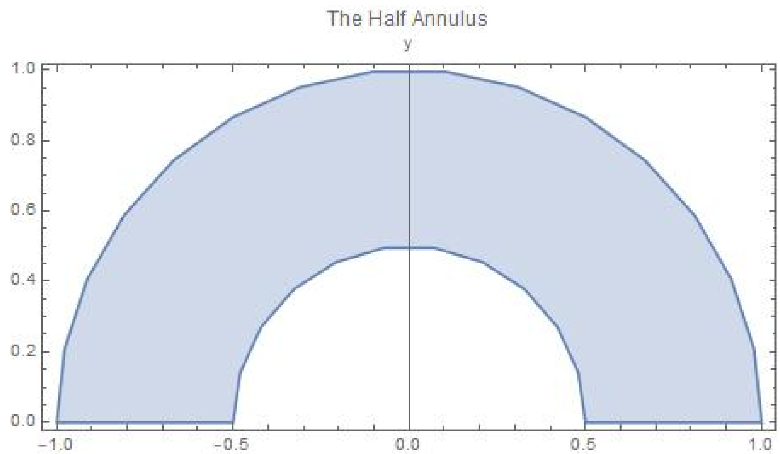
Task 5 Theory

Our half annulus can be defined as the region where $\frac{1}{2} \leq r \leq 1$ and $0 \leq \pi \leq 1$, plotted below:



Green's Theorem states that for most functions P(x,y) and Q(x,y):

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_C Pdx + Qdy$$

D is the region being integrated over and C is the positively oriented (counterclockwise) boundary of D. To compute the area of D, we usually integrate the function f(x,y)=1 over the region. However, we can also use Green's Theorem to compute area by choosing P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Any P and Q that satisfy this work and the area is equal to resulting line integral $\oint_C Pdx + Qdy$. For this problem, it is simplest to set P=0 and Q=x. This leaves us with the formula:

$$A = \oint_C xdy$$

The boundary of the annulus can be broken into 4 parts and parameterized as follows for $t_i \epsilon [0, 1]$:

i	$x(t_i)$	$y(t_i)$	$\frac{dy_i}{dt}$	$\vec{n}$
1	$cos(\pi t)$	$sin(\pi t)$	$\pi cos(\pi t)$	$\cos(\pi t)\hat{x} + sin(\pi t)\hat{y}$
2	$\frac{t}{2} - 1$	0	0	$-\hat{y}$
3	$-\frac{1}{2}cos(\pi t)$	$\frac{1}{2}sin(\pi t)$	$\frac{\pi}{2}cos(\pi t)$	$cos(\pi t)\hat{x} - sin(\pi t)\hat{y}$
4	$\frac{t}{2} + \frac{1}{2}$	0	0	$\hat{y}$

Curve 1 is the outer semicircle (oriented counterclockwise), curves 2 and 4 are the left and right edges along the x-axis, and curve 3 is the inner circle (oriented clockwise). When followed in order, they combine to make the boundary of the half annulus. Then our area $A = \oint_C xdy = \int_0^1 \sum_{i=1}^4 x_i(t)dy_i(t)dt$
$= \int_0^1 cos(\pi t) \cdot (\pi cos(\pi t)) + 0 \cdot (\frac{t}{2} - 1) - \frac{1}{2}cos(\pi t) \cdot (\frac{\pi}{2}cos(\pi t)) + 0 \cdot (\frac{t}{2} + \frac{1}{2})\ dt = \int_0^1 (\pi - \frac{\pi}{4}) \cdot cos^2(\pi t)dt$
$= \frac{3\pi}{4}[\frac{x}{2} - \frac{sin(2\pi x)}{4\pi}]_0^1 = \frac{3\pi}{4} \cdot (\frac{1}{2} - 0) = \underline{\frac{3\pi}{8} \approx 1.1781}.$

If we compute the area directly using the formula for the area of a half annulus (derived from the area of a circle), we get $A = \frac{\pi}{2} \cdot (1^2 - 0.5^2) = \frac{\pi}{2} \cdot \frac{3}{4} = \underline{\frac{3\pi}{8}}$. Therefore Green's Theorem works well for this example. For a general area, one can numerically compute and differentiate x and y then apply Green's Theorem. The unit normal vectors ($\vec{n}$) are also tabulated in closed form above and can generally be computed numerically, though they could be extremely inaccurate along a complicated boundary.