Topological completeness of S4

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Outline

The Basic Modal Language

Formal Languages

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We will first study a formal language which is called the basic modal language.

The Basic Modal Language: Syntax

The general Modal language helps us to formalize concepts of necessity - possibility, knowledge - belief, obligation - permission - prohibition, and time.[1]

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$$\mathcal{S} = \text{Set of symbols} = \{p, q, r, \ldots, \bot, \wedge, \neg, \Diamond, (,)\}$$

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The Basic Modal Language: Syntax (Cont'd)

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Formulas are special finite sequences (or strings) on the set of symbols.

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Some formulas: p, $\neg r$, $\Diamond\neg\bot$, $\neg(\bot \wedge (\Diamond p \wedge r))$, $\neg\neg\Diamond\neg q$.

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Some strings which are not formulas: $\bot\neg$, $pq \wedge \neg), \neg\neg\bot \wedge$.

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Writing parentheses is skipped, if the context is clear. For example, we may write $p \to \Box q$ instead of $(p \to \Box q)$.

The Kripke Semantics

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Examples

$(\mathbb{N}, \leq)$, $(\{x\}, \{(x,x)\})$ and $(\{x\}, \emptyset)$ are all examples of frames.

From Frames to Models

Let Φ denote the set of propositional variables, i.e.

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For a propositional variable p, $V(p) \subseteq W$. $V(p)$ should be thought of as points in W where p is 'true'.

Models: Example

Consider the frame $\mathfrak{F} = (W, R)$, where

$$W = \{1, 2, 3, 4\} \text{ and } R = \{(1, 2), (2, 3), (3, 4), (4, 2)\}.$$

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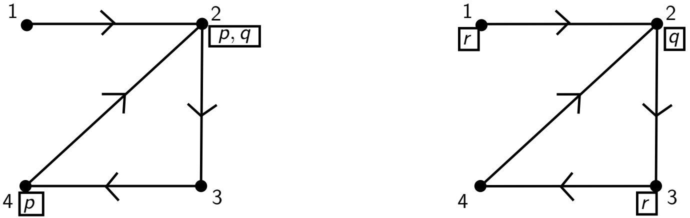

Figure: Two models based on the same frame $\mathfrak{F}$

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How do we talk about them?

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Recall that the set of propositional variables $(p, q, r, \ldots)$ is denoted by Φ.

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5. $\mathfrak{M}, w \vDash \Diamond\varphi$ iff there exists a $v \in W$ such that Rwv and $\mathfrak{M}, v \vDash \varphi$.

Truth and Satisfiability: An Example

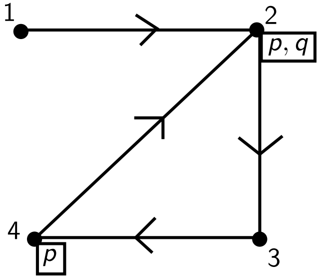

Figure: The model $\mathfrak{M}$

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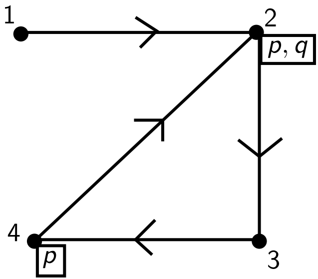

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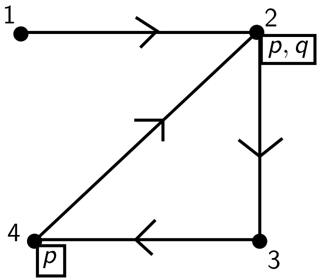

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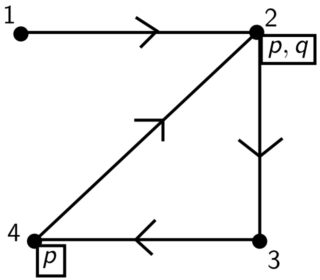

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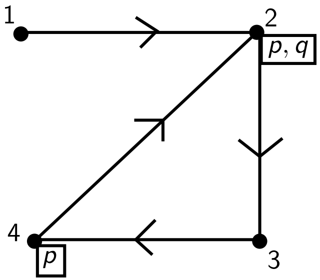

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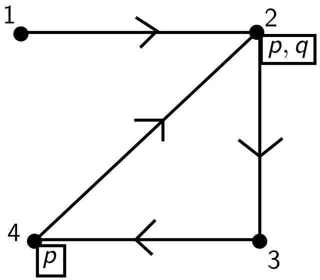

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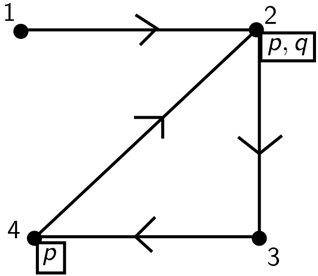

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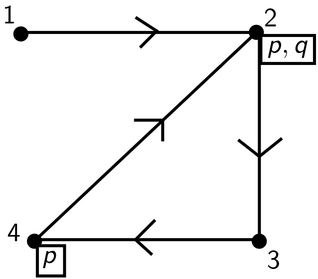

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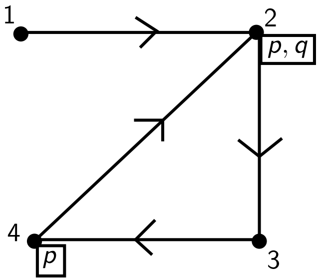

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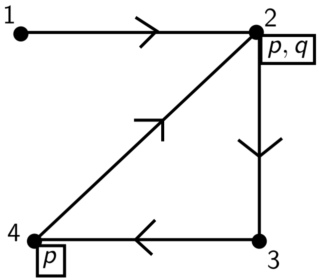

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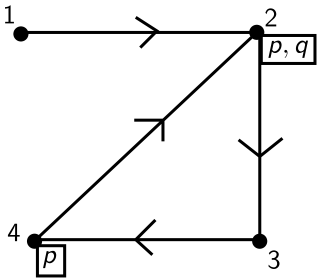

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Now we are able to express facts about the models using the formal language.

Unravelling the abbreviations

What would $\mathfrak{M}, w \vDash (\varphi \vee \psi)$ or $\mathfrak{M}, w \vDash \Box\varphi$ mean?

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\begin{aligned}
\mathfrak{M}, w \vDash (\varphi \vee \psi) &\Leftrightarrow \mathfrak{M}, w \vDash \neg(\neg\varphi \wedge \neg\psi) \\
&\Leftrightarrow \text{it's not that } \mathfrak{M}, w \vDash (\neg\varphi \wedge \neg\psi) \\
&\Leftrightarrow \text{it's not that both } \mathfrak{M}, w \vDash \neg\varphi \text{ and } \mathfrak{M}, w \vDash \neg\psi \\
&\Leftrightarrow \text{atleast one of } \mathfrak{M}, w \vDash \neg\varphi \text{ or } \mathfrak{M}, w \vDash \neg\psi \text{ doesn't hold} \\
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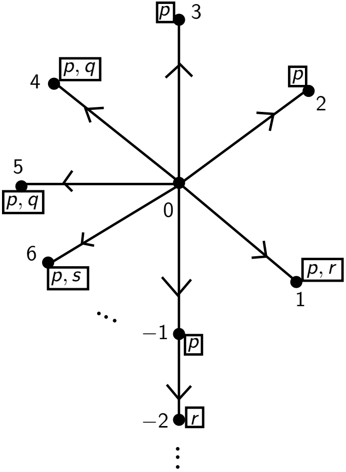
Unravelling the abbreviations: An Example



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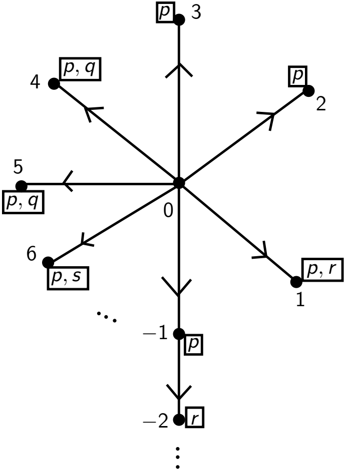

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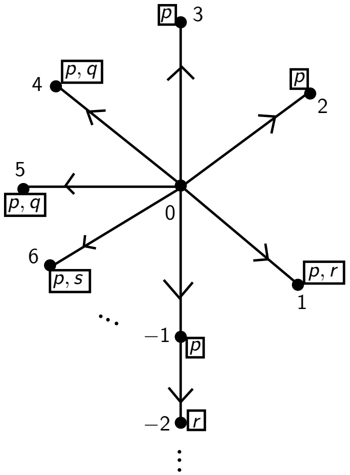

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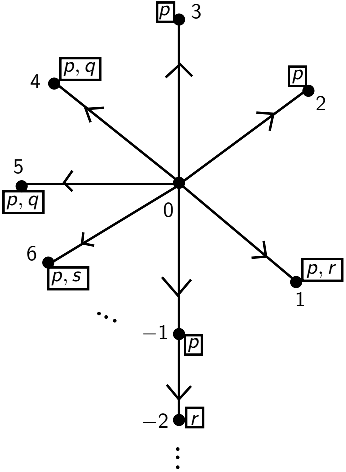

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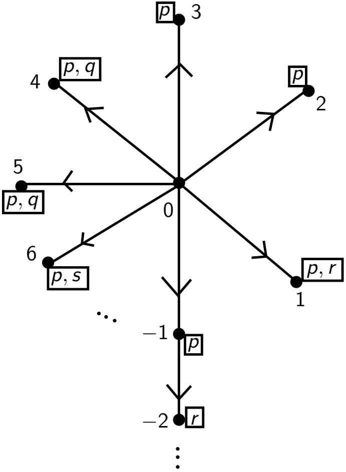

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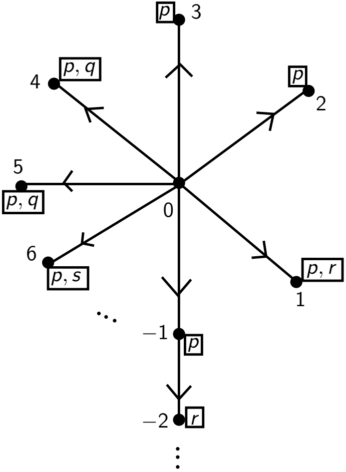

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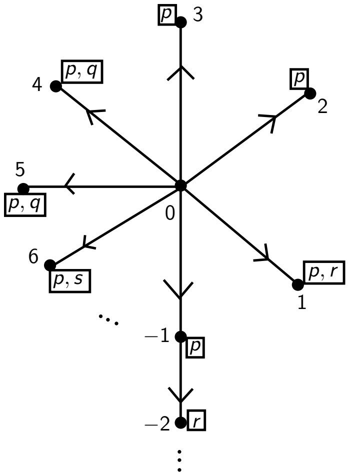

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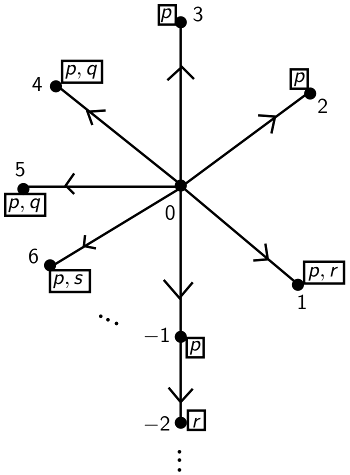

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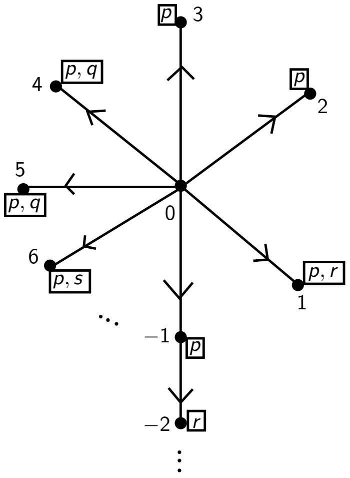

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Validity

Definition (Validity)

A formula φ is valid on a frame $\mathfrak{F} = (W, R)$ (notation $\mathfrak{F} \vDash \varphi$) if for all models $\mathfrak{M}$ based on $\mathfrak{F}$, we have $\mathfrak{M}, w \vDash \varphi$ for all states $w \in W$.

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Example

It can be checked that $p \to \Diamond p$ is valid on the class of all reflexive frames.

Kripke Completeness

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The formula

$$((\Box r \vee t) \wedge (\neg u \to \Box v)) \vee s$$

is a substitution instance of

$$((p \wedge q) \vee s),$$

as $((\Box r \vee t) \wedge (\neg u \to \Box v)) \vee s$ can be obtained from $((p \wedge q) \vee s)$ by uniformly substituting $\Box r \vee t$ for p, $\neg u \to \Box v$ for q and s for s.

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Propositional formulas are modal formulas which don't have an occurence of $\Diamond$ (or $\Box$).
Propositional tautologies are propositional formulas which are valid on every frame.

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The Basic Modal Language is an extension of the Sentential Language.

Examples

The formulas $p \vee \neg p$, $p \leftrightarrow \neg\neg p$, $(p \to q) \leftrightarrow (\neg q \to \neg p))$ are all examples of propositional tautologies.

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- generalization (i.e., if $\varphi \in \Lambda$, then $\Box\varphi \in \Lambda$).

If $\varphi \in \Lambda$, we say φ is a theorem of Λ (notation: $\vdash_\Lambda \varphi$).

Normal Modal Logics: Examples

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Thus, we have a smallest normal logic. We call it **K**.

For a collection of modal formulas Γ, the smallest normal logic containing Γ is denoted by $\mathbf{K}\Gamma$, which is the intersection of all normal logics which contain Γ.

This is called 'from the top down' approach.

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Consider the following construction:

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How does this help?

Soundness and Completeness

How can we relate normal logics to frames?

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Definition (Soundness)

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Remark

Thus, if Λ is sound with respect to F, then

$$\Lambda \subseteq \Lambda_{\mathsf{F}}.$$

Soundness and Completeness (Cont'd)

Example

The logic **K** is sound with respect to the class of all frames.

Soundness and Completeness (Cont'd)

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Key steps:

- Propositional tautologies, (K) and (Dual) are valid on the class of all frames.
- The property of being valid on the class of all frames is preserved under the rules of modus ponens, uniform substitution and generalization.

Soundness and Completeness (Cont'd)

Definition (Completeness)

A normal logic Λ is said to be complete with respect to a class of frames F, if every formula that is valid on F, is theorem of Λ, i.e.

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Thus, if Λ is complete with respect to F, then

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Thus, if a normal logic Λ is sound and complete with respect to a class of frames F, then we have,

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Some axioms:

$$\begin{array}{ll} (4) & \Diamond\Diamond p \to \Diamond p \\ (\mathrm{T}) & p \to \Diamond p \\ (\mathrm{B}) & p \to \Box\Diamond p \\ (\mathrm{D}) & \Box p \to \Diamond p \end{array}$$

It is customary to call **KT**, **KB**, **KT4** and **KT4B** as **T**, **B**, **S4** and **S5** respectively.

Soundness and Completeness (Cont'd)

K	the class of all frames
K4	the class of transitive frames
T	the class of reflexive frames
B	the class of symmetric frames
KD	the class of right-unbounded frames
S4	the class of reflexive, transitive frames
S5	the class of frames whose relation is an equivalence relation

Table: Some soundness and completeness results

Topological Semantics

Why move onto topology?

S4 has been defined to be the smallest normal logic containing the following axioms:

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Also, for an arbitrary subset Y of a topological space (X, τ), the following properties hold for the closure operator:

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We will see that in the topological semantics, $\Diamond$ and $\Box$ correspond to the closure and interior operators respectively.

A Topological Interpretation

Instead of frames and models, we will use the basic modal language to describe topological spaces.[3]

[3] Aiello, Pratt-Hartmann, van Bentham: Handbook of Spatial Logics (2007)

A Topological Interpretation

Instead of frames and models, we will use the basic modal language to describe topological spaces.[3]

- Frames will be replaced with topological spaces.
- Models will be replaced with topo-models.

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A Topological Interpretation

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Definition (Topo-models)

A topo-model is a 3-tuple (X, τ, v), where (X, τ) is a topological space and v is a function from Φ to $\mathcal{P}(X)$. Here v is said to be a valuation on X.

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Topo-models: An Example

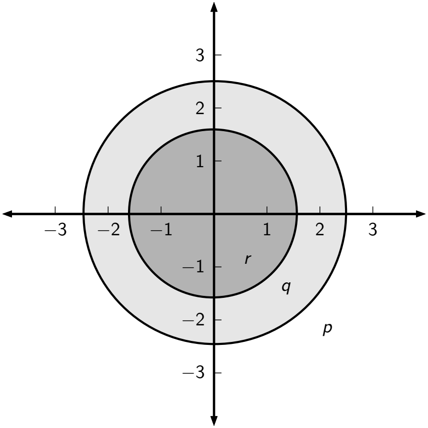

Figure: A topo-model based on $\mathbb{R}^2$

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Definition (Basic Topological Semantics)

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4. $M, x \vDash \Diamond\varphi$ iff for each $U \in \tau$ containing x, there exists a $y \in U$ such that $M, y \vDash \varphi$.

Remark

For any point x, if $M, x \vDash \varphi$, then $M, x \vDash \Diamond\varphi$.

An Example

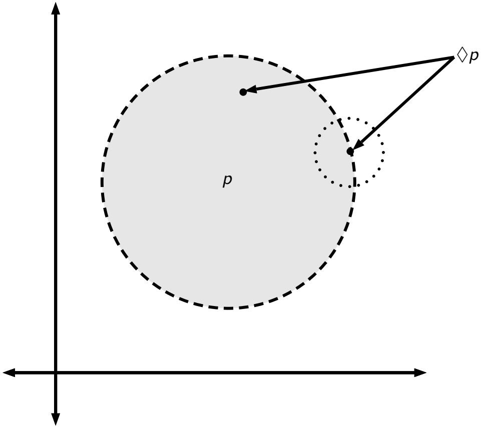

Figure: A topo-model based on $\mathbb{R}^2$

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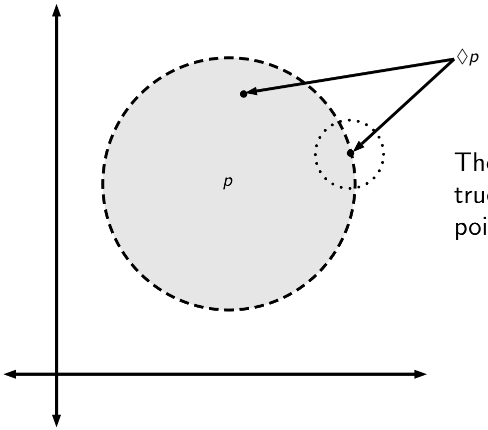

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The set of all points where $\Diamond p$ is true is the closure of the set of all points where p is true.

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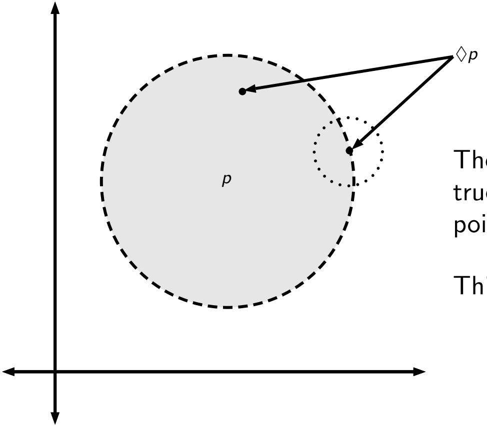

The set of all points where $\Diamond p$ is true is the closure of the set of all points where p is true.

This is not a coincidence.

Figure: A topo-model based on $\mathbb{R}^2$

$\Diamond$ as the Closure

Let $M = (X, \tau, v)$ be a topomodel. For a formula φ, let $[[\varphi]]$ denote all the points at which φ is true, i.e.

$$[[\varphi]] = \{x \in X \mid M, x \vDash \varphi\}.$$

Then, $y \in [[\Diamond\varphi]]$

$\Leftrightarrow$ for each $U \in \tau$ containing y, there exists some $z \in U$ such that $M, z \vDash \varphi$

$\Leftrightarrow$ for each $U \in \tau$ containing y, there exists some $z \in U$ such that $z \in [[\varphi]]$

$\Leftrightarrow$ for each $U \in \tau$ containing y, $U \cap [[\varphi]] \neq \emptyset$

$\Leftrightarrow$ $y \in$ Closure of $[[\varphi]]$.

Unravelling the Abbreviations

It can be checked that

- $M, x \models (\varphi \vee \psi)$ iff $M, x \models \varphi$ holds or $M, x \models \psi$ holds,
- $M, x \models (\varphi \rightarrow \psi)$ iff if $M, x \models \varphi$ holds, then $M, x \models \psi$ holds, and
- $M, x \models (\varphi \leftrightarrow \psi)$ iff either both $M, x \models \varphi$ and $M, x \models \psi$ hold, or both $M, x \not\models \varphi$ and $M, x \not\models \psi$ hold.

Unravelling the Abbreviations (Cont'd)

Also, $M, x \models \Box\varphi$

$\Leftrightarrow M, x \models \neg\Diamond\neg\varphi$

$\Leftrightarrow M, x \not\models \Diamond\neg\varphi$

$\Leftrightarrow$ it's not the case that for each $U \in \tau$ containing x, there exists a $y \in U$ such that $M, y \models \neg\varphi$

$\Leftrightarrow$ there exists some $U_0 \in \tau$ containing x, such that for each $z \in U_0$, we have $M, z \not\models \neg\varphi$

$\Leftrightarrow$ there exists some $U_0 \in \tau$ containing x, such that for each $z \in U_0$, we have $M, z \models \varphi$.

$\Box$ as the Interior

It can be checked that for a formula φ, we have

$$[[\Box\varphi]] = \text{ Interior of } [[\varphi]].$$

Also, we have the following:

$$\begin{aligned} [[\neg\varphi]] &= [[\varphi]]^c \\ [[\varphi \wedge \psi]] &= [[\varphi]] \cap [[\psi]] \\ [[\varphi \vee \psi]] &= [[\varphi]] \cup [[\psi]] \end{aligned}$$

Talking about spaces: An Example

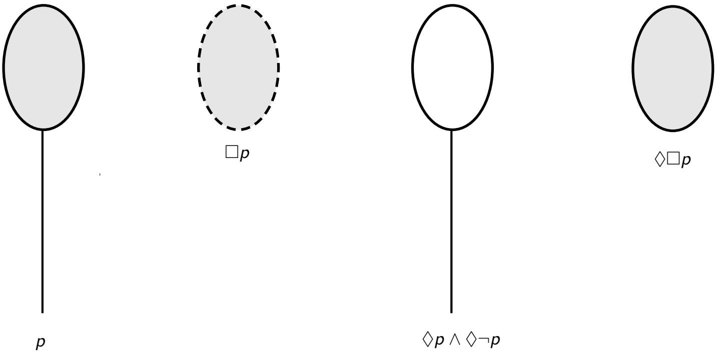

Figure: A spoon in $\mathbb{R}^2$

Validity

Definition (Validity)

A formula φ is valid on a topological space (X, τ) if φ is true at every point on every topo-model based on (X, τ) (notation: $(X, \tau) \vDash \varphi$).

A formula φ is valid on a class of topological spaces S if φ is valid on every member of S.

Validity: An Example

Example

The formula (Dual) given by

$$\Diamond p \leftrightarrow \neg\Box\neg p$$

which is just the abbreviation of

$$\Diamond p \leftrightarrow \neg\neg\Diamond\neg\neg p$$

is valid on the class of topological spaces,
as, for any topo-model,

- $M, x \vDash \Diamond p$ iff $x \in [[\Diamond p]]$ iff $x \in \mathrm{Cl}([[p]])$,
- $M, x \vDash \neg\neg\Diamond\neg\neg p$ iff $x \in [[\neg\neg\Diamond\neg\neg p]]$ iff $x \in \mathrm{Cl}([[p]]^{c\ c})^{c\ c}$.

Topological completeness of **S4**

Topological Soundess and Completeness

Definition (Topological Soundness)

A normal logic Λ is said to be sound with respect to a class of topological spaces S, if every theorem of Λ is valid on S, i.e.

$$\vdash_\Lambda \varphi \Rightarrow \mathsf{S} \vDash \varphi.$$

Definition (Topological Completeness)

A normal logic Λ is said to be complete with respect to a class of topological spaces S, if every formula that is valid on S, is theorem of Λ, i.e.

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Soundness of **S4**

Theorem

S4 is sound with respect to the class of all topological spaces.

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Key steps

- $$\begin{array}{ll} (\mathrm{K}) & \Box(p \to q) \to (\Box p \to \Box q), \\ (\mathrm{Dual}) & \Diamond p \leftrightarrow \neg\Box\neg p, \\ (\mathrm{T}) & p \to \Diamond p, \\ (4) & \Diamond\Diamond p \to \Diamond p. \end{array}$$

  and propositional validities are valid on the class of all topological spaces.

Soundness of **S4** (Cont'd)

Key steps (Cont'd)

- The property of being valid on the class of all topological spaces is preserved under the rules of modus ponens, uniform substitution and generalisation, i.e., on the class of all topological spaces

  1. if φ is valid and $\varphi \to \psi$ is valid, then ψ is valid (modus ponens),
  2. if φ is valid and ψ is a substitution instance of φ, then ψ is valid (uniform substitution), and
  3. if φ is valid, then $\Box\varphi$ is valid (generalisation).

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Hence, every theorem of **S4** is valid on the class of topological spaces.

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An equivalent definition of completeness is the following:

A normal logic Λ is complete with respect to a class of frames F, if every formula which is not in Λ, is not valid on F,

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Similarly, a normal logic Λ is complete with respect to a class of topological spaces S, if every formula which is not in Λ, is not valid on S,

i.e., if $\varphi \notin \Lambda$, then there exists a topo-model $M = (X, \tau, v)$ based on a topological space $(X, \tau) \in$ S and an $x \in X$ such that $M, x \nvDash \varphi$.

The Path to Completeness

It is known that **S4** is complete with respect to the class of reflexive, transitive frames (often called **S4**-frames).

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Using the model $\mathfrak{M} = (X, R, v)$, a topo-model $M = (X, \tau_R, v)$ will be constructed, such that for all formulas ψ,

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Consequently, $M, x_0 \nvDash \varphi$.

Upsets

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Let (X, R) be an **S4**-frame. A subset A of X is called an upset if for each $x, y \in X$, if $x \in A$ and Rxy holds, then $y \in A$.

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Upsets are subsets which are closed with respect to the relation R.

Upsets: Examples

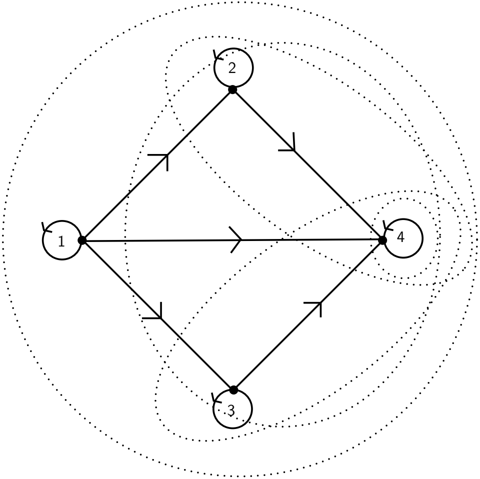

Figure: All the upsets (except $\emptyset$) of an **S4**-frame

Completeness of **S4**

Proposition

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$$\tau_R = \{ A \subseteq X \mid A \text{ is an upset}\},$$

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(X, τ_R) forms a topological space.

Lemma

Let $\mathfrak{M} = (X, R, v)$ be a model based on an **S4**-frame. Let M be the topomodel (X, τ_R, v). Then for all modal formulas φ and all $x \in X$ we have

$$\mathfrak{M}, x \vDash \varphi \text{ iff } M, x \vDash \varphi.$$

Completeness of **S4** (Cont'd)

Corollary

S4 is complete with respect to the class of all topological spaces.

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Steps

- For $\varphi \notin$ **S4**, there is a model $\mathfrak{M}$ based on an **S4**-frame (X, R), and $x_0 \in X$ such that $\mathfrak{M}, x_0 \not\models \varphi$.

- For the topo-model $M = (X, \tau_R, v)$, the previous lemma guarantees that $M, x_0 \not\models \varphi$.

Thus,

$$\mathbf{S4} = \{\text{Formulas that are valid on the class of all topological spaces}\}.$$

Goals for the even semester

- **McKinsey-Tarski Theorem**: **S4** is the logic of dense-in-itself, seperable metric-spaces.[4]

Many topological properties are not expressible in the basic modal language.

For example, we are not able to distinguish between the class of all topological spaces and the class of all dense-in-itself seperable metric spaces, only by looking at their corresponding modal logics which is **S4**.

- Study more expressive modal languages and interpretations which could capture these different properties of the spaces.

[4] McKinsey and Tarski: The Algebra of Topology, Annals of Mathematics (1944).

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Thank you!