4.4 Simplex Algorithm to Solve Minimization Problems

$$\begin{aligned} \min\ \ & z = 2x_1 - 3x_2 \\ \text{s.t.}\ \ & x_1 + x_2 \le 4 \\ & x_1 - x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

equivalent

Method 1

$$\begin{aligned} \max\ \ & -z = -2x_1 + 3x_2 \\ \text{s.t.}\ \ & x_1 + x_2 \le 4 \\ & x_1 - x_2 \le 6 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Initial Tableau

$-z$	x_1	x_2	s_1	s_2	rhs	BV
1	2	-3	0	0	0	$-z = 0$
0	1	1	1	0	4	$s_1 = 4$
0	1	-1	0	1	6	$s_2 = 6$

Optimal Tableau

Nonnegative
(Max problem)

$-z$	x_1	x_2	s_1	s_2	rhs	BV
1	5	0	3	0	12	$-z = 12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

<u>z = -12</u>

Method 2

$$\min \ z = 2x_1 - 3x_2$$
$$\text{s.t. } x_1 + x_2 \le 4$$
$$x_1 - x_2 \le 6$$
$$x_1, x_2 \ge 0$$

Initial Tableau

z	x_1	x_2	s_1	s_2	rhs	BV
1	-2	3	0	0	0	$z = 0$
0	1	1	1	0	4	$s_1 = 4$
0	1	-1	0	1	6	$s_2 = 6$

Optimal Tableau

Nonpositive

z	x_1	x_2	s_1	s_2	rhs	BV
1	-5	0	-3	0	-12	$z = -12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Optimal (Min Problem)

Nonpositive coefficient of NBV in Row 0

$$z - 5x_1 - 3s_1 = -12$$
$$z = -12 + 5x_1 + 3s_1$$

4.5 Alternative Optimal Solution

$\max \; z = 60x_1 + 35x_2 + 20x_3$

Optimal Tableau

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	BV
1	0	0	0	0	10	10	0	280	z = 280
0	0	-2	0	1	2	-8	0	24	$s_1 = 24$
0	0	-2	1	0	2	-4	0	8	$x_3 = 8$
0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2*$
0	0	1	0	0	0	0	1	5	$s_4 = 5$

Another Optimal Tableau

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	BV
1	0	0	0	0	10	10	0	280	z = 280
0	1.6	0	0	1	1.2	-5.6	0	27.2	$s_1 = 27.2$
0	1.6	0	1	0	1.2	-1.6	0	11.2	$x_3 = 11.2$
0	0.8	1	0	0	-0.4	1.2	0	1.6	$x_2 = 1.6$
0	-0.8	0	0	0	0	-1.2	1	3.4	$s_4 = 3.4$

4.6 Unbounded LPs

$$\max \ z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$
$$\text{s.t.} \ x_1 + x_2 - x_3 \le 5$$
$$6x_1 + 5x_2 - x_4 \le 10$$
$$x_1, x_2, x_3, x_4 \ge 0$$

Initial Tableau

z	x_1	x_2	x_3	x_4	s_1	s_2	rhs	BV
1	-36	-30	3	4	0	0	0	$z=0$
0	1	1	-1	0	1	0	5	$s_1=5$
0	6	5	0	-1	0	1	10	$s_2=10$

First Tableau

z	x_1	x_2	x_3	x_4	s_1	s_2	rhs	BV
1	0	0	3	-2	0	6	60	$z=60$
0	0	1/6	-1	1/6	1	-1/6	10/3	$s_1=10/3$
0	1	5/6	0	-1/6	0	1/6	5/3	$x_1=5/3$

Second Tableau

z	x_1	x_2	x_3	x_4	s_1	s_2	rhs	BV
1	0	2	-9	0	12	4	100	$z=100$
0	0	1	-6	1	6	-1	20	$x_4=20$
0	1	1	-1	0	1	0	5	$x_1=5$

Impossible to do ratio test

➡ Arbitrarily large z values

4.10 How to make Standard Form (Big M Method)

$$\min \ z = 2x_1 + 3x_2$$
$$\text{s.t.} \ 1/2\,x_1 + 1/4\,x_2 \le 4$$
$$x_1 + 3x_2 \ge 20$$
$$x_1 + x_2 = 10$$
$$x_1, x_2 \ge 0$$

$$z - 2x_1 - 3x_2 = 0$$
$$1/2\,x_1 + 1/4\,x_2 + s_1 = 4$$
$$x_1 + 3x_2 - e_2 = 20 \quad \textit{Excess variable}$$
$$x_1 + x_2 = 10 \quad \textit{Equality}$$
$$x_1, x_2, s_1, e_2 \ge 0$$

if $x_1, x_2 = 0$

How solve?

$$z - 2x_1 - 3x_2 = 0$$
$$1/2\,x_1 + 1/4\,x_2 + s_1 = 4$$
$$x_1 + 3x_2 - e_2 + a_2 = 20$$
$$x_1 + x_2 + a_3 = 10$$
$$x_1, x_2, s_1, e_2 \ge 0$$

Artificial variables a_2, a_3

But, artificial variables should be zero in the optimal solution.

4.11 Two-Phase Simplex Method

$$z - 2x_1 - 3x_2 = 0$$
$$1/2\, x_1 + 1/4\, x_2 + s_1 = 4$$
$$x_1 + 3x_2 - e_2 + a_2 = 20$$
$$x_1 + x_2 + a_3 = 10$$
$$x_1, x_2, s_1, e_2 \geq 0$$

Phase I LP

min $w' = a_2 + a_3$

s.t. $1/2\, x_1 + 1/4\, x_2 + s_1 = 4$

$x_1 + 3x_2 - e_2 + a_2 = 20$

$x_1 + x_2 + a_3 = 10$

New Row 0 $\quad w' + 2x_1 + 4x_2 - e_2 = 30$

*eliminate artificial variables from Row 0

Phase II LP

Eliminate <u>column of artificial variables</u> from optimal tableau of phase I and continue simplex method

Initial Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	-2	-3	0	0	0	0	$z = 0$
Row w'	0	1	2	4	0	-1	0	0	$w' = 30$
	0	0	1/2	1/4	1	0	0	0	$s_1 = 4$
	0	0	1	3	0	-1	1	0	$a_2 = 20$
	0	0	1	1	0	0	0	1	$a_3 = 10$

Next Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	-1	0	0	-1	1	0	$z = 20$
Row w'	0	1	2/3	0	0	1/3	-4/3	0	$w' = 10/3$
	0	0	5/12	0	1	1/12	-1/12	0	$s_1 = 7/3$
	0	0	1/3	1	0	-1/3	1/3	0	$x_2 = 20/3$
	0	0	2/3	0	0	1/3	-1/3	1	$a_3 = 10/3$

Optimal Tableau of Phase I

	z	w'	x_1	x_2	s_1	e_2	a_2	a_3	rhs
Row z	1	0	0	0	0	-1/2	1/2	3/2	$z = 25$
Row w'	0	1	0	0	0	0	-1	-1	$w' = 0$
	0	0	0	0	1	-1/8	1/8	-5/8	$s_1 = 1/4$
	0	0	0	1	0	-1/2	1/2	-1/2	$x_2 = 5$
	0	0	1	0	0	1/2	-1/2	3/2	$x_1 = 5$

Initial Tableau of **Phase II**

	z	w'	x_1	x_2	s_1	e_2	rhs
Row z	1	0	0	0	0	-1/2	$z = 25$
	0	0	0	0	1	-1/8	$s_1 = 1/4$
	0	0	0	1	0	-1/2	$x_2 = 5$
	0	0	1	0	0	1/2	$x_1 = 5$