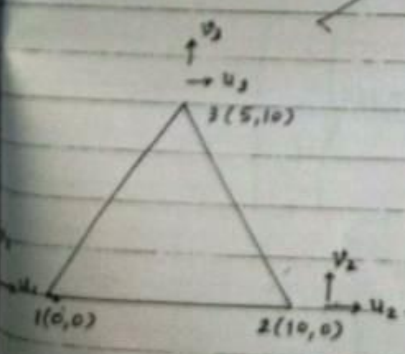
$\sigma_1 = 1646.50 \text{ kN/in}^2$ $\qquad$ $\sigma_2 = -366.34 \text{ kN/in}^2$

$$\tan 2\theta = \frac{-2\tau_{xy}}{\sigma_x - \sigma_y} = 1.6$$

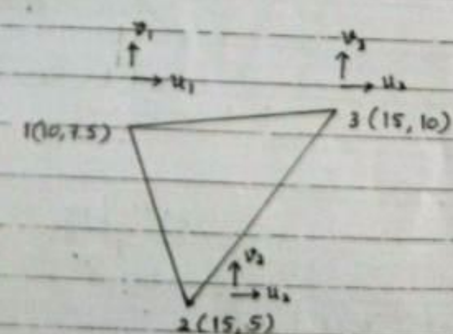
$\theta_1 = 29^\circ$ $\qquad$ $\theta_2 = 119^\circ$

$$\sigma_{vm} = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$$

$$= 1856.97 \text{ kN/in}^2$$



(a)



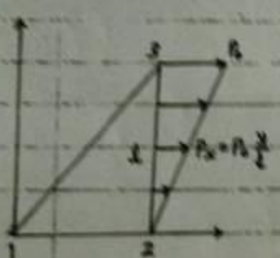
(b)

7. Determine the nodal forces for
(a) a linearly varying pressure
(b) the quadratic varying pressure
on the edge of the triangular element shown in fig. by evaluating surface intigral. Assume element thickness

(a)

$$\int_l u^T T t \, dl = \int_l [u \;\; v] \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} t \, dl$$

$$= \int_0^l [N_2 u_2 + N_3 u_3 \quad N_2 v_2 + N_3 v_3] \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} t \, dl$$



$$= u_2 t \int_0^l N_2 T_x \, dl + v_2 t \int_0^l N_2 T_y \, dl + u_3 t \int_0^l N_3 T_x \, dl + v_3 t \int_0^l N_3 T_y \, dl$$

$$= u_2 t \int_0^l (1 - y/l) \cdot P_0 \, y/l \, dy + v_2 (0) + u_3 t \int_0^l y/l \cdot P_0 \, y/l \, dy + v_3 (0)$$

$$= u_2 P_0 t \int_0^l \left(y/l - y^2/l^2 \right) dy + v_2 (0) + u_3 P_0 t \int_0^l y^2/l^2 \, dy + v_3 (0)$$

$$= u_2 P_0 t \left[y^2/2l - y^3/3l^2 \right]_0^l + v_2 (0) + u_3 P_0 t \left[y^3/3l^2 \right]_0^l + v_3 (0)$$

$$= u_2 P_0 t \left(l/2 - l/3 \right) + v_2 (0) + u_3 P_0 t \left(l/3 \right) + v_3 (0)$$

$$= \frac{P_0 t l}{6} u_2 + 0 \cdot v_2 + \frac{P_0 t l}{3} u_3 + 0 \cdot v_3$$

$$= [u_2 \;\; v_2 \;\; u_3 \;\; v_3] \frac{P_0 t l}{6} \begin{Bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{Bmatrix}$$

$$= \{q\}^T \{T^e\}$$

$$\therefore \{T^e\} = \frac{P_0 t l}{6} \begin{Bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{Bmatrix}$$

From case-(a) eq. (1)

$$\int_l u^T T\, t\, dl = u_2 t \int_0^l N_2 T_x\, dl + v_2 t \int_0^l N_2 T_y\, dl + u_3 t \int_0^l N_3 T_x\, dl + v_3 t \int_0^l N_3 T_y\, dl$$

$$= u_2 t \int_0^l (1 - y/l)\, p_0 \frac{y^2}{l^2}\, dy + v_2(0) + u_3 t \int_0^l \frac{y}{l}\, p_0 \frac{y^2}{l^2}\, dy + v_3(0)$$

$$= u_2 p_0 t \int_0^l \left(\frac{y^2}{l^2} - \frac{y^3}{l^3} \right) dy + v_2(0) + u_3 p_0 t \int_0^l \frac{y^3}{l^3}\, dy + v_3(0)$$

$$= u_2 p_0 t \left[\frac{y^3}{3l^2} - \frac{y^4}{4l^3} \right]_0^l + v_2(0) + u_3 p_0 t \left[\frac{y^4}{4l^3} \right]_0^l + v_3(0)$$

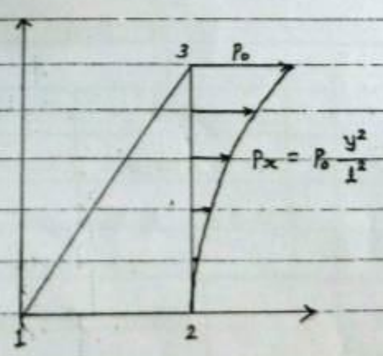
$$= u_2 p_0 t \left(l/3 - l/4 \right) + v_2(0) + u_3 p_0 t \left(l/4 \right) + v_3(0)$$

$$= \frac{p_0 t l}{12} u_2 + 0 \cdot v_2 + \frac{p_0 t l}{4} u_3 + 0 \cdot v_3$$

$$= \begin{bmatrix} u_2 & v_2 & u_3 & v_3 \end{bmatrix} \frac{p_0 t l}{12} \begin{Bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{Bmatrix}$$

$$= \{q\}^T \{T^e\}$$

$$\therefore \{T^e\} = \frac{p_0 t l}{12} \begin{Bmatrix} 1 \\ 0 \\ 3 \\ 0 \end{Bmatrix}$$



5. Determine nodal displacements, element stresses incl. principal stresses for the thin plate shown in fig.

Use $E = 210\,GPa$, $\nu = 0.3$.

Take different types of discretization as shown in fig. a

compare the results.

Note : You may use computer program or Excel work sheet for ma

operation.

$10\ N/mm^2 = 10 \times 250 \times 25\ N = 62500\ N$

$62500/2 = 31250\ N = 31.25\ kN$

$E = 210\ GPa = 210 \times 10^3\ MPa = 2.1 \times 10^8\ kN/m^2$

$\nu = 0.3$

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = 230.7692 \times 10^6 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

(a)

Element No.	Node-1	Node-2	Node-3
1	1	2	3
2	4	3	2

$$[A^1] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0 & 0.25 \end{bmatrix}$$

$2\Delta = 0.125$

$\therefore \Delta = 0.0625$

$$[A^1]^{-1} = \frac{1}{0.125}\begin{bmatrix} 0.125 & 0 & 0 \\ -0.25 & 0.25 & 0 \\ -0.5 & 0 & 0.5 \end{bmatrix}$$

$[d] = \frac{1}{0.125}$

$[k^e] = \Delta^e t^e$

$[k^1] = 23.0769$

$[A^2] =$

$2\Delta = 0.125$

$[B^2] = \frac{1}{0.125}$

$[k^2] = 23.07$

tresses incl
in fig.

wn in fig.

sheet for m

$$[B^1] = \frac{1}{0.125}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0.5 \\ -0.5 & -0.25 & 0 & 0.25 & 0.5 & 0 \end{bmatrix}$$

$$[k^e] = \Delta^e\, t^e\, [B^e]^T [D^e] [B^e]$$

$$[k^1] = 23.0769 \times 10^6 \begin{array}{c c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} & \\ \begin{bmatrix} 0.15 & 0.08125 & -0.0625 & -0.04375 & -0.0875 & -0.0375 \\ 0.08125 & 0.27188 & -0.0375 & -0.02188 & -0.04375 & -0.25 \\ -0.0625 & -0.0375 & 0.0625 & 0 & 0 & 0.0375 \\ -0.04375 & -0.02188 & 0 & 0.02188 & 0.04375 & 0 \\ -0.0875 & -0.04375 & 0 & 0.04375 & 0.0875 & 0 \\ -0.0375 & -0.25 & 0.0375 & 0 & 0 & 0.25 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \end{array}$$

$$[A^2] = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 1 & 0 & 0.25 \\ 1 & 0.5 & 0 \end{bmatrix} \qquad [A^2]^{-1} = \frac{1}{0.125}\begin{bmatrix} -0.125 & 0.125 & 0.125 \\ 0.25 & -0.25 & 0 \\ 0.5 & 0 & -0.5 \end{bmatrix}$$

$2\Delta = 0.125$

$$[B^2] = \frac{1}{0.125}\begin{bmatrix} 0.25 & 0 & -0.25 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & -0.5 \\ 0.5 & 0.25 & 0 & -0.25 & -0.5 & 0 \end{bmatrix}$$

$$[k^2] = 23.0769 \times 10^6 \begin{array}{c c} \begin{array}{cccccc} 7 & 8 & 5 & 6 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0.15 & 0.08125 & -0.0625 & -0.04375 & -0.0875 & -0.0375 \\ 0.08125 & 0.27188 & -0.0375 & -0.02188 & -0.04375 & -0.25 \\ -0.0625 & -0.0375 & 0.0625 & 0 & 0 & 0.0375 \\ -0.04375 & -0.02188 & 0 & 0.02188 & 0.04375 & 0 \\ -0.0875 & -0.04375 & 0 & 0.04375 & 0.0875 & 0 \\ -0.0375 & -0.25 & 0.0375 & 0 & 0 & 0.25 \end{bmatrix} & \begin{array}{c} 7 \\ 8 \\ 5 \\ 6 \\ 3 \\ 4 \end{array} \end{array}$$

$$[k]\{q\} = \{f\}$$

$$23.0769 \times 10^{6} \begin{array}{c} \begin{array}{cccc} 3 & 4 & 7 & 8 \end{array} \\ \begin{bmatrix} 0.15 & 0 & -0.0875 & -0.04375 \\ 0 & 0.22812 & -0.0375 & -0.25 \\ -0.0875 & -0.0375 & 0.15 & 0.08125 \\ -0.04375 & -0.25 & 0.08125 & 0.27188 \end{bmatrix} \end{array} \begin{array}{c} 3 \\ 4 \\ 7 \\ 8 \end{array} \begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 31.25 \\ -20 \\ 31.25 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} 16 \\ -31.52 \\ 29.57 \\ -35.25 \end{Bmatrix} \text{ m}$$

$$\{\sigma^e\} = [D^e][B^e]\{q^e\}$$

Element-1

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 7383.513 \\ 2215.054 \\ -5091.76 \end{Bmatrix} \text{ kN/m}^2 = \begin{Bmatrix} 7.3835 \\ 2.2151 \\ -5.0918 \end{Bmatrix} \text{ N/mm}^2$$

$\sigma_1 = 10.5093 \text{ N/mm}^2$

$\sigma_2 = -0.9107 \text{ N/mm}^2$

$\tan 2\theta = 1.9703$

$\theta_1 = 30.045^\circ$

$\theta_2 = 120.045^\circ$

Element - 2

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 20000.22 \\ 8273.68 \\ 8755.27 \end{Bmatrix} kN/m^2 = \begin{Bmatrix} 20 \\ 8.2376 \\ 6.8753 \end{Bmatrix} N/mm^2$$

$\sigma_1 = 22.58 \ N/mm^2$

$\sigma_2 = 5.43 \ N/mm^2$

$\tan 2\theta = -1.2157$

$\theta_1 = -22.323^\circ$

$\theta_2 = 67.233^\circ$

(b)

Element No.	Node-1	Node-2	Node-3
1	1	2	5
2	2	3	5
3	3	4	5
4	4	1	5

$$[A^1] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0.25 & 0.125 \end{bmatrix} \qquad [A^1]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0.0625 & 0 & 0 \\ -0.125 & 0.125 & 0 \\ -0.25 & -0.25 & 0.5 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^1] = \frac{1}{0.0625}\begin{bmatrix} -0.125 & 0 & 0.125 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.5 \\ -0.25 & -0.125 & -0.25 & 0.125 & 0.5 & 0 \end{bmatrix}$$

$$[A^2] = \begin{bmatrix} 1 & 0.5 & 0 \\ 1 & 0.5 & 0.25 \\ 1 & 0.25 & 0.125 \end{bmatrix} \qquad [A^2]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0 & -0.0625 & 0.125 \\ 0.125 & 0.125 & -0.25 \\ -0.25 & 0.25 & 0 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^2] = \frac{1}{0.0625}\begin{bmatrix} 0.125 & 0 & 0.125 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & 0.25 & 0 & 0 \\ -0.25 & 0.125 & 0.25 & 0.125 & 0 & -0.25 \end{bmatrix}$$

$$[A^3] = \begin{bmatrix} 1 & 0.5 & 0.25 \\ 1 & 0 & 0.25 \\ 1 & 0.25 & 0.125 \end{bmatrix} \qquad [A^3]^{-1} = \frac{1}{0.0625}\begin{bmatrix} -0.0625 & 0 & 0.125 \\ 0.125 & -0.125 & 0 \\ 0.25 & 0.25 & -0.5 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^3] = \frac{1}{0.0625}\begin{bmatrix} 0.125 & 0 & -0.125 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.25 & 0 & -0.5 \\ 0.25 & 0.125 & 0.25 & -0.125 & -0.5 & 0 \end{bmatrix}$$

$$[A^4] = \begin{bmatrix} 1 & 0 & \dots \\ 1 & 0 & \dots \\ 1 & 0.25 & \dots \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^4] = \frac{1}{0.0625}\begin{bmatrix} -0.12\dots \\ 0 \\ 0.25\dots \end{bmatrix}$$

$[k^e] = \Delta^e t^e [B\dots$

element-3

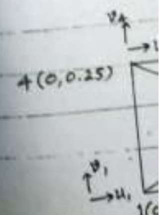
element-1

$[k^1] = [k^3] = 46.15384 \times 10\dots$

element-4

element-2

$[k^2] = [k^4] = 46.15384 \times 10^6\dots$



$$[A^4] = \begin{bmatrix} 1 & 0 & 0.25 \\ 1 & 0 & 0 \\ 1 & 0.25 & 0.125 \end{bmatrix} \qquad [A^4]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0 & 0.0625 & 0 \\ -0.125 & -0.125 & 0.25 \\ 0.25 & -0.25 & 0 \end{bmatrix}$$

$2A = 0.0625$

$$[B^4] = \frac{1}{0.0625}\begin{bmatrix} -0.125 & 0 & -0.125 & 0 & 0.25 & 0 \\ 0 & 0.25 & 0 & -0.25 & 0 & 0 \\ 0.25 & -0.125 & -0.25 & -0.125 & 0 & 0.25 \end{bmatrix}$$

$$[k^e] = \Delta^e\, t^e\, [B^e]^T [D^e] [B^e]$$

element-3: 5, 6, 7, 8, 9, 10

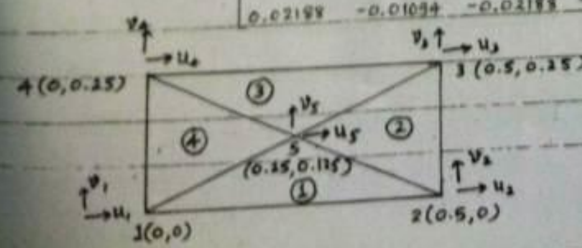
element-1: 1, 2, 3, 4, 9, 10

$$[k^1] = [k^3] = 46.15384 \times 10^6 \begin{bmatrix} 0.0375 & 0.02031 & 0.00625 & -0.00156 & -0.04375 & -0.01875 \\ 0.02031 & 0.06797 & 0.00156 & 0.05703 & -0.02188 & -0.125 \\ 0.00625 & 0.00156 & 0.0375 & -0.02031 & -0.04375 & 0.01875 \\ -0.00156 & 0.05703 & -0.02031 & 0.06797 & 0.02188 & -0.125 \\ -0.04375 & -0.02188 & -0.04375 & 0.02188 & 0.0875 & 0 \\ -0.01875 & -0.125 & 0.01875 & -0.125 & 0 & 0.25 \end{bmatrix} \begin{array}{cc} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \\ 9 & 9 \\ 10 & 10 \end{array}$$

element-4: 7, 8, 1, 2, 9, 10

element-2: 3, 4, 5, 6, 9, 10

$$[k^2] = [k^4] = 46.15384 \times 10^6 \begin{bmatrix} 0.0375 & -0.02031 & -0.00625 & -0.00156 & -0.03125 & 0.02188 \\ -0.02031 & 0.06797 & 0.00156 & -0.05703 & 0.01875 & -0.01094 \\ -0.00625 & 0.00156 & 0.0375 & 0.02031 & -0.03125 & -0.02188 \\ -0.00156 & -0.05703 & 0.02031 & 0.06797 & -0.01875 & -0.01094 \\ -0.03125 & 0.01875 & -0.03125 & -0.01875 & 0.0625 & 0 \\ 0.02188 & -0.01094 & -0.02188 & -0.01094 & 0 & 0.02188 \end{bmatrix} \begin{array}{cc} 3 & 7 \\ 4 & 8 \\ 5 & 1 \\ 6 & 2 \\ 9 & 9 \\ 10 & 10 \end{array}$$



$[k]\{q\} = \{f\}$

$$46.15384 \times 10^{6}
\begin{array}{c}
\begin{array}{cccccc} 3 & 4 & 5 & 6 & 9 & 10 \end{array} \\
\begin{bmatrix}
0.075 & -0.04063 & -0.00625 & -0.00156 & -0.075 & 0.04063 \\
-0.04063 & 0.13594 & 0.00156 & -0.05703 & 0.04063 & -0.13594 \\
-0.00625 & 0.00156 & 0.075 & 0.04063 & -0.075 & -0.04063 \\
-0.00156 & -0.05703 & 0.04063 & 0.13594 & -0.04063 & -0.13594 \\
-0.075 & 0.04063 & -0.075 & -0.04063 & 0.1 & 0 \\
0.04063 & -0.13594 & -0.04063 & -0.13594 & 0 & 0.54376
\end{bmatrix}
\end{array}
\begin{array}{c} 3 \\ 4 \\ 5 \\ 6 \\ 9 \\ 10 \end{array}
\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_5 \\ v_5 \end{Bmatrix}
=
\begin{Bmatrix} 31.2\ldots \\ -10 \\ \text{[illegible]} \\ 0 \\ \text{[illegible]} \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_5 \\ v_5 \end{Bmatrix} = 10^{-6} \begin{Bmatrix} 8.5455 \\ -46.2739 \\ 36.1204 \\ -48.4949 \\ 10.8657 \\ -21.632 \end{Bmatrix} m$$

$\{\sigma^e\} = [D^e][B^e]\{q^e\}$

Element-1

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 4777.6 \\ 3961.57 \\ -3214.95 \end{Bmatrix} kN/m^2 = \begin{Bmatrix} 4.7776 \\ 3.9616 \\ -3.2149 \end{Bmatrix} N/mm^2$$

$\sigma_1 = 7.6103 \ N/mm^2$

$\sigma_2 = 1.1288 \ N/mm^2$

$\tan 2\theta = 7.8794$

$\theta_1 = 41.38^\circ$

$\theta_2 = 131.38^\circ$

Element-2

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 4895.05 \\ 562.68 \\ 294.4 \end{Bmatrix} kN/m^2 = \begin{Bmatrix} 4.9851 \\ 0.5627 \\ 0.2944 \end{Bmatrix} N/mm^2$$

$\sigma_1 = 5.0046\ N/mm^2$ $\sigma_2 = 0.5432\ N/mm^2$

$\tan 2\theta = -0.1331$

$\theta_1 = -3.79^\circ$ $\theta_2 = 86.21^\circ$

Element-3

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 7611.2 \\ 86.3825 \\ -1592.53 \end{Bmatrix} kN/m^2 = \begin{Bmatrix} 7.6112 \\ 0.0864 \\ -1.5925 \end{Bmatrix} N/mm^2$$

$\sigma_1 = 7.9344\ N/mm^2$ $\sigma_2 = -0.2368\ N/mm^2$

$\tan 2\theta = 0.4233$

$\theta_1 = 11.47^\circ$ $\theta_2 = 101.47^\circ$

Element-4

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} 5014.95 \\ 1504.49 \\ -3494.4 \end{Bmatrix} kN/m^2 = \begin{Bmatrix} 5.0149 \\ 1.5045 \\ -3.4944 \end{Bmatrix} N/mm^2$$

$\sigma_1 = 7.1702\ N/mm^2$ $\sigma_2 = -0.6507\ N/mm^2$

$\tan 2\theta = 1.9908$

$\theta_1 = 31.66^\circ$ $\theta_2 = 121.66^\circ$

c) Element No. Node-1 Node-2 Node-3

Element No.	Node-1	Node-2	Node-3
1	1	5	6
2	1	2	5
3	2	3	5
4	3	4	5

$$[A^1] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25 & 0.25 \\ 1 & 0 & 0.25 \end{bmatrix} \qquad [A^1]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0.0625 & 0 & 0 \\ 0 & 0.25 & -0.25 \\ -0.25 & 0 & 0.25 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^1] = \frac{1}{0.0625}\begin{bmatrix} 0 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & 0 & 0 & 0.25 \\ -0.25 & 0 & 0 & 0.25 & 0.25 & -0.25 \end{bmatrix}$$

$$[A^2] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0.25 & 0 \\ 1 & 0.25 & 0.25 \end{bmatrix} \qquad [A^2]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0.0625 & 0 & 0 \\ -0.25 & 0.25 & 0 \\ 0 & -0.25 & 0.25 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^2] = \frac{1}{0.0625}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.25 & 0 & 0.25 \\ 0 & -0.25 & -0.25 & 0.25 & 0.25 & 0 \end{bmatrix}$$

$$[A^3] = \begin{bmatrix} 1 & 0.25 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0.25 & 0.25 \end{bmatrix} \qquad [A^3]^{-1} = \frac{1}{0.0625}\begin{bmatrix} 0.125 & -0.0625 & 0 \\ -0.25 & 0.25 & 0 \\ -0.25 & 0 & 0.25 \end{bmatrix}$$

$2\Delta = 0.0625$

$$[B^3] = \frac{1}{0.0625}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0 & 0 & 0 & 0.25 \\ -0.25 & -0.25 & 0 & 0.25 & 0.25 & 0 \end{bmatrix}$$

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Sat.

Boundary Element method, only the boundary surface of the body is discretised into the no of segment or Elements.

Q) Explain the steps involved in Finite Element Analysis

List of Explain The basic conceptual aspects of the FEM are presenting in following steps.

Step-1:- Discretize & select the Element type.

Step 2:- Select a displacement function. u, v, w ---

Step 3:- Define the strain / displacement & Stress-strain relationship.

Step 4:- Derive the Element stiffness matrix and Eqn

Step 5:- Assemble the Element Eqn to obtain the global or total Eqn & introduce boundary conditions.

Step 6:- Solve for unknown degree's of freedom (Generalized displacement's)

Step 7:- Solve for Element strain & stresses (Secondary unknown)

Step 8:- Interprete the results.

* Application of FEM :~

the FEM is used to analyse both structural & non structural problem typical structural area's include stress analysis including truss, & frame analysis & stress concentration problems

Reference book - Finite Element Method by Daryl. L. Logan

17 3 12 Sat

V.R.Prasad

FINITE ELEMENT METHOD.

INTRODUCTION:- finite Element method is a computer aided. mathematical technique. to obtain approximate numerical solution to the abstract of the Eqn that predicts the response of physical system under external influence's.

The FEM is a general form is used to analyse continuum structure. The direct stiffness method on other hand. is used to analyse skeleton structures.

The Basic concepts of the method is to discretise the structure into Elements. The structure is then consider as a assemblage of these Elements connected together at joints or node's. The displacement at node's are treated as primary unknown's these are obtained from the solution of Eqn which express the condition of minimization of total potential. energy of the structure.

Primary unknown :- Displacement

Secondary " → Stresses & strain

finally Secondary unknown are obtained

There are many numerical method available for structural analysis

1) finite difference Method.
2) Boundary Element method.

VIVA { In finite ~~Element~~ difference Method the basic Governing differential Eqn is discretised. In finite Element method the physical body or continuum is discretised.

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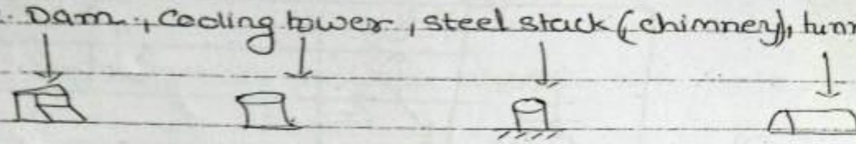
FINITE ELEMENT METHOD.

→ Skeletal structures → Beam, column, cable, arches, struts.
→ continuum structures
↓
2D, 3D.

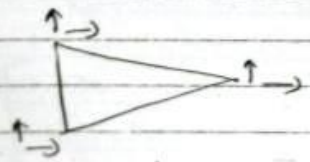
Eg: Dam, cooling tower, steel stack (chimney), tunne



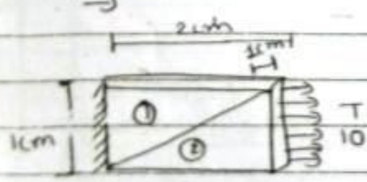
FEM is more applicable than Member approach method because it applied to all boundary condition.
FEM is a Extension of member approach method.

1) Discretization
2) Shape function
3) Stress / strain

14-3-12

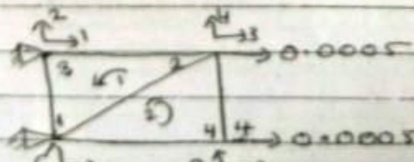


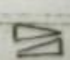
variational method
↓
minimium PE
↓
Stiffness matrix [k]



T (kN/m²)
10
Surface Traction

force = 10 × 0.01 × 0.01 = 0.0005 kN





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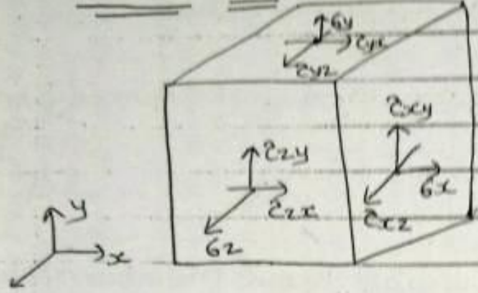
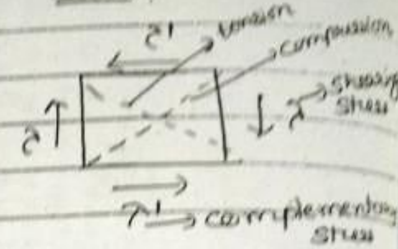
Stress - strain relationship in 3D.





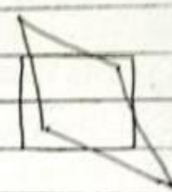
Shearing stress produce tension & compression

$\tau_{xy} = \tau_{yx}$

$\tau_{zx} = \tau_{xz}$

$\tau_{yz} = \tau_{zy}$

(Complementary shear stress)



$\left\{ \begin{array}{l} \nu \rightarrow \text{Mu poisson ratio.} \\ \vartheta \rightarrow v \end{array} \right\}$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \frac{E}{(1+\nu)(1+2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\varepsilon\}$$

$[D]$ = Stress / Strain matrix

= Constitute matrix

V.IMP. * PLANE STRESS

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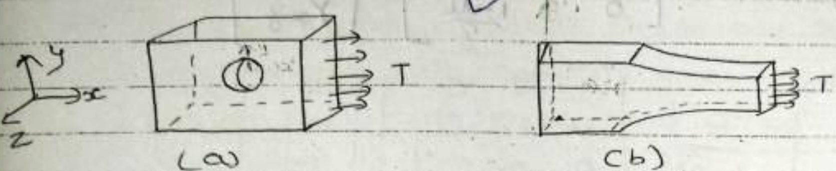


Fig 1 plane stress problem a) plate with hole.
b) Plate with fillet.

→ As plate are thin Element so idealised as a plane stress.

→ plane stress is to defined to be a state of stress in which the normal stress or shear stress directed to perpendicular to the plane are assumed to be zero ($\sigma_z = \tau_{xz} = \tau_{yz} = 0$)

→ for Eg:- fig 1(a) & 1(b) the plates in x,y plane shown subject to surface traction 'T' in the plane are under a state of plane stress.

i.e., the normal stress σ_z & shear stresses τ_{xz} τ_{yz} are assumed to be zero.

Generally member that are thin (z-dimension) whose loads acts only in the xy plane can be considered to be under plane stress.

for plane stress problem the stress-strain relation is given by

14/3/12 Wed

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\mu^2} \begin{bmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \frac{1-\mu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

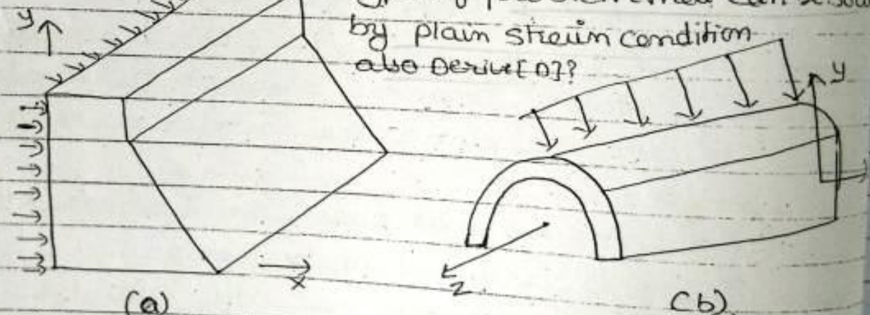
$$[\sigma] = [D]\{\varepsilon\}$$

$[D]$ = constitute matrix.

Thus for plane stress problem $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ & $\gamma_{xz} = \gamma_{yz} = 0$ & $\varepsilon_z \neq 0$ (strain).

* PLANE STRAIN

Q) What is plain strain condition? Discuss the different types of problem that can be solved by plain strain condition also Derive [D]?



(a) (b).

fig (a) plane strain problems.
a) dam subjected to horizontal loading.
b) Tunnel subjected to vertical load.

→ Plane strain is define to be a state of strain in which the strain to the xy plane ε_z and the shear strain γ_{xz}, & γ_{yz} are assumed to be zero.

→ The assumption of plane strain are realistics for long bodies with constant cross-sectional area subject to lateral loads and do not vary in longitudinal directional.

→ Some plane strain eg are shown in fig 2. In this eg only a unit thickness of the structure is considered because each unit thickness behaves identically except near the ends.

→ for plane strain problem the stress-strain relationship is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-\frac{2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$\{\sigma\} = [D]\{\epsilon\}$$

Thus for a plane strain problem $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$, & $\tau_{xz} = \tau_{yz} = 0$ & $\sigma_z \neq 0$.

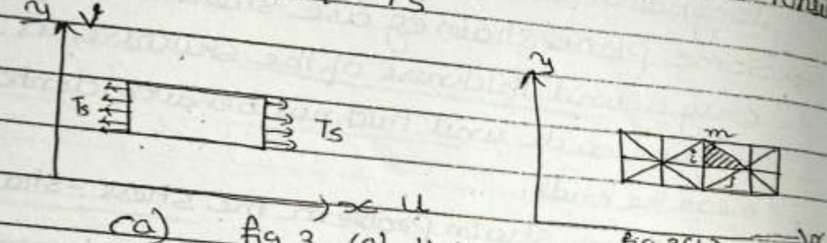
Q) Distinguish plain stress & plain strain problem? with suitable Eg? also Give their stress strain linking Matrices → [D]?

* Constant Strain Triangle (CST)

→ derivation of CST Element stiffness matrix and equations

14/3 wed.

consider the thin plate subject to tensile surface traction load T_s



(a) fig 3-(a) thin plate in tension

(b) fig 3-b Discretized plate using Δ element

Step-1 :- Select Element Type.

The plate is discretized using triangular Element each with nodes i, j, m. fig 3(b) each nodes has two degree of freedom as shown in fig -4.

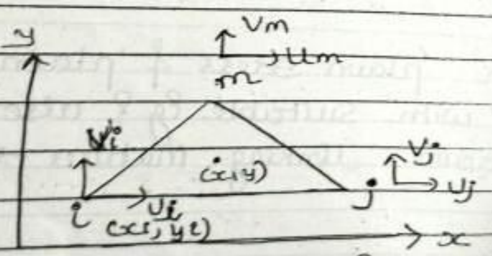


fig-4 Basic triangular Element showing degrees of freedom.

Thus the nodal displacement matrix.

$$\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \quad \text{—} \quad (1)$$

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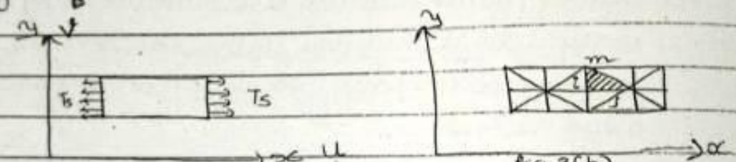
Constant Strain Triangle (CST).

→ derivation of CST Element stiffness matrix and Eqn.

consider the thin plate subject to tensile surface traction load T_s



(a) fig 3-(a) thin plate in tension

(b) fig 3-b Discretized plate using △ Element

Step-1 :- Select Element Type.

The plate is discretized using triangular Element each with nodes i, j, m. fig 3(b) each nodes has two degree of freedom as shown in fig -4.

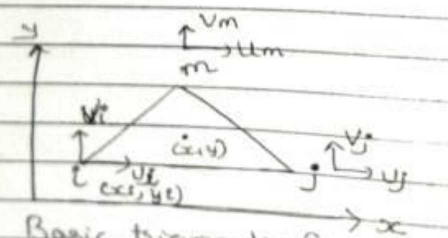


fig-4 Basic triangular Element showing degree of freedom

Thus the nodal displacement matrix

$$\{d\} = \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_m \\ v_m \end{Bmatrix} \quad \text{--- (1)}$$

two dimensional Element?

14.3.12 Wed.

Eqn. Step-2 Select displacement function

we select a linear displacement function for each element

$u(x,y) = a_1 + a_2 x + a_3 y$

$v(x,y) = a_4 + a_5 x + a_6 y$ — (2)

where $u(x,y)$ & $v(x,y)$ describe horizontal & vertical displacement at any interior point (x,y) Element

using Eqn-2 the generalized displacement function $\{\psi\}$ can be expressed as

$$\{\psi\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x & y \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

— (3)

To obtain constants a_1 to a_6 in Eqn (2) we begin by substituting the co-ordinates at nodal points in Eqn-(2)

$u_i = u(x_i\, y_i) = a_1 + a_2 x_i + a_3 y_i$

$v_i = v(x_i, y_i) = a_4 + a_5 x_i + a_6 y_i$

$u_j = u(x_j, y_j) = a_1 + a_2 x_j + a_3 y_j$

$v_j = u(x_j y_j) = a_4 + a_5 x_j + a_6 y_j$

$u_m = u(x_m\, y_m) = a_1 + a_2 x_m + a_3 y_m$

$v_m = v(x_m\, y_m) = a_4 + a_5 x_m + a_6 y_m$ (4)

we can solve for a_1, a_2, a_3, from Eqn (4) Express in matrix form

14 3 12
Wed

$$\begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad - (5)$$

$$\{u\} = [x]\{a\} \quad - (6)$$

$$\{a\} = [x]^{-1}\{u\} \quad - (6)$$

where

$$[x]^{-1} = \frac{1}{|x|} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \quad - (7)$$

where $\alpha_i = x_j y_m - y_j x_m$; $\alpha_j = y_i x_m - x_i y_m$

$\alpha_m = x_i y_j - y_i x_j$

$\beta_i = y_j - y_m \qquad \beta_j = y_m - y_i \qquad \beta_m = y_i - y_j$

$\gamma_i = x_m - x_j \qquad \gamma_j = x_i - x_m \qquad \gamma_m = x_j - x_i$ $\to$ (10)

and

$$|x| = \underbrace{\begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}}_{(8)} = \underbrace{x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)}_{(9)}$$

$$= 2A.$$

Here A is Area of triangle.

15-3-12 Thur: from Eqn-6. $\{a\} = [x]^{-1}\{u\}$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix} \quad - (11)$$

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Similarly using the last Eqn from (4)

$$\begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{bmatrix} V_i \\ V_j \\ V_m \end{bmatrix} \quad \text{--- (12)}$$

we derive the general the x-displacement functio $u(x,y)$ of $\{\psi\}$ in terms of co-ordinate variable. x & y, non-coordinate variable $\alpha_i, \alpha_j --- \gamma_m$. & unknown nodal displacement u_i, u_j & u_m

from Eqn-2

$$\{u\} = [1 \;\; x \;\; y] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \quad \text{--- (13)}$$

Substituting Eqn (11) into Eqn (13)

$$\{u\} = \frac{1}{2A} [1 \;\; x \;\; y] \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \quad \text{--- (14)}$$

Expanding Eqn -(14)

$$\{u\} = \frac{1}{2A} [1 \;\; x \;\; y] \begin{bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{bmatrix} \quad \text{--- (15)}$$

multiply two matrices in Eqn (15) & re-arranging

$$u(x,y) = \frac{1}{2A} \left[(\alpha_i + \beta_i x + \gamma_i y) u_i + (\alpha_j + \beta_j x + \gamma_j y) u_j + (\alpha_m + \beta_m x + \gamma_m y) u_m \right] \quad \text{--- (16)}$$

$\psi \rightarrow$ displacement within element

$d \rightarrow$ nodal displacement.

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Thurs

Similarly.

$$V(x,y) = \frac{1}{2A}\left[(\alpha i + \beta i x + \gamma i y)\, v_i + (\alpha j + \beta j x + \gamma j y)\, v_j + (\alpha m + \beta m x + \gamma m\, y]\, v_m\right] \quad \text{— (17)}$$

To Express Eqn (16) & (17) for u & V in Simplier form will defined.

$$N_i = \frac{1}{2A}(\alpha i + \beta i x + \gamma i y)$$

$$N_j = \frac{1}{2A}(\alpha j + \beta j x + \gamma j y).$$

$$N_m = \frac{1}{2A}(\alpha m + \beta m x + \gamma m y). \quad \text{— (18)}$$

Thus using (18) Eqn we can write Eqn (16) & (17) Eqn

shape fn

$$\begin{cases} U(x,y) = N_i U_i + N_j U_j + N_m U_m. \\ v(x,y) = N_i v_i + N_j v_j + N_m v_m \end{cases} \quad \text{— (19)}$$

Expressing Eqn (19) in matrix form.

$$\{\psi\} = \begin{Bmatrix} U(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{bmatrix} U_i \\ V_i \\ U_j \\ V_j \\ U_m \\ V_m \end{bmatrix} \quad \text{— (20)}$$

finally Expressing Eqn-20 abreviated form

$$\{\psi\} = [N]\{d\} \quad \text{— (21)}$$

$$\text{where } [N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \quad \text{— (22)}$$

$[N]$ = shape function

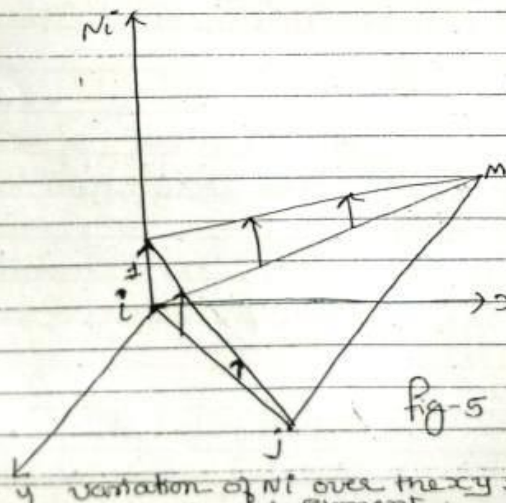
15 3 12
Thur

we have now Express general displacements as function of d in terms of the shape function $N_i, N_j, \& N_m$.

The shape function represents the shape of $\{\psi^n\}$ when plotted over the surface of the typical Element. for instance N_i represents the shape of the variable u when plotted over the surface of the Element for $U_i = 1$ & all othe degrees of freedom Equal to zero ie, $U_j = U_m = V_i = V_j = V_m = 0$

In addition, $U_{x_i y_i}$ must be Equal to U_i therefore we must have $N_i = 1$ $N_j = 0$ $N_m = 0$ at x_i, y_i

Similarly $U_{x_j y_j} = U_j$ therefore $N_i = 0$ $N_j = 1$ $N_m =$ at $x_j\, y_j$



variation of N_i over the xy surface of the typical Element

15 3 12
Thurs

fig 5 shows the shape variation of N_i plotted over the surface of a typical Element.
Note that N_i does not Equal to Zero Except along the line connecting and including nodes j & m
finally $N_i + N_j + N_m = 1$ for all x, y location of the surface of the Element so that
u & v will yield a constant value when rigid body occurs.

The Element must be able to translate uniform in either x or y-direction in the plane and to rotate without straining.

21-3-12

STEP-3

Define Strain / Displacement & Stress / strain relationships.

→ we express the Element of strain / stresses in terms of the unknown nodal displacements.
→ Element strains :- the strain associated with the two dimension Element is given by

u & v
∂u
$\{\varepsilon\} =$
Using
$\frac{\partial u}{\partial x}$
we
is a
Simil
Using
Expr
fun
$\frac{\partial N}{\partial}$
Si
u
$\frac{\partial u}{\partial x} =$

u & v ⇒ nodal displacement
∂u → change in displacement

21.3.12
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$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} \quad -(23)$$

Using Eqn → 19

$$\frac{\partial u}{\partial x} = \frac{\partial N_i}{\partial x} u_i + \frac{\partial N_j}{\partial x} u_j + \frac{\partial N_m}{\partial x} U_m \quad -(24)$$

we have use $\frac{\partial u_i}{\partial x} = 0$ coz $u_i = U(x_i y_i)$

is a constant value.

Similarly $\frac{\partial u_j}{\partial x} = 0 \quad \frac{\partial U_m}{\partial x} = 0 \quad -(25)$

Using Eqn (18) we can evaluate the Expression for the derivative of the shape function in Eqn (24) as follows.

$$\frac{\partial N_i}{\partial x} = \frac{1}{2A} \frac{\partial}{\partial x} (\alpha_i + \beta_i x + \gamma_i y) = \frac{\beta_i}{2A} \quad (26)$$

Similarly $\frac{\partial N_j}{\partial x} = \frac{\beta_j}{2A}$, $\frac{\partial N_m}{\partial x} = \frac{\beta_m}{2A}$ (27)

Using Eqn (26) & (27) in Eqn (24).

$$\frac{\partial u}{\partial x} = \frac{\beta_i}{2A} u_i + \frac{\beta_j}{2A} u_j + \frac{\beta_m}{2A} u_m \quad (28)$$

$$= \frac{1}{2A}\left[\beta_i u_i + \beta_j u_j + \beta_m U_m\right]$$

21 3 12
Wed

$$\frac{\partial v}{\partial y} = \frac{1}{2A}\left(\gamma_i v_i + \gamma_j v_j + \gamma_m v_m\right) \quad (29)$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{2A}\left[\gamma_i u_i + \beta_i v_i + \gamma_j u_j + \beta_j v_j + \gamma_m u_m + \beta_m v_m\right] \quad (29\text{-}a)$$

Using Eqn (28), (29), (29a) in Eqn (23)

$$\{\varepsilon\} = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_j \end{bmatrix}\begin{Bmatrix} u \\ v \\ \vdots \end{Bmatrix} \quad (30)$$

$$\{\varepsilon\} = [B]\{d\} \quad (31)$$

Strain, → nodal displacement matrix of B.

→ The Matrix [B] ⇒ constant ⇐ nodal displ

So Strain ⇒ constant x constant

⇒ constant.

So strain is constant through out the Element

CST

↳ Constant Strain triangle.

Solely → wholely

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The [B] is independent of the x & y co-ordinate it solely depends on the Element nodal co-ordinate as seen from Eqn (30) & (10)

The strain in Eqn (31) will be constant Hence the Element is called a constant Strain triangle

Part - II

Stress / Strain Relationship

In General the in plane Stress/strain relationship are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = [D] \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad - (32)$$

Where [D] = Constitutive Matrix.
Using Eqn (31) in (32) Eqn. we obtain in plane stresses in terms of the unknown nodal degree of freedom as

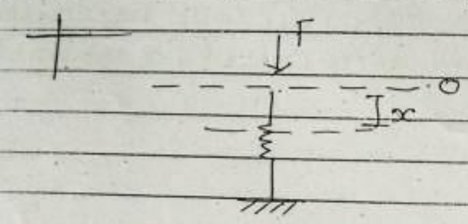
$$\{\sigma\} = [D][B]\{d\} \quad — (33)$$

where stresses σ are also Constant Everywhere within the Element.

Q) Using the theorem of min'm potential energy derive expression for Element stiffness matrix [k] for CST Element? 21/3/12 Wed.

★IMP STEP - 4

* Derive Element Stiffness matrix & Eqn.



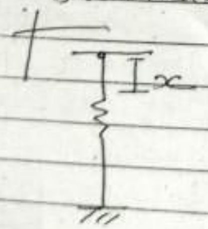
$T.E = PE + SE$

$= -Px + \frac{1}{2}kx^2$

When System in Equilibrium position then Potential energy is minimium.

→ Using principle of min'm potential Energy we can generate the Eqn for a typical CST Element.

→ for the basic plane stress Element the total potential Energy is now a function of nodal displacement $u_i, v_i, u_j, v_j, u_m, v_m$ such that



$P.E = Fx$

$\pi_p = \pi_p(x)$

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Wed.

$$\pi_p = \pi_p\,(u_i, v_i, u_j, v_j, u_m, v_m) \quad \text{(34)}$$

The total potential Energy is given by

$$\pi_p = \underbrace{U}_{\text{strain}} + \underbrace{\Omega_b + \Omega_s + \Omega_p}_{\text{P.E } \cancel{\text{strain}} \text{ Energy}} \quad \text{(35)}$$

$\Omega_b \Rightarrow$ body forces
$\Omega_s \Rightarrow$ Surface traction
$\Omega_p \Rightarrow$ point load.

Now,
Where strain Energy is given by

$$U = \frac{1}{2}\iiint \{\epsilon\}^T \{\sigma\}\, dv \quad \text{(36)}$$

(wrt to x, y, z)

OR Using Eqn (32).

$$U = \frac{1}{2}\iiint \{\epsilon\}^T [D] \{\epsilon\}\, dv \quad \text{(37)}$$

where $[D]^T = [D]$

The potential Energy of body force is given by

$$\Omega_b = -\iiint \{\psi\}^T \{X\}\, dv \quad \text{(38)}$$

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$\psi^n \Rightarrow$ General displacement fn.

$X \Rightarrow$ magnitude of body force. Per Unit vo
Or Weight density matrix

The potential Energy of point / concentra
load is given by

$$\Omega_P = -\{d\}^T\{P\} \quad --- \quad (39)$$

where $\{d\}$ represents usual nodal. dis
$\{P\}$ represents concentrated external loa

The potential energy of distributed load (Surfa
traction) moving through respective sur
displacement

$$\Omega_s = -\iint_s \{\psi_s^n\}^T \{T_s\}\, ds \quad -- \quad (40)$$

$\{x + y$ - direction$\}$
$dx\, \&\, dy$.

where $\{T_s\}$ represents the surface trac
$\{\psi_s^n\}$ represents the field of surface
displacement through which surface
traction acts, & ds, S represents the
surface over which the traction $\{T_s\}$ act

Similar

$\{\psi_s^n$
Shape fu
where
Using
Eqn

$\pi_p =$

$-\int$

$\iint_s$

The Nod
of Con
traction

$\pi_p = \frac{1}{2}$

from E
the
the
Eleme
$\{F\}$

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Similar to Eqn (21)

$\{\psi_s\} = [N_s]\{d\}$ where $[N_s]$ represents the shape function matrix evaluated along the surface where surface traction act

Using Eqn (21) $\{\psi\}$ & Eqn (31) for strain in Eqn (37) to (40) we have

$$\pi_p = \frac{1}{2}\iiint_V \{d\}^T[B]^T[D][B]\{d\}\,dv - \iiint_V \{d\}^T[N]^T\{X\}\,dv - \{d\}^T\{P\} - \iint_S \{d\}^T[N_s]^T\{T_s\}\,ds \qquad (41)$$

The Nodal displacements $\{d\}^T$ are independent of general x, y co-ordinate so $\{d\}$ can be taken out of the integrals of Eqn (41)

$$\pi_p = \frac{1}{2}\{d\}^T\left[\iiint_V [B]^T[D][B]\{d\}\,dv - \iiint_V [N]^T\{X\}\,dv - \{P\} - \iint_S [N_s]^T\{T_s\}\,ds\right] \qquad (42)$$

from Eqn (38) to (40) we can see that the last (3) terms of Eqn (42) represents the total load on the system $\{F\}$ or an Element

$$\{F\} = \iiint_V [N]^T\{X\}\,dv + \{P\} + \iint_S [N_s]^T[T_s]\,ds \qquad (43)$$

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Where the first, second, third terms of right side of Eqn (43) represents body force, concentrated nodal force & surface traction resp.

Using Eqn (43) in Eqn (42)

$$\pi_p = \frac{1}{2}\{d\}^T \iiint_V [B]^T[D][B]\,dv\,\{d\} - \{d\}^T\{F\} \quad (44)$$

Taking first variation or partial derivative of π_p w.r.t. nodal displacement we have

$$\frac{\partial \pi_p}{\partial \{d\}} = \left[\frac{1}{2}\iiint_V [B]^T[D][B]\,dv\right]\{d\} - \{F\} = 0 \quad (45)$$

Rewriting Eqn-(45) we have

$$\iiint [B]^T [D] [B]\,dv\,\{d\} = \{F\} \quad (46)$$

from Eqn $[K] = \iiint_V [B]^T[D][B]\,dv$ (47)

for an Element with constant thickness

$$[K] = t\iint [B]^T [D] [B]\,dx\,dy \quad (48)$$

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Where the integrand is not a function of x & y for the CST Element & thus can be taken outside the integral

$$[K] = tA\,[B]^T[D][B]. \quad (49)$$

Where A is given by Eqn (9), B is given by (31) & D is Given by Consitutive Matrix.

Note that when Eqn (43) is used to Evaluate the Surface & body forces. these Eqn are called Consistant load because they are derived from Energy approach.

for higher order Element Eqn (46) should be used. However for CST Element the body and Surface forces can be lumped at the nodes with Equivalent results & added to any concentrated nodal forces to obtain the Element force matrix

21/3/12 Wed.

STEP-5

*. Assemble the Element Eqn to obtain the global Eqn & introduce boundary condit

→ we obtain the global. the Structure stiffness matrix & Eqn by using direct stiffness method.

$$[K] = \sum_{e=1}^{N} [K^{(e)}] \quad -- \quad (50)$$

capital ← $\{F\} = [K]\{d\} \quad -- \quad (51)$

where in Eqn (50) all Element stiffness matrices $[K^{(e)}]$ are defined in terms of Global x, y co-ordinates, $\{d\}$ is now the total structure displacement matrix

$$\{F\} = \sum_{e=1}^{N} \{F^{(e)}\} \quad —— \quad (52)$$

is the column of Equivalent Global nodal loads obtain by lumping body forces & distributed loads at the proper nodes. Or by consistently using Eqn (43). In the formulation of Element stiffness matrix Eqn (4?) the matrix has been derived for a General Orientation in Global co-ordinates therefore no transformation from local to Global Eqn necessary.

STEP-6 Solve

we deter displacem Eqn give

STEP-7 Solve

Having S we obta Global using E the ma stresses Eqn

$\sigma_1 =$

$\sigma_2 =$

wher assum Elemen period is G

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STEP-6

Solve for a nodal displacements

we determine the unknown Global Structure nodal displacements by solving the system of Algebraic Eqn given by Eqn (51)

STEP-7 Solve for Element forces (Stresses)

Having solved for the nodal displacements we obtain the strains & stresses in the Global x & y directions in the Elements by using Eqn (31) to (33) finally we determine the max'm & min'm in plane principal stresses σ_1 & σ_2 by using transformation Eqn

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sigma_{max}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (53)$$

$$= \sigma_{min}$$

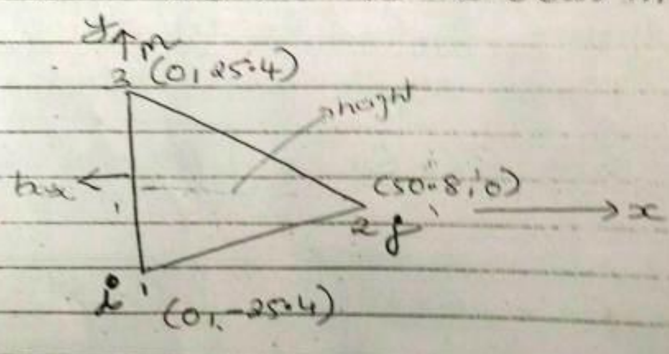
where these stresses are usually assume to act at the centroid of the Element The angle that one of the principal stress makes with the x-axis is Given by:

$\tan 2\theta_p = \dfrac{2 \tau_{xy}}{\sigma_x - \sigma_y}$ ——— 5.4

—— X —— X —— X

Problem-1 Evaluate the stiffness matrix for the element shown in figure-1 the co-ordinates are shown in unit of mm Assume plane stress condition Take $E = 2.07 \times 10^5$ $\nu = 0.25$, thickness 25.4 mm

Assume the element nodal displacement have been determined to be
$U_1 = 0$ $V_1 = 0.635$ mm $U_2 = 0.03048$ mm
$V_2 = 0$ $U_3 = 0$ $V_3 = 0.0635$ mm



Sol^n →

$$[K] = tA\,[B]^T[D][B]$$

$$[B] = \frac{1}{2A}\begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix}$$

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where

$$\beta_i = y_j - y_m = 0 - 25.4 = -25.4$$

$$\beta_j = y_m - y_i = 25.4 - (-25.4) = 50.8$$

$$\beta_m = y_i - y_j = -25.4 - 0 = -25.4.$$

$$\gamma_i = x_m - x_j = 0 - 50.8 = -50.8$$

$$\gamma_j = x_i - x_m = 0 - 0 = 0$$

$$\gamma_m = x_j - x_i = \cancel{25.4 - 0 = 25.4}\ \ 50.8 - 0 = 50.8$$

$$A = \frac{1}{2} bh = \frac{1}{2} \times 50.8 \times 50.8 = 1290.32\,mm^2$$

$$[B] = \frac{1}{2 \times 1290.32} \begin{bmatrix} -25.4 & 0 & 50.8 & 0 & -25.4 & 0 \\ 0 & -50.8 & 0 & 0 & 0 & 50.8 \\ -50.8 & -25.4 & 0 & 50.8 & 50.8 & -25.4 \end{bmatrix}$$

for the plane stress condition the constitutive matrix

$$[D] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

$$= \frac{2.07 \times 10^5}{1 - 0.25^2} \begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix}$$

22 3 12 Thur

$[k] = tA[B]^T[D][B]$.

$$[k] = \frac{25.4 \times 1290.32 \times 2.07\times10^5}{2^2 \times 1290.32^2 \times (1-0.25^2)} \begin{bmatrix} -25.4 & 0 & -50.8 \\ 0 & -50.8 & -25.4 \\ 50.8 & 0 & 0 \\ 0 & 0 & 50.8 \\ -25.4 & 0 & 50.8 \\ 0 & 50.8 & -25.4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0.25 & 0 \\ 0.25 & 1 & 0 \\ 0 & 0 & 0.375 \end{bmatrix} \begin{bmatrix} -25.4 & 0 & 50.8 & 0 & -25.4 & \cdots \\ 0 & -50.8 & 0 & 0 & 0 & \cdots \\ -50.8 & -25.4 & 0 & 50.8 & 50.8 & \cdots \end{bmatrix}$$

$$= 1086.63\times10^3 \begin{bmatrix} 1.613 & 0.806 & -1.29 & -0.968 & -0.323 & \cdots \\ 0.806 & 2.823 & -0.645 & -0.484 & -0.161 & \cdots \\ -1.29 & -0.645 & 2.581 & 0 & -1.29 & \cdots \\ -0.968 & -0.484 & 0 & 0.968 & 0.968 & \cdots \\ -0.323 & -0.161 & -1.29 & 0.968 & 1.613 & \cdots \\ 0.161 & -2.34 & 0.645 & -0.484 & -0.806 & \cdots \end{bmatrix}$$

To Evaluate the stresses

Using $\{\sigma\} = [D][B]\{d\}$.

22.3.12.
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$$\{\sigma\} = \frac{2.07\times10^5}{1-0.25^2} \times \frac{1}{2\times1290.32}\begin{bmatrix}1 & 0.25 & 0\\ 0.25 & 1 & 0\\ 0 & 0 & 0.375\end{bmatrix}$$

$$\begin{bmatrix}-25.4 & 0 & 50.8 & 0 & -25.4 & 0\\ 0 & -50.8 & 0 & 0 & 0 & 50.8\\ -50.8 & -25.4 & 0 & 50.8 & 50.8 & -25.4\end{bmatrix}\begin{Bmatrix}0.0\\ 0.0635\\ 0.03048\\ 0.0\\ 0.0\\ 0.0635\end{Bmatrix}$$

$$= 85.56\begin{Bmatrix}1.548\\ 0.387\\ -1.21\end{Bmatrix} N/mm^2$$

$$\{\sigma\} = \begin{Bmatrix}132.45\\ 33.11\\ -103.53\end{Bmatrix} N/mm^2.$$

$\sigma_x = 132.45$, $\sigma_y = 33.11$ $\tau_{xy} = -103.53$

principal stresses & principal Angles are obtained as.

$$\sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 82.78 + 114.82 = 197.6\ N/mm^2$$

$$\sigma_2 = 82.7 - 114.82 = -32.04\ N/mm^2$$

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x-\sigma_y}\right) = -32.18^\circ$$

2.1 One dimensional displacement function

$$f(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 - \quad + \alpha_{m+1} x^n$$

Eg for $n=2$

$$f(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \quad (1-D)$$

The co-efficients of the polynomial αs are known as generalized co-ordinates or generalized displacement amplitudes.

2.2 Two dimensional displacement function

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + --- + \alpha_m y^n$$

$$v = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + -- + \alpha_{2m} y$$

where u & v denote the component of displacement in x & y direction respectively

Since we are interested in displacement method of structural analysis, we assume simple functions to approximate the displacement for each Element These functions are called 'Displacement models' 'Displacement functions' 'Displacement fields'.

The unknown magnitudes of the displacement functions are the displacement at the nodal points.

Displacement model can be expressed as 1) Trigonometry functions.
2) Polynomial functions.

Generally Polynomial are selected as displacement models because they are easy to integrate or differentiate & thus simplify the mathematical manipulation.

Function a polynomial of arbitrary order permits a recognizable approximation to the true solution & it is easy to vary degree of approximation.

Thus in a bonded matrix all nonzero terms are clustered along a band. All co-efficients outside of the band width are zero & they do not have to be stored.

$b = (d+1)f$

b = bandwidth

d = largest difference betn node no in a Single Element

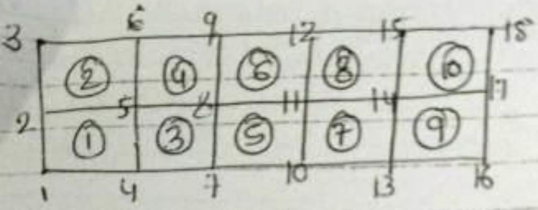
F = DOF at each node.

→ For example, for a plate as shown in fig different labelling system is shown band width for a system in fig (A) is 16 & that for system in fig (B) is 10

The proper labelling of the nodes thus reduces the computer storage.

2) Select displacement function

The basic theory of FEM is going to a part to whole we sub divide the given problem & represent the sol'n within each Element by a relatively simple function



$d = 4$
$f = 2$
$b = (4+1) \times 2 = 10$

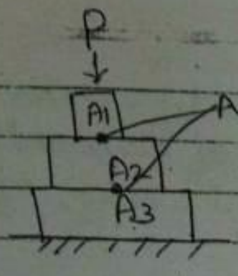
no. of DOF = $18 \times 2 = 36$

In order to facilitate manipulations in a Computer program we must have a systematic method of labelling elements & nodes.

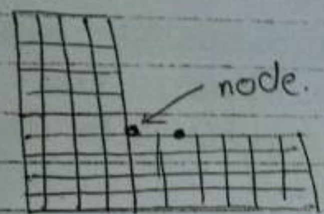
The labelling of nodes will be a trivial (small) operation if the node not did not influence the computational efficiency associated with obtaining a solution.

The set of linear Equations which arises when using the FEM has a large no of co-efficients which are zero.

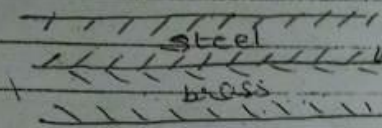
A listing of the Eqn's would show that all the non-zero values & some zero values fall betn two lines which can be constructed parallel to the main diagonal to the line is called Semi-band width.



(C) Stepped column



(D) Re-entreant corner

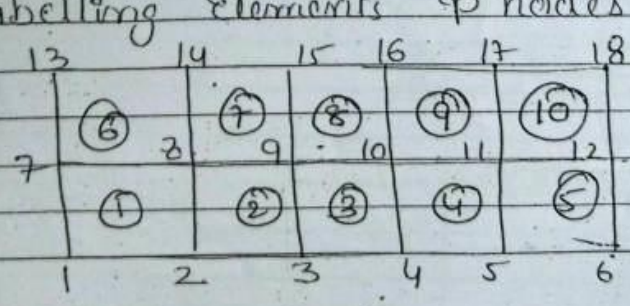


(E) change in material

* DICRETIZATION OF DISCONTINUITIES

The most obvious locations for nod or sub-division lines & planes are places wher abrupt / Sudden changes in Geometry, loading & material properties occur.
Examples of these natural sub-divisions are shown in fig for various 1-D & 2-D problem

(c) Labelling Elements & nodes



$d = 8-1 = 7$
$f = 2$
$b = (7+1) \times 2$
$= 16$

Discretization is the process of subdividing the Given body into no of Elements, which results into an equivalent body of finite Elements.

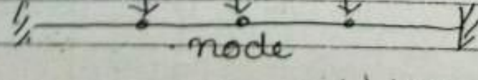
In the discretization, we have to decide the no size & type of the Elements in such a way that the original body is simmulated as closely as possible.

(A) Types of finite Elements.

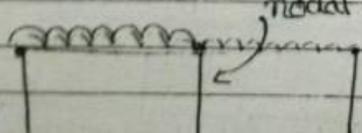
Selection of type of basically Element depends upon

1) Geometry of the body or structure.
2) upon the no of independent space co-ordinates (x, y & z) necessary to describe the problem various 1, 2D or 3D Element can be used fig shows different type of Elements in common use.

(B) The division of the continuum into Elements.

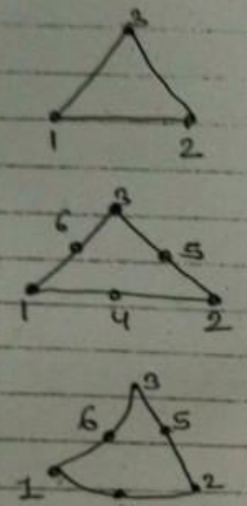


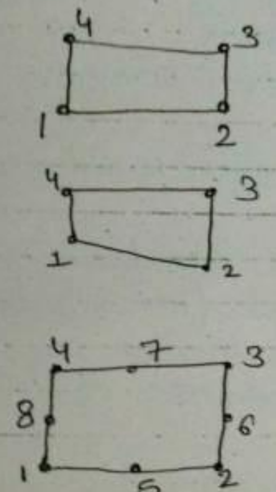
(A) point load



(B) change in loading

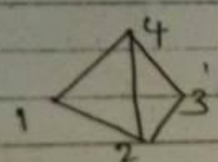
1 - D Elements → ——— line



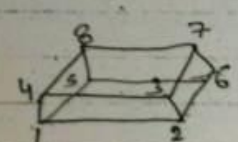


2D Elements
(Triangular)

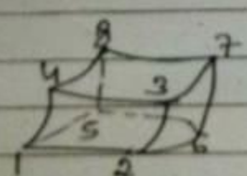
2D Elements
Quadrilateral.



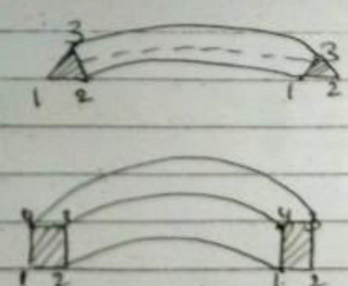
Tetrahedron



Hexa hedron



Curved Element



3D-Elements

Axisymmetric Elements.

typically associated with holes, fillet or other changes in geometry in a body.
2) Buckling
3) vibration analysis.

18-3-12

Non-structural problems include

1) heat transfer
2) fluid flow, including seepage through porous media
3) Distribution of Electric or magnetic potential.

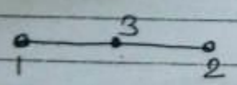
finally, some bio-mechanical engg problems typically include analysis of human-spine, Skull, heart, eye etc.

Q) Discuss with illustration discretisation process.

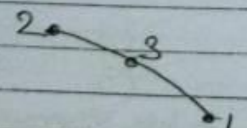
1) Discretize & Select the type of Element

TYPES OF ELEMENTS.

(A) Basic Element.



(B) H-Element

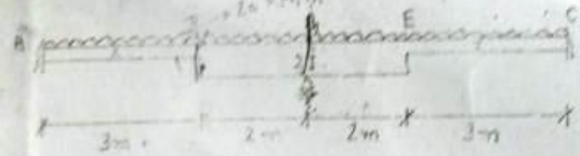


(C) Curved Element

PARTH DAMANI
ME CASAD
Roll No 01

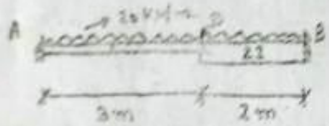
Problem No (1) STIFFNESS MEMBER APPROACH ⇒ "BEAM"

Check

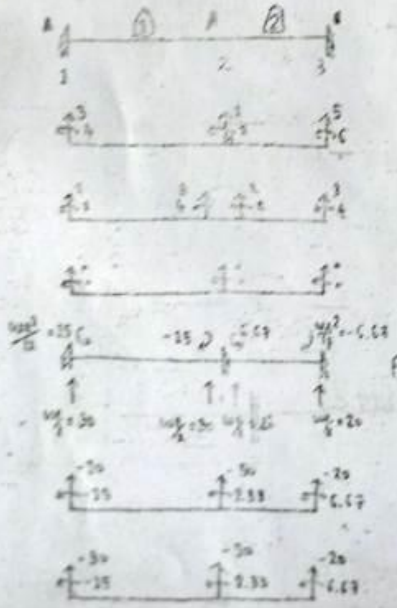


given beam

Now, By Using Symmetry..



given beam after applying Symmetry



Member and joint numbering

Global Nos Arrangement

Local Nos Arrangement

$[A_J]$

Fixed End Action

Reversed Action $[A_E]$

$[A_C] = [A_E] + [A_J]$

Now, Global & Local Nos. Arrangement...;

Member	Local Nos.	Global Nos.
1	1, 2, 3, 4	3, 4, 1, 2
2	1, 2, 3, 4	1, 2, 5, 6

∴ Now, $[A_C]^T = [A_E]^T + [A_J]^T$

$\therefore [A_C]^T = [A_E]^T \quad [\because [A_J]^T = [0]]$

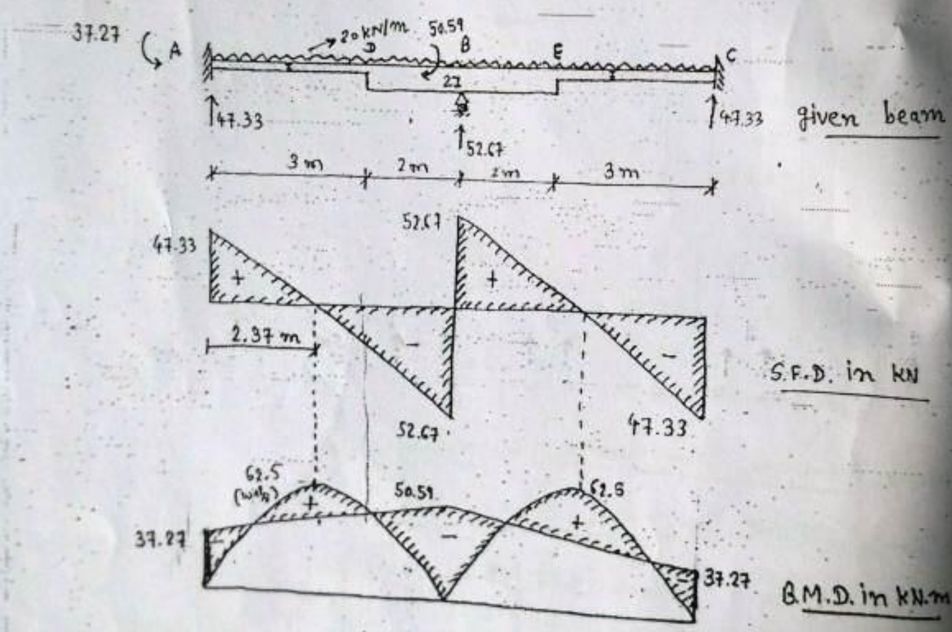
$\therefore [A_C]^T = [-50, 2.33 \,\vdots\, -30, -15, -20, 6.67]$

$[A_{FC}]^T \quad [A_{RC}]^T$

$$
= \begin{bmatrix} 20 \\ 6.67 \\ 20 \\ -6.67 \end{bmatrix} + \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \left[\begin{array}{cc|cc} 3 & 3 & -3 & 3 \\ 3 & 4 & -3 & 2 \\ \hline -3 & -3 & 3 & -3 \\ 3 & 2 & -3 & 4 \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \cdot EI \cdot \frac{1}{EI} \begin{bmatrix} -22.14 \\ 11.25 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array}
$$

$$
\therefore \begin{bmatrix} R_B \\ M_{BB} \\ R_B \\ M_{BD} \end{bmatrix} = \begin{bmatrix} -12.67 \\ -14.75 \\ 52.67 \\ -50.59 \end{bmatrix} \begin{array}{l} kN \\ kN.m \\ kN \\ kN\,m \end{array}
$$

→ Now, S.F. & B.M. Diagram:



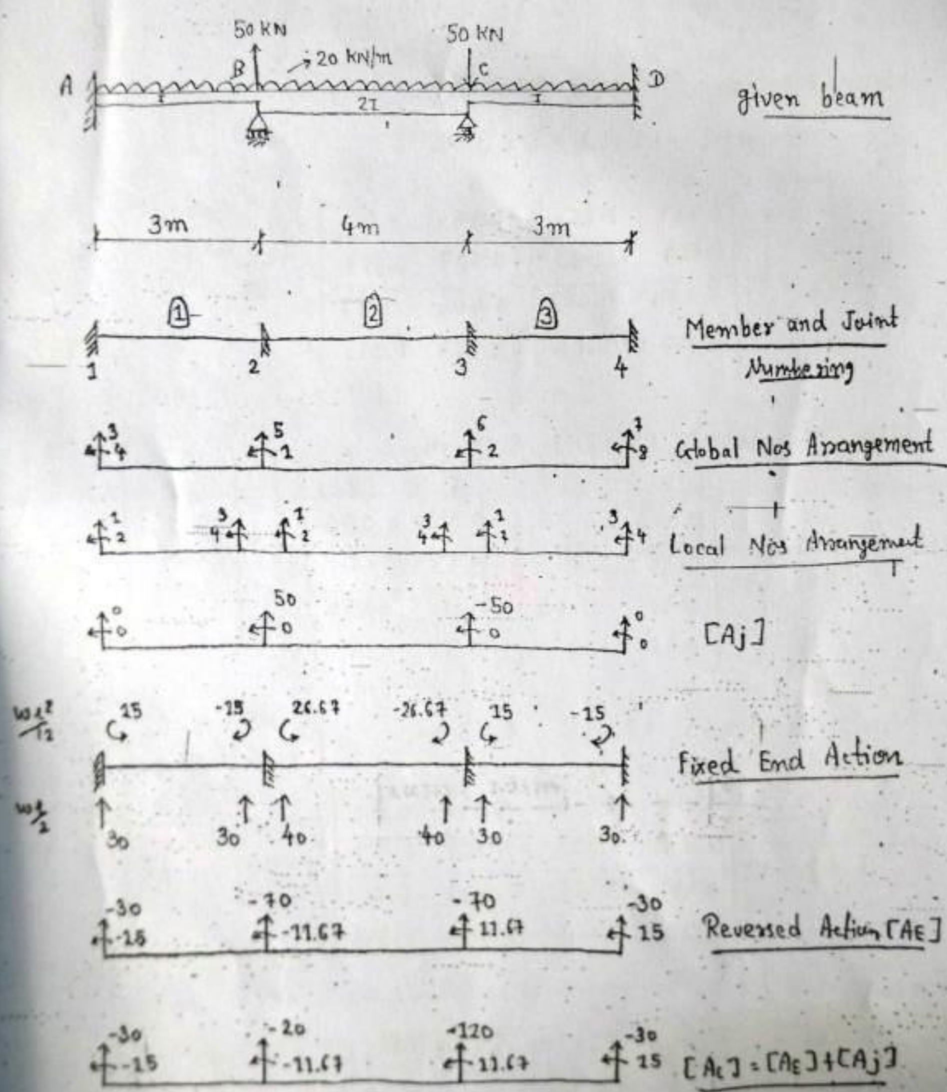
Point of Contraflexure:

$$\frac{47.33}{x} = \frac{52.67}{5-x} \Rightarrow x = 2.37\,m$$

$$47.33 \times 2.37 - 37.27 - 20 \times (2.37)^2/2 = 18.73$$

4.....

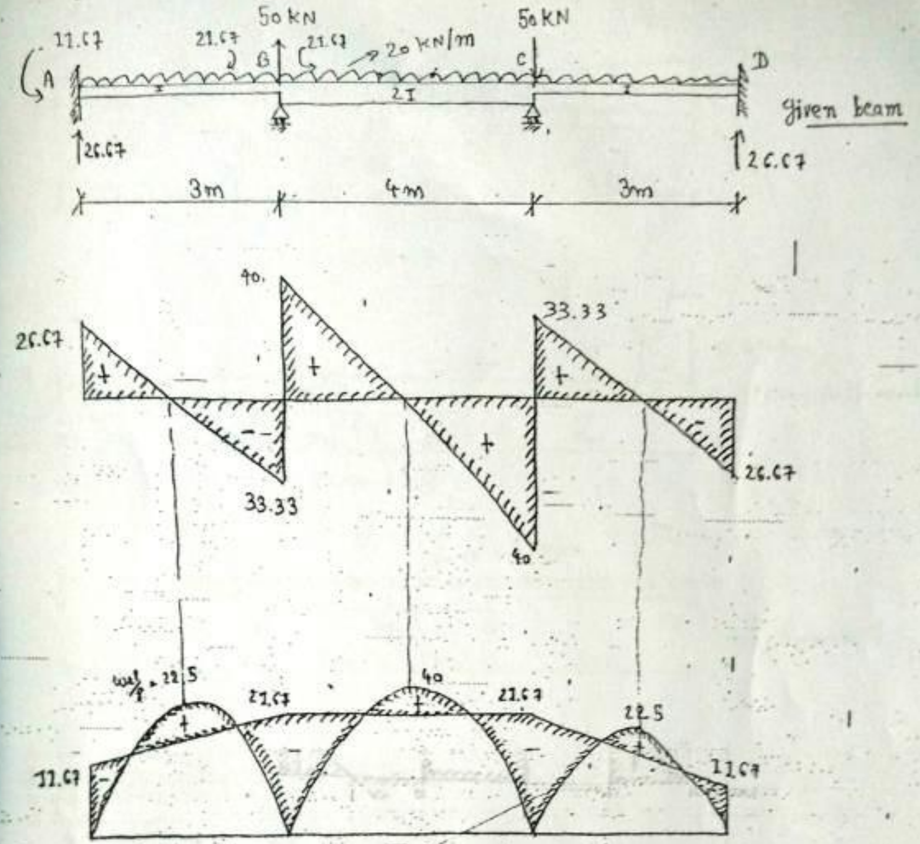
No: 3

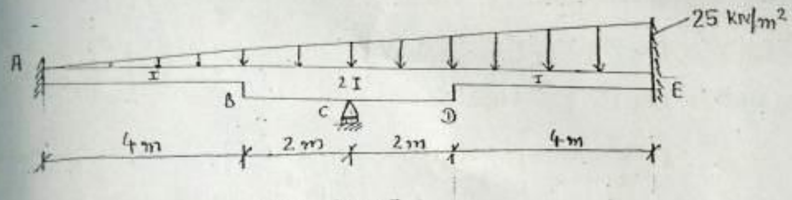


Now, Global & Local Nos. Arrangement:

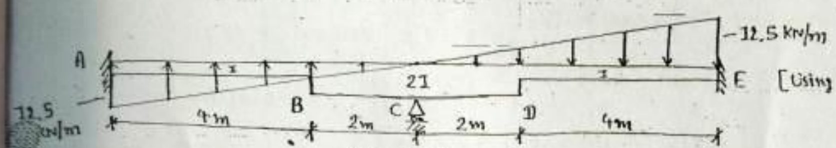
Member	Local Nos.	Global Nos..
1	1, 2, 3, 4	3, 4, 5, 1
2	1, 2, 3, 4	5, 1, 6, 2
3	1, 2, 3, 4	6, 2, 7, 8

S.F. & B.M. Diagram...



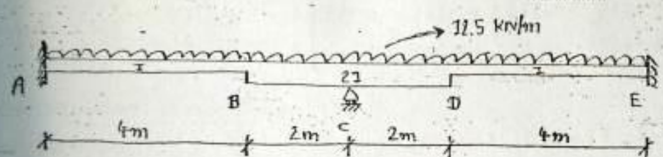


[Given beam]

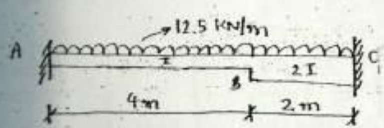


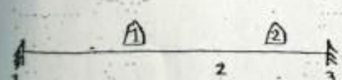
[Using AntiSymmetry]

[CASE-2]



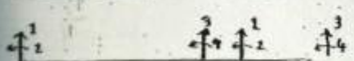
[Using Symmetry]



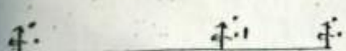


[Member & Joint numbering]

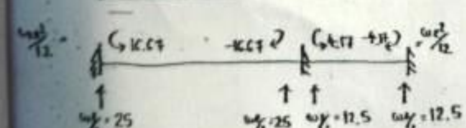
[Global Nos Arrangement]



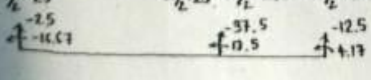
[Local Nos Arrangement]



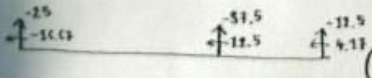
[Aj]



[Fixed End Action]



[Reversed Action [AE]]



[Ac] = [AE] + [Aj]

so, finally...;

$R_A = 35.30 - 29.88 = 5.42$ kN

$R_B = 39.71 - 29.88 = 9.83$ kN

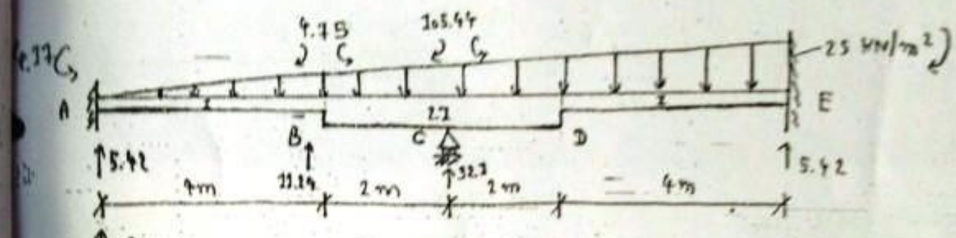
$M_{AB} = 33.44 - 29.27 = 4.17$ kN.m

$M_{BA} = 7.74 - 12.49 = -4.75$ kN.m

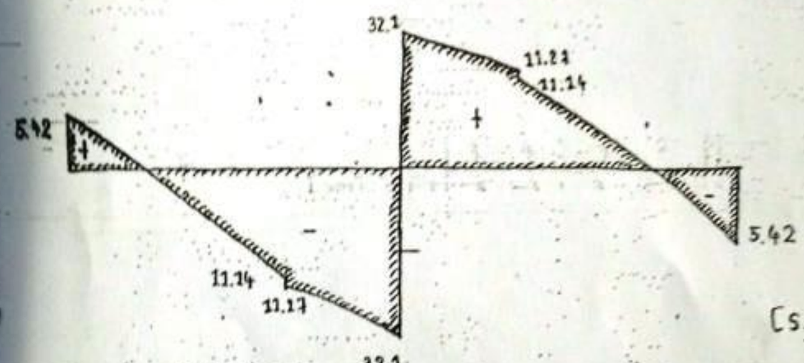
$M_{BC} = -7.76 + 12.47 = 4.71$ kN.m

$M_{CB} = -46.66 - 58.78 = -105.44$ kN m

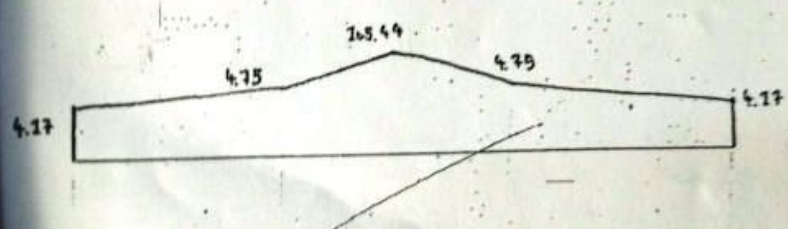
$R_C = 39.71 - 7.61 = 32.1$ kN



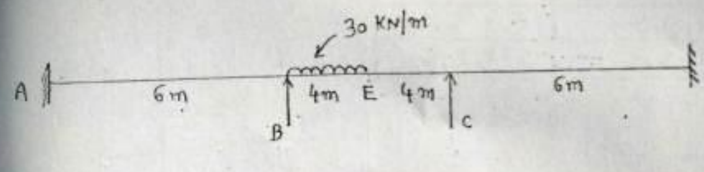
[Given Beam]



[S.F.D. in kN]

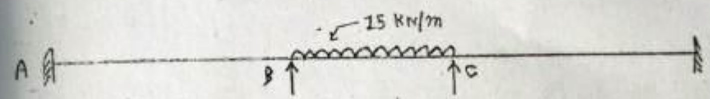


LEM : 5

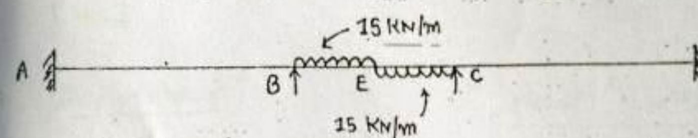


[Given Beam]

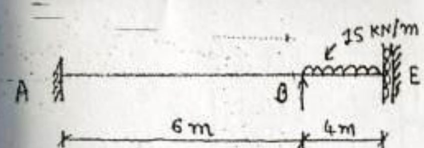
EI - Constant



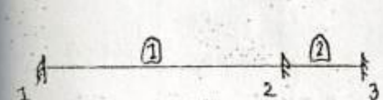
(i) Symmetry



(ii) Anti-Symmetry

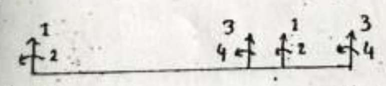


[Using Symmetry in case -(i).]

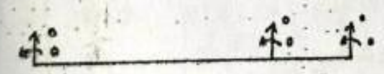


[Member & Joint numbering.]

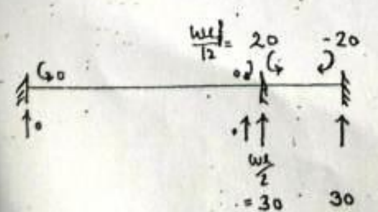
[Global Nos Arrangement]



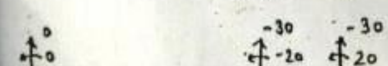
[Local Nos Arrangement]



[AJ].



[Fixed End Action]



[Reversed Action (AE)]

$\curvearrowleft = -36.1$

$AR_1 = -14.57 - 3.54 = -18.11$

$AR_3 = 74.57 + 37.07 = 111.64$

$AR_4 = 61.81 + 26.47 = 88.28$

1

$AM_2 = -29.05 - 7.05 = -36.1$

$AM_4 = -58.19 - 14.12 = -72.31$

$AM_1 = -14.57 - 3.54 = -18.11$

$AM_3 = 14.57 + 3.54 = 18.11$

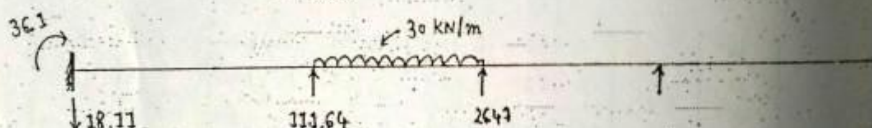
2

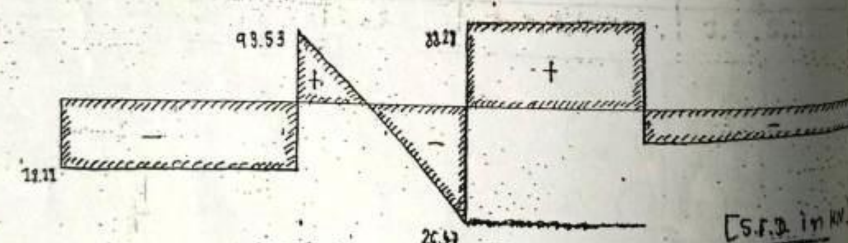
$AM_2 = 58.19 + 14.13 = 72.32$

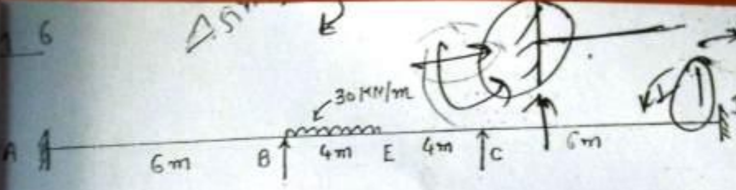
$AM_4 = 61.81 + 0 = 61.81$

$AM_1 = 60 + 33.53 = 93.53$

$AM_3 = 0 + 26.47 = 26.47$





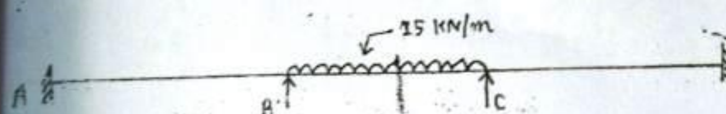


[Given Beam]

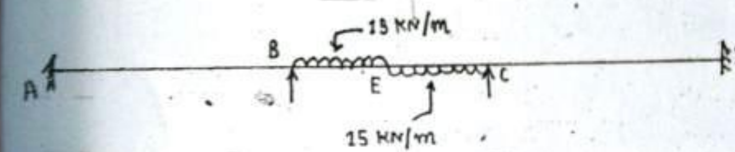
EI - Constant.

Support B sinks by 20 mm

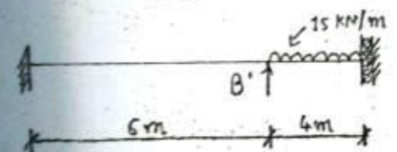
& Support C sinks by 10 mm



(i) Symmetry



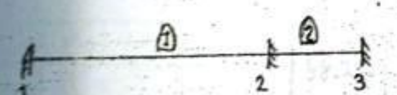
(ii) Anti-Symmetry



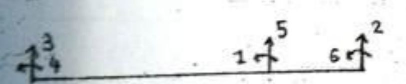
Support B sinks by $\frac{20+10}{2} = 15$ mm

$= 0.015$ m

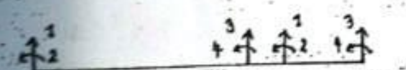
[using Symmetry in Case - (i)]



[Member & Joint Numbering]

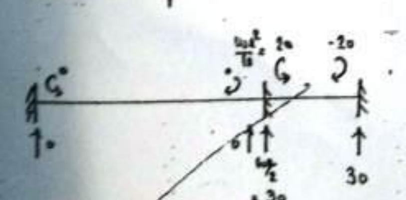


[Global Nos Arrangement]

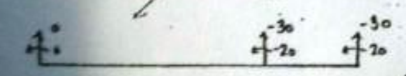


[Local Nos Arrangement]

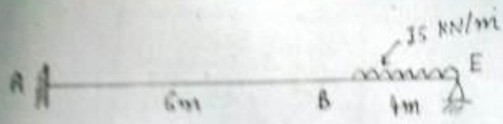
[AJ]



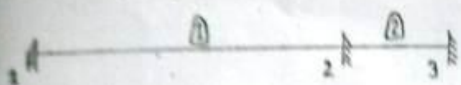
[Fixed End Action]



[Reversed Action [AE]]



[Using Symmetry in Case-(ii)]

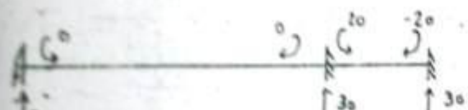


[Member & Joint Numbering]


[Global Nos Arrangement]


[Local Nos Arrangement]


[AJ]



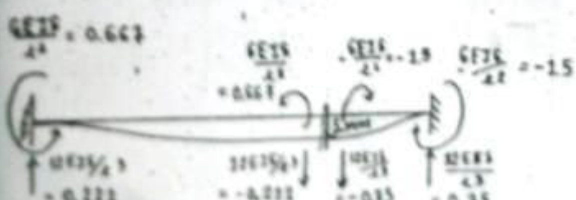
[Fixed End Action]

[Reverse End Action]

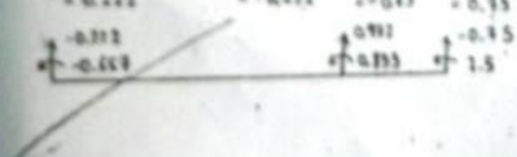
$\delta = 0.020 - 0.015$

$\therefore \delta = 0.005$ m

& Assume ; $EI = 800$ kN·m²

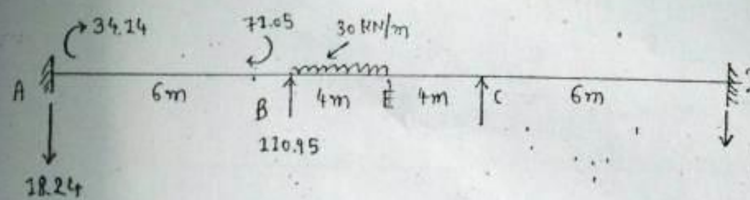


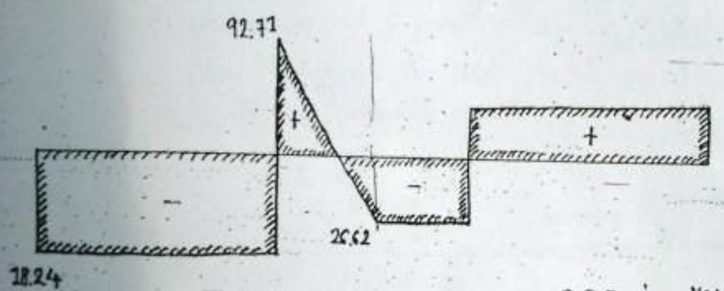
[Secondary effect]



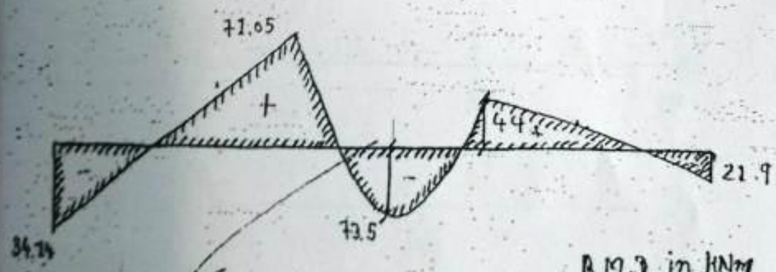
[Reversed Action due to Secondary effect]

[Final [AE]]

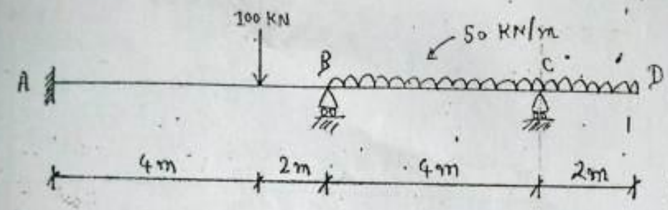




S.F.D. in KN

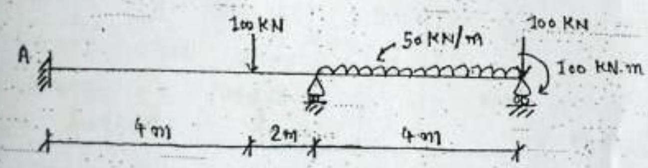


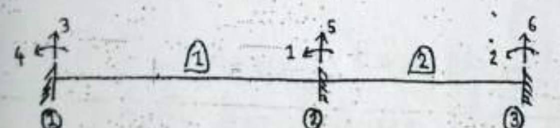
B.M.D. in KNm



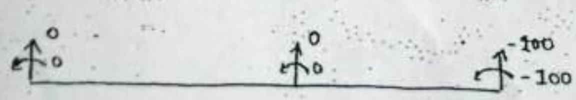
[given beam]

Settlement of Support 'B' By - 6 mm
Rotation of Support 'A'. By - 0.5° clockwise
In addition to load given on it...
Take EI = 20,000 KN.m²

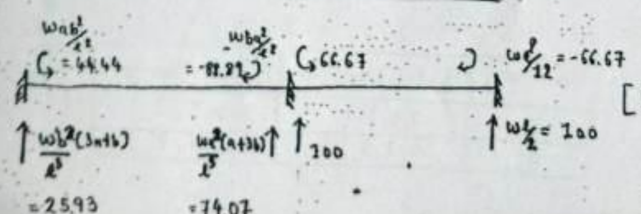




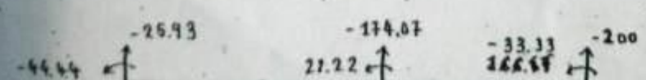
[Member & Joint Numbering]
[Global Numbering]



[AJ]



[Fixed End Acti



[AC] = [AF] + [AJ]

For Member-2,

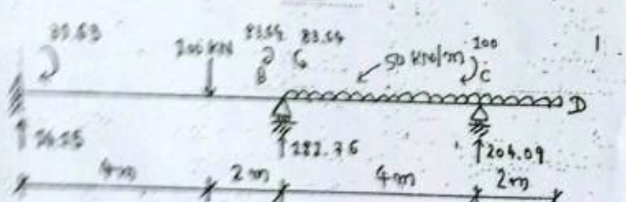
$$[AM]_2 = [AML]_2 + [SM]_2 \cdot [DM]_2$$

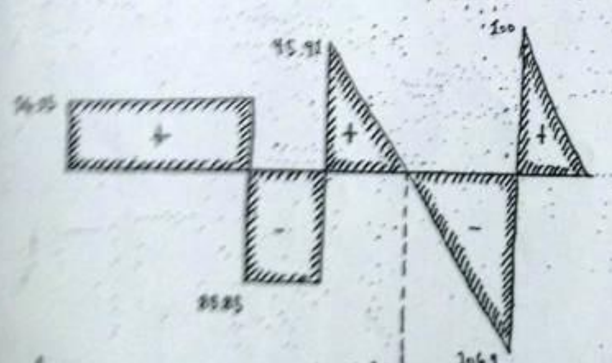
$$= \begin{bmatrix} 83.25 \\ 29.17 \\ 118.75 \\ -104.17 \end{bmatrix} + EI \begin{array}{c} \begin{array}{cccc} 5 & 1 & 6 & 2 \end{array} \\ \begin{bmatrix} 0.188 & 0.375 & -0.188 & 0.375 \\ 0.375 & 1 & -0.375 & 0.5 \\ -0.188 & -0.375 & 0.188 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 1 \\ 6 \\ 2 \end{array} \cdot \frac{1}{EI} \begin{bmatrix} 0 \\ 69.84 \\ 0 \\ -30.75 \end{bmatrix}$$

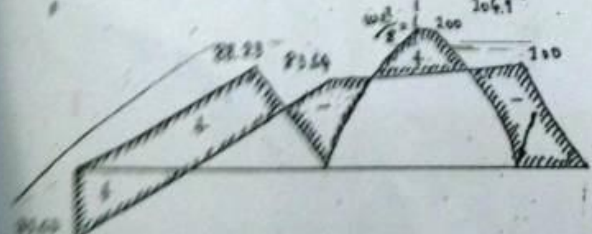
$$\therefore \begin{bmatrix} R_B \\ M_{BC} \\ R_C \\ M_{CB} \end{bmatrix} = \begin{bmatrix} 95.91 \\ 83.64 \\ 104.09 \\ -100 \end{bmatrix} \begin{array}{l} kN \\ kN.m \\ kN \\ kN.m \end{array}$$

Check ; $R_B = 85.85 + 95.91 = 181.76$ KN (✓ O.K.)

& $R_c = 104.9 + 100 = 204.9$ KN [Due to Over hanging]







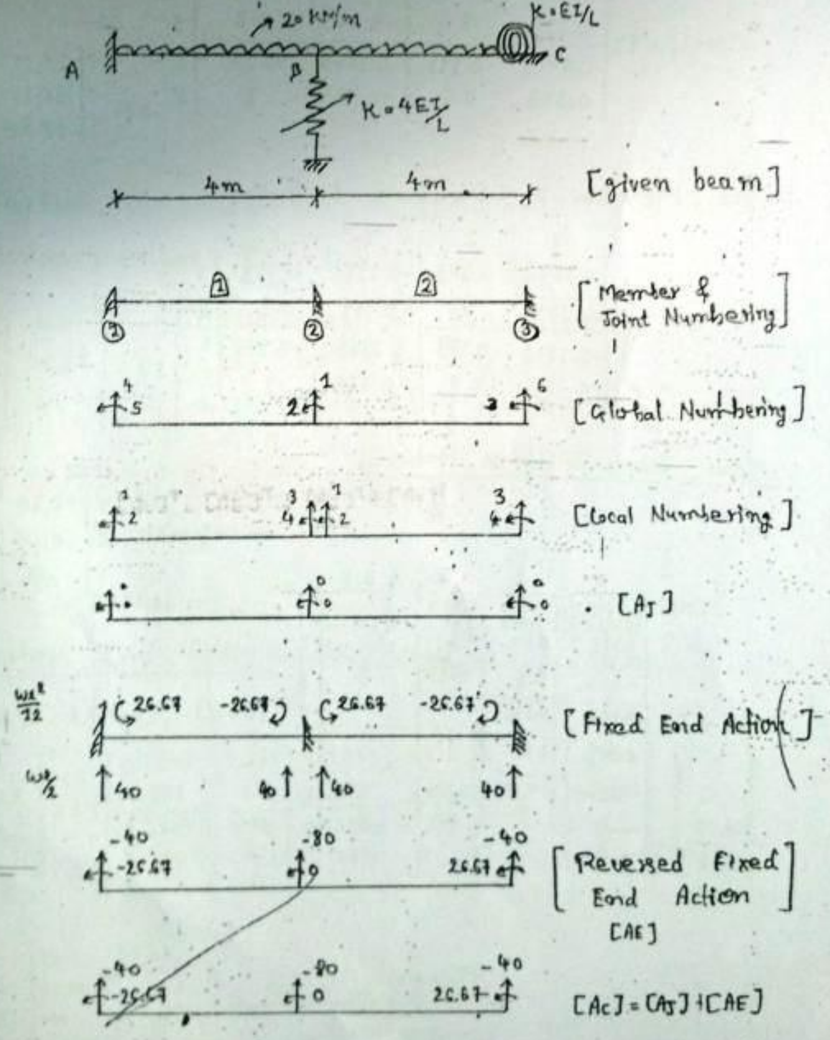
PROBLEM No. 8

Analyse the beam shown in fig. given below for following condition in addition to the loading...

i) Joint 'B' is having elastic support of stiffness $4EI/L$

ii) Joint 'C' is having rotational spring of stiffness EI/L.

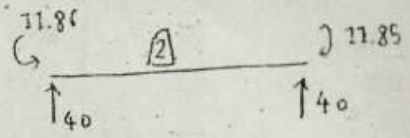
Take $EI = 20000$ kN.m^2

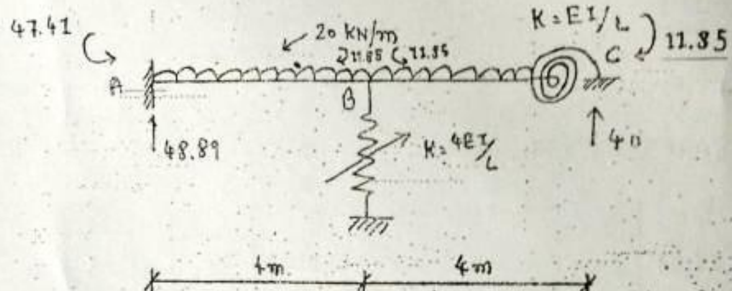


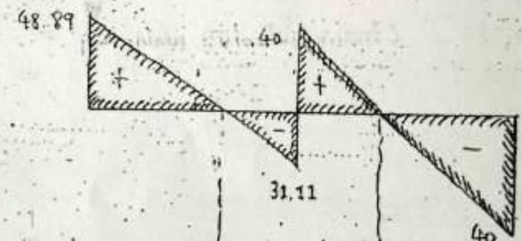
Member	Local Number	Global Num
1	1, 2, 3, 4	4, 5, 1, 2
2	1, 2, 3, 4	1, 2, 6, 3

$$
= \begin{bmatrix} 40 \\ 26.67 \\ 40 \\ -26.67 \end{bmatrix} + EI \begin{array}{c} \begin{bmatrix} 0.1875 & 0.375 & -0.1875 & 0.375 \\ 0.375 & 1 & -0.375 & 0.5 \\ -0.1875 & -0.375 & 0.1875 & -0.375 \\ 0.375 & 0.5 & -0.375 & 1 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 6 \\ 3 \end{array}
$$

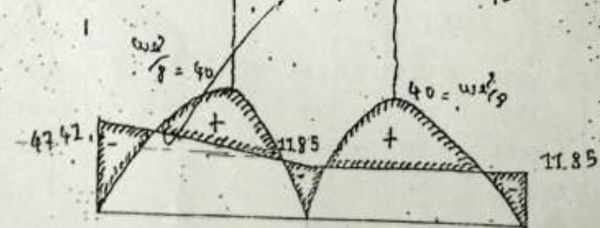
$$
\begin{bmatrix} R_B \\ M_{BC} \\ R_C \\ M_{CB} \end{bmatrix} = \begin{bmatrix} 40 \\ 11.86 \\ 40 \\ -11.85 \end{bmatrix} \begin{array}{l} KN \\ KN.m \\ KN \\ KN.m \end{array}
$$



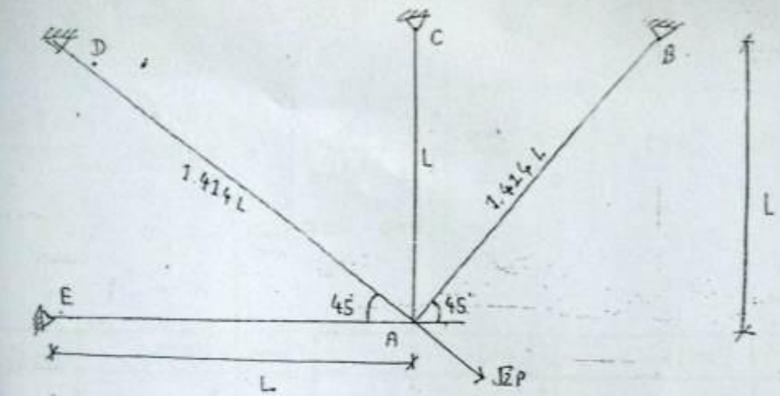




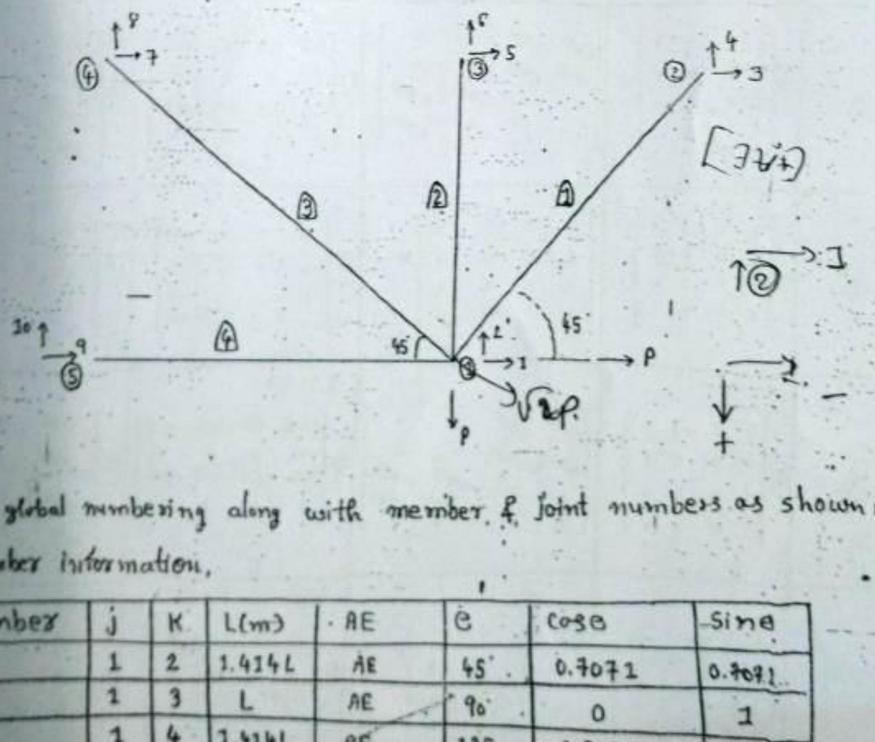
[S.F.D. in KN



[B.M.D in



[given truss]



...ous, The global numbering along with member & joint numbers as shown in

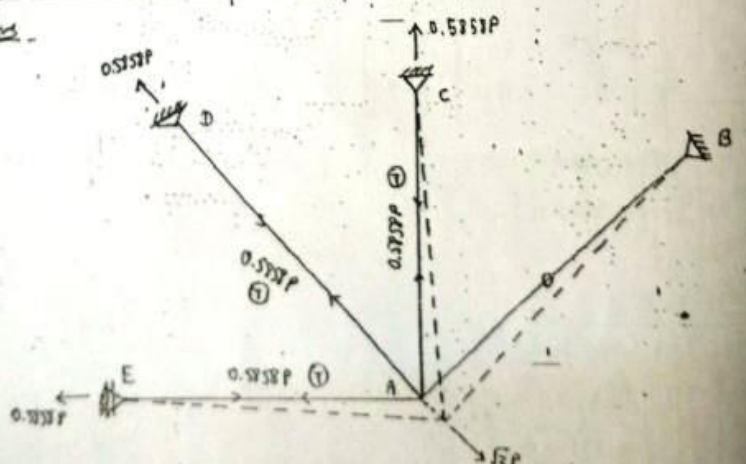
...ous, Member information,

Member	j	K	L(m)	AE	θ	cosθ	Sinθ
1	1	2	1.414L	AE	45°	0.7071	0.7071
2	1	3	L	AE	90°	0	1
3	1	4	1.414L	AE	135	-0.7071	0.7071
4	1	5	L	AE	0	1	0

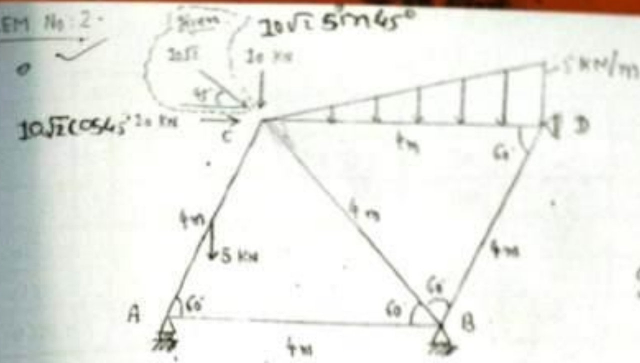
$[AM]_i = [AML]_i + [SM]_i \cdot [RT]_i \cdot [D_i]$; where, $[AML]_i$ = Null Matrix

$[AM]_i$	$[SM]_i . [RT]_i$	DOF	$[D]_i$	$[AM]_i$	Nature
$[AM]_1$	$\frac{EA}{L}\begin{bmatrix}0.5 & 0.5 & -0.5 & -0.5\\ 0 & 0 & 0 & 0\\ -0.5 & -0.5 & 0.5 & 0.5\\ 0 & 0 & 0 & 0\end{bmatrix}$	$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$	$\frac{L}{EA}\begin{bmatrix}0.5858P\\ -0.5858P\\ 0\\ 0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$	
$[AM]_2$	$\frac{EA}{L}\begin{bmatrix}0 & 1 & 0 & -1\\ 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 1\\ 0 & 0 & 0 & 0\end{bmatrix}$	1, 2, 5, 6	$\frac{L}{EA}\begin{bmatrix}0.5858P\\ -0.5858P\\ 0\\ 0\end{bmatrix}$	$\begin{bmatrix}-0.5858P\\ 0\\ 0.5858P\\ 0\end{bmatrix}$	↑ 0.5858P Tension ↓ 0.5858P
$[AM]_3$	$\frac{EA}{L}\begin{bmatrix}-0.5 & 0.5 & 0.5 & -0.5\\ 0 & 0 & 0 & 0\\ 0.5 & -0.5 & -0.5 & 0.5\\ 0 & 0 & 0 & 0\end{bmatrix}$	1, 2, 7, 8	$\frac{L}{EA}\begin{bmatrix}0.5858P\\ -0.5858P\\ 0\\ 0\end{bmatrix}$	$\begin{bmatrix}-0.5858P\\ 0\\ 0.5858P\\ 0\end{bmatrix}$	0.5858P Compressive force
$[AM]_4$	$\frac{EA}{L}\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 0 & 0 & 0\\ -1 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$	1, 2, 9, 10	$\frac{L}{EA}\begin{bmatrix}0.5858P\\ -0.5858P\\ 0\\ 0\end{bmatrix}$	$\begin{bmatrix}0.5858P\\ 0\\ -0.5858P\\ 0\end{bmatrix}$	← 0.5858P Tension

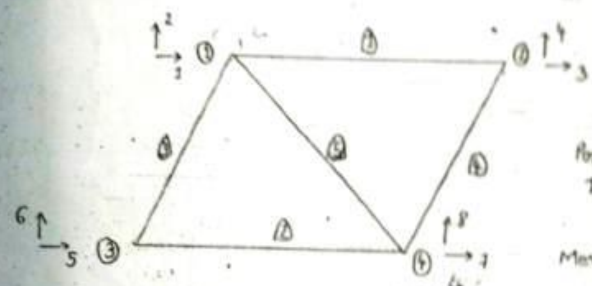
» Now,



PROBLEM No: 2

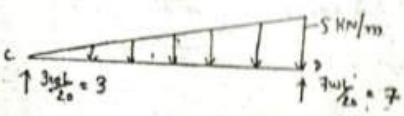


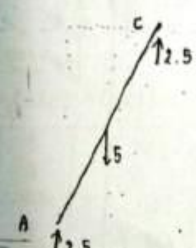
given truss



Possible
1,2,5

Member, joint & Global numbering..





$\uparrow$ −
$\rightarrow$ +

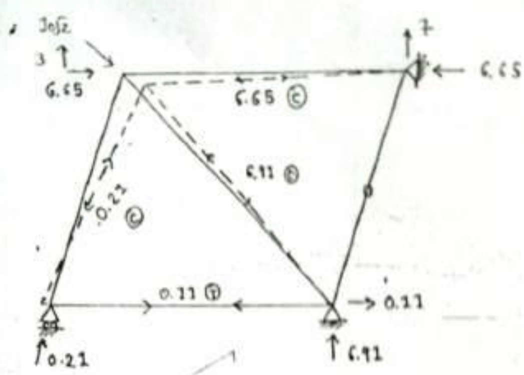
now; $[A_c]^T = [A_E]^T + [A_J]^T$

$= [0, -5.5, 0, -7, 0, -2.5, 0, 0]$

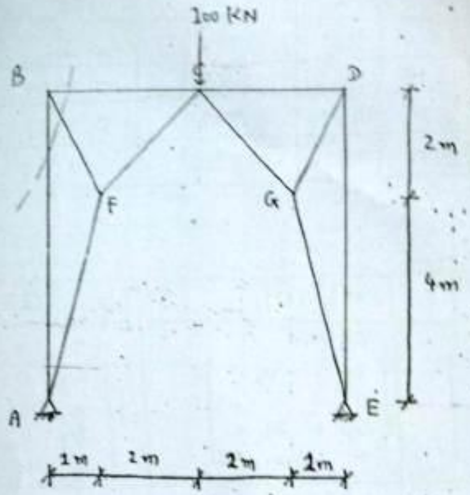
$+ [10, -10, 0, 0, 0, 0, 0, 0]$

$= [10, -15.5, 0, -7, 0, -2.5, 0, 0]$

55



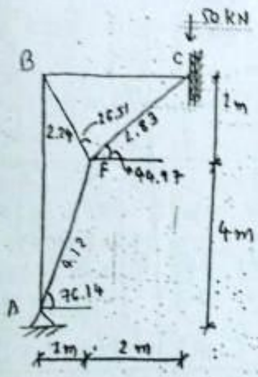
$3\sqrt{2}$
3
6.65
7
6.65
6.65 ©
6.91 ©
0.22 ©
0.11 T
0.11
0.21
6.91



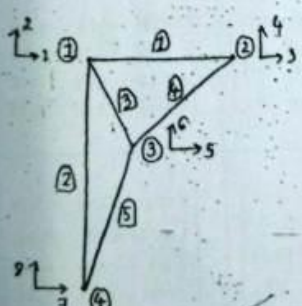
$A = 10\ Cm^2$

$E = 20000\ KN/Cm^2$

[given truss]



[Using Symmetry]



Member; Joint & global numbering.

Now; $[A_C]^T = [A_E]^T + [A_J]^T$

$= [A_J]^T$ $[\because [A_E]^T = [0]]$

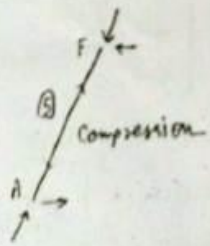
$= [0, 0, 0, -50, 0, 0 \mid 0, 0, 0]$

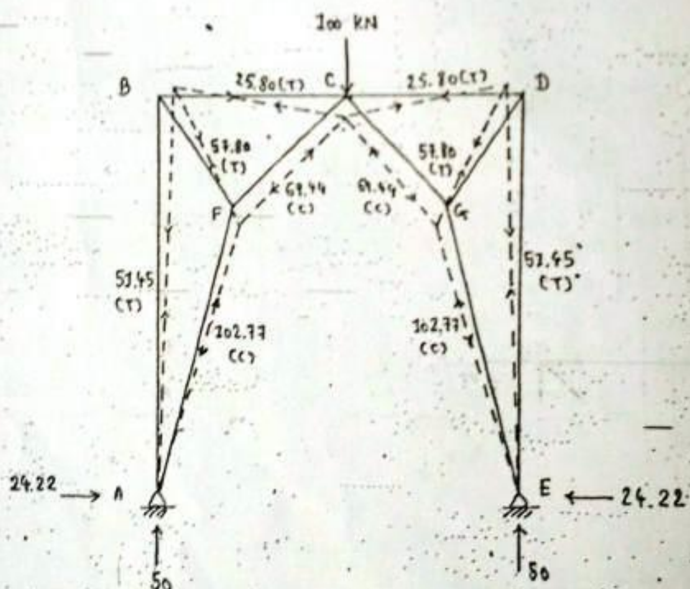
① ② ④ ⑤ ⑥ | ③ ⑦ ⑧

$[A_{DC}]$ $[A_{DR}]$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + EA \begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \left[\begin{array}{cc|cc} 0.06 & 0.24 & -0.06 & -0.24 \\ 0 & 0 & 0 & 0 \\ \hline -0.06 & -0.14 & 0.06 & 0.24 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 7 \\ 8 \\ 5 \\ 6 \end{array} \frac{1}{EA} \begin{bmatrix} 0 \\ 0 \\ -1225.09 \\ -121.95 \end{bmatrix}$$

$$\therefore \begin{bmatrix} R_{AH} \\ R_{AV} \\ R_{FH} \\ R_{FV} \end{bmatrix} = \begin{bmatrix} 102.77 \\ 0 \\ -102.77 \\ 0 \end{bmatrix}$$



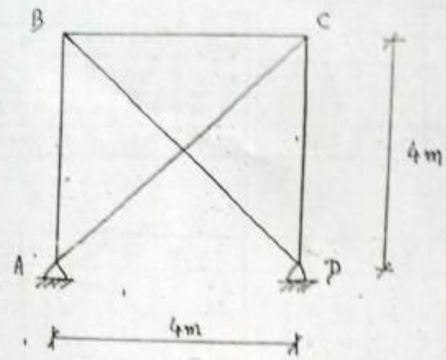


...EM No. 4

• Analyse the
condition Se...

(i) Temperature
(ii) Member A
Take EA =
A =
α =
Also sketch

ROBLEM No: 4

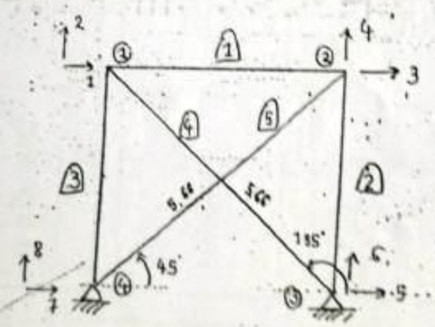


- Analyse the plane truss shown in Fig. given below for following condition Seperately...

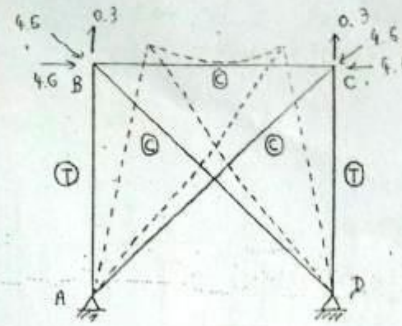
(i) Temperature rise of 30°C.
(ii) Member AC is short by 0.01 cm.

Take EA = 40,000 KN
A = 2 cm²
α = 0.000012/°C

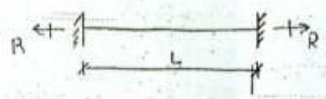
Also sketch deflected shape of truss in both Conditions..

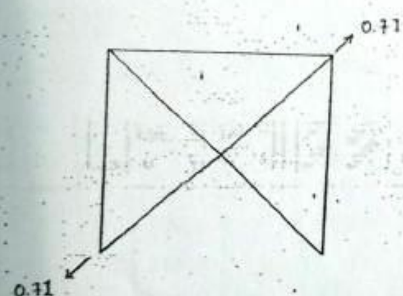


[Member & Joint Numbering & Possible displacement



Now, Case (ii) Member AC is short by 0.01 cm.



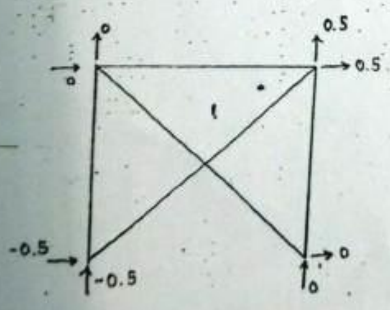


$R = \dfrac{EAe}{L}$

e = shorten length

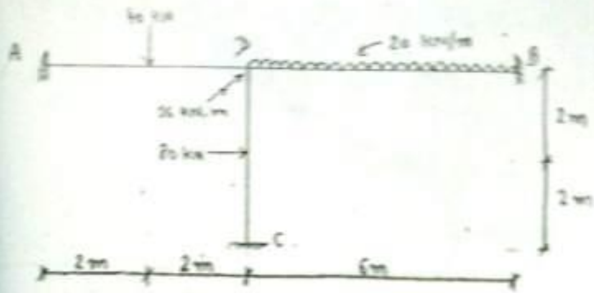
$= \dfrac{40{,}000 \times 0.01 \times 10^{-2}}{5.66}$

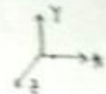
$= 0.71$



[Fixed End Action]

PROBLEM No: 3

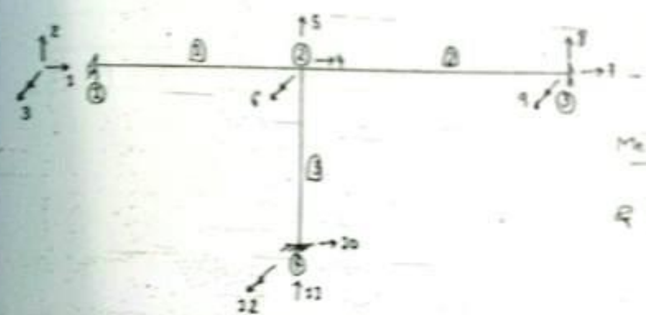




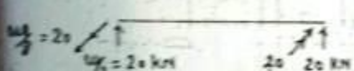
[given frame]

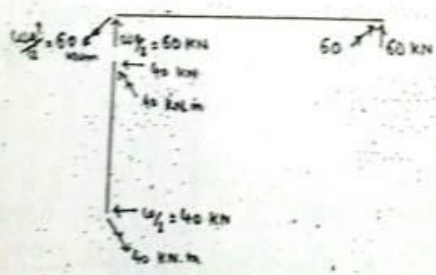
Consider Axial deformation...

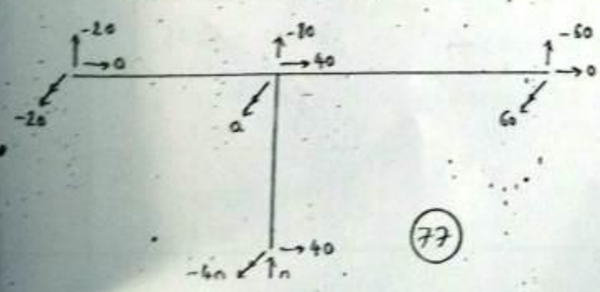
EI → Constant.



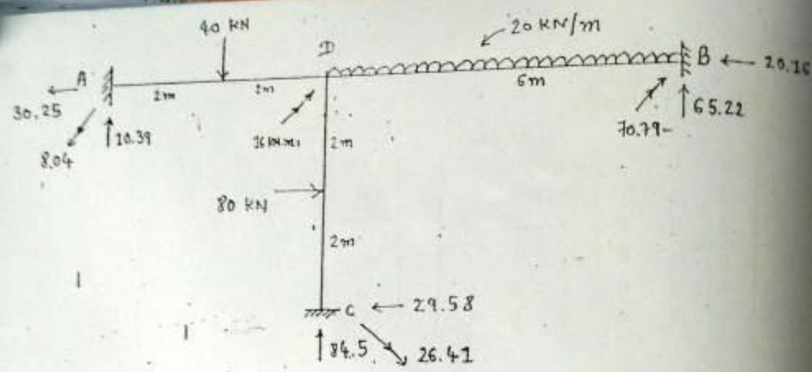
Member & Joint numbering & Possible displacement



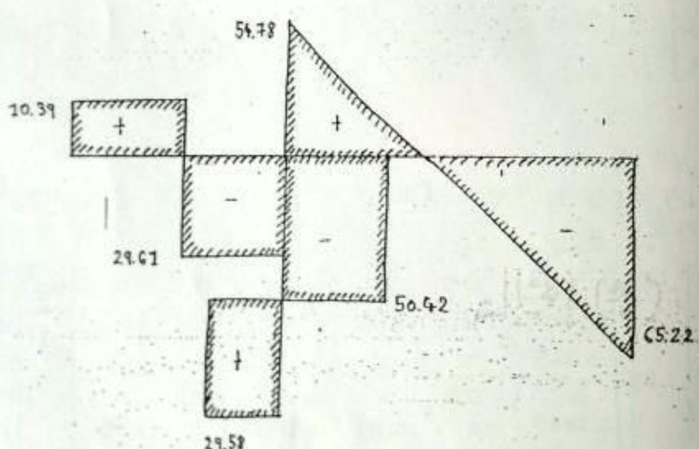




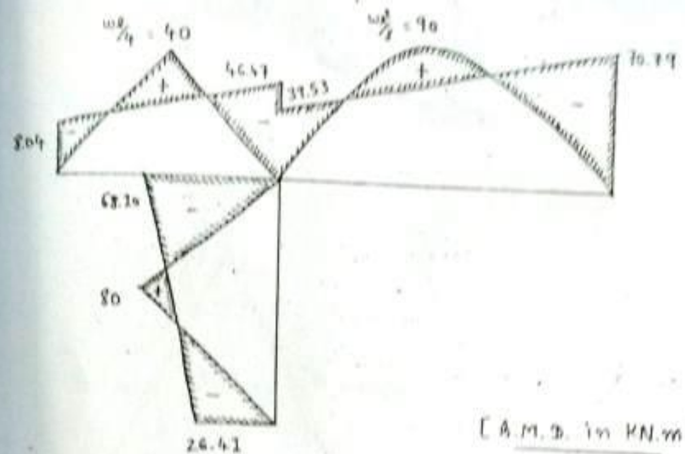
[AE]



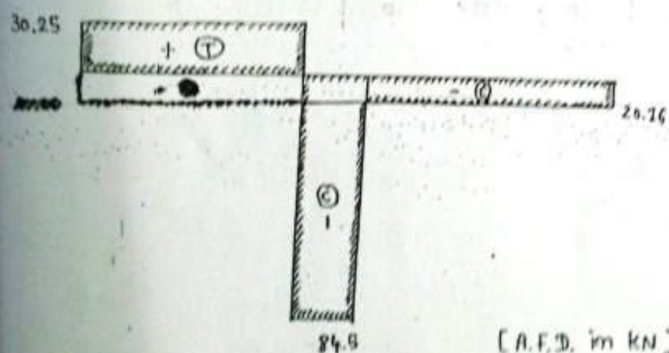
[Given Frame]



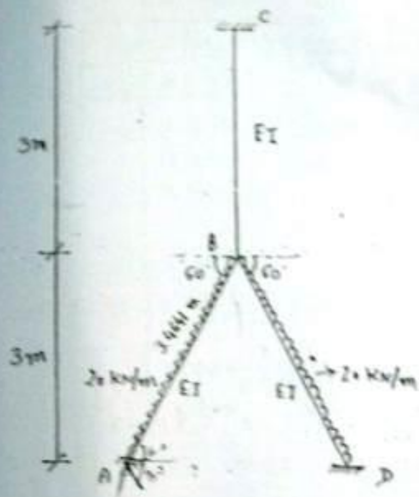
[S.F.D. in KN]



[A.M.D. in KN.m]



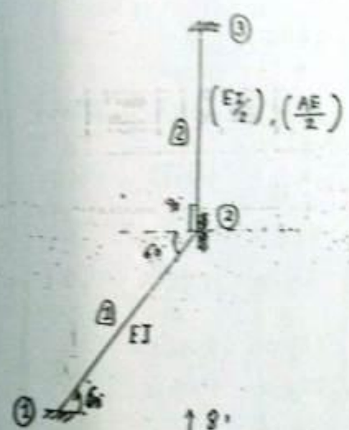
[A.F.D. in kN]



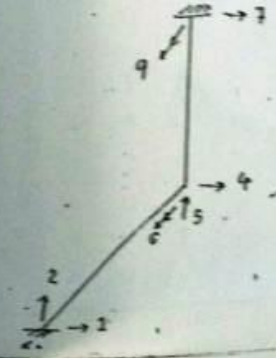
[given Frame]

EI → Constant

$\tan 60 = \frac{3}{x}$ | $x = 1.732$ m

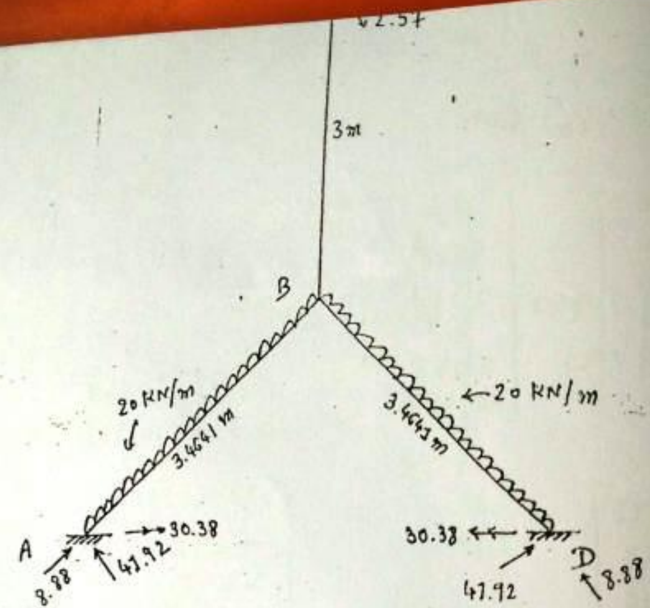


Member & Joint number
[Using Symmetry]

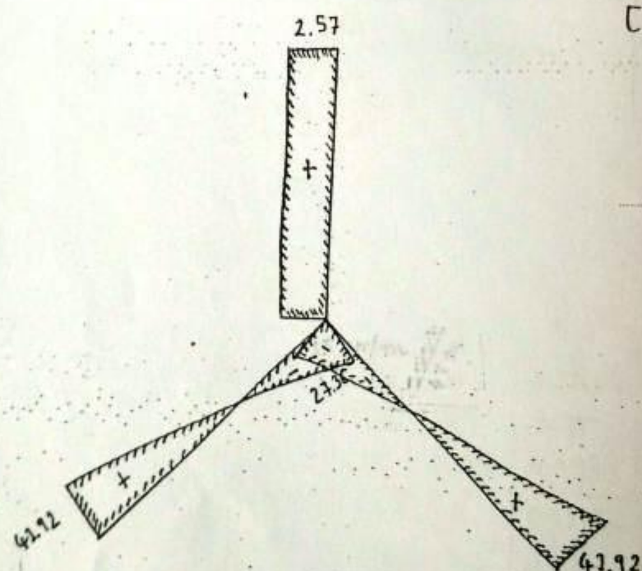


Global numbering

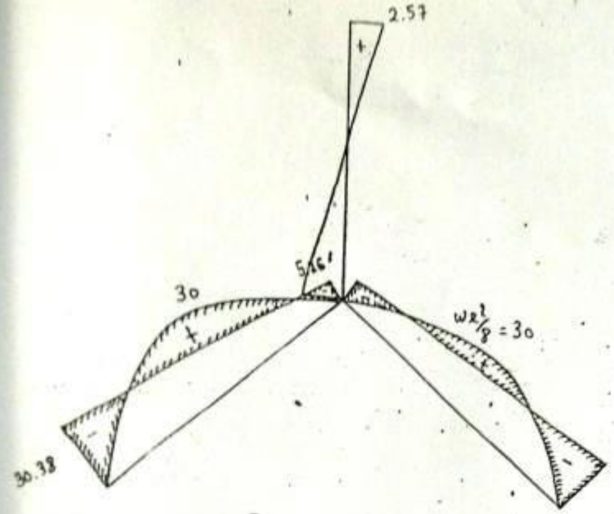
Possible displacement 5 & 6 only...



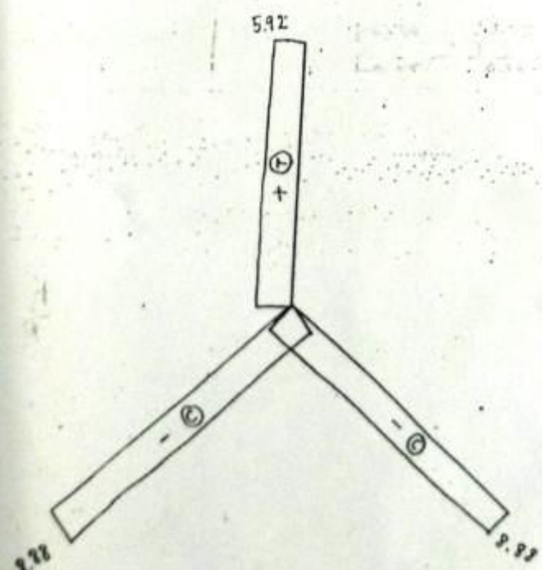
[Given Frame]



[S.F.D. in kN]



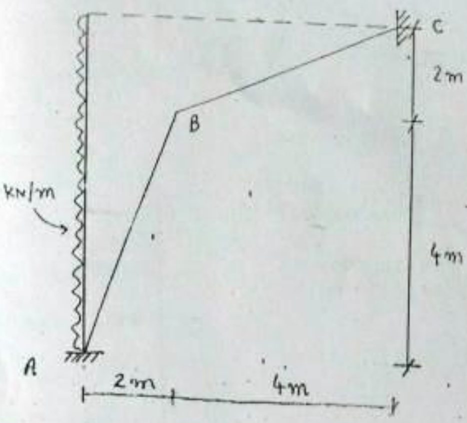
B.M.D. in kN.m



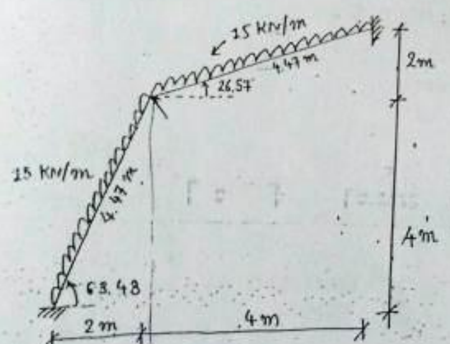
Axial Force dig. in kN

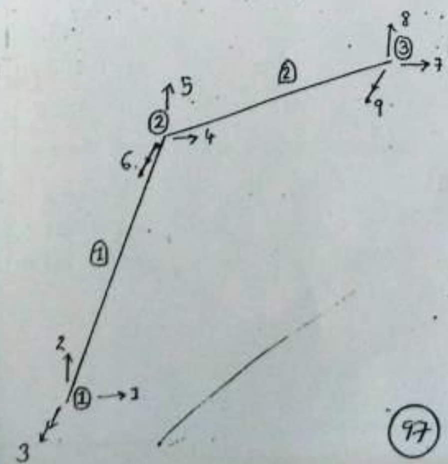
9.....

BLEM No : 3

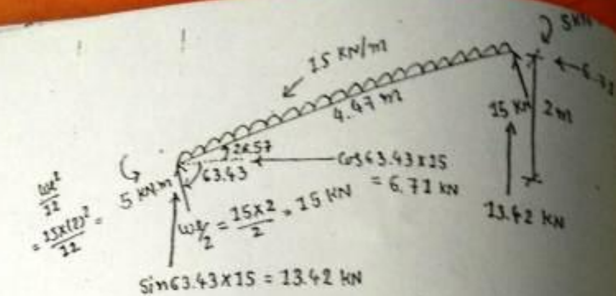


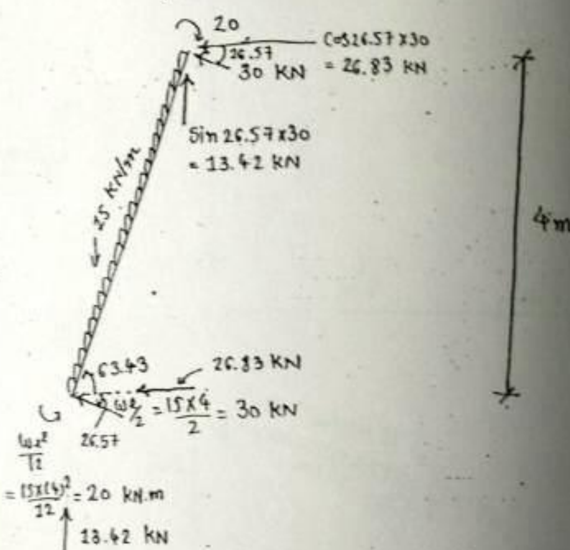
[given Frame]

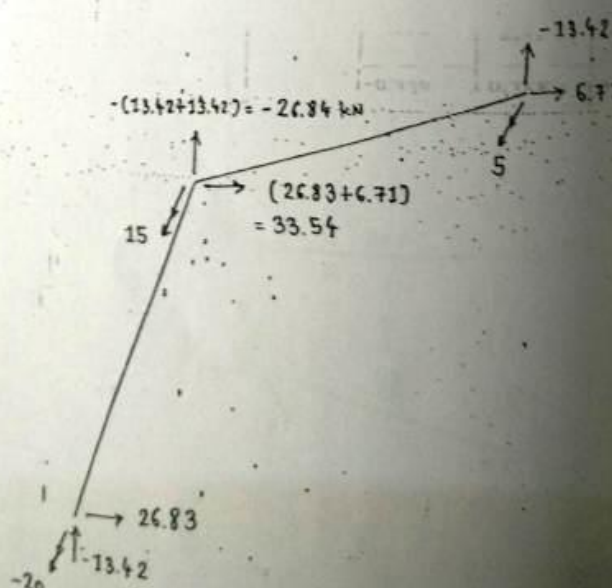




Member & Joint numbering & Global numbering



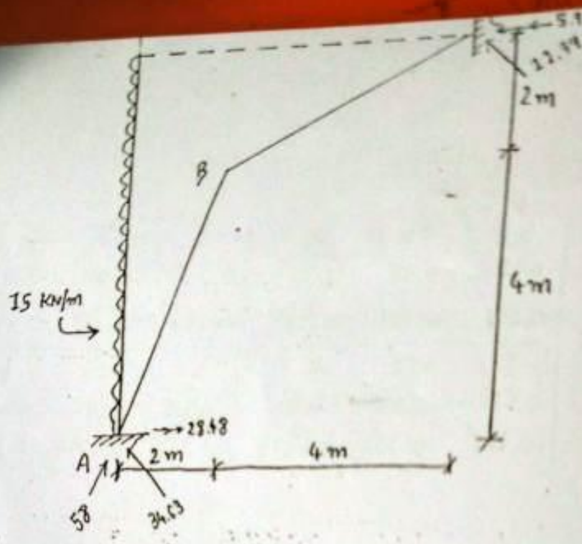




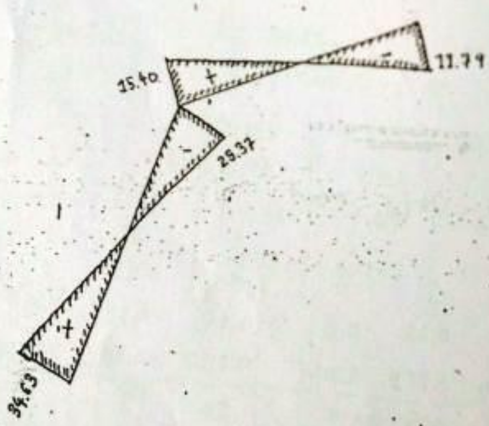
...mber information...

...mber	j	k	L...
1	1	2	4.4...
2	2	3	4.4...

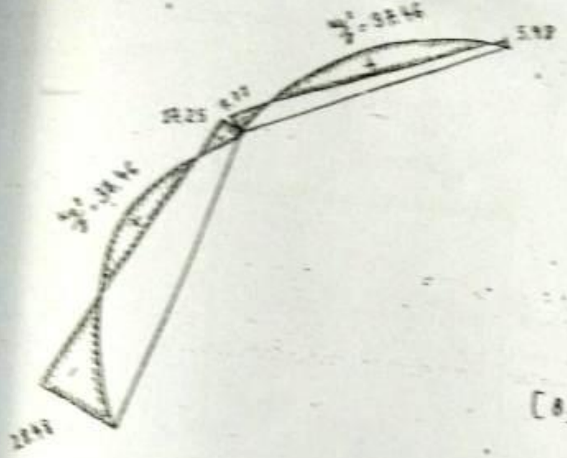
Member - 1, l = 4.47 m
& Assume AE...



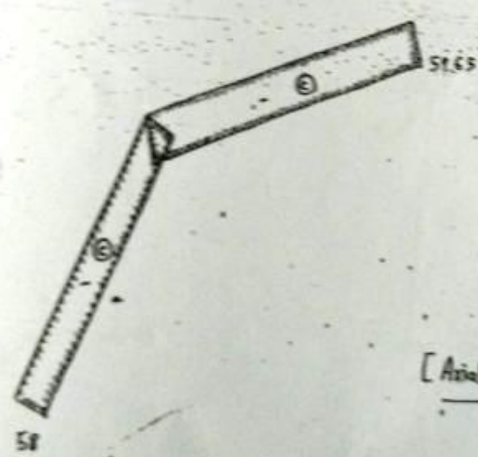
[Given Frame]



[S.F.D. In KN]



[B.M.D. in kN.m]

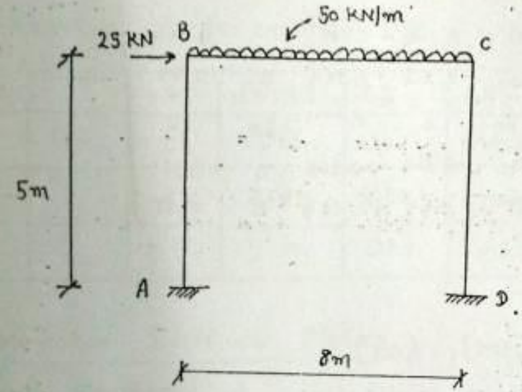


[Axial force dig. in kN]

PROBLEM No: 4

(109)

...lyse the plane frame shown in fig. below with
...nsidering Axial deformation
...eglecting Axial deformation.
...Neglecting Axial deformation in beam only..
...nd Compare result...

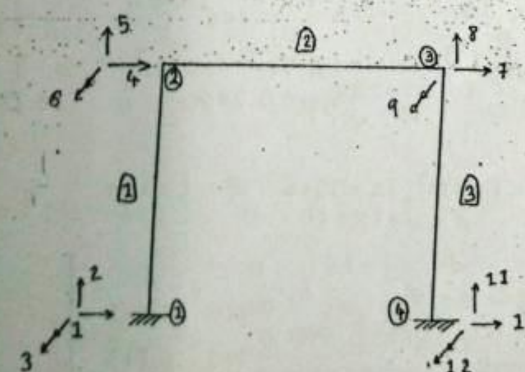


take beam & Columns are having a Same C/S area..

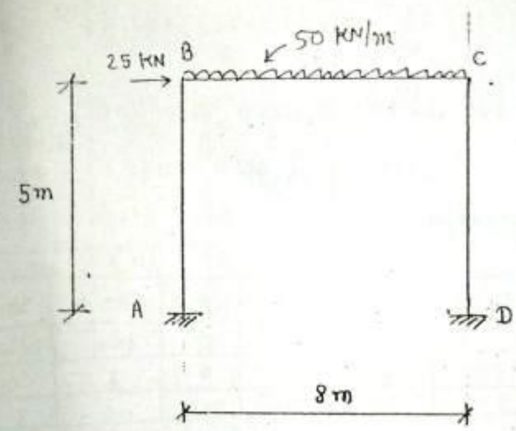
$AE = 600 \times 10^3$

$EI = 1100$

(i) Considering Axial deformation...

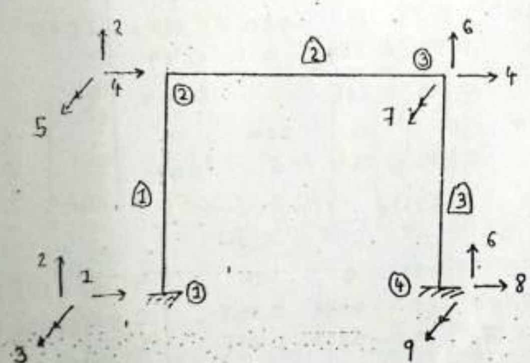


Member, Joint Numbering & Possible displacement



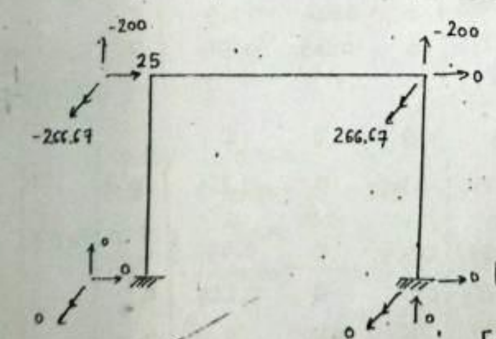
[Given frame]

(ii) Neglecting Axial Deformation



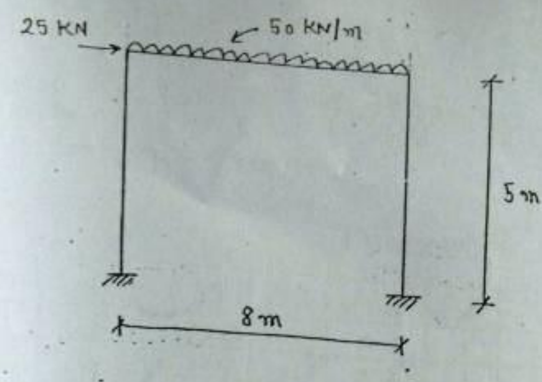
[Member & Joint Numbering & Possible Displacement

Note; Here Axial deformation Neglected So Here free degree of freedom means Possible displacements are Rotation at joint 2 3. means 5 [✓] & 7 [✓] .... & also 4 becaus it is a Independance Joint displacement

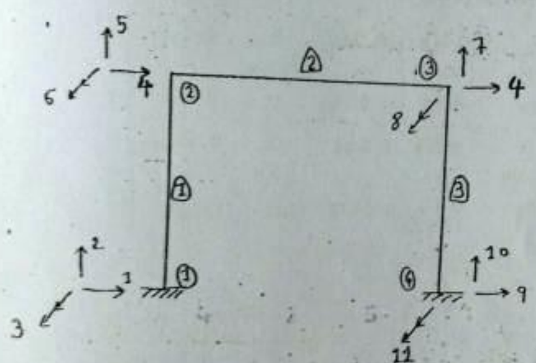


$[A_C] = [A_E] + [A_B]$

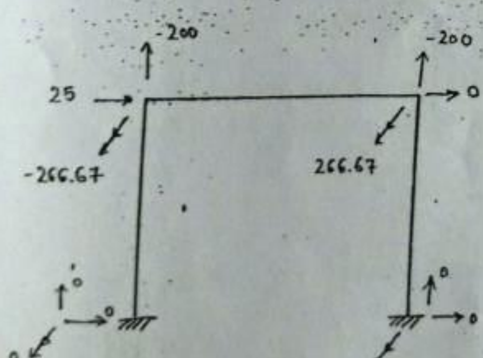
[Similar as in case (i)]



(iii) Neglecting Axial deformation in beam only.



[Member & Joint Numbering & Possible displacement 4, 5, 6, 7, 8

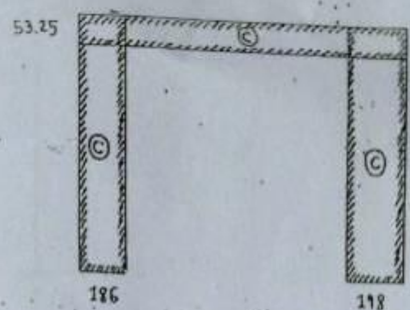


[AE] = [AE] + [AJ]

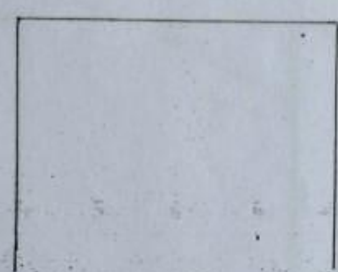
[Similar as Same as in case (i)]

17...

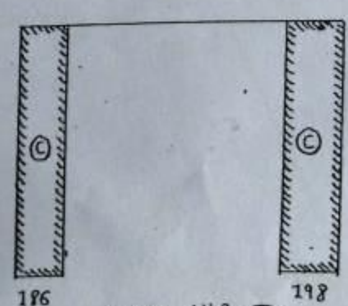
Axial Force diag. in kN



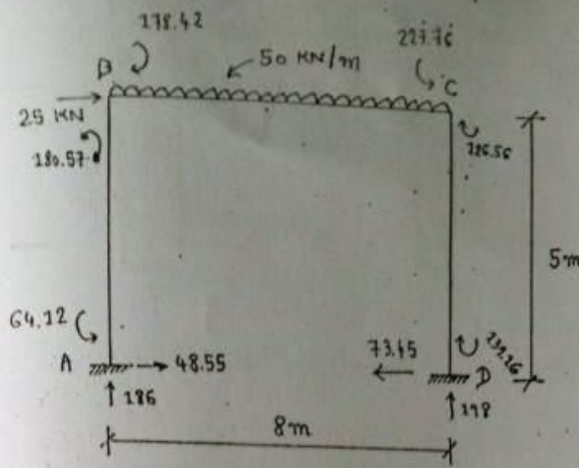
[case-i]

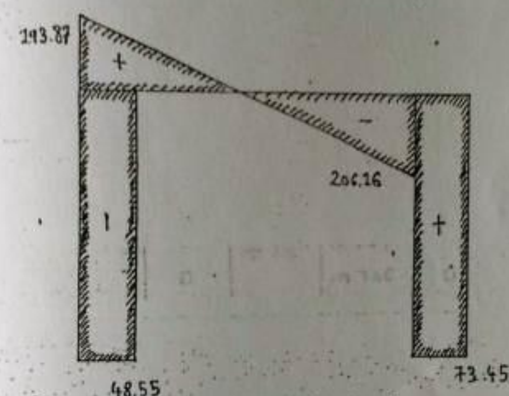


[case-ii]

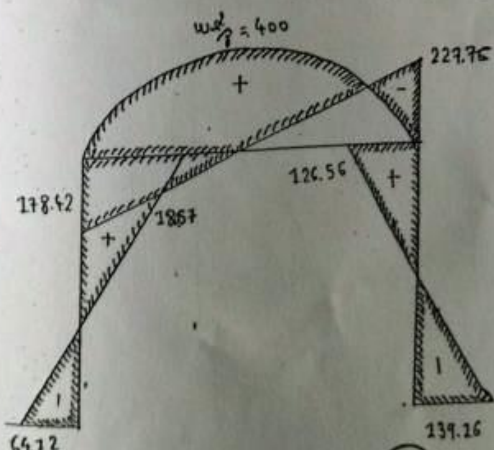


[case-iii]

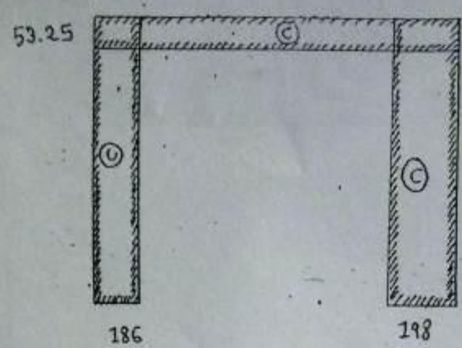




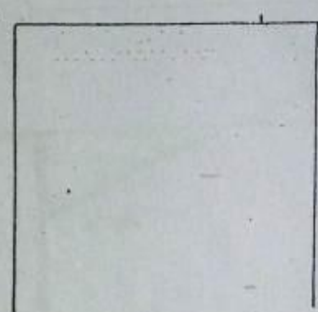
[S.F.D. in KN]



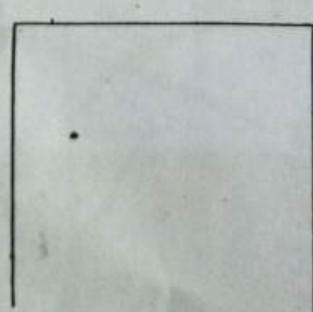
[B.M.D. in KNm]

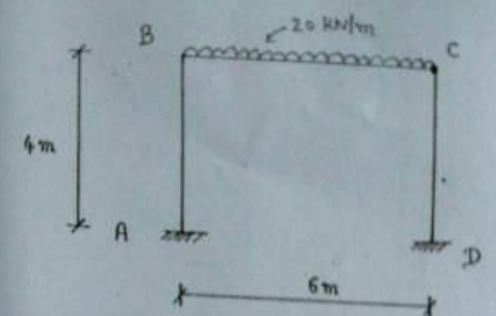


[Case-i] [Axial Force diag. in KN]



[Case-ii] [Axial Force diag. in KN]





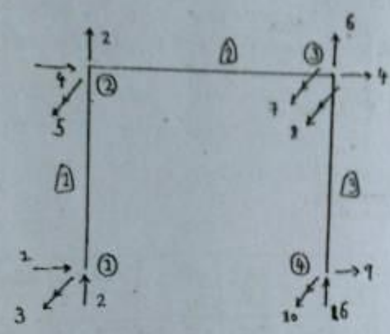
Joint 'C' as internal hinge

beam & Column has Same c/s area

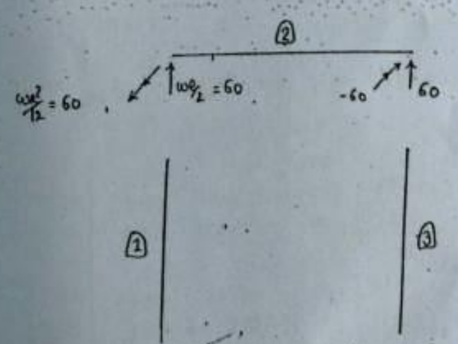
$EA = 600 \times 10^3$

$EI = 1100$

Now, Neglecting the Axial deformation



[Member & Joint Numbering & Possible displacement [4, 5, 7, 8]]



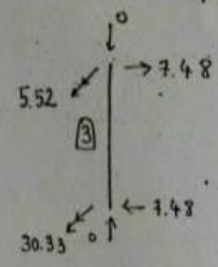
[Fixed End Action]

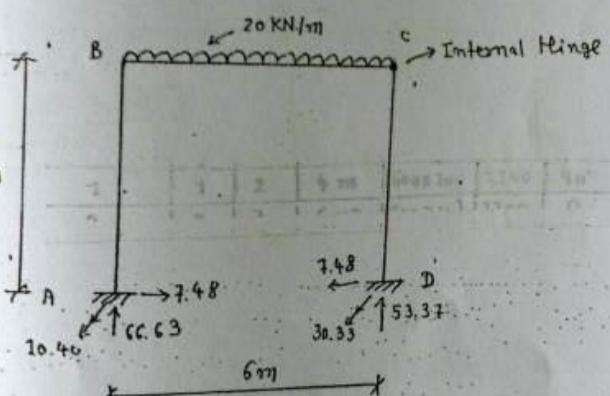
Member - 3;

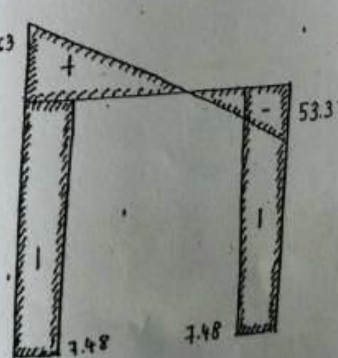
$[AM]_3 = [AML]_3 + [SM]_3 \cdot [RT]_3 \cdot [DF]_3$

$$
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 10^3
\begin{array}{c}
\begin{array}{cccccc} 9 & 6 & 10 & 4 & 5 & 8 \end{array} \\
\left[\begin{array}{ccc:ccc}
0 & 150 & 0 & 0 & -150 & 0 \\
-0.206 & 0 & 0.413 & 0.206 & 0 & 0.413 \\
-0.413 & 0 & 1.1 & 0.413 & 0 & 0.55 \\ \hdashline
0 & -150 & 0 & 0 & 150 & 0 \\
0.206 & 0 & -0.413 & -0.206 & 0 & -0.413 \\
-0.413 & 0 & 0.55 & 0.413 & 0 & 1
\end{array}\right]
\end{array}
\begin{array}{c} 9 \\ 6 \\ 10 \\ 4 \\ 6 \\ 8 \end{array}
\cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 14685 \\ 0 \\ -55.14 \end{bmatrix}
$$

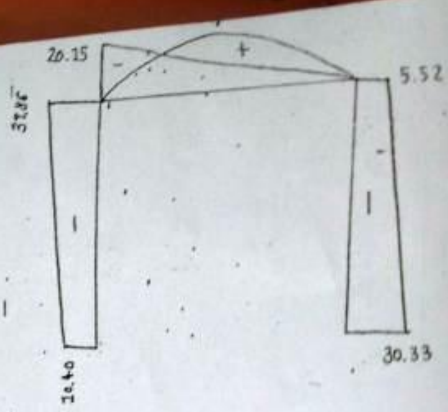
$$
\begin{bmatrix} R_{DH} \\ R_{DV} \\ M_{DC} \\ R_{CH} \\ R_{CV} \\ M_{CD} \end{bmatrix} = \begin{bmatrix} 0 \\ 7.48 \\ 30.33 \\ 0 \\ -7.48 \\ 5.52 \end{bmatrix} \begin{array}{l} KN \\ KN \\ KN.m \\ KN \\ KN \\ KN.m \end{array}
$$



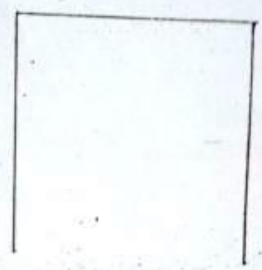




[S.F.D. in KN]

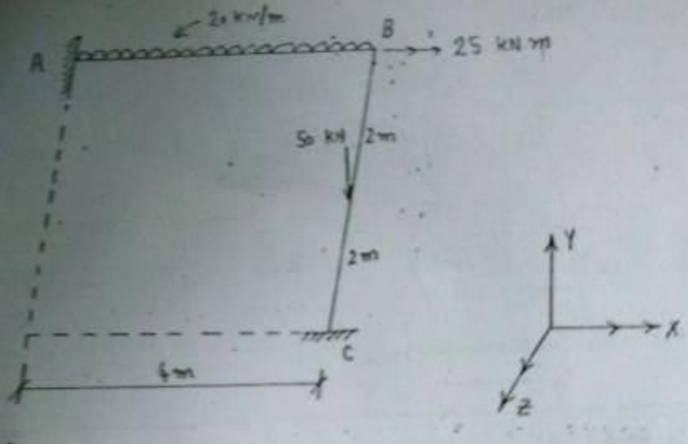


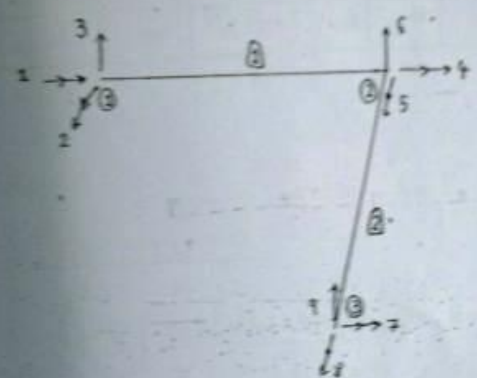
[B.M.D. in KNm]



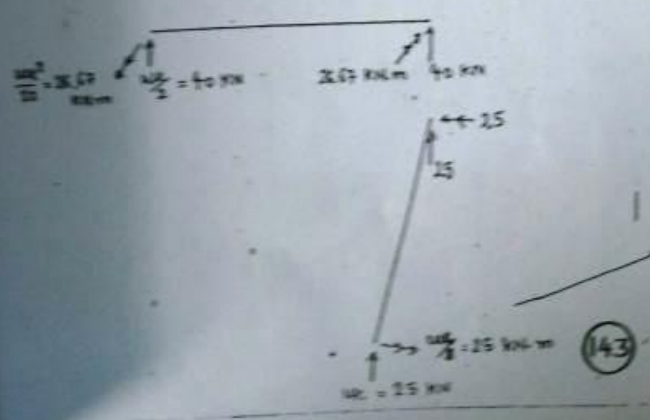
[Neglect Axial deformation.]

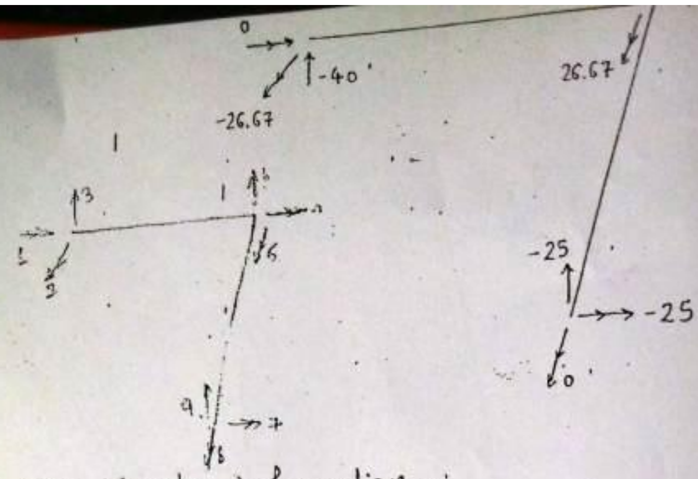
STIFFNESS MEMBER APPROACH → "GRID"





[Member, Joint & Global numbering]





$[A_E]$

Now, Member information--;

Sl No.	Member	j	K	L	EI	GJ	θ	Cos θ	sin θ
1	1	1	2	4 m	20×10^3	16×10^3	0	1	0
2	2	3	2	4 m	20×10^3	16×10^3	90	0	1

0.8 EI

Now, Combined Joint load Vector..;

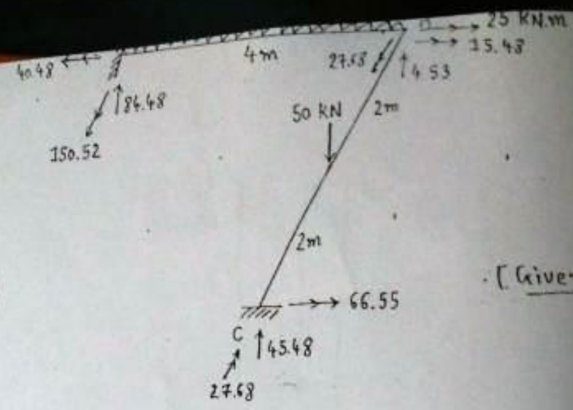
$[A_c]^T = [A_E]^T + [A_J]^T$

Now, $[A_E]^T = [\,0, -26.67, -40, 25, \cancel{-65}, 26.67, -65, -25, 0, -25\,]$

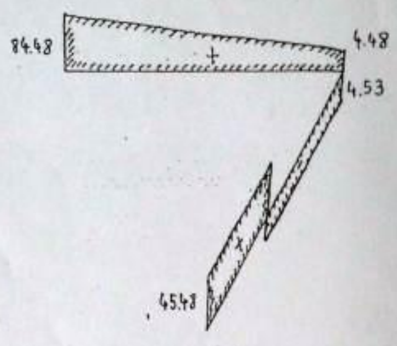
& $[A_J]^T = [\,0, 0, 0, 25, 0, 0, 0, 0, 0\,]$

$$\therefore [A_c]^T = [\,\underset{1}{0}, \underset{2}{-26.67}, \underset{3}{-40}, \underset{4}{50}, \underset{5}{26.67}, \underset{6}{-65}, \underset{7}{-25}, \underset{8}{0}, \underset{9}{-25}\,]$$

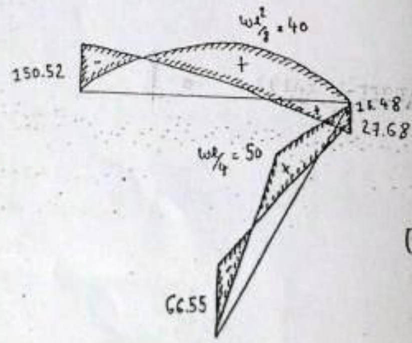
$$\therefore [A_c]^T = [\,\underbrace{50, 26.67, -65}_{[A_{FC}]^T},\ \vdots\ \underbrace{0, -26.67, -40, -25, 0, -25}_{[A_{RC}]^T}\,]$$



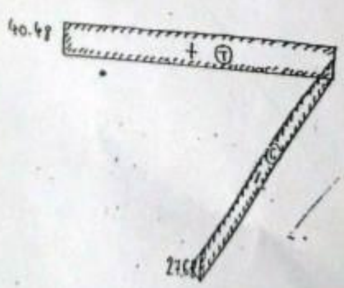
[Given Grid]



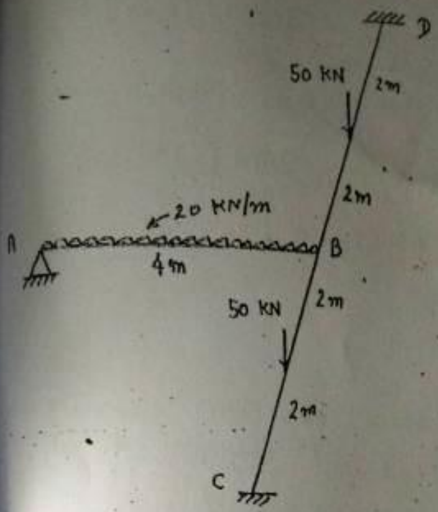
[S.F.D in KN]



[B.M.D. in KN.m]



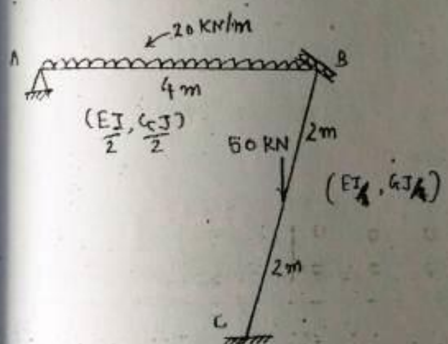
[T.M.D. in KN.m]



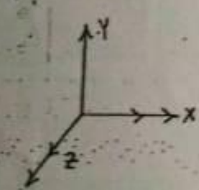
Given Grid

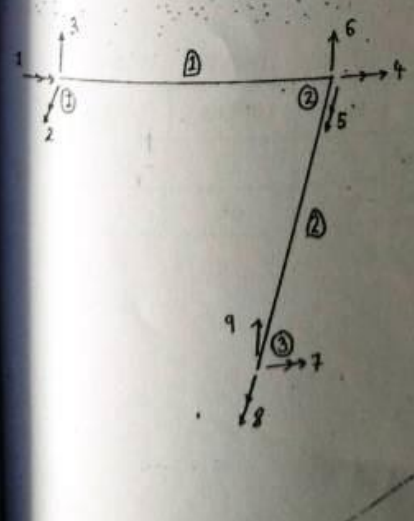
$EI = 20 \times 10^3$

$GJ = 16 \times 10^3$



[Using Symmetry]





[Member, Joint & Global numbering]

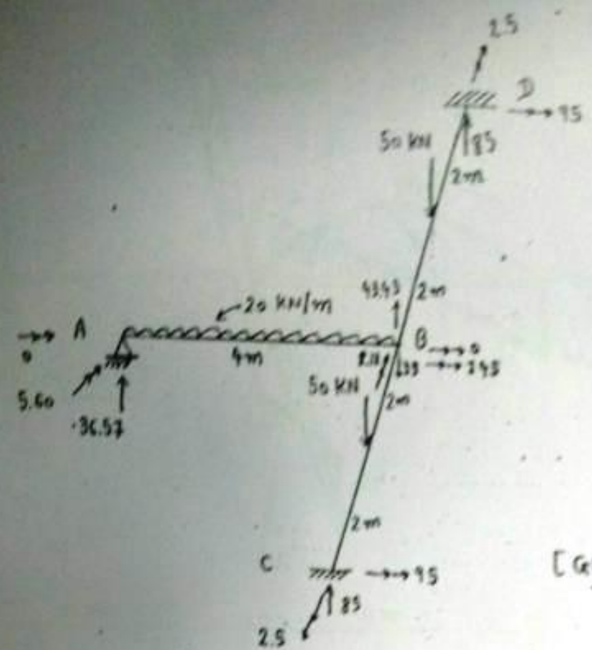
Possible displacement

$5 \rightarrow \theta_z$

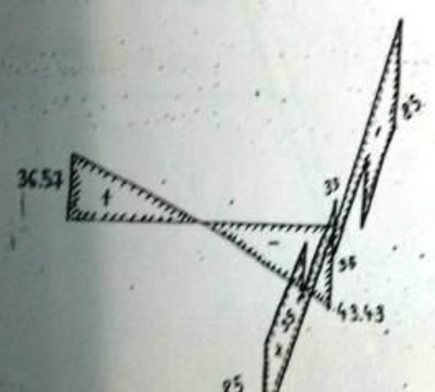
$6 \rightarrow \delta_y$

$1 \rightarrow \theta_x$

$2 \rightarrow \theta_z$

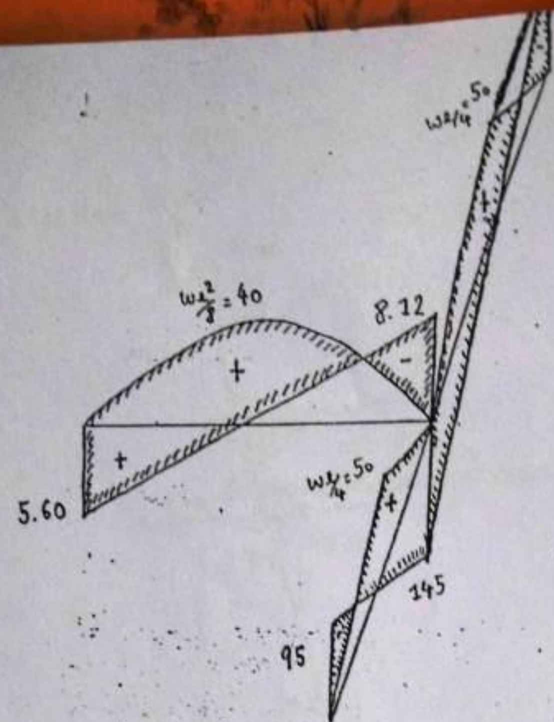


[Given Grid]

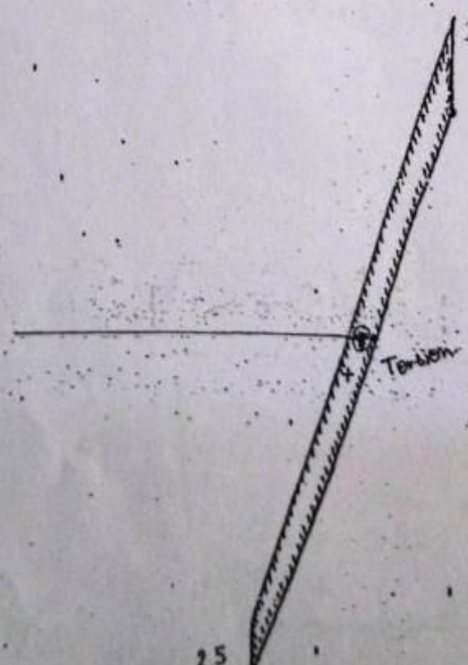


[S.F.D. In KN]

9....

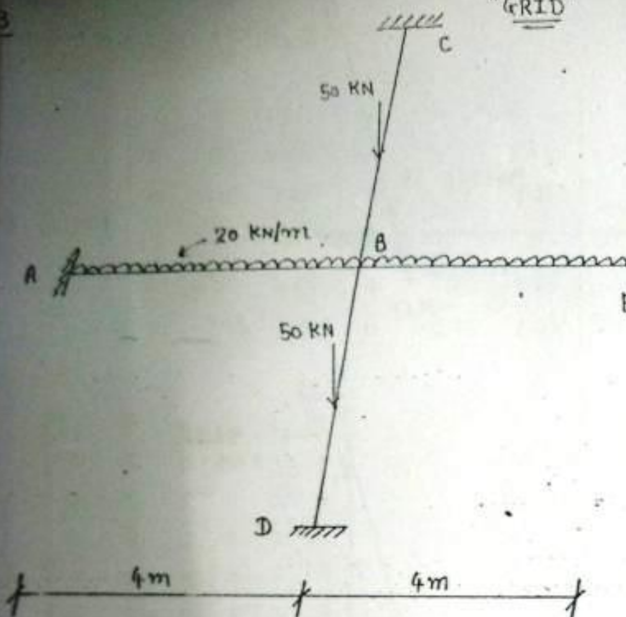


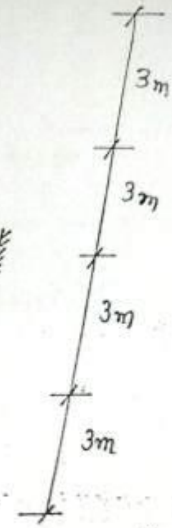
B.M.D. In KN.m



TMD in KN.m

"GRID"

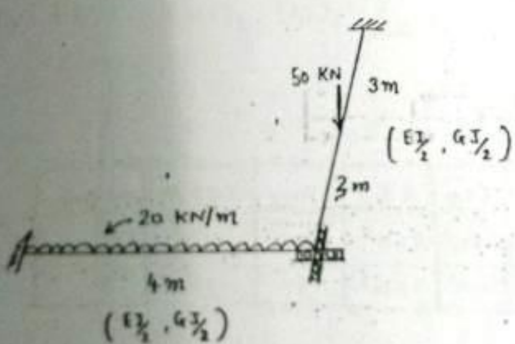




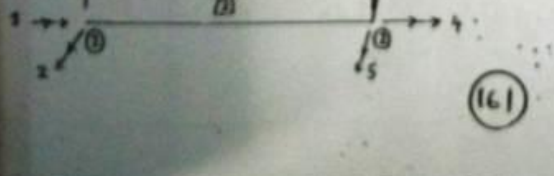
$EI = 20 \times 10^3$

$GJ = 16 \times 10^3$

[Given Grid]

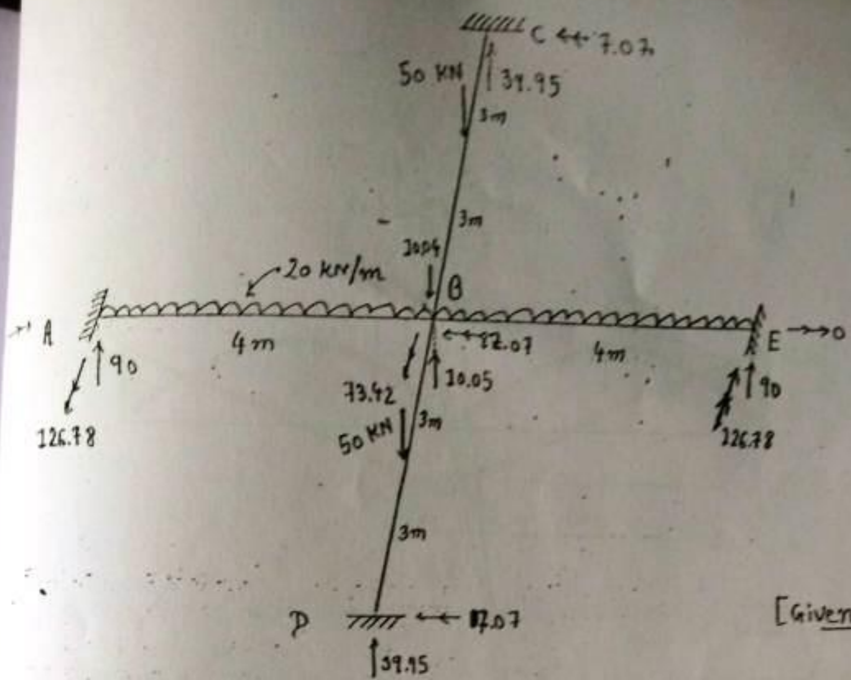


[Using Symmetry]

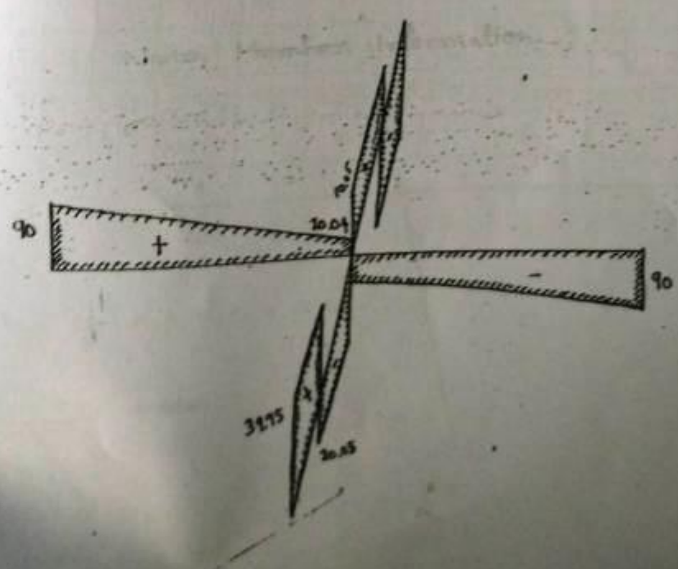


[Member, Joint & Global numbering]

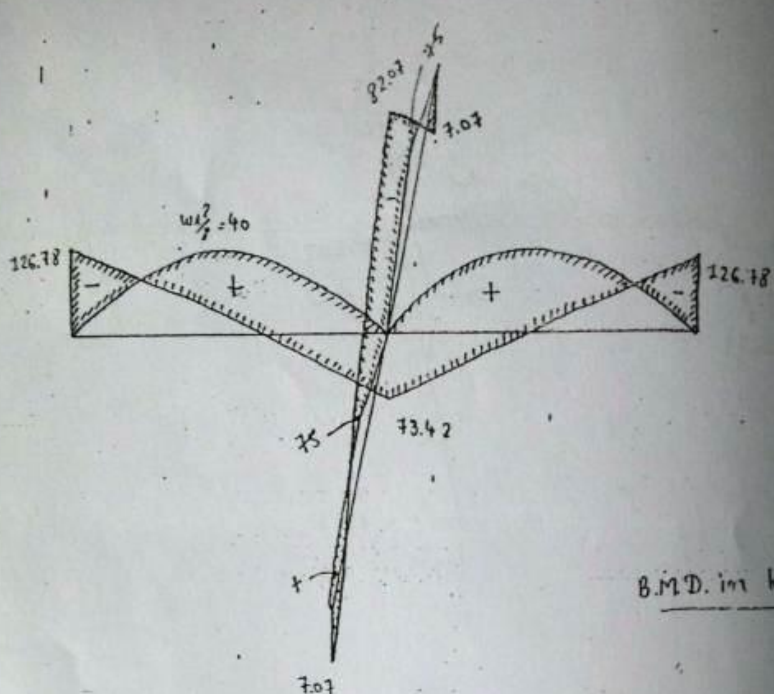
1....



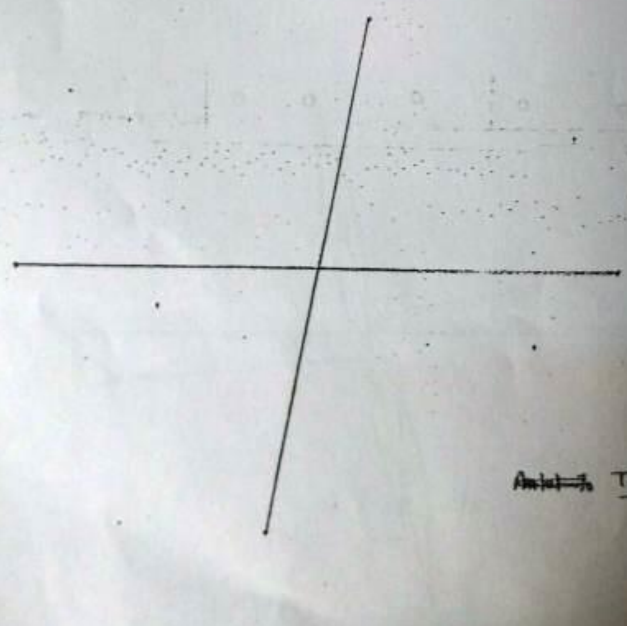
[Given Grid]



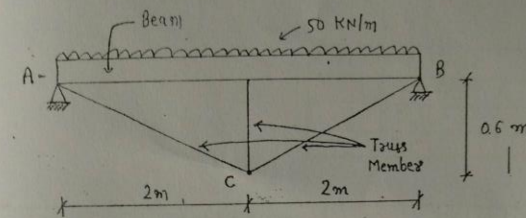
[S.F.D. in kN]



B.M.D. in kNm

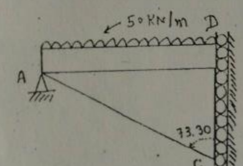


~~Axial~~ TMD in kNm



	Area	Moment of Inertia	Young Modulus
Beam	300 Cm 2	550 Cm 4	200 GPa
Truss	10 Cm 2		200 GPa

Here, we Using Symmetry...



Beam

$A = 300 \times 10^{-4}\ m^2$

$I = 550 \times 10^{-8}\ m^4$

$E = 200\ GPa$

$= 200 \times 10^{9}\ N/m^2$

$= 200 \times 10^{6}\ KN/m^2$

$\therefore AE = 6000 \times 10^{3}\ KN$

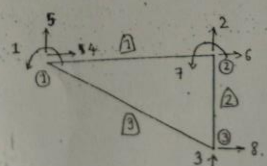
$\&\ EI = 1100\ KN.m^2$

truss

$A = 10\ Cm^2 = 10 \times 10^{-4}\ m^2$

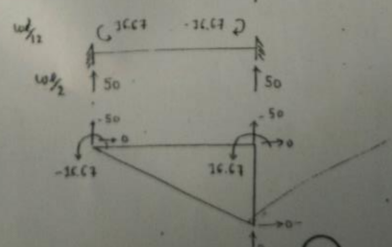
$E = 200 \times 10^{6}\ KN/m^2$

$\therefore AE = 200 \times 10^{3}\ KN$



[Member, & Joint Numbering..

& Possible displacement



[Fixed End Action]

$[A_C] = [A_E] + [A_J]$

$$= \left[\begin{array}{ccc|ccccc} -16.67 & -50 & 0 & 0 & -50 & 0 & 16.67 & 0 \end{array} \right]$$
$$\quad\; {}_{1} \quad\; {}_{2} \quad\; {}_{3} \quad\; {}_{4} \quad\; {}_{5} \quad\; {}_{6} \quad\; {}_{7} \quad\; {}_{8}$$
$[AFC]^T \mid [ARC]^T$

Now,

Member	Global Numbering
1	4, 5, 1, 6, 2, 7
2	8, 3, 6, 2
3	8, 3, 4, 5

Member	j	k	L(m)	EA	θ	cosθ	sinθ
2	3	2	0.6	100×10^3	90°	0	1
3	3	1	2.09	200×10^3	163.30	−0.9578	0.2874

Now, Member Stiffness Matrix;

For, Member-1 ; Here, RT is a Unit Matrix

$\therefore [SMS]_1 = [SM]_1$

$\therefore l = 2\ m$, $AE = 6000\times10^3$, $EI = 1100$

$$\therefore [SMS]_1 = 10^3 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 1 & 6 & 2 & 7 \end{array} \\ \left[\begin{array}{ccc|ccc} 3000 & 0 & 0 & -3000 & 0 & 0 \\ 0 & 1.65 & 1.65 & 0 & -1.65 & 1.65 \\ 0 & 1.65 & 2.2 & 0 & -1.65 & 1.1 \\ \hline -3000 & 0 & 0 & 3000 & 0 & 0 \\ 0 & -1.65 & -1.65 & 0 & 1.65 & -1.65 \\ 0 & 1.65 & 1.1 & 0 & -1.65 & 2.2 \end{array} \right] \end{array} \begin{array}{c} 4 \\ 5 \\ 1 \\ 6 \\ 2 \\ 7 \end{array}$$

Now, For Member-2; $l = 0.6\ m$. , $AE = 100\times10^3$ KN

$$\therefore [SM]_2 = 10^3 \left[\begin{array}{cc|cc} 166.67 & 0 & -166.67 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -166.67 & 0 & 166.67 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\& \ [RT]_2 = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right]$$

2.....

Now,

$$[SM]_2 \cdot [RT]_2 = \left[\begin{array}{cc|cc} 0 & 166.67 & 0 & -166.67 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -166.67 & 0 & 166.67 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now,

$$[SMS]_2 = [RT]_2^T \cdot [SM]_2 \cdot [RT]_2$$

$$\therefore [SMS]_2 = 10^3 \begin{array}{c} \begin{array}{cccc} 8 & 3 & 6 & 2 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 166.67 & 0 & -166.67 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -166.67 & 0 & 166.67 \end{array}\right] \end{array} \begin{array}{c} \\ 8 \\ 3 \\ 6 \\ 2 \end{array}$$

Now, Fig. Member-3', $l = 2.09\ m$ & $AE = 200 \times 10^3$

$$[SM]_3 = 10^3 \left[\begin{array}{cc|cc} 95.69 & 0 & -95.69 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -95.69 & 0 & 95.69 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now, &

$$[RT]_3 = \left[\begin{array}{cc|cc} -0.9578 & 0.2874 & 0 & 0 \\ -0.2874 & -0.9578 & 0 & 0 \\ \hline 0 & 0 & -0.9578 & 0.2874 \\ 0 & 0 & -0.2874 & -0.9578 \end{array}\right]$$

Now,

$$\therefore [SM]_3 \cdot [RT]_3 = 10^3 \left[\begin{array}{cc|cc} -91.65 & 27.50 & 91.65 & -27.50 \\ 0 & 0 & 0 & 0 \\ \hline 91.65 & -27.50 & -91.65 & 27.50 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore [SMS]_3 = 10^3 \begin{array}{c} \begin{array}{cccc} 8 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cc|cc} 87.78 & -26.34 & -87.78 & 26.34 \\ -26.34 & 7.90 & 26.34 & -7.90 \\ \hline -87.78 & 26.34 & 87.78 & -26.34 \\ 26.34 & -7.90 & -26.34 & 7.90 \end{array}\right] \end{array} \begin{array}{c} 8 \\ 3 \\ 4 \\ 5 \end{array}$$

Now,

$$\begin{array}{c} [SFF] \\ \\ [SJ] = 10^3 \\ \\ [SRF] \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4\ 5\ 6\ 7\ 8 \end{array} \\ \left[\begin{array}{ccc|c} 2.2 & -1.65 & 0 & \\ -1.65 & 1.65+166.67 & -166.67 & \\ 0 & -166.67 & 166.67+7.9 & \\ \hline 0 & 0 & 26.34 & \\ 1.65 & -1.65 & -7.90 & \\ 0 & 0 & 0 & \\ 1.1 & -1.65 & 0 & \\ 0 & 0 & -26.34 & \end{array}\right] \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

Now, Displacement, D;

$[D] = [SFF]^{-1} \cdot [AFC]$

$$= \frac{1}{10^3} \begin{bmatrix} 0.526 & 0.095 & 0.091 \\ 0.095 & 0.127 & 0.121 \\ 0.091 & 0.121 & 0.121 \end{bmatrix} \cdot \begin{bmatrix} -16.67 \\ -50 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -13.52 \\ -7.93 \\ -7.57 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \cdot \frac{1}{10^3}$$

4....

Now,

Reaction..;

$$[AR] = -[ARC] + [SRF].[D]$$

$$= \begin{bmatrix} 0 \\ 50 \\ 0 \\ -16.67 \\ 0 \end{bmatrix} + 10^3 \begin{bmatrix} 0 & 0 & 26.34 \\ 1.65 & -1.65 & -7.90 \\ 0 & 0 & 0 \\ 1.1 & -1.65 & 0 \\ 0 & 0 & -26.34 \end{bmatrix} \cdot \frac{1}{10^3} \begin{bmatrix} -13.52 \\ -7.93 \\ -7.57 \end{bmatrix}$$

$$\therefore \begin{bmatrix} R_{AH} \\ R_{AV} \\ R_{DH} \\ M_D \\ R_{CH} \end{bmatrix} = \begin{bmatrix} -199.39 \\ 100.58 \\ 0 \\ -18.46 \\ 199.39 \end{bmatrix} \begin{matrix} KN \\ KN \\ KN \\ KN.m \\ KN \end{matrix}$$

Now,

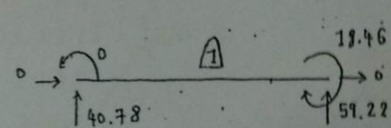
Member End Action..;

For. Member-1..;

$$[AM]_1 = [AML]_1 + [SM]_1 . [RT]_1 . [D_F]_1$$

$$= \begin{bmatrix} 0 \\ +50 \\ 16.67 \\ 0 \\ 50 \\ -16.67 \end{bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 4 & 5 & 1 & 6 & 2 & 7 \end{array} \\ \left[\begin{array}{ccc|ccc} 3000 & 0 & 0 & -3000 & 0 & 0 \\ 0 & 1.65 & 1.65 & 0 & -1.65 & 1.65 \\ 0 & 1.65 & 2.2 & 0 & -1.65 & 1.1 \\ \hline -3000 & 0 & 0 & 3000 & 0 & 0 \\ 0 & -1.65 & -1.65 & 0 & 1.65 & -1.65 \\ 0 & 1.65 & 1.1 & 0 & -1.65 & 2.2 \end{array}\right] \end{array} \begin{matrix} 4 \\ 5 \\ 1 \\ 6 \\ 2 \\ 7 \end{matrix} \cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ 0 \\ -13.1 \\ 0 \\ -7.9 \\ 0 \end{bmatrix}$$

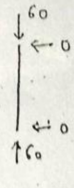
$$\therefore \begin{bmatrix} R_{AH} \\ R_{AV} \\ M_{AD} \\ R_{DH} \\ R_{DV} \\ M_{DA} \end{bmatrix} = \begin{bmatrix} 0 \\ 40.78 \\ 0.01 \\ 0 \\ 59.22 \\ -18.46 \end{bmatrix} \begin{matrix} KN \\ KN \\ KN.m \\ KN \\ KN \\ KN.m \end{matrix}$$



$$[AM]_2 = [AML]_2 + [SM]_2 \cdot [RT]_2 \cdot [DF]_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccc} 8 & 3 & 6 & 2 \end{array} \\ \left[\begin{array}{cc|cc} 0 & 166.67 & 0 & -166.67 \\ 0 & 0 & 0 & 0 \\ \hline 0 & -166.67 & 0 & 166.67 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 8 \\ 3 \\ 6 \\ 2 \end{array} \cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ -7.57 \\ 0 \\ -7.93 \end{bmatrix}$$

$$\therefore \begin{bmatrix} R_{BH} \\ R_{BV} \\ R_{DH} \\ R_{DV} \end{bmatrix} = \begin{bmatrix} 60 \\ 0 \\ -60 \\ 0 \end{bmatrix} \begin{array}{c} KN \\ KN \\ KN \\ KN \end{array}$$

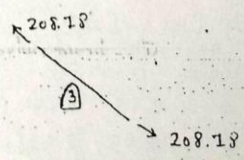


• For. Member-3;

$$[AM]_3 = [AML]_3 + [SM]_3 \cdot [RT]_3 \cdot [DF]_3$$

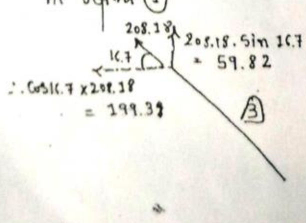
$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccc} 8 & 3 & 4 & 5 \end{array} \\ \left[\begin{array}{cc|cc} -91.65 & 27.50 & 91.65 & -27.50 \\ 0 & 0 & 0 & 0 \\ \hline 91.65 & -27.50 & -91.65 & 27.50 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{c} 8 \\ 3 \\ 4 \\ 5 \end{array} \cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ -7.57 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} R_{CH} \\ R_{CV} \\ R_{AH} \\ R_{AV} \end{bmatrix} = \begin{bmatrix} -208.18 \\ 0 \\ 208.18 \\ 0 \end{bmatrix} \begin{array}{c} KN \\ KN \\ KN \\ KN \end{array}$$



Check;

At Joint 1



$\therefore AR_1 = -199.39$ (✓ — OK.)

& $AR_2 = 59.82 + (AM_1)_1$

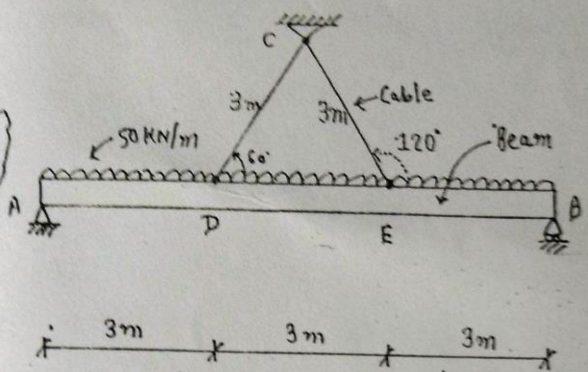
$= 59.82 + 40.78$

$= 100.6\ (\uparrow)$ (✓ — O.K.)

...BLEM No: 3

...e: The Cables Cannot ...st bending moment ...will Carry Axial ...es only

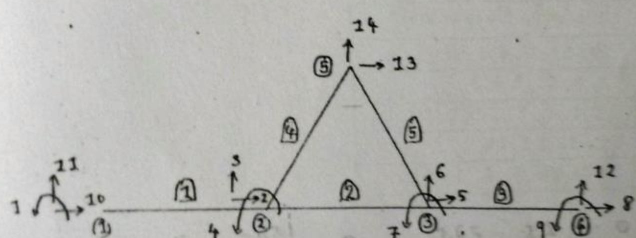
The Cables Cannot bending moment will carry Axial only



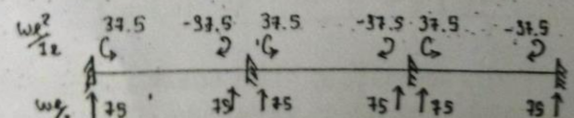
[given structure]

	A	I
Beam	30 Cm²	550 Cm⁴
Cable	2 Cm²	

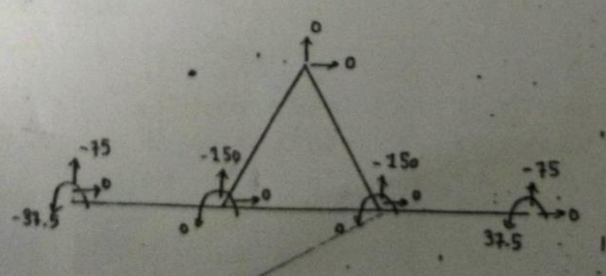
$E = 1200 \times 10^6$ kN/m²



[Joint & Member Numbering & Possible displacement]



[Fixed End Action]



$[A_C] = [A_E] + [A_S]$

$$= \left[\begin{array}{cccccccccc|ccccc} -37.5 & 0 & -150 & 0 & 0 & -150 & 0 & 0 & 37.5 & & 0 & -75 & -75 & 0 & 0 \end{array}\right]$$

1 2 3 4 5 6 7 8 9 | 10 11 12 13 14

$[FM_E]^T \;|\; [FM_R]^T$

Now, Member Information;

Beam

$A = 30 \times 10^{-4}\ m^2$

$I = 550 \times 10^{-8}\ m^4$

$E = 200 \times 10^{6}\ kN/m^2$

$EI = 1100\ kNm^2$

$AE = 600 \times 10^{3}\ kN$

Truss

$A = 2 \times 10^{-4}\ m^2$

$AE = 40 \times 10^{3}\ kN$

Member	Global Numbering
1	10, 11, 1, 2, 3, 4
2	2, 3, 4, 5, 6, 7
3	5, 6, 7, 8, 12, 9
4	2, 3, 13, 14
5	~~[illegible]~~, 5, 6, 13, 14

Member	j	k	L(m)	EA	L(m)	θ	cos θ	sin θ
4	2	5	3 m	40×10^3	3 m	60°	0.5	0.866
5	3	5	3 m	40×10^3	3 m	120°	-0.5	0.866

Now, for Member-1, 2 & 3, $AE = 600 \times 10^3$ kN & $EI = 1100$ kN.m²
& Here; it is a unit matrix $\therefore [SMS] = [SM]$

Now, for Member-1; $L = 3$ m.

$$\therefore [SMS]_1 = 10^3 \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccc|ccc} 200 & 0 & 0 & -200 & 0 & 0 \\ 0 & 0.489 & 0.733 & 0 & -0.489 & 0.733 \\ 0 & 0.733 & 1.467 & 0 & -0.733 & 0.733 \\ \hline -200 & 0 & 0 & 200 & 0 & 0 \\ 0 & -0.489 & -0.733 & 0 & 0.489 & -0.733 \\ 0 & 0.733 & 0.733 & 0 & -0.733 & 1.467 \end{array}\right] \end{array} \begin{array}{c} 10 \\ 11 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}$$

Now,

[SFF]

] = 10^3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
	1.467	0	-0.735	0.733	0	0	0	0	0					
	0	200+200 +3.333	5.772	0	-200	0	0	0	0					
	-0.733	5.772	0.489 +0.489 +1.0	-0.733 +0.733	0	-0.489	0.733	0	0					
	0.733	0	-0.733 +0.733	1.467 +1.467	0	-0.733	0.733	0	0					
	0	-200	0	0	200+200 +3.333	-5.772	0	-200	0					
	0	0	-0.489	-0.733	-5.772	0.489 +0.489+1.0	-0.733 +0.733	0	0.733					
	0	0	0.733	0.733	0	-0.733 +0.733	1.467 +1.467	0	0.733					
	0	0	0	0	-200	0	0	200	0					
	0	0	0	0	0	0.733	0.733	0	1.467					
	0	-200	0	0	0	0	0	0	0					
	0.733	0	-0.489	0.733	0	0	0	0	0					
	0	0	0	0	0	0.489	-0.733	0	-0.733					
	0	-3.333	-5.772	0	-3.333	5.772	0	0	0					
	0	-5.772	-10	0	5.772	-10	0	0	0					

[SRF]

Now, Displacement .. D;

] = [SFF]$^{-1}$. [AFC]

= $\frac{1}{EI}$ ×

0.817	-0.002	0.052	-0.219	-0.002	-0.013	0.046	-0.002	-0.017
-0.002	0.005	-0.003	0.001	0.005	0.003	0.001	0.005	-0.002
0.052	-0.003	0.098	-0.006	-0.003	0.002	-0.026	-0.003	0.012
-0.219	0.001	-0.006	0.432	0.002	0.027	-0.118	0.002	0.045
-0.002	0.005	-0.003	0.002	0.01	0.006	0.001	0.01	-0.003
-0.013	0.003	0.002	0.027	0.006	0.101	0.006	0.006	-0.054
0.046	0.001	-0.026	-0.118	0.001	0.006	0.431	0.001	-0.219
-0.002	0.005	-0.003	0.002	0.01	0.006	0.001	0.015	-0.003
-0.017	-0.002	0.012	0.045	-0.003	-0.054	-0.219	-0.003	0.818

×

-37.5
0
-150
0
0
-150
0
0
37.5

5....

$$[D] = \begin{array}{c} \begin{bmatrix} -37.13 \\ 0 \\ -16.5 \\ 6.75 \\ -0.49 \\ -17 \\ -6.94 \\ -0.49 \\ 37.61 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}$$

Now, Reaction ;

$$[AR] = -[ARC] + [SRF] . [D]$$

$$= \begin{bmatrix} 0 \\ 75 \\ 75 \\ 9 \\ 0 \end{bmatrix} + 10^3 \begin{bmatrix} 0 & -200 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.733 & 0 & -0.489 & 0.733 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.489 & -0.733 & 0 & -0.733 \\ 0 & -3.333 & -5.772 & 0 & -3.333 & 5.772 & 0 & 0 & 0 \\ 0 & -5.772 & -10 & 0 & 5.772 & -10 & 0 & 0 & 6 \end{bmatrix} \cdot \frac{1}{10^3} \begin{bmatrix} -37.13 \\ 0 \\ -16.5 \\ 6.75 \\ -0.49 \\ -17 \\ -6.94 \\ -0.49 \\ 37.61 \end{bmatrix}$$

$$\begin{bmatrix} R_{AH} \\ R_{AV} \\ R_{BV} \\ R_{DH} \\ R_{DV} \end{bmatrix} = \begin{bmatrix} 0 \\ 60.80 \\ 60.83 \\ -1.25 \\ 332.17 \end{bmatrix} \begin{array}{l} kN \\ kN \\ kN \\ kN \\ kN \end{array}$$

c....

$$
\begin{array}{c|ccccccccc|ccccc|}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\
\hline
[SFF] & 1.467 & 0 & -0.733 & 0.733 & 0 & 0 & 0 & 0 & 0 & & & & & \\
 & 0 & 200+200+3.333 & 5.772 & 0 & -200 & 0 & 0 & 0 & 0 & & & & & \\
 & -0.733 & 5.772 & 0.489+0.489+20 & -0.733+0.733 & 0 & -0.489 & 0.733 & 0 & 0 & & & & & \\
 & 0.733 & 0 & -0.733+0.733 & 1.467+1.467 & 0 & -0.733 & 0.733 & 0 & 0 & & & & & \\
 & 0 & -200 & 0 & 0 & 200+200+3.333 & -5.772 & 0 & -200 & 0 & & & & & \\
J = 10^3 & 0 & 0 & -0.489 & -0.733 & -5.772 & 0.489+0.489+10 & -0.733+0.733 & 0 & 0.733 & & & & & \\
 & 0 & 0 & 0.733 & 0.733 & 0 & -0.733+0.733 & 1.467+1.467 & 0 & 0.733 & & & & & \\
 & 0 & 0 & 0 & 0 & -200 & 0 & 0 & 200 & 0 & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0.733 & 0.733 & 0 & 1.467 & & & & & \\
\hline
 & 0 & -200 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\
 & 0.733 & 0 & -0.489 & 0.733 & 0 & 0 & 0 & 0 & 0 & & & & & \\
 & 0 & 0 & 0 & 0 & 0 & 0.481 & -0.733 & 0 & -0.733 & & & & & \\
 & 0 & -3.333 & -5.772 & 0 & -3.333 & 5.772 & 0 & 0 & 0 & & & & & \\
[SRF] & 0 & -5.772 & -10 & 0 & 5.772 & -10 & 0 & 0 & 0 & & & & & \\
\end{array}
$$

Now, Displacement .. D;

$= [SFF]^{-1} \cdot [AFC]$

$$
= \frac{1}{J 10^3}
\begin{bmatrix}
0.817 & -0.002 & 0.052 & -0.219 & -0.002 & -0.013 & 0.046 & -0.002 & -0.017 \\
-0.002 & 0.005 & -0.003 & 0.001 & 0.005 & 0.003 & 0.001 & 0.005 & -0.002 \\
0.052 & -0.003 & 0.098 & -0.006 & -0.003 & 0.002 & -0.026 & -0.003 & 0.012 \\
-0.219 & 0.001 & -0.006 & 0.432 & 0.002 & 0.027 & -0.118 & 0.002 & 0.045 \\
-0.002 & 0.005 & -0.003 & 0.002 & 0.01 & 0.006 & 0.001 & 0.01 & -0.003 \\
-0.013 & 0.003 & 0.002 & 0.027 & 0.006 & 0.101 & 0.006 & 0.006 & -0.054 \\
0.046 & 0.001 & -0.026 & -0.118 & 0.001 & 0.006 & 0.431 & 0.001 & -0.219 \\
-0.002 & 0.005 & -0.003 & 0.002 & 0.01 & 0.006 & 0.001 & 0.015 & -0.003 \\
-0.017 & -0.002 & 0.012 & 0.045 & -0.003 & -0.054 & -0.219 & -0.003 & 0.118
\end{bmatrix}
\cdot
\begin{bmatrix}
-37.5 \\ 0 \\ -150 \\ 0 \\ 0 \\ -150 \\ 0 \\ 0 \\ 37.5
\end{bmatrix}
$$

$$
[D] = \begin{array}{c|c}
\left[\begin{matrix} -37.13 \\ 0 \\ -16.5 \\ 6.75 \\ -0.49 \\ -17 \\ -6.94 \\ -0.49 \\ 37.61 \end{matrix}\right] &
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{matrix}
\end{array}
$$

Now, Reaction. ;

$$[AR] = -[ARC] + [SRF] \cdot [D]$$

$$
= \begin{bmatrix} 0 \\ 75 \\ 75 \\ 0 \\ 0 \end{bmatrix} + 10^3 \begin{bmatrix}
0 & -200 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0.733 & 0 & -0.489 & 0.733 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -0.489 & -0.733 & 0 & -0.733 \\
0 & -3.333 & -5.772 & 0 & -3.333 & 5.772 & 0 & 0 & 0 \\
0 & -5.772 & -10 & 0 & 5.772 & -10 & 0 & 0 & 0
\end{bmatrix} \cdot \frac{1}{10^3} \begin{bmatrix} -37.13 \\ 0 \\ -16.5 \\ 6.75 \\ -0.49 \\ -17 \\ -6.94 \\ -0.49 \\ 37.67 \end{bmatrix}
$$

$$
\begin{bmatrix} R_{AH} \\ R_{AV} \\ R_{BV} \\ R_{DH} \\ R_{DV} \end{bmatrix} = \begin{bmatrix} 0 \\ 60.80 \\ 60.83 \\ -1.25 \\ 332.17 \end{bmatrix} \begin{matrix} KN \\ KN \\ KN \\ KN \\ KN \end{matrix}
$$

Now,

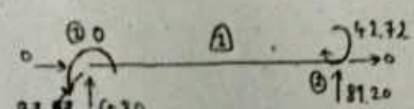
Member End Action;

For Member-1;

$[AM]_1 = [AML]_1 + [SM]_1 \cdot [RT]_1 \cdot [DF]_1$

$$= \begin{bmatrix} 0 \\ 75 \\ 37.5 \\ 0 \\ 75 \\ -37.5 \end{bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 10 & 11 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccc|ccc} 200 & 0 & 0 & -200 & 0 & 0 \\ 0 & 0.489 & 0.733 & 0 & -0.489 & 0.733 \\ 0 & 0.733 & 1.467 & 0 & -0.733 & 0.733 \\ \hline -200 & 0 & 0 & 200 & 0 & 0 \\ 0 & -0.489 & -0.733 & 0 & 0.489 & -0.733 \\ 0 & 0.733 & 0.733 & 0 & -0.733 & 1.467 \end{array}\right] \end{array} \begin{array}{c} 10 \\ 11 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ 0 \\ -37.13 \\ 0 \\ -26.5 \\ 6.75 \end{bmatrix}$$

$$\begin{bmatrix} R_{AH} \\ R_{AV} \\ M_{AD} \\ R_{BH} \\ R_{BV} \\ M_{DA} \end{bmatrix} = \begin{bmatrix} 0 \\ 60.80 \\ \cancel{37.57}0 \\ 0 \\ 89.20 \\ -42.72 \end{bmatrix} \begin{array}{l} KN \\ KN \\ KN\,m \\ KN \\ KN \\ KN\,m \end{array}$$

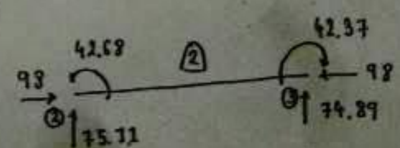


For Member-2;

$[AM]_2 = [AML]_2 + [SM]_2 \cdot [RT]_2 \cdot [DF]_2$

$$= \begin{bmatrix} 0 \\ 75 \\ 37.5 \\ 0 \\ 75 \\ -37.5 \end{bmatrix} + 10^3 \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccc|ccc} 200 & 0 & 0 & -200 & 0 & 0 \\ 0 & 0.489 & 0.733 & 0 & -0.489 & 0.733 \\ 0 & 0.733 & 1.467 & 0 & -0.733 & 0.733 \\ \hline -200 & 0 & 0 & 200 & 0 & 0 \\ 0 & -0.489 & -0.733 & 0 & 0.489 & -0.733 \\ 0 & 0.733 & 0.733 & 0 & -0.733 & 1.467 \end{array}\right] \end{array} \begin{array}{c} 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \cdot \frac{1}{10^3} \begin{bmatrix} 0 \\ -26.5 \\ 6.75 \\ -0.49 \\ -17 \\ -6.94 \end{bmatrix}$$

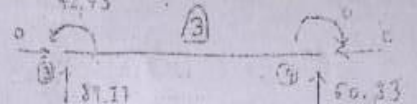
$$\therefore \begin{bmatrix} R_{DH} \\ R_{DV} \\ M_{DE} \\ R_{EH} \\ R_{EV} \\ M_{ED} \end{bmatrix} = \begin{bmatrix} 98 \\ 75.11 \\ 42.68 \\ -98 \\ 74.89 \\ -42.37 \end{bmatrix} \begin{array}{l} KN \\ KN \\ KN\,m \\ KN \\ KN \\ KN\,m \end{array}$$



7....

$$
= \begin{bmatrix} 0 \\ 75 \\ 37.5 \\ 0 \\ 75 \\ -37.5 \end{bmatrix} + 10^3
\begin{array}{c}
\begin{array}{cccccc} 5 & 6 & 7 & 8 & 12 & 9 \end{array} \\
\left[\begin{array}{ccc|ccc}
200 & 0 & 0 & -200 & 0 & 0 \\
0 & 0.489 & 0.733 & 0 & -0.489 & 0.733 \\
0 & 0.733 & 1.467 & 0 & 0.733 & 0.733 \\
\hline
-200 & 0 & 0 & 200 & 0 & 0 \\
0 & -0.489 & -0.733 & 0 & 0.489 & -0.733 \\
0 & 0.733 & 0.733 & 0 & -0.733 & 1.467
\end{array}\right]
\end{array}
\begin{array}{c} 5 \\ 6 \\ 7 \\ 8 \\ 12 \\ 9 \end{array}
\cdot \frac{1}{10^3}
\begin{bmatrix} -0.49 \\ -17 \\ -5.94 \\ -0.49 \\ [illegible] \\ [illegible] \end{bmatrix}
$$

$$
\begin{bmatrix} R_{EH} \\ R_{EV} \\ M_{E} \\ R_{DH} \\ R_{DV} \\ M_{DE} \end{bmatrix} = \begin{bmatrix} 0 \\ 87.17 \\ 42.43 \\ 0 \\ 60.83 \\ 0 \end{bmatrix} \begin{array}{l} \text{KN} \\ \text{KN} \\ \text{KN m} \\ \text{KN} \\ \text{KN} \\ \text{KN m} \end{array}
$$

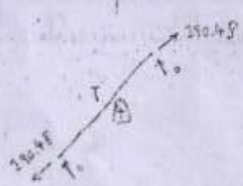


Now for Member - 4

$[AM]_4 = [AML]_4 + [SM]_4 \cdot [RT]_4 \cdot [DF]_4$

$$
= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 10^3
\begin{array}{c}
\begin{array}{cccc} 2 & 3 & 13 & 14 \end{array} \\
\left[\begin{array}{cc|cc}
6.665 & 11.544 & -6.665 & -11.544 \\
0 & 0 & 0 & 0 \\
\hline
-6.665 & -11.544 & 6.665 & 11.544 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} 2 \\ 3 \\ 13 \\ 14 \end{array}
\cdot \frac{1}{10^3}
\begin{bmatrix} 0 \\ -16.5 \\ 0 \\ 0 \end{bmatrix}
$$

$$
\begin{bmatrix} R_{BH} \\ R_{BV} \\ R_{CH} \\ R_{CV} \end{bmatrix} = \begin{bmatrix} -190.48 \\ 0 \\ 190.48 \\ 0 \end{bmatrix} \begin{array}{l} \text{KN} \\ \text{KN} \\ \text{KN} \\ \text{KN} \end{array}
$$



Now, for Member - 5

$[AM]_5 = [AML]_5 + [SM]_5 \cdot [RT]_5 \cdot [DF]_5$

$$
\begin{bmatrix} R_{EH} \\ R_{EV} \\ R_{CH} \\ R_{CV} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 10^3
\begin{array}{c}
\begin{array}{cccc} 5 & 6 & 13 & 14 \end{array} \\
\left[\begin{array}{cc|cc}
-6.665 & 11.544 & 6.665 & -11.544 \\
0 & 0 & 0 & 0 \\
\hline
6.665 & -11.544 & -6.665 & 11.544 \\
0 & 0 & 0 & 0
\end{array}\right]
\end{array}
\begin{array}{c} 5 \\ 6 \\ 13 \\ 14 \end{array}
\cdot \frac{1}{10^3}
\begin{bmatrix} -0.49 \\ 17 \\ 0 \\ 0 \end{bmatrix}
= \begin{bmatrix} -192.95 \\ 0 \\ 192.95 \\ 0 \end{bmatrix} \begin{array}{l} \text{KN} \\ \text{KN} \\ \text{KN} \\ \text{KN} \end{array}
$$

Member - 4

At Joint 3, -164.96 + 89.20 + 75.77 = 0 — O.K.

$\uparrow$ 190.48 sin 30 = 95 — O.K.

Member - 5

cos 30 · 192.95 = 167.13

At Joint, 3

166.13 - 89.17 - 76.77 = 0

8 — —