PROBLEM 1

$$E(\hat{\beta}_0) = E(\bar{y} - \beta_1 \bar{X})$$

Remember β_1 + $\bar{X}$ are constants so

$$E(\bar{y} - \beta_1 \bar{X}) = E(\bar{y}) - \beta_1 \bar{x}$$

$$= E\left(\sum_{i=1}^{n} y_i / n\right) - \beta_1 \bar{x}$$

$$= \frac{1}{n} E\left(\sum_{i=1}^{n} y_i\right) - \beta_1 \bar{x}$$

Since $y_i \; i=1,\dots,n$ are independent

$$E\left(\sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{n} E(y_i) = \sum_{i=1}^{n} [\beta_0 + \beta_1 x_i]$$

$\Rightarrow$

$$E(\hat{\beta}_0) = E(\bar{y} - \beta_1 \bar{x}) = \frac{1}{n} \sum_{i=1}^{n} [\beta_0 + \beta_1 x_i] - \beta_1 \bar{x}$$

$$= n\beta_0 / n + \frac{\beta_1 \sum x_i}{n} - \beta_1 \bar{x}$$

Now you finish it!

(SEE ATTACHED)

PROBLEM 1 Continued

$$E(\hat{\beta}_1) = E\left(\frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \right)$$

$$= \frac{1}{\sum_{i=1}^{n} (x_i - \bar{x})^2} E\left(\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \right)$$

$$= \frac{1}{\sum_{i=1}^{n} (x_i - \bar{x})^2} \sum_{i=1}^{n} (x_i - \bar{x}) E(y_i - \bar{y})$$

$$E(y_i - \bar{y}) = E(y_i) - \frac{1}{n} \sum_{i=1}^{n} E(y_i)$$

$$= \beta_0 + \beta_1 x_i - \frac{1}{n} \sum_{i=1}^{n} [\beta_0 + \beta_1 x_i]$$

$$= \beta_1 (x_i - \bar{x})$$

Now you finish it!

(SEE ATTACHED)

PROBLEM 1, PART 1

given $E(\hat{\beta}_0) = \frac{n\beta_0}{n} + \frac{\beta_1 \Sigma x_i}{n} - \beta_1 \bar{X}$

$$= \beta_0 + \frac{\beta_1 \Sigma x_i}{n} - \beta_1 \bar{X}$$

given $\bar{X} = \Sigma \frac{x_i}{n}$

$$E(\hat{\beta}_0) = \beta_0 + \beta_1 \bar{X} - \beta_1 \bar{X}$$

$$\boxed{E(\hat{\beta}_0) = \beta_0}$$

PROBLEM 1, PART 2

given $E(\hat{\beta}_1) = \frac{1}{\Sigma(x_i - \bar{x})^2} E(x_i - \bar{x}) E(y_i - \bar{y})$

also that $E(y_i - \bar{y}) = \beta_1 (x_i - \bar{x})$

$$E(\hat{\beta}_1) = \frac{1}{\Sigma \cancel{(x_i - \bar{x})^2}} \Sigma \cancel{(x_i - \bar{x})} \beta_1 \cancel{(x_i - \bar{x})}$$

$\boxed{E(\hat{\beta}_1) = \beta_1}$ UNBIASED

PROBLEM 2

$$\ell(\beta) = -n \log\left(\sqrt{2\pi\sigma^2}\right) - \frac{\sum_{i=1}^{n}(y_i - \beta x_i)^2}{2\sigma^2}$$

$$\frac{\delta \ell(\beta)}{\delta \beta} = \frac{2}{2\sigma^2} \sum_{i=1}^{n} x_i(y_i - \beta x_i)$$

Since σ^2 is constant we have the minimization results in

$$\sum_{i=1}^{n} x_i(y_i - \beta x_i) = 0$$

Now you finish it!

$$\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} \beta x_i^2 = 0$$

$$\beta_1 \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} x_i y_i$$

$$\boxed{\beta = \sum_{i=1}^{n} \frac{x_i y_i}{x_i^2}}$$

PROBLEM 3

$$\hat{S} = \frac{\sum (x_i - \bar{x})^2}{n-1}$$

Assume $\quad X_i \sim N(\mu, \sigma^2)$

$$E(\hat{S}) = \frac{1}{n-1} E\left[\sum_{i=1}^{n} (x_i^2 - 2x_i\bar{x} + \bar{x}^2)\right]$$

Now You Finish IT!

HINT: USE THE SLIDES

* CALCULATING BIAS
* VARIANCE

for help on this

(SEE ATTACHED)

PROBLEM 3

use $\bar{x} = \sum_{i=1}^{n} \frac{x_i}{n}$

or $n\bar{x} = \sum_{i=1}^{n} x_i$

$$E(\hat{S}) = \frac{1}{n-1} E\left[\sum_{i=1}^{n} x_i^2 - 2\bar{x}\sum_{i=1}^{n} x_i + n\bar{x}^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} x_i^2 - 2n\bar{x}^2 + n\bar{x}^2\right]$$

$$= \frac{1}{n-1} E\left[\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right]$$

$$= \frac{1}{n-1}\left[\sum_{i=1}^{n} E(x_i^2) - nE(\bar{x}^2)\right]$$

since $Var(x) = \sigma^2$ and $x \sim N(\mu, \sigma^2)$

$$\sigma^2 = E(x^2) - 2E(xE(x)) + E(x)^2$$

$$= E(x^2) - 2\mu E(x) + \mu^2$$

$$= E(x^2) - 2\mu^2 + \mu^2$$

$$E(x^2) = \sigma^2 + \mu^2$$

(over)

PROBLEM 3 (CONT.)

given $\bar{X} \sim N(\mu, \sigma^2 2n)$ or $Var(\bar{x}) = \frac{\sigma^2}{n}$

so,

$$Var(\bar{X}) = \frac{\sigma^2}{n} = E(\bar{X}^2) - 2E(\bar{x}E(\bar{x})) + E(\bar{x})^2$$

$$\frac{\sigma^2}{n} = E(\bar{X}^2) - 2\mu^2 + \mu^2$$

$$\therefore E(\bar{X}^2) = \frac{\sigma^2}{n} + \mu^2$$

substituting the values:

$$E(\hat{S}) = \frac{1}{n-1}\left[\sum_{i=1}^{n} E(X_i^2) - nE(\bar{X}^2)\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + \cancel{n\mu^2} - \frac{\cancel{n}\sigma^2}{\cancel{n}} - \cancel{n\mu^2}\right]$$

$$= \frac{1}{\cancel{n-1}}\left[\sigma^2 \cancel{(n-1)}\right]$$

$$\boxed{E(\hat{S}) = \sigma^2}$$

$\therefore$ the sample variance = estimated variance. It is <u>unbiased</u>