Derivation of the discrete-time constant turn rate and acceleration motion model

Daniel Svensson
Cruising & Highway Autonomous Driving
Zenuity AB
Göteborg, Sweden
daniel.svensson@zenuity.com

***Abstract*—For vehicle tracking in automotive applications there are a number of proposed motion models. One of those models is the constant turn rate and acceleration (CTRA) model. In the original paper where the model was introduced, the state prediction function was defined, but not the process noise. In this paper, a derivation of the process noise is made. For completeness, the discrete-time prediction model is also derived, using linearized discretization.**

***Index Terms*—target tracking, sensor fusion, motion models, covariance matrix, automotive, Kalman filtering**

I. Introduction

For both active safety systems and different levels of automated driving systems, it is of importance to have access to an accurate environmental description of the traffic surrounding the ego vehicle. That task is either put on a single-sensor system, or on a sensor fusion system that uses data from several sensors as input. Regardless of the cardinality of the sensor set, the central component of an environmental perception system is a target tracking function. Among the several components of such a module, there are two that are of extra importance: the motion model and the measurement model. In this paper, the former part is considered.

The motion model describes the future state of an object, given the current state [1], [2]. It is hence used in the prediction step of a recursive filter, e.g., a Kalman filter [3]. For best tracking accuracy, the motion model should be well aligned with the physical motion of the object that is being tracked. The motion model is also referred to as the process model, and has two main components: the prediction function and the process noise covariance matrix.

When tracking on-road vehicular objects, such as cars and trucks, the best motion models come from the family of coordinated turn (CT) models [4]–[6]. These models assume that the turn rate of the object is approximately constant, with perturbations captured by the process noise. In [5], different versions of CT models are compared. The three most interesting models are

1) Constant turn rate and velocity (CTRV)
2) Constant turn rate and acceleration (CTRA)
3) Constant curvature and acceleration (CCA).

The CTRV model is a model for horizontal turns, which is equal to what is often denoted as the CT model—especially for aircraft tracking [2]. The model has a five-dimensional state vector including position in two dimensions, speed, heading and yaw rate. By adding acceleration to the state vector, and assuming that the derivative of speed is constant, the CTRA model is obtained. Both CTRV and CTRA ignore the correlation between speed and yaw rate. A model that includes the correlations is the CCA model, which instead of yaw rate has curvature as a state. The geometrical interpretation of the models is that the CTRV and CCA models assume that the vehicle is moving along a circular trajectory, while the CTRA model assumes a clothoid trajectory.

The evaluation of the motion models in [5] is done for ego-vehicle state estimation using an unscented Kalman filter [7] with GPS and odometry information as input. The best trade-off between complexity and performance is concluded to be the CTRA model, which is the reason for only focusing on that model in this paper. The derivation of the CTRA process noise covariance herein is straightforward to apply also to the CTRV model, while the CCA model would be more cumbersome, given its complex nature.

In [5], the prediction function of the CTRA model is presented (albeit in a different form than the expression herein, due to a different coordinate system), but no derivation is made. Moreover, no process noise covariance matrix is described. Since both the prediction function and the covariance matrix are required for a complete motion model, there is a need for a derivation of a process noise. In this paper, the complete discrete-time CTRA process model is derived, starting from the time-continuous motion model with white noise.

II. Problem Formulation

A general time-continuous non-linear motion model, with additive noise, is defined as

$$\dot{x}(t) = f(x(t)) + \Gamma v(t), \tag{1}$$

where f is the prediction function, and $v(t)$ is a vector-valued continuous-time white noise process with power spectral density $\mathbf{Q}(t))$ [6]. Further Γ is a matrix that maps the noise process dimension to the state dimension. For the CTRA model, the continuous-time state vector is defined as:

$$\mathbf{x}(t) = \begin{bmatrix} x(t) & y(t) & s(t) & \varphi(t) & \omega(t) & a(t) \end{bmatrix}^T. \tag{2}$$

Here, $x(t)$ and $y(t)$ are longitudinal and lateral positions, respectively, $\varphi(t)$ is heading, $s(t)$ speed, $a(t)$ acceleration and $\omega(t)$ yaw rate (turn rate). An illustration of the coordinate system is given in Fig. 1. For CTRA, the prediction function is given by

$$f_{\text{CTRA}}(x(t)) = \begin{bmatrix} s(t)\cos(\varphi(t)) \\ s(t)\sin(\varphi(t)) \\ a(t) \\ \omega(t) \\ 0 \\ 0 \end{bmatrix}. \tag{3}$$

The intuition behind the model is thus that the changes in acceleration and turn rate are approximately zero, with perturbations captured by the process noise $v_{\text{CTRA}}(t) \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_{\text{CTRA}}(t))$, where

$$\mathbf{Q}_{\text{CTRA}}(t) = \begin{bmatrix} \sigma_{\dot{\omega}} & 0 \\ 0 & \sigma_{\dot{a}} \end{bmatrix}. \tag{4}$$

Here $\sigma_{\dot{a}}(t)$ and $\sigma_{\dot{\omega}}(t)$ are power spectral densities in jerk and rate of turn rate noise, respectively. Further,

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T. \tag{5}$$

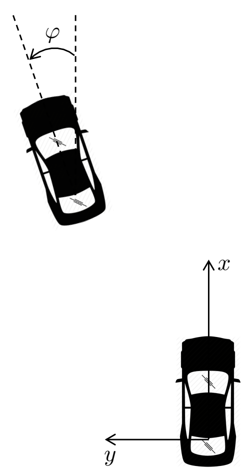

Fig. 1. Illustration of the coordinate system.

The problem addressed in this paper is to derive the corresponding discrete-time motion model

$$\mathbf{x}_{k+1}^{\text{CTRA}} = f_k^{\text{CTRA}}(\mathbf{x}_k^{\text{CTRA}}) + \mathbf{v}_k^{\text{CTRA}}, \tag{6}$$

where

$$\mathbf{x}_k^{\text{CTRA}} = \begin{bmatrix} x_k & y_k & s_k & \varphi_k & \omega_k & a_k \end{bmatrix}^T \tag{7}$$

is the discrete-time state vector at time step k, and $\mathbf{v}_k^{\text{CTRA}} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k^{\text{CTRA}}))$ is the discrete-time process noise.

III. Discrete-time CTRA model derivation

A. Prediction function

We start by discretizing the prediction function in (3). As described in [9], there are two ways of linearizing and discretizing a continuous-time motion model, called linearized discretization and discretized linearization, respectively. The first approach, where the non-linear time-continuous model is first discretized exactly, and then approximated by a Taylor expansion is more accurate, but is rarely applicable [9]. However, for the CTRA model, it can be applied, and the derivation herein hence relies on the linearized discretization approach. The linearization step is only necessary if an extended Kalman filter (EKF) [8] is to be applied. For sigma-point methods, the discretized prediction function is sufficient.

The discrete-time prediction step of a nonlinear filter, corresponding to the time-continuous prediction function $f(x)$ is given by [9]

$$x(t+T) = x(t) + \int_t^{t+T} \left(f(x(\tau)) + \Gamma v(\tau)\right) d\tau. \tag{8}$$

Since the process noise $v(t)$ is zero-mean, the discrete-time prediction function is given by

$$f_k^{\text{CTRA}}(\mathbf{x}_k^{\text{CTRA}}) = \int_t^{t+T} f_{\text{CTRA}}(x(\tau))\, d\tau. \tag{9}$$

From the definition of a discrete-time signal

$$x[k] \triangleq x(kT), \tag{10}$$

the discrete-time prediction step can be written as

$$x[k+1] = x[k] + \int_{kT}^{(k+1)T} f_{\text{CTRA}}(x(\tau))\, d\tau. \tag{11}$$

Throughout the paper, $x[k]$ and x_k are used interchangeably.

Let us start with the turn rate, ω_k, and acceleration, a. From (11) and (3), we get

$$\omega_{k+1} = \omega_k \tag{12}$$
$$a_{k+1} = a_k. \tag{13}$$

For the speed prediction, we obtain

$$\begin{aligned} s_{k+1} &= s_k + \int_{kT}^{kT+T} a(\tau) d\tau \quad (14) \\ &= s_k + a_k T. \quad (15) \end{aligned}$$

We here use that the acceleration is constant over a prediction interval $[kT, (k+1)T)$, according to the model. Similarly, for heading, the following holds true

$$\begin{aligned} \varphi_{k+1} &= \varphi_k + \int_{kT}^{kT+T} \omega(\tau) d\tau \quad (16) \\ &= \varphi_k + \omega_k T. \quad (17) \end{aligned}$$

We now have the discrete-time predictions for four of the states. The final two are a bit more cumbersome to obtain. Starting with longitudinal position, x_{k+1}, we have

$$x_{k+1} = x_k + \int_{kT}^{kT+T} s(\tau)\cos(\varphi(\tau)) d\tau. \tag{18}$$

Using partial integration, we obtain

$$x_{k+1} = x_k + \left[s(\tau) \frac{\sin(\varphi(\tau))}{\omega(\tau)} \right]_{kT}^{kT+T} \tag{19}$$

$$- \int_{kT}^{kT+T} a(\tau) \frac{\sin(\varphi(\tau))}{\omega(\tau)} d\tau$$

$$= x_k + \Delta x_k, \tag{20}$$

where

$$\Delta x_k = \left[s(\tau) \frac{\sin(\varphi(\tau))}{\omega(\tau)} \right]_{kT}^{kT+T} + \left[a(\tau) \frac{\cos(\varphi(\tau))}{\omega^2(\tau)} \right]_{kT}^{kT+T}. \tag{21}$$

Expanding the integration limits, we get

$$\Delta x_k = s(kT+T) \frac{\sin(\varphi(kT+T)}{\omega(kT+T)} - s(kT) \frac{\sin(\varphi(kT))}{\omega(kT)} + a(kT+T) \frac{\cos(\varphi(kT+T))}{\omega^2(kT+T)} - a(kT) \frac{\cos(\varphi(kT))}{\omega^2(kT)}. \tag{22}$$

By using that $a(kT+T) = a(kT) = a_k$ and $\omega(kT+T) = \omega(kT) = \omega_k$, we obtain

$$\Delta x_k = \frac{1}{\omega_k^2} \big(s_{k+1} \omega_k \sin(\varphi_{k+1}) + a_k \cos(\varphi_{k+1}) - s_k \omega_k \sin(\varphi_k) - a_k \cos(\varphi_k) \big). \tag{23}$$

Finally, the discrete-time prediction for the lateral position y_k is

$$y_{k+1} = y_k + \int_{kT}^{kT+T} s(\tau) \sin(\varphi(\tau))) d\tau. \tag{24}$$

Again using partial integration, we obtain

$$y_{k+1} = y_k - \left[s(\tau) \frac{\cos(\varphi(\tau))}{\omega(\tau)} \right]_{kT}^{kT+T} \tag{25}$$

$$+ \int_{kT}^{kT+T} a(\tau) \frac{\cos(\varphi(\tau))}{\omega(\tau)} d\tau$$

$$= y_k + \Delta y_k. \tag{26}$$

Here,

$$\Delta y_k = -s(kT+T) \frac{\cos(\varphi(kT+T)}{\omega(kT+T)} + s(kT) \frac{\cos(\varphi(kT))}{\omega(kT)} \tag{27}$$

$$+ a(kT+T) \frac{\sin(\varphi(kT+T))}{\omega^2(kT+T)} - a(kT) \frac{\sin(\varphi(kT))}{\omega^2(kT)}. \tag{28}$$

As for x_k, we insert $a(kT+T) = a(kT) = a_k$ and $\omega(kT+T) = \omega(kT) = \omega_k$ to get the final expression for Δy_k:

$$\Delta y_k = \frac{1}{\omega_k^2} \big(- s_{k+1} \omega_k \cos(\varphi_{k+1}) + a_k \sin(\varphi_{k+1}) + s_k \omega_k \cos(\varphi_k) - a_k \sin(\varphi_k) \big). \tag{29}$$

To sum up the above derivations, the discrete-time CTRA process model is

$$\mathbf{x}_{k+1}^{\text{CTRA}} = \mathbf{x}_k^{\text{CTRA}} + \begin{bmatrix} \Delta x_k \\ \Delta y_k \\ a_k T \\ \omega_k T \\ 0 \\ 0 \end{bmatrix} + \mathbf{v}_k, \tag{30}$$

where T is the prediction time and $\mathbf{v}_{k-1} \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k^{\text{CTRA}})$ is the discrete-time process noise, and where Δx_k and Δy_k are given by (23) and (29), respectively. The next step is to derive an expression for the covariance matrix $\mathbf{Q}_k^{\text{CTRA}}$.

B. Process noise covariance

In this section, a derivation of the covariance matrix $\mathbf{Q}_k^{\text{CTRA}}$ of the process noise $\mathbf{v}_k^{\text{CTRA}}$ is presented. The underlying assumption for the covariance matrix is that the acceleration and the yaw rate are nearly constant, and that changes in those dimensions are described by noise processes in acceleration derivative (jerk) and yaw rate derivative (rate of change of yaw rate). The noise processes are parameterized by $\sigma_{\dot{a}}^2$ and $\sigma_{\dot{\omega}}^2$, which are the power spectral densities of the rates of change in acceleration and yaw rate, respectively.

According to [9], there are five different ways of computing a discrete-time covariance matrix from a time-continuous one. The most exact one assumes that $v(t)$ (see (1)) is continuous white noise with covariance $\mathbf{Q}$. That is also the assumption used when deriving the process noise for the standard constant-velocity and constant-acceleration models (the T^3 and T^5 versions). In this derivation, we also assume that the continuous-time noise is white and Gaussian. As described in [9], the time-discrete covariance $\mathbf{Q}_k$ is given by

$$\mathbf{Q}_k^{\text{CTRA}} = \int_0^T e^{f'_{\text{CTRA}} \tau} \, \tilde{\mathbf{Q}}_{\text{CTRA}} \, e^{f'^{T}_{\text{CTRA}} \tau} d\tau, \tag{31}$$

where

$$f'_{\text{CTRA}} = \nabla f_{\text{CTRA}} \tag{32}$$

is the gradient of the motion model prediction function $f_{\text{CTRA}}(x(t))$, $e^{f'^{T}_{\text{CTRA}} \tau}$ is the transpose of $e^{f'_{\text{CTRA}} \tau}$, and where

$$\tilde{\mathbf{Q}}_{\text{CTRA}} = \Gamma \, \mathbf{Q}_{\text{CTRA}} \, \Gamma^T. \tag{33}$$

Here, and in the following expressions, the time variable t is excluded from the expressions for readability.

For the CTRA model, we obtain

$$f'_{\text{CTRA}} = \begin{bmatrix} 0 & 0 & \cos(\varphi) & -v\sin(\varphi) & 0 & 0 \\ 0 & 0 & \sin(\varphi) & v\cos(\varphi) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{34}$$

From the definition of the matrix exponential

$$e^{\mathbf{A}t} \triangleq \sum_{i=0}^{\infty} \frac{(\mathbf{A}t)^i}{i!} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \ldots, \tag{35}$$

and using that f'_{CTRA} is nilpotent, we get

$$e^{f'_{\text{CTRA}}\tau} = \mathbf{I} + \tau \nabla f_{\text{CTRA}} + \frac{\tau^2}{2}(\nabla f_{\text{CTRA}})^2, \tag{36}$$

where

$$(\nabla f_{\text{CTRA}})^2 = \nabla f_{\text{CTRA}} \, \nabla f_{\text{CTRA}} \tag{37}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & -s\sin(\varphi) & \cos(\varphi) \\ 0 & 0 & 0 & 0 & s\cos(\varphi) & \sin(\varphi) \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{38}$$

Combining (32), (34), (36) and (38), we obtain

$$e^{f'_{\text{CTRA}}\tau} = \begin{bmatrix} 1 & 0 & \tau\cos\varphi & -\tau s\sin\varphi & -\frac{\tau^2}{2}s\sin\varphi & \frac{\tau^2}{2}\cos\varphi \\ 0 & 1 & \tau\sin\varphi & \tau s\cos\varphi & \frac{\tau^2}{2}s\cos\varphi & \frac{\tau^2}{2}\sin\varphi \\ 0 & 0 & 1 & 0 & 0 & \tau \\ 0 & 0 & 0 & 1 & \tau & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{39}$$

Proceeding with the factors in (31), and using the result from (39), we have that

$$e^{f'_{\text{CTRA}}\tau}\tilde{\mathbf{Q}}_{\text{CTRA}} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sigma_{\dot{\omega}}^2\frac{\tau^2}{2}s\sin\varphi & \sigma_{\dot{a}}^2\frac{\tau^2}{2}\cos\varphi \\ 0 & 0 & 0 & 0 & \sigma_{\dot{\omega}}^2\frac{\tau^2}{2}s\cos\varphi & \sigma_{\dot{a}}^2\frac{\tau^2}{2}\sin\varphi \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{a}}^2\tau \\ 0 & 0 & 0 & 0 & \sigma_{\dot{\omega}}^2\tau & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\dot{\omega}}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\dot{a}}^2 \end{bmatrix}. \tag{40}$$

The complete product in (31) is hence given by (43).

By re-introducing the time-index, and by integrating (43), as given by (31), the discrete-time covariance matrix $\mathbf{Q}_k^{\text{CTRA}}$ is obtained, and is given by (44). Note that the units are m^2/s^5 for $\sigma_{\dot{a}}^2$ and rad^2/s^3 for $\sigma_{\dot{\omega}}^2$. Further note that the process noise only depends on the speed and the heading of the state vector.

IV. Practical considerations

In this section, we discuss two practical considerations that must be taken into account when using the motion model in real-world applications.

A. Small yaw rates

As seen in (23) and (29), the predictions of longitudinal and lateral positions include a division by the square of the yaw rate. For small yaw rates, this will lead to numerical instability and undefined behavior. As a remedy, for small values of ω_k, we replace the prediction equations with their limits,

$$\lim_{\omega_k \to 0} \Delta x_k = \left(s_k T + \frac{a_k T^2}{2}\right)\cos(\varphi_k) \tag{41}$$

$$\lim_{\omega_k \to 0} \Delta y_k = \left(s_k T + \frac{a_k T^2}{2}\right)\sin(\varphi_k). \tag{42}$$

By observing the above equations, it can be seen that they comprise a constant acceleration model prediction in the direction of the heading angle. The value of the threshold for when to use the approximation is a trade-off between accuracy and risk of numerical instability, and depends on the numerical precision of the computer.

In addition to prediction accuracy, the approximation also has an impact on the covariance matrix. When using an EKF, the cross-correlations between yaw rate and the remaining states are lost. For sensors that measure positions and potentially velocities, the measurement update will not decrease the uncertainty in yaw rate. This leads to unreasonably large yaw rate uncertainties, which affects the tracking performance. For sigma-point filters, e.g., CKF, the problem is alleviated, since some of the sigma points will be drawn outside of the approximation interval, for sufficiently large yaw rate uncertainties.

B. Potentially negative definite noise covariance matrix

Another consideration is that the matrix might have slightly negative eigenvalues when using single precision. One solution to this is to add a tiny diagonal matrix to the process noise covariance matrix, where the size of the diagonal components is proportional to the selected precision.

V. Summary

In this paper, the discrete-time constant turn rate and acceleration (CTRA) model is derived, including both the prediction equations and the process noise covariance matrix. The derivation starts from a time-continuous model with white process noise. The derivation of the prediction equation uses linearized discretization, and the process noise covariance matrix derivation follows the most exact out of five alternative methods for discrete-time process noise derivation presented in the literature. While the prediction equations have been presented earlier, a derivation of those equations has not been published. Moreover, no process noise covariance matrix has been derived and presented in the literature before.

References

[1] Y. Bar-Shalom, and X.R. Li, “Multitarget-Multisensor tracking: Principles and Techniques,” YBS Publishing, Storrs, CT, 1995

[2] S. Blackman, and R. Popoli, “Design and Analysis of Modern Tracking Systems,” Artech House, Norwood, 1999.

[3] R.E. Kalman, “A New Approach to Linear Filtering and Prediction Problems,”Transactions of the ASME–Journal of Basic Engineering, 82 (Series D): 35-45, 1960.

[4] F. Gustafsson, and A.J. Isaksson, “Best choice of coordinate system for tracking coordinated turns,” Proceedings of 35th IEEE Conference on Decision and Control, 1996.

[5] R. Schubert, E. Richter, and G. Wanielik, “Comparison and evaluation of advanced motion models for vehicle tracking,” Proceedings of the 11th International Conference on Information Fusion, 2008.

[6] X.R. Li, and V.P. Jilkov, “Survey of maneuvering target tracking. Part I: Dynamic models, ”IEEE Transactions on Aerospace and Electronic Systems, 39(4), 2003.

[7] S.J. Julier, and J.K. Uhlmann, “A New Extension of the Kalman Filter to Nonlinear Systems,” Signal Processing, Sensor Fusion, and Target Recognition VI, vol. 3068, Proc. SPIE, pp. 182–193, 1997

[8] A.H. Jazwinski, “Stochastic Processes and Filtering Theory,” Chapter 8, pp. 272–281, Academic, New York, 1970

[9] F. Gustafsson, “Statistical Sensor Fusion,” Studentlitteratur, 2018

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e^{f'_{\mathrm{CTRA}}\tau}\,\tilde{\mathbf{Q}}_{\mathrm{CTRA}}\,e^{f'^{T}_{\mathrm{CTRA}}\tau} =
\begin{bmatrix}
\left(\sigma_{\dot{\omega}}^2\, s^2 \sin^2\varphi + \sigma_{\dot{a}}^2 \cos^2\varphi\right)\frac{\tau^4}{4} & \left(\sigma_{\dot{a}}^2 - \sigma_{\dot{\omega}}^2\, s^2\right)\frac{\tau^4}{4}\sin\varphi\cos\varphi & \sigma_{\dot{a}}^2\,\frac{\tau^3}{2}\cos\varphi & -\sigma_{\dot{\omega}}^2\,\frac{\tau^3}{2}\, s\sin\varphi & -\sigma_{\dot{\omega}}^2\,\frac{\tau^2}{2}\, s\sin\varphi & \sigma_{\dot{a}}^2\,\frac{\tau^2}{2}\cos\varphi \\
\left(\sigma_{\dot{a}}^2 - \sigma_{\dot{\omega}}^2\, s^2\right)\frac{\tau^4}{4}\sin\varphi\cos\varphi & \left(\sigma_{\dot{\omega}}^2\, s^2 \cos^2\varphi + \sigma_{\dot{a}}^2 \sin^2\varphi\right)\frac{\tau^4}{4} & \sigma_{\dot{a}}^2\,\frac{\tau^3}{2}\sin\varphi & \sigma_{\dot{\omega}}^2\,\frac{\tau^3}{2}\, s\cos\varphi & \sigma_{\dot{\omega}}^2\,\frac{\tau^2}{2}\, s\cos\varphi & \sigma_{\dot{a}}^2\,\frac{\tau^2}{2}\sin\varphi \\
\sigma_{\dot{a}}^2\,\frac{\tau^3}{2}\cos\varphi & \sigma_{\dot{a}}^2\,\frac{\tau^3}{2}\sin\varphi & \sigma_{\dot{a}}^2\,\tau^2 & 0 & 0 & \sigma_{\dot{a}}^2\,\tau \\
-\sigma_{\dot{\omega}}^2\,\frac{\tau^3}{2}\, s\sin\varphi & \sigma_{\dot{\omega}}^2\,\frac{\tau^3}{2}\, s\cos\varphi & 0 & \sigma_{\dot{\omega}}^2\,\tau^2 & \sigma_{\dot{\omega}}^2\,\tau & 0 \\
-\sigma_{\dot{\omega}}^2\,\frac{\tau^2}{2}\, s\sin\varphi & \sigma_{\dot{\omega}}^2\,\frac{\tau^2}{2}\, s\cos\varphi & 0 & \sigma_{\dot{\omega}}^2\,\tau & \sigma_{\dot{\omega}}^2 & 0 \\
\sigma_{\dot{a}}^2\,\frac{\tau^2}{2}\cos\varphi & \sigma_{\dot{a}}^2\,\frac{\tau^2}{2}\sin\varphi & \sigma_{\dot{a}}^2\,\tau & 0 & 0 & \sigma_{\dot{a}}^2
\end{bmatrix}
\tag{43}
$$

$$
\mathbf{Q}_k^{\mathrm{CTRA}} =
\begin{bmatrix}
\left(\sigma_{\dot{\omega}}^2\, s_k^2 \sin^2\varphi_{\mathrm{k}} + \sigma_{\dot{a}}^2 \cos^2\varphi_{\mathrm{k}}\right)\frac{\mathrm{T}^5}{20} & \left(\sigma_{\dot{a}}^2 - \sigma_{\dot{\omega}}^2\, s_k^2\right)\frac{\mathrm{T}^5}{20}\sin\varphi_{\mathrm{k}}\cos\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^4}{8}\cos\varphi_{\mathrm{k}} & -\sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^4}{8}\, s_k\sin\varphi_{\mathrm{k}} & -\sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^3}{6}\, s_k\sin\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^3}{6}\cos\varphi_{\mathrm{k}} \\
\left(\sigma_{\dot{a}}^2 - \sigma_{\dot{\omega}}^2\, s_k^2\right)\frac{\mathrm{T}^5}{20}\sin\varphi_{\mathrm{k}}\cos\varphi_{\mathrm{k}} & \left(\sigma_{\dot{\omega}}^2\, s_k^2 \cos^2\varphi_{\mathrm{k}} + \sigma_{\dot{a}}^2 \sin^2\varphi_{\mathrm{k}}\right)\frac{\mathrm{T}^5}{20} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^4}{8}\sin\varphi_{\mathrm{k}} & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^4}{8}\, s_k\cos\varphi_{\mathrm{k}} & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^3}{6}\, s_k\cos\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^3}{6}\sin\varphi_{\mathrm{k}} \\
\sigma_{\dot{a}}^2\,\frac{\mathrm{T}^4}{8}\cos\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^4}{8}\sin\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^3}{3} & 0 & 0 & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^2}{2} \\
-\sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^4}{8}\, s_k\sin\varphi_{\mathrm{k}} & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^4}{8}\, s_k\cos\varphi_{\mathrm{k}} & 0 & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^3}{3} & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^2}{2} & 0 \\
-\sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^3}{6}\, s_k\sin\varphi_{\mathrm{k}} & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^3}{6}\, s_k\cos\varphi_{\mathrm{k}} & 0 & \sigma_{\dot{\omega}}^2\,\frac{\mathrm{T}^2}{2} & \sigma_{\dot{\omega}}^2\, T & 0 \\
\sigma_{\dot{a}}^2\,\frac{\mathrm{T}^3}{6}\cos\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^3}{6}\sin\varphi_{\mathrm{k}} & \sigma_{\dot{a}}^2\,\frac{\mathrm{T}^2}{2} & 0 & 0 & \sigma_{\dot{a}}^2\, T
\end{bmatrix}
\tag{44}
$$