APPM 5720 Homework 3

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1 Preliminaries

Consider a grid with $N + 1$ grid points $x_L = x_0 < x_1 < x_2 < \cdots < x_N = x_R$, defining N elements $\Omega_i = \{x \in [x_{i-1}, x_i]\}$, $i = 1, \ldots, N$. We seek to approximate a function $f(x)$ on $x \in [x_L, x_R]$ by L_2 projection onto the space of element-wise Legendre polynomials of degree q. This approximation takes the form

$$\int_{\Omega_i} P_l(r) \sum_{k=0}^{q} c(k, i) P_k(r) \, dx = \int_{\Omega_i} P_l(r) f(x) \, dx, \quad l = 0, \ldots, q.$$

where $c(i, k)$ are the coefficients of the L_2 projection. Here $r \in [-1, 1]$ is a local variable such that on element Ω_i the affine map satisfies $x(-1) = x_{i-1}, x(1) = x_i$. This affine map (explicitly) is

$$x(r) = \frac{1}{2}\left((x_i - x_{i-1})\, r + (x_i + x_{i-1})\right) \implies dx(r) = \frac{1}{2}(x_i - x_{i-1})\, dr.$$

Since the Legendre polynomials are orthogonal with respect to the weight functions $w(x) = 1$ and

$$\int_{-1}^{1} P_l(x) P_n(x) \, dx = \frac{2}{2n+1}\delta_{l,n},$$

the above system of equations reduces to

$$\int_{\Omega_i} P_l(r) \sum_{k=0}^{q} c(k, i) P_k(r) \, dx = \sum_{k=0}^{q} c(k, i) \int_{\Omega_i} P_l(r) P_k(r) \, dx$$

$$= \sum_{k=0}^{q} \frac{1}{2} c(k, i)\,(x_i - x_{i-1}) \int_{-1}^{1} P_l(r) P_k(r) \, dr$$

$$= \frac{1}{2} c(l, i)\,(x_i - x_{i-1})\, \frac{2}{2l+1}.$$

On the right hand side of our formulation, we approximate this using the Gauss-Lobato-Legendre weights and nodes w_j, r_j to obtain

$$\int_{\Omega_i} P_l(r)f(x)\,dx = \frac{1}{2}\left(x_i - x_{i-1}\right)\int_{-1}^{1} P_l(r)f(x(r))\,dr \approx \frac{1}{2}\left(x_i - x_{i-1}\right)\sum_{j=0}^{q} w_j P_l(r_j)f(x(r_j)).$$

We note that both sides have $1/2\left(x_i - x_{i-1}\right)$ so that our system of equations reduces to

$$c(l,i)\frac{2}{2l+1} = \sum_{j=0}^{q} w_j P_l(r_j)f(x(r_j)) \implies c(l,i) = \frac{2l+1}{2}\sum_{j=0}^{q} w_j P_l(r_j)f(x(r_j)), \quad l = 0,\dots,q.$$

That is, the set of N decoupled linear systems of equations of size $(q+1)\times(q+1)$ forms a diagonal matrix which can easily be solved without calling a routine that performs Gaussian elimination.

2 Results

We begin by considering a set of functions $f(x)$ and showing that with $q = 1, 2, \dots$, the approximation described above is increasingly accurate with an increasing number of elements as measured in the uniform and L_2 norm. In the following plots, note that $q = 1$ is in red and increasing q is colored according to the usual ordering of the color spectrum (i.e. ROYGBV applies). Let us first consider the function

$$f(x) = e^{(x-2)^2}, \quad x \in [0,4].$$

We note that this function is continuous and infinitely differentiable, and so we expect that our approximation scheme should increase in accuracy as we increase q and the number of elements. Indeed, this appears to be the case as demonstrated by the following plot of data. Consistently it appears taht for a fixed number of elements an increase in q decreases the error, and for fixed q an increase in number of elements (i.e. smaller average element sizes) decreases the overall error as expected.
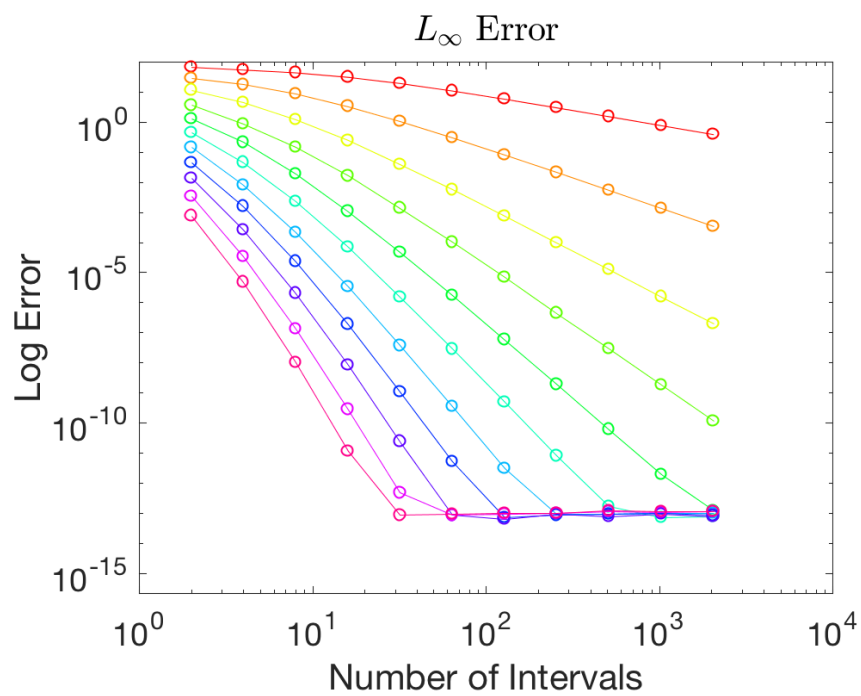
Figure 1: Uniform Norm Error of an Exponential

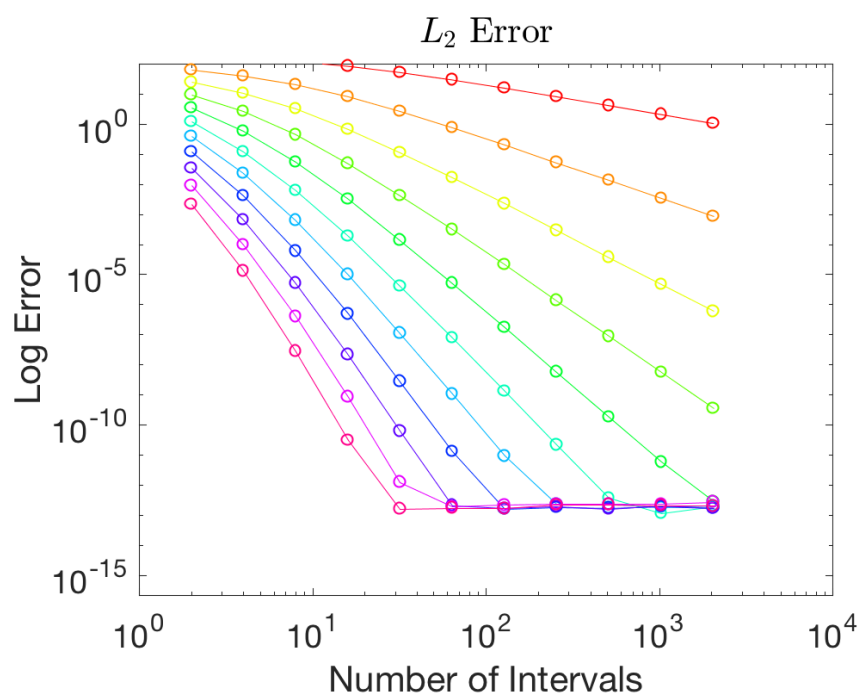The situation is much the same with respect to the L_2 error as depicted below.

Figure 2: Uniform Norm Error of an Exponential