APPM 5720 Homework 2

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1 Newton's Method

For this assignment, a basic perl script, `newtonS.p`, was modified by the group to take in `newtonS.f90.Template` and perform Newton's method on the set of sample functions

$$f(x) = e^x - 1\,, \quad g(x) = x^2,$$
$$h(x) = \sin(x)\,, \quad r(x) = e^{-1/x},$$

Equations obey grammar and punctuation rules.

where each function has a root at $x = 0$, save for $r(x)$ which is undefined at 0 though

$$\lim_{x \to 0^+} r(x) = \lim_{x \to 0^+} e^{-1/x} = 0\,.$$

We recall that Newton's method is quadratically convergent to a root under the assumption that the root is simple and the derivative is itself not 0 at the root of interest (and the second derivative is bounded). Let us examine each function individually.

NOTE: For cases where Newton's method required a significant number of iterations, only the first 15 iterations are used in the data tables that follow for the sake of brevity

1.1 $e^x - 1$

Try to start with a word both in headings and sentences.

This function meets the above assumptions for quadratic convergence. As such, it is observed that E_{n+1}/E_n^2 is approximately constant and Newton's method converges after seven iterations.

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^2
01	0.6321205588285577E+00	0.6321205588285577E+00
02	0.4869314375964384E+00	0.7703141921199606E+00
03	0.1894444060136254E+00	0.6154801567949590E+00
04	0.3031399383262888E-01	0.5198686660785536E+00
05	0.8848598680903540E-03	0.5005896024729687E+00
06	0.7820703579018673E-06	0.5000099574951582E+00
07	0.0000000000000000E+00	0.0000000000000000E+00

1.2 $f(x) = x^2$ Same as above.

We can see that this function has a double root at $x = 0$ and that the derivative of $f(x)$ is 0 at $x = 0$. Thus, using Newton's method we would expect linear convergence instead of quadratic convergence. Implementation of Newton's method shows that the ratio of errors from adjacent time steps, i.e. E_{n+1}/E_n, is a constant implying linear convergence.

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^2
01	0.5000000000000000E+00	0.5000000000000000E+00
02	0.5000000000000000E+00	0.1000000000000000E+01
03	0.5000000000000000E+00	0.2000000000000000E+01
04	0.5000000000000000E+00	0.4000000000000000E+01
05	0.5000000000000000E+00	0.8000000000000000E+01
06	0.5000000000000000E+00	0.1600000000000000E+02
07	0.5000000000000000E+00	0.3200000000000000E+02
08	0.5000000000000000E+00	0.6400000000000000E+02
09	0.5000000000000000E+00	0.1280000000000000E+03
10	0.5000000000000000E+00	0.2560000000000000E+03
11	0.5000000000000000E+00	0.5120000000000000E+03
12	0.5000000000000000E+00	0.1024000000000000E+04
13	0.5000000000000000E+00	0.2048000000000000E+04
14	0.5000000000000000E+00	0.4096000000000000E+04
15	0.5000000000000000E+00	0.8192000000000000E+04

Newton's method calculates the root after the $(k+1)^{th}$ iteration by the iterative formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

why?

Since we know that the root is a double root, we try modifying the formula by multiplying a factor of 2 to the 2nd term, i.e.

$$x_{n+1} = x_n - 2\frac{f(x_n)}{f'(x_n)}$$

This iterative formula ends up converging to the root $x = 0$ immediately on the first iteration irrespective of our initial guess! Say our initial guess is $x = x_0$. Then the first iteration would result in

$$\begin{aligned} x_1 &= x_0 - 2\frac{f(x_0)}{f'(x_0)} \\ &= x_0 - 2\frac{x_0^2}{2x_0} \\ &= 0 \end{aligned}$$

as demonstrated by the code which stopped after a single iteration.

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^2
01	0.1000000000000000E+01	0.1000000000000000E+01

1.3 $\sin(x)$

Because $\sin(x)$ only has single roots, one would expect Newton's method to converge quadratically as usual. When the code was run, it produced:

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^2
01	0.1523676741017902E+01	0.1523676741017902E+01
02	0.3877900367215199E+00	0.2545093892175937E+00
03	0.9689555030589873E-01	0.1639888607225156E+00
04	0.1090466714959705E-02	0.1904667166963419E-01
05	0.1299242582583448E-08	0.2081059578567191E-04
06	0.0000000000000000E+00	0.0000000000000000E+00

Neither ratio converges to a constant. However, when we check for cubic convergence, we get the following result:

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^3
01	0.1523676741017902E+01	0.1523676741017902E+01
02	0.3877900367215199E+00	0.1670363420049105E+00
03	0.9689555030589873E-01	0.2775395398051782E+00
04	0.1090466714959705E-02	0.3326792984270507E+00
05	0.1299242582583448E-08	0.3333333611137316E+00
06	0.0000000000000000E+00	0.0000000000000000E+00

As we can see, E_{n+1}/E_n^3 converges quickly to a constant. This means that Newton's method converges cubically for $f(x) = \sin(x)$. This might be because the sine function is almost linear near roots ($f''(x) = -\sin(x) = -f(x) = 0$ when $f(x) = 0$), so this should help speed up the convergence rate.

1.4 $e^{-1/x}$

We note that the derivative of $f(x) = e^{-1/x}$ is $f'(x) = \frac{e^{-1/x}}{x^2}$, which is undefined at 0. From this we can see that every derivative of $f(x) = e^{-1/x}$ is an exponential multiplied by a rational function with a singularity at 0 so that every derivative is undefined at the root. We then expect Newton's method to have extraordinarily slow convergence. A table containing the error ratios between successive iterations is below:

Iteration	E_{n+1}/E_n	E_{n+1}/E_n^2
01	0.1000000000000000E-03	0.1000000000000000E-03
02	0.9801000000000002E+00	0.9801000000000002E+04
03	0.9802980099999998E+00	0.1000202030405060E+05
04	0.9804920990079601E+00	0.1020506060403061E+05
05	0.9806823827241055E+00	0.1041012070006161E+05
06	0.9808689723299778E+00	0.1061720039624038E+05
07	0.9810519747073182E+00	0.1082629950052112E+05
08	0.9812314926443280E+00	0.1103741782460229E+05
09	0.9814076250301539E+00	0.1125055518381782E+05
10	0.9815804670383682E+00	0.1146571139703251E+05
11	0.9817501103001732E+00	0.1168288628654150E+05
12	0.9819166430679892E+00	0.1190207967797336E+05
13	0.9820801503700450E+00	0.1212329140019697E+05
14	0.9822407141565502E+00	0.1234652128523173E+05
15	0.9823984134379694E+00	0.1257176916816113E+05

We see that that the ratio of E_{n+1}/E_n is fairly constant and close to 1, which indicates very slow linear convergence as expected.